MichU
DeptE
CenREST
W
#95-10

Department of Economics

and

Center for Research on Economic and Social Theory

Working Paper Series

When is the Standard Analysis of Common Property Extraction Under Free Access Correct?---A Game-Theoretic Justification for Non Game-Theoretic Analyses

Robin Brooks, Jill Contraneo, Michael Murray, and Stephen Salant

October, 1995
Number 95-10

DEPARTMENT OF ECONOMICS
University of Michigan
Ann Arbor, MI 48109-1220

SUMNER AND LAURA FOSTER LIBRARY
JAN 17

When is the Standard Analysis of Common Property Extraction Under Free Access Correct?---A Game-Theoretic Justification for Non Game-Theoretic Analyses

Robin Brooks, Department of Economics, Bates College
Jill Controneo, Department of Economics, University of Chicago
Michael Murray, Department of Economics, Bates College
Stephen Salant, Department of Economics, University of Michigan

Abstract

Analyses of common property extraction under "free access" used to *assume* period-by-period rent dissipation thus avoiding the use of game theory; more modern analyses instead *deduce* the subgame perfect Nash equilibrium of the common property game and then investigate its free-access limit. Salant and Negri (1987) provided a troubling example where these two methodologies yield radically different predictions: while the older analysis predicts eventual extinction of the resource, the game-theoretic analysis predicts unlimited growth. We review and simplify their example and then provide weak conditions insuring that the two methodologies yield the same predictions.

JEL: C72, Q20
Keywords: common property, free access, Nash equilibrium, rent dissipation , subgame perfect equilibrium

WHEN IS THE STANDARD ANALYSIS OF COMMON PROPERTY EXTRACTION UNDER FREE ACCESS CORRECT? -- A GAME THEORETIC JUSTIFICATION FOR NON GAME-THEORETIC ANALYSES

Introduction

Analyses of common property extraction follow two distinct paths: analyses for small numbers of extractors that use game theoretic solution concepts and analyses for large numbers of extractors that assume period-by-period rent dissipation and make no use of game theory. Levhari and Mirman (1980) formulated the first dynamic game of common property extraction under restricted access; Eswaran and Lewis (1985), Mirman (1979), Reinganum and Stokey (1985) have also applied game theory to the restricted-access case.

Gordon (1954) first used the rent dissipation assumption to analyze common property extraction under free access. His approach has become the standard for analyzing the free-access case, appearing in both undergraduate (Hartwick and Olewiler (1986)) and graduate (Dasgupta and Heal 1979) textbooks. Indeed, even the pioneers of the game-theoretic models eschew analyzing the free-access case as the limit of Markov-perfect equilibria[1] as the number of extractors goes to infinity, in favor of using the rent dissipation assumption (Levhari, Michner, and Mirman (1981)).

The question naturally arises whether the game-theoretic analysis always gives the same predictions in the free-access limit as the older-style analysis. In an unpublished note, Salant and Negri (1987) provided the answer: an example where the predictions of the two methodologies differ. Indeed for certain parameter values of the Salant-Negri example, rent dissipation implies that the resource will become extinct while game theoretic-analysis provides a solution in which the exact opposite occurs--that the stock grows without bound.

Our purpose here is to provide conditions sufficient to ensure that the two methodologies for analyzing the free-access case yield the *same* predictions. Since these conditions strike us as easily satisfied, we tend to regard Salant and Negri's troubling example as pathological although highly valuable in illuminating what our sufficient conditions exclude. Others, however, do not regard the biological growth assumptions underlying Salant and Negri's example as pathological. For example, Rosen (1987) used precisely the same biological growth function as Salant-Negri in his competitive equilibrium model of holding animal stocks as assets. Our sufficient condition is violated in such cases and, if any are economically relevant, our analysis should be taken as a warning that assuming rent dissipation in the free access limit may be inappropriate.

In the next section, we reformulate Salant and Negri's example. In the free-access limit, there exists a Markov-perfect equilibrium in which neither industry profits (the form of rent dissipation assumed by Gordon) nor the marginal value of additional stock remaining (the form of rent dissipation assumed by Levhari, Michener, and Mirman) approaches zero. We then identify the source of the difficulty in the example and--using this observation--provide conditions sufficient in general for the two approaches to yield the same conclusion.

The Example

Consider a common property extracted under free access. Suppose the average cost in period t for any individual extractor is:

$$(1) \qquad A_t(y_t, x_t) = x_t + y_t$$

where y_t is the individual's current extraction as a fraction of the total current stock, and x_t is everyone else's current extraction as a fraction of the current stock. . Both x_t and y_t are constrained to be non-negative and $x_t + y_t \leq 1$.[2]

Let S_t denote the non-negative stock at the beginning of the current period t. Suppose that the stock S_{t+1} is related to S_t by

$$(2) \qquad S_{t+1} = gS_t - (x_t + y_t)\,S_t = (g - x_t - y_t)\,S_t \qquad \text{for } t \geq 0$$

where g is a constant growth factor.

The present discounted value of returns from all periods, if it converges, is:

$$(3) \qquad F(\{x_t\}, \{y_t\}) = \sum_{t=0}^{\infty} \beta^t\,[Py_t - y_t\,(x_t + y_t)]\,S_t$$

where β is the discount rate , $0 < \beta < 1$, and P is the price of the extracted resource, which in this example we assume to be constant over time.[3] A sufficient condition for the sum in (3) to converge is that

$$(4) \qquad x_t > b \equiv g - 1/\beta \qquad \text{for all } t.$$ [4]

We therefore assume (4). Condition (4) also ensures that a maximum exists for F with respect to the sequence $\{y_t\}$.[5]

Though many Markov-perfect equilibria may exist for this example, we simplify the analysis by confining our search to those equilibria in which x_t is independent of t. That is, we restrict attention to equilibria in which all other extractors extract a constant proportion x of the stock in every period.

When x_t is constant at some value x independent of time, we will show that the optimal y_t for any one extractor proves to be independent of time also ($y_t = y$ for all $t = 0, 1, \ldots$). Then there turn out to be two pure-strategy, symmetric, Markov-perfect equilibria in which strategies are independent of time. We find both of these equilibria by using a recursive approach and a simple graphical analysis.

By induction, the growth relationship (2) implies that

$$S_t = S_0 \prod_{j=0}^{t-1} (g - x - y_j) \qquad \text{for } t \geq 1.$$

Using this expression and the convenient notation $R(y_t) = Py_t - (x + y_t)y_t$ the discounted profit of a single extractor in response to the repeated extraction x of the others becomes

$$(5) \qquad F(x, \{y_t\}) = S_0\,[R(y_0) + \sum_{t=1}^{\infty} \beta^t R(y_t) \prod_{j=0}^{t-1} (g - x - y_j)]\,.$$

Clearly, the optimal sequence $\{y_t\}$ is independent of the initial stock size S_0. But since none of the other parameters of the problem (P, β, g, and x) changes with time, this implies that no matter what the single extractor does in period 0 , the extractor's problem in period 1 is exactly the same as it was in period 0 . In fact, in period 1 , the extractor's problem is to maximize his stream of profits discounted to period 1 :

$$(6) \qquad S_1\,[R(y_1) + \sum_{t=1}^{\infty} \beta^t R(y_{t+1}) \prod_{j=0}^{t-1} (g - x - y_{j+1})]$$

which is exactly the same as (5) except that each y_t has been replaced by y_{t+1} and S_0 has been replaced by S_1. Thus if y^* is an optimal value for y_0 in (5) it must also be optimal for y_1 in (6). More precisely, if there is an optimal sequence whose first element is y^*, then there is one whose first two elements are y^*. Cotroneo (1994) shows that any optimal value for y_0 and similarly for y_1 and y_2 etc. is unique.[6] It follows that we *must* have the optimal values for y_0, y_1, y_2 etc. all equal to the same y^*.

When we rewrite (5) with constant x and y, we have an ordinary geometric series:

$$F(x,y) = S_0 \sum_{t=0}^{\infty} R(y)\,(\beta g - \beta x - \beta y)^t = S_0(1 - \beta g + \beta x + \beta y)^{-1} R(y)$$

which permits easy derivation of the optimal y. (Recall that convergence is guaranteed by (4)). Analysis of the derivative of $F(x,y)$ with respect to y shows that the optimal y for given x (the "reaction function") is:[7]

$$(7) \qquad y^* = \begin{cases} -(x-b) + [(x-b(P-b)]^{1/2} & \text{if } x \le P \\ 0 & \text{if } x > P \end{cases}$$

where $b = g - 1/\beta$, as in condition (4). Figure 1 shows the optimal reaction function when $b > 0$ and $x \le P$. (If $x > P$, then rivals extract so much that average cost exceeds price no matter what extraction the individual chooses and his best response is to extract nothing. Since this situation cannot arise in a symmetric equilibrium, it is of no further interest.) In the case depicted in figure 1, $b > 0$. When x is either b or P then y^* is

zero, which accounts for the x intercepts in Fig. (1). Condition (4) and $x \le P$ ensure that P lies to the right of b.

Also shown in Fig. (1) is the line

$$(8) \qquad y = x/(n-1)$$

where n is the number of extractors. Symmetric Markov-perfect equilibria are found where this line intersects the reaction function for an individual extractor. If $b < 0$, the ray intersects the reaction function only once. If $b > 0$, then the ray intersects either twice or not at all. If for a given n the graphs of Eqs. (7) and (8) do not intersect, then n is not large enough for condition (4) to be met by a symmetric pattern of outputs. (There may be other Markov perfect equilibria; only symmetric ones are precluded by the non-intersection of the graphs.) As n approaches infinity, the ray approaches the horizontal axis yielding two Markof-perfect equilibria in the free-access limit. In both, y approaches zero. In one, x approaches P from below. In this case, as x approaches P, the payoffs both to the individual extractor and the industry approach zero, which matches the rent-dissipation assumptions of the traditional analysis. In the other, x approaches b from above. As x approaches b from above, the individual extractor's profits approach the finite, positive value $S_0(P-b)/\beta$, so the marginal value of stock approaches $(P-b)/\beta$), while the industry's payoff approaches infinity. This obviously violates the traditional assumption that rent is dissipated under free access. Both individual profits and the marginal value of stock fail to approach zero. Hence, there is a Markov-perfect symetric equilibrium the limit of which as the number of extractors approaches does *not* involve rent- dissipation.

Salant and Negri focus on this equilibrium in order to emphasize the disparity between the non game-theoretic and game-theoretic predictions. Gordon's rent-dissipation approach predicts that $x + y$ approaches P under free access and the period t stock (S_t $(g-x-y)^{t-1}S_0$) approaches $(g-P)^{t-1}S_0$. Thus, if $0 < g-P < 1$, then the stock approaches extinction as time t increases. In contrast, in equilibrium 1, the limiting extraction (as the number of extractors becomes large) is $b = g - 1/\beta$ so at time t the stock level is $(g - b)^{t-1}S_0$ $(1/\beta)^{t-1}S_0$ which grows without limit as t becomes large. Instead of extinction, game-theoretic analysis predicts that society will be overrun by stock.

Conditions Sufficient for the Traditional Approach to be Valid

Recall that when $b < 0$ in the Salant-Negri example, the *only* symmetric Markov-perfect equilibrium is equilibrium 2, the one in which rent is dissipated in the free-access limit. Notice that this equilibrium exists even for the sole owner case (n=1). Since $x = 0$ in that case, the stationary extraction occurs where the reaction function intersects the vertical axis. The peculiar equilibrium occurs only when $b > 0$.[8] In that case the reaction curve does not intersect the vertical axis, and as Gale showed, the sole owners discounted profit is unbounded.[9] Nonetheless for n sufficiently large, the ray cuts the reaction function twice, and two symmetric equilibria exist.

Although no solution to the sole owner's problem exists when $b > 0$, the common property externality gives firms an incentive to extract more today in the aggregate than a

single agent would. The scramble for the resource serves as a substitute for increased impatience and can cause the equilibrium to exist where the single-agent optimum does not.

If we require that the sole owners discounted profits be bounded, then in this particular example that implies that $b < 0$. In this case, there is only one symmetric Markov-perfect equilibrium, the one involving rent-dissipation in the free-access limit. In what follows, we show that this requirement (along with subsidiary assumptions) is sufficient to ensure more generally that the game theoretic approach and traditional approach both yield the same results.

These sufficient conditions ensure that in an infinite-horizon restricted access common property game with n extractors, if $Q_n^*(S)$ is the aggregate extraction corresponding to a symmetric Markov-perfect equilibrium and an initial stock level S , then as the number n becomes large, $Q_n^*(S)$ converges to an unique $Q_\infty^*(S)$. Moreover the limit $Q_\infty^*(S)$ dissipates all rent. This justifies the traditional analysis of dynamic problems with free access which makes no use of game theory.

In stationary problems, a very general way to express the wealth of an extractor as a function of his current extraction q , the current agrregate extraction of his rivals Q^- , and the current stock S is

$$\pi_n(q,S) = P(Q^- + q)q - A(Q^- + q, S)q + W_n(Q^- + q, S) \qquad (9)$$

where $P(Q)$ is this period's price, $A(Q,S)$ is this period's average cost, and $W_n(Q,S)$ is the wealth the extractor expects to earn from next period onward in current terms if the current stock is S and the aggregate current extraction by all n firms is Q. The function W_n is actually the composition $V_n(G(Q,S))$ of two functions G and V_n, where $G(Q,S)$ is a growth function giving next period's stock and $V_n(G(Q,S))$ is its value discounted to the current period.

For each number n of extractors, we assume there is a stationary, symmetric Markov-perfect equilibrium, and that $Q_n{}^*(S)$ is the corresponding aggregate extraction for the current period. Thus, if we substitute $(n-1)Q_n{}^*(S)/n$ for Q^- in (9), then (9) is maximized with respect to q when $q = Q_n{}^*(S)/n$.

We will show, under certain assumptions, that as n becomes large $Q_n{}^*(S)$ tends to an unique extraction $Q_\infty{}^*(S)$ satisfying $A(Q_\infty{}^*(S),S) = P(Q_\infty{}^*(S))$, i.e., average cost equals price, so all rent is dissipated. Hence, under these conditions, the limit of the Markov-perfect equilibria as the number of extractors becomes large, predicts the same aggregate extraction as the one given by Gordon's rent dissipation rule. Moreover, under additional assumptions, the $Q_n{}^*(S)$ and $Q_\infty{}^*(S)$ are continuous functions of S and convergence is uniform in S.

The most important assumption rules out examples like the one in the previous section: we assume that even in the case where there is only one extractor, the present value of all future extraction is a finite value bounded above by some number $V_{max}(S)$. This has the important consequence that in a symmetric equilibrium with n extractors, $V_n(G(Q,S))$ is bounded above by $V_{max}(S)/n$. An intuitive candidate for the upper bound $V_{max}(S)$ would be the value of future profits discounted to the current period if the resource were entirely owned and exploited by a sole owner.

Our other assumptions are that P, A, G, and V_n are non-negative, continuous, and have continuous first derivatives; $P(Q)$ is non-increasing; industry-wide total revenue $P(Q)Q$ is a weakly concave function of Q (non-increasing marginal revenue); average cost $A(Q,S)$ is a strictly increasing function of Q; industry-wide total cost $A(Q,S)Q$ is a strictly convex function of Q (strictly increasing marginal cost); next period's stock level $G(Q,S)$ is a weakly decreasing weakly concave function of this period's extraction[10] Q, and V_n is a weakly increasing weakly concave function of next period's stock level. We also assume that $P(S) < A(S,S)$, so it is not profitable to extract all the stock in this period, and that $P(0) > A(0,S)$, so it is profitable to extract at least a little of the stock (although it may be even more profitable to save it all for the future). We are, of course, only concerned with the case where the stock level is positive, $S > 0$.

Since V_n is weakly concave and weakly increasing, and $G(Q,S)$ is weakly concave and weakly decreasing in Q, it follows that the composite $W_n(Q,S) = V_n(G(Q,S))$ is weakly concave and weakly decreasing in Q. Since both G and V_n are differentiable, so is W_n.

In Markov-perfect equilibrium, the individual extractor will choose a value of q in the interval $0 \le q \le S-Q^-$ to maximize π_n given by (9). We therefore compute the derivative of π_n with respect to q:

$$(10) \quad \pi_n{}'(q,S) = P(Q^-+q)+P'(Q^-+q)q-A(Q^-+q,S)-A'(Q^-+q,S)q+W_n{}'(Q^-+q,S)$$

where the primes on π_n, A and W denote differentiation with respect to the first argument.

Given our curvature assumptions, π_n is a strictly concave function of q, so the right side of (10) is a strictly decreasing function of q. We will show that if extraction is symmetric, then for sufficiently large n we have $\pi_n'(0,S) > 0$ and $\pi_n'(S-Q^-,S) < 0$, so π will be maximized at an unique q^* with $0 < q^* < S-Q^-$. Since this is an interior point, we will then have $\pi_n'(q^*,S) = 0$.

For symmetric extraction, we have $q = Q/n$, so (10) becomes

(11) $\pi_n'(q,S) = P(Q) + P'(Q)Q/n - A(Q,S) - A'(Q,S)Q/n + W_n'(Q,S)$.

We will rewrite (11) in terms of industry-wide marginal revenue $M_R(Q)$ and industry-wide marginal cost $M_C(Q,S)$. Since total revenue is $P(Q)Q$, marginal revenue is $M_R(Q) = P(Q) + P'(Q)Q$. Thus $P'(Q)Q/n = M_R(Q)/n - P(Q)/n$. Similarly, $A'(Q,S)Q/n = M_C(Q,S)/n - A(Q,S)/n$. Substituting these into (11) gives

(12) $\pi_n'(q,S) = (P(Q)-A(Q,S))(n-1)/n + (M_R(Q)-M_C(Q,S))/n + W_n'(Q,S)$.

For fixed Q and S the first term on the right clearly goes to $P(Q)-A(Q,S)$ as n becomes large. The second term goes to 0, since its numerator is independent of n. We'll show that the last term also goes to 0.

Since $P(S) < A(S,S)$, and $S > 0$ the continuity of A and P implies that we may choose Q_{max} so that $0 < Q_{max} < S$ and $P(Q_{max}) < A(Q_{max},S)$ also. Then because $W_n(Q,S)$ is a weakly decreasing, weakly concave, and non-negative function of Q,

$$0 \geq W_n'(Q,S) \geq (W_n(S,S)-W_n(Q_{max},S))/(S-Q_{max}) \geq -W_n(Q_{max},S)/(S-Q_{max})$$

for any Q in the interval $0 \leq Q \leq Q_{max}$. But $0 \leq W_n(Q_{max},S) \leq V_{max}(S)/n$, so $W_n(Q_{max},S)$ goes to zero as n becomes large. Therefore $W_n'(Q,S)$ also goes to zero and the convergence is uniform on the closed interval $0 \leq Q \leq Q_{max}$.

Thus, by choosing n large enough, we may make $\pi_n'(q,S)$ arbitrarily close to $P(Q)-A(Q,S)$ for all Q in the interval $0 \leq Q \leq Q_{max}$. Since $P(0)-A(0,S) > 0$, it follows that for sufficiently large n we must have $\pi_n'(0,S) > 0$. Since $P(Q_{max})-A(Q_{max},S) < 0$, it also follows that for sufficiently large n, $\pi_n'(Q_{max}/n,S) < 0$. Hence, for large enough n the symmetric equilibrium value $Q_n^*(S)/n$ for q is in the interior of the interval $0 \leq q \leq Q_{max}$ and therefore satisfies

(13) $\pi_n'(Q_n^*(S)/n, S) = 0$.

Since π_n' is strictly decreasing in its first argument, (13) uniquely determines $Q_n^*(S)$.

The interval $0 \leq Q \leq Q_{max}$ is compact, so the sequence $\{Q_n^*(S)\}$ has a subsequence converging to some point $Q_\infty^*(S)$ in the interval. Since the right side of (11) converges uniformly to $P(Q)-A(Q,S)$, for all Q in the interval $0 \leq Q \leq Q_{max}$, it follows that $P(Q_\infty^*(S)) - A(Q_\infty^*(S),S) = 0$. But, since $P(Q)-A(Q,S)$ is a strictly decreasing function of Q, the point $Q_\infty^*(S)$ is unique. Thus every convergent subsequence of $\{Q_n^*(S)\}$ converges to $Q_\infty^*(S)$, and therefore (by compactness of the interval $0 \leq Q \leq Q_{max}$) the sequence $\{Q_n^*(S)\}$ itself converges to $Q_\infty^*(S)$.

Finally, by a standard argument, if S is confined to a compact interval $0 \le S \le S_{max}$ then the solution $Q_n{*}(S)$ to (5) is for each n a continuous function of S, and so is $Q_\infty{*}(S)$ [11]. If we make the additional reasonable assumption that for each S the sequence $\{Q_n{*}(S)\}$ is monotonically increasing, then by a theorem of Dini (Royden, 1963 page 140) convergence is uniform in S.

Conclusion

What counsel does this analysis suggest for modelers of common property resources extracted under free access? Quite simply, unless one's model allows infinite profits for a sole owner, one can confidently compute equilibria using the convenient assumption of period-by-period rent dissipation knowing that the computed dynamic paths are the same as would arise in the free access limit of Markov-perfect equilibria. Alternatively, if one's model permits infinite profits, one should be aware that the limit of Markov perfect equilibria may not coincide with the soluton obtained by assuming rent dissipation.

References.

Cotroneo, J. (1994) "A Real Analysis of the Common Property Extraction Problem," Senior Honors Thesis, Bates College, Lewiston ME.

Dasgupta, P. and G. Heal (1979) *Economic Theory and Exhaustible Resources.* Great

Britain: Cambridge University Press.

Eswaran , M. and T. Lewis (1985) "Exhaustible Resources and Alternative Solution Concepts," *Canadian Journal of Economics,* **18**, 459-73.

Fudenberg, D. and J. Tirole (1992) *Game Theory.* Cambridge. MIT Press.

Gale, D. (1967) "Optimal Development of a Multi-sector Economy," *Review of Economic Studies,* **34**, 1-18.

Gordon, H.S. (1954) "The Economic Theory of a Common Property Resource: The Fishery," *Journal of Political Economy,* **62**, 124-42.

Hartwick, J. and N. Olewiler (1986) *The Economics of Natural Resource Use.* New York. Harper and Row.

Levhari, D., R. Michener, and L. Mirman (1981) "Dynamic Programming Models of Fishing: Competition," *American Economic Review,* **71**, 649-61.

Levhari, D. and L. Mirman (1980) "The Great Fish War: An Example Using a Dynamic Cournot-Nash Solution Concept," *Bell Journal of Economics,* **11**, 322-34.

Mirman, L. (1979) "Dynamic Models of Fishing: A Heuristic Approach," in *Control Theory in Mathematical Economics,* ed. Pan-Tai Liu and Jon G. Sutinen. New York: Marcel Dekker, Inc.

Negri, D. (1986) "The Common Property Resource as a Dynamic Game," Ph.D. thesis, University of Michigan.

Negri, D. (1990) " 'Stragedy' of the Commons," *Natural Resource Modeling*, **4**, (1990), 521-37.

Reinganum, J. and N. Stokey (1985) "Oligopoly Extraction of a Common Property Resource: The Importance of the Period of Commitment in Dynamic Games," *International Economic Review*, **26**, 161-73.

Rosen, S. (1987), "Dynamic Animal Economics," *American Journal of Agricultural Economics*, **69**, 547-57

Royden, H. (1963) *Real Analysis*, New York. Macmillan.

Salant, S. and D. Negri (1987), "When is the Standard Analysis of Common Property Extraction under Free Access Incorrect," University of Michigan Discussion Paper.

Notes

[1]For a definition of Markov perfect equilibrium, see Fudenberg and Tirole (1992) p 551.

[2]This specification incorporates both congestion and stock externalities: each extractor has an average cost which is decreasing in the current stock (for a given number of, say, fish extracted) but increasing in the aggregate current extraction. Negri (1986,1990) provided the first analyses of extraction models with both of these externalities present.

[3]We further assume $0 < P < 1$ to preclude obvious solutions of extracting nothing at all ($P \leq 0$) or everything at once ($P \geq 1$).

[4]Gale (1967) discusses the special case of this condition for the instance in which there is a sole owner (i.e., $x_t = 0$). He proved that $F()$ converges in the sole owner case if and only if $g < 1/\beta$. See Cotroneo (1994) for a proof of the more general claim.

[5]Let Y_t be the closed and bounded interval from 0 to $1 - x_t$, which contains all y_t, and let Y be the Cartesian product of all the Y_t, where t ranges from zero to infinity. In order to show that a maximum exists for F with respect to the sequence $\{y_t\}$, it suffices to show that F is continuous in $\{y_t\}$ and that Y is compact. That F is uniformly convergent can be shown by comparing it to a geometric series and then applying the Weierstrass M-Test. Since the convergence is uniform, and each partial sum of F is a continuous function of $\{y_t\}$, then F is also a continuous function of $\{y_t\}$ by the Uniform Limit Theorem. Each Y_t, being closed and bounded, is compact by the Heine-Borel Theorem. It follows that Y is compact by the Tychonoff Theorem which states that the product of compact sets is compact. See Cotroneo (1994) for details.

[6]She does this by reformulating the maximization problem as a dynamic programming problem: Without loss of generality, let $S_0 = 1$. Then the maximum value for F satisfies the functional equation

$$V(x) = \underset{y_0}{Max}\ [R(y_0) + \beta(g - x - y_0)V(x)].$$

Uniqueness then follows from the strict concavity of R. See Cotroneo (1994) for details.

[7] It's easier (but equivalent) to maximize $\ln F = \ln R(y) - \ln(1 - \beta g + \beta x + \beta y) + \ln S_0$, with $0 \leq y \leq 1 - x$. Hence, at an interior optimum $R'(y)/R(y) = (1-\beta g + \beta x + \beta y)^{-1}$, or $[(P - x) - y] / [y(P - x) - y^2] = (1-\beta g + \beta x + \beta y)^{-1}$, where $b = g - 1/\beta$. Simplifying we obtain the equation $y^2 + 2(x - b)y - (P - x)(x - b) = 0$. Notice that $y = 0$ if either $x = P$ or $x = b$. Otherwise the quadratic equation yields two roots, only one of which is positive. This is the one reported in equation (7). Further analysis shows that the derivative of F is

negative when $y \quad 1 - x$, so the only possible end-point maximum is at $y = 0$. This occurs if and only if $x > P$.

[8]Rosen [1987, p. 550] regards the case where $b = g - 1/\beta$ is negative as "uninteresting for farm animals." Whenever $b < 0$ in his model, animal stocks are depleted as in an inexhaustible resource model and zero stocks are held after a transition phase. To have a steady state with strictly positive inventory holding requires $b > 0$. Rosen regards this as the relevant parameter range.

[9]See note 4 above.

[10]These assumptions on G admit simple constant growth rate models, like that in our example, as well as more sophisticated biological growth models that assume population pressures arising from limited environmental resources for the extant stock.

[11]To show continuity of Q_n^* at S_o let $\{S_k\}$ be a sequence converging to S_o. Then the sequence $\{Q_n^*(S_k)\}$ (n fixed, k varying) is confined to the compact interval $[0, S_{max}]$ and therefore has a subsequence converging to some point Q_{n0}^*. By continuity of π_n' this point satisfies $\pi_n'(Q_{n0}^*/n, S_0) = 0$. But by uniqueness of the solution to (5), this implies that $Q_{n0}^* \quad Q_n^*(S_0)$. Hence every convergent subsequence of $\{Q_n^*(S_k)\}$ converges to $Q_n^*(S_o)$ so the sequence $\{Q_n^*(S_k)\}$ converges to $Q_n^*(S_o)$, and therefore Q_n^* is continuous at S_0. Similarly, $Q_\infty^*(S)$ is uniquely characterized by $P(Q_\infty^*(S)) - A(Q_\infty^*(S), S) \quad 0$, so by the same argument Q_∞^* is also continuous.

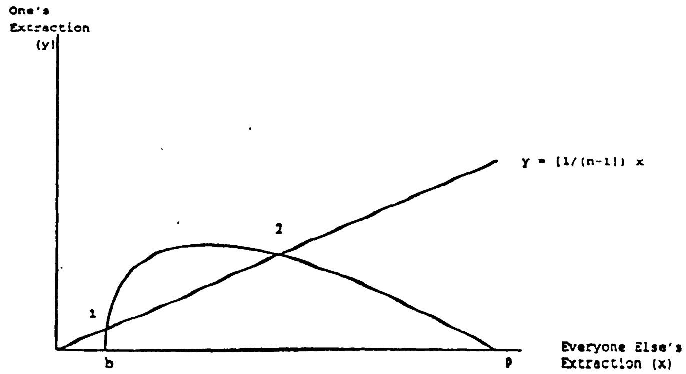

FIGURE 1

DEMCO