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A Median Choice Theorem

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Preliminary Draft

A Median Choice Theorem

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Abstract. This paper presents a Median Voter Theorem for a class of one-dimensional voting situations where individual preferences need not be single-peaked, but nonetheless satisfy a strong regularity condition. This condition arises when cartels with identical marginal costs vote on quotas [Cave and Salant, 1987] and also arises in the case of both agricultural marketing boards and prorationing boards restricting extraction from common properties.

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A MEDIAN CHOICE THEOREM

This paper presents a Median Voter Theorem for a class of one-dimensional voting situations where individual preferences need not be single peaked, but satisfy a hierarchical ordinal equivalence condition.

Let A be a one-dimensional space of alternatives. The voters are denoted $i = 1,...,L$, where L is odd. The ordinal preferences of voter i form a complete and transitive weak ordering R_i on the members of A. A **[strict] Condorcet winner** is a distinguished member a* of A such that more than half the voters [strictly] prefer a* to any other alternative.

It is well known that if each voter i has a most-preferred or "ideal" point $\mathbf{I_i}$, and the preferences R_i are single-peaked[1] the median ideal point is a Condorcet winner.

In this paper, individual preferences are not single-peaked, but their distribution satisfies a strong regularity condition. This structure emerges clearly in the system of marketing orders studied by Cave and Salant (1987).

That paper considers firms whose endowments of a homogeneous product can be sold on either an inelastic primary market or an infinitely elastic secondary market. A committee composed of such firms votes on the division of output between the two markets, selecting a uniform prorate level $F \in [0, 1]$. Each firm can sell up to F times its endowment on the primary market. As F falls below 1, firms with small endowments are the first to be constrained. The profit functions of constrained firms are multiples of each other [proportional to endowment]. All unconstrained firms prefer tighter constraints as long as they themselves are not constrained, since such constraints restrict the amount their rivals put on the market without limiting the market power of the unconstrained firms.

What is important for the structure of preferences is that each voter has an associated "cut-off" level. Alternatives above the cut-off are ranked in decreasing order, and the ordinal preferences of all voters coincide below their cut-off levels.

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[1] i.e., any cardinal representation of R_i is quasi-concave.

Formally, we posit the existence of a cut-off level $C_i \in A$ for each voter i. The voters are numbered in increasing order; $i < j$ implies $C_i \leq C_j$. The median voter under this numbering is denoted $m = (L+1)/2$.

Voter i's preferences, R_i, are assumed to satisfy three conditions.

Assumption 1: each voter has a unique ideal point I_i.

Assumption 2 ["ordinal nesting"]: R_i agrees with R_j below $\min\{C_i, C_j\}$:

(1) $a,b \leq \min\{C_i, C_j\}$ implies $[aR_ib \text{ iff } aR_jb]$.

Assumption 3: each R_i is monotone above C_i:

(2) $a \geq b \geq C_i$ implies bR_ia.

These assumptions ensure that I_m, the ideal point of the median voter, is a Condorcet winner. The discussion uses weak inequalities and breaks all ties in favor of I_m. If Assumption 3 is strengthened to strict monotonicity, I_m is the unique strict Condorcet winner.

Before proving this result, we note some properties of this preference structure.

Monotonicity implies that each voter's ideal point lies at or below his cut-off level:

(3) $I_i \leq C_i$.

Moreover, the ranking of voters by ideal points coincides with their ranking by cutoff levels:

(4) $i < j$ implies $I_i \leq I_j$.

To see this, note that $C_i \leq C_j$ by definition. If $I_j > C_i$, then $I_i \leq I_j$ by (3). Alternatively, if $I_j \leq C_i$, (3) implies that both voters' ideal points I_i and I_j lie below C_j. From (1), voters i and j must rank them in the same way. Therefore, in this case $I_i = I_j$.

Applying this argument repeatedly we obtain a **congruence property**:

(5) If $i < j$ and $I_j \leq C_i$, then $I_i = I_{i+1} = ... = I_j$.[2]

We now state our main result:

MEDIAN CHOICE THEOREM: if the preferences of the voters satisfy Assumptions 1-3, the ideal point of the median voter $[I_m]$ is a Condorcet winner.

Proof: Consider any $a < I_m$. By (1), a and I_m will be ranked in the same way by the majority coalition $\{m, ..., L\}$. Hence I_m defeats a in pairwise voting.

Now consider the only remaining possibility: that

$$C_{m-k} \leq I_m \leq \min \{C_{m-k+1}, a\}$$

All voters in the set $\{1, ..., m\text{-}k\}$ rank a and I_m according to monotone preferences (2), and therefore prefer I_m to a. By the congruence property (5), I_m is the ideal point of all voters in the set $\{m\text{-}k+1, ..., m\}$. Therefore, I_m is preferred to a by every member of the majority coalition $\{1, ..., m\}$. QED

Assumption 1 ensures that all voters whose preferences agree with those of the median voter strictly prefer I_m to the alternative a. It follows immediately that I_m is the unique strict Condorcet winner if assumption 3 is replaced by strict monotonicity.

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[2] Property (5) can be compared with Roberts' (1977) condition of "hierarchical adherence," defined for continuous cardinally comparable preferences over a two-dimensional space of alternatives. The two properties are logically independent.

There are other models in which voter preferences satisfy these assumptions. Consider a situation in which a public bad such as pollution is to be controlled, and where voters have different thresholds below which they are personally unaffected. Assumptions 1-3 amount to the conditions that personally affected people prefer lower levels of pollution, and that preferences below an individual's threshold level are determined by a common set of ethical and cost considerations. Since we only use ordinal preferences, this is not a statement about interpersonal comparisons or intensity of preference.

We do not suggest that this result solves either applied problem, but it certainly forms part of the solution. It predicts majority rule decisions and allows comparison with outcomes under other forms of organization, such as product allocation in the absence of marketing boards or private pollution abatement efforts.

The median choice result also facilitates analysis of changes in the underlying system that induces preferences. For instance, if the voters of the theorem are selected from a larger population according to some such mechanism as weighted majority rule, it is necessary to identify committee compositions with outcomes in order to analyze the process of committee selection. The same technique can be used to analyze changes in the distribution of cut-off and/or ideal points.

The foregoing analysis assumes sincere voting. It is well known that simple majority rule is vulnerable to manipulation. While we have no general results to offer in this regard, it can be argued that it does not benefit any voter to misrepresent his preferences, so long as the reported preferences satisfy Assumptions 1-3. Formally, this is captured in the following result.

NASH EQUILIBRIUM THEOREM: if a player misrepresents his cut-off and/or ideal points in such a way that Assumptions 1-3 remain satisfied, he will not prefer the resulting outcome to I_m.

Proof: First, it is clear that misrepresentation will not affect the result unless it changes the ideal point of the median voter. The median voter clearly cannot gain by distortion. Another voter $i \neq m$ can only affect the result if he shifts the identity of the median voter by pretending to be on the "other side" of m.

Suppose first that i < m; by misrepresentation i can obtain any outcome in the interval $[I_m, I_{m+1}]$. By claiming a cut-off point in the interior of this interval he becomes the new median voter and achieves his "pretended ideal." More extreme misrepresentation makes m-1 the new "median voter". In any case, i can only increase the voting outcome. The monotonicity property (2) ensures that i will not wish to do this.

Now consider i > m; by misrepresentation i can obtain any outcome in the interval $[I_{m-1}, I_m]$. By the ordinal nesting property [Assumption 2] i and m rank all points in this interval in the same way. Therefore, i prefers I_m to all other points in $[I_{m-1}, I_m]$, and cannot profit by distortion. QED

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