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ABSTRACT

This article has investigated the asymptotic distribution of the maximum likelihood estimator in a stochastic frontier function when the firms are all technically efficient. For such a situation, the true parameter vector is on the boundary of the parameter space, and the scores are linearly dependent. The maximum likelihood estimator is shown to be a mixture of certain truncated distributions. The maximum likelihood estimates for different parameters may have different rates of convergence. The model can be reparameterized into one with a regular likelihood function. The likelihood ratio test statistic has the usual mixture of chi-square distributions as in the regular case.

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Stochastic Frontier Function, Maximum Likelihood, Singular Information Matrix, Rates of Convergence, Asymptotic Distribution.

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1. Introduction

An econometric model, which has useful applications in production analysis, is the stochastic frontier function model of Aigner, Lovell and Schmidt [1]. The likelihood function of their model has an irregular feature at an interest point in the parameter space where the firms are all technically efficient. At such a point, the scores are linearly dependent, and the information matrix is singular.

The maximum likelihood method (ML) for the estimation of parametric models has been studied extensively when the information matrix is nonsingular. When the true parameter vector is in the interior of the parameter space, it is known that the ML estimator (MLE) is asymptotically normal (see, e.g., Rao [10]). The case where the true parameter vector is on the boundary of the parameter space has been analyzed in Moran [8], Chant [2] and Gourieroux, Holly and Monfort [4]. For the latter case, the MLE has the usual $N^{1/2}$-rate of convergence in distribution, and its limiting distribution is a mixture of truncated normal distributions. More recent developments on the theory of hypothesis testing of inequality constraints in econometrics can be found in Yancy, Judge, and Bock [20], Kodde and Palm [5], and Wolak [17,18,19].

Silvey [15] has argued that there is a connection between non-identifiability of a parameter vector and singularity of the information matrix. Rothenberg [11] proved that local non-identifiability of a parameter vector implies a singular information matrix, and if the true parameter vector is a regular point of the information matrix, the converse is true. Sargan [12] has constructed examples of simultaneous equation models which are identifiable, but the information matrices are singular. Sargan has analyzed the asymptotic distributions of some instrumental variable estimators for his models. The asymptotic distributions turn out to be nonnormal and are very complicated.

The irregularity in the stochastic frontier function model has been pointed out in Olsen et al [9], Waldman [16] and Schmidt and Lin [14]. Lee and Chester [6] have suggested some alternative test statistics and modifications of the classical Lagrange multiplier test and the Wold test, but there are no analyses on the asymptotic properties of the MLE at such a circumstance. Frontier function models have important empirical applications in applied econometrics. Literature surveys on the subject can be found in Forsund et al [3] and Schmidt [13]. Most recent developments are collected in Lewin and Lovell [7]. This article provides an asymptotic analysis of the maximum likelihood estimator when the irregularity occurs. It fills in a gap in the econometric literature of stochastic production function models.

2. An Irregularity in the Stochastic Frontier Function Model

The stochastic frontier function model introduced by Aigner, Lovell and Schmidt [1] is specified by

$$y_i = x_i\beta + u_i + v_i, \quad i = 1, \cdots, N, \tag{2.1}$$

where x_i is a k-dimensional vector of exogenous variables which contain a constant term; the disturbances u_i and v_i are independently distributed; $u_i \leq 0$ represents technical inefficiency; and v_i represents uncontrollable disturbance. A popular parametric distributional assumption is to assume that u_i is half normal with density function

$$h(u) = \frac{2}{\sqrt{2\pi}\sigma_1}\exp\left(-\frac{u^2}{2\sigma_1^2}\right), \quad u \leq 0, \tag{2.2}$$

and v_i is normal $N(0, \sigma_2^2)$. Furthermore, (u_i, v_i) are assumed to be independently and identically distributed for all i. We assume that the exogenous variables $\{x_i\}$ are uniformly bounded; the empirical distribution of $\{x_i\}$ converges in distribution to a limiting distribution, and $\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N x_i'x_i$ exists and is positive definite. For simplicity, the exogenous variables are in the deviation form such that $\sum_{i=1}^N x_{ij} = 0$, $j = 2, \ldots, k$, where $x_i = (x_{1i}, x_{2i}, \ldots, x_{ki})$ with $x_{1i} = 1$. Let $\delta = \sigma_1/\sigma_2$ and $\sigma^2 = \sigma_1^2 + \sigma_2^2$ as in Aigner et al. [1]. The parameter space Θ of $(\beta', \sigma^2, \delta)'$ in R^m, where $m = k+2$, is assumed to be compact. These regularity conditions are sufficient to justify our subsequent asymptotic analysis. While the variance σ^2 is always positive, the parameter δ is nonnegative but can be zero.

The log likelihood function for a random sample of size N is

$$\ln L(\theta) = N\ln 2 - \frac{N}{2}\ln(2\pi) - \frac{N}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^N (y_i - x_i\beta)^2 + \sum_{i=1}^N \ln\{1 - \Phi[\delta(y_i - x_i\beta)/\sigma]\}, \tag{2.3}$$

where $\theta = (\beta', \sigma^2, \delta)'$. Let $\epsilon_i = y_i - x_i\beta$ and $\lambda_i(\theta) = \phi(\delta\epsilon_i/\sigma)/[1 - \Phi(\delta\epsilon_i/\sigma)]$. The first order derivatives of (2.3) are

$$\frac{\partial \ln L(\theta)}{\partial \beta} = \frac{1}{\sigma^2}\sum_{i=1}^N x_i'(\epsilon_i + \delta\sigma\lambda_i(\theta)), \tag{2.4}$$

$$\frac{\partial \ln L(\theta)}{\partial \sigma^2} = -\frac{N}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^N \epsilon_i^2 + \frac{\delta}{2\sigma^3}\sum_{i=1}^N \lambda_i(\theta)\epsilon_i, \tag{2.5}$$

and

$$\frac{\partial \ln L(\theta)}{\partial \delta} = -\frac{1}{\sigma}\sum_{i=1}^N \lambda_i(\theta)\epsilon_i. \tag{2.6}$$

One of the interests in this model is to investigate whether all the firms are technically efficient or not. All the firms are technically efficient if and only if δ is zero. As δ is nonnegative, $\delta = 0$ is on the boundary of the parameter space. From (2.3), the right-hand derivatives of any finite order of the log likelihood function with respect to δ exists. The first order derivatives (2.4)-(2.6) at $\delta = 0$ are

$$\frac{\partial \ln L(\alpha, 0)}{\partial \beta} = \frac{1}{\sigma^2}\sum_{i=1}^N x_i'\epsilon_i, \tag{2.7}$$

$$\frac{\partial \ln L(\alpha, 0)}{\partial \sigma^2} = \frac{1}{2\sigma^2}\sum_{i=1}^N (\epsilon_i^2/\sigma^2 - 1), \tag{2.8}$$

and

$$\frac{\partial \ln L(\alpha, 0)}{\partial \delta} = -\frac{1}{\sigma}\sqrt{\frac{2}{\pi}}\sum_{i=1}^N \epsilon_i \tag{2.9}$$

because $\lambda_i(\alpha, 0) = \sqrt{2/\pi}$, where $\alpha = (\beta', \sigma^2)'$. Any irregularity of the derivatives in (2.7-2.9) is that they are linearly dependent as

$$\frac{\partial \ln L(\alpha, 0)}{\partial \theta'} S(\alpha) = 0, \quad \text{for all } \alpha, \tag{2.10}$$

where $S(\alpha) = (\sigma\sqrt{2/\pi}, 0', 1)'$. This implies that the information matrix at $\delta = 0$ is singular. Another irregularity of the likelihood function occurs on the second order derivatives of the log likelihood at $\delta = 0$. The formulas of the second order derivatives of the log likelihood function for the model can be found in Aigner et al. [1]. At $\delta = 0$, we observe that

$$\frac{\partial^2 \ln L(\alpha, 0)}{\partial \theta \partial \theta'} = \begin{pmatrix} -\frac{1}{\sigma^2}\sum_{i=1}^N x_i' x_i & -\frac{1}{\sigma^4}\sum_{i=1}^N x_i' \epsilon_i & \frac{1}{\sigma}\sqrt{\frac{2}{\pi}}\sum_{i=1}^N x_i' \\ * & \frac{N}{2\sigma^4} - \frac{1}{\sigma^6}\sum_{i=1}^N \epsilon_i^2 & \frac{1}{2\sigma^3}\sqrt{\frac{2}{\pi}}\sum_{i=1}^N \epsilon_i \\ * & * & -\frac{1}{\sigma^2}\frac{2}{\pi}\sum_{i=1}^N \epsilon_i^2 \end{pmatrix}.$$

It follows that

$$S'(\alpha)\frac{\partial^2 \ln L(\alpha, 0)}{\partial \theta \partial \theta'} S(\alpha) = \frac{2}{\pi}\sum_{i=1}^N (1 - \epsilon_i^2/\sigma^2). \tag{2.11}$$

The second irregular feature of the model is that

$$\left(\frac{\partial \ln L(\alpha, 0)}{\partial \beta'}, \frac{\partial \ln L(\alpha, 0)}{\partial \sigma^2}, S'(\alpha)\frac{\partial^2 \ln L(\alpha, 0)}{\partial \theta \partial \theta'} S(\alpha)\right) R(\alpha) = 0, \qquad \text{for all } \alpha, \tag{2.12}$$

where $R(\alpha) = (0', 4\sigma^2/\pi, 1)'$.

We note that even though the information matrix at $\delta = 0$ is singular, the parameters in this model are identifiable. As shown in Aigner et al. [1], the parameters in this model can be consistently estimated by using the first three sample moments of the distribution of y conditional on x and hence the parameters are identifiable. The problem of interest is to derive the asymptotic distribution of the maximum likelihood estimator when the true parameter vector has $\delta = 0$.

3. Simplification and Transformation

To derive the asymptotic distribution of the MLE, we will follow the approach of a Taylor series expansion in most of the asymptotic analyses. The two irregularity features (2.10) and (2.12) of the model complicate the picture. However, these irregularities can be simplified with some simple transformations. The following statements summarize the useful transformations.

Let $L(\alpha, \delta)$ denote a general likelihood function with a parameter vector α and a single parameter δ. The likelihood function is assumed to be continuously differentiable up to the third order.

Proposition 1. *Suppose that there exists a differentiable vector-value function $V(\alpha)$ such that*

$$\left(\frac{\partial lnL(\alpha,0)}{\partial \alpha'}, \frac{\partial lnL(\alpha,0)}{\partial \delta}\right)\begin{pmatrix}V(\alpha)\\ 1\end{pmatrix} = 0, \quad \text{for all } \alpha. \tag{3.1}$$

Define $L^*(\xi, \delta) = L(\xi + \delta V(\xi), \delta)$. Then

$$\frac{\partial lnL^*(\xi,0)}{\partial \xi} = \frac{\partial lnL(\xi,0)}{\partial \alpha}, \qquad \frac{\partial lnL^*(\xi,0)}{\partial \delta} = 0,$$

and $\frac{\partial^2 lnL^*(\xi,0)}{\partial \delta^2} = [V'(\xi), 1]\frac{\partial^2 lnL(\xi,0)}{\partial\theta\partial\theta'}\begin{pmatrix}V(\xi)\\ 1\end{pmatrix}$, *where* $\theta = (\alpha', \delta)'$.

Proposition 2. *Suppose that there exists a differentiable vector-value function $W(\alpha)$ such that*

$$\left(\frac{\partial lnL(\alpha,0)}{\partial \alpha'}, \frac{\partial^2 lnL(\alpha,0)}{\partial \delta^2}\right)\begin{pmatrix}W(\alpha)\\ 1\end{pmatrix} = 0, \quad \text{for all } \alpha. \tag{3.2}$$

Define $L^*(\xi, \delta) = L(\xi + \frac{\delta^2}{2}W(\xi), \delta)$. Then

$$\frac{\partial lnL^*(\xi,0)}{\partial \xi} = \frac{\partial lnL(\xi,0)}{\partial \alpha}, \quad \frac{\partial lnL^*(\xi,0)}{\partial \delta} = \frac{\partial lnL(\xi,0)}{\partial \delta}, \quad \text{and} \quad \frac{\partial^2 lnL^*(\xi,0)}{\partial \delta^2} = 0.$$

Proposition 3. *Suppose that* $\frac{\partial lnL(\alpha,0)}{\partial \delta} = 0$ *and* $\frac{\partial^2 lnL(\alpha,0)}{\partial \delta^2} = 0$. *Define $L^*(\alpha, \gamma) = L(\alpha, \gamma^{1/3})$. Then*

$$\frac{\partial L^*(\alpha,0)}{\partial \gamma} = \frac{1}{3!}\frac{\partial^3 lnL(\alpha,0)}{\partial \delta^3}.$$

All the proofs of these results are straightforward and can be found in the appendix. Proposition 1 says that the linear dependence of the first order derivatives of the log likelihood function in the form of (3.1) can be simplified to a zero score case after the transformation: $\alpha = \xi + \delta V(\xi)$ and $\delta = \delta$. If this transformation is one-to-one, it provides a useful reparameterization of the model as it simplifies the linear dependency of the scores of the likelihood function. The irregularity of the likelihood function in the form (3.2) involves the first and second order derivatives of the log likelihood function. Proposition 2 says that such an irregularity can be simplified to a zero second order derivative of the log likelihood function after the transformation: $\alpha = \xi + \frac{\delta^2}{2}W(\xi)$ and $\delta = \delta$. Proposition 3 says if the first and second order derivatives of the log likelihood function are zero, the third order derivative divided by 6 is the first order derivative of the log likelihood of a reparameterized model with the reparameterization $\gamma = \delta^3$.

By combining these transformations, the irregularity of our likelihood function of the stochastic function model can be simplified and eliminated. Let $l_1 = (1, 0, \ldots, 0)'$ be the first unit vector in R^k. Define the transformation

$$\xi_1 = \beta - \delta\sigma\sqrt{2/\pi}, \quad \xi_2 = \sigma^2, \quad \delta = \delta, \tag{3.3}$$

from $(\beta, \sigma^2, \delta)$ to (ξ, δ), where $\xi = (\xi_1', \xi_2)'$. This transformation is one-to-one. It follows that $(\beta', \sigma^2)' = \xi + \delta(\sqrt{2\xi_2/\pi}, 0')'$ which provides the transformation in Proposition 1 to simplify the irregularity in (2.10). Define $L_1^*(\xi, \delta) = L(\xi_1 + \delta\sqrt{2\xi_2/\pi}l_1, \xi_2, \delta)$ which is the likelihood on the parameter space of (ξ, δ). It follows from Proposition 1 that

$$\frac{\partial \ln L_1^*(\xi, 0)}{\partial \xi_1} = \frac{1}{\xi_2}\sum_{i=1}^N x_i'(y_i - x_i\xi_1), \quad \frac{\partial \ln L_1^*(\xi, 0)}{\partial \xi_2} = \frac{1}{2\xi_2}\sum_{i=1}^N \left\{\frac{1}{\xi_2}(y_i - x_i\xi_1)^2 - 1\right\}, \quad \frac{\partial \ln L_1^*(\xi, 0)}{\partial \delta} = 0, \tag{3.4}$$

and

$$\frac{\partial^2 \ln L_1^*(\xi, 0)}{\partial \delta^2} = \frac{2}{\pi}\sum_{i=1}^N \left\{1 - \frac{1}{\xi_2}(y_i - x_i\xi_1)^2\right\}. \tag{3.5}$$

These first and second order derivatives are linearly dependent as

$$\left(\frac{\partial \ln L_1^*(\xi, 0)}{\partial \xi'}, \frac{\partial^2 \ln L_1^*(\xi, 0)}{\partial \delta_2}\right)\begin{pmatrix} W(\xi) \\ 1 \end{pmatrix} = 0, \tag{3.6}$$

where $W(\xi) = (0', 4\xi_2/\pi)'$, which is equivalent to (2.12). Due to this dependency, these derivatives will not be sufficient to be used as the leading terms in the Taylor expansion to derive the asymptotic distribution of the ML estimates. Proposition 2 suggests the further transformation:

$$\eta_1 = \xi_1, \quad \eta_2 = (1 + 2\delta^2/\pi)^{-1}\xi_2, \quad \delta = \delta, \tag{3.7}$$

which is equivalent to the transformation $\xi = \eta + \frac{\delta^2}{2}(0', 4\eta_2/\pi)'$. The likelihood function of (η, δ) is

$$\begin{aligned} L_2^*(\eta, \delta) &= L_1^*(\eta_1, (1 + 2\delta^2/\pi)\eta_2, \delta) \\ &= L(\eta_1 + \delta[(2/\pi)(1 + 2\delta^2/\pi)\eta_2]^{1/2}l_1, (1 + 2\delta^2/\pi)\eta_2, \delta). \end{aligned} \tag{3.8}$$

Proposition 2 implies that

$$\frac{\partial \ln L_2^*(\eta, 0)}{\partial \eta_1} = \frac{1}{\eta_2}\sum_{i=1}^N x_i'(y_i - x_i\eta_1), \quad \frac{\partial \ln L_2^*(\eta, 0)}{\partial \eta_2} = \frac{1}{2\eta_2}\sum_{i=1}^N \left\{\frac{(y_i - x_i\eta_1)^2}{\eta_2} - 1\right\}, \tag{3.9}$$

$\frac{\partial \ln L_2^*(\eta, 0)}{\partial \delta} = 0$, and $\frac{\partial^2 \ln L_2^*(\eta, 0)}{\partial \delta^2} = 0$. From (3.8), by some tedious calculation,

$$\begin{aligned} \frac{\partial^3 \ln L_2^*(\eta, 0)}{\partial \delta^3} &= \sqrt{\frac{2}{\pi}}\left\{(1 - \frac{4}{\pi})\sum_{i=1}^N \frac{\epsilon_i^3}{\sigma^3} + \frac{12}{\pi}\sum_{i=1}^N \frac{\epsilon_i}{\sigma}\right\} \\ &= \sqrt{\frac{2}{\pi}}\left\{(1 - \frac{4}{\pi})\sum_{i=1}^N \frac{(y_i - x_i\eta_1)^3}{\eta_2^{3/2}} + \frac{12}{\pi}\sum_{i=1}^N \frac{(y_i - x_i\eta_1)}{\eta_2^{1/2}}\right\}. \end{aligned} \tag{3.10}$$

Finally, with the transformation $\gamma = \delta^3$, the likelihood function on (η, γ) is

$$L_3^*(\eta, \gamma) = L_2^*(\eta, \gamma^{1/3}) = L(\eta_1 + \gamma^{1/3}\{(2/\pi)(1 + 2\gamma^{2/3}/\pi)\eta_2\}^{1/2}l_1, (1 + 2\gamma^{2/3}/\pi)\eta_2, \gamma^{1/3}). \tag{3.11}$$

Proposition 3 implies that

$$\frac{\partial \ln L_3^*(\eta, 0)}{\partial \eta_1} = \frac{1}{\eta_2}\sum_{i=1}^N x_i'(y_i - x_i\eta_1), \quad \frac{\partial \ln L_3^*(\eta, 0)}{\partial \eta_2} = \frac{1}{2\eta_2}\sum_{i=1}^N \left\{\frac{(y_i - x_i\eta_1)^2}{\eta_2} - 1\right\}, \tag{3.12}$$

and

$$\frac{\partial \ln L_3^*(\eta,0)}{\partial\gamma} = \frac{1}{3!}\sqrt{\frac{2}{\pi}}\left\{(1-\frac{4}{\pi})\sum_{i=1}^{N}\frac{\epsilon_i^3}{\sigma^3}+\frac{12}{\pi}\sum_{i=1}^{N}\frac{\epsilon_i}{\sigma}\right\}$$
$$= \frac{1}{3!}\sqrt{\frac{2}{\pi}}\left\{(1-\frac{4}{\pi})\sum_{i=1}^{N}\frac{(y_i-x_i\eta_1)^3}{\eta_2^{3/2}}+\frac{12}{\pi}\sum_{i=1}^{N}\frac{(y_i-x_i\eta_1)}{\eta_2^{1/2}}\right\}. \quad (3.13)$$

The information matrix of $L_3^*(\eta,\gamma)$ at $(\eta',0)'$ is

$$I_N(\eta,0) = \begin{pmatrix} \frac{1}{\eta_2}\sum_{i=1}^N x_i'x_i & 0 & \frac{1}{\sqrt{2\pi\eta_2}}\sum_{i=1}^N x_i' \\ 0 & \frac{N}{2\eta_2^2} & 0 \\ \frac{1}{\sqrt{2\pi\eta_2}}\sum_{i=1}^N x_i & 0 & \frac{N}{6\pi}\left(5-\frac{16}{\pi}+\frac{32}{\pi^2}\right) \end{pmatrix}, \quad (3.14)$$

which is nonsingular. The likelihood function on the reparameterized parameter space of (η,γ) does not have irregularities.

4. Asymptotic Distribution of the Maximum Likelihood Estimator

Let $\beta = (\beta_1, \beta_2')'$ where β_1 is the intercept of the frontier model. The complete reparameterization of the model consists of the following transformations:

$$\eta_{11} = \beta_1 - \delta\sigma\sqrt{2/\pi}, \quad \eta_{12} = \beta_2, \quad \eta_2 = \sigma^2(1 + 2\delta^2/\pi)^{-1}, \quad \gamma = \delta^3, \tag{4.1}$$

where $\eta_1 = (\eta_{11}, \eta_{12}')'$. These define a mapping from $(\beta', \sigma^2, \delta)$ to (η, γ), which is one-to-one. We have from (4.1) that

$$\beta_1 = \eta_{11} + \gamma^{1/3}\eta_2^{1/2}(1 + 2\gamma^{2/3}/\pi)^{1/2}\sqrt{2/\pi}, \quad \beta_2 = \eta_{12}, \quad \sigma^2 = \eta_2(1 + 2\gamma^{2/3}/\pi), \quad \delta = \gamma^{1/3}. \tag{4.2}$$

Since the model is defined on $\delta \geq 0$ and, correspondingly, $\gamma \geq 0$, the corresponding MLE $(\tilde{\eta}', \tilde{\gamma})'$ maximizes the log likelihood function on the transformed parameter space of $(\eta', \gamma)'$ with $\gamma \geq 0$.

For any given sample of finite size, it is possible that the MLE may occur at the boundary with $\tilde{\gamma} = 0$. When $\tilde{\gamma} = 0$, $\tilde{\eta} = \hat{\eta} \equiv (\hat{\beta}_L', \hat{\sigma}^2)'$ where $\hat{\beta}_L$ and $\hat{\sigma}^2$ are the MLE of the normal linear regression model $y_i = x_i\beta + \epsilon_i$, $i = 1, \ldots, N$. As $\gamma \geq 0$, $\tilde{\gamma} = 0$ only if $\frac{\partial \ln L_3^*(\hat{\eta},0)}{\partial \gamma} \leq 0$. An analysis in Waldman [16] has shown that this necessary condition is also sufficient for $\tilde{\gamma} = 0$. Equivalently, from (3.13) $\tilde{\gamma} = 0$ if and only if $\sum_{i=1}^N \hat{\epsilon}_i^3 \geq 0$, where $\hat{\epsilon}_i = y_i - x_i\hat{\beta}$ is the least square residual. As $\tilde{\gamma} = 0$ may occur with positive probability, the asymptotic distribution of $\sqrt{(}\tilde{\eta}' - \eta', \tilde{\gamma})$ will be a mixture of certain distributions as in the general estimation of parametric models with inequality constraints (see, Moran [8] and Gourieroux et al. [4]). As the exogenous variables are in deviation form such that $\frac{1}{N}\sum_{i=1}^N x_i' = (1, 0, \ldots, 0)$. We assume that $\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N x_i'x_i = \begin{pmatrix} 1 & 0' \\ 0 & \Sigma \end{pmatrix}$ exists and is nonsingular.

When $\tilde{\gamma} > 0$, the MLE $(\tilde{\eta}', \tilde{\gamma})$ satisfies the first order conditions: $\frac{\partial \ln L_3^*(\tilde{\eta},\tilde{\gamma})}{\partial \eta} = 0$ and $\frac{\partial \ln L_3^*(\tilde{\eta},\tilde{\gamma})}{\partial \gamma} = 0$. By a Taylor expansion,

$$\sqrt{N}\begin{pmatrix} \tilde{\eta} - \eta \\ \tilde{\gamma} \end{pmatrix} = I_N^{-1}(\eta, 0)\begin{pmatrix} \frac{\partial \ln L_3^*(\eta,0)}{\partial \eta} \\ \frac{\partial \ln L_3^*(\eta,0)}{\partial \gamma} \end{pmatrix}. \tag{4.3}$$

The conditional asymptotic distribution of $\sqrt{N}(\tilde{\eta}' - \eta', \tilde{\gamma})$ converges in distribution to $F_1(t)$ where $F_1(t)$ is a m-dimensional truncated multivariate normal distribution defined on $-\infty < t_j < \infty$, $j = 1, \ldots, m-1$ and $t_m > 0$, and has in this region a density function equal to twice the density of a multivariate normal distribution with means zero, and covariance matrix $I^{-1}(\eta, 0)$ where

$$I(\eta, 0) = \lim_{N\to\infty}\frac{1}{N}I_N(\eta, 0) = \begin{pmatrix} \frac{1}{\eta_2} & 0 & 0 & \frac{1}{\sqrt{2\pi\eta_2}} \\ 0 & \frac{1}{\eta_2}\Sigma & 0 & 0 \\ 0 & 0 & \frac{1}{2\eta_2^2} & 0 \\ \frac{1}{\sqrt{2\pi\eta_2}} & 0 & 0 & \frac{1}{6\pi}(5 - \frac{16}{\pi} + \frac{32}{\pi^2}) \end{pmatrix}. \tag{4.4}$$

On the other hand, when $\tilde{\gamma} = 0$, the MLE $(\tilde{\eta}', 0)$ satisfies instead the conditions: $\frac{\partial \ln L_3^*(\tilde{\eta},0)}{\partial \eta} = 0$, and $\frac{\partial \ln L_3^*(\tilde{\eta},0)}{\partial \gamma} \leq 0$. From (3.12), it follows that asymptotically

$$\sqrt{N}(\tilde{\eta} - \eta) = \begin{pmatrix} \left[\frac{1}{N}\sum_{i=1}^N x_i'x_i\right]^{-1}\frac{1}{\sqrt{N}}\sum_{i=1}^N x_i'\epsilon_i \\ \frac{1}{\sqrt{N}}\sum_{i=1}^N(\epsilon_i^2 - \sigma^2) \end{pmatrix}. \tag{4.5}$$

As the exogenous variables are in deviation form, it follows from (3.13) that $\frac{\partial^2 \ln L_3^*(\bar{\eta},0)}{\partial \gamma \partial \eta_1'} \xrightarrow{P} -\frac{1}{\sqrt{2\pi}\sigma}l_1'$ and $\frac{\partial \ln L_3^*(\bar{\eta},0)}{\partial \gamma \partial \eta_2} \xrightarrow{P} 0$ for any consistent estimate $\bar{\eta}$ of η. These properties and the relations in (3.13) and (4.4) imply that by the mean value theorem

$$\begin{aligned}
&\frac{1}{\sqrt{N}}\frac{\partial \ln L_3^*(\hat{\eta},0)}{\partial \gamma} \\
&= \frac{1}{\sqrt{N}}\frac{\partial \ln L_3^*(\eta,0)}{\partial \gamma} + \frac{1}{N}\frac{\partial^2 \ln L_3^*(\bar{\eta},0)}{\partial \gamma \partial \eta'}\sqrt{N}(\hat{\eta}-\eta) \\
&= \frac{1}{3!}\sqrt{\frac{2}{\pi}}\left[\left(1-\frac{4}{\pi}\right)\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\frac{\epsilon_i}{\sigma}\right)^3 + \frac{12}{\pi}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{\epsilon_i}{\sigma}\right] - \frac{1}{\sqrt{2\pi}\sigma}l_1'\sqrt{N}(\hat{\beta}_L-\beta) + o_P(1) \\
&= \frac{1}{2}\sqrt{\frac{2}{\pi}}\left(1-\frac{4}{\pi}\right)\left[\frac{1}{3}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\frac{\epsilon_i}{\sigma}\right)^3 - \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{\epsilon_i}{\sigma}\right] + o_P(1).
\end{aligned} \tag{4.6}$$

Since ϵ_i is normal $N(0,\sigma^2)$, $E(\epsilon_i^4) = 3\sigma^4$ and $E(\epsilon_i^r) = 0$ for any odd r, we see from (4.4) and (4.5) that $\sqrt{N}(\hat{\eta}-\eta)$ and $\frac{1}{\sqrt{N}}\frac{\partial \ln L_3^*(\hat{\eta},0)}{\partial \gamma}$ are asymptotically independent. Therefore, $\sqrt{N}(\hat{\eta}-\eta)$ is asymptotically independent of the event that $\frac{1}{\sqrt{N}}\frac{\partial \ln L_3^*(\hat{\eta}',0)}{\partial \gamma} \leq 0$. Hence, conditional on $\tilde{\gamma}=0$, $\sqrt{N}(\tilde{\eta}'-\eta',\tilde{\gamma})$ converges in distribution to a distribution $F_2(t)$ of which its marginal distribution on the subspace $-\infty < t_i < \infty$, $i=1,\ldots,m-1$ is a multivariate normal distribution with means zero and covariance matrix Ω where

$$\Omega = \begin{pmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2\Sigma^{-1} & 0 \\ 0 & 0 & 2\sigma^4 \end{pmatrix},$$

and its marginal distribution on t_m is the indicator function of the set $[0,\infty)$, where $t=(t_1,\ldots,t_m)$. From (4.6), we see that $\frac{1}{\sqrt{N}}\frac{\partial \ln L_3^*(\hat{\eta},0)}{\partial \gamma}$ is aymptotically normally distributed with zero mean. Hence the probability of the event $\frac{1}{\sqrt{N}}\frac{\partial \ln L_3^*(\hat{\eta},0)}{\partial \gamma} \leq 0$ is asymptotically 0.5. It follows that the unconditional distribution of $\sqrt{N}(\tilde{\eta}'-\eta',\tilde{\gamma})$ converges in distribution to a mixture of distributions $\frac{1}{2}F_1(t)+\frac{1}{2}F_2(t)$. This implies, in particular, that $\sqrt{N}\tilde{\gamma}$ converges in distribution to a mixture of discrete and half-normal distributions $\frac{1}{2}F_n(t_m)+\frac{1}{2}F_o(t_m)$ where $F_o(t_m)$ is a degenerate distribution with unit mass at 0 and $F_n(t_m)$ is the half normal distribution of $N(0,3\pi/[1-4/\pi]^2)$ defined on $t_m>0$.

The asymptotic distribution of the MLE $(\tilde{\beta}',\tilde{\sigma}^2,\tilde{\delta})$ of $(\beta',\sigma^2,0)$ can be derived from (4.2). As $\sqrt{N}\tilde{\delta}^3 = \sqrt{N}\tilde{\gamma}$ is of order O(1), $\tilde{\delta}$ has a slow rate of convergence of order $O(1/N^{1/6})$. Conditional on $\tilde{\delta}>0$, the limiting distribution of $N^{1/6}\tilde{\delta}$ is the cubic root of the half normal $N(0,3\pi/[1-4/\pi]^2)$ variable. A mean value theorem applied to (4.2) implies that

$$\tilde{\beta}_1-\beta_1 = (\tilde{\eta}_{11}-\eta_{11}) + \frac{\partial\bar{\beta}_1}{\partial\eta_2}(\tilde{\eta}_2-\eta_2) + \frac{\partial\bar{\beta}_1}{\partial\gamma^{1/3}}\tilde{\gamma}^{1/3},$$

where $\partial\bar{\beta}_1/\partial\eta_2$ and $\partial\bar{\beta}_1/\partial\gamma^{1/3}$ are evaluated at a point lying between $(\tilde{\eta}',\tilde{\gamma}^{1/3})$ and $(\eta',0)$. The consistency of the estimates implies that $\text{plim}_{N\to\infty}\frac{\partial\bar{\beta}_1}{\partial\eta_2}=0$ and $\text{plim}_{N\to\infty}\partial\bar{\beta}_1/\partial\gamma^{1/3}=\sigma\sqrt{2/\pi}$. It follows that for $\tilde{\delta}>0$

$$\begin{aligned}
N^{1/6}(\tilde{\beta}_1-\beta_1) &= \frac{1}{N^{1/2-1/6}}N^{1/2}(\tilde{\eta}_{11}-\eta_{11}) + \frac{1}{N^{1/2-1/6}}\frac{\partial\bar{\beta}_1}{\partial\eta_2}N^{1/2}(\tilde{\eta}_2-\eta_2) + \frac{\partial\bar{\beta}_1}{\partial\gamma^{1/3}}(N^{1/2}\tilde{\gamma})^{1/3} \\
&= \left[(2\sigma^2/\pi)^{3/2}N^{1/2}\tilde{\gamma}\right]^{1/3} + o_P(1),
\end{aligned}$$

which converges in distribution to the cubic root of a half-normal variable $N(0,24\sigma^6/(4-\pi)^2)$. When $\tilde{\delta}=0$, $\tilde{\gamma}=0$ and

$$\begin{aligned}
N^{1/2}(\tilde{\beta}_1-\beta_1) &= N^{1/2}(\tilde{\eta}_{11}-\eta_{11}) + \frac{\partial\bar{\beta}_1}{\partial\eta_2}N^{1/2}(\tilde{\eta}_2-\eta_2) \\
&= N^{1/2}(\tilde{\eta}_{11}-\eta_{11}) + o_P(1) \\
&= N^{1/2}(\hat{\beta}_{1L}-\beta_1) + o_P(1),
\end{aligned}$$

which is asymptotically normal $N(0,\sigma^2)$. The MLE $\tilde{\beta}_1$ has different rates of convergence depending on the sign of $\tilde{\delta}$. From (4.2), (4.3) and (4.5),

$$N^{1/2}(\tilde{\beta}_2-\beta_2)=N^{1/2}(\tilde{\eta}_{12}-\eta_{12})\stackrel{D}{=}N^{1/2}(\hat{\beta}_{2L}-\beta_2).$$

The MLE $\tilde{\beta}_2$ has the rate of convergence of order $O(\frac{1}{N^{1/2}})$, independent of the sign of $\tilde{\delta}$, and has the same limiting distribution as the least squares estimate $\hat{\beta}_{2L}$ of β_2. A mean value theorem applied to σ^2 in (4.2) implies that

$$\tilde{\sigma}^2-\sigma^2=\frac{\partial\bar{\sigma}^2}{\partial\eta_2}(\tilde{\eta}_2-\eta_2)+\frac{\partial\bar{\sigma}^2}{\partial\gamma^{2/3}}\tilde{\gamma}^{2/3}.$$

Since $\text{plim}_{N\to\infty}\frac{\partial\bar{\sigma}^2}{\partial\eta_2}=1$ and $\text{plim}_{N\to\infty}\frac{\partial\bar{\sigma}^2}{\partial\gamma^{2/3}}=\frac{2}{\pi}\sigma^2$, for $\tilde{\delta}>0$,

$$\begin{aligned}N^{1/3}(\tilde{\sigma}^2-\sigma^2)&=\frac{1}{N^{1/2-1/3}}\frac{\partial\bar{\sigma}^2}{\partial\eta_2}N^{1/2}(\tilde{\eta}_2-\eta_2)+\frac{\partial\bar{\sigma}^2}{\partial\gamma^{2/3}}(N^{1/2}\tilde{\gamma})^{2/3}\\&=\left[(2\sigma^2/\pi)^{3/2}N^{1/2}\tilde{\gamma}\right]^{2/3}+o_P(1),\end{aligned}$$

which converges in distribution to the square of the cubic root of the half normal variable $N(0,24\sigma^6/(4-\pi)^2)$. On the other hand, conditional on $\tilde{\delta}=0$,

$$N^{1/2}(\tilde{\sigma}-\sigma^2)=\frac{\partial\bar{\sigma}^2}{\partial\eta_2}N^{1/2}(\tilde{\eta}_2-\eta_2)=N^{1/2}(\tilde{\eta}_2-\eta_2)+o_P(1),$$

which has the same limiting normal $N(0,2\sigma^4)$ distribution as the least square estimate $\hat{\sigma}_L^2$ of σ^2.

Finally we note that as the model can be reparameterized and $\ln L(\tilde{\beta},\tilde{\sigma}^2,\tilde{\delta})=\ln L_3^*(\tilde{\eta},\tilde{\gamma})$, $2[\ln L(\tilde{\beta},\tilde{\sigma}^2,\tilde{\delta})-\ln L(\hat{\beta}_L,\hat{\sigma}^2,0)]$ for the testing of $\delta=0$ has asymptotically a mixture of chi-square distributions $\frac{1}{2}\chi^2(0)+\frac{1}{2}\chi_c^2(1)$ where the distribution $\chi^2(0)$ is degenerate with a unit mass at zero and the distribution $\chi_c^2(1)$ is the square of a positively truncated standard normal variable $N(0,1)$ (Chant [2] and Gourieroux et al. [4]).

5. Conclusion

This article has analyzed the asymptotic distribution of a maximum likelihood estimator for a stochastic frontier function model. The disturbance of the stochastic frontier function model is the convolution of two independent random variables, namely, a normal variable and a half-normal variable. An irregularity of the likelihood function occurs at the point where all the firms are technically efficient. At that point, there are linear dependence on the first order derivatives and linear dependence across some of the first and second order derivatives of the log likelihood function. These irregularities can be simplified into identically zero first and second order derivatives of a reparameterized likelihood function. The asymptotic distribution of the MLE can then be derived from a simple Taylor series expansion. Except for the intercept term, the MLE of the regression coefficients of the stochastic frontier model turn out to be asymptotically normal as the ordinary least squares estimates. However, the remaining parameter estimates converge in distribution at much lower rates of convergence and the limiting distributions are nonnormal. Since the model can be reparameterized, the maximized log likelihood function provides a valid likelihood ratio test statistic as in the standard case with an inequality constraint.

We note that the asymptotic distribution for the MLE derived in this article is specific to the stochastic frontier model. Our asymptotic analysis and the simplified transformations have utilized the specific irregularities of the likelihood function of our model. We have not provided a general asymptotic theory which is applicable to any model with a singular information matrix. It is unlikely that a general theory can exist. However, the Taylor series expansion technique may remain useful for many circumstances.

Appendix

Proof of Proposition 1. As

$$\frac{\partial \ln L^*(\xi,\delta)}{\partial \xi} = \frac{\partial \ln L(\xi+\delta V(\xi),\delta)}{\partial \xi} = \left[I + \delta\frac{\partial V'(\xi)}{\partial \xi}\right]\frac{\partial \ln L(\xi+\delta V(\xi),\delta)}{\partial \alpha},$$

$$\frac{\partial \ln L^*(\xi,\delta)}{\partial \delta} = \frac{\partial \ln L(\xi+\delta V(\xi),\delta)}{\partial \alpha'}V(\xi) + \frac{\partial \ln L(\xi+\delta V(\xi),\delta)}{\partial \delta},$$

and

$$\frac{\partial^2 \ln L^*(\xi,\delta)}{\partial \delta^2} = (V'(\xi),1)\frac{\partial^2 \ln L(\xi+\delta V(\xi),\delta)}{\partial \theta \partial \theta'}\begin{pmatrix} V(\xi) \\ 1 \end{pmatrix},$$

where $\theta = (\alpha',\delta)'$, it follows that, at $\delta = 0$, $\frac{\partial \ln L^*(\xi,0)}{\partial \xi} = \frac{\partial \ln L(\xi,0)}{\partial \alpha}$,

$$\frac{\partial \ln L^*(\xi,0)}{\partial \delta} = \frac{\partial \ln L(\xi,0)}{\partial \alpha'}V(\xi) + \frac{\partial \ln L(\xi,0)}{\partial \delta} = 0,$$

and $\frac{\partial^2 \ln L^*(\xi,0)}{\partial \delta^2} = (V'(\xi),1)\frac{\partial^2 \ln L(\xi,0)}{\partial \theta \partial \theta'}\begin{pmatrix} V(\xi) \\ 1 \end{pmatrix}$. Q.E.D.

Proof of Proposition 2. For this case, we have

$$\frac{\partial \ln L^*(\xi,\delta)}{\partial \xi} = \left[I + \frac{\delta^2}{2}\frac{\partial W'(\xi)}{\partial \xi}\right]\frac{\partial \ln L(\xi+\frac{\delta^2}{2}W(\xi),\delta)}{\partial \alpha},$$

$$\frac{\partial \ln L^*(\xi,\delta)}{\partial \delta} = \delta\frac{\partial \ln L(\xi+\frac{\delta^2}{2}W(\xi),\delta)}{\partial \alpha'}W(\xi) + \frac{\partial \ln L(\xi+\frac{\delta^2}{2}W(\xi),\delta)}{\partial \delta},$$

and

$$\begin{aligned}\frac{\partial^2 \ln L^*(\xi,\delta)}{\partial \delta^2} =& \frac{\partial \ln L(\xi+\frac{\delta^2}{2}W(\xi),\delta)}{\partial \alpha'}W(\xi) \\ &+ \delta\left\{\delta W'(\xi)\frac{\partial^2 \ln L(\xi+\frac{\delta^2}{2}W(\xi),\delta)}{\partial \alpha \partial \alpha'} + \frac{\partial^2 \ln L(\xi+\frac{\delta^2}{2}W(\xi),\delta)}{\partial \delta \partial \alpha'}\right\}W(\xi) \\ &+ \left\{\delta W'(\xi)\frac{\partial^2 \ln L(\xi+\frac{\delta^2}{2}W(\xi),\delta)}{\partial \alpha \partial \delta} + \frac{\partial^2 \ln L(\xi+\frac{\delta^2}{2}W(\xi),\delta)}{\partial \delta^2}\right\}.\end{aligned}$$

It follows that, at $\delta = 0$, $\frac{\partial \ln L^*(\xi,0)}{\partial \xi} = \frac{\partial \ln L(\xi,0)}{\partial \alpha}$, $\frac{\partial \ln L^*(\xi,0)}{\partial \delta} = \frac{\partial \ln L(\xi,0)}{\partial \delta}$, and

$$\frac{\partial^2 \ln L^*(\xi,0)}{\partial \delta^2} = \frac{\partial \ln L(\xi,0)}{\partial \alpha'}W(\xi) + \frac{\partial^2 \ln L(\xi,0)}{\partial \delta^2} = 0.$$

Q.E.D.

Proof of Proposition 3. By the Taylor expansion up to the third order,

$$\begin{aligned}\frac{\partial \ln L(\alpha,\delta)}{\partial \delta} &= \frac{\partial \ln L(\alpha,0)}{\partial \delta} + \frac{\partial^2 \ln L(\alpha,0)}{\partial \delta^2}\delta + \frac{1}{2}\frac{\partial^3 \ln L(\alpha,\bar\delta)}{\partial \delta^3}\delta^2 \\ &= \frac{1}{2}\frac{\partial^3 \ln L(\alpha,\bar\delta)}{\partial \delta^3}\delta^2,\end{aligned}$$

where $\bar\delta$ lies between δ and 0, as the first two derivatives are zero. It follows that

$$\frac{\partial \ln L^*(\alpha,\gamma)}{\partial \gamma} = \frac{1}{3\gamma^{2/3}}\frac{\partial \ln L(\alpha,\gamma^{1/3})}{\partial \delta} = \frac{1}{3!}\frac{\partial^3 \ln L(\alpha,\bar\gamma^{1/3})}{\partial \delta^3},$$

where $\bar\gamma$ lies between γ and 0. Hence $\frac{\partial \ln L^*(\alpha,0)}{\partial \gamma} = \frac{1}{3!}\frac{\partial^3 \ln L(\alpha,0)}{\partial \delta^3}$. Q.E.D.

REFERENCES

1. Aigner, D.J., C.A.K. Lovell and P. Schmidt. Formulation and estimation of stochastic frontier production function models. *Journal of Econometrics*, 6 (1977): 21-37.
2. Chant, D. On asymptotic tests of composite hypotheses. *Biometrika*, 61 (1974): 291-298.
3. Forsund, F.R., C.A.K. Lovell and P. Schmidt. A survey of frontier production functions and of their relationship to efficiency measurement. *Journal of Econometrics*, 13 (1980): 5-25.
4. Gourieroux, C., A. Holly and A. Monfort. Likelihood ratio test, Wald test, and Kuhn-Tucker test in linear models with inequality constraints on the regression parameters. *Econometrica*, 50 (1982): 63-80.
5. Kodde, D.A., and F.C. Palm. Wald criterion for jointly testing equality and inequality restrictions. *Econometrica*, 54 (1986): 1243-1248.
6. Lee, L.F. and A. Chesher. Specification testing when score test statistics are identically zero. *Journal of Econometrics*, 31 (1986): 121-149.
7. Lewin, A.Y. and C.A. Knox Lovell. *Frontier analysis: parametric and nonparametric approaches*, Annals, *Journal of Econometrics*, Vol. 46, No. 1/2. Amsterdam: North Holland, 1990.
8. Moran, P.A.P. Maximum-likelihood estimation in non-standard conditions. *Proceeding of Cambridge Philos. Society.*, 70 (1971): 441-450.
9. Olson, J.A. P. Schmidt and D.M. Waldman. A monte carlo study of estimators of stochastic frontier production functions. *Journal of Econometrics*, 13 (1980): 67-82.
10. Rao, C.R. *Linear statistical inference and its applications.* 2nd edition, John Wiley and Sons: New York, 1973.
11. Rothenberg, T.J. Identification in parametric models. *Econometrica*, 39 (1971): 577-591.
12. Sargan, J.D. Identification and lack of identification. *Econometrica*, 51 (1983): 1605-1633.
13. Schmidt, P. Frontier production functions. *Econometric Reviews* 4 (1985): 289-328.
14. Schmidt, P. and T.-F. Lin. Simple tests of alternative specification in stochastic frontier models. *Journal of Econometrics*, 24 (1984): 349-361.
15. Silvey, S.D. *Statistical inference.* Chapman and Hall: London, 1975.
16. Waldman, D.M. A stationary point for the stochastic frontier likelihood. *Journal of Econometrics*, 18 (1982): 275-279.
17. Wolak, F.A. Local and global testing of linear and nonlinear inequality constraints in nonlinear econometric models. *Econometric Theory*, 5 (1989): 1-35.
18. Wolak, F.A. Testing inequality constraints in linear econometric models. *Journal of Econometrics*, 41 (1989): 205-235.
19. Wolak, F.A. The local nature of hypothesis tests involving inequality constraints in nonlinear models. *Econometrica*, 59 (1991): 981-995
20. Yancy, T.A., G.G. Judge, and M.E. Bock. Testing multiple equality and inequality hypothesis in economics. *Economic Letters*, 4 (1981): 249-255.

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