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Pastures of Plenty: When is the Standard Analysis of Common Property Extraction Under Free Access Incorrect?

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Pastures of Plenty: When is the Standard Analysis of Common Property Extraction under Free Access Incorrect?

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Abstract. There are two ways to calculate the dynamic path of aggregate extraction when there is free access to a common property resource: (1) calculate the rent-dissipating aggregate extraction in any period (as a function of the stock) and then derive the dynamic path using the transition equation and initial condition or (2) examine the path of aggregate extraction in a subgame perfect equilibrium as the number of players expands without bound. The latter approach is theoretically correct but often intractable. The former approach, which has been widely used for more than thirty years, is tractable and generally presumed to yield the identical aggregate extraction path. We show by example that this presumption is erroneous. We then provide conditions which suffice for the traditional approach to be correct.

Keywords. common property, free access, Nash equilibrium, rent dissipation, subgame perfect equilibrium

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[1] Resource and Tecnology Division, Economic Research Service.

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1. Introduction

It is well-known that when a small number of extractors draws from a common pool (the so-called "restricted-access" case), strategic interactions over time are involved which require game-theoretic analysis. Levhari and Mirman [1980] formulated the first dynamic game of common-property extraction under restricted access and computed its subgame perfect equilibrium. Investigations of the perfect equilibria of other restricted-access models have been conducted by Eswaran and Lewis [1985], Mirman [1979], and Reinganum and Stokey [1985].

When the number of agents is very large (the so-called "free-access" case), however, dynamic strategic analysis is never used. Analysis of the free-access case dates back to Gordon's [1954] classic article in this *Journal*—and is undoubtedly familiar to most readers: a transition equation determines the stock tomorrow as a function of the stock today and the current aggregate extraction. Current aggregate extraction is in turn deduced from the assumption that—for each date and stock—unrestricted entry dissipates industry profits. The evolution of the stock is then traced from the initial state onward and its steady-state (assuming one exists) is determined.[2]

This analysis of common property dynamics under free access is now taught at both the undergraduate [Hartwick and Olewiler, p.275] and graduate levels [Dasgupta and Heal, p.61 and p.121-126]. Gould [1972, p.1036-7] provides a particularly concise treatment of Gordon's model using the "catch locus"—the rent-dissipating catch associated with each size stock (assuming a fixed output price).

We would like to thank Ted Bergstrom, Larry Blume, Dick Porter, and Joe Swierzbinski for their helpful comments.

[2] Others have interpreted Gordon's analysis as applicable only to steady states (not to momentary equilibria). Under this interpretation as well, our example shows that his analysis is not generally correct.

The pioneers of game-theoretic analysis of the restricted-access case have also considered the free-access case [Levhari, Michener, and Mirman, 1981]. In a characteristically lucid article, these authors extended previous analyses by endogenizing the output price and integrating a variety of disparate results from the free-access literature.

Surprisingly, however, these authors did not treat the free-access case as the limit of the subgame perfect equilibria as the number of players goes to infinity. Instead, they *assumed* that—under free access—no individual extractor places positive value on carrying additional stock into the future. Whenever this is the case dynamic analysis is, of course, unnecessary.

The purpose of this paper is to investigate when industry profit and the shadow value of the resource *in fact* approach zero in a perfect equilibrium as the number of extractors approaches infinity.

In the next section, we consider a symmetric perfect equilibrium in a common property example. The example demonstrates that certain assumptions central to the free-access literature are in general incorrect.

In our example, industry profits in a given period do not approach zero as the number of extractors increases without bound—contrary to the assumption underlying Gordon's analysis. Nor does the marginal value of additional stock remaining approach zero as the number of extractors goes to infinity—contrary to the assumption underlying Levhari, Michner, and Mirman's analysis. As a result, aggregate extraction in the perfect equilibrium does not converge pointwise to the "catch locus" derived using the rent-dissipation assumption. As a further consequence, the stock in the perfect equilibrium may differ over time rather dramatically from the steady-state stock deduced using the traditional analysis. Indeed, for some values of the exogenous parameters, the stock increases without bound in the perfect equilibrium under free access even though Gordon's analysis implies that the stock would converge to a steady state of zero (extinction).

Given this example, there is a need to clarify when the traditional approach to free-access problems can be used without risk of error.

Section 2 considers the example and verifies that a specified strategy combination forms a subgame perfect equilibrium. Section 3 discusses individual and aggregate behavior in this equilibrium and gives the intuition underlying several of its surprising properties. Having shown by example that the assumptions underlying the traditional analysis are incorrect unless qualified, we consider in Section 4 the appropriate qualifications: we provide conditions sufficient (but not necessary) for the traditional analysis to be correct, show that they are violated by our example, and relate our example to the literature on the nonexistence of a solution to certain single-agent optimization problems with an unbounded horizon. Section 5 concludes the paper.

2. The Example

Traditional Analysis of the Example

Consider a common property resource under free access. Assume there exists both a congestion and a stock externality [3] so that each extractor has an average cost which is decreasing in the remaining stock (S) but increasing in aggregate current extraction (Q). To illustrate, suppose the average cost of extractor i is:

$$A(Q_t, S_t) = \frac{\gamma Q_t}{S_t}, \tag{1}$$

where

Q_t denotes current aggregate extraction,

S_t denotes the remaining stock at the beginning of the period, and

γ is an exogenous parameter.

Suppose the sequence of real prices, $\{P_t\}$, is constant over time (at P) and that the stock regenerates as follows: $S_{t+1} = gS_t - Q_t$. If the initial stock is S_0, how will aggregate extraction proceed over time?

The accepted answer to this question is the following: if Q_t is aggregate extraction from the pool in period t, the industry profit will be $Q_t\{P - A(Q_t, S_t)\}$. Since free access will insure that aggregate profits are dissipated in each period, extraction in period t must be Q_t^*, the implicit solution to $P - A(Q_t, S) = 0$. In our example,

$$Q_t^* = \frac{S_t P}{\gamma} = \frac{1}{a} S_t,$$

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[3] This terminology was coined by Smith (1968).

where $\frac{P}{\gamma} = \frac{1}{a}$. The stock available at time t ($t = 0, 1, 2, \ldots$) is, therefore, $S_t = S_0(g - 1/a)^t$. If $g - 1/a > 1$, the stock grows without bound. If $0 < g - 1/a < 1$, the resource becomes extinct ($\lim_{t\to\infty} S_t = 0$). If $g - 1/a = 1$, any initial stock is a steady-state stock ($\lim_{t\to\infty} S_t = S_0$).

A Perfect Equilibrium of the Example

We now consider a particular set of strategies and verify that it forms a perfect equilibrium of this same example. Let β denote the discount factor. Let $q^i(S) = [\frac{1-\beta C}{a(n+1)}]S$ denote firm i's extraction strategy, where

$$C = \frac{(n+1)^2 x - 2n\beta + (n+1)y^{1/2}}{2n^2\beta^2}, \tag{2}$$

$$x = a - \beta[ag - 1], \text{ and} \tag{3}$$

$$y = (n+1)^2 x^2 - 4\beta n x. \tag{4}$$

It will simplify notation to define $\frac{1-\beta C}{a(n+1)}$ as k; hence $q^i(S) = kS$. Denote the aggregate extraction of the *other* extractors (excluding i) in the proposed solution as $Q^{-i}(S)$, where $Q^{-i}(S) = (n-1)kS$.

It is asserted that the set of strategies $\{q^i(S)\}_{i=1,n}$ forms a subgame perfect equilibrium provided the exogenous parameters satisfy the following restrictions:[4]

$$\begin{aligned} 0 <&\beta < 1 \\ &a \geq 1 \\ 1/\beta <&g < \frac{(a+\beta)(n+1)^2 - 4n\beta}{(n+1)^2 a\beta}. \end{aligned} \tag{5}$$

To verify that the set of strategies $\{q^i(S)\}_{i=1,n}$ forms a subgame perfect equilibrium, we need merely show that $q^i(\cdot)$ is an optimal policy for extractor i ($i = 1, 2, \ldots, n$) if the other players extract in aggregate $Q^{-i}(\cdot)$ from any date ν and state S onward. Since the

[4] The role of these restrictions will be clarified below. It should be noted that the interval restricting g is nonempty if and only if $n > 1$.

proposed equilibrium is symmetric, we need merely check the optimality of a single agent's strategy. That is, we must verify that by setting $q_t^i = q(S_t)$ agent i will

$$\begin{aligned}
&\text{maximize} \quad \sum_{t=\nu}^{\infty} q_t^i [1 - \frac{a(q_t^i + (n-1)kS_t)}{S_t}] &\quad (6)\\
&\qquad q_t^i \geq 0 \\
&\qquad S_t \geq 0 \\
&\text{subject to} \quad S_{t+1} = gS_t - q_t^i - (n-1)kS_t \\
&\qquad S_\nu = S, \quad \text{for } S \geq 0,\ t = \nu, \nu+1, \ldots \text{ and } \nu = 1, 2, \ldots.
\end{aligned}$$

To verify that the proposed policy $(q^i(S))$ solves this stationary, discrete-time, discounted optimization problem we will employ dynamic-programming methods. But we must exercise some care because in this example the "current-return function" is bounded neither from above nor below. Specifically the current-return function is

$$\begin{aligned}
\pi(q, S) &= q[1 - \frac{a(q + (n-1)kS)}{S}] \\
&= q\{1 - (n-1)ka - \frac{aq}{S}\}. \qquad (7)
\end{aligned}$$

Clearly, if q is set at a positive number and S approaches zero from above, the function approaches *minus infinity.* On the other hand, if the ratio $\frac{q}{S} = \theta$ is set such that the second factor in (7) is positive, then π approaches plus infinity as q approaches *plus infinity.*

When the current-return function in a discounted dynamic-programming problem is unbounded, we *cannot* verify that a proposed solution is optimal *merely* by checking that (1) there exists a value function $(V^i(S))$ which is a fixed point of the functional equation associated with the dynamic-programming problem and (2) the proposed optimal policy $q^i(S)$ maximizes the current return plus this future return.

A simple example illustrates that such a check is insufficient when the current-return function is unbounded. Consider a current-return function where the payoffs from "extraction" are linear in the quantity extracted, $\pi(q, S) = q$. Let the dynamics be governed by proportional growth of the stock, $S_{t+1} = gS_t - q_t$. The corresponding dynamic programming formulation of the recursive problem is

$$V(S) = \max_{0 \leq q \leq S} q + \beta V(gS - q).$$

Let the stock grow at the discount rate, $g = \frac{1}{\beta}$. Consider the value function $V(S) = \frac{2}{\beta}S$. We can show that this value function solves the functional equation and that this optimum is achieved at $q(S) = 0$.

$$V(S) = \max_{0 \leq q \leq S} \; q + \beta[\frac{2}{\beta}(gS - q)].$$

By noting that the functional equation is negatively sloped in q (with slope equal to -1) it is apparent that the decision rule $q = 0$ maximizes the right-hand side of this functional equation. Substituting $q = 0$ and $g = \frac{1}{\beta}$ we obtain $\frac{2}{\beta}S$. Thus, the proposed value function and the associated optimal decision rule solve the functional equation. However, it is easy to see that $q = 0$ is not an optimal solution to the infinite-horizon problem. Indeed, any policy with positive extraction in some period dominates the policy $q_t = 0$ since the payoffs are increasing in q_t.

Conditions sufficient for a proposed solution to such maximization problems to be optimal have been derived for problems where the current-return function is bounded from below but not above [see Bertsekas, p. 222-59]. We utilize *these conditions* in our demonstration that the proposed $q^i(S)$ is optimal for the i^{th} agent.

To use them, we construct a new but related maximization problem whose objective function is bounded from below. This related problem has the identical constraint set and transition equation as the original problem. Hence any program feasible in one problem will be feasible in the other. However, its current-return function is instead MAX $[0, \pi(q, S)]$, where the "MAX" operator means the larger of the two arguments. Since $\pi(q, S)$ is continuous and unbounded from above, the current-return function to the related problem must inherit each property. However, the current-return function of the related problem *is* bounded from below (by zero)—making Bertsekas' conditions applicable. We use these conditions to verify that the proposed policy is optimal in the related problem. Since the current-return function in the original problem *can never be larger than in the related problem*, any program which is both optimal in the related problem and also achieves the same payoff in both problems must also be optimal in the original problem. Hence $\{q^i(S)\}_{i=1,n}$ forms a symmetric subgame perfect equilibrium.

To develop this argument in more detail, consider the following dynamic-programming problem.

$$V^i(S) = \max\{\text{MAX}[0, q(1 - \frac{a(q + (n-1)kS)}{S}] + \beta V^i(gS - q - (n-1)kS)\} \tag{8}$$

$$\text{subject to } 0 \leq q \leq [g - (n-1)k]S.$$

We will prove that $q^i(S) = kS$ is optimal for this (the "related") problem.

Since the current-return function in this problem is bounded from below,[5] the proposed solution is optimal if and only if (1) some function ($V^i(S)$) solves the functional equation, (2) the proposed optimal policy ($q^i(S) = kS$) maximizes the right-hand side of equation (8) for the given $V^i(S)$ and any $S \geq 0$, and (3) the given value function equals the wealth which would result if $q^i(\cdot)$ was played indefinitely against the proposed equilibrium strategies of the other players ($Q^{-i}(\cdot)$) starting from an initial stock, S. It is this *additional* condition (3) which must be checked in maximization problems where the current return function is bounded only from below.[6]

To begin, we verify that $V^i(S) = CS$ solves the functional equation in (8) and that the right-hand side of (8) is maximized at $q^i(S) = kS$. Substituting, we obtain

$$\begin{aligned} V^i(S) = \max\{&\text{MAX}[\beta C(gS - q - (n-1)kS), \\ &q(1 - \frac{a(q + (n-1)kS)}{S}) + \beta C(gS - q - (n-1)kS)]\} \end{aligned} \tag{9}$$

$$\text{subject to } 0 \leq q \leq [g - (n-1)k]S.$$

Consider the two arguments of the MAX operator. In principle, at the optimum the first argument can be strictly larger than the second or, alternatively, the second argument can be at least as large as the first. In the latter case, the right-hand side of (9) can be replaced

[5] The current-return function which is bounded from below satisfies "assumption N" on page 251 of Bertsekas [1976]. The necessary and sufficient conditions for optimality are given in Corollary 9.2 on page 259.

[6] Returning to the illustration of an unbounded decision problem on p.5-6, note that playing the candidate strategy, $q(X) = 0$ from initial stock S does *not* achieve the value $\frac{1}{\beta}S$. Hence, the proposed program is *not* optimal.

by the maximized value of the second argument. To verify that the first argument can never be strictly larger at the optimum, assume the contrary. Then the optimum would have to occur at $q = 0$ since the first argument is strictly decreasing in q; but then the second argument would have the same value as the first—contradicting the premise that the first is strictly larger.[7] Hence the value function and optimizer satisfying (9) must also solve:

$$V^i(S) = \max q(1 - \frac{a(q + (n-1)kS)}{S}) + \beta C(gS - q - (n-1)kS)$$
$$\text{subject to } 0 \leq q \leq [g - (n-1)k]S. \tag{10}$$

Let $F(q, S)$ denote the objective function on the right-hand side of the functional equation. Differentiating, we obtain:

$$\frac{\partial F}{\partial q} = 1 - \frac{a(q + (n-1)kS)}{S} - \frac{qa}{S} - \beta C$$
$$\frac{\partial^2 F}{\partial q^2} = -\frac{2a}{S} < 0. \tag{11}$$

Since the objective function is continuous and the constraint set is compact, an optimum exists. Since the objective function is strictly concave in q, only one solution exists to the following Kuhn-Tucker conditions:

$$q = 0 \text{ and } \frac{\partial F}{\partial q}(0, S) \leq 0, \tag{12}$$

$$q - [gS - (n-1)kS] = 0 \text{ and } \frac{\partial F}{\partial q}(gS - (n-1)kS, S) \geq 0, \tag{13}$$

$$0 < q^* < gS - (n-1)kS \text{ and } \frac{\partial F}{\partial q}(q^*, S) = 0. \tag{14}$$

The optimum occurs at neither corner provided the exogenous parameters satisfy (5). Substituting (11) into (12) and using the definition of k, it is clear if $(1 - \beta C) > 0$, then $q = 0$ is never optimal. Substituting (11) into (13) and simplifying, it is clear that if $\beta(g - nk) > 0$, then $q = gS - (n-1)k$ is never optimal.[8] Hence the optimal policy is always interior and satisfies (14).

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[7] For the proof that $C > 0$, see part IV of the Appendix.

[8] Part IV of the Appendix confirms that $1 - \beta C > 0$ while Part III confirms that $\beta(g - nk) > 0$; the proofs require that the exogenous parameters satisfy (5).

Expanding (14), we obtain:

$$\frac{\partial F}{\partial q}(q^*, S) = 0 \Rightarrow q^* = \frac{(1-\beta C)}{a(n+1)} S = kS.$$

This establishes that $q^i(S) = kS$ does maximize the right-hand side of the functional equation.

Substituting this optimizer back into the objective function, we obtain:

$$kS(1 - \frac{a(kS + (n-1)kS)}{S}) + \beta C(gS - kS - (n-1)kS) = [k(1-ank) + \beta C(g-nk)]S. \quad (15)$$

To verify that this expression is equal to $V^i(S) = CS$ for all $S \geq 0$, we must show that

$$C = k(1 - ank) + \beta C(g - nk). \quad (16)$$

Writing $k = \frac{1-\beta C}{a(n+1)}$ yields a quadratic equation with the value of C (defined in (2)) as one of its roots.[9]

Finally, we must verify that if firm i uses the rule $q^*(X) = kX$ and everyone else uses the rule $Q^{-i}(X) = (n-1)kX$ the wealth generated by firm i is CS, where S is the initial stock. Let $\beta^t \pi^i_t$ denote the present value of profit to firm i in period t:

$$\beta^t \pi^i_t = \beta^t k(1 - ank)S_t$$

$$\text{where } S_t = (g - nk)^t S, \text{ for } t = 0, 1, 2, \ldots.$$

Hence

$$\sum_{t=0}^{\infty} \beta^t \pi^i_t = \sum_{t=0}^{\infty} k(1 - ank)[\beta(g - nk)]^t S. \quad (17)$$

Since $0 < \beta(g - nk) < 1$, [10] this infinite series converges. Denote the discounted sum (wealth) as W:

$$W = [\frac{k(1 - ank)}{1 - \beta(g - nk)}]S. \quad (18)$$

Since C satisfies (16), we can rewrite (18) as $W = CS$, as was to be proved.

We have verified, therefore, that $q^i(S) = kS$ is optimal for the "related problem". Since $q^i(S) > 0$, however, this program must also be optimal for the original problem. Hence, we have proved that the set of strategies $\{q^i(S)\}_{i=1,n}$ forms a subgame perfect equilibrium.

[9] The upper bound on g in (5) was chosen to assure that the roots are real. That is, if $g > \frac{(a+\beta)(n+1)^2 - 4n\beta}{(n+1)^2 a\beta}$, the discriminant of the quadratic equation (16) would be negative and both roots would be imaginary. For details see Part I of the Appendix.

[10] See Part III of the Appendix.

Limiting Results Under Free Access

To consider the free-access limit, we let n approach infinity. From (5), it is clear that the upper bound of g expands while the lower bound does not change. Hence the free-access equilibrium is well-defined if

$$0 < \beta < 1$$
$$a \geq 1$$
$$1/\beta < g < 1/a + 1/\beta. \tag{5'}$$

It can be shown [11] that

$$\lim_{n\to\infty} C = \frac{a - \beta(g-1)}{\beta^2} > 0. \tag{19}$$

Hence, free entry does not reduce the wealth of an individual extractor to zero; nor does it reduce to zero the shadow value (C) of the stock remaining. Since aggregate extraction can be expressed in terms of C, we can calculate its limiting value:

$$\lim_{n\to\infty} nkS = \lim_{n\to\infty} \frac{n(1-\beta C)}{(n+1)a} = (g - 1/\beta)S > 0. \tag{20}$$

This differs from the aggregate extraction (S/a) deduced from the assumption of rent dissipation.[12] Indeed, from (5') it is clear that:

$$(g - 1/\beta)S < S/a. \tag{21}$$

Hence, the traditional approach overestimates aggregate extraction and overestimates rent dissipation for this particular example.

Finally, consider the stock remaining at the beginning of period t in the perfect equilibrium:

$$S_t = S_0(g - nk)^t = S_0(g - [g - 1/\beta])^t = S_0/\beta^t \tag{22}$$

for $t = 0, 1, \ldots$ Hence, the stock remaining diverges in the free-access perfect equilibrium. Recall that the traditional analysis implied *extinction* of the resource would occur for some parameter values $(1/a < g < 1 + 1/a)$. Hence, if

$$\text{MAX}(1/a, 1/\beta) < g < \text{MIN}(1 + 1/a, 1/a + 1/\beta)$$

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[11] See Part II of the Appendix.

[12] Recall the discussion on p.3-4.

or, equivalently,

$$1/\beta < g < 1 + 1/a, \tag{23}$$

then the free-access equilibrium exists and the stock remaining goes to infinity—while the traditional analysis implies extinction. *Given this example, some general result is needed to clarify when the traditional analysis of free-access problems can be used without error.*

3. Discussion

It is useful to begin discussion of this equilibrium by reviewing the problem of a sole owner of the resource. If his extraction policy is optimal then it will generate wealth $\hat{V}(S)$ and $\hat{V}(S)$ will solve the following functional equation:

$$\hat{V}(S) = \max_{0 \leq q \leq S} q[1 - \frac{aq}{S}] + \beta \hat{V}(gS - q). \tag{24}$$

It is well-known [Gale, 1967] that while an optimum exists for this single-agent decision problem if $g < 1/\beta$, no optimum exists if $g > 1/\beta$.[13]

Nonexistence can be established as follows. It can be verified that any solution must be of the form $\hat{V}(S) = \hat{C}S$. For $g < 1/\beta$, there is a unique solution of that form. For $g > 1/\beta$, however, there is no such solution. If we regard the sole-owner's problem as a one-player game ($n = 1$), the upper bound on g in (5) reduces to $1/\beta$ and hence the interval of admissible g parameters is empty. This in turn implies that the only roots of the quadratic equation (16) defining C are imaginary.

Since no solution exists to the sole-owner's problem, the supremum cannot be achieved by any feasible program. In addition, *there exists no finite supremum to this problem.* For (19) implies that even if the sole owner merely mimicked the aggregate extraction of the industry under free access, his wealth would be unbounded.[14] The significance of this observation will become clear in the next section.

[13] No optimum exists for $g = 1/\beta$ as well. For then, $C = 1/\beta$, $k = 0$, and therefore $\beta(g - nk) = 1$. Hence there exists a value function which solves the functional equation but—since the geometric series defining wealth diverges—the value cannot be achieved.

[14] More generally, suppose the sole owner used some linear extraction rule $q = \sigma S$. From (17), sole-owner profit is strictly positive in each period if $\sigma(1 - a\sigma) > 0$ or, equivalently, if $\sigma > 0$ and $\sigma < 1/a$. The sum of discounted profits diverges if $\beta(g - \sigma) > 1$ or, equivalently, if $\sigma < g - 1/\beta$. Since $g - 1/\beta < 1/a$ if (5) holds, the wealth of the sole owner diverges to plus infinity if $0 < \sigma < g - 1/\beta$.

Although no solution to the sole owner's problem exists for $g > 1/\beta$, the multi-agent equilibrium does exist when g satisfies (5). The inefficiency of the commons gives every firm an incentive to extract more today than a single agent would. The scramble for the resource is like increased impatience and causes the equilibrium to exist when the single-agent optimum does not.

In the free-access equilibrium, each individual gets positive value (CS) from carrying stock into the future. This can only happen if industry wealth is unbounded so that as n increases the wealth *per firm* (W) does not go to zero.

Because each individual has an incentive to leave some of the resource in reserve, aggregate extraction does not fully dissipate profits, extraction is smaller than Gordon's calculations would suggest and the remaining stock can *grow without limit*—rather than disappear in its entirety.

The behavior of individual wealth and aggregate extraction in the equilibrium as g increases (for $g > 1/\beta$) deserves comment. As g increases, one might expect that wealth would increase and aggregate extraction would decrease. This would indeed occur *if the other players did not alter their strategies* in response to the increased fertility of the common property. But the responses of the other players involve so strong an increase in extraction that in the new equilibrium—even though the property is more fertile—each firm earns a smaller wealth (C falls) and extracts a larger fraction of the stock (k rises) in each period.

4. Conditions Sufficient for the Traditional Approach to be Valid

Having learned from our counterexample, we can now provide weak sufficient conditions which insure that the sequence of aggregate extraction rules in a perfect equilibrium $\{Q_n(S)\}_{n=1}^{\infty}$ converges pointwise to the rent-dissipating extraction-rule, $Q^*(S)$.[15]

The rent-dissipating extraction rule implicitly solves $P = A(Q, S)$. The aggregate extraction rule in a perfect equilibrium implicitly solves the following equation:

$$P = \frac{1}{n}M(Q,S) + \frac{(n-1)}{n}A(Q,S) - \beta\tilde{V}_n'(Q,S) \tag{25}$$

[15] Standard common property examples illustrate that even when pointwise convergence occurs, uniform convergence does not. See Negri [1986, p.43 and 59] for two such examples.

where $M(Q,S)$ and $A(Q,S)$ denote, respectively, the marginal and average cost curves and $\tilde{V}_n'(Q,S)$ is the change in wealth an extractor would receive from the next period onward if aggregate current extraction marginally increased above Q and the stock at the beginning of the current period is S (i.e. the prime denotes differentiation with respect to the *first* variable). The function $\tilde{V}_n(Q,S)$ is related to $V_n(S)$ as follows: let $T(Q,S)$ be the stock next period given that the current stock is S and the current extraction is Q. Since current extraction and next period's stock must be nonnegative, $0 \leq Q \leq \bar{Q}(S)$ where $T(\bar{Q}(S),S) = 0$ implicitly defines $\bar{Q}(S)$. Define $\tilde{V}_n(Q,S) = V_n(T(Q,S))$. Assume that $V_n(\cdot)$ [respectively, $T(\cdot,S)$] is weakly concave and strictly increasing [respectively, weakly concave and strictly decreasing]. Hence $\tilde{V}_n(Q,S)$ is weakly concave and strictly decreasing in Q.[16] Moreover, assume $V_n(Q,S)$ and $T(Q,S)$ are differentiable with respect to Q and that the derivative of each is continuous.

To verify (25), note that in a perfect equilibrium, extractor i sets q^i to

$$\max_{q^i}[P - A(q^i + Q^{-i}, S)]q^i + \beta\tilde{V}_n^i(q^i + Q^{-i}, S).$$

Since $\tilde{V}_n^i$ is differentiable with respect to its first argument, we obtain the following first-order condition if q^i is interior:

$$P = A(q^i + Q^{-i}, S) + q^i A'(q^i + Q^{-i}, S) - \beta\tilde{V}_n^{i\prime}(q^i + Q^{-i}, S).$$

In a symmetric equilibrium, $q^i = \frac{Q}{n}, Q^{-i} = (\frac{n-1}{n})Q$ and $\tilde{V}_n^i(Q,S) = \tilde{V}_n(Q,S)$. Substituting, we obtain

$$P = A(Q,S) + \frac{Q}{n}A'(Q,S) - \beta\tilde{V}_n'(Q,S).$$

Since total cost can be written as $QA(Q,S)$, marginal cost is

$$M(Q,S) = A(Q,S) + QA'(Q,S).$$

Substituting, we obtain the desired result:

$$P = \frac{1}{n}M(Q,S) + \frac{(n-1)}{n}A(Q,S) - \beta\tilde{V}_n'(Q,S). \tag{26}$$

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[16] For examples where the value function in a perfect equilibrium problem fails to be concave, see Mirman [1979] and Arvan [1985].

Denote the right-hand side of (26) as $f_n(Q,S)$ and the implicit solution as $Q_n(S)$. Since $A(Q,S)$ and $M(Q,S)$ are positive, continuous and strictly increasing functions of Q, $f_n(Q,S)$ inherits these properties and any solution to (26) will be unique. Assume that $A(\bar{Q}(S),S) > P$ and $A(0,S) = M(0,S) < P$ (as in our example). Then for sufficiently large n a solution $Q_n(S)$ must exist and $0 < Q_n(S) < \bar{Q}(S)$.

To prove that $\{Q_n(S)\}$ converges to $Q^*(S)$, we make one crucial assumption which is strong enough to rule out the example in Section 2 yet weak enough to be widely satisfied: we assume that the supremum of the sole owner's wealth is finite. This suffices for the demonstration that $\{f_n(Q,S)\}$ converges uniformly to $A(Q,S)$ on compact sets which in turn assures that the sequence of implicit solutions of (26) converges to the rent-dissipating extraction.[17]

To develop this argument in more detail, note that the right-hand side of (26) is the sum of three functions of Q. Since $\frac{1}{n}M(Q,S)$ converges uniformly on compact sets to zero and $\frac{(n-1)}{n}A(Q,S)$, converges uniformly on compact sets to $A(Q,S)$, the sum converges on compact sets to $A(Q,S)$ if $\tilde{V}_n'(Q,S)$ converges uniformly to zero for $0 \leq Q \leq \bar{Q}(S)$.

In the example of Section 2, $\lim_{n\to\infty} \tilde{V}_n'(Q,S) = -C < 0$. Hence *in general* $\tilde{V}_n'(Q,S)$ does not converge to zero. Recall, however, that no finite supremum exists in that example for the sole-owner's optimization problem. Suppose we *restrict* attention to that class of problems where the supremum to the sole-owner's problem is finite.[18] Denote this supremum—which depends on the initial stock—as $J_{sole}(S)$. We exploit the fact that in a symmetric perfect-equilibrium with n extractors, each extractor's wealth is bounded by $1/n^{th}$ of this supremum: $\tilde{V}_n(Q,S) \leq J_{sole}(S)/n$ for any $0 \leq Q \leq \bar{Q}(S), S \geq 0$, and $n = 1, 2, \ldots$.

We now argue that $\tilde{V}_n'(Q,S)$ approaches the zero function uniformly for $0 \leq Q \leq \bar{Q}(S)$. Since

$$\tilde{V}_n(Q,S) = V_n(T(Q,S)),$$

[17] The traditional approach can therefore be used in examples where restricted-access equilibria exist but the sole-owner's optimum does not exist—provided the supremum is finite.

[18] This weak restriction will be shown to be *sufficient* for the traditional approach to be correct. However, it is not necessary. The traditional approach sometimes yields the correct aggregate extraction rule even when the supremum of the sole-owner's problem is not finite. Indeed, there exists a second perfect equilibrium to the example of Section 2 (associated with the second root of the quadratic equation (16)) which illustrates this point. For details, see Negri [1986, p.43].

$$\frac{\partial \tilde{V}_n}{\partial Q}(Q,S) = \tilde{V}_n'(Q,S) = \lim_{h\to 0} \frac{V_n(T(Q,S)) - V_n(T(Q+h,S))}{-h}.$$

For *any* h, the numerator of this expression is positive but smaller than $J_{sole}(S)/n$. Since $J_{sole}(S)$ is finite by assumption, this upper bound converges to zero as n increases. Hence, the slope of the chord goes to zero for any h. Thus, $\tilde{V}_n'(Q,S)$ converges pointwise to zero. Moreover, the *magnitude* of $\tilde{V}_n'(Q,S)$ is largest at $\bar{Q}(S)$ since this derivative is negative and nonincreasing. Since the derivative converges to zero at this point, it converges *uniformly* for $0 \leq Q \leq \bar{Q}(S)$. It follows that $\{f_n(Q,S)\}$ converges uniformly on compact sets to $A(Q,S)$.

To conclude the proof,[19] consider *any* convergent subsequence of aggregate outputs $\{Q_n(S)\}$ satisfying $f_n(Q_n(S),S) = P$. Denote its limit as $\hat{Q}(S)$. Since $A(Q,S)$ is continuous in Q and $\{f_n(Q,S)\}$ converges uniformly on compact sets, $\{f_n(Q_n(S),S)\}$ converges to $A(\hat{Q}(S),S)$.[20] Since $f_n(Q_n(S),S) = P$, $A(\hat{Q}(S),S) = P$. This uniquely defines the limit point since the average cost curve is strictly increasing. Thus, *every* convergent subsequence must have this same limit point—and this limit point is what we previously referred to as the rent-dissipating aggregate extraction: $\hat{Q}(S) = Q^*(S)$. Since aggregate extraction is bounded, the sequence $\{Q_n(S)\}$ must itself converge to $Q^*(S)$—as was to be proved.

5. Conclusions

We have reexamined the traditional way of analyzing common-property resources under free access which dates back to Gordon [1954]. An example was used to show that this approach is not generally correct and can be misleading. The example underlined that no one had previously analyzed the conditions under which the traditional approach is valid.

[19] We wish to thank Larry Blume for his help in structuring this argument.

[20] To restate the assertion more precisely, for any $\epsilon > 0$, there is an N such that if $n > N$, $\left|f_n(Q_n(S),S) - A(\hat{Q}(S),S)\right| < \epsilon$. From the triangle inequality, $\left|f_n(Q_n(S),S) - A(\hat{Q}(S),S)\right| \leq \left|A(Q_n(S),S) - A(\hat{Q}(S),S)\right| + |f_n(Q_n(S),S) - A(Q_n(S),S)|$. Consider the two terms on the right-hand side of this inequality. Continuity of the average-cost curve assures that for any $\epsilon > 0$, the first term can be made smaller than $\epsilon/2$ for $n > M_1$. The uniform convergence of $f_n(\cdot,S)$ on compact sets assures that the second term can be made smaller than $\epsilon/2$ for $n > M_2$. Hence for any $\epsilon > 0$, their sum can be made smaller than ϵ by choosing $n > MAX(M_1, M_2)$. Since their sum bounds the distance on the left-hand side of the inequality, the left-hand side will also then be smaller than ϵ and the assertion in the text is proved.

We derived easily-checked sufficient conditions under which the traditional approach can be used. Although the traditional approach cannot be used in our example (among others), most cases of interest satisfy our conditions. In these cases, the traditional approach—which is a tractable shortcut in the analysis of the free-access case—is valid.

APPENDIX

I

We begin by showing that the expression for C in (2) solves (16). Substituting $\frac{1-\beta C}{a(n+1)}$ for k in equation (16) and simplifying, we obtain

$$n^2\beta^2C^2 + [2n\beta - (n+1)^2x]C + 1 = 0 \qquad (A1)$$

where x is defined in (3). The roots of this quadratic are imaginary for $(n+1)^2x^2 - 4n\beta x < 0$ which occurs when x is in the interval $0 < x < \frac{4n\beta}{(n+1)^2}$. As (3) reflects, g is a linear function of x. Hence the roots in (A1) are imaginary if g lies in the following interval:

$$\frac{(a+\beta)(n+1)^2 - 4n\beta}{a\beta(n+1)^2} < g < \frac{a+\beta}{a\beta}.$$

Note that the lower bound equals $1/\beta$ when $n = 1$, increases in n, and approaches the upper bound in the limit as n goes to infinity.

The roots of (A1) are real if $g \leq \frac{(a+\beta)(n+1)^2-4n\beta}{a\beta(n+1)^2}$. In that case, $y \geq 0$ and the two roots of (A1) are:

$$\frac{(n+1)^2x - 2n\beta \pm (n+1)y^{1/2}}{2n^2\beta^2}. \qquad (A2)$$

C (defined in (2)) is the larger of these two roots.

II

We now determine the limiting value of C (the larger root) as the number of agents goes to infinity. Substituting for y in (2), dividing both the numerator and denominator by n^2, introducing the term $\frac{n+1}{n^2}$ under the square root sign, and taking the limit, we obtain:

$$\lim_{n\to\infty} C = \lim_{n\to\infty} \frac{1}{2\beta^2}\left[\frac{(n+1)^2x}{n^2} - \frac{2n\beta}{n^2} - \left[\frac{(n+1)^4x^2}{n^4} - \frac{4n\beta x(n+1)^2}{n^4}\right]^{1/2}\right] = \frac{x}{\beta^2}. \qquad (A3)$$

This result is reported as (19).

III

The infinite series in (17) converges monotonically if $0 < \beta(g - nk) < 1$. This convergence criteria is satisfied for $n > 1$ and the parameter values in (5). This can be verified by showing that $\beta(g-nk)$ is a) equal to 1 for $g = 1/\beta$, b) strictly decreasing in g in the interval

$$1/\beta < g < \frac{(a+\beta)(n+1)^2 - 4n\beta}{a\beta(n+1)^2}$$

and c) positive for g equal to the upper end of this interval.

a. Let $g = 1/\beta$. (2)-(4) imply that $x = \beta, y = \beta^2(n-1)^2, C = 1/\beta, k = 0$ and $\beta(g-nk) = 1$.

b. $$\frac{\partial}{\partial g}\beta(g-nk) = 1 + \frac{n\beta}{a(n+1)^2}\left[\frac{-(n+1)^2 a + (n+1)ay^{-1/2}[2n\beta - (n+1)^2 x]}{2n^2\beta}\right]. \quad (A4)$$

Simplifying (A4), it follows that the derivative is negative if

$$(n-1) + y^{-1/2}[2n\beta - (n+1)^2 x] < 0. \quad (A5)$$

To simplify (A5), we must first confirm that the term in brackets is negative. From (5),

$$g < \frac{(n+1)^2(a+\beta) - 4n\beta}{a\beta(n+1)^2} < \frac{(n+1)^2(a+\beta) - 2n\beta}{a\beta(n+1)^2},$$

which implies that the bracketed term in (A5) is negative. Isolating y on the left hand side of (A5)and then squaring both sides, we conclude that if

$$\frac{1}{y} > \frac{(1-n)^2}{(n+1)^2[(n+1)^2x^2 - 4n\beta x] + 4n^2\beta^2}, \quad (A6)$$

then the derivative in (A4) is negative. Substituting y for the term in brackets and collecting terms in y gives

$$y > -n\beta^2. \quad (A7)$$

As shown in Part I, $y > 0$ if g is in the interval in (5). Hence the derivative in (A4) is negative as was to be proved.

c. Finally we show that $\beta(g-nk) > 0$ for g at the upper end of the interval in (5). For that g, $x = \frac{4n\beta}{(n+1)^2}, y = 0, C = 1/n\beta$ and $k = \frac{(n-1)}{an(n+1)}$. Substituting for k and g, $\beta(g-nk)$ is positive if

$$g - nk = \frac{(n+1)^2(a+\beta) - 4n\beta}{(n+1)^2 a\beta} - \frac{n-1}{a(n+1)} > 0 \quad (A8)$$

or, alternatively, if

$$(n^2+1)a + 2n(a-\beta) + 2\beta > 0. \quad (A9)$$

If (5) is satisfied, $a \geq 1$ and the left hand side of (A9) is indeed strictly positive. Hence, $\beta(g - nk) > 0$ as was to be proved.

IV

It remains to be shown that $C > 0$ and $1 - \beta C > 0$ if g is in the interval in (5). This can be verified by showing that a) C is strictly decreasing in g, b) $C > 0$ if g is at the upper end of the interval, and c) $1 - \beta C = 0$ if g is at the lower end.

$$a. \quad \frac{\partial C}{\partial g} = \frac{-(n+1)^2 a + (n+1) y^{-1/2} a[2n\beta - (n+1)^2 x]}{2n^2\beta} < 0 \qquad (A10)$$

since $y > 0$ and the term in brackets is negative (see the discussion following (A5)).

b. If g is at the upper end of the interval in (5), $C = 1/n\beta > 0$ (see III c).

c. If g is at the lower end of the interval in (5), $x = \beta, y = \beta^2(n-1)^2, C = 1/\beta$ and therefore $1 - \beta C = 0$.

Hence $C > 0$ and $1 - \beta C > 0$ as was to be proved.

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