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Abstract. We consider a public utility that offers its service at two different times. Capacity in place can be used in both periods. We study the effects of a change from uniform pricing throughout the day to peak-load pricing, when the utility is constrained to operate with a fixed rate of return on capital. We show that there are plausible circumstances in which the introduction of peak-load pricing reduces the price of the service *both* in peak and off-peak times. We show further that peak-load pricing can lead either to greater or to smaller capacity than uniform pricing. We are able to find simple expressions that determine the size and direction of each of these effects. We also provide a straightforward criterion for determining whether a particular individual gains or loses from peak-load pricing. Some of the results are extended under different assumptions about preferences, technology and market structure.

Peak-Load Pricing–With and Without Constrained Rate of Return

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Consider a public utility that provides its product at two different times, morning and afternoon. Capacity in place can be used in both periods, but the amount consumed in either period must be no larger than capacity. Assume that demand will be greater in the afternoon than in the morning if the same price is charged in both periods. Accordingly, let us define the afternoon to be the *peak* and morning the *off-peak* demand period. The product might be electricity, or telephone service.

Suppose that in the past, the utility charged the same price in the morning as in the afternoon. This might be because time-of-day metering was too expensive, or it might be that government regulations prohibited differential pricing. What will happen to prices if the utility is now able to peak-load price? Will equilibrium capacity increase or decrease? Which consumers will gain and which will lose? We will answer these three questions for a utility that is constrained to operate with a fixed rate of return on capital. Such a constraint might be enforced by a regulatory commission, or it might be a competitive rate of return enforced by the threat of entry. We also briefly examine the case in which the utility services are provided by an unconstrained profit-maximizing monopolist.

Economists like to write and discuss papers with "surprising results". We have two such results, and finding these results was the spur that led us to write this paper. In the process of explaining and generalizing the result, we worked out the comparative statics of peak-load pricing in a neat and decisive way. It is these explicit comparative statics that we regard as the main contribution of this paper.

Our first surprise is this. Suppose that a utility is constrained to make a fixed rate of return on capacity and is not able to charge different prices in the two periods. If this utility is now allowed to charge different prices, what will happen to prices in the peak and off-peak periods? The "straightman" in each of us is supposed to reply, "The off-peak price will fall and the peak price will rise." We show that under some reasonable (and easily expressed) conditions on demand the utility will actually reduce prices *both* in the peak and in the off-peak period.

Our second surprise is to show that industry capacity may increase with the introduction of time-of-day pricing. We derive a simple condition that determines when capacity will increase or decrease. Our answer to the welfare question offers no surprises, but we are able to provide a simple criterion for determining whether an individual has positive or negative willingness to pay for having time-of-day pricing.

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elegant analysis of peak-load pricing under several alternative assumptions about market structure and regulation. They too have a surprise: under certain assumptions about elasticities of demand and the regulatory rule, a utility will charge a higher price in the *off*-peak than in the peak period. Our analysis differs from that of Bailey and White in three important respects. One is that we focus attention on the comparative statics of moving from uniform to peak-load pricing, while Bailey and White study only time-of-day prices without comparing those prices to uniform prices. The second difference is that we allow the possibility that there may be substitutability or complementarity between peak and off-peak consumption; Bailey and White confine their attention to the case where demand in each period depends only on the price in that period. The third difference is that we do not impose a particular regulatory criterion, or even require that prices be optimal according to any objective function. Our results hold for any prices that satisfy a rate-of-return constraint.

The first section presents our model. The next three sections answer the questions: what happens to prices, what happens to capacity, and who benefits when the utility moves to peak-load prices? The next two sections relate our analysis to the existing literature. We consider in Section 5 the possibility that the off-peak price will be higher than the peak price, as in Bailey and White. Price reversals can occur in our model, and our comparative static results still hold. In Section 6 we treat the case of independent demands, which has often been assumed in previous papers. We then extend our results to more general specifications of the production technology. In Section 8 we briefly examine the results when the firm is an unconstrained profit-maximizing monopolist. The concluding section relates our theoretical results to some of the empirical literature on time-of-day demands for electricity and local telephone calling.

1. The Basic Model

Technology

A public utility produces x_M units of its product in the morning and x_A units in the afternoon. The larger of these two quantities can not exceed "capacity," which we denote by K. In the long run, the amount of capacity can be varied at constant unit cost, $r > 0$.[2] In addition to capacity costs, the utility faces "user costs" of u_A per unit of output in the afternoon, and u_M per unit of output in the morning. Long-run total costs of a firm with capacity K that produces x_A in the afternoon and x_M in the morning are therefore $rK + u_A x_A + u_M x_M$. We allow user costs to differ by time of day, but assume that $r + u_A > u_M$.

[1] Related work includes Baumol and Bradford (1970); Bailey (1972, 1973); Klevorick (1971). The literature is surveyed by Brown and Sibley (1986).

[2] We consider more general production technologies in Section 7.

Let p_M and p_A denote the prices charged in the morning and the afternoon and let x_M and x_A denote total consumption in the morning and in the afternoon. Assume that preferences are weakly separable between utility services and other goods, and that preferences over utility services are homothetic. Specifically, the utility function of each consumer, i, is of the form,

$$U_i(y^i, f(x^i_M, x^i_A))$$

where y^i is i's consumption of "other goods". Although the functions U_i may be different for different consumers, we assume that the aggregator functions $f(x^i_M, x^i_A)$ are the same for all consumers and that they are homothetic, twice differentiable, and strictly quasi-concave. The assumption of homothetic separability with identical aggregators greatly simplifies analysis because it ensures that the ratio of aggregate demand for afternoon consumption to aggregate demand for morning consumption is determined by the ratio of afternoon price to morning price. As it happens, this assumption is common in the empirical literature on peak-load pricing. [3] Alternatively, instead of assuming homotheticity, we could, like Bailey and White, assume that demand in each period is independent of price in the other period. We explore this alternative later in the paper.

Let "other goods" be the *numeraire* and let aggregate demands for morning and afternoon use of the utility be functions, $x_A(p_A, p_M)$ and $x_M(p_A, p_M)$. At an interior maximum, an optimizing consumer will choose a consumption bundle such that her marginal rate of substitution between afternoon and morning consumption equals the price ratio, p_A/p_M. We define the price ratio as $\rho = p_A/p_M$. Since the functions f are homothetic and strictly quasi-concave, it must be that person i's marginal rate of substitution between afternoon and morning consumption is determined by the ratio, x^i_A/x^i_M, and is a strictly monotone decreasing function of this ratio. Then for all i,

$$MRS(x^i_M/x^i_A) = \frac{f_2(x^i_M, x^i_A)}{f_1(x^i_M, x^i_A)} = \frac{p_A}{p_M}. \tag{1}$$

Strict quasi-concavity of f implies that as the ratio x^i_M/x^i_A ranges from 0 to infinity, $MRS(x^i_M/x^i_A)$ decreases monotonically over a real interval, R_ρ. Since the function, $MRS(x^i_M/x^i_A)$, is monotonic, it has an inverse. That is, for any price ratio, $p_A/p_M \in R_\rho$, there is a unique ratio x^i_M/x^i_A such that $MRS(x^i_M/x^i_A) = p_A/p_M$. Indeed, since all individuals have the same aggregator function, f, and all face the same price ratio, p_A/p_M, it must be that x^i_A/x^i_M is the same for all i. Therefore the ratio, $x_A(p_A, p_M)/x_M(p_A, p_M)$, is determined by the price ratio p_A/p_M. These facts allow us to make the following definition.

Definition. *Define the function, $\chi(\rho)$, with domain R_ρ to be the function that is implicitly determined by the equation $MRS(\chi(\rho)) = \rho$. That is, $\chi(\rho)$ is the ratio of demand for afternoon consumption to demand for morning consumption, when their price ratio is ρ.*

[3] See, *e.g.*, Parks and Weitzel (1984); Caves *et al.* (1984), and Hausman *et al.* (1979).

We will also want to apply standard definitions of expenditure shares and elasticity of substitution to the aggregator function, f.

Definition. *Let the expenditure shares of afternoon and morning consumption be respectively*

$$\theta_A = \frac{p_A x_A}{p_M x_M + p_A x_A} \quad \text{and} \quad \theta_M = \frac{p_M x_M}{p_M x_M + p_A x_A}.$$

Definition. *Denote the elasticity of substitution between afternoon and morning consumption by* $\sigma(\rho) = -d\ln\chi(\rho)/d\ln\rho$.

Notice that since $MRS(\rho)$ is a monotone decreasing function and $\chi(\rho)$ is its inverse function, it must be that $\chi(\rho)$ is monotone decreasing. Therefore $\sigma(\rho) > 0$ for all ρ.

We assume that if prices are the same in both periods, demand in the afternoon will exceed demand in the morning. This assumption is expressed in our notation as $\chi(1) > 1$. The assumption that the afternoon is the peak-load period at uniform prices does not exclude the possibility that at some prices morning demand might be higher.

Pareto Efficiency

Where marginal costs are well-defined, a necessary condition for Pareto efficiency is that consumers' marginal rates of substitution between morning and afternoon consumption equal the ratio of marginal costs. So long as demand in the afternoon exceeds demand in the morning, an additional bit of service can be provided in the morning without changing capacity while an extra bit provided in the afternoon requires a corresponding increase in capacity. Therefore the marginal cost of utility service in the morning is u_M and the marginal cost of utility service in the afternoon is $r + u_A$ where r is the "rental cost" of capacity. Let $\rho^* = \frac{r+u_A}{u_M}$. If the ratio of afternoon price to morning price is ρ^*, then afternoon demand will equal or exceed morning demand if and only if $\chi(\rho^*) \geq 1$. Therefore if $\chi(\rho^*) \geq 1$, Pareto efficiency requires that $\rho = \rho^*$. Since, by assumption, $u_A + r > u_M$, it must be that $\rho^* > 1$.

There is a second possibility. It might be that $\chi(\rho^*) < 1$. Then at a price ratio of ρ^*, demand in the morning would exceed demand in the afternoon and so if these demands were met, the ratio of marginal costs would be not $\frac{u_A+r}{u_M}$, but rather $\frac{u_A}{u_M+r}$. In this case, Pareto efficiency will require that the amount of utility service supplied in the morning equals the amount supplied in the afternoon. When this is the case, each consumer's marginal rate of substitution between morning and afternoon consumption will be ρ^{**}, where $\chi(\rho^{**}) = 1$. Given our maintained assumption that $\chi(1) > 1$, it follows from the continuity and monotonicity of the function χ that if $\chi(\rho^*) < 1$, then there exists such a ρ^{**} and $1 < \rho^{**} < \rho^*$.

Thus, with no metering costs, Pareto efficiency requires either that demand in the two periods is equalized or that $\rho = \rho^* = (r + u_A)/u_M$. The former case applies if $\chi(\rho^*) < 1$ and the latter if $\chi(\rho^*) > 1$.

A Constrained Rate of Return and Equilibrium

In the next several sections, we assume that the public utility is constrained to operate at a fixed rate of return on capital. This constraint might be enforced by a regulatory agency, or it might be an equilibrium rate of return that is enforced by potential competition. The capital base on which the utility is allowed to earn this rate of return is proportional to its capacity. Let c_K denote the return per unit of capacity that will yield the allowable rate of return on capital. If the regulatory agency seeks a Pareto optimal outcome, it will set $c_K = r$. But it might, for various reasons, allow a rate of return $c_K > r$.[4]

The constraint on the rate of return is expressed by:

$$p_A x_A + p_M x_M - u_A x_A - u_M x_M = c_K K. \tag{2}$$

We assume that the utility produces to meet demand in each period. This means that the prices and quantities chosen must satisfy equation (1). We also assume that the utility uses its full capacity at least some time during the day, so that when $x_A > x_M$, it must be that $x_A = K$. These assumptions restrict the set of possible equilibria to a "one-dimensional continuum" determined by the parameter ρ. That is to say:

Lemma 1. *For any* $\rho \in R_\rho$ *such that* $\chi(\rho) \geq 1$, *there is exactly one set of equilibrium prices and quantities,* p_A, p_M, x_A, *and* x_M, *that satisfies equation (2) and equates capacity to peak-load demand.*

Proof. In equation (2), substitute x_A for K, substitute ρp_M for p_A, divide both sides by x_M, and rearrange. The resulting expression is

$$p_M(\rho\frac{x_A}{x_M} + 1) = (c_K + u_A)\frac{x_A}{x_M} + u_M. \tag{3}$$

But $x_A/x_M = \chi(\rho)$. It follows from equation (3) that

$$p_M = \frac{(c_K + u_A)\chi(\rho) + u_M}{1 + \rho\chi(\rho)}.$$

Therefore we see that ρ uniquely determines p_M. Since $p_A = \rho p_M$, both prices are determined by their ratio. Then quantities are determined by $x_A = x(p_A, p_M)$ and $x_M = x_M(p_A, p_M)$. ∎

In consequence of Lemma 1, we can study the comparative statics effects of moving from uniform pricing to peak-load pricing by studying the derivatives of equilibrium prices and quantities with respect to the variable ρ. Exactly what the utility will do when it is allowed to use time-of-day pricing will depend on the kind of regulation or market structure that applies in the industry. Because the rate-of-return constraint in equation (2) allows

[4] See, *e.g.*, Averch and Johnson (1962).

us to determine the equilibrium price relations (Lemma 1), we do not need to specify a particular constrained objective function for the utility. Our results hold for a number of different supply-side specifications. For example, both an Averch-Johnson regulator—who allows the firm to maximize profits subject to a maximum rate of return—and a Ramsey regulator—who requires a declining-cost firm to maximize welfare subject to a minimum rate of return—are special cases of our analysis.

Normally we would expect the utility to move relative prices in the direction that would tend to equalize demands in the two periods. But, as Bailey and White point out, under some circumstances a profit-constrained monopolist will choose to make peak-period prices *lower* than off-peak prices. Our comparative statics results can be applied to either case. When the ratio of peak to off-peak prices is increased, we will say that prices have moved toward peak-load pricing. Bailey and White call the other situation a price reversal; we discuss reversals in Section 5. Even in the "normal" case, an increase in the *ratio* of peak to off-peak price does not necessarily imply an increase in the peak-period price. In the next section, we show just when it happens that the equilibrium prices in both periods fall as ρ rises.

2. Does Peak-Load Pricing Make Peak Prices Rise or Fall?

With time-of-day pricing, it is possible to allocate capacity more efficiently between morning and afternoon use. Since the rate of return on capacity is fixed, this gain in efficiency may lead to a decline in the equilibrium prices in both periods. To see how this works, it helps to think about two special cases that are easy enough to "solve in one's head."

Two Easy Cases

In each of these special cases, we assume that user costs are zero and the allowable rate of return on capacity is c_K.

First consider the extreme case of easy substitution between morning and afternoon consumption where utility is linear in both periods' consumption but afternoon consumption is "twice as good as" morning consumption. Then the aggregator function for utility services is:

$$f(x_A, x_M) = 2x_A + x_M.$$

If there is a uniform price p in both periods, the only demand for utility services will be in the afternoon. The zero-profit constraint requires that the entire cost of capacity be repaid by afternoon usage, so that $p = c_K$. On the other hand, peak-load pricing would equalize morning and afternoon demands. This happens when $p_A = 2p_M$. At these prices, consumers are indifferent between using the service in the morning and afternoon, and consumption in both periods can be set equal to capacity. In effect, peak-load pricing allows the firm to sell its entire capacity twice, once in the morning and once in the afternoon. The profit constraint is satisfied when $p_A + p_M = c_K$. Since $p_A = 2p_M$, it must be that with peak-load pricing, $p_A = 2c_K/3$, and $p_M = c_K/3$. Moving from uniform pricing to peak-load pricing results in lower prices in both periods.

Now consider the opposite extreme-a case of perfect complements, where, at any price, consumers always want to consume exactly twice as much in the afternoon as in the morning. Let

$$f(x_A, x_M) = \min\{x_A, 2x_M\}.$$

At any price, consumers will choose $x_A/x_M = 2$. No matter what prices it chooses, the utility can sell all of its capacity in the afternoon and only half of its capacity in the morning. Therefore the zero profit condition will be satisfied for any pair of prices, p_A and p_M, for which $p_A + p_M/2 = c_K$ would satisfy the profit constraint. In this example, raising the price of peak use relative to the price of off-peak use requires an increase in the price of peak use.

The Case of Zero User Costs

More generally, assuming that $u_A = u_M = 0$ and that full capacity is used in the afternoon, equation (2) simplifies to $p_A x_A + p_M x_M = c_K x_A$. Multiply both sides of this equation by θ_A/x_A to obtain $p_A = \theta_A c_K$. It follows that an increase in the price ratio ρ will make afternoon consumption go up or down depending on whether the expenditure share θ_A is an increasing or decreasing function of ρ. A familiar result from production theory is that θ_A is an increasing (decreasing) function of ρ if and only if the elasticity of substitution σ is less than (greater than) one. Therefore as ρ is increased, the price of afternoon consumption will rise if $\sigma < 1$, fall if $\sigma > 1$, and stay constant if $\sigma = 1$.

A General Answer

Let us define the ratio of net return on morning sales to price of morning consumption as[5]

$$L_M = (p_M - u_M)/p_M = 1 - u_M/p_M.$$

Lemma 2, which is proved in the appendix, has explicit formulae for the change in price in each period as the price ratio, ρ, is changed.

Lemma 2. *Assume technology is as described above and that profits are constrained as in equation (2). Let preferences be weakly separable between utility services and other goods and let the aggregator functions for utility services be linearly homogeneous and the same for all consumers. Then for all ρ such that $\chi(\rho) > 1$,*

$$\frac{d \ln p_A}{d \ln \rho} = \theta_M(1 - L_M \sigma(\rho)) \tag{4}$$

[5] In much of the discussion below, we implicitly assume that $L_M \geq 0$. But our equations apply whether L_M is positive or negative Thus it is possible to consider cases where regulators require the utility to set the morning price below variable user cost Wenders (1976) shows that a profit-maximizing utility with a regulated rate of return on installed capital may set the off-peak price below marginal cost in order to encourage the expansion of capital intensive base-load capacity

and

$$\frac{d\ln p_M}{d\ln \rho} = \frac{d\ln p_A}{d\ln \rho} - 1 = -(\theta_A + L_M\sigma(\rho)\theta_M). \tag{5}$$

From equation (4), it is apparent that the sign of $dp_A/d\rho$ is the same as that of $1-L_M\sigma$. From equation (5), we see that $dp_M/d\rho$ is always negative. Therefore Lemma 2 allows us to claim:

Theorem 1. *Under the assumptions of Lemma 2, moving toward peak-load pricing results in: (a) lower prices in both peak and off-peak times if the elasticity of substitution between peak and off-peak consumption is greater than $p_M/(p_M - u_M)$, (b) higher prices in peak times and lower prices in off-peak times if the elasticity of substitution is less than $p_M/(p_M - u_M)$.*

3. Does Peak-Load Pricing Increase or Decrease Industry Capacity?

It may seem reasonable to expect that peak-load pricing will reduce the demand for capacity by utilities (see, *e.g.*, Berlin *et al.* (1974); Nemetz and Hankey (1989); Caves *et al.* (1984)). But it isn't necessarily so. There are two forces at work here. Peak-load pricing allows more efficient use of capacity, since less capacity is idle off-peak. This means that less capacity is required to generate a given amount of the composite commodity, *utility services*. On the other hand utility services become cheaper, which tends to increase the demand for utility services. It turns out that which effect exerts the stronger force on equilibrium capacity depends in a simple way on the elasticity of demand for utility services.

The weakly separable functional form that we have assumed for utility allows us to define a composite commodity, x, such that the quantity of x is $f(x_M, x_A)$ and the "price" p of the composite commodity is just equal to the value of the expenditure function,

$$p = e(p_M, p_A) = \min_{f(x_M, x_A)=1} p_M x_M + p_A x_A. \tag{6}$$

Since the equilibrium conditions determine p_A and p_M as functions of ρ, we can define

$$p(\rho) = e(p_M(\rho), p_A(\rho)). \tag{7}$$

Applying standard results from duality theory, we find that

$$\frac{d\ln p(\rho)}{d\ln \rho} = \theta_A \frac{d\ln p_A}{d\ln \rho} + (1-\theta_A)\frac{d\ln p_M}{d\ln \rho}. \tag{8}$$

Substituting from equations (4) and (5) into equation (8) we find that:

$$\frac{d\ln p(\rho)}{d\ln \rho} = -L_M\sigma(\rho)(1-\theta_A). \tag{9}$$

From equation (9) we are able to conclude the following:

Lemma 3. *Under the assumptions of Lemma 2, if demand in the afternoon exceeds demand in the morning at the price ratio $\rho = p_A/p_M$, then moving toward peak-load pricing must lower the price of the composite good.*

This result is not surprising, since an increase in ρ tends to equalize demand in the morning and afternoon, thus allowing more "efficient" production of the composite commodity. But whether a reduction in its price results in an increase or a decrease in expenditure on the commodity depends on the magnitude of the elasticity of demand.

Let total demand for the composite commodity, x, be $D_x(p)$ where p is the price of the composite commodity. Denote the price elasticity for the composite commodity by η. Total revenue, $p_A x_A + p_M x_M$, from the sales of utility services is equal to $pD_x(p)$. Therefore when $\chi(\rho) > 1$, so that $K = x_A$, we can write

$$\theta_A = \frac{p_A K}{p(\rho) D_x(p(\rho))}. \tag{10}$$

Logarithmically differentiating both sides of (10) and making substitutions, we are able to prove the following (see the appendix):

Lemma 4. *Under the assumptions of Lemma 2,*

$$\frac{d\ln K}{d\ln \rho} = -\theta_M\sigma(1 + L_M\eta). \tag{11}$$

From Lemmas 3 and 4, we deduce:

Theorem 2. *Under the assumptions of Lemma 2, moving toward peak-load pricing will lower the price of the composite good, utility services, and will increase or decrease the equilibrium capacity depending on whether the absolute value of the price elasticity of demand for the composite good is greater or smaller than $p_M/(p_M - u_M)$.*

In case $u_M = u_A = 0$, it is pretty easy to interpret this result. Since the composite price p falls when prices move toward peak-load pricing, total consumer expenditures must increase if the aggregate elasticity is greater than one. But the rate of return on capacity is constrained to stay constant, and since user costs are zero it must be that in equilibrium the extra consumer expenditures are spent on more capacity. If instead user costs are positive, then lowering the morning price increases off-peak utilization of the capacity which increases the total off-peak user costs. Only when the demand elasticity is large enough relative to the user cost effect ($-\eta > 1/L_M$) will an increase in total expenditure require a higher equilibrium capacity.

4. Who Gains and Who Loses From Peak-Load Pricing?

In the last section we showed that when preferences over time of use are homothetic and identical, moving from uniform pricing toward time-of-day pricing will reduce the cost of the composite good, "utility services", for every consumer. All consumers benefit from the change. Now suppose that preferences differ between individuals. If the prices of morning and afternoon consumption both fall as the system is moved toward peak-load pricing, then of course all consumers will benefit. But if the price of afternoon consumption rises and the price of morning consumption falls, then those for whom an especially large proportion of consumption is in the afternoon might be worse off.

To analyze these effects, we allow different consumers to have different aggregator functions, $f_i(x_M, x_A)$. We assume utility functions are of the form, $U_i(y^i, f^i(x_M, x_A))$, and that the functions $f^i(x_M, x_A)$ are homogeneous of degree one. We also assume that the ratio χ of *total* afternoon demand to total morning demand is determined by the ratio ρ of the afternoon price to the morning price.[6] Then, just as in the earlier sections, we can define the elasticity of substitution to be $\sigma(\rho) = -d\ln\chi(\rho)/d\ln\rho$.

The response of the equilibrium prices p_M and p_A to changes in ρ is still described by equations (4) and (5). This makes it easy to figure out whether a consumer is a net gainer from the price change. Since the aggregator functions f_i are assumed to be homothetic, all we need to do to find out whether i is a gainer or a loser is to see whether the unit cost to i of producing one unit of the aggregate $f_i(x_A, x_M)$ has gone up or down. This cost is measured by the expenditure function,

$$p_i(\rho) = e_i(p_M(\rho), p_A(\rho)) = \min_{f_i(x_M, x_A)=1} \{p_M(\rho)x_M + p_A(\rho)x_A\}.$$

From standard duality results,

$$\frac{d\ln p_i(\rho)}{d\ln\rho} = \theta^i_A \frac{d\ln p_A}{d\ln\rho} + \theta^i_M \frac{d\ln p_M}{d\ln\rho}. \tag{12}$$

where θ^i_A and θ^i_M are the afternoon's and morning's shares of i's expenditures on utility services. Substituting from equations (4) and (5) into (12) and rearranging terms, we find that

$$\frac{d\ln p^i(\rho)}{d\ln\rho} = \theta_M(1 - \sigma(\rho)L_M) - \theta^i_M. \tag{13}$$

From equation (13) we can see the following:

Lemma 5. *If preferences are as modelled in this section and technology and market structure are as modelled in Section 1, then a movement toward peak-load pricing will benefit consumer i if*

$$\frac{\theta^i_M}{\theta_M} > 1 - \sigma(\rho)L_M \tag{14}$$

and will make consumer i worse off if the inequality is reversed.

Lemma 5 gives us an exact description of who gains and who loses from peak-load pricing. From Lemma 5, we can also deduce some simple rules of thumb for determining losers and gainers. In particular, we have

Theorem 3. *Under the conditions of Lemma 5, a move toward peak-load pricing will:*

(a) *yield a Pareto improvement if $\sigma L_M > 1$;*

(b) *benefit customer i for any substitution elasticity σ if i has a higher than average off-peak expenditure share ($\theta^i_M > \theta_M$) and if $L_M \geq 0$.*

5. Price Reversals

We have so far assumed that if a public utility is enabled to use time-of-day pricing, it will increase the ratio of peak to off-peak price. Bailey and White (1974) show that this isn't necessarily so. In one of their cases, an Averch–Johnson monopolist sets prices in the two periods, but is constrained to a fixed rate of return, c_K, on capacity. The rate, c_K, exceeds the market rate of return on capital. They show that if demand in each period depends only on the price in that period, the monopolist may choose to charge a *lower* price for peak-load use than it does for off-peak use.

This also happens in our case of homothetically separable consumers. Since the Averch–Johnson monopolist is constrained to make no more than the rate of return c_K on his investment, and since c_K exceeds the market rate of return, the monopolist's objective is equivalent to maximizing his capacity subject to his rate of return on capacity being at least c_K. (Like Bailey and White, we assume that the monopolist does not include "useless" capital in its rate base.)

Lemma 4, above, tells us how capacity changes with ρ. Indeed

$$\frac{d\ln K}{d\ln\rho} = -\theta_M\sigma(1 + L_M\eta)$$

where $L_M = 1 - u_M/p_M$ and η is the elasticity of demand for the composite commodity, utility services. Thus, for example, in the case where $u_M = 0$, capacity will be a decreasing function of ρ if demand for utility services is inelastic. If this is the case, the Averch–Johnson monopolist who starts out with equal prices in the morning and the afternoon will want to reduce the ratio, ρ, of peak to off-peak prices. The underlying reason is rather simple. We have not allowed the monopolist to increase his rate base by adding useless capital. The only method available to him for increasing the size of his rate base is to use his capital "wastefully" by using a "perverse" time-of-day pricing structure. The inefficient time-of-day pricing structure makes the cost of the composite commodity higher for consumers, but this does not reduce the monopolist's revenue when demand is inelastic.

We can combine this observation with Lemma 2 to make a complete catalog of what can happen if an Averch–Johnson monopolist acquires the ability to use time-of-day pricing.

[6] If, for example, utility takes the quasi-linear form, $U_i(y^i, f^i(x_M, x_A)) = y^i + f^i(x_M, x_A)$, this assumption will be satisfied.

A summary of the results is presented in Table 1. According to Lemma 2, in the simple case where $u_M = 0$, the off-peak price will always move in the opposite direction from ρ and the peak price will move in the same or the opposite direction depending on whether the elasticity of substitution, σ, between morning and afternoon consumption, is smaller or greater than 1. As we demonstrated above, if $\eta < -1$, then the monopolist will increase ρ, and the comparative statics results presented in previous sections apply directly. If $\eta > -1$, then ρ will decrease. Therefore, we see from equation (5) that the price of morning utility services will increase. From equation (4), we see that if, in addition, $\sigma < 1$, then the price of afternoon utility services will also increase, while if $\sigma > 1$, the price of afternoon utility services will fall.

Thus, it is possible that the off-peak price will be higher than the peak-period price, and that *both* prices will be higher than the uniform price. We have shown that our surprising result holds even when there is a price reversal, although of course the signs of the derivatives change.

6. Independent Demands.

In their treatment of peak-load pricing, Bailey and White assume that demand in each time period depends only on its own price and not on the price in the other period. Our model allows the possibility that demand in each period may depend on price in the other period. Instead we make an alternative simplifying assumption, namely that preferences over utility consumption in the two periods are homothetically separable with identical aggregator functions for different people. Neither assumption is strictly more general than the other. In order for demands in each period to depend only on that period's price, it must be that utility functions are linear in consumption of other goods and additively separable for each period's utility consumption. That is, utility must have the *quasilinear* form: $U(y^i, x^i_A, x^i_M) = y^i + f^i_A(x^i_A) + f^i_M(x^i_M)$. Our model assumes that consumers have utility functions of the form $U_i(y^i, f(x^i_M, x^i_A))$ where U_i does not have to be linear in y_i and where f is homothetic, but does not have to be additively separable.[7]

The kind of comparative statics analysis that we have done for the case of homothetic separability can also be applied to the case of independent demands. Here we sketch the outlines of how this analysis goes. To simplify the exposition, we confine our attention to the case where $u_A = u_M = 0$. We can prove the following theorem.

Theorem 4. *Suppose that demand for utility services in each period depends only on the price in that period. Assume that $u_A = u_M = 0$, and that the firm is constrained to a fixed rate of return, c_K on capacity. If the utility moves from equal pricing to time-of-day pricing by reducing the price of off-peak (morning) consumption, then the price of peak (afternoon) consumption will increase if morning demand is inelastic and decrease if morning demand is elastic.*

[7] The intersection of the class of utility functions dealt with by us and by Bailey and White is the class of utility functions that are linear in consumption of other goods, and additively separable and homothetic in utility consumption. By Bergson's theorem (Burk (Bergson), 1936) on homothetic additively separable functions, this class is exactly the set of utility functions of the form $U(y^i, x^i_A, x^i_M) = y^i + \alpha(x^i_A)^\rho + \mu(x^i_M)^\rho$ where $\rho \leq 1$.

A reduction in the off-peak price is accompanied by an increase in afternoon price if and only if morning demand is inelastic. But afternoon demand is assumed to depend only on the afternoon price, and capacity is equal to the afternoon quantity demanded. If the demand curve is assumed to slope down, then it must be that a reduction in the off-peak price leads to a decrease in capacity if morning demand is inelastic. The converse also holds. Therefore we have:

Theorem 5. *Given the assumptions of Theorem 4, if the utility moves from uniform pricing toward peak-load pricing by reducing the off-peak price, then capacity will decrease (increase) if morning demand is inelastic (elastic).*

7. More General Technologies

Different Types of Plant in Peak and Off-Peak Periods

If a utility company does not use its full capacity in both the morning and the afternoon, there may be cost advantages in having one type of plant that is used all day and a second type of plant that is used only for the afternoon peak load. Such an arrangement would be particularly appealing if it is possible to reduce user costs by building a more expensive type of plant. Then it may be desirable to employ a high-capacity-cost, low-user-cost plant for all-day use and a cheaper plant with higher user costs for peak-load use.[8] [9]

There is a nice isomorphism between models of this type and the model we have already treated. Let us suppose that a utility has access to two kinds of capacity, one of which it uses all day and one of which it uses only in the afternoon. Then the utility uses $K_M = x_M$ units of the first kind of capacity and $K_A = x_A - x_M$ units of the second type. Let r_M and r_A be the costs per unit of capacity of the two types and let u_M and u_A be the corresponding user costs. Then total costs to the utility of supplying (x_M, x_A) are:

$$r_M x_M + r_A(x_A - x_M) + u_M x_M + u_A x_A = (r_A + u_A)x_A + (r_M - r_A + u_M)x_M. \quad (15).$$

Compare this expression to total costs in the model where there is only one kind of capacity. In this case, total costs are just

$$(r_A + u_A)x_A + u_M x_M. \quad (16)$$

[8] This type of technology has been discussed by Turvey (1968), Joskow (1976), and Wenders (1976), among others.

[9] A closely related possibility is that as technological change occurs, a utility company finds itself with some old capacity with high user costs and some newer capacity with lower user costs. The company may then choose to use the capacity with high user costs only at peak-load times. We will not explicitly deal with this case here, but will confine our attention to long-run equilibrium analysis. But certainly it would be possible to extend our results in this direction.

Looking at expressions (15) and (16), we see that we could write the costs for (15) in the form of (16) by defining $\tilde{u}_M = r_M - r_A + u_M$. Indeed all of our analysis of the previous sections goes through for this model if the utility is constrained to earn zero profits. If an above-market rate of return is allowed, all of our previous results apply if the regulator reinterprets the user cost in the earlier model to be $\tilde{u}_M$, which is the actual user cost *plus* the difference between the unit cost of capacity which is used all day and the unit cost of capacity which is used only in the afternoon, and allows an above-market return that is proportional to peak-load capacity, K_A.[10] In retrospect, it is easy to see why this must be. In the earlier model the only marginal cost of selling more utility services in off-peak periods was the user cost. In the current model, the long-run cost of expanding morning output includes both the user cost and the cost of replacing one kind of capacity with the other.[11]

Notice that our results would be formally correct even if it turned out that $\tilde{u}_M$ were negative. But if this were the case, we would have to be careful in interpreting the theorems since we are accustomed to assuming that marginal costs are nonnegative. As it happens, a very plausible assumption guarantees that $\tilde{u}_M \geq 0$. It is reasonable to assume that $c_M > c_A$, that is for the all-day or base-load technology to have high capacity cost (but low user cost, so that $c_M + u_M \leq c_A + u_A$). Then it is sufficient that $u_M \geq 0$ for $\tilde{u}_M = c_M - c_A + u_M \geq 0$.

Nonconstant Returns in Provision of Capacity

Our results also extend to cases where there are increasing or decreasing returns to scale in the provision of capacity. We confine our attention to the case where there is a single kind of capacity and where user costs are zero and where the firm is constrained to make zero profits.[12]

We write $c_K(K)$ to denote the *average* cost of capacity when capacity is K and we define $\epsilon_K(K)$ to be the elasticity of average cost with respect to K. That is, $\epsilon_K(K) \equiv d\ln c_K(K)/d\ln K$. Then $\epsilon_K(K)$ will be positive if there is increasing average cost and negative if there is decreasing average cost. Assuming that the marginal cost of capacity is always nonnegative, we must have $\epsilon_K(K) > -1$ for all K.

[10] If an above-market rate of return is allowed on *all* capital, the analysis is complicated because total capital is unlikely to remain proportional to peak-load capacity as prices change, which is what we need for the isomorphism.

[11] There is another simple isomorphism that is worth pointing out. Formally, in either of these models, there is no reason to distinguish between capacity costs in the peak period and user costs in the peak period. Either model would be isomorphic to a model with zero user costs in the afternoon and with afternoon capacity costs $\bar{r}_A = r_A + u_M$. Williamson (1966) has noticed this simplification in the modelling of long-run utility costs.

[12] These results apply as well to the case of a firm regulated to earn profits that are a fixed proportion of capacity costs.

Given the assumption that user costs are zero and that the average cost of capacity is now $c_K(K)$, the equilibrium condition in equation (2) can be rewritten as

$$p_A + \frac{p_M x_M}{x_A} = c_K(K). \tag{17}$$

Equation (17) is equivalent to

$$p_A = c_K(K)\theta_A. \tag{18}$$

Logarithmically differentiate to obtain:

$$\frac{d\ln p_A}{d\ln\rho} = \frac{d\ln\theta_A}{d\ln\rho} + \epsilon_K(K)\frac{d\ln K}{d\ln\rho}. \tag{19}$$

Using results and methods from previous sections, we have:

Theorem 6. *Under the assumptions of this section,*

$$\frac{d\ln K}{d\ln\rho} = -\theta_M\sigma(\rho)\frac{1+\eta}{1-\eta\epsilon_K(K)}. \tag{20}$$

It is trivial to show that $1 - \eta\epsilon_K(K) > 0$ is necessary for an equilibrium to exist; if the inequality is reversed, then the economies of scale in capacity and the demand responsiveness of the composite good demand would feed on each other to drive equilibrium output to infinity. Therefore, even when costs are non-constant, the result of Theorem 2 continues to hold: equilibrium capacity will increase or decrease depending on whether the absolute value of the price elasticity is greater or smaller than one.

The effect of nonconstant costs is to magnify or diminish the equilibrium capacity change, as the cost elasticity is negative or positive. For instance, if $\epsilon_K < 0$ then the denominator of (20) is less than one, and the change in K is magnified. If demand is elastic ($\eta < -1$), capacity will increase more the larger are the cost economies (negative ϵ_K). This is so because the price efficiency gain from peak-load pricing is amplified by the cost efficiency gain as scale economies are achieved (leading to even lower prices; see below).

Substitute equation (20) into (19), to find:

Theorem 7. *Under the assumptions of Theorem 6,*

$$\frac{d\ln p_A}{d\ln\rho} = \theta_M\left(1 - \sigma(\rho)\frac{1+\epsilon_K(K)}{1-\eta\epsilon_K(K)}\right) \tag{21}$$

and

$$\frac{d\ln p_M}{d\ln\rho} = -\left(\theta_A + \theta_M\sigma(\rho)\frac{1+\epsilon_K(K)}{1-\eta\epsilon_K(K)}\right). \tag{22}$$

Similar results can be found or the case of nonconstant capacity costs and nonzero user off-peak user costs u_M. In particular, we have[13]

[13] Proof available from the authors on request.

Theorem 8. *Under the assumptions of Theorem 1, with the additional assumption that unit capacity costs may be a non-constant function of capacity, $c_K(K)$, moving towards peak-load pricing results in*

$$\frac{d\ln p_A}{d\ln\rho} = \theta_M\left(1-\sigma(\rho)\frac{L_M+\hat{\epsilon}_K(K)}{1-\eta\hat{\epsilon}_K(K)}\right) \tag{23}$$

$$\frac{d\ln p_M}{d\ln\rho} = -\left(\theta_A+\theta_M\sigma(\rho)\frac{L_M+\hat{\epsilon}_K(K)}{1-\eta\hat{\epsilon}_K(K)}\right) \tag{24}$$

$$\frac{d\ln K}{d\ln\rho} = -\sigma(\rho)\theta_M\left(\frac{1+L_M\eta}{1-\eta\hat{\epsilon}_K(K)}\right) \tag{25}$$

where

$$\hat{\epsilon}_K(K) = \epsilon_K(K)\frac{(p_M-u_M)x_M+p_Ax_A}{p_Mx_M+p_Ax_A}. \tag{26}$$

8. An Unconstrained Monopolist

To this point we have considered equilibrium prices and quantities only when the firm operates under a profit constraint. Some analyses of public utility pricing assume that the firm is able to behave as an unconstrained monopolist. In this section we examine the prices and quantities set by an unconstrained monopolist under uniform and peak-load pricing.[14] In the simplest case of independent demands and concave profit functions, we obtain the "normal" result that peak-load pricing leads to an increase in the peak period price, and a decrease in equilibrium capacity. With more general preferences it is again possible that both peak and off-peak prices can be lower than the uniform price.

Recall that with a weakly separable utility function, we can define a composite commodity, x, equal to $f(x_M,x_A)$, for which there is a composite price, $p(p_M,p_A)$, defined at equation (6). The consumer's problem can be broken into two stages. In the first stage, the consumer allocates income between y and x; then in the second stage the consumer chooses x_M and x_A to maximize $f(x_M,x_A)$ subject to the expenditure on utility services set in the first stage. We saw in equation (1) that the second stage allocation of expenditure between peak and off-peak services implied

$$\frac{f_2(x_M,x_A)}{f_1(x_M,x_A)} = \frac{p_A}{p_M}.$$

[14] Peak-load pricing by an unconstrained monopolist is a form of third-degree price discrimination. The welfare effects of third-degree price discrimination were first analyzed by Pigou (1932), and have been recently analyzed under varying conditions by Schmalensee (1981), Varian (1985), Hausman and MacKie-Mason (1988), and Schwartz (1989). However, none of these authors develop comparative statics for price or capacity.

It is convenient to think of the unregulated monopolist as a seller of the composite good, x, which is produced according to the production function $f(x_M,x_A)$. The monopolist chooses input levels of the "factors" x_M and x_A to maximize profits,

$$\max\ p(f(x_M,x_A))f(x_M,x_A) - u_Mx_M - (u_A+r)x_A.$$

Taking the ratio of the first-order conditions and using equation (1) we find that the equilibrium price ratio will be

$$\frac{p_A}{p_M} = \frac{u_A+r}{u_M}.$$

Then, since we have assumed that $u_A+r > u_M$, we can state the following result.

Theorem 9. *Assume that technology is as described above, and that the firm is an unregulated monopolist. Let preferences be as described in Lemma 2. Then the profit-maximizing time-of-day prices satisfy $p_A > p_M$.*

We have found that a price reversal is not possible. Yet Bailey and White showed that price reversals could occur with an unregulated monopolist under their assumptions. The difference in results is not due to their restriction that demands be independent, since our result also holds for independent demands. Rather, it is the homotheticity restriction that rules out a reversal when prices are set by an unconstrained monopolist. With independent demands, the monopoly prices satisfy the usual inverse-elasticity mark-up on marginal cost. Since peak-period marginal cost is higher, a necessary condition for a pricing reversal is that the off-peak demand elasticity be lower. It can be shown, however, that with homothetic utility and independent demands, the demand elasticities in each period must be identical. A similar argument shows that a reversal is impossible for interdependent but homothetic demands as well, because the sums of own and cross-price elasticities are identical for the two periods.[15] [16]

We next ask whether the peak-period price increases or decreases if the monopolist changes from uniform to time-of-day pricing. We do this only for the case of homothetic preferences with independent demands and concave profit functions. The result is straightforward:

Theorem 10. *Assume the conditions of Theorem 9 hold. Assume further that demand for service in each period depends only on the price in that period, and that the utility's profit functions for each period viewed as separate submarkets are concave. Then the*

[15] The restrictions on the demand elasticities can be derived by recalling that with homothetic preferences, the demand ratio is a function of the price ratio, $x_A/x_M = \chi(\rho)$. Demand elasticities are obtained by logarithmically differentiating, and using the properties of the expenditure function to equate cross-partials.

[16] We found that price reversals *were* possible with homothetic demands for a monopolist with a constrained rate of return. In that case, a reversal is possible with equal demand elasticities when the profit constraint is tight enough, because the firm will substantially lower the peak-period price to increase the capacity base.

peak-period price is higher than the uniform price, and the off-peak price is lower than the uniform price.

When profits are not concave in price, Nahata, Ostaszewski and Sahoo (1989) have shown that a third-degree price-discriminating monopolist may either raise both prices above the uniform price, or lower both prices, thus returning us to the situation described in Section 2 for a rate-of-return regulated monopolist. Naturally, in the case when demands are interdependent, it is also possible that both peak and off-peak prices can either fall or rise.

Since the demand for peak and off-peak services are independent, the effect of time-of-day pricing on equilibrium capacity follows directly from the conclusion that the peak price is higher than the uniform price.

Theorem 11. *Under the conditions of Theorem 10, equilibrium capacity falls if an unconstrained monopolist moves from uniform pricing to time-of-day pricing.*

9. Remarks

As public utilities and their regulators decide whether to move towards time-of-use pricing, they will need to know the answers to the questions we have posed: what will happen to the prices in the different periods; what will happen to equilibrium capacity; and who will benefit, who will lose? We have provided simple results that determine the answers to these questions when the firm is constrained to earn a fixed rate of return on installed capital, and faces demands determined by separable and homothetic preferences. We also extended some of the results to cases in which the firm is an unconstrained monopolist, or in which demands are nonhomothetic and independent.

Most of the empirical research on time-of-day pricing for residential electricity has assumed that preferences are separable and homothetic; see, *e.g.*, Caves *et al.* (1984), Howrey and Varian (1984), Parks and Weitzel (1984), Caves and Christensen (1980a, 1980b), Atkinson (1979), and Hausman *et al.* (1979).[17] Thus, our analysis is quite relevant for the application of the estimated price and substitution elasticities from this literature to predict equilibrium prices and capacity from a change to time-of-day pricing.

What does the empirical literature say about the critical parameters that we have identified, in particular the elasticity of substitution between peak and off-peak services and the demand elasticity for the composite utility services commodity? In the time-of-day electricity literature, most estimates of the substitution elasticity are quite low; the range of 0.10 to 0.14 reported in Caves *et al.* (1984) is typical. This suggests that our first surprising result—that the peak-period price may be lower than the uniform price—is unlikely for residential electricity demand. However, as noted by Acton (1982), through no fault of the researchers most of the estimates rely on the use of sophisticated econometric analyses of underidentified experimental data, and thus are somewhat suspect until data sets with greater price variation are produced. Further, most of the studies have examined demand over no more than one year. Yet the long-run substitutability of off-peak for peak demand will depend in large part on changes in the consumer portfolios of electricity-using appliances, and on technological advances in time-shifting of demand.

Most of the data sets on which time-of-day electricity demands have been estimated have not provided sufficient detail about the household's income and other expenditures to estimate the price elasticity of the electricity composite good (Parks and Weitzel (1984) discuss this problem). If we look back to studies of electricity demand before time-of-day pricing we find that the long-run elasticity of demand was typically estimated to be greater than one in magnitude; Taylor (1977) surveys a number of such studies. Our results show that if the long-run demand elasticity is greater than one, then equilibrium capacity under time-of-day pricing may actually be greater than under uniform pricing. With more efficient pricing and elastic demand for utility services, consumers will want to spend more on those services; with a constrained rate of return on capacity, equilibrium may require that capacity be expanded.

There has been much recent interest in time-of-day pricing for local telephone use. However, there have been few published studies of local telephone time-of-day demand.[18] It is premature to draw conclusions about the long-run elasticity of substitution and the price elasticity of a household's composite local telephone demand.

Our analysis indicates the importance of having good estimates of a few preference parameters if we wish to forecast the price, capacity and distributional effects of peak-load pricing. There have been few attempts to estimate these parameters for local telephone usage; many attempts to estimate the substitution elasticity for electricity usage were hampered by data limitations. Future empirical research in this area should seek to obtain good estimates of these parameters. Design of future experiments should also take our results into account. For instance, most experiments to date have been designed with the implicit assumption that the peak price would be higher, and the off-peak price lower than the pre-existing uniform price.[19] Should a long-run *equilibrium require one of the* different price configurations that we have identified, the elasticity estimates derived from these experiments will be useful only to the extent that they are good predictors well outside the range of the experimental design. Such an assumption may be warranted for electricity demand (although long-run substitution elasticities are likely to be higher than the mostly short-run elasticities estimated to date), but we have too little evidence to draw any conclusions yet for local telephone demand.

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[17] Baker *et al.* (1989) and Hausman and Trimble (1984) do not impose the homotheticity restriction.

[18] MacKie-Mason is currently undertaking such a study

[19] See Manning *et al.* (1979) for a detailed discussion of the design of the Los Angeles Electric Rate Demonstration project, which is typical in this regard.

Appendix

The results in this appendix are all proved under the assumptions made for Lemma 2 of the text, unless otherwise indicated.

Fact 1. *Where $\sigma(\rho)$ is the (negative of the) elasticity of substitution between x_M and x_A, at the price ratio, ρ,*

$$\frac{d\ln\theta_A}{d\ln\rho} = \theta_M(1-\sigma(\rho)).$$

Proof. The utility function is of the form $U_i(y^i, f(x_M, x_A))$, where $f(x_M, x_A)$ is strictly quasi-concave and homogeneous of degree one. Consumer optimization therefore implies that:

$$\frac{f_2(x^i_M, x^i_A)}{f_1(x^i_M, x^i_A)} = \rho \tag{A.1}$$

where $\rho = p_A/p_M$.

Since f is strictly quasi-concave, the slope of an isoquant is uniquely determined by the ratio x_A/x_M. Therefore the equilibrium condition (A.1) uniquely determines the ratio $x_A/x_M = \chi(\rho)$. Evidently

$$\rho\chi(\rho) = p_A x_A / p_M x_M = \theta_A/(1-\theta_A). \tag{A.2}$$

It follows that $\theta_A = \rho\chi(\rho)/(1+\rho\chi(\rho))$. Therefore equilibrium values of θ_A are uniquely determined by ρ, and we can define the function $\theta_A(\rho)$ accordingly.

Most of the results are more conveniently derived in the form of elasticities, so we work largely with logarithmic derivatives. From (A.2), it follows that

$$\ln\theta_A - \ln(1-\theta_A) = \ln\rho + \ln\chi(\rho). \tag{A.3}$$

The elasticity of substitution is $\sigma(\rho) = -d\ln\chi(\rho)/d\ln(\rho)$. Differentiating both sides of equation (A.3) with respect to $\ln\rho$, we have

$$\frac{d\ln\theta_A}{d\ln\rho}\left(1+\frac{\theta_A}{1-\theta_A}\right) = 1-\sigma(\rho). \tag{A.4}$$

Multiply both sides of (A.4) by $(1-\theta_A)$ and recall that $(1-\theta_A) = \theta_M$ to obtain Fact 1. ∎

Proof of Lemma 2.

If we multiply both sides of text equation (2) by θ_A/x_A, we have:

$$p_A = \left(c_K + u_A + u_M\frac{x_M}{x_A}\right)\theta_A. \tag{A.5}$$

From the fact that $\theta_M = 1-\theta_A$, it follows that $\frac{x_M}{x_A}\theta_A = \rho(1-\theta_A)$. Therefore equation (A.5) is equivalent to

$$p_A = (c_K + u_A - u_M\rho)\theta_A + u_M\rho. \tag{A.6}$$

Differentiating, we have:

$$\frac{dp_A}{d\rho} = (c_K + u_A - u_M\rho)\frac{d\theta_A}{d\rho} + u_M(1-\theta_A).$$

Then multiplying both sides by ρ/p_A and simplifying, we have:

$$\frac{d\ln p_A}{d\ln\rho} = (c_K + u_A - u_M\rho)\frac{\theta_A}{p_A}\frac{d\ln\theta_A}{d\ln\rho} + \frac{u_M}{p_A}\rho(1-\theta_A). \tag{A.7}$$

From (A.6) and the definition of L_M, we see that

$$(c_K + u_A - u_M\rho)\frac{\theta_A}{p_A} = 1 - \frac{u_M}{p_A}\rho = 1 - \frac{u_M}{p_M} = L_M. \tag{A.8}$$

Therefore, using Fact 1 and (A.8) we can simplify (A.7) to

$$\frac{d\ln p_A}{d\ln\rho} = L_M\theta_M(1-\sigma(\rho)) + \frac{u_M}{p_A}\rho(1-\theta_A). \tag{A.9}$$

Since $\frac{u_M}{p_A}\rho(1-\theta_A) = (1-L_M)\theta_M$, we can further simplify (A.9) to

$$\frac{d\ln p_A}{d\ln\rho} = \theta_M(1-L_M\sigma(\rho)).$$

This is the first equation claimed in Lemma 2. The second equation is a trivial consequence of the first. ∎

Proof of Lemma 4.

Logarithmically differentiate text equation (10) to get

$$\begin{aligned}\frac{d\ln K}{d\ln\rho} &= \frac{d\ln\theta_A}{d\ln\rho} - \frac{d\ln p_A}{d\ln\rho} + \frac{d\ln p}{d\ln\rho} + \frac{d\ln D_x}{d\ln p}\frac{d\ln p}{d\ln\rho} \\ &= \frac{d\ln\theta_A}{d\ln\rho} - \frac{d\ln p_A}{d\ln\rho} + \frac{d\ln p}{d\ln\rho}(1+\eta).\end{aligned} \tag{A.10}$$

Substituting the results of Fact 1, Lemma 2, and text equation (9), yields the result. ∎

Proof of Theorem 4.

The constraint on the profit rate can be written as $R(p_M) = (c_K - p_A)x_A(p_A)$ where $R(p_M) = p_M x_M(p_M)$. Differentiate both sides of this equation to obtain $R'(p_M) = [(c_K -$

$p_A)x'_A(p_A) - x_A(p_A)]\frac{dp_A}{dp_M}$. But $(c_K - p_A)x'_A(p_A) - x_A(p_A) < 0$. Therefore $\frac{dp_A}{dp_M}$ is of the opposite sign from $R'(p_M)$. Now $R'(p_M)$ is positive (negative) if demand in the morning is inelastic (elastic). Therefore $\frac{dp_A}{dp_M}$ is negative (positive) if demand in the morning is inelastic (elastic). It follows that if morning price is reduced, then afternoon price will increase (decrease) if morning demand is inelastic (elastic). ∎

Proof of Theorem 6.

Since we assume that user costs are zero but capacity costs are non-constant, we can rewrite (A.5) as

$$p_A = c_K(K)\theta_A. \tag{A.11}$$

Logarithmically differentiating,

$$\begin{aligned}\frac{d\ln p_A}{d\ln\rho} &= \frac{d\ln\theta_A}{d\ln\rho} + \frac{d\ln c_K(K)}{d\ln K}\frac{d\ln K}{d\ln\rho} \\ &= \theta_M(1-\sigma(\rho)) + \epsilon_K(K)\frac{d\ln K}{d\ln\rho}\end{aligned} \tag{A.12}$$

where the second equality follows from using Fact 1 and the definition of the cost elasticity, ϵ_K. Then, since $\rho = p_A/p_M$, we know that

$$\frac{d\ln p_M}{d\ln\rho} = \frac{d\ln p_A}{d\ln\rho} - 1, \tag{A.13}$$

so

$$\frac{d\ln p_M}{d\ln\rho} = -\theta_A - \sigma\theta_M + \epsilon_K(K)\frac{d\ln K}{d\ln\rho} \tag{A.14}$$

by substituting (A.12). Then we can rewrite text equation (8) as

$$\begin{aligned}\frac{d\ln p(\rho)}{d\ln\rho} &= \theta_A\left[\frac{d\ln p_A}{d\ln\rho} - \frac{d\ln p_M}{d\ln\rho}\right] + \frac{d\ln p_M}{d\ln\rho} \\ &= \theta_A + \frac{d\ln p_M}{d\ln\rho} \\ &= -\sigma\theta_M + \epsilon_K(K)\frac{d\ln K}{d\ln\rho}\end{aligned} \tag{A.15}$$

where the second equality follows from (A.13) and the third equality from substituting (A.14).

We now need to find the change in K. We can begin with equation (A.10). Substituting in (A.12) gives

$$\begin{aligned}\frac{d\ln K}{d\ln\rho} &= -\epsilon_K(K)\frac{d\ln K}{d\ln\rho} + (1+\eta)\frac{d\ln p(\rho)}{d\ln\rho} \\ &= -\sigma\theta_M(1+\eta) + \eta\epsilon_K(K)\frac{d\ln K}{d\ln\rho} \\ &= -\sigma\theta_M\frac{1+\eta}{1-\eta\epsilon_K(K)}\end{aligned} \tag{A.16}$$

using (A.15) to get the second equality and rearranging to obtain the third. ∎

Proof of Theorem 10.

Let the optimal uniform price be $\bar{p}$. Suppose that the uniform price is higher than both optimal time-of-day prices. Since demands are independent, charging instead a uniform price equal to the time-of-day peak-period price would maximize profits from that service. With a concave profit function for off-peak services, off-peak profits would also increase by moving the uniform price closer to the time-of-day optimum. Thus, the uniform price cannot be higher than the time-of-day peak-period price. Strict inequality is established by noting that marginal peak-period profit is zero at the optimal peak-period price, but marginal off-peak profits are negative at that price, so the uniform price will be set lower than the peak-period price. A similar argument works for a uniform price below p_M, so $p_M < \bar{p} < p_A$. ∎

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Table 1. Summary of Price Results			
	Change relative to a uniform price		
	Peak price	Off-peak price	Capacity
1. No price reversal: $p_A > p_M$			
a. $\sigma < \frac{p_M}{p_M - u_M}$	higher	lower	
b. $\sigma > \frac{p_M}{p_M - u_M}$	lower	lower	
c. $\lvert\eta\rvert > \frac{p_M}{p_M - u_M}$			higher
2. Price reversal: $p_A < p_M$			
a. $\sigma < \frac{p_M}{p_M - u_M}$	lower	higher	
b. $\sigma > \frac{p_M}{p_M - u_M}$	higher	higher	
c. $\lvert\eta\rvert > \frac{p_M}{p_M - u_M}$			lower

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