INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust Nonlinear Control 2015; **25**:2515–2531
Published online 1 August 2014 in Wiley Online Library (wileyonlinelibrary.com). DOI: 10.1002/rnc.3224

The minimum gain lemma

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SUMMARY

This paper focuses on the newly developed notion of minimum gain and the corresponding Large Gain Theorem. The Large Gain Theorem is an input–output stability result particularly well suited to unstable plants connected in feedback with stable or unstable controllers. This paper aims to facilitate the practical application of these results. An altered definition of minimum gain broadens the applicability of the Large Gain Theorem, and the novel Minimum Gain Lemma provides LMI conditions that imply and are often equivalent to a minimum gain for LTI systems. Numerical examples are provided to clarify the differences between the existing and proposed definitions of minimum gain, highlight the utility of the newly established Minimum Gain Lemma, and demonstrate how the paper's contributions may be employed in practice.

Received 9 January 2014; Revised 29 April 2014; Accepted 18 June 2014

KEY WORDS: minimum gain; Large Gain Theorem; input–output stability; linear matrix inequality; robust stability

1. INTRODUCTION

A variety of techniques can be used to assess the closed-loop stability of a plant connected in negative feedback with a controller. In a SISO LTI context, analysis can often be performed in the frequency domain. For instance, the Nyquist stability criteria can be used to determine stability, while robustness of the closed loop to perturbations can be characterized in terms of gain and phase margins. Although frequency-domain-based criteria remain in common use, input–output stability has proven vital to the analysis of closed-loop stability. A system is input–output stable if it maps L_2 inputs to L_2 outputs. In theoretical terms, the importance of this notion is clear; by describing systems in terms of mappings between function spaces, control engineers may capitalize on the tools of functional analysis. These resources have proven especially useful in the fields of robust, nonlinear, and optimal control. For instance, the Passivity Theorem [1, 2] is often employed when plant passivity is robust with respect to perturbations. When dealing with nonlinear plants, frequency-domain properties cease to be meaningful, and it is often natural to work with input–output stability criteria. Similarly, when performance objectives are stated in the time domain, input–output stability notions may be preferred.

Although existing input–output stability results have proven useful in many applications, obstacles remain that inhibit their use. For instance, engineers must often design controllers for open-loop unstable plants, which prohibit the application of the two most popular input–output stability results, the Small Gain and Passivity Theorems. The abundance of unstable systems that must be controlled demands the further development and exploration of input–output stability results

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adapted to such problems. Inspired by the conditions for instability developed in [1, 3, 158-162], the concept of minimum gain and the corresponding Large Gain Theorem was introduced in [4] as input–output stability analysis tools tailor-made for unstable systems. A system's gain is the supremum of the ratio between the norm of the output and input taken over all nonzero inputs. As might be expected, Zahedzadeh *et al*. [4] defined the minimum gain of a system to be the infimum of the same ratio. The Large Gain Theorem states that if a plant has a finite, nonzero minimum gain, then any controller with adequately large minimum gain will stabilize their closed-loop negative feedback interconnection.

Despite directly addressing a demand in the literature, the promising results of Zahedzadeh *et al.* [4] are not yet in common use. The reason for this is evident: methods to systematically determine minimum gains and impose them upon controllers to ensure stability via the Large Gain Theorem do not yet exist. It is the objective of this paper to assuage these problems. To begin, a more flexible, but similar, definition of minimum gain is suggested in order to broaden the applicability of the Large Gain Theorem. The proposed definition explicitly incorporates initial conditions, which is in contrast to the original definition where the plant and controller were required to have null initial conditions. The most important alteration to the definition of minimum gain was inspired by noting that the Small Gain Theorem is applicable even when only an upper bound on the gain is known. The modified Large Gain Theorem is likewise applicable with only a lower bound. In accordance with this observation, the proposed definition of minimum gain replaces the infimum with a lower bound. Together, the alterations admit stable LTI systems with positive minimum gains, further broadening the applicability of the Large Gain Theorem, which previously could not be applied to such systems.

In seeking a systematic approach to identify minimum gain, inspiration was imparted by the Dissipativity Lemma, developed in [5] and preceded by [6, 7], which provides LMI conditions equivalent to QSR-dissipativity for stable, LTI systems. The Dissipativity Theorem [8] has enjoyed improved useability because of the readily verifiable conditions of the Dissipativity Lemma. One of its corollaries, the Conic Sector Lemma [9, 10], has enabled the development of nearly $\mathcal{H}_2$-optimal controller design procedures with guaranteed input–output stability for conic plants [11, 12] and the design of linear-quadratic-Gaussian controllers [10] with robust stability guarantees. Because the definition of minimum gain suggested here is a sub-case of QSR-dissipativity, it is desirable to employ the Dissipativity Lemma, but considering plants of known minimum gain, rather than conic ones. Because the Dissipativity Lemma of Gupta [5] is only applicable to interconnections of stable systems, it is ill-suited to this application because the Large Gain Lemma is chiefly of service when considering negative feedback interconnections involving one or more unstable systems. Conversely, Vasegh and Ghaderi [13] present LMI conditions implying a lower bound on the minimum gain of an unstable LTI system. However, as it dealt with the original definition of minimum gain, Vasegh and Ghaderi [13] does not consider nonzero initial conditions or stable systems with nonzero minimum gains. Likewise, earlier results for dissipative systems, [6, 7], rely on assumptions of reachability or other properties not required here. The primary contribution of this paper is derived with techniques distinct from those employed in the related work of [5–7, 13], rendering a minimally restrictive result that is applicable to stable and unstable systems; the Minimum Gain Lemma provides LMI conditions equivalent to minimum gain for LTI systems, enabling the development of future analysis and synthesis methods.

The novel contributions of this paper are a modified definition of minimum gain, a correspondingly modified version of the Large Gain Theorem, featuring broadened applicability, and the Minimum Gain Lemma, which facilitates the practical application of this theorem by providing easily verifiable LMI conditions equivalent to minimum gain for LTI systems. Additionally, three numerical examples highlight the subtle features of minimum gain, the Large Gain Theorem, and their application. These developments promise to enable future analysis and synthesis techniques capitalizing on the stability guarantees of the Large Gain Theorem.

The paper is organized as follows. Section 2 introduces notation and definitions. Section 3 presents the Minimum Gain Lemma and its proof, which are preceded by the solution of an optimization problem upon which they rely. Three numerical examples are provided in Section 4. In order to clarify the changes to the definition of minimum gain and demonstrate the practical advantages incurred, the first example revisits an example originally presented in [4]. The second example presents a simple system for which the minimum gain is calculated directly from the definition, thus verifying the Minimum Gain Lemma and numerical solvers used to implement it. The system also proves well-suited to simple, effective controller synthesis. The final example presents a system for which analytical calculation of minimum gain is prohibitively difficult. In this case, the Minimum Gain Lemma is used to robustly establish the system's minimum gain through exploration of the parameter space. Motivated by the abundance of trajectory tracking problems in engineering applications, the effectiveness of minimum gain controllers is explored for such a problem. Having computed minimum gains for three systems in Section 4.1, one is used as a controller in trajectory tracking problems where perturbed versions of the other two are plants. Section 5 summarizes the results.

2. PRELIMINARIES

To begin, $\mathbf{M} < 0$ indicates that the matrix $\mathbf{M}$ is negative definite. Positive definiteness, positive semi-definiteness, and negative semi-definiteness are denoted correspondingly. Asterisks replace duplicate blocks in symmetric matrices. The nth standard basis vector is denoted $\mathbf{e}_n$. The expression $\mathcal{C}^n(v_1, \ldots, v_k)$ denotes the space of functions that are n-times continuously differentiable with respect to variables $v_1, \ldots, v_k$. The partial derivative of $f(v_1, \ldots, v_k)$ with respect to v_i is often denoted f_{v_i} for brevity.

Let $|\cdot|$ denote the standard Euclidean norm. Recall that $\mathbf{y} \in L_2$ if $\|\mathbf{y}\|_2^2 = \int_0^\infty |\mathbf{y}(t)|^2\, dt < \infty$, and $\mathbf{y} \in L_{2e}$ if $\|\mathbf{y}\|_{2T}^2 = \int_0^\infty |\mathbf{y}_T|^2\, dt < \infty$, $T \in \mathbb{R}^+$ where $\mathbf{y}_T(t) = \mathbf{y}(t)$ for $0 \leqslant t \leqslant T$ and $\mathbf{y}_T(t) = \mathbf{0}$, $t > T$. With basic notation established, the notion of minimum gain may be presented. The forthcoming definition varies from the original one presented in [4]. In Section 4, the alterations will serve to accommodate for nonzero initial conditions and stable LTI systems. In fact, the increased flexibility garnered by these alterations is a significant contribution of this paper.

Definition 2.1 (Minimum Gain)
A causal system, $\mathcal{G} : L_{2e} \to L_{2e}$, has minimum gain $0 \leqslant \nu < \infty$ if there exists β, depending only on the initial conditions, such that

$$\|\mathcal{G}\mathbf{u}\|_{2T} - \nu\|\mathbf{u}\|_{2T} \geqslant \beta,\ \forall \mathbf{u} \in L_{2e},\ \forall T \in \mathbb{R}^+. \tag{1}$$

Minimum gain properties can be used in conjunction with the Large Gain Theorem to demonstrate input–output stability. As stated here, the theorem involves the altered definition of minimum gain.

Theorem 2.1
(Large Gain Theorem) Consider the negative feedback interconnection of two systems, $\mathcal{G}_1 : L_{2e} \to L_{2e}$, and $\mathcal{G}_2 : L_{2e} \to L_{2e}$, defined as

$$\begin{aligned} &\mathbf{y} = \begin{bmatrix} \mathbf{y}_1^\mathsf{T} & \mathbf{y}_2^\mathsf{T} \end{bmatrix}^\mathsf{T}, && \mathbf{y}_1 = \mathcal{G}_1\mathbf{u}_1, && \mathbf{y}_2 = \mathcal{G}_2\mathbf{u}_2, \\ &\boldsymbol{r} = \begin{bmatrix} \boldsymbol{r}_1^\mathsf{T} & \boldsymbol{r}_2^\mathsf{T} \end{bmatrix}^\mathsf{T}, && \mathbf{u}_1 = \boldsymbol{r}_1 - \mathbf{y}_2, && \mathbf{u}_2 = \boldsymbol{r}_2 + \mathbf{y}_1. \end{aligned}$$

If $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively, have minimum gains equal to ν_1 and ν_2, satisfying

$$1 < \nu_1\nu_2 < \infty,$$

then the closed-loop system, $\mathbf{y} = \mathcal{G}\boldsymbol{r}$, is input–output stable.

A proof of Theorem 2.1, following in the vein of [4], is presented in order to ensure that this version of the Large Gain Theorem holds, as it incorporates the proposed alterations to the definition of minimum gain.

DOI: 10.1002/rnc

Proof
As is standard with input–output stability proofs, the norm of the output will be bounded in terms of the norm of the input, provided the conditions stated in Theorem 2.1 are satisfied. To begin, the triangle inequality implies that

$$\begin{aligned}\|\mathbf{y}_1\|_{2T} &= \|\mathbf{u}_2 - \boldsymbol{r}_2\|_{2T} \leqslant \|\boldsymbol{r}_2\|_{2T} + \|\mathbf{u}_2\|_{2T} \text{ and}\\ \|\mathbf{y}_2\|_{2T} &= \|\boldsymbol{r}_1 - \mathbf{u}_1\|_{2T} \leqslant \|\boldsymbol{r}_1\|_{2T} + \|\mathbf{u}_1\|_{2T}.\end{aligned} \tag{2}$$

Applying the definition of minimum gain, and the assumed minimum gains of each system, there exist $\beta_1 \in \mathbb{R}$ and $\beta_2 \in \mathbb{R}$, which depend only on the initial conditions, such that

$$\begin{aligned}\nu_1\|\mathbf{u}_1\|_{2T} &\leqslant \|\mathbf{y}_1\|_{2T} - \beta_1 \text{ and}\\ \nu_2\|\mathbf{u}_2\|_{2T} &\leqslant \|\mathbf{y}_2\|_{2T} - \beta_2.\end{aligned}$$

Combining this with (2) and re-arranging implies that

$$\begin{aligned}(\nu_1\nu_2 - 1)\|\mathbf{y}_1\|_{2T} &\leqslant \nu_1\nu_2\|\boldsymbol{r}_2\|_{2T} + \nu_1\|\boldsymbol{r}_1\|_{2T} - \beta_1 - \nu_1\beta_2 \text{ and}\\ (\nu_1\nu_2 - 1)\|\mathbf{y}_2\|_{2T} &\leqslant \nu_1\nu_2\|\boldsymbol{r}_1\|_{2T} + \nu_2\|\boldsymbol{r}_2\|_{2T} - \beta_2 - \nu_2\beta_1.\end{aligned}$$

From here, upon recalling that $\nu_1\nu_2 > 1$, it is readily seen that

$$\begin{aligned}\|\mathbf{y}\|_{2T} &\leqslant \|\mathbf{y}_1\|_{2T} + \|\mathbf{y}_2\|_{2T}\\
&\leqslant \frac{1}{\nu_1\nu_2 - 1}\left((\nu_1\nu_2 + \nu_1)\|\boldsymbol{r}_1\|_{2T} + (\nu_1\nu_2 + \nu_2)\|\boldsymbol{r}_2\|_{2T}\right) + \beta\\
&\leqslant \frac{\nu_1\nu_2 + \max\{\nu_1, \nu_2\}}{\nu_1\nu_2 - 1}\left(\|\boldsymbol{r}_1\|_{2T} + \|\boldsymbol{r}_2\|_{2T}\right) + \beta\\
&\leqslant \frac{\nu_1\nu_2 + \max\{\nu_1, \nu_2\}}{\nu_1\nu_2 - 1}\sqrt{\left(\|\boldsymbol{r}_1\|_{2T} + \|\boldsymbol{r}_2\|_{2T}\right)^2 + \left(\|\boldsymbol{r}_1\|_{2T} - \|\boldsymbol{r}_2\|_{2T}\right)^2} + \beta\\
&= \frac{\nu_1\nu_2 + \max\{\nu_1, \nu_2\}}{\nu_1\nu_2 - 1}\sqrt{2\left(\|\boldsymbol{r}_1\|_{2T}^2 + \|\boldsymbol{r}_2\|_{2T}^2\right)} + \beta\\
&\leqslant \kappa\|\boldsymbol{r}\|_{2T} + \beta,\end{aligned}$$

where $\kappa = \frac{\sqrt{2}(\nu_1\nu_2 + \max\{\nu_1, \nu_2\})}{\nu_1\nu_2 - 1}$ and $\beta = -\frac{(1+\nu_2)\beta_1 + (1+\nu_1)\beta_2}{\nu_1\nu_2 - 1}$. Hence, $\mathcal{G}$ maps any input $\boldsymbol{r} \in L_2$ to an output, $\mathbf{y} \in L_2$, implying the closed-loop system is input–output stable. □

3. THE MINIMUM GAIN LEMMA

At present, a lack of methods to systematically identify a plant's minimum gain and impose a desired minimum gain upon a controller inhibits the practical application of the Large Gain Theorem. This section introduces LMIs equivalent to a lower bound on the minimum gain of LTI systems. This result is intended to facilitate the employment of the Large Gain Theorem in practice.

3.1. A related optimization problem

The proof of this paper's main result will hinge upon the existence of a solution to a related optimization problem, discussed here. The solution to this problem is found using the Hamilton–Jacobi equation. Although this material was introduced in control literature as early as 1960 with [14], the statement presented here is based on [15, pp. 165].

DOI: 10.1002/rnc

Theorem 3.1
(Hamilton–Jacobi equation [15, pp. 165]) Consider the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$ and objective

$$\mathcal{V}(\mathbf{x}_0, \mathbf{u}, t_0, t_f) = \int_{t_0}^{t_f} \ell(\mathbf{x}, \mathbf{u}, t)dt, \tag{3}$$

where $\mathbf{f} \in \mathcal{C}^1(\mathbf{x}, \mathbf{u}, t)$ and $\ell \in \mathcal{C}^1(\mathbf{x}, \mathbf{u}, t)$. Suppose further that there exists a unique $\bar{\mathbf{u}}(\mathbf{x}, \boldsymbol{\lambda}, t) \in \mathcal{C}^1(\mathbf{x}, \boldsymbol{\lambda}, t)$, which minimizes

$$\mathcal{I}(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, t) = \ell(\mathbf{x}, \mathbf{u}, t) + \boldsymbol{\lambda}^{\mathsf{T}}\mathbf{f}(\mathbf{x}, \mathbf{u}, t), \tag{4}$$

with respect to $\mathbf{u}$ and that $\mathcal{V}^*(\mathbf{x}, t, t_f)$ solves

$$\begin{aligned} -\mathcal{V}_t^* &= \ell(\mathbf{x}, \bar{\mathbf{u}}(\mathbf{x}, \mathcal{V}_{\mathbf{x}}^*, t), t) + (\mathcal{V}_{\mathbf{x}}^*)^{\mathsf{T}}\mathbf{f}(\mathbf{x}, \bar{\mathbf{u}}(\mathbf{x}, \mathcal{V}_{\mathbf{x}}^*, t), t) \\ \text{where } &\mathcal{V}^*(\mathbf{x}(t_f), t_f, t_f) = 0. \end{aligned} \tag{5}$$

Then the minimum of (3) is $\mathcal{V}^*(\mathbf{x}_0, t_0, t_f)$, and it is attained for the control input

$$\mathbf{u}(\mathbf{x}, t) = \bar{\mathbf{u}}(\mathbf{x}, \mathcal{V}_{\mathbf{x}}^*, t).$$

The following theorem is presented to allow a brief proof of this paper's main result, the Minimum Gain Lemma, in Section 3.2.

Theorem 3.2
Consider an LTI system, $\mathcal{G} : L_{2e} \to L_{2e}$, with minimum gain $0 < \nu < \infty$ and state-space realization $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$. If $\text{span}(\mathbf{C}) \subseteq \text{span}(\mathbf{D})$ and $\mathbf{D}^{\mathsf{T}}\mathbf{D} - \nu^2\mathbf{1} > 0$, then for $t \in (0, \infty]$ there exists some finite, symmetric matrix, $\boldsymbol{\Pi}(t, t_f)$, satisfying

$$\boldsymbol{\Pi}(t, t_f) \leqslant 0, \tag{6}$$

$$\boldsymbol{\Pi}(t_f, t_f) = 0, \text{ and} \tag{7}$$

$$\begin{aligned} -\boldsymbol{\Pi}_t(t, t_f) = {}& \mathbf{C}^{\mathsf{T}}\mathbf{C} + \boldsymbol{\Pi}(t, t_f)\mathbf{A} + \mathbf{A}^{\mathsf{T}}\boldsymbol{\Pi}(t, t_f) \\ &- \left(\mathbf{C}^{\mathsf{T}}\mathbf{D} + \boldsymbol{\Pi}(t, t_f)\mathbf{B}\right)\left(\mathbf{D}^{\mathsf{T}}\mathbf{D} - \nu^2\mathbf{1}\right)^{-1}\left(\mathbf{D}^{\mathsf{T}}\mathbf{C} + \mathbf{B}^{\mathsf{T}}\boldsymbol{\Pi}(t, t_f)\right). \end{aligned} \tag{8}$$

Moreover, $\boldsymbol{\Pi}(t, \infty) = \boldsymbol{\Pi}$ is constant and satisfies

$$-\boldsymbol{\Pi}\mathbf{A} - \mathbf{A}^{\mathsf{T}}\boldsymbol{\Pi} - \mathbf{C}^{\mathsf{T}}\mathbf{C} = -\left(\mathbf{C}^{\mathsf{T}}\mathbf{D} + \boldsymbol{\Pi}\mathbf{B}\right)\mathbf{R}^{-1}\left(\mathbf{D}^{\mathsf{T}}\mathbf{C} + \mathbf{B}^{\mathsf{T}}\boldsymbol{\Pi}\right). \tag{9}$$

When $t_f < \infty$, (7) and (8) are guaranteed to have a finite solution for t adequately close to t_f, because of basic results in DEs, for instance [16, Theorem 1.2, pp. 3]. The solution is assumed to be symmetric without loss of generality because if a non-symmetric matrix satisfies these equations, then so does its symmetric part. The crux of this proof is to show that a solution exists for any $t \in [t_0, t_f)$, that $t_f = \infty$ is acceptable, and that the solution is negative semi-definite. Theorem 3.1 will be used to show that the solvability of (8) is tied to the solution of a slightly unusual LQ regulator problem via the Hamilton-Jacobi-Bellman equation. Unlike the cases treated in [14], [15, pp. 181–199] and [17, pp. 7–26], the related objective involves a matrix that is not positive semi-definite and no assumptions of controllability are made. This departure from well-established results demands a careful treatment in order to show that the large gain assumption will compensate for this difference.

A distinguishing feature of the treatment here is the assumption that $\text{span}(\mathbf{C}) \subseteq \text{span}(\mathbf{D})$. This assumption is made to show that $\boldsymbol{\Pi}(t, t_f) \leqslant 0$ and to establish an upper bound on the cost; two results that are typically dealt with separately. The structures of LQ regulator problems are commonly chosen so that the expression determining $\boldsymbol{\Pi}(t, t_f)$ varies monotonically in t_f, allowing

DOI: 10.1002/rnc

it's definiteness to be determined given an initial value, as in [15, pp. 191–192]. This is not possible here, but the span assumption alleviates the need for such an argument. Often, assumptions of controllability or stability admit upper bounds on the cost, as in [15, pp. 192], or the classic text, [17, pp. 22-24]. These requirements are unnecessary when assuming that span($\mathbf{C}$) $\subseteq$ span($\mathbf{D}$). Moreover, this is a natural assumption in the context of this paper, as it will be shown to hold for all square, LTI, minimum gain systems.

Proof

Consider applying Theorem 3.1, with

$$\begin{aligned}&\mathbf{f}(\mathbf{x},\mathbf{u},t)=\mathbf{f}(\mathbf{x},\mathbf{u})=\mathbf{A}\mathbf{x}+\mathbf{B}\mathbf{u}, \text{ and}\\ &\ell(\mathbf{x},\mathbf{u},t)=\ell(\mathbf{x},\mathbf{u})=|\mathcal{G}\mathbf{u}|^2-|\nu\mathbf{u}|^2=|\mathbf{C}\mathbf{x}+\mathbf{D}\mathbf{u}|^2-|\nu\mathbf{u}|^2.\end{aligned}$$

This selection is admissible in Theorem 3.1 because both functions are polynomials in all arguments, and are hence in $\mathcal{C}^1(\mathbf{x},\mathbf{u},t)$. It is straightforward to show that a unique continuously differentiable minimizer of (4) exists for this selection by furnishing the required function. Consider

$$\bar{\mathbf{u}}(\mathbf{x},\boldsymbol{\lambda},t)=-(\mathbf{D}^{\mathsf{T}}\mathbf{D}-\nu^2\mathbf{1})^{-1}\left(\mathbf{D}^{\mathsf{T}}\mathbf{C}\mathbf{x}+\tfrac{1}{2}\mathbf{B}^{\mathsf{T}}\boldsymbol{\lambda}\right).$$

Because of its linearity, $\bar{\mathbf{u}}(\mathbf{x},\boldsymbol{\lambda},t)\in\mathcal{C}^1(\mathbf{x},\boldsymbol{\lambda},t)$. To see that $\bar{\mathbf{u}}=\bar{\mathbf{u}}(\mathbf{x},\boldsymbol{\lambda},t)$ is the unique minimizer, substitute $\mathbf{u}\in L_{2e}\setminus\{\bar{\mathbf{u}}\}$ in (4) and observe that

$$\begin{aligned}\mathcal{I}(\mathbf{x},\mathbf{u},\boldsymbol{\lambda},t)&=\mathbf{x}^{\mathsf{T}}\mathbf{C}^{\mathsf{T}}\mathbf{C}\mathbf{x}+2\left(\mathbf{D}^{\mathsf{T}}\mathbf{C}\mathbf{x}\right)^{\mathsf{T}}\mathbf{u}+\mathbf{u}^{\mathsf{T}}\left(\mathbf{D}^{\mathsf{T}}\mathbf{D}-\nu^2\mathbf{1}\right)\mathbf{u}+\boldsymbol{\lambda}^{\mathsf{T}}\mathbf{A}\mathbf{x}+\boldsymbol{\lambda}^{\mathsf{T}}\mathbf{B}\mathbf{u}\\&=\mathcal{I}(\mathbf{x},\bar{\mathbf{u}},\boldsymbol{\lambda},t)+(\mathbf{u}-\bar{\mathbf{u}})^{\mathsf{T}}\left(\mathbf{D}^{\mathsf{T}}\mathbf{D}-\nu^2\mathbf{1}\right)(\mathbf{u}-\bar{\mathbf{u}})>\mathcal{I}(\mathbf{x},\bar{\mathbf{u}},\boldsymbol{\lambda},t).\end{aligned}$$

For $t<t_f$ adequately close to t_f, selecting $\mathcal{V}^*(\mathbf{x},t,t_f)=\mathbf{x}^{\mathsf{T}}\boldsymbol{\Pi}(t,t_f)\mathbf{x}$, where $\boldsymbol{\Pi}(t,t_f)=\boldsymbol{\Pi}^{\mathsf{T}}(t,t_f)$ satisfies (7) and (8) implies that $\mathcal{V}^*(\mathbf{x},t,t_f)$ satisfies (5). To verify this, note that

$$\begin{aligned}&\mathbf{x}^{\mathsf{T}}\boldsymbol{\Pi}(t_f,t_f)\mathbf{x}=0, \text{ and}\\ &\ell(\mathbf{x},\bar{\mathbf{u}},t)+(\mathcal{V}^*_{\mathbf{x}})^{\mathsf{T}}\mathbf{f}(\mathbf{x},\bar{\mathbf{u}},t)=|\mathbf{C}\mathbf{x}+\mathbf{D}\bar{\mathbf{u}}|^2-|\nu\bar{\mathbf{u}}|^2+2\mathbf{x}^{\mathsf{T}}\boldsymbol{\Pi}\left(\mathbf{A}\mathbf{x}+\mathbf{B}\bar{\mathbf{u}}\right)\\&\qquad=\mathbf{x}^{\mathsf{T}}\left(\mathbf{C}^{\mathsf{T}}\mathbf{C}+\boldsymbol{\Pi}\mathbf{A}+\mathbf{A}^{\mathsf{T}}\boldsymbol{\Pi}\right)\mathbf{x}+\left(2\mathbf{x}^{\mathsf{T}}\mathbf{C}^{\mathsf{T}}\mathbf{D}+\bar{\mathbf{u}}^{\mathsf{T}}\mathbf{R}+2\mathbf{x}^{\mathsf{T}}\boldsymbol{\Pi}\mathbf{B}\right)\bar{\mathbf{u}}\\&\qquad=\mathbf{x}^{\mathsf{T}}\left(\mathbf{C}^{\mathsf{T}}\mathbf{C}+\boldsymbol{\Pi}\mathbf{A}+\mathbf{A}^{\mathsf{T}}\boldsymbol{\Pi}-\left(\mathbf{C}^{\mathsf{T}}\mathbf{D}+\boldsymbol{\Pi}\mathbf{B}\right)\mathbf{R}^{-1}\left(\mathbf{D}^{\mathsf{T}}\mathbf{C}+\mathbf{B}^{\mathsf{T}}\boldsymbol{\Pi}\right)\right)\mathbf{x}\\&\qquad=\mathbf{x}^{\mathsf{T}}(-\boldsymbol{\Pi}_t)\mathbf{x}=-\frac{\partial}{\partial t}\left(\mathbf{x}^{\mathsf{T}}\boldsymbol{\Pi}\mathbf{x}\right),\end{aligned}$$

where, for the sake of brevity, $\bar{\mathbf{u}}=\bar{\mathbf{u}}(\mathbf{x},\mathcal{V}^*_{\mathbf{x}},t)$, $\boldsymbol{\Pi}=\boldsymbol{\Pi}(t,t_f)$, and $\mathbf{R}=\mathbf{D}^{\mathsf{T}}\mathbf{D}-\nu^2\mathbf{1}$. Theorem 3.1 now implies that

$$\mathbf{x}_0^{\mathsf{T}}\boldsymbol{\Pi}(t_0,t_f)\mathbf{x}_0=\mathcal{V}^*(\mathbf{x}_0,t_0,t_f)=\inf_{\mathbf{u}\in L_{2e}}\mathcal{V}(\mathbf{x}_0,\mathbf{u},t_0,t_f), \tag{10}$$

for $t_0<t_f$ adequately close to t_f. To extend this result to all $t_0\in[0,t_f)$, the objective and hence the entries of $\boldsymbol{\Pi}(t,t_f)$ must be bounded. Selecting $T=t_f-t_0$ and exploiting the time invariance of $\ell(\mathbf{x},\mathbf{u})$ yields a lower bound:

$$\mathbf{x}_0^{\mathsf{T}}\boldsymbol{\Pi}(t,t_f)\mathbf{x}_0=\mathcal{V}^*(\mathbf{x}_0,t_0,t_f)=\mathcal{V}^*(\mathbf{x}_0,0,T)=\inf_{\mathbf{u}\in L_{2e}}\left(\|\mathcal{G}\mathbf{u}\|_{2T}^2-\|\nu\mathbf{u}\|_{2T}^2\right)\geqslant\tilde{\beta}.$$

To find an upper bound for the cost, while simultaneously demonstrating that $\boldsymbol{\Pi}(t,t_f)\leqslant 0$, note that because span($\mathbf{C}$) $\subseteq$ span($\mathbf{D}$), the input may be selected such that $\mathbf{D}\mathbf{u}(t)=-\mathbf{C}\mathbf{x}(t)$, implying

$$\mathbf{x}_0^{\mathsf{T}}\boldsymbol{\Pi}(t,t_f)\mathbf{x}_0\leqslant\mathcal{V}(\mathbf{x}_0,\mathbf{u},t_0,t_f)=\int_{t_0}^{t_f}\left(|\mathbf{C}\mathbf{x}+\mathbf{D}\mathbf{u}|^2-|\nu\mathbf{u}|^2\right)dt=-\int_{t_0}^{t_f}|\nu\mathbf{u}|^2\,dt\leqslant 0.$$

DOI: 10.1002/rnc

Knowing that no terms in $\mathbf{\Pi}$ can diverge, [16, Theorem 1.4, pp. 3] extends the existence of solutions to $t_0 \in [0, t_f)$. The boundedness of $\mathcal{V}^*(\mathbf{x}, t, t_f)$ further implies that $\mathbf{\Pi}(t, \infty)$ is well-defined. Moreover, for any $\mathbf{x_0} \in \mathbb{R}$, and $t \in \mathbb{R}^+$

$$\begin{aligned}\mathbf{x}^{\mathsf{T}}\mathbf{\Pi}(t,\infty)\mathbf{x} &= \inf_{\mathbf{u}\in L_{2e}} \int_t^{\infty} \ell(\mathbf{x},\mathbf{u})dt = \lim_{t_f\to\infty} \inf_{\mathbf{u}\in L_{2e}} \int_0^{t_f-t} \ell(\mathbf{x},\mathbf{u})dt = \inf_{\mathbf{u}\in L_{2e}} \int_0^{\infty} \ell(\mathbf{x},\mathbf{u})dt \\ &= \mathbf{x}^{\mathsf{T}}\mathbf{\Pi}(0,\infty)\mathbf{x},\end{aligned}$$

Therefore, $\mathbf{\Pi}(t, \infty) = \mathbf{\Pi}$ is constant in t and is seen to satisfy (9) upon re-arranging (8). □

3.2. The minimum gain lemma and its proof

Having now established the necessary supporting results, the Minimum Gain Lemma and its proof may be presented.

Theorem 3.3
(The Minimum Gain Lemma) Consider an LTI system, $\mathcal{G} : L_{2e} \to L_{2e}$, with state-space realization $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$. The following equivalent statements are sufficient conditions for $\mathcal{G}$ to have minimum gain $0 \leqslant \nu < \infty$:

1. There exist matrices $\mathbf{P} = \mathbf{P}^{\mathsf{T}} \geqslant 0$, $\mathbf{L}$, and $\mathbf{W}$ such that

$$\mathbf{PA} + \mathbf{A}^{\mathsf{T}}\mathbf{P} - \mathbf{C}^{\mathsf{T}}\mathbf{C} = -\mathbf{L}^{\mathsf{T}}\mathbf{L}, \tag{11a}$$

$$\mathbf{PB} - \mathbf{C}^{\mathsf{T}}\mathbf{D} = -\mathbf{L}^{\mathsf{T}}\mathbf{W}, \text{ and} \tag{11b}$$

$$\nu^2\mathbf{1} - \mathbf{D}^{\mathsf{T}}\mathbf{D} = -\mathbf{W}^{\mathsf{T}}\mathbf{W}. \tag{11c}$$

2. There exists a matrix $\mathbf{P} = \mathbf{P}^{\mathsf{T}} \geqslant 0$ such that

$$\mathbf{M}(\mathbf{P}, \nu) = \begin{bmatrix} \mathbf{PA} + \mathbf{A}^{\mathsf{T}}\mathbf{P} - \mathbf{C}^{\mathsf{T}}\mathbf{C} & \mathbf{PB} - \mathbf{C}^{\mathsf{T}}\mathbf{D} \\ * & \nu^2\mathbf{1} - \mathbf{D}^{\mathsf{T}}\mathbf{D} \end{bmatrix} \leqslant 0. \tag{12}$$

Further, if $\mathcal{G}$ is a square system or $\operatorname{span}(\mathbf{C}) \subseteq \operatorname{span}(\mathbf{D})$, then these are necessary conditions for $\mathcal{G}$ to have minimum gain $0 \leqslant \nu < \infty$.

It is tempting to classify this result as a sub-case of the Dissipativity Lemma [5], which inspired its creation. However, the Dissipativity Lemma involves only stable, minimal state-space realizations of LTI systems, while the Large Gain Theorem and Minimum Gain are primarily intended for use with unstable systems. The necessity proof presents a further departure because the equivalent portion of the Dissipativity Lemma's proof involves a power function that is convex and quadratic in terms of the states. A similar argument cannot be followed here because the existence of such a function is not guaranteed for minimum gain systems.

Proof

To verify that Statements 1 and 2 are equivalent, note that if Statement 1 holds, then

$$\mathbf{M}(\mathbf{P}, \nu) = \begin{bmatrix} -\mathbf{L}^{\mathsf{T}}\mathbf{L} & -\mathbf{L}^{\mathsf{T}}\mathbf{W} \\ * & -\mathbf{W}^{\mathsf{T}}\mathbf{W} \end{bmatrix} = -\begin{bmatrix} \mathbf{L}^{\mathsf{T}} \\ \mathbf{W}^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} \mathbf{L} & \mathbf{W} \end{bmatrix} \leqslant 0,$$

implying Statement 2. Likewise, if Statement 2 holds, then $\mathbf{M}(\mathbf{P}, \nu) \leqslant 0$ implies the existence of the required matrices, $\mathbf{L}$ and $\mathbf{W}$. When demonstrating sufficiency and necessity, it is now adequate to consider only Statement 2, because Statements 1 and 2 are known to be equivalent.

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Assuming Statement 2 holds, basic manipulations reveal that the system has a minimum gain, ν:

$$\begin{aligned}\|\mathcal{G}\mathbf{u}\|_{2T}^2 - \nu^2\|\mathbf{u}\|_{2T}^2 &= \int_0^T \left(|\mathbf{Cx}+\mathbf{Du}|^2 - \nu^2\,|\mathbf{u}|^2 + \frac{d}{dt}\left(\mathbf{x}^{\mathsf T}\mathbf{Px}\right) - 2\mathbf{x}^{\mathsf T}\mathbf{P}\left(\mathbf{Ax}+\mathbf{Bu}\right)\right) dt \\ &= \int_0^T \left(-\left[\,\mathbf{x}^{\mathsf T}\;\mathbf{u}^{\mathsf T}\,\right]\mathbf{M}(\mathbf{P},\nu)\begin{bmatrix}\mathbf{x}\\ \mathbf{u}\end{bmatrix} + \frac{d}{dt}\left(\mathbf{x}^{\mathsf T}\mathbf{Px}\right)\right) dt \\ &\geqslant -\,\mathbf{x}_0^{\mathsf T}\mathbf{P}\mathbf{x}_0 = \tilde{\beta}.\end{aligned}$$

The remainder of this proof is devoted to showing the necessity of Statement 2. The desired matrix, $\mathbf{P}$, will be constructed in terms of the optimization problem defined in Theorem 3.2. This theorem demands that span($\mathbf{C}$) $\subseteq$ span($\mathbf{D}$). If $\mathcal{G}$ is square, then this requirement need not be explicitly verified as it is implied when assuming that $\mathbf{R} = \mathbf{D}^{\mathsf T}\mathbf{D} - \nu^2\mathbf{1} > 0$, which is another requirement of Theorem 3.2. In this case, span($\mathbf{D}$) $= \mathbb{R}^n$ because otherwise $\mathbf{Dx} = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{x}^{\mathsf T}\mathbf{Rx} = -\nu^2\,|\mathbf{x}|^2 < 0$ contradicting the assumption that $\mathbf{R} > 0$. It then holds that span($\mathbf{D}$) $= \mathbb{R}^n \supseteq$ span($\mathbf{C}$).

As previously mentioned, Theorem 3.2 also demands that $\mathbf{R} = \mathbf{D}^{\mathsf T}\mathbf{D} - \nu^2\mathbf{1} > 0$. To fulfill this requirement, it will then be shown that $\nu > 0$ implies $\mathbf{R} \geqslant 0$. $\mathbf{P}$ will initially be found assuming that $\mathbf{R}$ is non-singular, in order to apply Theorem 3.2. In the singular case, $\mathbf{P}$ can be constructed in terms of matrices found for the non-singular case.

Case 1: $\mathbf{R} \ngeqslant 0$
The assumption of this case directly implies that $\mathbf{w}^{\mathsf T}\mathbf{Rw} < 0$ for some $\mathbf{w} \neq \mathbf{0}$. Consider the response to initial states $\mathbf{x}_0 = \mathbf{0}$ and input $\mathbf{u}(t) = \mathbf{w}$. Note that

$$\left[\,\mathbf{x}_0^{\mathsf T}\;\mathbf{w}^{\mathsf T}\,\right]\begin{bmatrix}\mathbf{C}^{\mathsf T}\mathbf{C} & \mathbf{C}^{\mathsf T}\mathbf{D}\\ * & \mathbf{R}\end{bmatrix}\begin{bmatrix}\mathbf{x}_0\\ \mathbf{w}\end{bmatrix} = \mathbf{w}^{\mathsf T}\mathbf{Rw} < 0.$$

The linear system's continuity implies that there exists some adequately small $T_{\mathbf{w}} > 0$ such that

$$\|\mathcal{G}\mathbf{w}\|_{2T}^2 - \nu^2\|\mathbf{w}\|_{2T}^2 = \int_0^{T_{\mathbf{w}}}\left[\,\mathbf{x}^{\mathsf T}\;\mathbf{w}^{\mathsf T}\,\right]\begin{bmatrix}\mathbf{C}^{\mathsf T}\mathbf{C} & \mathbf{C}^{\mathsf T}\mathbf{D}\\ * & \mathbf{R}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{w}\end{bmatrix} dt < 0.$$

Employing the linearity of $\mathcal{G}$, if the minimum gain of $\mathcal{G}$ is ν, then this yields a contradiction:

$$-\infty < \tilde{\beta} \leqslant \lim_{\kappa\to\infty}\left(\|\mathcal{G}(\kappa\mathbf{w})\|_{2T}^2 - \nu^2\|\kappa\mathbf{w}\|_{2T}^2\right) = \lim_{\kappa\to\infty}\kappa^2\left(\|\mathcal{G}(\mathbf{w})\|_{2T}^2 - \nu^2\|\mathbf{w}\|_{2T}^2\right) = -\infty.$$

Therefore $\mathbf{R} \geqslant 0$ must hold.
Case 2: $\mathbf{R} \geqslant 0$ and $\mathbf{R}$ is non-singular
The assumptions of this case directly imply that $\mathbf{R} > 0$. Consider $\mathbf{\Pi} = \mathbf{\Pi}(t,\infty)$ as defined in Theorem 3.2. After recalling the definition of $\mathbf{R}$ and applying (9), it can be seen that Statement 2 holds:

$$\begin{aligned}\mathbf{M}(-\mathbf{\Pi},\nu) &= \begin{bmatrix}-\mathbf{\Pi A} - \mathbf{A}^{\mathsf T}\mathbf{\Pi} - \mathbf{C}^{\mathsf T}\mathbf{C} & -\mathbf{\Pi B} - \mathbf{C}^{\mathsf T}\mathbf{D}\\ * & \nu^2\mathbf{1} - \mathbf{D}^{\mathsf T}\mathbf{D}\end{bmatrix}\\ &= \begin{bmatrix}-(\mathbf{C}^{\mathsf T}\mathbf{D} + \mathbf{\Pi B})\mathbf{R}^{-1}(\mathbf{D}^{\mathsf T}\mathbf{C} + \mathbf{B}^{\mathsf T}\mathbf{\Pi}) & -(\mathbf{\Pi B} + \mathbf{C}^{\mathsf T}\mathbf{D})\mathbf{R}^{-\frac{1}{2}}\mathbf{R}^{\frac{1}{2}}\\ * & -\mathbf{R}^{\frac{1}{2}}\mathbf{R}^{\frac{1}{2}}\end{bmatrix} \leqslant 0.\end{aligned}$$

Case 3: $\mathbf{R} \geqslant 0$ and $\mathbf{R}$ is singular
A sequence, $\{\nu_n\}_{n=1}^{\infty}$, may be chosen to monotonically increase toward the actual minimum gain of the system, ν. For $n \in \mathbb{N}$, $\mathcal{G}$ may also be said to have minimum gain ν_n because

$$\|\mathcal{G}\mathbf{u}_T\|_2 - \nu_n\|\mathbf{u}_T\|_2 > \|\mathcal{G}\mathbf{u}_T\|_2 - \nu\|\mathbf{u}_T\|_2 \geqslant \beta.$$

DOI: 10.1002/rnc

However, $\mathbf{R}_n = \mathbf{D}^\mathsf{T}\mathbf{D} - \nu_n^2\mathbf{1} > \mathbf{R} \geqslant 0$ must be non-singular. For $n \in \mathbb{N}$, the previous arguments therefore imply the existence of the matrices $\mathbf{\Pi}_n = \mathbf{\Pi}_n^\mathsf{T} \leqslant 0$ in Theorem 3.2 such that

$$\mathcal{V}_n^*(\mathbf{x}_0, 0, \infty) = \mathcal{V}^*(\mathbf{x}_0, 0, \infty)|_{\nu=\nu_n} = \mathbf{x}_0^\mathsf{T}\mathbf{\Pi}_n\mathbf{x}_0.$$

For $n \in \mathbb{N}$, $\mathbf{u} \in L_{2e}$ and any initial state, $\mathbf{x}_0$

$$\begin{aligned} &\mathcal{V}_n(\mathbf{x}_0, \mathbf{u}, 0, \infty) \geqslant \mathcal{V}_{n+1}(\mathbf{x}_0, \mathbf{u}, 0, \infty) \geqslant \mathcal{V}_0(\mathbf{x}_0, \mathbf{u}, 0, \infty), \\ &\Rightarrow \mathcal{V}_n^*(\mathbf{x}_0, 0, \infty) \geqslant \mathcal{V}_{n+1}^*(\mathbf{x}_0, 0, \infty) \geqslant \mathcal{V}_0^*(\mathbf{x}_0, 0, \infty), \\ &\Rightarrow \mathbf{x}_0^\mathsf{T}\mathbf{\Pi}_n\mathbf{x}_0 \geqslant \mathbf{x}_0^\mathsf{T}\mathbf{\Pi}_{n+1}\mathbf{x}_0 \geqslant \tilde{\beta}. \end{aligned} \tag{13}$$

This demonstrates that the sequence $\{\mathbf{\Pi}_n\}_{n=1}^{\infty}$ is monotonically decreasing, bounded below, and therefore converges to some finite $\mathbf{\Pi}_0$. A more detailed justification for the convergence of monotonic bounded sequences of matrices is found in [15, pp. 191]. Setting $\mathbf{P}_0 = -\mathbf{\Pi}_0$, it can be seen that $\mathbf{P}_0 = \mathbf{P}_0^\mathsf{T} \geqslant 0$ because $\mathbf{\Pi}_n = \mathbf{\Pi}_n^\mathsf{T} \leqslant 0$ for $n \in \mathbb{N}$. Similarly, $\mathbf{M}(\mathbf{P}_0\nu) \leqslant 0$ because $\mathbf{M}(-\mathbf{\Pi}_n\nu_n) \leqslant 0$ for $n \in \mathbb{N}$. Hence, Statement 2 holds. □

4. NUMERICAL EXAMPLES

The principal contributions of this paper are an altered definition of minimum gain, accounting for non-zero initial conditions, and an LMI characterization of this property, allowing simplified stability analysis involving LTI systems. The numerical examples presented here demonstrate the application of these results and highlight their utility. Section 4.1 presents calculations of minimum gain for three illustrative examples, while Section 4.2 employs these results in order to robustly ensure input–output stability in two trajectory tracking examples.

4.1. Minimum gain calculations

4.1.1. A contextualizing example. A system first examined in [4] will be revisited in order to compare the previous and proposed definitions of minimum gain. Example 3.2 in [4] concerned a system $H : L_{2e} \to L_{2e}$ defined as

$$H(u(t)) = \int_0^t g(t-\tau)u(\tau)d\tau, \text{ for } u \in L_{2e},$$

where $g(t)$ had the Laplace transform

$$G_1(s) = \frac{s+1}{s-1}. \tag{14}$$

Using the original definition, and assuming a unit-less control input, the minimum gain of H was calculated to be

$$\nu(H) = \inf_{u \in L_2 \setminus \{0\}} \frac{\|Hu\|_{2T}}{\|u\|_{2T}} = 1 \text{ (s}^{-1}\text{)}.$$

When considering the transfer function defined in (14), H is only the zero-state response. If it is desirable to account for initial conditions, then it is preferable to study $\mathcal{G}_1 : L_{2e} \to L_{2e}$, defined in state-space form as

$$\begin{aligned} \dot{x} &= x + u, \\ y &= 2x + u, \end{aligned}$$

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where $x \in L_{2e}$ is the state, $u \in L_{2e}$ is the input, and $y \in L_{2e}$ is the output. Unfortunately, when using the original definition of minimum gain, it is found that if $x_0 \neq 0$

$$0 \leqslant \nu(\mathcal{G}_1) = \inf_{u \in L_2 \setminus \{\mathbf{0}\}} \frac{\|\mathcal{G}_1 u\|_{2T}}{\|u\|_{2T}} \leqslant \frac{\|\mathcal{G}_1(-2x)\|_{2T}}{\|-2x\|_{2T}} = 0.$$

This did not hold for $\nu(H)$ in [4] because the input used earlier, $u = -2x$, is identically equal to zero when $x_0 = 0$. Because $\nu(\mathcal{G}_1) = 0$ when using the original definition with zero initial conditions, the Large Gain Theorem could not be applied to establish input–output stability for negative feedback interconnections involving $\mathcal{G}_1$. It is this observation that inspired Definition 2.1. Noting that $\mathbf{M}(2,1) = \mathbf{0}$, Theorem 3.3 verifies that when employing Definition 2.1, $\mathcal{G}_1$ has minimum gain less than or equal to 1, agreeing with the result found for the zero-state response in [4].

One benefit of applying input–output stability results is that they may engender robustness with respect to changes in plant parameters. As such, a perturbed system, $\tilde{\mathcal{G}}_1 : L_{2e} \to L_{2e}$, is considered with transfer function

$$\tilde{G}_1(s) = \frac{s + (1+\delta_2)}{s - (1+\delta_1)},$$

where $-\Delta \leqslant \delta_1 \leqslant \Delta$ and $-\Delta \leqslant \delta_2 \leqslant \Delta$ for some $0 \leqslant \Delta < 1$. The state-space formulation of $\tilde{\mathcal{G}}_1$ is

$$\dot{x} = (1+\delta_1)x + u,$$
$$y = (2 + \delta_1 + \delta_2)x + u,$$

where y, u, and x are as defined in the preceding text.

Before using the Minimum Gain Lemma, note that the minimum gain of $\tilde{\mathcal{G}}_1$ satisfies $0 \leqslant \nu \leqslant 1$ because the nominal system is recovered when $\delta_1 = \delta_2 = 0$, and $\mathcal{G}_1$ has a minimum gain of 1. Further, the bottom-right entry of $\mathbf{M}(p,\nu)$ equals $\nu^2 - 1$, and must be negative for $\mathbf{M}(p,\nu) \leqslant 0$. Setting $p_M = (\nu^2 - 1)(1+\delta_1) + (2 + \delta_1 + \delta_2) \geqslant 0$, recalling that

$$\det(\mathbf{M}) = \det(\mathbf{M}_{22}) \det\left(\mathbf{M}_{11} - \mathbf{M}_{12}\mathbf{M}_{22}^{-1}\mathbf{M}_{21}\right),$$

and taking the Schur complement then implies $\mathbf{M}(p_M,\nu) \leqslant 0$ if and only if $\det(\mathbf{M}(p_M,\nu)) \geqslant 0$. This quantity may be expressed as

$$\det(\mathbf{M}(p_M,\nu)) = \left[1-\nu^2\right]\left[(\delta_2+1)^2 - \nu^2(1+\delta_1)^2\right].$$

It can now be seen that $\tilde{\mathcal{G}}_1$ has a minimum gain of $\nu = \frac{1-\Delta}{1+\Delta}$ because

$$\det\left(\mathbf{M}\left(p_M, \frac{1-\Delta}{1+\Delta}\right)\right) \geqslant \left[1-\nu^2\right]\left[(1-\Delta)^2 - \nu^2(1+\Delta)^2\right] = 0.$$

4.1.2. A confirmatory example. In this section, the minimum gain of a simple, one-state system will be calculated analytically. This example is presented to provide confirmation of the Large Gain LMI. More importantly, this system's parameters may be tuned to yield any desired minimum gain, providing a simple state-space structure for controller design. In Section 4.2, controllers designed using this system will be shown to yield desirable closed-loop responses while providing robust stability guarantees.

The system of interest $\mathcal{G}_2 : L_{2e} \to L_{2e}$ has the following state-space realization

$$\dot{x} = ax + u, \tag{15}$$

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$$y = k(x + u), \tag{16}$$

where $x \in L_{2e}$ is the state, $u \in L_{2e}$ is the input, $y \in L_{2e}$ is the output, and $a \in \mathbb{R}$ and $k \in \mathbb{R} \setminus \{0\}$ are system parameters.

In order to calculate the minimum gain, $0 \leqslant \nu$, a lower bound is sought in terms of u^2 by expressing y^2 as

$$y^2 = k^2\left(u^2 + 2ux + x^2\right) = k^2\left((1-\lambda)u^2 + \lambda u^2 + 2ux + x^2\right), \tag{17}$$

where $1 \geqslant \lambda \geqslant 0$. Because $x^2 \geqslant 0$, it is only the $2ux$ term in (17) that prevents the minimum gain from equaling k for all selections of a. Intuitively, a portion of the u^2 term must be devoted to canceling out the potentially negative $2ux$, while the remainder contributes to the minimum gain. The constant λ is introduced to reflect this partition. Different selections of λ will be required as the parameter a varies, resulting in different minimum gains. The required proportion is found by using (15) to write u in terms of x and $\dot{x}$:

$$\begin{aligned} y^2 &= k^2\left((1-\lambda)u^2 + (\dot{x} - ax)(\lambda\dot{x} + (2 - a\lambda)x) + x^2\right) \\ &= k^2\left((1-\lambda)u^2 + \lambda\dot{x}^2 + (1-\lambda a)\frac{d}{dt}(x^2) + \left(1 - 2a + \lambda a^2\right)x^2\right). \end{aligned}$$

Integrating shows that for any $T > 0$,

$$\|y\|_{2T}^2 = k^2\left((1-\lambda)\|u\|_{2T}^2 + \lambda\|\dot{x}\|_{2T}^2 + (1-\lambda a)\left(x^2(T) - x^2(0)\right) + (1 - 2a + \lambda a^2)\|x\|_{2T}^2\right). \tag{18}$$

If $a \leqslant \frac{1}{2}$, then setting $\lambda = 0$ in (18) shows that

$$\|y\|_{2T}^2 \geqslant k^2\|u\|_{2T}^2 - k^2x^2(0).$$

By the definition of minimum gain, this implies that $\nu = |k|$ if $a \leqslant \frac{1}{2}$. Likewise, if $\frac{1}{2} < a < 1$, then $\nu = \frac{|k|(1-a)}{a}$ because setting $\lambda = \frac{2a-1}{a^2} > 0$ in (18) yields

$$\|y^2\|_{2T} \geqslant \frac{k^2(1-a)^2}{a^2}\|u\|_{2T}^2 - \frac{k^2(1-a)}{a}x^2(0). \tag{19}$$

However, if $a \geqslant 1$, then such an argument cannot be followed. In fact, it is possible to demonstrate that the system does not have positive minimum gain in this case. It follows from the quadratic formula that if $a > 1$, then for any $\lambda < 1$, there exists a corresponding $\gamma > 0$ such that

$$\lambda\gamma^2 + 2\gamma(1-\lambda a) + \left(1 - 2a + \lambda a^2\right) < 0.$$

If the system has a positive minimum gain, ν, then selecting $u = (\gamma - a)x$ and $k = \frac{\nu}{\sqrt{1-\lambda}} > 0$ implies that

$$\begin{aligned} \lim_{T\to\infty}\left(\|y\|_{2T}^2 - \nu^2\|u\|_{2T}^2\right) &= \lim_{T\to\infty}\left(\|y\|_{2T}^2 - k^2(1-\lambda)\|u\|_{2T}^2\right) \\ &= \lim_{T\to\infty}\left(k^2x^2(0)\left(\lambda\gamma^2 + 2\gamma(1-\lambda a) + (1 - 2a + \lambda a^2)\right)\|e^{\gamma t}\|_{2T}\right) \\ &= -\infty. \end{aligned}$$

Comparing the aforementioned equation to the definition of minimum gain yields a contradiction.

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Similar arguments can be used for $a < 1$ to show that the highest possible minimum gains are

$$\nu = \begin{cases} |k| & \text{if } a \leqslant \frac{1}{2} \\ \frac{|k|(1-a)}{a} & \text{if } \frac{1}{2} < a < 1 \\ 0 & \text{if } 1 \leqslant a. \end{cases} \tag{20}$$

As a verification of the Minimum Gain Lemma, the largest possible ν satisfying (12) were calculated using MATLAB, YALMIP [18], SeDuMi [19], and SDPT3 [20] for various selections of a and k. Figure 1 shows that the minimum gains calculated numerically agreed with the analytical results. Further, this confirms the accuracy of the numerical solvers, which will be used to calculate minimum gains in Section 4.1.3, where analytical results are unavailable. These results also illustrate a significant departure from those that would have been found when verifying the original definition of minimum gain; stable, strictly proper, linear systems were shown to have zero minimum gain in [4], but Figure 1 displays positive minimum gains for many systems where $a < 0$, which correspond to stable systems. According to the proposed definition of minimum gain, stable systems may have strictly positive minimum gains, thereby admitting the use of the Large Gain Theorem in stability analysis, and broadening its applicability.

4.1.3. A representative example. Although the previous examples provide context for and verification of the contributions of this paper, the forthcoming example is likely more reflective of how the Minimum Gain Lemma will be used in practice, with more complicated systems. Exploration of the parameter space, rather than analytical arguments, is used to show that the system has a particular minimum gain, for reasonable variations in the nominal parameters.

The system, $\mathcal{G}_3 : L_{2e} \to L_{2e}$, is defined in terms of its transfer function,

$$G_3(s) = \frac{g(s-d)(s-e)(s-f)}{(s-a)(s-b)(s-c)},$$

where the nominal parameter values are listed in Table I. These values were chosen to yield an unstable system with all zeros in the open left half-plane. While the selection of an unstable system

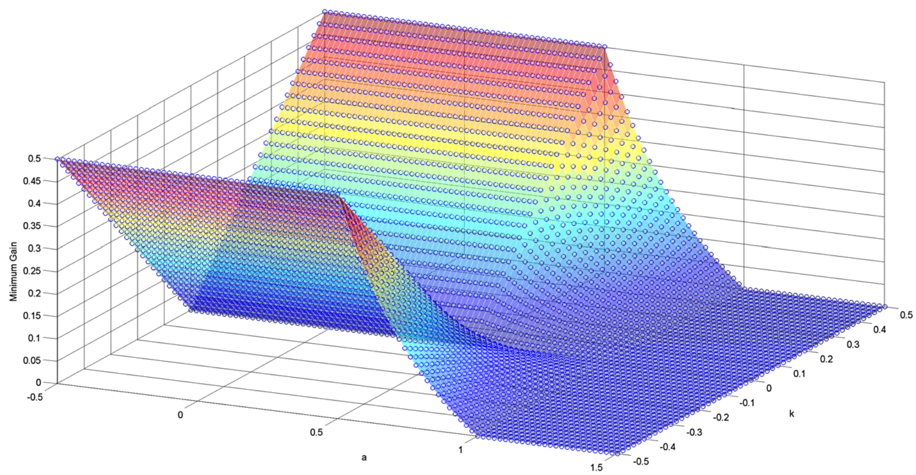

Figure 1. The minimum gain calculated for various system parameters. Circles indicate values calculated numerically while the surface inscribes the analytic results.

Table I. Nominal parameters of $\mathcal{G}_3$.

Parameter	a	b	c	d	e	f	g
Nominal	1	2	3	−1	−7	−9	1
Perturbed	1.1	1.9	3.15	−1.1	−6.6	−9.4	0.9

DOI: 10.1002/rnc

is made to present a more interesting control problem, the selection of zeros is vital because LTI systems with zeros in the closed right half-plane necessarily have zero minimum gain. This was shown for the original definition of minimum gain in [4]. The proof in [4] relies on the selection of an input that cancels the LTI system's unstable zeros and can be adapted to the new definition by selecting an input that cancels both unstable poles and unstable zeros.

Upon computing a state-space realization of this system, the highest possible minimum gain satisfying (12) was found to be $\nu = 1$ for the nominal system using `MATLAB`, `YALMIP` [18], `SeDuMi` [19], and `SDPT3` [20]. Ideally, stability guarantees produced using the Large Gain Lemma will be robust with respect to changes in plant parameters. As such, a minimum gain that remained valid through the entire parameter space was sought via a grid search. The highest possible minimum gain was calculated for each possible combination of parameters, where all parameters but g were checked at intervals of 5% change, up to a maximum variation of $\pm 10\%$ each. More refined searches, where parameters varied individually, but were checked for changes of 0.5%, were also performed. The minimum gain was found to be $\nu = 1 \pm 9 \times 10^{-6}$, where 10^{-6} was the tolerance selected for the numerical methods employed.

It was not necessary to include variations of g in these searches because its effect on the minimum gain is easily decoupled from the rest. Suppose that for $g = 1$ and a given selection of the other parameters, the minimum gain of $\mathcal{G}_3$ was calculated to be ν. Then the minimum gain is $|1+\delta|\nu$ for a perturbed system, $\tilde{\mathcal{G}}_3$ where $g = 1 + \delta$ for some $\delta \in \mathbb{R}$, because for any input, $\mathbf{u} \in L_{2e}$, and any time, $t > 0$, there exists some $\beta \in \mathbb{R}$ such that

$$\|\tilde{\mathcal{G}}_3\mathbf{u}\|_{2T} = \|(1+\delta)\mathcal{G}_3\mathbf{u}\|_{2T} = |1+\delta|\|\mathcal{G}_3\mathbf{u}\|_{2T} \geqslant |1+\delta|\nu\|\mathbf{u}\|_{2T} + \beta.$$

From here, it was concluded that $\mathcal{G}_3$ had a minimum gain of $\nu = 0.9$, allowing for variations from the nominal parameters by $\pm 10\%$.

4.2. *Stability and trajectory tracking*

Next, the utility of this paper's contributions is highlighted in the context of trajectory tracking. Controllers adhering to the structure of $\mathcal{G}_2$ in Section 4.1.2 will be used to robustly stabilize systems $\mathcal{G}_1$ and $\mathcal{G}_3$ from Sections 4.1.1 and 4.1.3. As the simple, one-state controller provides reasonable closed-loop responses for both plants, this illustrates the ease and effectiveness of employing the Minimum Gain LMI and Large Gain Theorem in stability analysis. Further, it will be shown that both stable and unstable controllers may be used to stabilize unstable plants with positive minimum gains.

Negative feedback interconnections of the controller, $\mathcal{G}_c = \mathcal{G}_2$ and either plant, $\mathcal{G}_p = \mathcal{G}_1$ or $\mathcal{G}_p = \mathcal{G}_3$, were considered, as illustrated in Figure 2. In each experiment, the controller parameters were tuned so that, given all plant and controller states initially set to 0.5, the nominal systems' outputs converged to zero within 2(s) without experiencing overshoot, as can be seen in Figure 3.

The controller parameters selected are displayed in Table II. Table II also displays the controllers' minimum gains, calculated following (20). The plant minimum gains displayed in this table were calculated following Section 4.1 allowing $\pm 30\%$ variation in the parameters of $\mathcal{G}_1$ (that is, setting $\Delta = 0.3$) and allowing $\pm 10\%$ variation in the parameters of $\mathcal{G}_3$. The final column of Table II verifies that the controllers had adequately high minimum gains, showing that input–output stability was robustly guaranteed by the Large Gain Theorem.

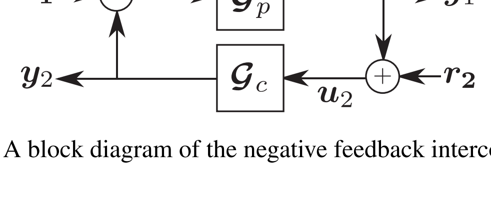

Figure 2. A block diagram of the negative feedback interconnection.

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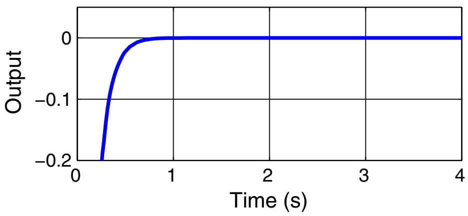
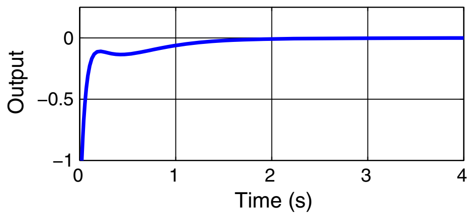


(a) Response of plant $\mathcal{G}_1$ and controller $\mathcal{G}_2$.

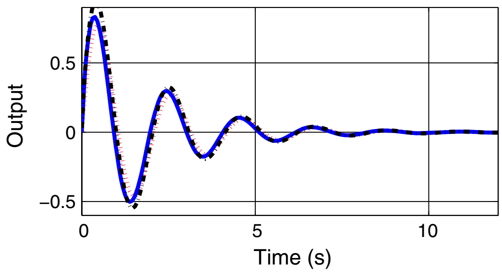


(b) Response of plant $\mathcal{G}_3$ and controller $\mathcal{G}_2$.

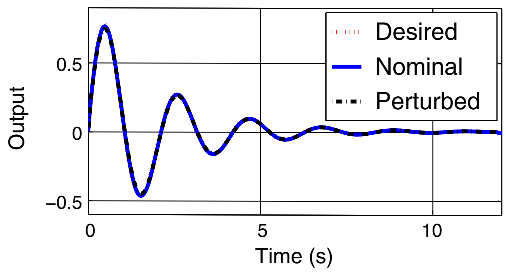
Figure 3. Responses to initial conditions observed during controller design.

Table II. Controller parameters.

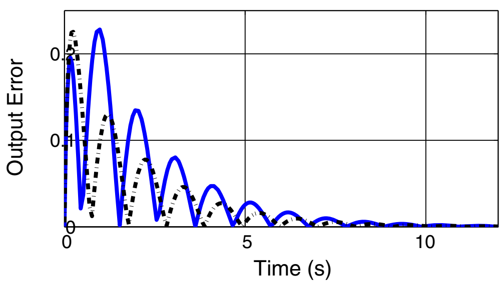
Parameter	a	k	ν_c	ν_p	$\nu_p \nu_c$
Value tuned for $\mathcal{G}_1$	-3	-2	2	0.538	$1.08 > 1$
Value tuned for $\mathcal{G}_3$	-2	-5	5	0.9	$4.5 > 1$



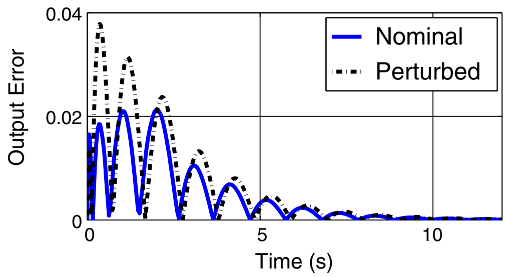
(a) Trajectory of plant $\mathcal{G}_1$ and controller $\mathcal{G}_2$.



(b) Trajectory of plant $\mathcal{G}_3$ and controller $\mathcal{G}_2$.



(c) Error in trajectory of plant $\mathcal{G}_1$ and controller $\mathcal{G}_2$.



(d) Error in trajectory of plant $\mathcal{G}_3$ and controller $\mathcal{G}_2$.

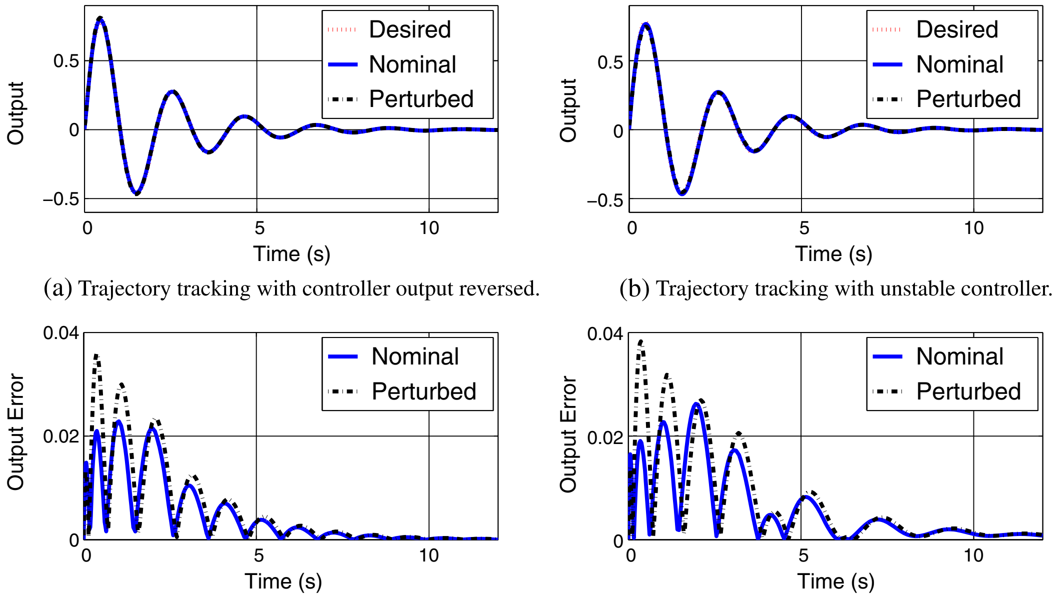
Figure 4. Tracking a decaying sine wave using the nominal and perturbed plants.

In order to test the vulnerability of performance to changes in parameters, the controllers were used to track a decaying sine wave with the nominal and perturbed plants. The reference signal was

$$r_1 = 0, \ r_2 = -\mathrm{e}^{-0.5t} \sin(3t).$$

For $\mathcal{G}_1$, it was taken that $\delta_1 = \delta_2 = 0.3$. The perturbed parameters of $\mathcal{G}_3$ can be found in Table I. As can be seen in Figure 4, even with these relatively large changes in plant parameters, the controllers' performance was quite good, and input–output stability was achieved.

It is also of interest to note that stability was achieved during tuning for a surprising variety of controller parameters. Perhaps the most unintuitive observation is that reversing the controller's output, that is, multiplying k by -1, does not affect the stability of the system, although it may result in degraded performance. Mathematically, this is true because $\|\mathcal{G}\mathbf{u}\|_{2T} = \|-\mathcal{G}\mathbf{u}\|_{2T}$ implies that $\mathcal{G} : L_{2e} \to L_{2e}$ and $-\mathcal{G} : L_{2e} \to L_{2e}$ share the same minimum gain and therefore satisfy the conditions of the Large Gain Theorem for the same negative feedback interconnections. As an illustration,



(a) Trajectory tracking with controller output reversed. (b) Trajectory tracking with unstable controller.

(c) Error in trajectory with controller output reversed. (d) Error in trajectory with unstable controller $\mathcal{G}_2$.

Figure 5. Tracking a decaying sine wave using modified controllers and plant $\mathcal{G}_3$.

Figure 5 presents the results of trajectory tracking performed with plant $\mathcal{G}_3$ when the controller is reversed, to be $-\mathcal{G}_2$. Of further interest is the fact that negative feedback interconnections involving plants and controllers, which are both unstable, may result in stable closed-loop systems if the Large Gain Theorem is satisfied. For instance, if $a = 0.75$ and $k = -5$ for $\mathcal{G}_2$, and this system is used in negative feedback interconnection to control $\mathcal{G}_3$, then closed-loop input–output stability is guaranteed because $\nu_p \nu_c = (0.9)\left(\frac{|-5|(1-0.75)}{0.75}\right) = 3 > 1$. The result of trajectory tracking with this closed-loop system is also displayed in Figure 5. These results highlight the versatility of the Large Gain Theorem, and thereby the utility of the Minimum Gain Lemma, which facilitates its use. Moreover, the broad range of qualitatively different controllers that can be employed promises that these tools will enable the pursuit of varied design objectives while still maintaining stability guarantees via the Large Gain Theorem.

5. CONCLUSION

When considering interconnections of unstable plants and/or controllers, well-established results, such as the Passivity and Small Gain Theorems, often are inapplicable. The resulting shortage of readily-applicable resources makes the control of such systems more challenging than the control of their open-loop stable counterparts. The results presented here contribute to reducing this deficit by building upon the stability criteria of Zahedzadeh *et al*. [4]. Although not yet widely employed, the notion of minimum gain and the corresponding Large Gain Theorem presented in [4] are potentially invaluable tools when dealing with unstable systems. This paper serves to broaden the applicability of these results and facilitate their practical application.

The definition of minimum gain is fine-tuned in Section 2. In this section, it is also shown that the Large Gain Lemma still holds when accounting for these alterations. Moreover, the lemma becomes more readily applicable; it may only be applied to systems with strictly positive minimum gain, and the alterations are made to account for nonzero initial conditions and admit stable LTI systems, which had zero minimum gain according to the original definition. Paramount to these improvements, the novel Minimum Gain Lemma provides LMI conditions equivalent to minimum gains for LTI systems. These provide a systematic way to determine the minimum gain of LTI systems, and could be applied to linearizations of nonlinear ones. Further, the Minimum Gain Lemma invites application in the synthesis of controllers providing guaranteed closed-loop, input–output stability, as have similar results, such as the Conic Sector Lemma [9, 10].

The numerical examples in this paper highlight the immediate utility of the contributions of this paper and demonstrate how they may be applied. In Sections 4.1 and 4.2, the Minimum Gain Lemma is used in stability analysis. Two unstable plants are shown to posses a minimum gain, allowing for reasonable perturbations in their parameters. For one plant, the conditions of the Minimum Gain Lemma are verified analytically, while they are verified numerically for the other. The minimum gain is determined with and without the use of the Minimum Gain Lemma for a third system, as a verification of the Lemma and the numerical solvers employed. This system serves as a controller for both plants, eliciting good responses in trajectory-tracking problems and robustly ensuring closed-loop input–output stability via the Large Gain Theorem. In fact, multiple selections of controller parameters that engender adequately high minimum gains are shown to illicit similarly effective results, despite resulting in qualitatively different controllers, such as stable and unstable controllers, or those with opposite outputs. This exposes the robustness of the Large Gain Lemma.

Although this paper focuses on employing the notion of minimum gain, the Minimum Gain Lemma, and the Large Gain Theorem in stability analysis, its contributions are potentially useful for closed-loop controller synthesis. In $\mathcal{H}_\infty$ control, the Bounded Real Lemma is used to synthesize stabilizing controllers while ensuring desired closed-loop gains [21, pp. 215–262], for instance [22–25]. Via the Small Gain Theorem [1, 2], this yields closed-loop systems that are robust with respect to uncertainties with adequately low gains. The next step in working with the Minimum Gain Lemma will be to develop similar methods, but synthesizing stabilizing controllers that ensure desired closed-loop minimum gains. By the Large Gain Theorem, these would yield closed-loop systems that are robust with respect to uncertainties with adequately high minimum gains. These uncertainties could incorporate unstable behavior, providing a distinct advantage in many applications.

Fundamentally, this paper provides the building blocks for executing analyses and implementing synthesis methods based on the Large Gain Theorem. The versatility of this theorem is improved by the altered definition of minimum gain. The value of these improvements is borne out by a series of numerical examples, and their potential is underscored by their applicability to systems involving unstable, nonlinear plants, for which novel approaches to stability analysis and controller synthesis are in demand.

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DOI: 10.1002/rnc