Fefferman's Hypersurface Measure and Volume Approximation Problems

by

Purvi Gupta

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in the University of Michigan
2015

Doctoral Committee:

Professor David E. Barrett, Chair
Professor Daniel M. Burns Jr.
Professor Lizhen Ji
Professor Mattias Jonsson
Associate Professor Venkatesh K. Nagar

A C K N O W L E D G M E N T S

It gives me great joy to acknowledge the toil, patience and perseverance of all those who have made this thesis happen. Foremost among them is my thesis adviser, David Barrett, who has been a constant source of encouragement, ideas and inspiration. His generosity and good temper cannot be exaggerated, and his style of mathematics will forever influence my work. I have also benefited vastly from Dan Burns' wisdom, breadth of knowledge and accessibility. I am grateful to him, Mattias Jonsson, Lizhen Ji and Venky Nagar for agreeing to be on my doctoral committee and for their invaluable feedback on this thesis. I am also obliged to Debraj Chakrabarti, Sönmez Şahotoğlu, Rasul Shafikov, Norm Levenberg and Federico Piazzon for raising pertinent questions related to the material in this thesis — it has greatly improved my understanding of the subject.

My mathematical growth has been fostered by wonderful teachers, among whom I must mention my Masters' adviser, Gautam Bharali, who not only introduced me to Several Complex Variables, but continues to impart the kind of advice that gets only better with age. I am also indebted to the math community at the University of Michigan, especially my graduate colleagues Tengren Zhang, Jeff Calder, Xiaolei Zhao, and the various members of the student analysis seminar group for hours of edifying conversations. In the time of need, I also received sound professional advice and encouragement from Debraj Chakrabarti, Ilya Kossovskiy, Andrea Nicoara and Mei-Chi Shaw, for which I am grateful.

On a personal note, I must thank all those who have made my long graduate years enjoyable: Tengren for paradoxically being a great source of motivation and distraction, Zhibek for her equanimity and shared sense of fun (not to mention her doomed attempts to teach me Russian), Rob and Rohini for heartily providing nourishment — literal and otherwise, Chee and Laura for helping me acclimatize to Ann Arbor so swiftly, and Arun, Felipe, Grace, Jake, Yuchong, Xiaolei, William and others for their friendly words. To this I must add the gratitude I owe to Rishika, Shruti, Venu, Hannah, Aditi, Amita, Jonathan and Prasanna whose warm e-mails and phone calls have sustained me through harsh Michigan winters.

Finally, there are those who have already earned a lifetime of gratitude; my parents who have always treated my choices with the right amount of respect, forbearance, and skepticism, my sister and brother-in-law for their unending affection and hospitality, my baby nephew, in whose anticipation I discovered the germ of this work, and Sandeep, whose ability to mitigate my worries is simply overwhelming. This thesis would indeed amount to nothing without all their support.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF NOTATION

$\mathbb{N}_+$ the set of positive natural numbers

$\mathbb{R}$, $\mathbb{R}^d$ the real line, the d-dimensional real Euclidean space

$\mathbb{C}$, $\mathbb{C}^d$ the complex plane, the d-dimensional complex Euclidean space

$\mathbb{C}^*$ the punctured plane $\mathbb{C} \setminus \{0\}$

$|\cdot|$ the absolute value of a complex number

$||\cdot||$ the Euclidean norm in $\mathbb{R}^d$, $d > 1$

$\mathrm{dist}(\cdot, \cdot)$ the Euclidean distance between two compact sets or a point and a compact set

$\omega_{\mathbb{C}^d}$ the standard volume form on $\mathbb{C}^d$

Note. ω or ω_d denotes the Lebesgue measure on $\mathbb{C}^d$

σ_Ω Fefferman's hypersurface area form on the boundary of Ω

s_Ω the Euclidean surface area form on the boundary of the smooth domain Ω

Note. s denotes the Euclidean surface area measure, in Chapter 3

$\mathbb{B}^d(z; r)$ the Euclidean ball in $\mathbb{C}^d$ with center z and radius r

$\mathbb{B}^d = \mathbb{B}^d(0; 1)$ the unit Euclidean ball in $\mathbb{C}^d$

$\mathbb{D} = \mathbb{B}^1$ the unit Euclidean disk in $\mathbb{C}$

$\mathrm{vol}(A)$ the Lebesgue measure of A

Note. $\mathrm{vol}_3(A)$ is used in 3 to emphasize that the volume is being evaluated in $\mathbb{C} \times \mathbb{R} = \mathbb{R}^3$

$\mathfrak{b}_d$ the volume of $\mathbb{B}^d$

$f_x,\ f_{xy}$ abbreviation for $\dfrac{\partial f}{\partial x}, \dfrac{\partial^2 f}{\partial x \partial y}$, where x, y can be real or complex

∇f the gradient of a real-valued function f

f^* the pull-back operator induced by f on differential forms and measures

$\mathrm{J}_{\mathbb{R}}f(x)$ the real Jacobian of f at x

$\mathrm{J}_{\mathbb{C}}f(x)$ the complex Jacobian of f at x

$\mathrm{Hess}_{\mathbb{R}}f(x)$ the real Hessian of f at x

$\mathrm{Hess}_{\mathbb{C}}f(x)$ the complex Hessian of f at x, see (2.1)

$T_z^{\mathbb{R}}M$ the real tangent space to M at z

$T_z^{\mathbb{C}}M$ the complex tangent space to M at z, see Remark 2.1.2

$\mathscr{L}_z\rho$ the Levi form of ρ at z, see Remark 2.1.2

$\mathcal{C}(D)$ the space of continuous functions on D

$\mathcal{C}^k(D)$ the space of functions that are k-times continuously differentiable in some open neighborhood of D

$\mathcal{O}(\Omega)$ the space of holomorphic functions on the open set Ω

$\mathcal{A}(\Omega)$ the space of holomorphic functions on Ω that are continuous on $\overline{\Omega}$

$L^2(X,\mu)$ the space of square-integrable functions on the measure space (X,μ)

$L^2_{(p,q)}(\Omega)$ the space of (p,q) differential forms with L^2 coefficients on Ω

ℓ^2 the space of square-summable complex-valued sequences

K_Ω the Bergman kernel of Ω, see Definition 2.3.5

$\chi_{_A}$ the indicator function of the set A

$\mathrm{int}_B A$ the interior of A in B in the relative topology, when $A \subset B$

v^{tr} the transpose of v

$\lfloor x \rfloor$ the floor function or integer part of x

$\cdot_{_{\mathbb{H}}}$ multiplication in the Heisenberg group, see (3.17)

$d_{_{\mathbb{H}}}$ the Korányi metric in the Heisenberg group, see (3.18)

$\dim_{\mathrm{HF}}$ the Hausdorff-Fefferman dimension, see Definition 4.1.1

Remark. Some chapter-specific notation can also be found in Sections 3.2 and 3.5.1.

ABSTRACT

Fefferman's Hypersurface Measure and Volume Approximation Problems

by

Purvi Gupta

Chair: Professor David E. Barrett

In this thesis, we give some alternate characterizations of Fefferman's hypersurface measure on the boundary of a strongly pseudoconvex domain in complex Euclidean space. Our results exhibit a common theme: we connect Fefferman's measure to the limiting behavior of the volumes of the gap between a domain and its (suitably chosen) approximants. In one approach, these approximants are polyhedral objects with increasing complexity — a construction inspired by similar results in convex geometry. In our second approach, the super-level sets of the Bergman kernel is the choice of approximants. In both these cases, we provide examples of some (non-strongly) pseudoconvex domains where these alternate characterizations lead to boundary measures that are invariant under volume-preserving biholomorphisms, thus extending the scope of Fefferman's original definition.

CHAPTER 1

Introduction

The transformation properties of Euclidean quantities play an important role in complex analysis. For example, if s_{arc} and ω_d denote the arc-length measure in $\mathbb{C}$ and the Lebesgue measure in $\mathbb{C}^d$, respectively, the identities (under suitable conditions)

$$F^* s_{\mathrm{arc}} = |\det \mathrm{J}_{\mathbb{C}} F| s_{\mathrm{arc}}, \quad (1.1)$$

$$F^* \omega_d = |\det \mathrm{J}_{\mathbb{C}} F|^{2d} \omega_d, \quad (1.2)$$

for a biholomorphism F, lead to the construction of biholomorphically invariant objects such as the Szegő projection in $\mathbb{C}$, and the Bergman metric and the Bergman projection in $\mathbb{C}^d$. Such a transformation law is lacking for the standard Euclidean surface area measure (or the $(2d-1)$-dimensional Hausdorff measure) on the boundary of a domain in $\mathbb{C}^d$, $d > 1$. In his paper *Parabolic invariant theory in complex analysis* (1979, see [12]), Fefferman observed that the boundary of a strongly pseudoconvex domain does, in fact, support a measure that satisfies a (1.1)-type condition, when acted upon by biholomorphisms. This measure has been used to study Szegő projections on CR-manifolds ([17]), volume-preserving CR invariants, isoperimetric problems (see [15] and [5]) and invariant metrics ([6]).

As strong pseudoconvexity is a biholomorphically invariant version of strong convexity (see Part 3. of Proposition 2.1.3), it is natural to ask whether an analogue of Fefferman's measure exists in the affine setting. It turns out that such a measure has indeed been studied in convex geometry. In 1923, Blaschke ([7]) introduced a measure on the boundary of a strongly convex domain that transforms well under affine maps. In particular, it is invariant under equi-affine (volume-preserving affine) maps. This initiated a project of characterizing Blaschke's measure in ways that did not rely on the smoothness of the convex body in question. Many of these methods rely on the volume-approxmation approach, a classical example of which is the following observation: for a sufficiently regular domain $D \subset \mathbb{R}^d$,

$$\lim_{\delta \to 0} \frac{\mathrm{vol}(D) - \mathrm{vol}(D[\delta])}{\delta} = \int_{\partial D} ds_D,$$

where $D[\delta] := \{x \in D : \mathrm{dist}(x, \partial D) > \delta\}$ and s_D is the standard Euclidean or $(d-1)$-Hausdorff measure on ∂D (compare this to Result 2.2.2). This thesis, in a similar vein, establishes some alternate characterizations of the Fefferman hypersurface measure, so as to expand the class of domains that fall under its purview.

In Chapter 2, we rigorously introduce the objects of our study. First, we give Fefferman's definition of a measure (on strongly pseudoconvex hypersurfaces) that transforms tractably under biholomorpisms. This is followed by an introduction to some analogous ideas from convex geometry that have motivated the work in this thesis. In particular, we will single out characterizations of the Blaschke surface area measure that seem to resonate with our goal. Chapter 2 also contains some relevant material on the theory of Hilbert spaces with reproducing kernels, as these play a vital role both in the methods and the goals of this work.

In Chapter 3 , we connect Fefferman's hypersurface measure to the question of polyhedral approximation of strongly pseudoconvex bodies. We discuss why it does not seem reasonable to work with the full set of analytic polyhedra for this purpose, and include a description — in the guise of Theorem 3.1.1 — of some classes of polyhedral objects for which such a connection can be made. To give a specific example, our result implies that Fefferman's measure on $\partial\Omega$ can be completely described in terms of asymptotic estimates on the volume approximation of Ω by sets of the form

$$\{z \in \Omega : |K_\Omega(z, w_j)| < m_j, j = 1, ..., n\} \qquad w_j \in \partial\Omega, m_j > 0, \tag{1.3}$$

where K_Ω is the Bergman kernel of Ω. The rest of the chapter is devoted to the proof of our main theorem and its corollaries. Although the proof is technical at times, an intuitive idea of the combinatorics involved therein can be gleaned from the contents of Section 3.4, where we cover the model case (Siegel upper half space) of our setup. This particular case connects up to a tiling problem on the Heisenberg group, which is not only of independent interest but also seems indispensable in our proof. We elaborate on this in the final section of this chapter.

Lastly, in Chapter 4, we take motivation from (1.3) to construct measures on the boundaries of d-dimensional domains in terms of the super-level sets of their Bergman kernel. Our construction recovers the Fefferman measure in the case of strongly pseudoconvex domains, and produces some interesting results for products of balls. These examples illustrate the possibility of finding lower-dimensional or ‘Hausdorff’ Fefferman measures on the boundaries of non-smooth domains. In view of our goal to produce invariant quantities, we end this chapter, with a heuristic approach to constructing Hardy spaces using these Hausdorff-Fefferman measures.

CHAPTER 2

Background material

2.1 Fefferman's hypersurface measure

We first introduce a notion of convexity that is natural to complex analysis.

Definition 2.1.1. Let $\Omega \subset \mathbb{C}^d$ be given by $\{\rho < 0\}$, where ρ is a $\mathcal{C}^2$-smooth function in a neighborhood of $\overline{\Omega}$ and $\nabla\rho \neq 0$ on $\partial\Omega$. Ω is called *strongly pseudoconvex* if

$$\overline{v}^{\mathrm{tr}} \cdot \mathrm{Hess}_{\mathbb{C}}\rho(z) \cdot v = \sum_{1\leq j,k\leq d} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z_k}}(z) v_j \overline{v_k} > 0 \quad \text{when } z \in \partial\Omega \text{ and } \sum_{j=1}^{d} \frac{\partial \rho}{\partial z_j}(z) v_j = 0. \tag{2.1}$$

A domain (possibly non-smooth) $\Omega \subset \mathbb{C}^d$ is called *pseudoconvex* if it can be exhausted by strongly pseudoconvex domains, i.e, $\Omega = \cup_{j\in\mathbb{R}}\Omega_j$ with each Ω_j strongly pseudoconvex and $\Omega_j \subseteq \Omega_k$ for $j < k$.

Remark 2.1.2. The space of vectors v satisfying the final condition in (2.1) is called the *complex tangent space of* $\partial\Omega$ *at* z and is denoted by $T_z^{\mathbb{C}}\partial\Omega$. It is the maximal complex subspace of $T_z^{\mathbb{R}}\partial\Omega$ — the space of vectors v such that $z + v$ is tangent to $\partial\Omega$ at z. The *Levi form of* ρ *at* z, $\mathscr{L}_z\rho$, is the Hermitian form $\mathrm{Hess}_{\mathbb{C}}\rho(z)$ restricted to $T_z^{\mathbb{C}}\partial\Omega$.

The following characterizations of strongly pseudoconvex domains give some geometric insight, and are used often in this dissertation. For a proof, see [16, Section 1.5].

Proposition 2.1.3. *Let* $\Omega \subset \mathbb{C}^d$ *be a* $\mathcal{C}^2$*-smooth bounded domain. Then, the following conditions are equivalent.*

1. Ω *is strongly pseudoconvex.*

2. *There exists a defining function* $\rho : \mathbb{C}^d \to \mathbb{R}$ *for* Ω *such that*

$$\overline{v}^{\mathrm{tr}} \cdot \mathrm{Hess}_{\mathbb{C}}\rho(z) \cdot v = \sum_{1\leq j,k\leq n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z_k}}(z) v_j \overline{v_k} > 0 \qquad \forall z, v \in \mathbb{C}^d. \tag{2.2}$$

A function satisfying (2.2) *is called* strictly plurisubharmonic.

3. *Ω is locally convexifiable, i.e., for every $z \in \partial\Omega$, there is a neighborhood U_z containing z and a biholomorphic map h_z on U_z such that $h_z(U_z \cap \Omega)$ is a strongly convex domain (see Definition 2.2.1).*

Related to the Levi form is the *Fefferman Monge-Ampére operator* M on the space of $\mathcal{C}^2$-smooth real-valued functions on $\mathbb{C}^d$, defined by

$$M[\rho] = -\det \begin{pmatrix} \rho & \rho_{\overline{z_k}} \\ \rho_{z_j} & \rho_{z_j\overline{z_k}} \end{pmatrix}_{1\le j,k\le d}.$$

Its properties are central to our discussion and we collect them as

Lemma 2.1.4. *Let $\rho : \mathbb{C}^d \to \mathbb{R}$ be $\mathcal{C}^2$-smooth. Then,*

$(a.)$ $M[\eta\rho] = \eta^{d+1}M[\rho]$ *when $\rho = 0$, for a $\mathcal{C}^2$-smooth η.*

$(b.)$ *Let $F : \mathbb{C}^d \to \mathbb{C}^d$ be a biholomorphism. Then, $M[\rho \circ F] = |\,\mathrm{J}_{\mathbb{C}}F|^2(M[\rho] \circ F)$.*

$(c.)$ *If $\Omega \subset \mathbb{C}^d$ is strongly pseudoconvex, then $M[\rho] > 0$ on $\partial\Omega$ for any defining function ρ of Ω such that $\Omega = \{z \in \mathbb{C}^d : \rho(z) < 0\}$.*

Proof. For claim $(a.)$, let $w \in \mathbb{C}^d$ be a point where ρ vanishes. Then, we may assume $\eta(w) \neq 0$, else the claim follows trivially. We can now write

$$-\det \begin{pmatrix} \eta\rho & (\eta\rho)_{\overline{z_k}} \\ (\eta\rho)_{z_j} & (\eta\rho)_{z_j\overline{z_k}} \end{pmatrix}_{z=w} = -\det \begin{pmatrix} 0 & \eta\rho_{\overline{z_k}} \\ \eta\rho_{z_j} & \eta_{z_j}\rho_{\overline{z_k}} + \eta_{\overline{z_k}}\rho_{z_j} + \eta\rho_{z_j\overline{z_k}} \end{pmatrix}_{z=w}.$$

For each j, we can multiply the first row by $\frac{\eta_{z_j}(w)}{\eta(w)}$ and subtract it from the j-th row, and then multiply the first column by $\frac{\eta_{\overline{z_k}}(w)}{\eta(w)}$ and subtract it from the k-th column, to obtain $\eta^{d+1}M[\rho](w)$.

To prove $(b.)$, we set $R(z_0, z) = |z_0|^{\frac{2}{d+1}}\rho(z)$ for $(z_0, z) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^d$, and lift F to a biholomorphic map $\mathscr{F} : \mathbb{C}^* \times \mathbb{C}^d \to \mathbb{C}^* \times \mathbb{C}^d$ by setting

$$\mathscr{F}(z_0, z) = (z_0/\det \mathrm{J}_{\mathbb{C}}F(z), F(z)).$$

Thus, for $\tilde{R} := R \circ \mathscr{F}$, we get $\mathrm{Hess}_{\mathbb{C}}\tilde{R} = \mathrm{J}_{\mathbb{C}}\mathscr{F} \cdot \mathrm{Hess}_{\mathbb{C}}R \cdot \overline{\mathrm{J}_{\mathbb{C}}\mathscr{F}}^{\mathrm{tr}}$. As $\det \mathrm{J}_{\mathbb{C}}\mathscr{F} \equiv 1$, $\mathrm{Hess}_{\mathbb{C}}\tilde{R}(z_0, z) = \mathrm{Hess}_{\mathbb{C}}R(F(z_0, z))$. On substituting, we find that

$$\tilde{R}(z_0, z) = |z_0|^{\frac{2}{d+1}}|\det \mathrm{J}_{\mathbb{C}}F(z)|^{\frac{-2}{d+1}}(\rho \circ F)(z).$$

Moreover, $\mathrm{Hess}_{\mathbb{C}} R(z_0, z) = c_d M[\rho](z)$ for some dimensional constant c_d. Using part 1.,

$$M[\rho] \circ F = M[|\,\mathrm{J}_{\mathbb{C}} F|^{\frac{-2}{d+1}}(\rho \circ F)] = |\,\mathrm{J}_{\mathbb{C}} F|^{-2} M[\rho \circ F].$$

The proof of $(c.)$ relies on the fact that if $A : \mathbb{C}^d \to \mathbb{C}^d$ is an isometry, i.e., $Az = Uz + b$ for some unitary U and $b \in \mathbb{C}^d$, then

$$\begin{aligned} M[\rho \circ A] &= -\det\left(\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}\begin{pmatrix} \rho \circ A & \rho_{\overline{z_k}} \circ A \\ \rho_{z_j} \circ A & \rho_{z_j\overline{z_k}} \circ A \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \overline{U}^{\mathrm{tr}} \end{pmatrix}\right) \\ &= M[\rho] \circ A. \end{aligned}$$

Therefore, if $z \in \partial\Omega$, after an isometry A we may assume that $z = 0$ and $\rho_{z_j}(0) = 0$, $1 \leq j, \leq d-1$, i.e., $\{z_d = 0\} = T_0^{\mathbb{C}}\partial\Omega$, to obtain that

$$M[\rho](z) = -\det\begin{pmatrix} 0 & 0 & \cdots & \rho_{\overline{z_d}}(z) \\ 0 & & & * \\ \vdots & & L_z\rho & \vdots \\ \rho_{z_d}(z) & * & \cdots & * \end{pmatrix} = |\rho_{z_d}(z)|^2 \det \mathscr{L}_z\rho > 0.$$

where $L_z\rho$ is the matrix representation of the Levi form $\mathscr{L}_z\rho$ on $T_z^{\mathbb{C}}\partial\Omega = \{z_d = 0\}$.

□

Based on these facts, Fefferman defined a $(2d-1)$-form on $\partial\Omega$, characterized by the equation

$$\sigma_\Omega \wedge d\rho = 4^{\frac{d}{d+1}} M[\rho]^{\frac{1}{d+1}} \omega_{\mathbb{C}^d},$$

where $\omega_{\mathbb{C}^d}$ is the standard volume form on $\mathbb{C}^d$. In view of the following lemma, we call σ_Ω the *Fefferman hypersurface area measure* on $\partial\Omega$.

Lemma 2.1.5. *Let $\Omega \subset \mathbb{C}^d$ be a strongly pseudoconvex domain. Then, σ_Ω does not depend on the choice of ρ, and for any biholomorphism F on Ω that extends $\mathcal{C}^1$-smoothly to a neighborhood of $\overline{\Omega}$,*

$$F^*\sigma_{F(\Omega)} = |\det \mathrm{J}_{\mathbb{C}} F|^{\frac{2d}{d+1}} \sigma_\Omega. \tag{T1}$$

Proof. Based on the characterizing property of σ_Ω, we have

$$\sigma_\Omega = 4^{\frac{d}{d+1}} \frac{M[\rho]^{\frac{1}{d+1}}}{||\nabla\rho||} s_\Omega,$$

where s_Ω is the standard Euclidean surface area measure on $\partial\Omega$ and ρ is any defining function of Ω. Independence from ρ follows from part a. of Lemma 2.1.4 and the fact that $\nabla(\eta\rho) = \eta\nabla\rho$ when $\rho = 0$. Property (T1) follows from part b. of Lemma 2.1.4 and the transformation property of the standard volume form in $\mathbb{C}^d$. $\square$

2.2 Blaschke's surface measure

A set $D \subset \mathbb{R}^d$ is called *convex* if, for any two points in D, the line segment joining those points also lies in D. A compact, convex set with non-empty interior is called a *convex body*. If the boundary of a convex body D contains no line segments, it is called *strictly convex*. This notion is closely related to (but weaker than) the concept of strong convexity.

Definition 2.2.1. Let $D \subset \Omega$ be given by $\{r < 0\}$, where r is a $\mathcal{C}^2$-smooth function defined in a neighborhood of $\overline{D}$ and has a non-vanishing gradient on ∂D. D is *strongly convex* if

$$v^{\mathrm{tr}} \cdot \mathrm{Hess}_{\mathbb{R}} r(x) \cdot v = \sum_{1 \le j,k \le d} \frac{\partial^2 \rho}{\partial x_j \partial \overline{x_k}}(z) v_j v_k > 0 \quad \text{for } x \in \partial\Omega \text{ and } v \in T_x^{\mathbb{R}} \partial D. \tag{2.3}$$

If $D \subset \mathbb{R}^d$ is a $\mathcal{C}^2$-smooth convex body, the *Blaschke surface area measure* on ∂D is given by

$$\mu_D = \kappa^{\frac{1}{d+1}} s_D,$$

where κ and s_D are the Gaussian curvature function and the Euclidean surface area form on ∂K, respectively. Under an affine transformation $A : \mathbb{R}^d \to \mathbb{R}^d$,

$$A^* \mu_{A(D)} = |\det \mathrm{J}_{\mathbb{R}} A|^{\frac{d-1}{d+1}} \mu_D.$$

This feature of μ_D makes it very suitable for the purpose of affine geometry, and inspired several mathematicians to extend the notion of affine surface area to arbitrary convex bodies (see [23] for details).

We discuss the Schütt-Werner approach (see [28]) which is based on a modification of Dupin's notion of a floating body. For a convex body $D \subset \mathbb{R}^d$, the *convex floating body* D_δ is the intersection of all the halfspaces in $\mathbb{R}^d$ whose hyperplanes cut off a set of volume δ from D. For any $x \in \partial D$, $\Delta(x, \delta)$ denotes the height of the slice of volume δ cut off by a hyperplane orthogonal to the normal to ∂D at x.

Result 2.2.2 (Schütt-Werner)**.** *For any d-dimensional convex body,*

$$\lim_{\delta\to\infty} c_d \frac{\mathrm{vol}(D) - \mathrm{vol}(D_\delta)}{\delta^{\frac{2}{d+1}}} = \int_{\partial D} \lim_{\delta\to 0} c_d \frac{\Delta(x,\delta)}{\delta^{\frac{2}{d+1}}} d\mathcal{H}^{d-1}(x),$$

where c_d is a dimensional constant and $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure on ∂D. The limit under the integral exists almost everywhere, and coincides with $\kappa^{\frac{1}{d+1}}$ when D is $\mathcal{C}^2$-smooth.

This results clearly relies on the natural regularity of convex bodies — a feature that pseudoconvex bodies may lack in general. Nonetheless, a floating body approach to Fefferman's hypersurface measure has been explored by D. Barrett in [4].

In his paper, Barrett also asks whether Fefferman's measure can be connected to the question of polyhedral approximations of pseudoconvex bodies. He raises this question in view of several such results in the convex setting where the Blaschke measure appears in the asymptotics for the approximation of convex bodies by convex polyhedra with increasing complexity (see [14, Chap. 1.10] for a survey). Of particular interest to us is a result due to Gruber [13] who showed that if $D \subset \mathbb{R}^d$ is a $\mathcal{C}^2$-smooth strongly convex body, then

$$\inf\{\mathrm{vol}(P \setminus \Omega) : P \in \mathcal{P}_n^c\} \sim \frac{1}{2}\,\mathrm{div}_{d-1}\left(\int_{\partial K} \mu_K\right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}}, \tag{2.4}$$

as $n \to \infty$, where $\mathcal{P}_n^c$ is the class of all polyhedra that circumscribe K and have at most n facets, and div_{d-1} is a dimensional constant. Ludwig [24] later showed that, if the approximating polyhedra are from $\mathcal{P}_n$, the class of *all* polyhedra with at most n facets, then

$$\inf\{\mathrm{vol}(\Omega \Delta P) : P \in \mathcal{P}_n\} \sim \frac{1}{2}\,\mathrm{ldiv}_{d-1}\left(\int_{\partial K} \mu_K\right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}} \tag{2.5}$$

as $n \to \infty$, where Δ denotes the symmetric difference between sets and ldiv_{d-1} is a dimensional constant. Later, Böröczky [20] proved both these formulae for general smooth convex bodies.

We end this section by noting that Result 2.2.2 and the formulae (2.4) and (2.5) do not capture the full potential of these methods, especially for non-smooth domains. To illustrate our point, we state a result that motivates much of Chapter 4 of this thesis.

Result 2.2.3 (Schütt, [27])**.** *Let P be a d-dimensional polyhedron. Then,*

$$\lim_{\delta\to 0} \frac{\mathrm{vol}(P) - \mathrm{vol}(P_\delta)}{\delta(\ln(1/\delta))^{d-1}} = \frac{1}{d!}\frac{1}{d^{d-1}}\Phi_d(P),$$

where $\Phi_d(P) := \#\{$*flags* $F_0 \subset F_1 \subset \cdots \subset F_{d-1}\}$*,* F_j *being a* j*-dimensional face of* P*.*

2.3 Reproducing Kernel Hilbert Spaces

In this section we present some background material on reproducing kernel Hilbert spaces that will play an important role in the rest of this dissertation. For a more exhaustive treatment, we direct the interested reader to Aronszajn's classical treatise (see [2]) on this subject.

Definition 2.3.1. Let $\Omega \subset \mathbb{C}^d$ be a domain and $H \subseteq \mathcal{C}(\Omega)$ a separable $\mathbb{C}$-Hilbert space of functions on Ω with scalar product $(\cdot,\cdot)_H$ and norm $||\cdot||_H$. H is called a reproducing kernel Hilbert space (RKHS) on Ω if

$$\text{for all } x \in \Omega, \text{ the evaluation functional } k_x : x \mapsto f(x) \text{ is bounded on } H. \tag{B1}$$

The Riesz representation theorem then guarantees that for any $x \in \Omega$, there is a unique $k_x(y) \in H$ such that $f(x) = (f(y), k_x(y))_H$. The uniquely determined function $K(x,y) := \overline{k_x(y)} : \Omega \times \Omega \to \mathbb{C}$ is called the reproducing kernel of Ω with respect to H.

Remark 2.3.2. It is often useful to assume the following stronger condition on H:

$$\text{for any compact } J \subset \Omega, \exists\, C_J > 0 \text{ with } \sup_J |f(x)| \leq C_J ||f||_H,\ \forall f \in H. \tag{B2}$$

Proposition 2.3.3. *Suppose* H *is an RKHS on* Ω *satisfying property* (B2)*, and for every* $x \in \Omega$ *there is an* $f \in H$ *such that* $f(x) \neq 0$*. Then, the reproducing kernel* K *satisfies*

(a) $K(y,x) = \overline{K(x,y)}$ *for all* $x, y \in \Omega$*.*

(b) $K(x,x) > 0$ *for all* $x \in \Omega$*.*

(c) *For any complete orthonormal basis* $\{\phi_j\}$ *of* H*,* $K(x,y) = \sum_{j=1}^{\infty} \phi_j(x)\overline{\phi_j(y)}$*, where the right-hand side converges uniformly on compacts in* $\Omega \times \Omega$*.*

(d) *For every* $x \in \Omega$*,*

$$K(x,x) = \sup_{f \in H} \frac{|f(x)|^2}{||f||_H^2} = \sup_{||f||_H = 1} |f(x)|^2 = \frac{1}{\inf_{f(x)=1} ||f||_H^2}.$$

If the infimum on the right-hand side is achieved at $f_x \in H$*,* $K(x,y) = \overline{f_x(y)}/||f_x||_H^2$*.*

Proof. (a) As $\overline{K(x,\cdot)} \in H$

$$\overline{K(x,y)} = (\overline{K(x,\cdot)}, \overline{K(y,\cdot)})_H = \overline{(\overline{K(y,\cdot)}, \overline{K(x,\cdot)})_H} = \overline{\overline{K(y,x)}}.$$

From part (a), $K(x,x) = \overline{K(x,x)}$. We thus get, $K(x,x) = (\overline{K(x,\cdot)}, \overline{K(x,\cdot)})_H = ||\overline{K(x,\cdot)}||_H^2 > 0$. This establishes (b).

For the proof of (c), let $||\{a_j\}||_{\ell^2} := \left(\sum_{j=1}^{\infty} |a_j|^2\right)^{\frac{1}{2}}$. Then, for any compact $J \subset \Omega$,

$$\begin{aligned} \sup_J \left(\sum_{j=1}^{\infty} |\phi_j(x)|^2\right)^{\frac{1}{2}} = \sup_J ||\{\phi_j(x)\}_{j=1}^{\infty}||_{\ell^2} &= \sup_{\substack{||\{a_j\}||_{\ell^2}=1 \\ x\in J}} \left|\sum_{j=1}^{\infty} a_j\phi_j(x)\right| \\ &= \sup_{\substack{||f||_H=1 \\ x\in J}} |f(x)| \leq C_J, \end{aligned}$$

by the Riesz-Fisher and Riesz representation theorems, and condition (B2). Now,

$$\sum_{j=1}^{\infty} |\phi_j(x)\overline{\phi_j(y)}| \leq \left(\sum_{j=1}^{\infty} |\phi_j(x)|^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{\infty} |\phi_j(y)|^2\right)^{\frac{1}{2}}.$$

Thus, $\tilde{K}(x,y) := \sum_{j=1}^{\infty} \phi_j(x)\overline{\phi_j(y)}$ converges uniformly when $x, y \in K$. Moreover, our computation also shows that $\{\phi_j(x)\}_{j=1}^{\infty} \in \ell^2$ and, thus, $\overline{\tilde{K}(x,\cdot)} \in \mathcal{A}^2(\Omega)$ for all $x \in \Omega$. Lastly, for any $f \in H$,

$$\left(f(\cdot), \overline{\tilde{K}(x,\cdot)}\right)_H = \sum_{j=1}^{\infty} \phi_j(x)(f(\cdot), \phi_j(\cdot))_H = f(x),$$

in the sense of convergence in the norm topology. But this implies pointwise convergence due to condition (B2). Thus, $\tilde{K}$ has the two characterizing properties of a reproducing kernel, and must be K.

The first part of (d) follows from (c) as

$$K(x,x) = \sum_{j=1}^{\infty} |\phi_j(x)|^2 = \left(\sup_{||\{a_j\}||_{\ell^2}=1} \left|\sum_{j=1}^{\infty} \phi_j(x)a_j\right|\right)^2 = \sup_{||f||_H=1} |f(x)|^2.$$

Here, we have used the Riesz-Fisher representation theorem. The other expressions for $K(x,x)$ follow.

Now, let $g_x(y) := \overline{K(x,y)}/K(x,x)$. Observe that $g_x \in H$ for all $x \in \Omega$, and $g_x(x) = 1$. Thus, $||g_x||_H \geq ||f_x||_H$, by the definition of f_x. On the other hand, by the definition of K

and Hölder's inequality,

$$||g_x||_H = \frac{1}{||K(x,\cdot)||_H} = \frac{f_x(x)}{||K(x,\cdot)||_H} = \frac{(f_x(\cdot), K(x,\cdot))_H}{||K(x,\cdot)||_H} \leq ||f_x||_H$$

Thus, $||g_x||_H = ||f_x||_H$. But, for any $x \in \Omega$, $\{f \in H : f(x) = 1\}$ is a closed, convex subset of the Hilbert space H. Since such a set always has a unique norm-minimizing element, $f_x(y) = g_x(y) = \overline{K(x,y)}/K(x,x) = \overline{K(x,y)}||f_x||_H^2$. □

2.3.1 The Bergman kernel

An important example of an RKHS is the space of square-integrable holomorphic functions on bounded domains in $\mathbb{C}^d$. Specifically, for a bounded domain $\Omega \subset \mathbb{C}^d$, the Bergman space of Ω is

$$\mathcal{A}^2(\Omega) = \left\{ f \in \mathcal{O}(\Omega) : ||f||^2_{\mathcal{A}^2} := \int_\Omega |f|^2 d\omega < \infty \right\},$$

where ω is the Lebesgue measure on $\mathbb{C}^d$. This is an RKHS due to the following lemma.

Lemma 2.3.4. *For a compact $J \subset \Omega$ and $f \in \mathcal{A}^2(\Omega)$,*

$$\sup_J\{|f(z)|\} \leq \frac{\mathfrak{b}_d}{\operatorname{dist}(J,\partial\Omega)^d}||f||_{\mathcal{A}^2},$$

where $\mathfrak{b}_d$ is the volume of the unit ball in $\mathbb{C}^d$.

Proof. Let $r < \operatorname{dist}(J,\partial\Omega)$. Then, for any $z \in J$, $\mathbb{B}^d(z;r) \subset \Omega$. Therefore, by the mean-value property for harmonic functions, for any $z \in J$ and $f \in \mathcal{A}^2(\Omega)$,

$$|f(z)| \leq \frac{1}{\operatorname{vol}(\mathbb{B}^d(z;r))}\left|\int_{\mathbb{B}^d(z;r)} f(w)d\omega(w)\right| \leq (\operatorname{vol}(\mathbb{B}^d(z;r)))^{-\frac{1}{2}}||f||_{\mathcal{A}^2} \equiv \frac{\mathfrak{b}_d}{r^n}||f||_{\mathcal{A}^2}.$$

□

Thus, the Bergman space of a bounded domain satisfies all the hypotheses of Proposition 2.3.3, yielding a reproducing kernel that displays properties (a), (b), (c) and (d) stated in Proposition 2.3.3.

Definition 2.3.5. The Bergman kernel of Ω, K_Ω, is the reproducing kernel of the RKHS $\mathcal{A}^2(\Omega)$.

From (a) in Proposition 2.3.3, it follows that $K_\Omega(z,w)$ is holomorphic in z and anti-holomorphic in w. An extremely important feature of this kernel is its behavior under biholomorphic transformations:

Proposition 2.3.6. *Let $F : \Omega_1 \to \Omega_2$ be a biholomorphism between bounded domains in $\mathbb{C}^d$. Then,*

$$\det \mathrm{J}_{\mathbb{C}} F(z) \cdot K_{\Omega_2}(F(z), F(w)) \cdot \overline{\det \mathrm{J}_{\mathbb{C}} F(w)} = K_{\Omega_1}(z, w),$$

for all $z, w \in \Omega$.

Proof. Suppose $f \in \mathcal{A}^2(\Omega_1)$. Note that, by a change of variables,

$$\int_{\Omega_2} \left| \frac{f(F^{-1}(\tilde{w}))}{\det \mathrm{J}_{\mathbb{C}} F(F^{-1}(\tilde{w}))} \right|^2 d\omega(\tilde{w}) = \int_{\Omega_1} |f(w)|^2 d\omega(w) < \infty.$$

Thus, $f \in \mathcal{A}^2(\Omega_2)$ if and only if $(f \circ F) \det \mathrm{J}_{\mathbb{C}} F \in \mathcal{A}^2(\Omega_1)$. Now, for $\tilde{w} = F(w)$, we get

$$\begin{aligned}
&\int_{\Omega_1} f(w) \det \mathrm{J}_{\mathbb{C}} F(z) K_{\Omega_2}(F(z), F(w)) \overline{\det \mathrm{J}_{\mathbb{C}} F(w)} d\omega(w) \\
&= \int_{\Omega_2} f(F^{-1}(\tilde{w})) \det \mathrm{J}_{\mathbb{C}} F(z) K_{\Omega_2}(F(z), \tilde{w}) \overline{\det \mathrm{J}_{\mathbb{C}} F(F^{-1}(\tilde{w}))} \det \mathrm{J}_{\mathbb{R}} F^{-1}(\tilde{w}) d\omega(w') \\
&= \det \mathrm{J}_{\mathbb{C}} F(z) \int_{\Omega_2} f(F^{-1}(\tilde{w})) K_{\Omega_2}(F(z), \tilde{w}) \frac{\overline{\det \mathrm{J}_{\mathbb{C}} F(F^{-1}(\tilde{w}))}}{|\det \mathrm{J}_{\mathbb{C}} F(F^{-1}(\tilde{w}))|^2} d\omega(w') \\
&= \det \mathrm{J}_{\mathbb{C}} F(z) \frac{f(F^{-1}(F(z)))}{\det \mathrm{J}_{\mathbb{C}} F(F^{-1}(F(z)))} = f(z),
\end{aligned} \tag{2.6}$$

where the penultimate equality follows from the fact that $f \circ F^{-1} / \det \mathrm{J}_{\mathbb{C}} F \circ F^{-1} \in \mathcal{A}^2(\Omega_2)$, and the reproducing property of K_{Ω_2}. As K_Ω is uniquely determined by the properties $\overline{K_\Omega(z, \cdot)} \in \mathcal{A}^2(\Omega)$ for all $z \in \Omega$, and K_Ω reproduces $\mathcal{A}^2(\Omega)$, we have the claim. □

The above proposition shows, in particular, that the Bergman kernel of a domain is invariant under volume-preserving biholomorphisms. This suggests a connection to Fefferman's hypersurface measure — a theme we will explore in subsequent chapters.

2.3.2 The Szegő kernel

Another example of an RKHS that plays an important role in complex analysis is obtained by considering square-integrable functions with respect to hypersurface measures rather than volume measures on $\mathbb{C}^d$. This leads to the theory of Hardy spaces and their Szegő kernels. Here, we discuss some general conditions that would grant the existence of a 'good' Szegő kernel.

Let Ω be a domain in $\mathbb{C}^d$ and σ be a measure on $\partial\Omega$ such that $L^2(\partial\Omega, \sigma)$ is a separable Hilbert space containing $\mathcal{C}(\partial\Omega)$. Let E be some closed subspace of $L^2(\partial\Omega, \sigma)$ containing the space $\mathcal{A}(\Omega) := \mathcal{C}(\overline{\Omega}) \cap \mathcal{O}(\Omega)$ (with elements identified with their boundary values). Suppose there exists a linear operator $P : E \to \mathcal{O}(\Omega)$ such that

1. P is injective;
2. $P(f) = f$ for any $f \in \mathcal{A}(\Omega)$; and
3. for any compact $J \subset \Omega$, there is a $C_J > 0$ such that for any $f \in E$,

$$\sup_J |P(f)(z)| \leq C_J ||f||_{L^2(\partial\Omega,\sigma)}.$$

Let $H^2(\sigma) :=$ the closure of $\mathcal{A}(\Omega)$ in $L^2(\partial\Omega, \sigma)$.

Proposition 2.3.7. *There exists a unique $S_\sigma : \Omega \times \partial\Omega \to \mathbb{C}$ such that*

$i.$ $\overline{S_\sigma(z,\cdot)} \in H^2(\sigma)$ for all $z \in \Omega$; and

$ii.$ $f(z) = \int_{\partial\Omega} f(w) S_\sigma(z,w) d\sigma$, where $z \in \Omega$ and $f \in \mathcal{A}(\Omega)$.

S_σ admits a unique extension to $\Omega \times \Omega$ that satisfies properties $(a) - (d)$ in Proposition 2.3.3.

Moreover, if $F : \Omega_1 \to \Omega_2$ is a biholomorphim between bounded domains such that $F \in \mathcal{C}^1(\overline{\Omega}_1)$, $(\det \mathrm{J}_{\mathbb{C}} F)^\beta$ is well-defined in $\mathcal{A}(\Omega)$, and $F^*\sigma_2 = |\det \mathrm{J}_{\mathbb{C}} F|^{2\beta}\sigma_1$ for boundary measures σ_j, $j = 1, 2$, that admit Szegő kernels as above, then

$$\det \mathrm{J}_{\mathbb{C}} F(z)^{\,\beta} \cdot S_{\sigma_2}(F(z), F(w)) \cdot \overline{\det \mathrm{J}_{\mathbb{C}} F(w)}^{\,\beta} = S_{\sigma_1}(z,w), \quad (z,w) \in \Omega \times \overline{\Omega}. \quad \text{(T2)}$$

Proof. P converts $H^2(\sigma)$ into a function space on Ω, thus allowing for the theory discussed at the beginning of this section to be applicable. Specifically, we set

$$\begin{aligned} H &:= P(H^2(\sigma)) \\ (f,g)_H &:= \int_{\partial\Omega} (P^{-1}f)(\overline{P^{-1}g}) d\sigma, \quad \forall f, g \in H. \end{aligned}$$

Assumption 3. on P corresponds to condition (B2) on H, and so we obtain a reproducing kernel $S_H : \Omega \times \Omega \to \mathbb{C}$, where $s_z(\cdot) := \overline{S_H(z,\cdot)} \in H$, so that

$$(f, s_z)_H = \int_{\partial\Omega} (P^{-1}f)(w)\overline{(P^{-1}s_z)(w)} d\sigma(w) = f(z) \qquad \forall f \in H.$$

Let

$$S_\sigma(z,\cdot) := \overline{(P^{-1}s_z)(\cdot)} \in \overline{H^2}(\sigma), \tag{2.7}$$

where $\overline{H^2}(\sigma) = \{f \in L^2(\partial\Omega, \sigma) : \overline{f} \in H^2(\sigma)\}$. Properties $i.$ and $ii.$, and the uniqueness follow from the general theory discussed above.

To extend S_σ to $\Omega \times \overline{\Omega}$, simply set

$$S_\sigma(z, w) := \begin{cases} S_{_H}(z, w), \text{ if } z, w \in \Omega; \\ S_\sigma(z, w)(\text{as in (2.7)}), \text{ if } z \in \Omega, w \in \partial\Omega. \end{cases}$$

Then, part (a) of Proposition 2.3.3 gives $\overline{S_\sigma(z, w)} = S_\sigma(w, z)$ for $z, w \in \Omega$.

For the final claim, we follow the proof of Proposition 2.3.6. We observe that given the condition on F,

- $f \in \mathcal{A}(\Omega_2) \iff (f \circ F)(\det \mathrm{J}_{\mathbb{C}} F)^\beta \in \mathcal{A}(\Omega_1)$; and
- $f \in L^2(\partial\Omega_2, \sigma_2) \iff (f \circ F)(\det \mathrm{J}_{\mathbb{C}} F) \in L^2(\partial\Omega_1, \sigma_1)$.

Thus, $f \in H^2(\sigma_2)$ if and only if $f \in H^2(\sigma_1)$. Hence, by a computation almost identical to (2.6), we have that for any $z \in \Omega_1$ and $w \in \partial\Omega_1$,

$$\det \mathrm{J}_{\mathbb{C}} F(z)^{\;\beta} \cdot S_{\sigma_2}(F(z), F(w)) \cdot \overline{\det \mathrm{J}_{\mathbb{C}} F(w)}^{\;\beta} = S_{\sigma_1}(z, w).$$

Now, if $h \in H^2(\sigma)$ and $g \in \mathcal{A}(\Omega)$, then $P(hg) = P(h)g$. Thus, the above relation also holds when $(z, w) \in \Omega \times \Omega$, by the definition of S_σ on $\Omega \times \Omega$. □

Remark 2.3.8. We emphasize that the construction of S_σ does not depend on the choice of P, because P is identity on a dense subset of $H^2(\sigma)$, and due to condition (B2), convergence in $H^2(\sigma)$ implies uniform convergence on compacts in Ω. Thus, if P_1 and P_2 are two such linear operators, $P_1(f) = P_2(f)$ for any $f \in H^2(\sigma)$.

A classical example of the above construction is when $\Omega \subset \mathbb{C}^d$ is a $\mathcal{C}^2$-smooth bounded domain and σ is the standard surface-area measure s_Ω on $\partial\Omega$. The Poisson kernel plays the role of P. The details of this construction can be found in [29]. Another situation where this theory applies is in the case of polydisks, where P is set as the product of the Poisson kernels on the disk in each variable (see [26] for details). In the final chapter of this dissertation, we will consider another example where this theory can be of use. In particular, we will discuss situations in which the Szegő kernel displays property (T2).

CHAPTER 3

Polyhedral Approximations of Pseudoconvex Domains

In this chapter, we explore the connection between Fefferman's hypersurface measure on (the boundary of) a pseudoconvex domain and the complexity of its polyhedral approximations. Our approach is directly inspired by the asymptotic expressions (2.4) and (2.5) discovered by Gruber and Ludwig, respectively, as discussed in Section 2.2. In complex analysis, a natural notion of polyhedron is that of an analytic polyhedron. In $\Omega \subset\subset \mathbb{C}^d$, an *analytic polyhedron* is a finite union of relatively compact components of any set of the form

$$P = \{z \in \Omega : |f_j(z)| < 1,\ j = 1, ..., n\},$$

where $f_1, ..., f_n$ are holomorphic functions in Ω. The natural notion of complexity for an analytic polyhedron, P, is its order — i.e., the number of inequalities that define P. This setup, however, is not suited for our purpose as demonstrated by a result due to Bishop (Lemma 5.3.8 in [19]) which says that any pseudoconvex domain in $\mathbb{C}^d$ can be approximated arbitrarily well (in terms of the volume of the gap) by analytic polyhedra of order at most $2d$. The following example indicates where the problem lies:

Example 1. Let $\Omega = \mathbb{D}$ be the unit disc in $\mathbb{C}$. Consider the lemniscate-bound domains

$$P_n := \left\{ z \in \mathbb{D} : |f_n(z)| = \prod_{k=0}^{2n-1} \left| z - \exp(\tfrac{k\pi i}{n}) \right| > \tfrac{\pi}{n} \right\}.$$

Each P_n has order 1 and satisfies $\{|z| < 1 - \pi/n\} \subset P_n \subset \{|z| < 1 - \sqrt{3}\pi/2n\}$. Thus, for all $n \geq 1$,

$$\inf\{\mathrm{vol}(\mathbb{D} \setminus P) : P \text{ is an analytic polyhedron of order at most } n\} = 0.$$

If we, instead, declare the complexity of P_n to be $2n$ — i.e., the number of zeros of f_n, then, since $\lim_{n\to\infty} n \cdot \mathrm{vol}(\mathbb{D} \setminus P_n) < \infty$, we can expect results similar to (2.4) and (2.5).

3.1 Statements of results

Hereafter, we work in $\mathbb{C}^2$. Example 1 leads us to a special class of polyhedral objects. For any fixed $f \in \mathcal{C}(\overline{\Omega} \times \partial\Omega)$, let $\mathcal{P}_n(f)$ be the collection of all relatively compact sets in Ω of the form

$$P = \left\{ z \in \Omega : |f(w^j, z)| > \delta_j, j = 1, ..., n \right\},$$

where, $w^1, ..., w^n \in \partial\Omega$ and $\delta_1, ..., \delta_n > 0$. We present a class of functions f for which asymptotic results such as (2.4) and (2.5) can be obtained:

Theorem 3.1.1. *Let $\Omega \subset\subset \mathbb{C}^2$ be a $\mathcal{C}^4$-smooth strongly pseudoconvex domain. Suppose $f \in \mathcal{C}(\overline{\Omega} \times \partial\Omega)$ is such that*

(i) $f(z, w) = 0$ *if and only if* $z = w \in \partial\Omega$, *and*

(ii) *there exist* $\nu \in \mathbb{N}_+$, $\eta > 1$ *and* $\tau > 0$ *such that*

$$(\text{❋}) \qquad f(z, w) = a(z, w)\mathfrak{p}(z, w)^\nu + O\left(\mathfrak{p}(z, w)^{\eta\nu}\right)$$

on $\Omega_\tau := \{(z, w) \in \overline{\Omega} \times \partial\Omega : |z - w| \leq \tau\}$, where $\mathfrak{p}$ is the Levi polynomial of some strictly plurisubharmonic defining function of Ω (see Section 3.2) and a is some continuous non-vanishing function on Ω_τ.

Then, there exists a constant $l_{\mathrm{kor}} > 0$, independent of Ω, such that

$$\inf\{\mathrm{vol}(\Omega \setminus P) : P \in \mathcal{P}_n(f)\} \sim \frac{1}{2} l_{\mathrm{kor}} \left(\int_{\partial\Omega} \sigma_\Omega \right)^{\frac{3}{2}} \frac{1}{\sqrt{n}}, \tag{3.1}$$

as $n \to \infty$.

Remark 3.1.2. For Ω as above, let $\mathrm{LP}(\Omega)$ denote the class of $f \in \mathcal{C}(\overline{\Omega} \times \partial\Omega)$ that satisfy conditions (i) and (ii) of Theorem 3.1.1. Then, $\mathrm{LP}(\Omega)$ is invariant under biholomrphisms that extend ($\mathcal{C}^2$-)smoothly to the boundary.

Remark 3.1.3. The transformation law for the Fefferman measure and a heuristic argument on the unit ball allows us to guess what the left-hand side in (3.1) would be for higher-dimensional domains:

$$\frac{1}{2} l_{\mathrm{kor},\, d-1} \left(\int_{\partial\Omega} \sigma_\Omega \right)^{\frac{d+1}{d}} \frac{1}{n^{1/d+1}},$$

where is $l_{\text{kor},\, d-1}$ is a dimensional constant. This strongly resembles (2.4) and (2.5) if we set $d' := 2d+1$. This prognosticates the importance of viewing the tangent space to $\partial\Omega$ at any point as the $(2d-1)$-dimensional Heisenberg group (thus, making its Hausdorff dimension $2d = d' - 1$).

(⋇) is a natural condition when working with strongly pseudoconvex domains. We exhibit its scope by making special choices of $f \in \mathrm{LP}(\Omega)$ that yield analytic polyhedra.

Corollary 3.1.4. *Let Ω be as in Theorem 3.1.1. Then,* (3.1) *holds when f is a Henkin-Ramirez generating map of Ω. (see Section 3.2).*

Remark 3.1.5. It is natural to ask whether Theorem 3.1.1 can be obtained for holomorphic generating maps (of Cauchy-Fantappié kernels) that satisfy condition (i) but don't necessarily satisfy condition (⋇). The Cauchy-Leray map (see Section 3.2) on strongly convex domains is one such example. To understand this scenario, we define

$$B_f(w,\delta) := \{z \in \partial\Omega : |f(z,w)| < \delta\},\ w \in \partial\Omega,\ \delta > 0,$$

where f satisfies condition (i) in Theorem 3.1.1. Further, let

$$\phi_f : w \mapsto \limsup_{\delta\to 0} \frac{\sup_{y\in B_{\mathfrak{p}}(w,\delta)} \inf\{\delta' : y \in B_f(w,\delta')\}}{\inf_{y\notin B_{\mathfrak{p}}(w,\delta)} \inf\{\delta' : y \in B_f(w,\delta')\}},$$

where $\mathfrak{p}$ is as in Theorem 3.1.1. The definition of ϕ_f is inspired by the notion of quasiconformality (see [9, Section 6.5]), and captures the infinitesimal shape of the holomorphic discs $\{f(z,w) = \delta\} \cap \Omega$, as $|\delta| \to 0$. In particular, $\phi_f \equiv 1$ for f satisfying (⋇). Our proof of Theorem 3.1.1 indicates that for a general generating map, f, the above procedure will yield a measure on $\partial\Omega$ whose Radon-Nikodym derivative with respect to the Fefferman measure exists and is a continuous function of ϕ_f.

Well-known estimates on the Bergman kernel ([11]) yield a corollary to Theorem 3.1.1 that suggests a way of extending (3.1) to more general domains (see Section 3.7 for some elaboration).

Corollary 3.1.6. *Let Ω be a smooth strongly pseudoconvex domain and K_Ω denote its Bergman kernel function. Let $\mathcal{BP}_n$ be the collection of all analytic polyhedra in Ω of the form*

$$P = \left\{z \in \Omega : |K_\Omega(w^j, z)| < m_j, j = 1, ..., n\right\},$$

where, $w^1, ..., w^n \in \partial\Omega$ *and* $m_1, ..., m_n > 0$*. Then,*

$$\inf\{\mathrm{vol}(\Omega \setminus P) : P \in \mathcal{BP}_n\} \sim \frac{1}{2} l_{\mathrm{kor}} \left(\int_{\partial\Omega} \sigma_\Omega \right)^{\frac{3}{2}} \frac{1}{\sqrt{n}}, \tag{3.2}$$

as $n \to \infty$*.*

In the same vein, the expansion for the Szegő kernel (see [8]) gives the following result.

Corollary 3.1.7. *Corollary 3.1.6 holds when* K_Ω *is replaced by* S_Ω*, the Szegő kernel function of* Ω *with respect to any smooth multiple of the surface area measure.*

3.2 Preliminaries

In this chapter, we will always work in $\mathbb{C}^2$, and employ the following notation:

- $z = (z_1, z_2) = (x_1 + iy_1, x_2 + iy_2)$, $w = (w_1, w_2) = (u_1 + iv_1, u_2 + iv_2)$ for points;
- $\left\langle \cdot, \cdot \right\rangle$ for the complex pairing between a co-vector and a vector;
- "$\,'\,$" to indicate projection onto $\{y_2 = 0\} = \mathbb{C} \times \mathbb{R}$;
- A^{res} for $(A\big|_{\{y_2=0\}})' : \mathbb{C} \times \mathbb{R} \to \mathbb{C} \times \mathbb{R}$, where $A : \mathbb{C}^2 \to \mathbb{C}^2$,
- $\mathrm{vol}_3(B)$ for the Lebesgue measure of the set $B \subset \mathbb{C} \times \mathbb{R}$, and

We reintroduce the polyhedral objects of our study.

Definition 3.2.1. Let $\Omega \subset \mathbb{C}^2$ be a domain and $f \in \mathcal{C}(\overline{\Omega} \times \partial\Omega)$. Given a compact set $J \subset \partial\Omega$, an *f-polyhedron over J* is any set of the form

$$P = \{z \in \Omega : |f(w^j, z)| > \delta_j, j = 1, ..., n\}, \quad (w^j, \delta_j) \in \partial\Omega \times (0, \infty),$$

such that $J \subset \partial\Omega \setminus \overline{P}$ and for every $j \in \{1, ..., n\}$, $|f(w^j, z)| < \delta_j$ for some $z \in J$. If Ω is bounded, then an f-polyhedron over $\partial\Omega$ is simply called an *f-polyhedron*. We call

- each (w^j, δ_j) a *source-size pair* of P;
- each $C(w^j, \delta_j; f) := \{z \in \overline{\Omega} : |f(w^j, z)| \leq \delta_j\}$ a *cut* of P;
- each $F(w^j, \delta_j; f) := \{z \in \overline{\Omega} : |f(w^j, z)| = \delta_j, |f(w^l, z)| \geq \delta_l,\ l \neq j\}$ a *facet* of P;
- $(w^1, ..., w^n)$ and $(\delta_1, ..., \delta_n)$ the *source-tuple* and *size-tuple* of P, respectively.

We emphasize that, by definition, the cuts of an f-polyhedron over J cover J, and each of its cuts intersects J non-trivially.

Remarks. When there is no ambiguity in the choice of f, we drop any reference to it from our notation for cuts and facets. Repetitions are permitted when listing the sources of an f-polyhedron. Thus, P — as in Definition 3.2.1 — has at most n facets.

Notation. Let Ω, f, P and J be as in Definition 3.2.1 above.

- $\delta(P) := \max_{1\leq j\leq n}\{\delta_j : (\delta_1, ..., \delta_n)$ is the size-tuple of $P\}$.
- $\mathcal{P}_n(f) :=$ the collection of all f-polyhedra in Ω with at most n facets.
- $\mathcal{P}_n(J; f) :=$ the collection of all f-polyhedra over J with at most n facets.
- $\mathcal{P}_n(J \subset H; f) := \{P \in \mathcal{P}_n(J; f) : \partial\Omega \setminus \overline{P} \subset H\}$, where $H \subset \partial\Omega$ is a compact superset of J.
- $v(\Omega; \mathscr{P}) := \inf\{\text{vol}(\Omega \setminus P) : P \in \mathscr{P}\}$, for any sub-collection $\mathscr{P} \subset \mathcal{P}_n(J; f)$.

We now recall some standard concepts (see [16, Ch. 1]) in the theory of integral representation kernels in $\mathbb{C}^d$ (focusing on $d = 2$). For a bounded domain $\Omega \subset \mathbb{C}^2$, a $\mathcal{C}^1$-smooth function $g(z, w) = (g_1(z, w), g_2(z, w))$ on $\Omega \times \partial\Omega$ is called a *Leray map* for Ω if

$$\mathfrak{g}(z, w) := g_1(z, w)(z_1 - w_1) + g_2(z, w)(z_2 - w_2) \neq 0 \quad \text{for all } (z, w) \in \Omega \times \partial\Omega.$$

The *Cauchy-Fantappié form* generated by g is given by

$$\text{CF}(g)(z, w) = \frac{g_1(z, w) \wedge \partial_{\overline{w}} g_2(z, w) \wedge dw - g_2(z, w) \wedge \partial_{\overline{w}} g_1(z, w) \wedge dw}{\mathfrak{g}(z, w)^2},$$

where $dw = dw_1 \wedge dw_2$. Indulging in non-standard terminology, we call $\mathfrak{g}$ the *generating map* of $\text{CF}(g)$.

Cauchy Fantappié forms act as reproducing kernels: if Ω has piecewise $\mathcal{C}^1$-boundary, then

$$f(z) = \frac{1}{(2\pi i)^2} \int_{\partial\Omega} f(w) \wedge CF(g)(z, w), \quad z \in \Omega,$$

where $f \in \mathcal{C}(\overline{\Omega})$ is holomorphic in Ω. It has been of interest to construct Leray maps such that $CF(g)(z, w)$ is holomorphic in $z \in \Omega$. For strongly pseudoconvex domains, it is enough to directly construct a generating map that is holomorphic in z. Henkin and

Ramirez constructed such maps (see [25, §3] for details) for $\mathcal{C}^2$-smooth strongly pseudoconvex domains, based on

$$\mathfrak{p}(z,w) \;=\; \sum_{j=1}^{2}\frac{\partial\rho}{\partial z_j}(w)(z_j-w_j)+\frac{1}{2}\sum_{j,k=1}^{2}\frac{\partial^2\rho}{\partial z_j\partial z_k}(w)(z_j-w_j)(z_k-w_k),$$

where ρ is a defining function of Ω. $\mathfrak{p}$ is called the *Levi polynomial* of ρ. The corresponding Cauchy Fantappié kernels are called Henkin-Ramirez reproducing kernels. If Ω is a $\mathcal{C}^1$-smooth $\mathbb{C}$-linearly convex domain, i.e., the complement of Ω is a union of complex hyperplanes, a simpler holomorphic (in z) generating map is given by the *Cauchy-Leray map* of a defining function ρ:

$$\mathfrak{l}(z,w)=\sum_{j=1}^{2}\frac{\partial\rho}{\partial z_j}(w)(z_j-w_j).$$

3.3 Some Technical Lemmas

Here, we restrict our attention to Jordan measurable domains $\Omega\subset\mathbb{C}^2$. J and H are compact subsets of $\partial\Omega$ such that $J\subset\operatorname{int}_{\partial\Omega}H$. We will concern ourselves with f-polyhedra that lie 'above' J but are constrained by H. We first prove a lemma that will allow us to work locally.

Lemma 3.3.1. *Let Ω, J and H be as above. Suppose $f\in\mathcal{C}(\overline{\Omega}\times H)$ satisfies*

- (a) $\{z\in\overline{\Omega}: f(z,w)=0\}=\{w\}$*, for any $w\in H$,*
- (b) *For some $\delta_0>0$ and $c>0$, $C(w,\delta;f)\supseteq C(w,c\delta;g)$, for all $w\in H$ and $\delta<\delta_0$, where $g\in\mathcal{C}(\overline{\Omega}\times H)$ satisfies (a) and $C(w,\delta;g)$ is Jordan measurable for each $w\in H$ and $\delta<c\delta_0$.*

Then, for $P_m\in\mathcal{P}_m(J\subset H;f)$ such that $\lim_{m\to\infty}\operatorname{vol}(\Omega\backslash P_m)=0$, we have that $\lim_{m\to\infty}\delta(P_m)=0$.

Proof. It suffices to show that for each $\delta<\delta_0$, there is a $b>0$ such that $\operatorname{vol}(C(w,\delta;f))>b$ for all $w\in H$. By condition (b), it is enough to show this for the cuts of g. As g satisfies condition (a), $\operatorname{vol}\big(C(w,\delta;g)\big)>0$ for each $w\in H$. Therefore, if we can establish the continuity of $w\mapsto\operatorname{vol}\big(C(w,\delta;g)\big)$ on the compact set H, we will be done.

Fix a $\delta\in(0,c\delta_0)$. Let $\chi_w:=\chi_{C(w,\delta;g)}$, where χ_A denotes the indicator function of A. For a given $w\in H$, consider a sequence of points $\{w^n\}_{n\in\mathbb{N}}\subset H$ that converges to w as

$n \to \infty$. Then,

$$\lim_{n\to\infty} \chi_{w^n}(z) = \chi_w(z) \qquad \text{for a.e. } z \in \overline{\Omega}. \tag{3.3}$$

To see this, consider a $z \in \overline{\Omega}$ such that $\chi_w(z) = 0$. Suppose, there is a subsequence $\{w^{n_j}\}_{j\in\mathbb{N}} \subset \{w^n\}_{n\in\mathbb{N}}$ such that $\chi_{w^{n_j}}(z) = 1$. Then, $|g(w^{n_j}, z)| \le \delta$ but $\lim_{j\to\infty} |g(w^{n_j}, z)| = |g(w, z)| \ge \delta$. This is only possible if $g(w, z) = \delta$. An analogous argument holds if $\chi_w(z) = 1$. Thus, $z \in \partial C(w, \delta; g)$. Due to assumption (b), this is a null set. Thus, (3.3) is true and we invoke Lebesgue's dominated convergence theorem to conclude that

$$\text{vol}\big(C(w^n, \delta; g)\big) = \int_{\overline{\Omega}} \chi_{w^n} d\omega \xrightarrow{n\to\infty} \int_{\overline{\Omega}} \chi_w d\omega = \text{vol}\big(C(w, \delta; g)\big),$$

where $\delta < c\delta_0$ and ω is the Lebesgue measure on $\mathbb{C}^d$. □

Next, we prove a lemma that permits us to concentrate on a single representative of $\text{LP}(\Omega)$.

Lemma 3.3.2. *Let Ω, J and H be as above. Suppose $f, g \in \mathcal{C}(\overline{\Omega} \times H)$ are such that*

(i) $\{z \in \overline{\Omega} : f(z, w) = 0\} = \{z \in \overline{\Omega} : g(z, w) = 0\} = \{w\}$, *for any fixed $w \in H$, and*

(ii) *there exist constants $\varepsilon \in (0, 1/3)$ and $\tau > 0$, such that*

$$|f(z, w) - g(z, w)| \le \varepsilon(|g(z, w) + |f(z, w)||) \tag{3.4}$$

on $\{(z, w) \in \overline{\Omega} \times H : ||z - w|| \le \tau\}$.

Further, assume that the cuts of g are Jordan measurable and satisfy a doubling property as follows

⊛ *there is a $\delta_g > 0$ and a continuous $\mathcal{E} : [0, 16] \to \mathbb{R}$ so that, for any $m \in \mathbb{N}_+$, $(w^j, \delta_j) \in H \times (0, \delta_g),\ 1 \le j \le m$, and $t \in [0, 16]$,*

$$\text{vol}\left(\bigcup_{j=1}^m C(w^j, (1+t)\delta_j)\right) \le \mathcal{E}(t) \cdot \text{vol}\left(\bigcup_{j=1}^m C(w^j, \delta_j)\right).$$

Then, for every $\beta > 0$,

$$\limsup_{n\to\infty} n^\beta v_n(f) \quad \le \quad D_1(\varepsilon) \limsup_{n\to\infty} n^\beta v_n(g); \tag{3.5}$$

$$\liminf_{n\to\infty} n^\beta v_n(f) \quad \ge \quad D_2(\varepsilon)^{-1} \liminf_{n\to\infty} n^\beta v_n(g), \tag{3.6}$$

where $v_n(h) := v\left(\Omega; \mathcal{P}_n(J \subset H; h)\right)$*, and* D_1, D_2 *depend on* $\mathcal{E}$ *and satisfy* $\lim_{\varepsilon\to 0} D_j(\varepsilon) = \lim_{t\to 0} \mathcal{E}(t)$.

Proof. Observe that if $\hat{\varepsilon} := \frac{1+\varepsilon}{1-\varepsilon}$, then inequality (3.4) may be transcribed as

$$|f(z,w)| \leq \hat{\varepsilon}|g(z,w)| \text{ and } |g(z,w)| \leq \hat{\varepsilon}|f(z,w)| \tag{3.7}$$

on $\{(z,w) \in \overline{\Omega} \times H : ||z-w|| \leq \tau\}$. Hence, for any $w \in H$ and $\delta > 0$,

$$C(w,\delta;f) \subseteq \mathbb{B}_2(w;\tau) \quad \Rightarrow \quad C(w,\delta;f) \subseteq C\left(w,\hat{\varepsilon}\delta;g\right); \tag{3.8}$$

$$C(w,\delta;g) \subseteq \mathbb{B}_2(w;\tau) \quad \Rightarrow \quad C(w,\delta;g) \subseteq C\left(w,\hat{\varepsilon}\delta;f\right). \tag{3.9}$$

We first show that

$$\limsup_{n\to\infty} n^\beta v_n(f) \leq \mathcal{E}\left(\frac{(1+\varepsilon)^2}{(1-\varepsilon)^2} - 1\right) \limsup_{n\to\infty} n^\beta v_n(g). \tag{3.10}$$

Let $\xi > 1$. Assume that $L_{\text{sup}} := \limsup_{n\to\infty} n^\beta v_n(g)$, is finite. Then, there is an $n_\xi \in \mathbb{N}_+$ such that for each $n \geq n_\xi$, we can pick a $Q_n \in \mathcal{P}_n(J \subset H; g)$ satisfying

$$\text{vol}(\Omega \setminus Q_n) \leq \xi L_{\text{sup}} n^{-\beta}. \tag{3.11}$$

As the cuts of g are Jordan measurable, Lemma 3.3.1 implies that $\delta(Q_n) \to 0$ as $n \to \infty$. Consequently, n_ξ can be chosen so that (3.11) continues to hold, and for all source-size pairs (w,δ) of Q_n, $n \geq n_\xi$, we have that

(a) $\delta < \delta_g$ (see condition ⊛ on g);

(b) $C(w,\delta;g) \subset \mathbb{B}_2(w;\tau)$ and $C(w,4\delta;g) \cap \partial\Omega \subset H$; and

(c) $C(w,2\delta;f) \subset \mathbb{B}_2(w;\tau)$.

The second part of (b) is possible as each cut of Q_n is compelled to intersect J non-trivially, by definition. For a fixed source-size pair (w,δ) of Q_n, we have, due to (3.9) and (3.8),

$$C(w,\delta;g) \subseteq C(w,\hat{\varepsilon}\delta;f) \subseteq C\left(w,\hat{\varepsilon}^2\delta;g\right).$$

The second inclusion is valid as $\hat{\varepsilon}\delta \leq 2\delta$, thus permitting the use of (3.8), given (c).

We can now approximate Q_n by an f-polyhedron by setting

$$\widetilde{Q_n} := \left\{z \in \Omega : |g(z,w)| > \hat{\varepsilon}^2\delta, (w,\delta) \text{ is a source-size pair of } Q_n\right\};$$

$$P_n := \{z \in \Omega : |f(z,w)| > \hat{\varepsilon}\delta, (w,\delta) \text{ is a source-size pair of } Q_n\}.$$

Our assumptions imply that $\widetilde{Q_n}$ and P_n are in $\mathcal{P}_n(J \subset H; g)$ and $\mathcal{P}_n(J \subset H; f)$, respectively. From the above inclusions, we have that $\widetilde{Q_n} \subseteq P_n \subseteq Q_n$, $n \geq n_\xi$. Hence, by property ⊛ of g and (3.11), we see that

$$\begin{aligned} n^\beta v_n(f) \leq n^\beta \operatorname{vol}(\Omega \setminus P_n) &\leq n^\beta \operatorname{vol}\left(\Omega \setminus \widetilde{Q_n}\right) \\ &\leq \mathcal{E}\left(\hat{\varepsilon}^2 - 1\right) n^\beta \operatorname{vol}(\Omega \setminus Q_n) \\ &\leq \xi \mathcal{E}\left(\hat{\varepsilon}^2 - 1\right) L_{\sup}, \end{aligned}$$

for $n \geq n_\xi$. As $\xi > 0$ was arbitrary and $\hat{\varepsilon} = \frac{1+\varepsilon}{1-\varepsilon}$, (3.10) follows.

To complete this proof, we show that

$$\liminf_{n \to \infty} n^\beta v_n(f) \geq \mathcal{E}\left(\frac{(1+\varepsilon)^4}{(1-\varepsilon)^4} - 1\right)^{-1} \liminf_{n \to \infty} n^\beta v_n(g). \tag{3.12}$$

For this, fix a $\xi > 1$, and assume that $L_{\inf} := \liminf_{n \to \infty} n^\beta v_n(g)$, is finite. Thus, there is an $n_\xi \in \mathbb{N}_+$ such that

$$v_n(g) \geq \frac{1}{\xi} L_{\inf} n^{-\beta}; \text{ for } n \geq n_\xi. \tag{3.13}$$

For each n, we pick an $R_n \in \mathcal{P}_n(J \subset H; f)$ that satisfies

$$v(\Omega \setminus R_n) \leq \xi v_n(f). \tag{3.14}$$

Now, we may also assume that $\liminf_{n \to \infty} n^\beta v_n(f) < \infty$ (else, there is nothing to prove), thus obtaining that $v_n(f) \to 0$ for infinitely many $n \in \mathbb{N}_+$. But, as $v_n(f)$ is decreasing in n, we get that $v_n(f) \to 0$ for all $n \in \mathbb{N}_+$. Now, due to (3.9), it is possible to choose δ small enough so that

$$C\left(w, \frac{\delta}{\hat{\varepsilon}}; g\right) \subseteq C(w, \delta; f),$$

for each $w \in H$. As the cuts of g are Jordan measurable (there is no such assumption on the cuts of f), we invoke Lemma 3.3.1 to conclude that $\delta(R_n) \to 0$ as $n \to \infty$. As before, we find a new n_ξ such that (3.13) continues to hold, and for all $n \geq n_\xi$ and all source-size pairs (w, δ) of R_n, we have

(a') $\delta < \delta_g$ (see condition ⊛ on g);

(b') $C(w, 4\delta; f) \subset \mathbb{B}_2(w; \tau)$ and $C(w, 4\delta; f) \cap \partial\Omega \subset H$; and

(c') $C(w, 2\delta; g) \subset \mathbb{B}_2(w; \tau)$.

Then, as before

$$C\left(w, \frac{\delta}{\hat{\varepsilon}}; g\right) \subseteq C(w, \delta; f) \subseteq C\left(w, \hat{\varepsilon}\delta; g\right) \subseteq C\left(w, \hat{\varepsilon}^2\delta; f\right) \subseteq C\left(w, \hat{\varepsilon}^3\delta; g\right). \qquad (3.15)$$

We now approximate R_n with an n-faceted g-polyhedron, using

$$\begin{aligned} \widetilde{R_n} : &= \left\{z \in \Omega : |f(z,w)| > \hat{\varepsilon}^2\delta, (w,\delta) \text{ is a source-size pair of } R_n\right\}; \\ S_n : &= \left\{z \in \Omega : |g(z,w)| > \hat{\varepsilon}\delta, (w,\delta) \text{ is a source-size pair of } R_n\right\}. \end{aligned}$$

Our assumptions are designed to ensure that $\widetilde{R_n} \in \mathcal{P}_n(J \subset H; f)$ and $S_n \in \mathcal{P}_n(J \subset H; g)$. From the above inclusions, we have that

$$\widetilde{R_n} \subseteq S_n \subseteq R_n,\ n \geq n_\xi.$$

Moreover, the first and last inclusions in (3.15) and the assumption ⊛ on g (note that $\hat{\varepsilon}^4 < 16$) imply that

$$\begin{aligned} & \operatorname{vol}\left(\Omega \setminus \widetilde{R_n}\right) - \operatorname{vol}(\Omega \setminus R_n) \\ \leq\ & \operatorname{vol}\left(\bigcup_{(w,\delta)\in\Lambda_n} C\left(w, \hat{\varepsilon}^3\delta; g\right) - \bigcup_{(w,\delta)\in\Lambda_n} C\left(w, \frac{\delta}{\hat{\varepsilon}}; g\right)\right) \\ \leq\ & \mathcal{E}\left(\hat{\varepsilon}^4 - 1\right) \operatorname{vol}(\Omega \setminus R_n), \end{aligned} \qquad (3.16)$$

where Λ_n is the set of source-size pairs of R_n.

Therefore, using (3.16) and (3.14), we see that

$$\begin{aligned} \frac{1}{\xi} L_{\inf} n^{-\beta} \leq v_n(g) &\leq \operatorname{vol}(\Omega \setminus S_n) \leq \operatorname{vol}\left(\Omega \setminus \widetilde{R_n}\right) \\ &\leq \mathcal{E}\left(\hat{\varepsilon}^4 - 1\right) \operatorname{vol}(\Omega \setminus R_n) \\ &\leq \mathcal{E}\left(\hat{\varepsilon}^4 - 1\right) \xi v_n(f). \end{aligned}$$

Therefore,

$$n^\beta v_n(f) \geq \xi^{-2} \mathcal{E}\left(\hat{\varepsilon}^4 - 1\right)^{-1} L_{\inf}, \quad n \geq n_\xi.$$

As $\xi > 0$ was arbitrary and $\hat{\varepsilon} = \frac{1+\varepsilon}{1-\varepsilon}$, (3.12) follows. □

Remark 3.3.3. In practice, f and g may only be defined on $(\overline{\Omega} \cap U) \times H$ for some open set $U \subset \mathbb{C}^2$ containing a τ-neighborhood of H, while satisfying the analogous version of condition (i) there. As the remaining hypothesis (and indeed the result itself) depends only on the values of f and g on an arbitrarily thin tubular neighborhood of H in $\overline{\Omega}$, we may replace f (and, similarly, g) by f_{e} to invoke Lemma 3.3.2, where

$$f_{\mathsf{e}} := f(z,w)\varsigma(||z-w||^2) + ||z-w||^2(1-\varsigma(||z-w||^2))$$

for some non-negative $\varsigma \in \mathcal{C}^\infty(\mathbb{R})$ such that $\varsigma(x) = 1$ when $x \leq \tau^2/2$ and $\varsigma(x) = 0$ when $x \geq \tau^2$. We will do so without comment, when necessary.

3.4 Approximating Model Domains

As a first step, we examine volume approximations of the Siegel domain by a particular class of analytic polyhedra. This problem enjoys a connection with Laguerre-type tilings of the Heisenberg surface equipped with the Korányi metric (see the appendix for further details).

Let $\mathcal{S} := \{(z_1, x_2 + iy_2) \in \mathbb{C}^2 : y_2 > |z_1|^2\}$ and $f_{\mathcal{S}}(z,w) = z_2 - \overline{w_2} - 2iz_1\overline{w_1}$. We view $\mathbb{C} \times \mathbb{R}$ as the first Heisenberg group, $\mathbb{H}$, with group law

$$(z_1, x_2) \cdot_{\mathbb{H}} (w_1, u_2) = (z_1 + w_1, x_2 + u_2 + 2\,\mathrm{Im}(z_1\overline{w}_1)) \tag{3.17}$$

and the left-invariant Korányi gauge metric (see [9, Sec. 2.2])

$$d_{\mathbb{H}}((z_1, x_2), (w_1, u_2)) := ||(w_1, u_2)^{-1} \cdot_{\mathbb{H}} (z_1, x_2)||_{\mathbb{H}}, \tag{3.18}$$

where $||(z_1, x_2)||_{\mathbb{H}}^4 := |z_1|^4 + x_2^2$. Observe that, for any cut $C(w,\delta) = C(w,\delta; f_{\mathcal{S}})$, $w \in \partial\mathcal{S}$, $C(w,\delta)'$ is the set

$$K(w', \sqrt{\delta}) = \{(z_1, x_2) \in \mathbb{C} \times \mathbb{R} : |z_1 - w_1|^4 + (x_2 - u_2 + 2\,\mathrm{Im}(z_1\overline{w_1}))^2 \leq \delta^2\}, \tag{3.19}$$

which is the ball of radius $\sqrt{\delta}$ centered at w', in the Korányi metric.

Notation. We will use the following notation in this section:

- $I^r := \{(x_1 + iy_1, x_2) \in \mathbb{C} \times \mathbb{R} : 0 \leq x_1 \leq r,\ 0 \leq y_1 \leq r,\ 0 \leq x_2 \leq r^2\}$, $r > 0$.
- $\hat{I}^r := I^{2r} - \left(\frac{r}{2} + i\frac{r}{2}, \frac{3r^2}{2}\right)$, $r > 0$. $I^r \subset \hat{I}^r$ and they are concentric.
- $v_n(J \subset H) := v(\mathcal{S}; \mathcal{P}_n(J \subset H; f_{\mathcal{S}}))$, for $J \subset H \subset \partial\mathcal{S}$. If $J \subset H \subset \mathbb{C} \times \mathbb{R}$, $v_n(J \subset H)$ is meaningful in view of the obvious correspondence between $\mathbb{C} \times \mathbb{R}$ and $\partial\mathcal{S}$.

Lemma 3.4.1. *Let $I = I^1$ and $\hat{I} = \hat{I}^1$. There exists a positive constant $l_{\text{kor}} > 0$ such that*

$$v_n(I \subset \hat{I}) \sim \frac{l_{\text{kor}}}{\sqrt{n}}$$

as $n \to \infty$.

Proof. Simple calculations show that

$$\text{vol}(C(w, \delta)) = \frac{2\pi}{3}\delta^3 \tag{3.20}$$

$$\text{vol}(K(w', \sqrt{\delta})) = \frac{\pi^2}{2}\delta^2 \tag{3.21}$$

for all $w \in \partial\mathcal{S}$ and $\delta > 0$.

We utilize a special tiling in $\mathbb{C} \times \mathbb{R}$. Let $k \in \mathbb{N}_+$ and consider the following points in $\mathbb{C} \times \mathbb{R}$:

$$v_{pqr} := \left(\frac{p}{k} + i\frac{q}{k}, \frac{r}{k^2}\right), \ (p, q, r) \in \Sigma_k,$$

where $\Sigma_k := \{(p, q, r) \in \mathbb{Z}^3 : -2q \leq r \leq k^2 - 1 + 2p, \ 0 \leq p, q \leq k - 1\}$. Observe that $\text{card}(\Sigma_k) = k^4 + 2k^3 - 2k^2$. Now, we set $E_{pqr} := v_{pqr} \cdot_{\mathbb{H}} I^{\frac{1}{k}}$, and note that $I \subset \cup_{\Sigma_k} E_{pqr} \subset \hat{I}$, for all $k \in \mathbb{N}_+$.

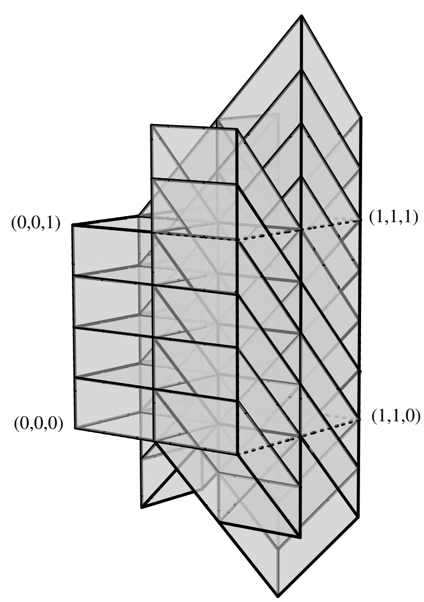

Figure 3.1: The 24 tiles E_{pqr} when $k = 2$.

1. We first show that there is a constant $\alpha_1 > 0$ such that

$$v_n(I \subset \hat{I}) \le \frac{\alpha_1}{\sqrt{n}} \tag{3.22}$$

for all $n \in \mathbb{N}_+$.

For this, let

$$u_{pqr} := \text{center of } E_{pqr} = v_{pqr} \cdot_{\mathbb{H}} \left(\frac{1}{2k} + i\frac{1}{2k}, \frac{1}{2k^2} \right), \ (p,q,r) \in \Sigma_k, \ k \in \mathbb{N}_+.$$

Then, the Korányi ball $K\left(u_{pqr}, \frac{\sqrt[4]{5}}{\sqrt[4]{2}k}\right)$ (see (3.19)) contains E_{pqr} and is contained in $\hat{I}$. Hence, if $w_{pqr} \in \partial\mathcal{S}$ is such that $w'_{pqr} = u_{pqr}$, the cuts

$$C\left(w_{pqr}, \frac{\sqrt{5}}{\sqrt{2}k^2}; f_{\mathcal{S}}\right), \ (p,q,r) \in \Sigma_k,$$

define P_k, an $f_{\mathcal{S}}$-polyhedron over I with $k^4 + 2k^3 - 2k^2$ facets. In fact, $P_k \in \mathcal{P}_{k^4+2k^3-2k^2}(I \subset \hat{I}; f_{\mathcal{S}})$, for all $k \in \mathbb{N}_+$. Therefore, using (3.20)

$$\begin{aligned} v_{k^4+2k^3-2k^2}(I \subset \hat{I}) &\le \operatorname{vol}(\mathcal{S} \setminus P_k) \\ &\le \operatorname{vol}\left(\bigcup_{\Sigma_k} C\left(w_{pqr}, \frac{\sqrt{5}}{\sqrt{2}k^2}\right)\right) \\ &\le \frac{2\pi}{3}\left(\frac{\sqrt{5}}{\sqrt{2}k^2}\right)^3 (k^4+2k^3-2k^2) = \frac{5\sqrt{5}\pi}{3\sqrt{2}} \frac{(k^4+2k^3-2k^2)}{k^6}, \end{aligned}$$

$k \in \mathbb{N}_+$. Now, for a given $n \in \mathbb{N}_+$, choose k such that $k^4 + 2k^3 - 2k^2 \le n \le (k+1)^4 + 2(k+1)^3 - 2(k+1)^2$. Then, one can easily find a $\alpha_1 > 0$ such that

$$\begin{aligned} v_n(I \subset \hat{I})\sqrt{n} &\le v_{k^4+2k^3-2k^2}(I \subset \hat{I})\sqrt{(k+1)^4 + 2(k+1)^3 - 2(k+1)^2} \\ &\le \frac{5\sqrt{5}\pi}{3\sqrt{2}} \frac{(k^4+2k^3-2k^2)\sqrt{(k+1)^4 + 2(k+1)^3 - 2(k+1)^2}}{k^6} \\ &\le \alpha_1. \end{aligned}$$

2. Next, we show that there is an $\alpha_2 > 0$ such that

$$v_n(I \subset \hat{I}) \ge \frac{\alpha_2}{\sqrt{n}} \tag{3.23}$$

for $n \in \mathbb{N}_+$.

If finitely many Korányi balls of radii $\sqrt{\rho_1}, ..., \sqrt{\rho_k}$ cover I, then (3.21) yields

$$(\sqrt{\rho}_1)^4 + \cdots + (\sqrt{\rho}_k)^4 \geq \frac{2}{\pi^2} \operatorname{vol}_3(I) = \frac{2}{\pi^2}. \tag{3.24}$$

We will also need the following mean inequality (a consequence of Jensen's inequality)

$$\left(\frac{\rho_1^{d+1} + \cdots + \rho_k^{d+1}}{k}\right)^{\frac{1}{d+1}} \geq \left(\frac{\rho_1^{d-1} + \cdots + \rho_k^{d-1}}{k}\right)^{\frac{1}{d-1}}, \tag{3.25}$$

for positive ρ_j, $1 \leq j \leq k$, and $d > 1$.

Now, fix a positive $\xi < 1$. Let $P_n \in \mathcal{P}_n(I \subset \hat{I}; f_{\mathcal{S}})$ be such that

$$\operatorname{vol}(\mathcal{S} \setminus P_n) \leq \frac{1}{\xi} v_n(I \subset \hat{I}). \tag{3.26}$$

Let $C_j(n)$ and $K_j(n)$, $j = 1, ..., n$, be the cuts and their projections, respectively, of P_n. Now, $\mathcal{K}_n := \{K_j(n), j = 1, ..., n\}$ is a finite covering of I, so by the Wiener covering lemma (see [22, Lemma 4.1.1] for a proof that generalizes to metric spaces), we can find disjoint Korányi balls $K_1, ..., K_k \in \mathcal{K}_n$, of radii $\sqrt{\rho_1}, ..., \sqrt{\rho_k}$, such that $\cup_{K \in \mathcal{K}_n} K \subset \cup_{1 \leq j \leq k} 3K_j$, where, for $j = 1, ..., k$, $3K_j$ has the same centre as K_j but thrice its radius. Let C_j denote the cut that projects to K_j, $j = 1, ..., k$. It follows from (3.26), (3.20) and the inequalities (3.25) (for $d = 5$) and (3.24) that

$$\begin{aligned}
v_n(I \subset \hat{I})\sqrt{n} &\geq \xi \operatorname{vol}\left(\bigcup_{j=1}^{k} C_j\right)\sqrt{k} \\
&= \xi \left(\sum_{i=1}^{k} \operatorname{vol}(C_j)\right)\sqrt{k} = \xi \frac{2\pi}{3}\left(\rho_1^3 + \cdots + \rho_k^3\right)\sqrt{k} \\
&= \xi \frac{2\pi}{3^7}\left((9\rho_1)^3 + \cdots + (9\rho_k)^3\right)\sqrt{k} \\
&= \xi \frac{2\pi}{3^7}\left((3\sqrt{\rho_1})^6 + \cdots + (3\sqrt{\rho_k})^6\right)k^{\frac{2}{4}} \\
&\geq \xi \frac{2\pi}{3^7}\left((3\sqrt{\rho_1})^4 + \cdots + (3\sqrt{\rho_k})^4\right)^{\frac{6}{4}} \\
&\geq \xi \frac{4\sqrt{2}}{\pi^2 3^7} \operatorname{vol}_3(I)^{\frac{3}{2}} = \xi \frac{4\sqrt{2}}{\pi^2 3^7} > 0, \text{ for } n = n_0, n_0 + 1, ...
\end{aligned}$$

As $\xi < 1$ was arbitrary, we have proved (3.23).

3. Define

$$l_{\text{kor}} = \liminf_{n \to \infty} v_n(I \subset \hat{I})\sqrt{n}.$$

By (3.23) and (3.22), $0 < l_{\mathrm{kor}} < \infty$. We now show that

$$l_{\mathrm{kor}} = \lim_{n\to\infty} v_n(I \subset \hat{I})\sqrt{n}. \tag{3.27}$$

For this, it suffices to show that for every $\xi > 1$, if $n_0 \in \mathbb{N}_+$ is chosen so that

$$v_{n_0}(I \subset \hat{I})\sqrt{n_0} \leq \xi l_{\mathrm{kor}} \tag{3.28}$$

then,

$$v_n(I \subset \hat{I})\sqrt{n} \leq \xi^4 l_{\mathrm{kor}} \tag{3.29}$$

for n sufficiently large.

Now, let $P_{n_0} \in \mathcal{P}_{n_0}(I \subset \hat{I}; f_{\mathcal{S}})$ be such that

$$\mathrm{vol}(\mathcal{S} \setminus P_{n_0}) \leq \xi v_{n_0}(I \subset \hat{I}).$$

For any $w \in \partial\mathcal{S}$ and $k \in \mathbb{N}_+$, let $A_{w,k} : \mathbb{C}^2 \to \mathbb{C}^2$ be the biholomorphism

$$(z_1, z_2) \mapsto \left(w_1 + \frac{1}{k}z_1, w_2 + \frac{1}{k^2}z_2 - \frac{2i}{k}z_1\overline{w_1}\right).$$

Then, $A_{w,k}$ has the following properties:

- $A^{\mathrm{res}}_{w,k}(z') = w' \cdot_{\mathbb{H}} (\frac{1}{k}z_1, \frac{1}{k^2}x_2)$;
- $A_{w,k}(\mathcal{S}) = \mathcal{S}$;
- $A_{w,k}(P_{n_0}) \in \mathcal{P}_{n_0}(w' \cdot_{\mathbb{H}} I^{\frac{1}{k}} \subset w' \cdot_{\mathbb{H}} \hat{I}^{\frac{1}{k}}; f_{\mathcal{S}})$; and
- $\mathrm{vol}(\mathcal{S} \setminus A_{w,k}(P_{n_0})) \leq \xi \frac{v_{n_0}(I\subset\hat{I})}{k^6}$.

As a consequence,

$$P := \bigcup_{\Sigma_k} A_{v_{pqr},k}(P_{n_0})$$

satisfies the following conditions:

- $P \in \mathcal{P}_{n_0(k^4+2k^3-2k^2)}(I \subset \hat{I}; f_{\mathcal{S}})$
- $\mathrm{vol}(\mathcal{S} \setminus P) \leq \xi v_{n_0}(I \subset \hat{I})\frac{k^4+2k^3-2k^2}{k^6}$.

Hence, by assumption (3.28),

$$\begin{aligned} v_{n_0(k^4+2k^3-2k^2)}(I \subset \hat{I})\sqrt{n_0(k^4+2k^3-2k^2)} &\leq \xi v_{n_0}(I \subset \hat{I})\sqrt{n_0}\frac{(k^4+2k^3-2k^2)^{\frac{3}{2}}}{k^6} \\ &\leq \xi^2 v_{n_0}(I \subset \hat{I})\sqrt{n_0} \leq \xi^3 l_{\mathrm{kor}}, \end{aligned} \tag{3.30}$$

for sufficiently large k. Choose k_0 so that (3.30) holds and $\frac{(k+1)^4+2(k+1)^3-2(k+1)^2}{k^4+2k^3-2k^2} \leq \xi^2$ for $k > k_0$. For $n \geq n_0(k_0^4 + 2k_0^3 - 2k_0^2)$, let k be such that $n_0(k^4 + 2k^3 - 2k^2) \leq n \leq n_0((k+1)^4 + 2(k+1)^3 - 2(k+1)^2)$. Consequently,

$$\begin{aligned} v_n(I \subset \hat{I})\sqrt{n} &\leq v_{n_0(k^4+2k^3-2k^2)}(I \subset \hat{I})\sqrt{n_0((k+1)^4 + 2(k+1)^3 - 2(k+1)^2)} \\ &\leq \xi^3 l_{\mathrm{kor}} \sqrt{\frac{(k+1)^4 + 2(k+1)^3 - 2(k+1)^2}{k^4 + 2k^3 - 2k^2}} \leq \xi^4 l_{\mathrm{kor}}, \end{aligned}$$

by (3.30). We have proved (3.29) and, therefore, our claim (3.27). □

Our choice of the unit square in the above lemma facilitates the computation for polyhedra lying above more general Jordan measurable sets in the boundary of $\mathcal{S}$.

Lemma 3.4.2. *Let $J, H \subset \partial\mathcal{S}$ be compact and Jordan measurable with $J \subset \mathrm{int}_{\partial\mathcal{S}} H$. Then*

$$v_n(J \subset H) \sim \mathrm{vol}_3(J')^{\frac{3}{2}} l_{\mathrm{kor}} \frac{1}{\sqrt{n}}$$

as $n \to \infty$.

Proof. **1.** We first show that

$$\limsup_{n\to\infty} v_n(J \subset H)\sqrt{n} \leq l_{\mathrm{kor}} \,\mathrm{vol}_3(J')^{\frac{3}{2}}. \tag{3.31}$$

Let $\xi > 1$ be fixed. As J is Jordan measurable, we can find m points $v^1, ..., v^m \in \mathbb{C}\times\mathbb{R}$ and some $r > 0$, such that

$$J' \subset \bigcup_1^m \left(v^j \cdot_{\mathbb{H}} I^r\right) \subset \bigcup_1^m \left(v^j \cdot_{\mathbb{H}} \hat{I}^r\right) \subset H \tag{3.32}$$

and

$$m \,\mathrm{vol}_3(I^r) \leq \xi \,\mathrm{vol}_3(J'). \tag{3.33}$$

Now, observe that

$$\sqrt{k}\, \frac{v_k(v^j \cdot_{\mathbb{H}} I^r \subset v^j \cdot_{\mathbb{H}} \hat{I}^r)}{\mathrm{vol}_3(I^r)^{\frac{3}{2}}} = \sqrt{k}\, \frac{v_k(I^r \subset \hat{I}^r)}{\mathrm{vol}_3(I^r)^{\frac{3}{2}}} = \sqrt{k}\, v_k(I \subset \hat{I}). \tag{3.34}$$

Thus, due to (3.32), Lemma 3.4.1, (3.34) and (3.33), we have

$$\begin{aligned} v_{km}(J \subset H)\sqrt{km} &\leq \sum_{j=1}^{m} v_k(v^j \cdot_{\mathbb{H}} I^r \subset v^j \cdot_{\mathbb{H}} \hat{I}^r)\sqrt{k}\sqrt{m} \\ &\leq \xi l_{\mathrm{kor}} \operatorname{vol}_3(I^r)^{\frac{3}{2}} m^{\frac{3}{2}} \\ &\leq \xi^{\frac{5}{2}} l_{\mathrm{kor}} \operatorname{vol}_3(J')^{\frac{3}{2}} \end{aligned} \tag{3.35}$$

for k sufficiently large. Choose $k_0 \in \mathbb{N}_+$ such that for $k \geq k_0$, (3.35) holds and $\sqrt{(k+1)/k} \leq \xi$. For sufficiently large n, we can find a $k \geq k_0$ such that $mk \leq n \leq m(k+1)$. Hence,

$$\begin{aligned} v_n(J \subset H)\sqrt{n} &\leq v_{km}(J \subset H)\sqrt{(k+1)m} \\ &\leq \xi^{\frac{5}{2}} l_{\mathrm{kor}} \operatorname{vol}_3(J')^{\frac{3}{2}} \sqrt{\frac{k+1}{k}} \\ &\leq \xi^{\frac{7}{2}} l_{\mathrm{kor}} \operatorname{vol}_3(J')^{\frac{3}{2}}. \end{aligned}$$

As $\xi > 1$ was arbitrarily fixed, we have proved (3.31).

2. It remains to show that

$$\liminf_{n\to\infty} v_n(J \subset H)\sqrt{n} \geq l_{\mathrm{kor}} \operatorname{vol}_3(J')^{\frac{3}{2}}. \tag{3.36}$$

Once again, fix a $\xi > 1$. The Jordan measurability of J ensures that there are pairwise disjoint sets $I_1, ..., I_m$, where $I_j = v^j \cdot_{\mathbb{H}} I^{r_j}$ for some $r_j > 0$ and $v^j \in \mathbb{C} \times \mathbb{R}$, $1 \leq j \leq m$, such that

$$\bigcup_{1}^{m} I_j \subset J' \text{ and } \bigcup_{1}^{m} \hat{I}_j \subset J', \tag{3.37}$$

where $\hat{I}_j = v^j \cdot_{\mathbb{H}} \hat{I}^{r_j}$, and

$$\operatorname{vol}_3(J') \leq \xi \sum_{j=1}^{m} \operatorname{vol}_3(I_j). \tag{3.38}$$

Choose a $P_n \in \mathcal{P}_n(J \subset H; f_{\mathcal{S}})$ such that $v(\mathcal{S} \setminus P_n) \leq \xi v_n(J \subset H)$ and let n_j denote the number of cuts of P_n whose projections intersect I_j and are contained in $\hat{I}_j$. By part **1.**, $v_n(J \subset H) \to 0$ as $n \to \infty$. Thus, recalling (3.20), $\delta(P_n) \to 0$ as $n \to \infty$. So, we may choose n so large that the projections of these n_j cuts, in fact, cover I_j and no two cuts of P whose projections intersect two different I_j's intersect. Therefore,

$$n_1 + \cdots + n_m \leq n. \tag{3.39}$$

By Lemma 3.4.1 and (3.34), there is an $n_0 \in \mathbb{N}_+$ such that

$$v_k(I_j \subset \hat{I}_j) \geq \frac{1}{\xi} l_{\text{kor}} \operatorname{vol}_3(I_j)^{\frac{3}{2}} \frac{1}{\sqrt{k}} \tag{3.40}$$

for $k \geq n_0$ and $j = 1, ..., m$. We may further increase n to ensure that

$$n_j \geq n_0 \text{ for } j = 1, ..., m.$$

Consequently, by (3.37) and (3.40), we have,

$$v_n(J \subset H) \geq \frac{1}{\xi} \sum_{j=1}^{m} v_{n_j}(I_j \subset \hat{I}_j) \geq \frac{l_{\text{kor}}}{\xi^2} \sum_{j=1}^{m} \frac{\operatorname{vol}_3(I_j)^{\frac{3}{2}}}{\sqrt{n_j}}.$$

Now, Hölder's inequality yields,

$$\sum_{j=1}^{m} \operatorname{vol}_3(I_j) = \sum_{j=1}^{m} \left(\frac{\operatorname{vol}_3(I_j)}{n_j^{1/3}} \right) n_j^{1/3} \leq \left(\sum_{j=1}^{m} \frac{\operatorname{vol}_3(I_j)^{3/2}}{n_j^{1/2}} \right)^{\frac{2}{3}} \left(\sum_{j=1}^{m} n_j \right)^{\frac{1}{3}}.$$

Using this, (3.38) and (3.39), we obtain

$$v_n(J \subset H) \geq \frac{l_{\text{kor}}}{\xi^2} \left(\sum_{j=1}^{m} \operatorname{vol}_3(I_j) \right)^{\frac{3}{2}} \left(\frac{1}{\sum_1^m n_j} \right)^{\frac{1}{2}} \geq \frac{l_{\text{kor}}}{\xi^{7/2}} \operatorname{vol}_3(J')^{\frac{3}{2}} \frac{1}{\sqrt{n}}$$

for n sufficiently large. As the choice of $\xi > 1$ was arbitrary, (3.36) now stands proved. □

As a final remark, we extend the above lemma to a class of slightly more general model domains in order to illustrate the effect of the Levi-determinant on our asymptotic formula.

Corollary 3.4.3. *Let $\mathcal{S}_\lambda := \{(z_1, x_2 + iy_2) \in \mathbb{C}^2 : y_2 > \lambda |z_1|^2\}$ and $f_{\mathcal{S}_\lambda}(z, w) = \lambda(z_2 - \overline{w_2}) - 2i\lambda^2(z_1\overline{w_1})$. Let $J, H \subset \partial\mathcal{S}_\lambda$ be compact and Jordan measurable with $J \subset \operatorname{int}_{\partial\mathcal{S}_\lambda} H$. Then*

$$v_n(\mathcal{S}_\lambda; J \subset H) := v(\mathcal{S}; \mathcal{P}_n(J \subset H; f_{\mathcal{S}_\lambda})) \sim \lambda^{\frac{1}{2}} \operatorname{vol}_3(J')^{\frac{3}{2}} l_{\text{kor}} \frac{1}{\sqrt{n}}$$

as $n \to \infty$.

Proof. Let $\Xi : \mathbb{C}^2 \to \mathbb{C}^2$ be the biholomorphism $\Xi : (z_1, z_2) \mapsto (\lambda z_1, \lambda z_2)$. Then, $\mathcal{S} = \Xi(\mathcal{S}_\lambda)$ and $f_{\mathcal{S}_\lambda}(z, w) = f_{\mathcal{S}}(\Xi(z), \Xi(w))$. Therefore, there is a bijective correspondence between $\mathcal{P}_n(J \subset H; f_{\mathcal{S}_\lambda})$ and $\mathcal{P}_n(\Xi J \subset \Xi H; f_{\mathcal{S}})$ given by $P \mapsto \Xi P$. Now, as $\det(\mathrm{J}_{\mathbb{R}}\Xi) \equiv$

λ^4 and $\det(\mathrm{J}_{\mathbb{R}}\Xi^{\mathrm{res}}) \equiv \lambda^3$, we have

$$\frac{v_n(\mathcal{S}_\lambda; J \subset H)}{\mathrm{vol}_3(J')^{\frac{3}{2}}} = \frac{\lambda^{-4} v_n(\mathcal{S}; \Xi J \subset \Xi H)}{\lambda^{-\frac{9}{2}} \mathrm{vol}_3(\Xi^{\mathrm{res}} J')^{\frac{3}{2}}} \sim \lambda^{\frac{1}{2}} l_{\mathrm{kor}} \frac{1}{\sqrt{n}}.$$

□

3.5 Local Estimates Via Model Domains

Lemma 3.3.2 suggests a way to locally compare the volume-minimizing approximations drawn from two different classes of f-polyhedra which exhibit some comparability. In this section, we set up a local correspondence between Ω and a model domain $\mathcal{S}_\lambda$, pull back the special cuts given by $f_{\mathcal{S}_\lambda}$ (see Section 3.4) via this correspondence, and establish a (3.4)-type relationship between the pulled-back cuts and those coming from the Levi polynomial of a defining function of Ω. First, we note a useful estimate on the Levi polynomial.

Lemma 3.5.1. *Let Ω be a $\mathcal{C}^2$-smooth strongly pseudoconvex domain. Suppose $\rho \in \mathcal{C}^2(\mathbb{C}^2)$ is a strictly plurisubharmonic defining function of Ω. Then, there exist constants $\mathscr{C} > 0$ and $\tau > 0$ such that*

$$|z-w|^2 \leq \mathscr{C} |\mathfrak{p}(z,w)|, \tag{3.41}$$

on Ω_τ, where $\mathfrak{p}$ is the Levi polynomial of ρ.

Proof. The second-order Taylor expansion of ρ about $w \in \partial\Omega$ gives:

$$-2\,\mathrm{Re}\,\mathfrak{p}(z,w) = -\rho(z) + \sum_{j,k=1}^{2} \frac{\partial^2 \rho(w)}{\partial z_j \partial \overline{z_k}} (z_j - w_j)(\overline{z_k} - \overline{w_k}) + o(|z-w|^2),$$

The strict plurisubharmonicity of ρ implies the existence of a $c > 0$ so that

$$\sum_{j,k=1}^{2} \frac{\partial^2 \rho(w)}{\partial z_j \partial \overline{z_k}} (z_j - w_j)(\overline{z_k} - \overline{w_k}) \geq c|z-w|^2, \quad (z,w) \in \overline{\Omega} \times \overline{\Omega}.$$

The result follows quite easily from this. □

3.5.1 Special Darboux Coordinates

Notation. As we are now going to construct a non-holomorphic transformation, we need to alternate between the real and complex notation. Here are some clarifications.

- We will use z (and similarly w) to denote both $(z_1, z_2) = (x_1 + iy_1, x_2 + iy_2) \in \mathbb{C}^2$ and $(x_1, y_1, x_2, y_2) \in \mathbb{R}^4$. The usage will be clear from the context. In the same vein, by z' we mean either $(z_1, x_2) = (x_1 + iy_1, x_2) \in \mathbb{C} \times \mathbb{R}$ or $(x_1, y_1, x_2) \in \mathbb{R}^3$.

- Recall that $\left\langle \theta, z \right\rangle$ denotes the pairing between a complex covector and a complex vector. When θ is a real covector, we write $\left\langle\left\langle \theta, z \right\rangle\right\rangle$ to stress that z, here, is a tuple in $\mathbb{R}^4$.

Fix a $\lambda > 0$. For reasons that will become clear in the next section, we consider a special $\mathcal{C}^4$-smooth strongly pseudoconvex domain Ω such that $0 \in \partial\Omega$ and for a neighborhood U of the origin, there is a convex function function $\rho : U \to \mathbb{R}$ such that $\Omega \cap U = \{z \in U : \rho(z) < 0\}$ and

$$\rho(z) = -\operatorname{Im} z_2 + \lambda|z_1|^2 + 2\operatorname{Re}(\mu z_1 \overline{z_2}) + \upsilon|z_2|^2 + o(|z|^2). \tag{3.42}$$

We may shrink U to find a convex function $F := F_\rho : U' \to \mathbb{R}$ that satisfies $\rho(z_1, x_2, F(z_1, x_2)) = 0$. ρ and F_ρ are both $\mathcal{C}^4$-smooth and $-i(\partial\rho - \overline{\partial}\rho)$ is a $\mathcal{C}^3$-smooth contact form on $\partial\Omega \cap U$. The domain $\mathcal{S}_\lambda$ from Section 3.4 is such a domain with $\rho^\lambda(z) = -\operatorname{Im} z_2 + \lambda|z_1|^2$ and $F_{\rho^\lambda}(z_1, x_2) = \lambda|z_1|^2$.

Darboux's theorem in contact geometry (see [1, Appendix 4]) says that any two equidimensional contact structures are locally contactomorphic. We seek local diffeomorphisms between Ω and $\mathcal{S}_\lambda$ that extend to local contactomorphisms between $(\partial\Omega, -i(\partial\rho - \overline{\partial}\rho))$ and $(\partial\mathcal{S}_\lambda, -i(\partial\rho^\lambda - \overline{\partial}\rho^\lambda))$, and satisfy estimates essential to our goal. We carry out this construction over the next three lemmas, working intially on $\mathbb{R}^3$ instead of $\partial\Omega$. For this, if $\mathsf{gr}_\rho : U' \to U$ maps (x_1, y_1, x_2) to $(x_1, y_1, x_2, F_\rho(x_1, y_1, x_2))$, we set

$$\begin{aligned}
\theta_\rho &:= (\mathsf{gr}_\rho)^* \left(\frac{\partial\rho - \overline{\partial}\rho}{i}\right) \\
&= \frac{-1}{\rho_{y_2}}\Big((\rho_{y_2}\rho_{y_1} + \rho_{x_1}\rho_{x_2})dx_1 - (\rho_{y_2}\rho_{x_1} - \rho_{y_1}\rho_{x_2})dy_1 + (\rho_{y_2}^2 + \rho_{x_2}^2)dx_2\Big),
\end{aligned}$$

where, by the partial derivatives of ρ we mean their pull-backs to U' via gr_ρ.

Lemma 3.5.2. *Let Ω be defined by* (3.42). *There is an open subset $(0 \in)V \subset U' \subset \mathbb{R}^3$ and a $\mathcal{C}^2$-smooth diffeomorphism $\mathfrak{d} = (\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3) : V \to \mathbb{R}^3$ with $\mathfrak{d}(0) = 0$ satisfying*

- $\mathfrak{d}^*\theta_{\rho^\lambda}(z') = \mathfrak{a}(z')\theta_\rho(z')$ *for all $z' \in V$, and some $\mathfrak{a} \in \mathcal{C}(V)$ with $\mathfrak{a}(0) = 1$; and*

- $|\det \mathrm{J}_\mathbb{R}\mathfrak{d}(0)| = 1$.

Proof. We proceed with the understanding that when refering to functions defined a priori on U (such as ρ or its derivatives) we implicitly mean their pull-backs to U' via gr_ρ.

Now, consider the following $\mathcal{C}^3$-smooth vector field in $\ker\theta_\rho$ on U':

$$v = \frac{\partial\rho}{\partial x_2}\frac{\partial}{\partial x_1} - \frac{\partial\rho}{\partial y_2}\frac{\partial}{\partial y_1} - \frac{\partial\rho}{\partial x_1}\frac{\partial}{\partial x_2}.$$

We let $\gamma^t(z') := \gamma(z';t) = (\gamma_1(z';t), \gamma_2(z';t), \gamma_3(z';t))$ be the flow of v such that $\gamma(z';0) = z'$. Note that $\gamma(z';t)$ is $\mathcal{C}^3$-smooth in z' and $\mathcal{C}^4$-smooth in s. Differentiating the initial value problem for the flow, we have

$$\mathrm{J}_{\mathbb{R}}\gamma^0 \equiv \mathrm{Id}\,. \text{ and } \mathrm{Hess}_{\mathbb{R}}\gamma^0 \equiv 0. \tag{3.43}$$

Observe that the map

$$\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3) : z' = (x_1, y_1, x_2) \mapsto \gamma(x_1, 0, x_2; y_1),$$

is defined on some neighborhood, $U_1' \subset U'$, of the origin. Moreover, dropping the arguments, switching to our shorthand notation, and denoting $f \circ \Gamma$ by $\widetilde{f}$, we have

$$\mathrm{J}_{\mathbb{R}}\Gamma = \begin{pmatrix} \Gamma_{1x_1} & \widetilde{\rho_{x_2}} & \Gamma_{1x_2} \\ \Gamma_{2x_1} & -\widetilde{\rho_{y_2}} & \Gamma_{2x_2} \\ \Gamma_{3x_1} & -\widetilde{\rho_{x_1}} & \Gamma_{3x_2} \end{pmatrix},$$

and

$$(\mathrm{J}_{\mathbb{R}}\Gamma)^{-1} = \begin{pmatrix} \frac{\widetilde{\rho_{x_1}}\Gamma_{2x_2} - \widetilde{\rho_{y_2}}\Gamma_{3x_2}}{\det \mathrm{J}_{\mathbb{R}}\Gamma} & \frac{-\widetilde{\rho_{x_1}}\Gamma_{1x_2} - \widetilde{\rho_{x_2}}\Gamma_{3x_2}}{\det \mathrm{J}_{\mathbb{R}}\Gamma} & \frac{\widetilde{\rho_{y_2}}\Gamma_{1x_2} + \widetilde{\rho_{x_2}}\Gamma_{2x_2}}{\det \mathrm{J}_{\mathbb{R}}\Gamma} \\ \frac{\Gamma_{2x_2}\Gamma_{3x_1} - \Gamma_{2x_1}\Gamma_{3x_2}}{\det \mathrm{J}_{\mathbb{R}}\Gamma} & \frac{-\Gamma_{1x_2}\Gamma_{3x_1} + \Gamma_{1x_1}\Gamma_{3x_2}}{\det \mathrm{J}_{\mathbb{R}}\Gamma} & \frac{\Gamma_{1x_2}\Gamma_{2x_1} - \Gamma_{1x_1}\Gamma_{2x_2}}{\det \mathrm{J}_{\mathbb{R}}\Gamma} \\ \frac{\widetilde{\rho_{y_2}}\Gamma_{3x_1} - \widetilde{\rho_{x_1}}\Gamma_{2x_1}}{\det \mathrm{J}_{\mathbb{R}}\Gamma} & \frac{\widetilde{\rho_{x_2}}\Gamma_{3x_1} + \widetilde{\rho_{x_1}}\Gamma_{1x_1}}{\det \mathrm{J}_{\mathbb{R}}\Gamma} & \frac{-\widetilde{\rho_{y_2}}\Gamma_{1x_1} - \widetilde{\rho_{x_2}}\Gamma_{2x_1}}{\det \mathrm{J}_{\mathbb{R}}\Gamma} \end{pmatrix},$$

wherever $\mathrm{J}_{\mathbb{R}}\Gamma$ is invertible. In particular, $\mathrm{J}_{\mathbb{R}}\Gamma(0) = (\mathrm{J}_{\mathbb{R}}\Gamma)^{-1}(0) = \mathrm{Id}\,.$ We may, therefore, locally invert Γ (as a $\mathcal{C}^3$-smooth function) in some neighborhood $W_1 \subset U_1'$ of 0. Let

$$(X_1, Y_1, X_2) = \Gamma^{-1}(x_1, y_1, x_2).$$

Γ is constructed to 'straighten' v — i.e., $\mathrm{J}_{\mathbb{R}}\Gamma(\frac{\partial}{\partial Y_1}) = v$. So, if we view X_1 and X_2 as $\mathcal{C}^3$-smooth functions on $W := \Gamma(W_1) \cap U'$, they are linearly independent and $v(X_1) \equiv v(X_2) \equiv 0$. Thus, $dX_1 \wedge dX_2 \neq 0$ everywhere on W and $dX_1(v) \equiv dX_2(v) \equiv \theta_\rho(v) \equiv 0$

on W. So, it must be the case that

$$\theta_\rho(\cdot) = \mathfrak{w}_1(\cdot)dX_1(\cdot) + \mathfrak{w}_2(\cdot)dX_2(\cdot),$$

for some $\mathfrak{w}_1, \mathfrak{w}_2 \in \mathcal{C}^2(W)$. Substituting the expressions for θ_ρ, dX_1 and dX_2 (the latter two can be read off the matrix $(\mathrm{J}_{\mathbb{R}}\Gamma)^{-1}$ above), we get

$$\begin{aligned}\mathfrak{w}_1 \quad &= \quad \frac{-\Gamma_{1x_1}\widetilde{\rho_{y_2}}(\rho_{y_1}\rho_{y_2} + \rho_{x_1}\rho_{x_2}) - \Gamma_{3x_1}\widetilde{\rho_{y_2}}(\rho_{x_2}^2 + \rho_{y_2}^2)}{\rho_{y_2}\widetilde{\rho_{y_2}}} \\ &\qquad + \frac{\Gamma_{2x_1}(\widetilde{\rho_{x_1}}(\rho_{y_2}^2 + \rho_{x_2}^2) - \widetilde{\rho_{x_2}}(\rho_{y_1}\rho_{y_2} + \rho_{x_1}\rho_{x_2}))}{\rho_{y_2}\widetilde{\rho_{y_2}}}\end{aligned}$$

and

$$\begin{aligned}\mathfrak{w}_2 \quad &= \quad \frac{-\Gamma_{1x_2}\widetilde{\rho_{y_2}}(\rho_{y_1}\rho_{y_2} + \rho_{x_1}\rho_{x_2}) - \Gamma_{3x_2}\widetilde{\rho_{y_2}}(\rho_{x_2}^2 + \rho_{y_2}^2)}{\rho_{y_2}\widetilde{\rho_{y_2}}} \\ &\qquad + \frac{\Gamma_{2x_2}(\widetilde{\rho_{x_1}}(\rho_{y_2}^2 + \rho_{x_2}^2) - \widetilde{\rho_{x_2}}(\rho_{y_1}\rho_{y_2} + \rho_{x_1}\rho_{x_2}))}{\rho_{y_2}\widetilde{\rho_{y_2}}},\end{aligned}$$

where, once again, $\widetilde{f} := f \circ \Gamma$. Observe that $\mathfrak{w}_1(0) = 0$ and $\mathfrak{w}_2(0) = 1$. Thus, for some neighborhood, $V \subset W$, of the origin, $\mathfrak{w}_2 \neq 0$ and

$$\theta_\rho = \mathfrak{w}_2(Y_1 dX_1 + dX_2),$$

where $Y_1 := \mathfrak{w}_1/\mathfrak{w}_2$. Finally, set

$$\mathfrak{a} := \frac{1}{\mathfrak{w}_2},\ \mathfrak{d}_1 := X_1,\ \mathfrak{d}_2 := -\frac{Y_1}{4\lambda} \text{ and } \mathfrak{d}_3 := X_2 + \frac{X_1 Y_1}{2}.$$

Then, on V,

$$\mathfrak{a}\theta_\rho = -2\lambda\mathfrak{d}_2 d\mathfrak{d}_1 + 2\lambda\mathfrak{d}_1 d\mathfrak{d}_2 + d\mathfrak{d}_3 = \mathfrak{d}^*(\theta_{\rho^\lambda}) \tag{3.44}$$

and $a(0) = 1$.

Refering to (3.43) and the formulae for $\mathfrak{w}_1$, $\mathfrak{w}_2$ and $(\mathrm{J}_{\mathbb{R}}\Gamma)^{-1}$, we get

$$\mathrm{J}_{\mathbb{R}}\mathfrak{d}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{\operatorname{Im}\mu}{2\lambda} \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.45}$$

We have, thus, constructed the required map. □

We now show that the contact transformation constructed above satisfies an estimate crucial to our analysis.

Lemma 3.5.3. *Let $\mathfrak{d}$ and V be as in the proof of Lemma 3.5.2 and $\mathcal{V} \Subset V$ be a neighborhood of the origin. Then, there is an $\mathscr{E}_1 \in \mathcal{C}(\mathcal{V})$ with $\lim_{w'\to 0} \mathscr{E}_1(w') = 0$ and a $\mathscr{C}_1 > 0$ such that, for all $w' \in \mathcal{V}$ and $z' \in \mathbb{R}^3$,*

$$\begin{aligned}
&|(z'-w')^{\mathrm{tr}} \cdot \mathrm{Hess}_{\mathbb{R}} \mathfrak{d}_3(w') \cdot (z'-w')| \\
&\qquad \leq \mathscr{E}_1(w')|z'-w'|^2 + \mathscr{C}_1(|z_1-w_1||x_2-u_2| + |x_2-u_2|^2). \quad (3.46)
\end{aligned}$$

Proof. Recall that $\mathfrak{d}_3 = X_2 + \frac{X_1 Y_1}{2}$. We refer to the construction in the proof of Lemma 3.5.2 and collect the following data:

$$\begin{aligned}
&(X_1)_{x_1}(0) = 1,\ (X_1)_{y_1}(0) = 0; \\
&(Y_1)_{x_1}(0) = 0,\ (Y_1)_{y_1}(0) = -4\lambda; \\
&(X_2)_{x_1x_1}(0) = 0,\ (X_2)_{x_1y_1}(0) = 2\lambda = (X_2)_{y_1x_1}(0),\ (X_2)_{y_2y_2}(0) = 0.
\end{aligned}$$

Next, we write out the relevant terms.

$$\begin{aligned}
&(z'-w')^{\mathrm{tr}} \cdot \mathrm{Hess}_{\mathbb{R}} \mathfrak{d}_3(w') \cdot (z'-w') \\
&= \Big(X_{2_{x_1x_1}}(w') + X_{1_{x_1}}(w') Y_{1_{x_1}}(w')\Big)(x_1-u_1)^2 \\
&\quad + \frac{1}{2}\Big(Y_1(w') X_{1_{x_1x_1}}(w') + X_1(w') Y_{1_{x_1x_1}}(w')\Big)(x_1-u_1)^2 \\
&\quad + \Big(2X_{2_{x_1y_1}}(w') + X_{1_{x_1}}(w') Y_{1_{y_1}}(w') + X_{1_{y_1}}(w') Y_{1_{x_1}}(w')\Big)(x_1-u_1)(y_1-v_1) \\
&\quad + \Big(Y_1(w') X_{1_{x_1y_1}}(w') + X_1(w') Y_{1_{x_1y_1}}(w')\Big)(x_1-u_1)(y_1-v_1) \\
&\quad + \Big(X_{2_{y_1y_1}}(w') + X_{1_{y_1}}(w') Y_{1_{y_1}}(w')\Big)(y_1-v_1)^2 \\
&\quad + \frac{1}{2}\Big(Y_1(w') X_{1_{y_1y_1}}(w') + X_1(w') Y_{1_{y_1y_1}}(w')\Big)(y_1-v_1)^2 \\
&\quad + 2\mathfrak{d}_{3_{x_1x_2}}(w')(x_1-u_1)(x_2-u_2) + 2\mathfrak{d}_{3_{y_1x_2}}(w')(y_1-v_1)(x_2-u_2) \\
&\quad + \mathfrak{d}_{3_{x_2x_2}}(w')(x_2-u_2)^2.
\end{aligned}$$

Now, the coefficients of $(x_1-u_1)^2$, $(x_1-u_1)(y_1-v_1)$ and $(y_1-u_1)^2$ in the above expansion all vanish at the origin (see data listed above). Thus, we have that the estimate (3.46). ☐

All that remains is to extend the above transformation to Ω. For this, let V be as in

Lemma 3.5.2 and $\mathcal{G}_\rho : V \times \mathbb{R} \to \mathbb{C}^2$ be the map

$$(x_1, y_1, x_2, y_2) \mapsto (x_1, y_1, x_2, F_\rho(x_1, y_1, x_2) + y_2).$$

$\mathcal{G}_\rho$ is evidently a $\mathcal{C}^4$-smooth diffeomorphism with $\mathcal{G}(V \times (0, t]) \subset \Omega$ for some $t > 0$. We note the following facts about $\mathcal{G}_\rho$:

- $\mathrm{J}_\mathbb{R}\mathcal{G}_\rho(0) = \mathrm{Id}\,.$ and $\mathrm{J}_\mathbb{R}\mathcal{G}_\rho^{\mathrm{res}}(0) = \mathrm{Id}\,..$
- $(\mathcal{G}_\rho)^*(\partial\rho + \overline{\partial}\rho) = \left(\dfrac{\partial \rho}{\partial y_2} \circ \mathcal{G}_\rho\right) dy_2$ and $(\mathcal{G}_\rho)^* \left(\frac{\partial\rho - \overline{\partial}\rho}{i}\right) = \theta_\rho$ on $V \times \{0\}$.

Lemma 3.5.4. *There is a neighborhood $\mathcal{U}$ of the origin and a $\mathcal{C}^2$-smooth diffeomorphism $\Psi : \mathcal{U} \to \mathbb{C}^2$ such that*

- $\Psi(0) = 0$, $\Psi(\Omega \cap \mathcal{U}) = \mathcal{S}_\lambda \cap \Psi(\mathcal{U})$ *and* $\Psi(\partial\Omega \cap \mathcal{U}) = \partial\mathcal{S}_\lambda \cap \Psi(\mathcal{U})$*;*
- $\det \mathrm{J}_\mathbb{R}\Psi(0) = 1$ *and* $\det \mathrm{J}_\mathbb{R}\Psi^{\mathrm{res}}(0) = 1$*; and*
- *if $\mathfrak{l}_\rho$ and $\mathfrak{l}_\lambda$ denote the Cauchy-Leray map of ρ and ρ^λ, respectively, then*

$$\begin{aligned} &\left|\mathfrak{l}_\rho(z, w) - \mathfrak{l}_\lambda(\Psi(z), \Psi(w))\right| \\ &\leq (\mathscr{E}(w) + \mathscr{D}(z - w)) \left(|\mathfrak{l}_\lambda(\Psi(z), \Psi(w)) + |z - w|^2\right) + \mathscr{C}|\mathfrak{l}_\lambda(\Psi(z), \Psi(w))|^2, \end{aligned} \tag{3.47}$$

on $\{(z, w) \in \overline{\Omega} \times \mathcal{U} : |z - w| \leq \tau\}$, for some choice of $\mathscr{E} \in \mathcal{C}(U)$ with $\lim_{w\to 0} \mathscr{E}(w) = 0$, $\mathscr{D}(\zeta) = o(1)$ as $|\zeta| \to 0$, and constants $\mathscr{C}, \tau > 0$.

Proof. Let $\Psi = (\Psi_1, \Psi_2) := \mathcal{G}_T \circ (\mathfrak{d}, \mathrm{Id}\,.) \circ \mathcal{G}_\rho^{-1}$, where $\mathrm{Id}\,.$ is the identity map on $\mathbb{R}$, and $\mathcal{U} \Subset \mathcal{G}_\rho(V \times [-t, t])$ is a neighborhood of the origin. We use the notation $(\Psi_1, \Psi_2) = (\psi_1 + i\psi_2, \psi_3 + i\psi_4)$. The regularity and mapping properties of Ψ follow from its definition. Since $\mathrm{id.}^*(-dy_2) = -dy_2$ and $\mathfrak{d}^*(\theta_{\rho^\lambda}) = \mathfrak{a}\theta_\rho$ on $\{y_2 = 0\}$,

$$\Psi^*(\partial\rho^\lambda + \overline{\partial}\rho^\lambda) = \mathfrak{a}_1(\partial\rho + \overline{\partial}\rho)$$

and

$$\Psi^* \left(\frac{\partial\rho^\lambda - \overline{\partial}\rho^\lambda}{i}\right) = \mathfrak{a}_2 \left(\frac{\partial\rho - \overline{\partial}\rho}{i}\right),$$

on $\partial\Omega \cap \mathcal{U}$, where $\mathfrak{a}_1(x_1, y_1, x_2, y_2) = -\dfrac{\partial\rho}{\partial y_2}(\mathcal{G}_\rho(x_1, y_1, x_2, y_2))$ and $\mathfrak{a}_2(x_1, y_1, x_2, y_2) =$

$\mathfrak{a}(x_1, y_1, x_2)$. Therefore, for all $w \in \partial\Omega \cap \mathcal{U}$ and $z \in \mathbb{C}^2$,

$$\begin{aligned}
&2\Big\langle \partial\rho^\lambda(\Psi(w)), \mathrm{J}_{\mathbb{C}}\Psi(w)(z-w)\Big\rangle \\
&= 2\,\mathrm{Re}\Big\langle \partial\rho^\lambda(\Psi(w)), \mathrm{J}_{\mathbb{C}}\Psi(w)(z-w)\Big\rangle + 2i\,\mathrm{Im}\Big\langle \partial\rho^\lambda(\Psi(w)), \mathrm{J}_{\mathbb{C}}\Psi(w)(z-w)\Big\rangle \\
&= \Big\langle\Big\langle (\partial\rho^\lambda + \overline{\partial}\rho^\lambda)(\Psi(w)), \mathrm{J}_{\mathbb{C}}\Psi(w)(z-w)\Big\rangle\Big\rangle \\
&\qquad + i\Big\langle\Big\langle \frac{\partial\rho^\lambda - \overline{\partial}\rho^\lambda}{i}(\Psi(w)), \mathrm{J}_{\mathbb{C}}\Psi(w)(z-w)\Big\rangle\Big\rangle \\
&= \Big\langle\Big\langle \Psi^*(\partial\rho^\lambda + \overline{\partial}\rho^\lambda)(w), z-w\Big\rangle\Big\rangle + i\Big\langle\Big\langle \Psi^*\left(\frac{\partial\rho^\lambda - \overline{\partial}\rho^\lambda}{i}\right)(w), z-w\Big\rangle\Big\rangle \\
&= \mathfrak{a}_1(w)\Big\langle\Big\langle (\partial\rho + \overline{\partial}\rho)(w), z-w\Big\rangle\Big\rangle + i\mathfrak{a}_2(w)\Big\langle\Big\langle \left(\frac{\partial\rho - \overline{\partial}\rho}{i}\right)(w), z-w\Big\rangle\Big\rangle \\
&= 2\mathfrak{a}_1(w)\,\mathrm{Re}\Big\langle \partial\rho(w), z-w\Big\rangle + 2i\mathfrak{a}_2(w)\,\mathrm{Im}\Big\langle \partial\rho(w), z-w\Big\rangle.
\end{aligned}$$

Now, since $\rho^\lambda := \lambda|z_1|^2 - y_2$, $\dfrac{\partial\rho^\lambda}{\partial z_1}(\Psi(z)) = \lambda\overline{\Psi_1(z)}$ and $\dfrac{\partial\rho^\lambda}{\partial z_2}(\Psi(z)) = \dfrac{i}{2}$. Therefore, there is a $\tau_1 > 0$ such that on $\{(z, w) \in \mathbb{R}^4 \times \mathcal{U} : ||z-w|| \leq \tau_1\}$,

$$\begin{aligned}
&\Big|\Big\langle \partial\rho^\lambda(\Psi(w)), \Psi(z) - \Psi(w) - \mathrm{J}_{\mathbb{C}}\Psi(w)(z-w)\Big\rangle\Big| \\
&\qquad \leq c|\Psi_1(w)| \cdot ||z-w||^2 + \frac{1}{2}R_1(z-w) + R_2(z-w), \qquad (3.48)
\end{aligned}$$

where, $c > 0$, $R_1(z-w) = |(z-w)^{\mathrm{tr}} \cdot (\mathrm{Hess}_{\mathbb{R}}\psi_3(w) + \mathrm{Hess}_{\mathbb{R}}\psi_4(w)) \cdot (z-w)|$, and $R_2(\zeta) = o(|\zeta|^2)$ as $|\zeta| \to 0$. Observe that $\psi_3(z', y_2) = \mathfrak{d}_3(z')$ and $\psi_4(z', y_2) = \mathfrak{d}_1(z')^2 + \mathfrak{d}_2(z')^2 + y_2 - F(z')$. As,

$$\begin{aligned}
\psi_{4_{x_1x_1}}(w) &= 2\sum_{j=1}^{2}(\mathfrak{d}_{j_{x_1}}(w')^2 + \mathfrak{d}_j(w')\mathfrak{d}_{j_{x_1x_1}}(w')) - F_{x_1x_1}(w'), \\
\psi_{4_{y_1y_1}}(w) &= 2\sum_{j=1}^{2}(\mathfrak{d}_{j_{y_1}}(w')^2 + \mathfrak{d}_j(w')\mathfrak{d}_{j_{y_1y_1}}(w')) - F_{y_1y_1}(w') \text{ and} \\
\psi_{4_{x_1y_1}}(w) &= 2\sum_{j=1}^{2}(\mathfrak{d}_{j_{x_1}}(w')\mathfrak{d}_{j_{y_1}}(w') + \mathfrak{d}_j(w')\mathfrak{d}_{j_{x_1y_1}}(w')) - F_{x_1y_1}(w')
\end{aligned}$$

all vanish at $w = 0$, we have, for all $(z, w) \in \mathbb{R}^4 \times \mathcal{U}$,

$$\begin{aligned}
&|(z-w)^{\mathrm{tr}} \cdot \mathrm{Hess}_{\mathbb{R}}\psi_4(w) \cdot (z-w)| \\
&\qquad \leq \mathscr{E}_2(w)||z-w||^2 + \mathscr{C}_2(|z_1-w_1||z_2-w_2| + |z_2-w_2|^2), \quad (3.49)
\end{aligned}$$

where $\mathscr{E}_1 \in \mathcal{C}(\mathcal{U})$ with $\lim_{w\to 0} \mathscr{E}_1(w) = 0$, and $\mathscr{C}_1 > 0$ is a constant. Combining (3.48), (3.46) and (3.49) (and adding $c|\Psi_1|$, $\mathscr{E}_1$ and $\mathscr{E}_2$), we have that

$$\begin{aligned} \mathcal{A} &:= \left|\left\langle \partial\rho^\lambda(\Psi(w)), \Psi(z) - \Psi(w) - \mathrm{J}_{\mathbb{C}}\Psi(w)(z-w)\right\rangle\right| \\ &\leq (\mathscr{E}_3(w) + \mathscr{D}_3(z-w))||z-w||^2 + \mathscr{C}_3(|z_1-w_1||z_2-w_2| + |z_2-w_2|^2), \end{aligned} \tag{3.50}$$

on $\{(z,w) \in \mathbb{R}^4 \times \mathcal{U} : ||z-w|| \leq \tau_3\}$, for some $\mathscr{E}_3 \in \mathcal{C}(\mathcal{U})$ with $\lim_{w\to 0} \mathscr{E}_3(w) = 0$, $\mathscr{D}_3(\zeta) = o(1)$ as $|\zeta| \to 0$, and constants $\mathscr{C}_3, \tau_3 > 0$.

Next, we have that

$$\begin{aligned} |\Psi_2(z) - \Psi_2(w)| &= 2\left|\left\langle \partial\rho^\lambda(\Psi(w)), \Psi(z) - \Psi(w)\right\rangle - \overline{\Psi_1(z)}(\Psi_1(z) - \Psi_1(w))\right| \\ &\leq \mathscr{C}_4\left|\left\langle \partial\rho^\lambda(\Psi(w)), \Psi(z) - \Psi(w)\right\rangle\right| + \mathscr{E}_4(w)||z-w||, \end{aligned} \tag{3.51}$$

on $\{(z,w) \in \mathbb{R}^4 \times \mathcal{U} : ||z-w|| \leq \tau_4\}$, for some choice of $\mathscr{E}_4$, $\mathscr{C}_4$ and τ_4 as before. Also, if $\Psi^{-1} = (\hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3, \hat{\psi}_4)$, then $\mathrm{J}_{\mathbb{R}}\hat{\psi}_3(0) = (0,0,1,0)$ and $\mathrm{J}_{\mathbb{R}}\hat{\psi}_4(0) = (0,0,0,1)$. So, we are permitted to conclude that

$$|z_2 - w_2| \leq \mathscr{C}_4|\Psi_2(z) - \Psi_2(w)| + (\mathscr{E}_5(w) + \mathscr{D}_5(z-w))||z-w||, \tag{3.52}$$

on $\{(z,w) \in \mathbb{R}^4 \times \mathcal{U} : ||z-w|| \leq \tau_5\}$, for some $\mathscr{E}_5$, $\mathscr{C}_5$, $\mathscr{D}_5$ and τ_5 as before.

Finally, as $a_1(0) = a_2(0) = 1$, (3.48), (3.50), (3.51) and (3.52) combine to give an $\mathscr{E}$, $\mathscr{C}$, $\mathscr{D}$ and τ with the required properties, such that

$$\begin{aligned} &|\mathfrak{l}_\rho(z,w) - \mathfrak{l}_\lambda(\Psi(z), \Psi(w))| \\ &\leq \left|\left\langle \partial\rho(w), z-w\right\rangle - \left\langle \partial\rho^\lambda(\Psi(w)), \mathrm{J}_{\mathbb{C}}\Psi(w)(z-w)\right\rangle\right| + \mathcal{A}, \\ &\leq (\mathscr{E}(w) + \mathscr{D}(z-w))\left(|\mathfrak{l}_\lambda(\Psi(z),\Psi(w))| + ||z-w||^2\right) + \mathscr{C}|\mathfrak{l}_\lambda(\Psi(z),\Psi(w))|^2. \end{aligned}$$

on $\{(z,w) \in \mathbb{R}^4 \times \mathcal{U} : ||z-w|| \leq \tau\}$. □

3.5.2 Convexification

In this section, we return to general strongly pseudoconvex domains. Assume $0 \in \partial\Omega$ and the outward unit normal vector to $\partial\Omega$ at 0 is $(0,-i)$. Let ρ be a $\mathcal{C}^2$-smooth strictly plurisubharmonic defining function of Ω such that $||\nabla\rho(0)|| = 1$. Now, ρ has the following second-order Taylor expansion about the origin:

$$\rho(w) = \operatorname{Im}\left(-w_2 + i\sum_{j,k=1}^{2}\frac{\partial^2\rho(0)}{\partial z_j\partial z_k}w_jw_k\right) + \sum_{j,k=1}^{2}\frac{\partial^2\rho(0)}{\partial z_j\partial\overline{z_k}}w_j\overline{w_k} + o(|w|^2).$$

Using a classical trick, attributed to Narasimhan, we convexify Ω near the origin via the map Φ given by:

$$\begin{aligned} w_1 &\mapsto \Phi_1(w) = w_1 \\ w_2 &\mapsto \Phi_2(w) = w_2 - i\sum_{j,k=1}^{2}\frac{\partial^2\rho(0)}{\partial z_j\partial z_k}w_jw_k. \end{aligned}$$

Owing to the inverse function theorem, Φ is a local biholomorphism on some neighborhood U of 0. We may further shrink U so that the strong convexity of $\Phi(\partial\Omega\cap U)$ at 0 propagates to all of $\Psi(\partial\Omega\cap U)$. We collect the following key observations:

- $\mathrm{J}_{\mathbb{R}}\Phi(0) = \mathrm{Id}$. and $\mathrm{J}_{\mathbb{R}}\Phi^{\mathrm{res}}(0) = \mathrm{Id}$.;
- If $\hat\rho := \rho\circ\Phi^{-1}$, then $\hat\rho(w) = -\operatorname{Im} w_2 + \sum_{j,k=1}^{2}\frac{\partial^2\rho(0)}{\partial z_j\partial\overline{z_k}}w_j\overline{w_k} + o(|w|^2)$.
- If $\mathfrak{p}$ is the Levi-polynomial of ρ and $\mathfrak{l}_{\hat\rho}(z,w)$ is the Cauchy-Leray map of $\hat\rho$, then, for any neighborhood $\mathcal{U} \Subset U$ of the origin, there is a $\tau > 0$ such that, on $\{(z,w)\in\mathbb{C}^2\times\mathcal{U} : |z-w|\leq\tau\}$,

$$\begin{aligned} &|\mathfrak{p}(z,w) - \mathfrak{l}_{\hat\rho}(\Phi(z),\Phi(w))| \qquad (3.53)\\ &\leq \left|\Big\langle\partial\rho(w),(z-w)\Big\rangle - \Big\langle\partial\hat\rho(\Phi(w)),\mathrm{J}_{\mathbb{C}}\Phi(w)(z-w)\Big\rangle\right| \\ &\quad + \frac{1}{2}\left|\sum_{j,k=1}^{2}\left(\frac{\partial^2\rho(w)}{\partial z_j\partial z_k} + 2i\frac{\partial\hat\rho(\Phi(w))}{\partial w_2}\frac{\partial^2\rho(0)}{\partial z_j\partial z_k}\right)(z_j-w_j)(z_k-w_k)\right| \\ &\leq \left|\Big\langle\partial\rho(w),(z-w)\Big\rangle - \Big\langle\Phi^*(\partial\hat\rho)(w),(z-w)\Big\rangle\right| \\ &\quad + \frac{1}{2}\left|\sum_{j,k=1}^{2}\left(\frac{\partial^2\rho(0)}{\partial z_j\partial z_k} + o(1) + (-1+o(|w|))\frac{\partial^2\rho(0)}{\partial z_j\partial z_k}\right)(z_j-w_j)(z_k-w_k)\right| \\ &\leq \mathscr{E}(w)|z-w|^2, \end{aligned}$$

for some $\mathscr{E}\in\mathcal{C}(U)$ with $\lim_{w\to0}\mathscr{E}(w) = 0$.

3.5.3 Main Local Estimate

We combine the maps constructed above:

Lemma 3.5.5. *Fix an $\varepsilon > 0$. Let $\Omega \subset \mathbb{C}^2$ be a $\mathcal{C}^4$-smooth strongly pseudoconvex domain and ρ a strictly plurisubharmonic defining function of Ω. Assume that $0 \in \partial\Omega$, $\nabla\rho(0) = (0,0,0,-1)$ and $M[\rho](0) = \lambda$. Then, there exists a neighborhood U of the origin, a $\mathcal{C}^2$-smooth origin-preserving diffeomorphism Θ on U that carries $\overline{\Omega} \cap U$ onto $\overline{\mathcal{S}_\lambda} \cap \Theta(U)$, and a constant $\tau > 0$ such that*

- $1 - \varepsilon \leq \dfrac{\text{vol}(\Theta(V))}{\text{vol}(V)} \leq \dfrac{1}{1-\varepsilon}$*, for every Jordan measurable $V \subset U$;*
- $1 - \varepsilon \leq \dfrac{\text{vol}_3(\Theta(J)')}{\text{vol}_3(J')} \leq \dfrac{1}{1-\varepsilon}$*, for every Jordan measurable $J \subset \partial\Omega \cap U$; and*
- *if P is the Levi polynomial of ρ and $\mathfrak{l}_\lambda$ is the Cauchy-Leray map of ρ^λ, then*

$$|\mathfrak{p}(z,w) - \mathfrak{l}_\lambda(\Theta(z),\Theta(w))| \leq \varepsilon(|\mathfrak{p}(z,w)| + |\mathfrak{l}_\lambda(\Theta(z),\Theta(w))|)$$

on $\{(z,w) \in (\overline{\Omega} \cap U) \times H : |z-w| \leq \tau\}$, where $H \subset \partial\Omega \cap U$ is compact.

Proof. The needed map is $\Psi \circ \Phi$ (see Sections 3.5.1 and 3.5.2). The mapping and volume distortion properties follow from those of Ψ and Φ. The estimate is a combination of (3.53), (3.47) and (3.41). □

The following lemma is an application of Lemma 3.3.2 and gives us a local version of our main theorem.

Lemma 3.5.6. *Let Ω, f and ρ be as in Theorem 3.1.1. Fix an $\varepsilon \in (0, 1/3)$ and a point $q \in \partial\Omega$. Then, there exists a neighborhood $U_{q,\varepsilon}$ of q such that for every Jordan measurable pair $J, H \subset \partial\Omega \cap U_{q,\varepsilon}$ such that $J \subset \text{int}_{\partial\Omega} H$,*

$$(1-\varepsilon)^{31}\, l_{\text{kor}} \frac{\lambda(q)^{\frac{1}{2}} s(J)^{\frac{3}{2}}}{\sqrt{n}} \leq v(\Omega; \mathcal{P}_n(J \subset H; f)) \leq (1-\varepsilon)^{-19}\, l_{\text{kor}} \frac{\lambda(q)^{\frac{1}{2}} s(J)^{\frac{3}{2}}}{\sqrt{n}}$$

for sufficiently large n, where $\lambda(q) := \dfrac{4M[\rho](q)}{||\nabla\rho(q)||^3}$ and s is the Euclidean surface area measure on $\partial\Omega$.

Proof. Let ρ be the strictly plurisubharmonic defining function of Ω for which (⋇) in Theorem 3.1.1 holds. Let $A : \mathbb{C}^2 \to \mathbb{C}^2$ be an isometry that takes q to the origin and the outer unit normal at q to $(0, -i||\nabla\rho(q)||)$. Set $\hat{\rho}(z) := ||\nabla\rho(q)||^{-1}\rho(A^{-1}z)$. Then, $\hat{\rho}$ satisfies the hypotheses of Lemma 3.5.5, with $M(\hat{\rho})(0) = \lambda(q)$. Moreover, the Levi polynomial $\hat{\mathfrak{p}}$ of $\hat{\rho}$ satisfies

$$||\nabla\rho(q)||\hat{\mathfrak{p}}(Az, Aw) = \mathfrak{p}(z, w). \tag{3.54}$$

Suppose Θ and U are the map and neighborhood, respectively, granted by Lemma 3.5.5. Set $V_q := A^{-1}(U)$ and $\Theta_q := \Theta \circ A$. Note that Θ_q maps $\overline{\Omega}$ to $\overline{\mathcal{S}}_{\lambda(q)}$ locally near q. We define

$$\begin{aligned}
\tilde{f}(z,w) &:= \frac{f(z,w)}{||\nabla\rho(q)||^{\nu}};\\
g(z,w) &:= f_{\mathcal{S}_{\lambda(q)}}(\Theta_q z, \Theta_q w);\ \text{and}\\
\tilde{g}(z,w) &:= a(w,w)\left(\frac{2i}{\lambda(q)} f_{\mathcal{S}_{\lambda(q)}}(\Theta_q z, \Theta_q w)\right)^{\nu}\\
&= a(w,w)\mathfrak{l}_{\lambda(q)}(\Theta_q z, \Theta_q w)^{\nu}\ \text{(see Section 3.4).}
\end{aligned}$$

Observe that, when defined,

$$C(w,\delta,\tilde{f}) = C\left(w, ||\nabla\rho(q)||^{\nu}\delta; f\right);\ \text{and} \tag{3.55}$$

$$C(w,\delta,\tilde{g}) = C\left(w, \frac{\lambda(q)}{2}\left(\frac{\delta}{|a(w,w)|}\right)^{\frac{1}{\nu}}; g\right). \tag{3.56}$$

Thus, for our point of interest, there is little difference between f and $\tilde{f}$ (and g and $\tilde{g}$).

Keeping this observation in mind, we will apply Lemma 3.3.2 to $\tilde{f}, \tilde{g} \in \mathcal{C}(\overline{\Omega}\times(V_q\cap\partial\Omega))$ (see Remark 3.3.3). By (✻), there exist $\tau_1 \in (0,\tau]$ and $l > 0$ such that

$$|\mathfrak{p}(z,w)|^{\nu} \leq l|\tilde{f}(z,w)| \quad \text{on } \Omega_{\tau_1}. \tag{3.57}$$

Now, fix an $\varepsilon \in (0, 1/3)$. Let $\hat{\varepsilon} :=$

$$\frac{\varepsilon}{2}\min\left\{\frac{||\nabla\rho(q)||^{\nu}}{l}, \frac{(2\nu||\nabla\rho(q)||^{\nu}\max_{\Omega_\tau}|a(z,w)|)^{-1}}{l}, \left(2\nu\frac{\max_{\Omega_\tau}|a(z,w)|}{\min_{\partial\Omega}|a(w,w)|}\right)^{-1}, \min_{\partial\Omega}|a(w,w)|\right\}. \tag{3.58}$$

By (✻), we can find a $\tau_2 \in (0, \tau_1]$ such that

$$|\tilde{f}(z,w) - a(z,w)\hat{\mathfrak{p}}(Az,Aw)^{\nu}| = \frac{|f(z,w) - a(z,w)\mathfrak{p}(z,w)^{\nu}|}{||\nabla\rho(q)||^{\nu}} \leq \frac{\hat{\varepsilon}}{||\nabla\rho(q)||^{\nu}}|\mathfrak{p}(z,w)|^{\nu} \quad \text{on } \Omega_{\tau_2}. \tag{3.59}$$

By Lemma 3.5.5, (3.54), and the continuity of a on $\overline{\Omega}_\tau$, we shrink τ_2 so that on $\Omega_{\tau_2} \cap$

$\left(\overline{\Omega} \times V_q\right)$,

$$\begin{aligned}
&|a(z,w)\hat{\mathfrak{p}}(Az,Aw)^\nu - a(z,w)\mathfrak{l}_{\lambda(q)}\big(\Theta_q z, \Theta_q w\big)^\nu| \\
&\leq |a(z,w)| \left(|\hat{\mathfrak{p}}(Az,Aw) - \mathfrak{l}_{\lambda(q)}\big(\Theta_q z, \Theta_q w\big)|\right) \nu \max\{|\hat{\mathfrak{p}}(Az,Aw)|, |\mathfrak{l}_{\lambda(q)}\big(\Theta_q z, \Theta_q w\big)|\}^{\nu-1} \\
&\leq |a(z,w)|\hat{\varepsilon} \left(|\hat{\mathfrak{p}}(Az,Aw)| + |\mathfrak{l}_{\lambda(q)}\big(\Theta_q z, \Theta_q w\big)|\right) \nu \max\{|\hat{\mathfrak{p}}(Az,Aw)|, |\mathfrak{l}_{\lambda(q)}\big(\Theta_q z, \Theta_q w\big)|\}^{\nu-1} \\
&\leq 2\nu\hat{\varepsilon}|a(z,w)| \left(|\hat{\mathfrak{p}}(Az,Aw)|^\nu + |\mathfrak{l}_{\lambda(q)}\big(\Theta_q z, \Theta_q w\big)|^\nu\right) \\
&\leq \hat{\varepsilon}\left(2\nu||\nabla\rho(q)||^\nu \max_{\Omega_\tau} |a(z,w)|\right) |\mathfrak{p}(z,w)|^\nu + \hat{\varepsilon}\left(2\nu \frac{\max_{\Omega_\tau} |a(z,w)|}{\min_{\partial\Omega} |a(w,w)|}\right) |\tilde{g}(z,w)|, \qquad (3.60)
\end{aligned}$$

and

$$\begin{aligned}
|a(z,w)\mathfrak{l}_{\lambda(q)}\big(\Theta_q z, \Theta_q w\big)^\nu - \tilde{g}(z,w)| &= |a(z,w) - a(w,w)| \cdot |\mathfrak{l}_{\lambda(q)}\big(\Theta_q z, \Theta_q w\big)|^\nu \\
&\leq \frac{\hat{\varepsilon}}{\min_{\partial\Omega} |a(w,w)|} |\tilde{g}(z,w)|. \qquad (3.61)
\end{aligned}$$

Adding (3.59), (3.60) and (3.61), and recalling (3.58) and (3.57), we get

$$|\tilde{f}(z,w) - \tilde{g}(z,w)| \leq \varepsilon\left(|\tilde{f}(z,w)| + |\tilde{g}(z,w)|\right) \qquad \text{on } \Omega_{\tau_2} \cap \left(\overline{\Omega} \times V_q\right).$$

We now need to show that $\tilde{g}$ satisfies the remaining hypotheses of Lemma 3.3.2. But these are conditions on the cuts of $\tilde{g}$, which are identical to the cuts of g (by (3.56)). So, we work with g instead. Let $U_{q,\varepsilon} \Subset V_q$ be an open neighborhood of q, and $\delta_0 > 0$ be such that $C(w,\delta;g) \subset V_q$ for all $w \in U_{q,\varepsilon} \cap \partial\Omega$ and $\delta < \delta_0$. Then,

$$\Theta_q = \Theta \circ A : C(w,\delta;g) \twoheadrightarrow C\big(\Theta_q w, \delta; f_{\mathcal{S}_{\lambda(q)}}\big), \qquad (3.62)$$

for $w \in U_{q,\varepsilon} \cap \partial\Omega$ and $\delta < \delta_0$. Therefore, exploiting Lemma 3.8.1, we get

1. $C(w,\delta;g)$ is Jordan measurable for all $w \in U_{q,\varepsilon} \cap \partial\Omega$ and $\delta < \delta_0$;

2. If $w^1, ..., w^m \in U_{q,\varepsilon} \cap \partial\Omega$, $m \in \mathbb{N}_+$, then

$$\begin{aligned}
\mathrm{vol}\left(\bigcup_{j=1}^m C(w^j,(1+t)\delta;g)\right) &\leq \frac{1}{1-\varepsilon}\,\mathrm{vol}\left(\bigcup_{j=1}^m C(\Theta_q w^j,(1+t)\delta; f_{\mathcal{S}_{\lambda(q)}})\right) \\
&\leq \frac{(1+t)^3}{1-\varepsilon}\,\mathrm{vol}\left(\bigcup_{j=1}^m C(\Theta_q w^j,\delta; f_{\mathcal{S}_{\lambda(q)}})\right) \\
&\leq \frac{1}{(1-\varepsilon)^2}(1+t)^3\,\mathrm{vol}\left(\bigcup_{j=1}^m C(w^j,\delta;g)\right),
\end{aligned}$$

for all $t \in (0, 16)$ and $\delta_j \leq \delta_0/16$, $j = 1, ..., m$. Thus, g satisfies the doubling property ⊛ with quantifiers $\delta_g = \delta_0/16$ and $\mathcal{E}(t) = (1-\varepsilon)^{-2}(1+t)^3$.

Lastly, we further shrink $U_{q,\varepsilon}$ — if necessary — to ensure that

(‡) for any s-measurable set $J \subset (U_{q,\varepsilon} \cap \partial\Omega)$,

$$1 - \varepsilon \leq \frac{s(J)}{[J'']} \leq \frac{1}{1-\varepsilon},$$

where J'' denotes the projection of J onto the tangent plane to $\partial\Omega$ at q and $[J''] := \mathrm{vol}_3(A(J)')$.

We are now ready to estimate. Consider Jordan measurable compact sets $J \subset H \subset (U_{q,\varepsilon} \cap \partial\Omega)$ such that $J \subset \mathrm{int}_{\partial\Omega} H$. By (3.55), (3.10), (3.56), the volume-distortion properties of Θ_q — see Lemma 3.5.5 and recall that A is an isometry — and property (‡), we have that

$$\begin{aligned}
&\limsup_{n\to\infty} \sqrt{n}\, v(\Omega; \mathcal{P}_n(J \subset H; f)) = \limsup_{n\to\infty} \sqrt{n}\, v(\Omega; \mathcal{P}_n(J \subset H; \tilde{f}))\\
&\leq \frac{1}{(1-\varepsilon)^2}\left(1 + \frac{(1+\varepsilon)^2}{(1-\varepsilon)^2} - 1\right)^3 \limsup_{n\to\infty} \sqrt{n}\, v(\Omega; \mathcal{P}_n(J \subset H; \tilde{g}))\\
&= \frac{1}{(1-\varepsilon)^2}\left(1 + \frac{(1+\varepsilon)^2}{(1-\varepsilon)^2} - 1\right)^3 \limsup_{n\to\infty} \sqrt{n}\, v(\Omega; \mathcal{P}_n(J \subset H; g))\\
&\leq (1-\varepsilon)^{-14} \limsup_{n\to\infty} \sqrt{n}\,(1-\varepsilon)^{-1} v(\mathcal{S}_{\lambda(q)}; \mathcal{P}_n(\Theta_q J \subset \Theta_q H; f_{\mathcal{S}_{\lambda(q)}}))\\
&\leq (1-\varepsilon)^{-15}\, l_{\mathrm{kor}} \lambda(q)^{\frac{1}{2}}\, \mathrm{vol}_3((\Theta_q J)')^{\frac{3}{2}}\\
&\leq (1-\varepsilon)^{-\frac{33}{2}}\, l_{\mathrm{kor}} \lambda(q)^{\frac{1}{2}} [J'']^{\frac{3}{2}} \leq (1-\varepsilon)^{-18}\, l_{\mathrm{kor}} \lambda(q)^{\frac{1}{2}} s(J)^{\frac{3}{2}}.
\end{aligned}$$

By a similar argument, but now using (3.12) from the proof of Lemma 3.3.2, we get that

$$\lim_{n\to\infty} \sqrt{n}\, v(\Omega; \mathcal{P}_n(J \subset H; f)) \geq (1-\varepsilon)^{30}\, l_{\mathrm{kor}} \lambda(q)^{\frac{1}{2}} s(J)^{\frac{3}{2}}.$$

Therefore, for large enough n, we get the desired estimates. □

3.6 Proof of Theorem 3.1.1

Proof of Theorem 3.1.1. Fix an $\varepsilon \in (0, 1/3)$. There exists a tiling $\{L_j\}_{1\leq j\leq m}$ of $\partial\Omega$ consisting of Jordan measurable compact sets with non-empty interior such that

- for each $j = 1, ..., m$, there is a $q_j \in L_j$ for which $L_j \subset U_{q_j,\varepsilon}$, where the latter comes from Lemma 3.5.6;

- $(1-\varepsilon)\lambda(q) \le \lambda(q_j) \le (1-\varepsilon)^{-1}\lambda(q)$, for all $q \in L_j$.

Then, recalling that $\lambda(q) = \dfrac{4M[\rho](q)}{||\nabla\rho(q)||^3}$, we obtain estimates as follows:

$$4^{-\frac{1}{3}} \int_{\partial\Omega} \sigma_\Omega = \int_{\partial\Omega} 4^{\frac{1}{3}} M[\rho](q)^{\frac{1}{3}} \frac{ds(q)}{||\nabla\rho(q)||} = \sum_{j=1}^{m} \int_{L_j} \lambda(q)^{\frac{1}{3}} ds(q)$$
$$\begin{cases} \le (1-\varepsilon)^{-1} \displaystyle\sum_{j=1}^{m} \lambda(q_j)^{\frac{1}{3}} s(L_j) \\ \ge (1-\varepsilon) \displaystyle\sum_{j=1}^{m} \lambda(q_j)^{\frac{1}{3}} s(L_j). \end{cases} \tag{3.63}$$

We extend this tiling to a thin tubular neighborhood N of $\partial\Omega$ in the obvious way, denoting the tile corresponding to L_j by $\hat{L}_j$. Lastly, for all $j = 1, ..., m$, we choose compact Jordan measurable sets J_j and H_j such that $J_j \subset \mathrm{int}_{\partial\Omega} L_j \subset \mathrm{int}_{\partial\Omega} H_j \subset U_{q_j,\varepsilon}$ and

$$s(J_j) \ge (1-\varepsilon) s(L_j). \tag{3.64}$$

1. We first estimate $v(\Omega; \mathcal{P}_n(f))$ from above. For $j = 1, ..., m$, choose $P^j \in \mathcal{P}_{n_j}(L_j \subset H_j; f)$ such that $\mathrm{vol}(\Omega \setminus P^j) \le (1-\varepsilon)^{-1} v(\Omega; \mathcal{P}_{n_j}(L_j \subset H_j; f))$. Let P denote the intersection of all these P^j's. Then, P is an f-polyhedron with at most $n_1 + \cdots + n_m$ facets. Thus, by Lemma 3.5.6, for sufficiently large $n_1, ..., n_m$,

$$\begin{aligned} \mathrm{vol}(\Omega \setminus P) &\le (1-\varepsilon)^{-1} \sum_{j=1}^{m} v(\Omega; \mathcal{P}_{n_j}(L_j \subset H_j; f)) \\ &\le (1-\varepsilon)^{-20}\, l_{\mathrm{kor}} \sum_{j=1}^{m} \frac{\lambda(q_j)^{\frac{1}{2}} s(L_j)^{\frac{3}{2}}}{\sqrt{n_j}} \\ &= (1-\varepsilon)^{-20}\, l_{\mathrm{kor}} \sum_{j=1}^{m} \lambda(q_j)^{\frac{1}{3}} s(L_j) \left(\frac{\lambda(q_j)^{\frac{1}{3}} s(L_j)}{n_j} \right)^{\frac{1}{2}}. \end{aligned} \tag{3.65}$$

Now, fix an $n \in \mathbb{N}_+$. Suppose, we set

$$n_j = \left\lfloor \frac{\lambda(q_j)^{\frac{1}{3}} s(L_j)}{\sum_{j=1}^{m} \lambda(q_j)^{\frac{1}{3}} s(L_j)} n \right\rfloor, \; j = 1, ..., m. \tag{3.66}$$

Then,

$$n_1 + \cdots + n_m \le n; \tag{3.67}$$

and

$$(1-\varepsilon)\frac{\lambda(q_j)^{\frac{1}{3}}s(L_j)}{\sum_{j=1}^m \lambda(q_j)^{\frac{1}{3}}s(L_j)}n \le n_j \tag{3.68}$$

if n is large. We use (3.67), substitute (3.68) in (3.65) and invoke (3.63) to get

$$\begin{aligned} v(\Omega;\mathcal{P}_n(f)) &\le (1-\varepsilon)^{-21}\, l_{\text{kor}} \left(\sum_{j=1}^m \lambda(q_j)^{\frac{1}{3}}s(L_j)\right)^{\frac{3}{2}} \frac{1}{\sqrt{n}} \\ &\le (1-\varepsilon)^{-24}\, l_{\text{kor}} \left(\int_{\partial\Omega} \sigma_\Omega\right)^{\frac{3}{2}} \frac{1}{\sqrt{n}}, \end{aligned} \tag{3.69}$$

for n sufficiently large.

2. Next, we produce a lower bound for $v(\Omega;\mathcal{P}_f(n))$. Choose a $P_n \in \mathcal{P}_n(f)$ such that $\text{vol}(\Omega \setminus P_n) \le (1-\varepsilon)^{-1}v(\Omega;\mathcal{P}_f(n))$. Let n_j be the number of cuts of P_n that cover J_j. As $\lim_{n\to\infty}\delta(P_n) = 0$ due to Lemma 3.3.1 and the upper bound on $v(\Omega;\mathcal{P}_f(n))$ obtained above, we can choose n sufficiently large so that

- The n_j cuts that cover J_j lie in $\widehat{L}_j$.
- Each n_j is large enough so that the bounds in Lemma 3.5.6 hold.

Thus, invoking Lemma 3.5.6 and using (3.64), we have that

$$\begin{aligned} \text{vol}(\Omega \setminus P_n) \ge \sum_{j=1}^m \text{vol}\left(\widehat{L}_j \setminus P_n\right) &\ge \sum_{j=1}^m v(\Omega;\mathcal{P}_{n_j}(J_j \subset L_j; f)) \\ &\ge (1-\varepsilon)^{31}\, l_{\text{kor}} \sum_{j=1}^m \frac{\lambda(q)^{\frac{1}{2}}s(J_j)^{\frac{3}{2}}}{\sqrt{n_j}} \\ &\ge (1-\varepsilon)^{33}\, l_{\text{kor}} \sum_{j=1}^m \frac{\lambda(q)^{\frac{1}{2}}s(L_j)^{\frac{3}{2}}}{\sqrt{n_j}}. \end{aligned}$$

Now, Hölder's inequality gives

$$\sum_{j=1}^m \lambda(q)^{\frac{1}{3}}s(L_j) = \sum_{j=1}^m \left(\frac{\lambda(q)s(L_j)^3}{n_j}\right)^{\frac{1}{3}} n_j^{\frac{1}{3}} \le \left(\sum_{j=1}^m \frac{\lambda(q)^{\frac{1}{2}}s(L_j)^{\frac{3}{2}}}{\sqrt{n_j}}\right)^{\frac{2}{3}} \left(\sum_{j=1}^m n_j\right)^{\frac{1}{3}}$$

Thus, using one of the estimates in (3.63),

$$\begin{aligned} \mathrm{vol}(\Omega \setminus P_n) &\geq (1-\varepsilon)^{33}\, l_{\mathrm{kor}} \left(\sum_{j=1}^{m} \lambda(q)^{\frac{1}{3}} s(L_j) \right)^{\frac{3}{2}} \frac{1}{\sqrt{n_1 + \cdots + n_m}} \\ &\geq (1-\varepsilon)^{35}\, l_{\mathrm{kor}} \left(\frac{1}{4^{1/3}} \int_{\partial\Omega} \sigma_\Omega \right)^{\frac{3}{2}} \frac{1}{\sqrt{n}}. \end{aligned}$$

By our choice of P_n,

$$v(\Omega; \mathcal{P}_n(f)) \geq (1-\varepsilon)^{36}\, \frac{1}{2}\, l_{\mathrm{kor}} \left(\int_{\partial\Omega} \sigma_\Omega \right)^{\frac{3}{2}} \frac{1}{\sqrt{n}}, \tag{3.70}$$

for all n sufficiently large.

Finally, we combine (3.70) and (3.69), and recall that $\varepsilon \in (0, 1/3)$ was arbitrary, to declare the proof of Theorem 3.1.1 complete.

□

Proof of Corollary 3.1.4. Let ρ be a $\mathcal{C}^4$-smooth strictly plurisubharmonic defining function of Ω. A Henkin-Ramirez generating map enjoys the following properties (see [25, Prop. 3.1] for a complete description):

1. $\mathfrak{g}$ is defined and $\mathcal{C}^3$-smooth on some neighborhood of $\overline{\Omega} \times \partial\Omega$;
2. $\mathfrak{g}(\cdot, w)$ is holomorphic in Ω for each $w \in \partial\Omega$;
3. for $(z, w) \in \overline{\Omega} \times \partial\Omega$, $\mathfrak{g}(z, w) = 0$ if, and only if, $z = w$; and
4. there is a $\tau > 0$ and a function $a \in \mathcal{C}^3(\Omega_\tau)$ with $|a| \geq \frac{2}{3}$ so that $\mathfrak{g} = a\mathfrak{p}$ on Ω_τ, where $\mathfrak{p}$ is the Levi polynomial of ρ.

This is precisely the set-up needed to invoke Theorem 3.1.1.

□

Proof of Corollary 3.1.6. By Theorem 2 in [11], there is a $\tau > 0$ and a non-zero $a \in \mathcal{C}(\partial\Omega)$, such that

$$K_\Omega(z, w) = \frac{a(w)}{\mathfrak{p}(z, w)^3} + O\big(\mathfrak{p}(z, w)^{-\nu}\big), \ \ \nu \in (0, 3), \tag{3.71}$$

on Ω_τ, where $\mathfrak{p}$ is the Levi polynomial of some strictly plurisubharmonic defining function of Ω. One would like to apply Theorem 3.1.1 to $f = K_\Omega^{-1}$. As K_Ω may vanish when $(z, w) \notin \Omega_\tau$, we use a cut-off function (see Remark 3.3.3) to obtain a $\mathfrak{K} \in \mathcal{C}(\overline{\Omega} \times \partial\Omega)$ such that $\mathfrak{K} = 0$ precisely on the set $\{(z, w) : z = w \in \partial\Omega\}$ and $\mathfrak{K} = K_\Omega^{-1}$ on Ω_τ. Then, there is an $m > 0$, such that for n sufficiently large,

$$\{z \in \Omega : |K_\Omega(z, w^j)| < m_j,\ j = 1, ..., n\} = \{z \in \Omega : |\mathfrak{K}(z, w^j)| > 1/m_j,\ j = 1, ..., n\} \\ \in \mathcal{P}_n(\mathfrak{K}),$$

where $w^1, ..., w^n \in \partial\Omega$ and $m_1, ..., m_n > m$. But, by Lemma 3.3.1, if n is sufficiently large,

$$\inf\{\text{vol}(\Omega \setminus P) : P \in \mathcal{P}_n(\mathfrak{K})) = \inf\{\text{vol}(\Omega \setminus P) : P \in \mathcal{P}_n(\mathfrak{K}), \delta(P) < 1/m\}.$$

Thus, $\inf\{\text{vol}(\Omega \setminus P : P \in \mathcal{BP}_n\} \leq v(\Omega; \mathcal{P}_n(\mathfrak{K}))$. The reverse inequality follows from a similar argument. As Theorem 3.1.1 applies to $\mathfrak{K}$-polyhedra (due to (3.71)), the claimed asymptotic result holds.

Alternately, we can avoid constructing $\mathfrak{K}$ by observing that the statement and proof of Theorem 3.1.1 are not adversely affected if we allow f to be a $\mathbb{P}^1$-valued function.

□

Proof of Corollary 3.1.7. This proof follows along the same lines as the proof of Corollary 3.1.6, with (3.71) replaced by the following formula (which can be deduced from Boutet de Monvel and Sjöstrand's formulae in [8]):

$$S_\Omega(z, w) = \frac{a(z, w)}{\mathfrak{p}(z, w)^2} + O\big(\mathfrak{p}(z, w)^{-\nu}\big), \quad \nu \in (0, 2), \tag{3.72}$$

on Ω_τ, where $\mathfrak{p}$ is the Levi polynomial of some strictly plurisubharmonic defining function of Ω.

□

3.7 Some Remarks

Although the techniques used in the proof of Theorem 3.1.1 are exclusive to strongly pseudoconvex domains, we suspect that the result can be generalized to a larger class of domains. As evidence, we mention three situations for which Corollary 3.1.6 holds.

- Suppose $\Omega \subset \mathbb{C}^2$ is a smooth domain which is strongly pseudoconvex at all but m points in $\partial\Omega$. Further, suppose that $K_\Omega : \overline{\Omega} \times \overline{\Omega} \to \mathbb{P}^1$ is a continuous function that takes the value ∞ precisely on the diagonal of $\partial\Omega \times \partial\Omega$. Let $\varepsilon \in (0, 1/3)$, and $\mathcal{C}(\varepsilon)$ be a collection of m disjoint K_Ω-cuts that contain a weakly pseudoconvex point each, and $\text{vol}(C) < \varepsilon/m^{3/2}$ for each $C \in \mathcal{C}(\varepsilon)$. Let $\partial\Omega(\varepsilon) := \partial\Omega \setminus \bigcup_{\mathcal{C}(\varepsilon)}(C \cap \partial\Omega)$. We construct a tiling of $\partial\Omega(\varepsilon)$ as in Section 3.6. Repeating the computations in Section 3.6, (3.69) and (3.70) yield

$$(1-\varepsilon)^{36}\,\frac{1}{2}\,l_{\mathrm{kor}}\left(\int_{\partial\Omega(\varepsilon)}\sigma_\Omega\right)^{\frac{3}{2}}\frac{1}{\sqrt{n+m}}$$
$$\leq v(\Omega;\mathcal{BP}_{n+m})\leq(1-\varepsilon)^{-24}\,l_{\mathrm{kor}}\left(\int_{\partial\Omega(\varepsilon)}\sigma_\Omega\right)^{\frac{3}{2}}\frac{1}{\sqrt{n}}+\frac{\varepsilon}{\sqrt{m}},\quad n \text{ large.}$$

Applying the Cauchy-Schwartz inequality, and shrinking ε to zero, we get the result.

- $\Omega=\{(z_1,z_2)\in\mathbb{C}^2:|z_1|^2+|z_2|^{2p}<1\}$, $p>1$. The locus of weakly pseudoconvex points is the curve $\mathcal{W}=\{(\exp(i\theta),0):\theta\in[0,2\pi)\}$. As in the previous example, it suffices to cover $\mathcal{W}$ by n cuts with total volume at most $O(\varepsilon)/\sqrt{n}$. The Bergman kernel of Ω (see [10]) is

$$K_\Omega(z,w)=\frac{2}{p\pi^2}\sum_{j=1}^{2}c_j\frac{(1-z_1\overline{w_1})^{-2+\frac{j}{p}}}{\left((1-z_1\overline{w_1})^{\frac{1}{p}}-(z_2\overline{w_2})^2\right)^{1+j}},\quad c_1=(p-1)/2, c_2=1.$$

  Thus, $C=\{z\in\Omega:|K_\Omega(z,(1,0))|>m\}=\left\{z\in\Omega:|z-1|<cm^{-\frac{p}{2p+1}}\right\}$, for some constant c depending only on p. Now, $\mathrm{vol}(C)\sim m^{-1}$, and using the symmetry $(z_1,z_2)\mapsto(\exp(i\theta)z_1,z_2)$, we can cover $\mathcal{W}$ by $O\left(\lceil m^{p/(2p+1)}\rceil\right)$ many such cuts. Hence, the claim.

- $\Omega=\mathbb{D}^2$. Then,

$$\lim_{n\to\infty}\sqrt{n}\,\inf\{\mathrm{vol}(\Omega\setminus P:P\in\mathcal{BP}_n\}=0.$$

  Although, Ω is non-smooth, since its boundary is Levi-flat almost everywhere, we can interpret $\int_{\partial\Omega}\sigma_\Omega$ as zero (see [4, Sec. 4] for more evidence).

Thus, for a smooth domain, it is reasonable to ask whether a control on the size of the locus of weakly pseudoconvex points is enough to grant (3.1)-type results. For non-smooth domains, the existence of the limit on the left-hand side of (3.1) is of interest.

3.8 Power Diagrams in the Heisenberg Group

3.8.1 The Euclidean Plane

Let $T(a;r) \subset \mathbb{R}^2$ be a circle of radius r centered at $a \in \mathbb{R}^2$. The *power* of a point $z = (x,y) \in \mathbb{R}^2$ with respect to $T = T(a;r)$ is the function

$$\mathrm{pow}(z,T) = |z-a|^2 - r^2.$$

Note that if z is outside the disk bounded by T, then $\mathrm{pow}(z,T)$ is the square of the length of a line segment from P to a point of tangency with T. Thus, it is a generalized distance between z and T. For a collection, $\mathscr{T}$, of circles in the plane, the *power diagram* or *Laguerre diagram* of $\mathscr{T}$ is the collection of all

$$\mathrm{cell}(T) = \{z \in \mathbb{R}^2 : \mathrm{pow}(T,z) < \mathrm{pow}(T^*,z), \forall T^* \in \mathscr{T} \setminus \{T\}\},\ T \in \mathscr{T}.$$

If $\mathscr{T}$ consists of equiradial circles, the power diagram reduces to the Dirichlet-Voronoi diagram of the centers of the circles. In general, the power diagram of any $\mathscr{T}$ gives a convex tiling of the plane.

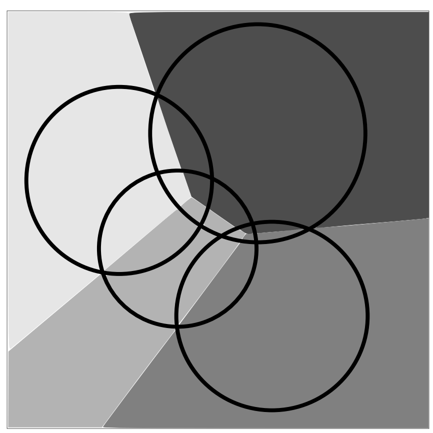

Figure 3.2: A power diagram in the plane.

Power diagrams occur naturally and have found several applications (see [3], for instance). From the point of view of polyhedral approximations, power diagrams (in $\mathbb{R}^{d-1}$) are intimately related to the constant ldiv_{d-1} in (2.5) (see [24] and [21] for explicit details).

3.8.2 The Heisenberg Group

Let $G(0;\delta) = \{z' \in \mathbb{H} : |z_1|^4 + (x_2)^2 < \delta^4\}$ be a Korányi sphere in $\mathbb{H}$ (see (3.19)). We define the *horizontal power* of a point $z' \in \mathbb{H}$ with respect to $G = G(0;\delta)$ as

$$\mathrm{hpow}(G, z') = \begin{cases} |z_1|^2 - \sqrt{\delta^4 - (x_2)^2}, & \text{if } |x_2|^2 \leq \delta; \\ \infty, & \text{otherwise.} \end{cases}$$

Note that $G_c := G \cap \{x_2 = c\}$ is a (possibly empty) circle in the $\{x_2 = c\}$ plane, and $\mathrm{hpow}(G, (z_1, x_2)) = \mathrm{pow}(G_{x_2}, z_1)$, where the right-hand side — being a generalized distance — is set as ∞ when G_{x_2} is empty. hpow is then extended to all Korányi spheres to be left-invariant under $\cdot_{\mathbb{H}}$ (defined in Section 3.4). For a collection $\mathscr{G}$ of Korányi spheres in $\mathbb{H}$, let

$$K_{\mathscr{G}} := \bigcup_{\partial K \in \mathscr{G}} K,$$

i.e., the union of all the Korányi balls bounded by the spheres in $\mathscr{G}$. We define the *horizontal power diagram* of $\mathscr{G}$ to be the collection of all

$$\Delta(G) = \big\{z' \in K_{\mathscr{G}} : \mathrm{hpow}(G, z') < \mathrm{hpow}(G^*, z'), \forall G^* \in \mathscr{G} \setminus \{G\}\big\},\ G \in \mathscr{G}.$$

Then, $\Delta(G) \subset$ the Korányi ball bounded by G, for all $G \in \mathscr{G}$.

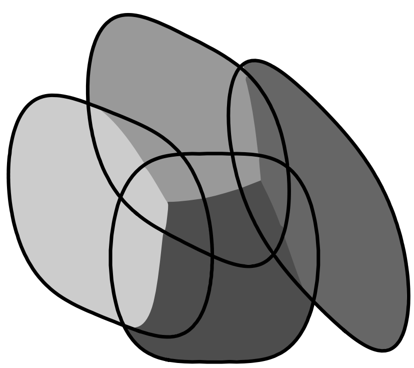

Figure 3.3: A $\{x_1 = 0\}$-slice of a horizontal power diagram in $\mathbb{H}$.

We now give two reasons why this concept is useful for us. Let

$$\begin{aligned} &\mathrm{dil}_\xi : (z_1, x_2) \mapsto (\xi z_1, \xi^2 x_2), \\ &\mathrm{dil}_{w',\xi} : z' \mapsto w' \cdot_{\mathbb{H}} \mathrm{dil}_\xi(-w' \cdot_{\mathbb{H}} z') \end{aligned}$$

be the dilations in $\mathbb{H}$ centered at the origin and w', respectively. Then,

1. $\operatorname{dil}_{w',\xi}(G(w',\delta)) = G(w',\xi\delta)$,

2. $\operatorname{hpow}(G(w',\delta), \operatorname{dil}_{w',\xi}(z')) = \xi^2 \operatorname{hpow}(G(w',\xi^{-1}\delta), z')$, and

3. if $\mathscr{G}$ is given by the center-radius pairs $\{(a_1,\delta_1),...,(a_m,\delta_m)\}$, then, $\operatorname{dil}_{a_j,\xi}\Delta\big(G(a_j;\delta_j)\big) \cap \operatorname{dil}_{a_k,\xi}\Delta\big(G(a_k;\delta_k)\big) = \emptyset$, for all $1 \leq j \neq k \leq m$ and $\xi \leq 1$.

Now, consider the Siegel domain $\mathcal{S}$ and the function $f_{\mathcal{S}}$ studied in Section 3.4. The cuts of any $f_{\mathcal{S}}$-polyhedron P over $J \subset \partial S$ project to a collection of Korányi balls in $\mathbb{C}\times\mathbb{R}$ that form a covering of J'. The (open) facets of P project to the horizontal power diagram of the corresponding set of spheres $\mathscr{G}_P$. This perspective facilitates the proof of

Lemma 3.8.1. *The cuts of $f_{\mathcal{S}_\lambda}$, $\lambda > 0$, are Jordan measurable and satisfy the doubling property ⊛ for any $\delta_{f_{\mathcal{S}_\lambda}} > 0$ and $\mathcal{E}(t) = (1+t)^3$.*

Proof. The Jordan measurability of the cuts is obvious. Now, without loss of generality, we may assume $\lambda = 1$ (the map $(z,w) \mapsto (\lambda z, \lambda w)$ can be used to handle the other cases). Let $H \subset \partial\mathcal{S}$ be a compact set, $\{w^j\}_{1\leq j\leq m} \subset H$, $\{\delta_j\}_{1\leq j\leq m} \subset (0,\infty)$ and $t > 0$. For $j = 1,...,m$, let

$$C_j(t) := C(w_j, (1+t)\delta_j; f_{\mathcal{S}}),$$
$$v^j = (w^j)' = (w_1^j, u_2^j),$$

and (see (3.19))

$$K_j(t) := C_j(t)' = K\left(v^j; \sqrt{(1+t)\delta_j}\right).$$

Consider $\mathscr{G} = \{\partial K_j(t) : 1 \leq j \leq m\}$ and the corresponding horizontal power diagram $\{\Delta_j(t) = \Delta(\partial K_j(t)) : 1 \leq j \leq m\}$. Then, setting $dz' = dx_1 dy_1 dx_2$, we have, by a change of variables and (1), (2) and (3) above, that

$$\begin{aligned}
&\operatorname{vol}\left(\bigcup_{j=1}^{m} C_j(t)\right)\\
=\;& \int\limits_{\cup_{j=1}^m K_j(t)} \max_{1\le j\le m}\left\{\operatorname{Re}\sqrt{\delta_j^2-(x_2-u_2^j+2\operatorname{Im} z_1\overline{w_1}^j)-|z_1-w_1^j|^2}\right\}dz'\\
=\;& \int\limits_{\cup_{j=1}^m K_j(t)} \max_{1\le j\le m}\{-\operatorname{hpow}(\partial K_j(t),z')\}dz'\\
=\;& -\sum_{j=1}^{m}\int\limits_{\Delta_j(t)} \operatorname{hpow}(\partial K_j(t),z')dz'\\
=\;& -(1+t)^2\sum_{j=1}^{m}\int_{\operatorname{dil}_{v^j,\frac{1}{\sqrt{1+t}}}(\Delta_j(t))} \operatorname{hpow}\left(\partial K_j(t),\operatorname{dil}_{v^j,\sqrt{1+t}}(\zeta)\right)d\zeta\\
=\;& -(1+t)^3\sum_{j=1}^{m}\int_{\operatorname{dil}_{v^j,\frac{1}{\sqrt{1+t}}}(\Delta_j(t))} \operatorname{hpow}\left(\partial K_j(0),\zeta\right)d\zeta\\
\le\;& (1+t)^3\int_{\cup_{j=1}^m K_j(0)} \max\left\{-\operatorname{hpow}\left(\partial K_j(0),\zeta\right):1\le j\le m\right\}d\zeta\\
=\;& (1+t)^3\operatorname{vol}\left(\bigcup_{j=1}^{m} C_j(0)\right),\ \forall t\ge 0.
\end{aligned}$$

□

The computations in the above proof also show that

$$l_{\mathrm{kor}}=\lim_{n\to\infty}\inf\left\{-\sum_{G\in\mathscr{G}}\int_{\Delta(G)}\operatorname{hpow}(G,z')dz' : I\subset K_{\mathscr{G}},\ \#(\mathscr{G})\le n\right\},$$

where I is the unit square in $\mathbb{C}\times\mathbb{R}$ (see Section 3.4). Our proof of Lemma 3.4.1 yields bounds for l_{kor} as follows:

$$0.0003\approx\frac{4\sqrt{2}}{\pi^2 3^7}\le l_{\mathrm{kor}}\le\frac{5\sqrt{5}\pi}{3\sqrt{2}}\approx 8.2788.$$

It would be interesting to know if computations, similar to the ones carried out by Böröczky and Ludwig in [21] for ldiv_2, can be done to find the exact value of l_{kor}.

CHAPTER 4

Hausdorff-Fefferman Measures

In the previous chapter, we characterized the Fefferman measure on the boundary of a strongly pseudoconvex domain in $\mathbb{C}^2$ in terms of its Bergman kernel. Our results therein rely on the knowledge of off-diagonal estimates on the Bergman kernel, which are generally harder to obtain than on-diagonal estimates. With this in view, we discuss another construction involving the (diagonal) Bergman kernel that generates a blueprint for Fefferman-type measures on more general domains.

4.1 The Hausdorff-Fefferman dimension

Let $\Omega \subset \mathbb{C}^d$ be a bounded domain with Bergman kernel K_Ω. In this section, we will abuse notation to denote $K_\Omega(z,z)$ by $K_\Omega(z)$ for $z \in \Omega$. We propose the following definition:

Definition 4.1.1. For $M > 0$, let $\Omega_M := \{z \in \Omega : K_\Omega(z) > M\}$. Suppose

$$\sup\{\alpha > 0 : \liminf_{M\to\infty} M^{\frac{1}{\alpha}} \operatorname{vol}(\Omega_M) = \infty\} = \inf\{\alpha > 0 : \limsup_{M\to\infty} M^{\frac{1}{\alpha}} \operatorname{vol}(\Omega_M) = 0\}. \tag{4.1}$$

Then, the *Hausdorff-Fefferman dimension* of Ω, $\dim_{\mathrm{HF}}(\Omega)$, is said to exist and is defined as either of the quantities in (4.1). Note that, by definition, $\dim_{\mathrm{HF}}(\Omega) > 0$.

Hereafter, unless stated otherwise, our domains admit a Hausdorff-Fefferman dimension. We first collect some simple facts about $\dim_{\mathrm{HF}}$.

Proposition 4.1.2. *Let $\Omega \subset \mathbb{C}^d$ be a $\mathcal{C}^1$-smooth bounded domain. Then,* $\dim_{\mathrm{HF}}(\Omega) \leq d+1$.

Proof. Let $z \in \Omega$ and $\operatorname{dist}(z, \partial\Omega)$ denote the Euclidean distance of z from $\partial\Omega$. This proof relies on the well-known inequality

$$K_\Omega(z) \leq \frac{\text{const.}}{\operatorname{dist}(z,\partial\Omega)^{d+1}}, \text{ for all } z \in \Omega,$$

which is obtained by rolling a ball of fixed radius in Ω along $\partial\Omega$. Thus, $\{z \in \Omega : K_\Omega(z) > M\} \subseteq \{z \in \Omega : \mathrm{dist}(z, \partial\Omega) < (\text{const.})M^{1/(d+1)}\}$. The regularity assumption on Ω yields

$$\mathrm{vol}(\Omega_M) \leq \mathrm{vol}\{z \in \Omega : \mathrm{dist}(z, \partial\Omega) < (\text{const.})M^{1/(d+1)}\} \sim \frac{1}{M^{1/(d+1)}} \quad \text{as } M \to \infty.$$

Hence, the claim. □

Proposition 4.1.3. *Let $\Omega^j \subset \mathbb{C}^{d_j}$ be a bounded domain such that $\lim_{z\to w} K_{\Omega^j}(z) = \infty$, for all $w \in \partial\Omega^j$, $j = 1, ..., k$. Then, $\dim_{\mathrm{HF}}(\Omega^1 \times \cdots \times \Omega^k) \geq \max\{\dim_{\mathrm{HF}}(\Omega^j) : 1 \leq j \leq k\}$.*

Proof. Let $k = 2$. It is known that $K_{\Omega^1\times\Omega^2}\big((z, w)\big) = K_{\Omega^1}(z)K_{\Omega^2}(w)$. We may, thus, write

$$(\Omega^1 \times \Omega^2)_M = \bigcup_{w\in\Omega^2} \left\{(z, w) : z \in \Omega^1_{M/K_{\Omega_2}(w)}\right\} \supset \bigcup_{w\in\Omega^2} \left\{(z, w) : z \in \Omega^1_{M/k_2}\right\},$$

where $k_2 := \min_{w\in\overline{\Omega^2}} K_{\Omega_2}(w)$. Thus, for all $\alpha > 0$,

$$M^{1/\alpha}\, \mathrm{vol}\left((\Omega^1 \times \Omega^2)_M\right) \geq M^{1/\alpha}\, \mathrm{vol}\left(\Omega^2\right) \mathrm{vol}\left(\Omega^1_{M/k_2}\right).$$

As k_2 and $\mathrm{vol}(\Omega^2)$ are independent of M,

$$\left\{\alpha : \limsup_{M\to\infty} M^{\frac{1}{\alpha}}\, \mathrm{vol}\left((\Omega^1 \times \Omega^2)_M\right) = 0\right\} \subseteq \left\{\alpha : \limsup_{M\to\infty} M^{\frac{1}{\alpha}}\, \mathrm{vol}(\Omega^1_M) = 0\right\}.$$

Repeating the argument with Ω^2_M instead, we get that $\dim_{\mathrm{HF}}(\Omega^1\times\Omega^2) \geq \max\{\dim_{\mathrm{HF}}(\Omega^j) : j = 1, 2\}$. The argument for general $k \in \mathbb{N}_+$ follows from the fact that if $\Omega^1, ..., \Omega^k$ satisfy the hypothesis of the proposition, then so do $\Omega^1 \times \cdots \times \Omega^{k-1}$ and Ω^k. □

Proposition 4.1.4. *Let $F : \Omega^1 \to \Omega^2$ be a biholomorphism such that $a \leq |\det(\mathrm{J}_{\mathbb{C}}F)| \leq b$ for some $a, b > 0$. If Ω^1 admits a Hausdorff-Fefferman dimension, then so does Ω^2, and $\dim_{\mathrm{HF}}(\Omega^2) = \dim_{\mathrm{HF}}(\Omega^1)$.*

Proof. Let $K_j(z) := K_{\Omega^j}(z)$ for $z \in \Omega^j$, $j = 1, 2$. Observe that

$$\begin{aligned} F^{-1}(\Omega^2_M) = \{F^{-1}(w) \in \Omega^1 : K_2(w) > M\} &= \{z \in \Omega^1 : K_2(F(z)) > M\} \\ &= \{z \in \Omega^1 : K_1(z) > M|\det \mathrm{J}_{\mathbb{C}}F(z)|^2\} \\ &\subseteq \{z \in \Omega^1 : K_1(z) > Ma^2\}. \end{aligned} \tag{4.2}$$

Therefore,

$$\begin{aligned}\mathrm{vol}(\Omega^2_M) &= \int_{F^{-1}(\Omega^2_M)} |\det \mathrm{J}_{\mathbb{C}} F(z)|^2 \,\omega_{\mathbb{C}^d}(z) \\ &\leq \int_{\Omega^1_{Ma^2}} |\det \mathrm{J}_{\mathbb{C}} F(z)|^2 \,\omega_{\mathbb{C}^d}(z) \leq b^2 \,\mathrm{vol}(\Omega^1_{Ma^2}). \qquad (4.3)\end{aligned}$$

As a and b are independent of M, we get that $\dim_{\mathrm{HF}}(\Omega^2) \leq \dim_{\mathrm{HF}}(\Omega^1)$. The reverse inequality also holds as $F^{-1} : \Omega^2 \to \Omega^1$ satisfies the hypothesis of the claim. □

We will now use known estimates and formulas for the Bergman kernel to compute the Hausdorff-Fefferman dimension of some domains.

Example 2. Let $\Omega \subset \mathbb{C}^d$ be a $\mathcal{C}^1$-smooth domain such that $\overline{\partial} : L^2_{(0,0)}(\Omega) \to L^2_{(0,1)}(\Omega)$ has closed range. Further, let $p \in \partial\Omega$ be such that $\partial\Omega$ is $\mathcal{C}^2$-smooth in a neighborhood of p and $\partial\Omega$ is strongly pseudoconvex at p. Then, $\dim_{\mathrm{HF}}(\Omega) = d + 1$.

Proof. As proved in Proposition 4.1.2, $\dim_{\mathrm{HF}}(\Omega) \leq d + 1$. By Hörmander's theorem on the boundary behavior of the (diagonal) Bergman kernel (see Theorem 3.5.1 in [18]), there exists a neigborhood $U \subset \partial\Omega$ of p and a continuous function $f : U \to \mathbb{R}$ such that

$$\mathrm{dist}(z, \partial\Omega)^{d+1} K_\Omega(z) \to f(z_0), \quad z \to z_0 \in U.$$

Thus, for any $V \Subset U$, there is a $c > 0$, such that $\{z \in \Omega : K_\Omega(z) > M\} \supseteq \{z \in \Omega : \mathrm{dist}(z, V) < cM^{1/(d+1)}\}$. We get,

$$\mathrm{vol}(\Omega_M) \geq \frac{c' s(V)}{M^{1/(d+1)}},$$

where $c' > 0$ is a constant and $s(V)$ is the Euclidean surface area of V. This gives the required lower bound on $\dim_{\mathrm{HF}}(\Omega)$. □

Example 3. Let $\mathbb{B}^d$ be the unit ball in $\mathbb{C}^d$. If $\Omega \subset \mathbb{C}^k$ is a domain such that

(a) Ω is *Bergman exhaustive*, i.e., $\lim_{z\to w} K_\Omega(z) = +\infty$, for all $w \in \partial\Omega$, and

(b) $\int_{\Omega\setminus\Omega_M} \sqrt{K_\Omega(z)}\, \omega_{\mathbb{C}^d}(z) = o(M^\eta)$ as $M \to \infty$, for every $\eta > 0$,

then $\dim_{\mathrm{HF}}(\mathbb{B}^d \times \Omega) = \max\{d + 1, \dim_{\mathrm{HF}}(\Omega)\}$.

Remark 4.1.5. An elementary example of a domain that satisfies condition (b) is $\mathbb{B}^d$, $d \geq 1$. Moreover, if $\Omega^j \subset \mathbb{C}^{d_j}$, $j = 1, ..., k$, are domains that satisfy conditions (a) and (b) in Example 3, then so does $\Omega^1 \times \cdots \times \Omega^k$. Thus, in particular, $\dim_{\mathrm{HF}}(\mathbb{B}^{d_1} \times \cdots \mathbb{B}^{d_l}) = \max_{1\leq j\leq l}\{d_j + 1\}$.

Proof. We observe that for $\mathfrak{b}_d = \text{vol}(\mathbb{B}^d)$,

$$K_{\mathbb{B}^d \times \Omega}\big((z,w)\big) = \frac{1}{\mathfrak{b}_d(1-||z||^2)^{d+1}} K_\Omega(w).$$

Thus, we may write

$$\big(\mathbb{B}^d \times \Omega\big)_M = \{(z,w) : z \in \mathbb{B}^d, w \in \Omega_{M\mathfrak{b}_d}\} \bigcup \{(z,w) : z \in (\mathbb{B}^d)_{M/K_\Omega(w)}, w \in \Omega \backslash \Omega_{M\mathfrak{b}_d}\}. \tag{4.4}$$

Now, fix an $\alpha > \max\{d+1, \dim_{\text{HF}}(\Omega)\}$ and let $\eta = \frac{1}{d+1} - \frac{1}{\alpha}$. Then, by the definition of $\dim_{\text{HF}}$ and the hypothesis on Ω, given $\varepsilon > 0$, there is an $M_\varepsilon > 0$ such that $\text{vol}(\Omega_{M\mathfrak{b}_d}) < \varepsilon M^{-1/\alpha}$ and $\int_{\Omega\backslash\Omega_{M\mathfrak{b}_d}} K_\Omega(w)^{1/(d+1)}\, \omega_{\mathbb{C}^d}(w) \leq \int_{\Omega\backslash\Omega_{M\mathfrak{b}_d}} \sqrt{K_\Omega(w)}\, \omega_{\mathbb{C}^d}(w) < \varepsilon M^\eta$, for all $M \geq M_\varepsilon$. Using the decomposition in (4.4) and the fact that $\text{vol}(\mathbb{B}^d_M) \leq C_d/M^{1/(d+1)}$ for some dimensional constant C_d, we get

$$\begin{aligned}
\text{vol}\left((\mathbb{B}^d \times \Omega)_M\right) &= \text{vol}(\mathbb{B}^d)\,\text{vol}(\Omega_{M\mathfrak{b}_d}) + \int_{\Omega\backslash\Omega_{M\mathfrak{b}_d}} \text{vol}\left(\mathbb{B}^d_{M/K_\Omega(w)}\right)\, \omega_{\mathbb{C}^d}(w) \\
&\leq \text{vol}(\mathbb{B}^d)\frac{\varepsilon}{M^{1/\alpha}} + \frac{C_d}{M^{1/(d+1)}} \int_{\Omega\backslash\Omega_{M\mathfrak{b}_d}} K_\Omega(w)^{1/(d+1)}\, \omega_{\mathbb{C}^d}(w) \\
&< \left(\text{vol}(\mathbb{B}^d) + C_d\right) \frac{\varepsilon}{M^{1/\alpha}},
\end{aligned}$$

for $M \geq M_\varepsilon$. Thus, $\dim_{\text{HF}}(\mathbb{B}^d \times \Omega) \leq \alpha$ for all $\alpha > \max\{d+1, \dim_{\text{HF}}(\Omega)\}$. The lower bound follows from Proposition 4.1.3. □

Remark 4.1.6. To see how the Hausdorff-Fefferman dimension distinguishes domains, we observe that in $\mathbb{C}^3$, $\mathbb{B}^3$, $\mathbb{B}^1 \times \mathbb{B}^2$ and $\mathbb{B}^1 \times \mathbb{B}^1 \times \mathbb{B}^1$ have Hausdorff-Fefferman dimensions 4, 3 and 2, respectively.

4.2 Hausdorff-Fefferman gauge functions and measures

In analogy with Hausdorff measures, we would like to use the Hausdorff-Fefferman dimension of Ω to construct Fefferman-type measures on $\partial\Omega$. Under such a scheme, the total measure of $\partial\Omega$ would be $\lim_{M\to\infty} M^{1/\dim_{\text{HF}}(\Omega)}\, \text{vol}(\Omega_M)$. However, the following example shows why this can fail to yield anything meaningful even for some simple domains.

Example 4. Let $\Omega = \mathbb{D} \times \mathbb{D} \subset \mathbb{C}^2$. By Remark 4.1.5, we know that $\dim_{\text{HF}}(\Omega) = 2$. Now, Ω_M is the disjoint union of $\{(z,w) \in \mathbb{D} \times \mathbb{D} : K_{\mathbb{D}}(w) > M\pi\}$ and $\{(z,w) \in \mathbb{D} \times \mathbb{D} :$

$K_{\mathbb{D}}(w) \le M\pi, K_{\mathbb{D}}(z) > M/K_{\mathbb{D}}(w)\}$. Therefore, for $M > 1$,

$$\begin{aligned}\operatorname{vol}(\Omega_M) &= \operatorname{vol}(\mathbb{D}_{M\pi})\operatorname{vol}(\mathbb{D}) + \int_{|w|^2 \le 1-\frac{1}{\sqrt{M\pi}}} \frac{1}{\sqrt{M}(1-|w|^2)}\,\omega_{\mathbb{C}^d}(w) \\ &= \frac{\pi^2}{\sqrt{M\pi}} + \frac{2\pi}{\sqrt{M}}\int_0^{\sqrt{1-1/\sqrt{M\pi}}} \frac{r}{1-r^2}dr = \frac{\pi^2}{\sqrt{M\pi}} + \frac{\pi\ln\sqrt{M\pi}}{\sqrt{M}}.\end{aligned}$$

Thus, $\lim_{M\to\infty} M^{\frac{1}{2}}\operatorname{vol}(\Omega_M) = \infty$.

In view of the logarithmic term seen in Example 4, we expand the notion of the Hausdorff-Fefferman dimension in the following manner.

Definition 4.2.1. Let $\Omega \subset \mathbb{C}^d$ be a bounded domain. Any increasing $d_\Omega \in \mathcal{C}((0,\infty))$ is called a *Hausdorff-Fefferman gauge function (or an* HF*-gauge function) of* Ω if

$$\lim_{M\to\infty} d_\Omega(M)\operatorname{vol}(\Omega_M) \text{ exists, and is positive and finite.}$$

Remark. One can always choose $d_\Omega(M) = \operatorname{vol}(\Omega_M)^{-1}$, but it is preferable to find a $d_\Omega(M)$ that comes from local qualitative data at the boundary.

Definition 4.2.2. Let Ω and d_Ω be as in Definition 4.2.1, and $\omega_{\mathbb{C}^d}$ be viewed as a measure on $\overline{\Omega}$. The *Hausdorff-Fefferman measure* on $\partial\Omega$ (corresponding to d_Ω) is defined as

$$\widetilde{\sigma}_\Omega(A) := \text{ weak-}* \text{ limit of } d_\Omega(M)\chi_{\Omega_M}\omega_{\mathbb{C}^d} \text{ as } M \to \infty,$$

when it exists, where χ_A denotes the indicator function of A.

Remark. The weak-$*$ limit above is in the space $C(\overline{\Omega})^*$ — the space of bounded linear functionals on $\mathcal{C}(\overline{\Omega})$. By the Riesz representation theorem, $\widetilde{\sigma}_\Omega$ is a finite, positive, regular, Borel measure on $\overline{\Omega}$ — in fact, the support of $\widetilde{\sigma}_\Omega$ is contained in $\partial\Omega$, but may be strictly smaller, as we will see later.

We will now work with domains that admit a Hasudorff-Fefferman measure on their boundaries. The next couple of propositions justify our nomenclature for $\widetilde{\sigma}_\Omega$.

Proposition 4.2.3. *Here $\approx_{\text{const.}}$ denotes equality up to a constant factor, and all volume and hypersurface forms are viewed as measures.*

1. *If $\Omega \subset\subset \mathbb{C}^d$ is a strongly pseudoconvex domain, then for $d_\Omega(M) = M^{1/(d+1)}$, $\widetilde{\sigma}_{\partial\Omega} \approx_{\text{const.}} \sigma_{\partial\Omega}$.*

2. *If* $\Omega = \mathbb{B}^d \times \mathbb{B}^d$*, then for* $d_\Omega(M) = \frac{M^{1/(d+1)}}{\ln(M)}$*,* $\widetilde{\sigma}_{\partial\Omega}$ *is supported on* $\partial\mathbb{B}^d \times \partial\mathbb{B}^d$ *and* $\widetilde{\sigma}_{\partial\Omega} \approx_{\text{const.}} s_{\mathbb{B}^d} \cdot s_{\mathbb{B}^d}$*, where* $s_{\mathbb{B}^d}$ *is the standard surface area on* $\partial\mathbb{B}^d$*.*

3. *If* $\Omega = \mathbb{B}^{d_1} \times \mathbb{B}^{d_2}$*, with* $d_1 > d_2$*, then for* $d_\Omega(M) = M^{1/(d_1+1)}$*,* $\widetilde{\sigma}_{\partial\Omega}$ *is supported on* $\partial\mathbb{B}^{d_1} \times \mathbb{B}^{d_2}$ *and* $\widetilde{\sigma}_{\partial\Omega} \approx_{\text{const.}} h \cdot s_{\mathbb{B}^{d_1}} \cdot \omega_{\mathbb{C}^{d_2}}$*, where* $h(z,w) = K_{\mathbb{B}^{d_2}}^{\frac{1}{d_1+1}}$*.*

Proof. Let $\Omega \subset\subset \mathbb{C}^d$ be strongly pseudoconvex. As the range of $\overline{\partial} : L^2_{(0,0)}(\Omega) \to L^2_{(0,1)}(\Omega)$ is closed, we have, by our computations in Example 2, that $d_\Omega(M) = M^{\frac{1}{d+1}}$ is indeed an HF-gauge function for Ω. To compute $\widetilde{\sigma}_\Omega$ with respect to this $d_\Omega(M)$, we recall Hormänder's estimate:

$$\lim_{z\to z_0 \in \partial\Omega} r(z)^{d+1} K_\Omega(z) = \frac{d!}{\pi^d} M[r](z_0), \quad \forall z_0 \in \partial\Omega,$$

where r is a $\mathcal{C}^2$-smooth defining function for Ω such that $r(z) = \operatorname{dist}(z, \partial\Omega)$ in some fixed neighbourhood of $\partial\Omega$ in Ω, and $M[r]$ is the Fefferman Monge-Ampére operator defined in Section 2.1. Thus, setting $n(z) := \left(\frac{M[r](z)}{\mathfrak{b}_d M}\right)^{\frac{1}{d+1}}$ and $\nu(z)$ to be the outward unit normal vector at $z \in \partial\Omega$, we have for any $f \in \mathcal{C}(\overline{\Omega})$, $\varepsilon > 0$, an M large enough so that

$$\begin{aligned}\{z - r\nu(z) \in \Omega : z \in \partial\Omega, r \in (0, n(z)(1-\varepsilon))\} &\subseteq \Omega_M \\ &\subseteq \{z - r\nu(z) \in \Omega : z \in \partial\Omega, r \in (0, n(z)(1+\varepsilon))\},\end{aligned}$$

and

$$|f(z - r\nu(z)) - f(z)| < \varepsilon, \quad \forall z \in \partial\Omega, r \in [0, n(z)(1+\varepsilon)].$$

We, therefore, obtain

$$\begin{aligned} d_\Omega(M) \int_{\Omega_M} f\, \omega_{\mathbb{C}^d} &< M^{\frac{1}{d+1}} \int_{\partial\Omega} (f(z) + \varepsilon)(n(z)(1+\varepsilon))\, s_\Omega \\ &= (4^d \mathfrak{b}_d)^{\frac{-1}{d+1}} (1+\varepsilon) \int_{\partial\Omega} (f(z)+\varepsilon)\, \sigma_\Omega(z). \end{aligned}$$

Similarly, $d_\Omega(M) \int_{\Omega_M} f\, \omega_{\mathbb{C}^d} > (4^d \mathfrak{b}_d)^{\frac{-1}{d+1}} (1-\varepsilon) \int_{\partial\Omega} (f(z) - \varepsilon)\, \sigma_\Omega(z)$. Therefore,

$$\widetilde{\sigma}_\Omega = (4^d \mathfrak{b}_d)^{\frac{-1}{d+1}} \sigma_\Omega \text{ (as measures).}$$

Next, suppose $\Omega = \mathbb{B}^{d_1} \times \mathbb{B}^{d_2}$, $d_1 \geq d_2$. To simplify exposition, we let $K_{d_j} := K_{\mathbb{B}^{d_j}}$, $j = 1, 2$. As in Example 3, we may write

$$\operatorname{vol}(\Omega_M) = \operatorname{vol}(\mathbb{B}^{d_1}) \operatorname{vol}(\mathbb{B}^{d_2}_{M\mathfrak{b}_{d_1}}) + \int_{\mathbb{B}^{d_2} \setminus (\mathbb{B}^{d_2})_{M\mathfrak{b}_{d_1}}} \operatorname{vol}\left(\mathbb{B}^{d_1}_{M/K_{d_2}(w)}\right) \omega_{\mathbb{C}^d}(w)$$

On expanding, we find that

$$\begin{aligned}\operatorname{vol}(\mathbb{B}^{d_1})\operatorname{vol}(\mathbb{B}^{d_2}_{M\mathfrak{b}_{d_1}}) &= \mathfrak{b}_{d_1}\mathfrak{b}_{d_2}\left(1-\left(1-(M\mathfrak{b}_{d_1}\mathfrak{b}_{d_2})^{-\frac{1}{(d_2+1)}}\right)^{d_2}\right)\\ &= \frac{d_2(\mathfrak{b}_{d_1}\mathfrak{b}_{d_2})^{d_2/(d_2+1)}}{M^{1/(d_2+1)}}+o(M^{-1/(d_2+1)}), \qquad (4.5)\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{B}^{d_2}\setminus(\mathbb{B}^{d_2})_{M\mathfrak{b}_{d_1}}}\operatorname{vol}\left(\mathbb{B}^{d_1}_{M/K_{d_2}(w)}\right)\omega_{\mathbb{C}^d}(w)\\ &= \mathfrak{b}_{d_1}\int_{\mathbb{B}^{d_2}\setminus(\mathbb{B}^{d_2})_{M\mathfrak{b}_{d_1}}}\left(1-\left(1-\left(\frac{K_{d_2}(w)}{\mathfrak{b}_{d_1}M}\right)^{\frac{1}{d_1+1}}\right)\right)^{d_1}\omega_{\mathbb{C}^d}(w)\\ &= \mathfrak{b}_{d_1}\sum_{r=1}^{d_1}(-1)^{r+1}\binom{d_1}{r}(\mathfrak{b}_{d_1}M)^{\frac{-r}{d_1+1}}I\left[M\mathfrak{b}_{d_1};d_1;d_2;r\right], \qquad (4.6)\end{aligned}$$

where

$$I[M;d_1;d_2;r] := \int_{\{w\in\mathbb{B}^{d_2}:K_{d_2}(w)\le M\}}K_{d_2}(w)^{\frac{r}{d_1+1}}\,\omega_{\mathbb{C}^d}(w).$$

Now, we observe that

$$\begin{aligned}I[M;d_1;d_2;r] &= (\mathfrak{b}_{d_2})^{\frac{-r}{d_1+1}}\int_{||w||^2\le 1-(\mathfrak{b}_{d_2}M)^{-1/(d_2+1)}}(1-||w||^2)^{\frac{-r(d_2+1)}{d_1+1}}\,\omega_{\mathbb{C}^d}(w)\\ &= (\mathfrak{b}_{d_2})^{\frac{-r}{d_1+1}}\left(2d_2\mathfrak{b}_{d_2}\int_0^{\sqrt{1-\frac{1}{(\mathfrak{b}_{d_2}M)^{1/(d_2+1)}}}}\frac{t^{2d_2-1}}{(1-t^2)^{\frac{r(d_2+1)}{d_1+1}}}dt\right)\\ &= \frac{d_2\mathfrak{b}_{d_2}}{(\mathfrak{b}_{d_2})^{\frac{r}{d_1+1}}}\beta\left[1-\frac{1}{(\mathfrak{b}_{d_2}M)^{1/(d_2+1)}};d_2,1-\frac{d_2+1}{d_1+1}r\right],\end{aligned}$$

where $\beta[z;a,b]$ is the incomplete beta function $\int_0^z t^{a-1}(1-t)^{b-1}dt$. Now, for $x\in(0,1)$, as $x\to 0$,

$$\beta[1-x;a,b] = \begin{cases} C_{a,b}\,x^{-b}+o(x^{-b}), & \text{if } b<0;\\ \ln\frac{1}{x}+C_a+O(x), & \text{if } b=0;\\ \beta(a,b)+O(x^b), & \text{if } 0<b<1,\end{cases}$$

where $C_{a,b}, C_a > 0$ are independent of x. Thus, as $M \to \infty$, $I[M; d_1; d_2; r]$

$$= \begin{cases} \widetilde{C}_{d_1,d_2,r}\, M^{\frac{1-(d_2+1)r/(d_1+1)}{d_2+1}} + o(M^{1-\frac{d_2+1}{d_1+1}r}), & \text{if } \frac{d_2+1}{d_1+1}r > 1; \\ \frac{d_2}{d_2+1}(\mathfrak{b}_{d_2})^{d_2/(d_2+1)} \ln M + \widetilde{C}_{d_2} + O(M^{-1/(d_1+1)}), & \text{if } \frac{d_2+1}{d_1+1}r = 1; \\ \frac{d_2 \mathfrak{b}_{d_2}}{(\mathfrak{b}_{d_2})^{\frac{r}{d_1+1}}} \beta\left(d_2, 1 - \frac{d_2+1}{d_1+1}r\right) + O(M^{1-\frac{d_2+1}{d_1+1}r}), & \text{if } \frac{d_2+1}{d_1+1}r \in (0,1), \end{cases} \tag{4.7}$$

where $\widetilde{C}_{d_1,d_2,r}, \widetilde{C}_{d_2} > 0$ are independent of M.

Our goal is to determine the asymptotic behavior of the sum in (4.6), as $M \to \infty$.

Case i. $d_1 = d_2$. When $r > 1$, $\frac{d_2+1}{d_1+1}r = r > 1$. Thus, from (4.7),

$$M^{-r/d_1+1} I\left[M\mathfrak{b}_{d_1}; d_1; d_2; r\right] \sim \frac{1}{M^{\frac{2r-1}{d_1+1}}} = o(M^{-1/(d_1+1)}), \quad \text{as } M \to \infty. \tag{4.8}$$

On the other hand, when $r = 1$, we get that

$$M^{-r/d_1+1} I\left[M\mathfrak{b}_{d_1}; d_1; d_2; r\right] \sim \frac{\ln(M)}{M^{\frac{1}{d_1+1}}}, \quad \text{as } M \to \infty. \tag{4.9}$$

Combining (4.5), (4.6), (4.8) and (4.9), we get that $d_\Omega(M) := \frac{M^{1/(d+1)}}{\ln(M)}$ is an HF-gauge function for $\Omega = \mathbb{B}^d \times \mathbb{B}^d$, and collecting the various constants,

$$\lim_{M\to\infty} d_\Omega(M)\,\mathrm{vol}(\Omega_M) = \frac{d^2}{d+1}(\mathfrak{b}_d)^{\frac{2d}{d+1}}.$$

Next, we compute $\widetilde{\sigma}_\Omega$ with respect to $d_\Omega = \frac{M^{1/(d+1)}}{\ln(M)}$. For $\eta \in (0,1)$, let

$$\begin{aligned} R_\eta &:= \{(z,w) \in \mathbb{B}^d \times \mathbb{B}^d : \min\{||z||, ||w||\} > \eta\}; \\ |R|_{\eta,M} &:= \{(|z|,|w|) \in \mathbb{R}^2 : (z,w) \in \Omega_M \cap R_\eta\}. \end{aligned}$$

Due to rotational symmetry in each variable, $\mathrm{vol}(\Omega_M \cap R_\eta) = (2d\mathfrak{b}_d)^2\,\mathrm{vol}(|R|_{\eta,M})$. Now, for a fixed η, when $M > \mathfrak{b}_d^{-2}(1-\eta^2)^{-2d-2}$,

$$\begin{aligned} \mathrm{vol}\left(\Omega_M \setminus R_\eta\right) &= 2\mathfrak{b}_d \int_{\{||w||<\eta\}} \left(1 - \left(1 - \frac{1}{(\mathfrak{b}_d^2 M)^{1/(d+1)}(1-||w||^2)}\right)^d\right) \omega_{\mathbb{C}^d}(w) \\ &\sim M^{\frac{-1}{d+1}} \qquad \text{as } M \to \infty. \end{aligned}$$

Therefore, for any $f \in \mathcal{C}(\overline{\Omega})$ and $\eta \in (0,1)$,

$$\lim_{M\to\infty} d_\Omega(M) \int_{\Omega_M} f\, \omega_{\mathbb{C}^d} = \lim_{M\to\infty} d_\Omega(M) \int_{\Omega_M \cap R_\eta} f\, \omega_{\mathbb{C}^d}.$$

In particular, $\lim_{M\to\infty} d_\Omega(M)\operatorname{vol}(\Omega_M) = \lim_{M\to\infty} d_\Omega(M)\operatorname{vol}(\Omega_M \cap R_\eta)$.

Now, fix an $\varepsilon > 0$. Then, for η close enough to 1, we have that

$$|f(r\theta, r'\theta') - f(\theta,\theta')| < \varepsilon \quad \text{for any } r, r' \in (\eta, 1) \text{ and } \theta, \theta' \in \partial\mathbb{B}^d.$$

Therefore,

$$\begin{aligned} d_\Omega(M) \int_{\Omega_M \cap R_\eta} f\, \omega_{\mathbb{C}^d} &= d_\Omega(M) \int_{|R|_{\eta,M}} \int_{\partial\mathbb{B}^d} \int_{\partial\mathbb{B}^d} f(r\theta, r'\theta')(rr')^{2d-1} s_{\mathbb{B}^d}(\theta) s_{\mathbb{B}^d}(\theta') dr dr' \\ &< \left(\varepsilon + \int_{\partial\mathbb{B}^d\times\partial\mathbb{B}^d} f(\theta,\theta') s_{\mathbb{B}^d}(\theta) s_{\mathbb{B}^d}(\theta')\right) d_\Omega(M)\operatorname{vol}(|R|_{\eta,M}) \\ &= \left(\int_{(\partial\mathbb{B}^d)^2} f\, s_{\mathbb{B}^d} s_{\mathbb{B}^d} + \varepsilon\right) d_\Omega(M) \frac{\operatorname{vol}(\Omega_M \cap R_\eta)}{(2d\mathfrak{b}_d)^2}. \end{aligned}$$

Similarly,

$$d_\Omega(M) \int_{\Omega_M \cap R_\eta} f\, \omega_{\mathbb{C}^d} > \left(\int_{(\partial\mathbb{B}^d)^2} f\, s_{\mathbb{B}^d} s_{\mathbb{B}^d} - \varepsilon\right) (1-\eta)^{2d-1} d_\Omega(M) \frac{\operatorname{vol}(\Omega_M \cap R_\eta)}{(2d\mathfrak{b}_d)^2}.$$

Thus,

$$\widetilde{\sigma}_{\mathbb{B}^d\times\mathbb{B}^d} = \frac{d^2}{d+1} \frac{(\mathfrak{b}_d)^{\frac{2d}{d+1}}}{(2d\mathfrak{b}_d)^2} s_{\mathbb{B}^d} s_{\mathbb{B}^d} = \frac{(\mathfrak{b}_d)^{\frac{-2}{d+1}}}{4(d+1)} s_{\mathbb{B}^d} s_{\mathbb{B}^d} \text{ (as measures).}$$

Case ii. $d_1 > d_2$. If $\frac{d_2+1}{d_1+1} r \geq 1$, then $r > 1$. Thus, by (4.7), we have, as $M \to \infty$,

$$\frac{I\,[M\mathfrak{b}_{d_1}; d_1; d_2; r]}{M^{r/d_1+1}} \sim \begin{cases} \frac{1}{M^{\frac{1-(d_2+1)r/(d_1+1)}{d_2+1} + \frac{r}{d_1+1}}} = o(M^{-1/(d_1+1)}), & \text{if } \frac{d_2+1}{d_1+1} r > 1; \\ \frac{\ln(M)}{M^{r/d_1+1}} = o(M^{-1/(d_1+1)}), & \text{if } \frac{d_2+1}{d_1+1} r = 1. \end{cases} \tag{4.10}$$

On the other hand, if $\frac{d_2+1}{d_1+1} r < 1$, we get,

$$\frac{I\,[M\mathfrak{b}_{d_1}; d_1; d_2; r]}{M^{r/d_1+1}} = \begin{cases} O(M^{-r/(d_1+1)}) = o(M^{-1/(d_1+1)}), & \text{if } r > 1; \\ \frac{d_2 \mathfrak{b}_{d_2}}{(\mathfrak{b}_{d_2})^{\frac{r}{d_1+1}}} \frac{\beta\left(d_2, 1 - \frac{d_2+1}{d_1+1} r\right)}{M^{1/(d_1+1)}} + o(M^{-1/(d_1+1)}), & \text{if } r = 1. \end{cases} \tag{4.11}$$

Once again, combining (4.5), (4.6), (4.10) and (4.11), we conclude that $d_\Omega(M) = \frac{1}{M^{1/(d_1+1)}}$

acts as an HF-gauge function for $\Omega = \mathbb{B}^{d_1} \times \mathbb{B}^{d_2}$ as long as $d_2 < d_1$. Moreover,

$$\lim_{M\to\infty} d_\Omega(M)\,\mathrm{vol}(\Omega_M) = d_1 d_2 (\mathfrak{b}_{d_1})^{\frac{d_1}{d_1+1}} (\mathfrak{b}_{d_2})^{\frac{d_2}{d_2+1}} \beta\left(d_2, 1 - \frac{d_2+1}{d_1+1}\right).$$

In order to compute $\widetilde{\sigma}_\Omega$, we set, for an $\eta \in (0,1)$,

$$\begin{aligned} A_\eta &:= \{(z,w) \in \mathbb{B}^{d_1} \times \mathbb{B}^{d_2} : ||z|| > \eta\}; \\ |A|_{\eta,M}(w) &:= \{|z| \in \mathbb{R} : (z,w) \in \Omega_M \cap A_\eta\}. \end{aligned}$$

We record the fact that for $w \in \mathbb{B}^{d_2}$,

$$\begin{aligned} \mathfrak{I}(|A|_{\eta,M}(w)) &:= \int_{|A|_{\eta,M}(w)} r^{2d_1-1} dr \qquad (4.12) \\ &= \begin{cases} \frac{1}{2d_1}\left(1 - \left(1 - \left(\frac{K_{d_2}(w)}{\mathfrak{b}_{d_1} M}\right)^{\frac{1}{d_1+1}}\right)^{d_1}\right), & ||w||^2 \le 1 - \frac{(\mathfrak{b}_{d_1}\mathfrak{b}_{d_2})^{\frac{-1}{d_2+1}}}{(M(1-\eta^2)^{d_1+1})^{\frac{1}{d_2+1}}}; \\ \frac{1}{2d_1}(1-\eta^{2d_1}), & \text{otherwise}. \end{cases} \end{aligned}$$

Now, for a fixed $\eta \in (0,1)$,

$$\begin{aligned} \mathrm{vol}(\Omega_M \setminus A_\eta) &= \mathfrak{b}_{d_2} \int_{\{||z|| \le \eta\}} \left(1 - \left(1 - \frac{1}{(\mathfrak{b}_{d_1}\mathfrak{b}_{d_2} M (1-||z||^2)^{d_1+1})^{\frac{1}{d_2+1}}}\right)^{d_2}\right) \omega_{\mathbb{C}^d}(z) \\ &\sim M^{\frac{-1}{d_2+1}} \qquad \text{as } M \to \infty. \end{aligned}$$

Therefore, for any $f \in \mathcal{C}(\overline{\Omega})$ and $\eta \in (0,1)$,

$$\lim_{M\to\infty} d_\Omega(M) \int_{\Omega_M} f\, \omega_{\mathbb{C}^d} = \lim_{M\to\infty} d_\Omega(M) \int_{\Omega_M \cap A_\eta} f\, \omega_{\mathbb{C}^d}.$$

In particular, $\lim_{M\to\infty} d_\Omega(M)\,\mathrm{vol}(\Omega_M) = \lim_{M\to\infty} d_\Omega(M)\,\mathrm{vol}(\Omega_M \cap A_\eta)$. Now, for any fixed $\varepsilon > 0$, we may choose η close enough to 1, so that

$$|f(r\theta, w) - f(\theta, w)| < \varepsilon \quad \text{for any } r \in (\eta, 1),\ \theta \in \partial\mathbb{B}^{d_1} \text{ and } w \in \mathbb{B}^{d_2}.$$

Hence, for a fixed η and large M,

$$\begin{aligned} d_\Omega(M) \int_{\Omega_M \cap A_\eta} f\, \omega_{\mathbb{C}^d} &= d_\Omega(M) \int_{\mathbb{B}^{d_2}} \int_{|A|_{\eta,M}(w)} \int_{\partial\mathbb{B}^{d_1}} f(r\theta, w) s_{\mathbb{B}^d}(\theta) r^{2d_1-1} dr\, \omega_{\mathbb{C}^d}(w) \\ &< M^{\frac{1}{d_1+1}} \int_{\mathbb{B}^{d_2}} \left(\int_{\partial\mathbb{B}^{d_1}} (\varepsilon + f(\theta, w)) s_{\mathbb{B}^d}(\theta)\right) \mathfrak{I}(|A|_{\eta,M}(w))\, \omega_{\mathbb{C}^d}(w). \end{aligned}$$

Here, we write $\mathbb{B}^{d_2}$ as $B_1 \cup B_2$, where $B_1 = \{w \in \mathbb{C}^{d_2} : ||w|| \leq 1 - (\mathfrak{b}_{d_1}\mathfrak{b}_{d_2}M(1-\eta^2)^{d_1+1})^{\frac{-1}{d_2+1}}\}$ and $B_2 = \mathbb{B}^{d_2} \setminus B_1$. Then, as $\mathfrak{I}(|A|_{\eta,M}(w)) \equiv \frac{1}{2d_1}(1-\eta^{2d_1})$ on B_1 (see (4.12)), for any fixed function h continuous in w,

$$\int_{B_1} h(w)\mathfrak{I}(|A|_{\eta,M}(w))\, \omega_{\mathbb{C}^d}(w) = O(M^{-1/(d_2+1)}).$$

On the other hand,

$$\begin{aligned}
&\int_{B_2} h(w)\mathfrak{I}(|A|_{\eta,M}(w))\, \omega_{\mathbb{C}^d}(w) \\
&\quad = \int_{B_2} h(w)\frac{1}{2d_1}\left(1 - \left(1 - \left(\frac{K_{d_2}(w)}{\mathfrak{b}_{d_1}M}\right)^{\frac{1}{d_1+1}}\right)^{d_1}\right)\, \omega_{\mathbb{C}^d}(w) \\
&\quad = \frac{1}{2}\left(\mathfrak{b}_{d_1}M\right)^{\frac{-1}{d_1+1}} \int_{B_2} h(w)K_{d_2}(w)^{\frac{1}{d_1+1}}\, \omega_{\mathbb{C}^d}(w) + o(M^{\frac{-1}{d_1+1}}).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\lim_{M\to\infty}\left(M^{\frac{1}{d_1+1}}\int_{\Omega_M\cap A_\eta} f\, \omega_{\mathbb{C}^d}\right) \\
&\qquad < \frac{1}{2}\left(\mathfrak{b}_{d_1}M\right)^{\frac{-1}{d_1+1}}\int_{\mathbb{B}^{d_2}}\int_{\partial\mathbb{B}^{d_1}} (\varepsilon + f(\theta,w))K_{d_2}(w)^{\frac{1}{d_1+1}} s_{\mathbb{B}^d}(\theta)\, \omega_{\mathbb{C}^d}(w).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\lim_{M\to\infty}\left(M^{\frac{1}{d_1+1}}\int_{\Omega_M\cap A_\eta} f\, \omega_{\mathbb{C}^d}\right) \\
&\qquad > \frac{1}{2}\left(\mathfrak{b}_{d_1}M\right)^{\frac{-1}{d_1+1}}\int_{\mathbb{B}^{d_2}}\int_{\partial\mathbb{B}^{d_1}} (f(\theta,w) - \varepsilon)K_{d_2}(w)^{\frac{1}{d_1+1}}\, s_{\mathbb{B}^d}(\theta)\, \omega_{\mathbb{C}^d}(w).
\end{aligned}$$

We must now note that $K_{d_2}^{\frac{1}{d_1+1}} \approx_{\text{const.}} (1-||w||^2)^{-\frac{d_2+1}{d_1+1}}$ is integrable on $\mathbb{B}^{d_2}$. Thus, we can let $\eta \to 1$, to obtain that

$$\widetilde{\sigma}_\Omega = \frac{1}{2}\left(\mathfrak{b}_{d_1}\right)^{\frac{-1}{d_1+1}} K_{d_2}^{\frac{1}{d_1+1}}\, s_{\mathbb{B}^d}\, \omega_{\mathbb{C}^d} \text{ (as measures).}$$

□

Remarks 4.2.4. The above computations can be extended to show that if $\Omega = \mathbb{B}^{d_1} \times \cdots \times \mathbb{B}^{d_k}$, where $d_1 = \cdots = d_r > d_{r+1} \geq d_{r+2} \geq \cdots d_k$, then for $d_\Omega(M) = M^{1/(d_1+1)}\ln(M)^{1-r}$, $\widetilde{\sigma}_\Omega$ is supported on $(\partial\mathbb{B}^d)^r \times \mathbb{B}^{d_{r+1}} \cdots \times \mathbb{B}^{d_k}$ and $\widetilde{\sigma}_\Omega \approx_{\text{const.}} h_{r+1}\cdots h_k \cdot (s_{\mathbb{B}^{d_1}})^r \cdot \omega_{\mathbb{C}^{d_{r+1}}}\cdots\omega_{\mathbb{C}^{d_k}}$, where $h_j(z_1,...,z_k) = K_{d_j}^{\frac{1}{d_1+1}}(z_j)$.

We now present a transfomation law for the measures constructed in this section. The extra hypotheses in the statement of our result help us avoid domains whose HF-gauge functions have (long-term) oscillatory behavior.

Proposition 4.2.5. *Let $\Omega^1, \Omega^2 \subset\subset \mathbb{C}^d$ be domains, and $F : \Omega^1 \to \Omega^2$ a biholomorphism such that $F \in \mathcal{C}^1(\overline{\Omega^1})$ and $\mathrm{J}_{\mathbb{C}}F$ is non-vanishing. Suppose*

(i) $\delta := \dim_{\mathrm{HF}}(\Omega^1) < \infty$*;*

(ii) *for any* $a > 0$, $\liminf_{M\to\infty} \dfrac{\mathrm{vol}(\Omega^1_M)}{\mathrm{vol}(\Omega^1_{aM})} = \limsup_{M\to\infty} \dfrac{\mathrm{vol}(\Omega^1_M)}{\mathrm{vol}(\Omega^1_{aM})}$*; and*

(iii) $\liminf_{M\to\infty} \dfrac{\mathrm{vol}(\Omega^1_M)}{\mathrm{vol}(\Omega^2_M)} = \limsup_{M\to\infty} \dfrac{\mathrm{vol}(\Omega^1_M)}{\mathrm{vol}(\Omega^2_M)}$.

Then,

$$F^*\widetilde{\sigma}_{\Omega^2} \approx_{\text{const.}} |\det \mathrm{J}_{\mathbb{C}}F|^{2\left(1-\frac{1}{\delta}\right)}\widetilde{\sigma}_{\Omega^1},$$

where the constant implicit in $\approx_{\text{const.}}$ depends on the choice of HF-gauge functions for Ω^1 and Ω^2.

We isolate a lemma that indicates the necessity of conditions (i) and (ii) in Proposition 4.2.5.

Lemma 4.2.6. *Let $\Omega \subset\subset \mathbb{C}^d$ be such that $\delta := \dim_{\mathrm{HF}}(\Omega) \in (0,\infty)$ and condition (ii) of Propostion 4.2.5 holds. Then, for any $a > 0$,*

$$\lim_{M\to\infty} \frac{\mathrm{vol}(\Omega_M)}{\mathrm{vol}(\Omega_{Ma})} = a^{\frac{1}{\delta}}.$$

Proof. Set $h(M) := M^{1/\delta}\,\mathrm{vol}(\Omega_M)$. Note that

$$\ell_a := \lim_{M\to\infty} \frac{h(M)}{h(aM)} = \lim_{M\to\infty} \frac{M^{\frac{1}{\delta}}\,\mathrm{vol}(\Omega_M)}{(aM)^{\frac{1}{\delta}}\,\mathrm{vol}(\Omega_{aM})} = a^{-\frac{1}{\delta}} \lim_{M\to\infty} \frac{\mathrm{vol}(\Omega_M)}{\mathrm{vol}(\Omega_{Ma})}.$$

Thus, $\ell_a \in [0,\infty]$, by condition (ii).

Now, by the definition of $\dim_{\mathrm{HF}}$ and d_Ω, we know that for any $\varepsilon > 0$,

$$\begin{aligned} \lim_{M\to\infty} M^{\frac{1}{\delta-\varepsilon}}\,\mathrm{vol}(\Omega_M) &= \infty \\ \lim_{M\to\infty} M^{\frac{1}{\delta+\varepsilon}}\,\mathrm{vol}(\Omega_M) &= 0. \end{aligned}$$

Therefore,

$$\lim_{M\to\infty} M^{\frac{\varepsilon}{(\delta-\varepsilon)\delta}} h(M) = \infty \text{ and } \lim_{M\to\infty} M^{\frac{-\varepsilon}{(\delta+\varepsilon)\delta}} h(M) = 0. \tag{4.13}$$

Now, fix an $a > 1$. Suppose $\ell_a > 1$. Then, there is an $s > 0$ and an $M > 0$, such that $h(M') > a^s h(aM')$ for all $M' \geq M$. Therefore, the sequence $\{s_j := (a^j M)^s h(a^j M)\}_{j \in \mathbb{N}_+}$ is a strictly decreasing sequence of positive numbers that converges to ∞ (see the first part of (4.13)). This is a contradiction.

If $\ell_a < 1$, then, once again, for some $s > 0$ and $M > 0$, $h(M') < a^{-s} h(aM')$ for all $M' \geq M$. Therefore, the sequence $\{t_j := (a^j M)^{-s} h(a^j M)\}_{j \in \mathbb{N}_+}$ is a strictly increasing sequence of positive numbers that converges to 0 (the second part of (4.13) is invoked here). This, too, is a contradiction. Therefore, $\ell_a = 1$ when $a > 1$. When $a < 1$, we simply note that $\ell_a = 1/\ell_{\frac{1}{a}} = 1$, since $1/a > 1$. $\square$

Proof of Proposition 4.2.5. Fix $d_j := d_{\Omega^j}$ — a choice of HF-gauge function for Ω_j, $j = 1, 2$. We first show that $\lim_{M \to \infty} d_1(M)/d_2(M)$ exists and lies in $(0, \infty)$. For this, observe that by the condition on F, we can find $a, b > 0$ such that $a \leq |\det(\mathrm{J}_{\mathbb{C}} F)| \leq b$. Thus, by Proposition 4.1.4, $\dim_{\mathrm{HF}}(\Omega^2) = \delta$. We set $h_j(M) := M^{1/\delta} \operatorname{vol}(\Omega^j_M)$. Then,

$$\frac{d_1(M)}{d_2(M)} = \frac{d_1(M) \operatorname{vol}(\Omega^1_M)}{d_2(M) \operatorname{vol}(\Omega^2_M)} \times \frac{\operatorname{vol}(\Omega^2_M)}{\operatorname{vol}(\Omega^1_M)}. \tag{4.14}$$

By definition, $\lim_{M \to \infty} d_j(M) \operatorname{vol}(\Omega^j_M) \in (0, \infty)$. So, it suffices to show that $\lim_{M \to \infty} \frac{\operatorname{vol}(\Omega^2_M)}{\operatorname{vol}(\Omega^1_M)}$ is non-zero and finite (see condition (iii) for existence). Now, from the proof of Proposition 4.1.4 (see (4.3), in particular) we get

$$a^2 \operatorname{vol}(\Omega^1_{Mb^2}) \leq \operatorname{vol}(\Omega^2_M) \leq b^2 \operatorname{vol}(\Omega^1_{Ma^2}), \quad M \in (0, \infty).$$

Thus,

$$a^2 \frac{\operatorname{vol}(\Omega^1_{Mb^2})}{\operatorname{vol}(\Omega^1_M)} \leq \frac{\operatorname{vol}(\Omega^2_M)}{\operatorname{vol}(\Omega^1_M)} \leq b^2 \frac{\operatorname{vol}(\Omega^1_{Ma^2})}{\operatorname{vol}(\Omega^1_M)}, \quad M \in (0, \infty). \tag{4.15}$$

Thus, by Lemma 4.2.6, we have that $\operatorname{vol}(\Omega^2_M)/\operatorname{vol}(\Omega^1_M)$ is bounded above and below as $M \to \infty$. Combining (4.14), (4.15) and (iii),

$$L := \lim_{M \to \infty} \frac{d_2(M)}{d_1(M)} \text{ exists and is in } (0, \infty). \tag{4.16}$$

Now, in order to prove the transformation law, we first show that $F^* \widetilde{\sigma}_{\Omega^2} << \widetilde{\sigma}_{\Omega^1}$. For this, we set

$$\sigma^j_M := d_j(M) \chi_{\Omega^j_M} \omega_{\mathbb{C}^d}, \qquad j = 1, 2.$$

We also recall that if a bounded family of positive Borel measures $\{\ell_M\}_{M > 0}$ on a metric space X converges weakly to a finite positive measure σ on X, then

$$\lim_{M\to\infty} \ell_M(C) = \sigma(C) \text{ for every continuity set } C \text{ — i.e., } \sigma(\partial C) = 0 \text{ — of } X. \tag{4.17}$$

Now, let $A \subset \overline{\Omega^1}$ be such that $\widetilde{\sigma}_{\Omega^1}(A) = 0$, and $\varepsilon > 0$. By the sparseness of discontinuity sets (see [30, Page 7]) and the regularity of $\widetilde{\sigma}_{\Omega^1}$, we can find open sets V_ε in $\overline{\Omega^1}$ containing A such that $\widetilde{\sigma}_{\Omega^1}(V_\varepsilon) < \varepsilon$, and V_ε are continuity sets for $\widetilde{\sigma}_{\Omega^1}$ and $F^*\widetilde{\sigma}_{\Omega^2}$. By (4.17),

$$\lim_{M\to\infty} \ell^1_M(V_\varepsilon) = \widetilde{\sigma}_{\Omega^1}(V_\varepsilon) < \varepsilon.$$

By (4.2) in the proof of Proposition 4.1.4, we observe that

$$F^{-1}(F(V_\varepsilon) \cap \Omega^2_M) \subset V_\varepsilon \cap \Omega^1_{Ma^2}.$$

Hence,

$$\begin{aligned} F^*\sigma^2_M(V_\varepsilon) &\leq b^2 \frac{d_2(M)}{d_1(Ma^2)} \sigma^1_{Ma^2}(V_\varepsilon) \\ &= b^2 \frac{d_2(M)}{d_1(M)} \frac{d_1(M)\operatorname{vol}(\Omega^1_M)}{d_1(Ma^2)\operatorname{vol}(\Omega^1_{Ma^2})} \frac{\operatorname{vol}(\Omega^1_{Ma^2})}{\operatorname{vol}(\Omega^1_M)} \sigma^1_{Ma^2}(V_\varepsilon). \end{aligned}$$

As $d_2(M)/d_1(M)$, $d_1(M)\operatorname{vol}(\Omega^1_M)$ and $\operatorname{vol}(\Omega^2_{Ma^2})/\operatorname{vol}(\Omega^1_M)$ all admit finite, non-zero limits as $M \to \infty$, we get that $F^*\sigma^2_M(V_\varepsilon) < c\varepsilon$ for large enough M, and some constant $c > 0$ independent of ε and M. By (4.17), $F^*\widetilde{\sigma}_{\Omega^2}(V_\varepsilon) = \lim_{m\to\infty} F^*\sigma^2_M(V_\varepsilon) < c\varepsilon$. By outer regularity, $F^*\widetilde{\sigma}_{\Omega^2}(A) = 0$.

In view of the Radon-Nikodym theorem, our conclusion above shows that there exists a $\widetilde{\sigma}_{\Omega^1}$-measurable function G on $\partial\Omega^1$ such that $F^*(\widetilde{\sigma}_{\Omega^2}) = G \cdot \widetilde{\sigma}_{\Omega^1}$ on $\partial\Omega^1$. Let $x_0 \in \partial\Omega^1$. By the sparseness of discontinuity sets, we may find a decreasing sequence of neighborhoods U_ε of x_0 that are continuity sets with respect to both $\widetilde{\sigma}_{\Omega^1}$ and $F^*\widetilde{\sigma}_{\Omega^2}$ and satisfy

$$|\det \mathrm{J}_{\mathbb{C}}F(z) - \det \mathrm{J}_{\mathbb{C}}F(x_0)| < \varepsilon \; \forall x \in U_\varepsilon.$$

Now, we observe that

$$\begin{aligned} F^{-1}(\Omega^2_M \cap F(U_\varepsilon)) &= \{F^{-1}(w) \in \Omega^1 \cap U_\varepsilon : K_2(w) > M\} \\ &= \{z \in \Omega^1 \cap U_\varepsilon : K_2(F(z)) > M\} \\ &= \{z \in \Omega^1 \cap U_\varepsilon : K_1(z) > M|\det \mathrm{J}_{\mathbb{C}}F(z)|^2\} \\ &\subseteq \{z \in \Omega^1 \cap U_\varepsilon : K_1(z) > M(|\det \mathrm{J}_{\mathbb{C}}F(x_0)| - \varepsilon)^2\}. \end{aligned}$$

As in (4.3), we get that

$$F^*\sigma^2_M(U_\varepsilon) \le (|\det \mathrm{J}_{\mathbb{C}}F(x_0)|+\varepsilon)^2 \frac{d_2(M)}{d_1(M(|\det \mathrm{J}_{\mathbb{C}}F(x_0)|-\varepsilon)^2)} \sigma^1_{M(|\det \mathrm{J}_{\mathbb{C}}F(x_0)|-\varepsilon)^2}(U_\varepsilon). \tag{4.18}$$

In a similar manner, we get

$$F^*\sigma^2_M(U_\varepsilon) \ge (|\det \mathrm{J}_{\mathbb{C}}F(x_0)|-\varepsilon)^2 \frac{d_2(M)}{d_1(M(|\det \mathrm{J}_{\mathbb{C}}F(x_0)|+\varepsilon)^2)} \sigma^1_{M(|\det \mathrm{J}_{\mathbb{C}}F(x_0)|+\varepsilon)^2}(U_\varepsilon). \tag{4.19}$$

Now, taking limits as $M \to \infty$ on both sides of (4.18) and (4.19), and observing that

$$\lim_{M\to\infty} \frac{d_2(M)}{d_1(cM)} = \lim_{M\to\infty} \left(\frac{d_2(M)}{d_1(M)} \frac{d_1(M)\,\mathrm{vol}(\Omega^1_M)}{d_1(cM)\,\mathrm{vol}(\Omega^1_{cM})} \frac{\mathrm{vol}(\Omega^1_{cM})}{\mathrm{vol}(\Omega^1_M)} \right) = c^{-1/\delta} L,$$

due to (4.16), the defining property of d_1 and Lemma 4.2.6, we get that

$$L\left(\frac{|\det \mathrm{J}_{\mathbb{C}}F(x_0)|-\varepsilon}{(|\det \mathrm{J}_{\mathbb{C}}F(x_0)|+\varepsilon)^{-1/\delta}} \right)^2 \le \frac{F^*\widetilde{\sigma}_{\Omega^2}(U_\varepsilon)}{\widetilde{\sigma}_{\Omega_1}(U_\varepsilon)} \le L\left(\frac{|\det \mathrm{J}_{\mathbb{C}}F(x_0)|+\varepsilon}{(|\det \mathrm{J}_{\mathbb{C}}F(x_0)|-\varepsilon)^{-1/\delta}} \right)^2.$$

Therefore, as $\varepsilon \to 0$, we get that $\frac{F^*\widetilde{\sigma}_{\Omega^2}}{\widetilde{\sigma}_{\Omega_1}} = L|\det \mathrm{J}_{\mathbb{C}}F|^{2\left(1-\frac{1}{\delta}\right)}$ almost everywhere with respect to $\widetilde{\sigma}_{\Omega^1}$, where $L = \lim\limits_{M\to\infty} \frac{d_2(M)}{d_1(M)}$. □

Remark 4.2.7. If F is a constant-Jacobian biholomorphism, then condition (iii) follows from conditions (i) and (ii), as can be deduced from (4.15).

4.3 Further directions: Hausdorff-Fefferman Hardy spaces

Let $\Omega \subset \mathbb{C}^d$ be a bounded domain. We fix an HF-gauge function d_Ω of Ω. For any function ϕ on Ω, $M' > M > 0$, set

$$\begin{aligned} \ell_M(\phi) &:= d_\Omega(M) \int_{\Omega_M} \phi\, \omega_{\mathbb{C}^d}; \\ \ell_{M,M'}(\phi) &:= d_\Omega(M) \int_{\Omega_M \setminus \Omega_{M'}} \phi\, \omega_{\mathbb{C}^d}. \end{aligned}$$

Based on our constructions in this chapter, we propose two notions of Hausdorff-Fefferman Hardy spaces.

Definition 4.3.1. We define the Hardy-Smirnov HF-space of Ω as follows:

$$\mathrm{EHF}^2(\Omega) \quad := \quad \left\{ f \in \mathcal{O}(\Omega) : ||f||^2_{\mathrm{EHF}^2} := \limsup_{M\to\infty} \ell_M(|f|^2) < \infty \right\}.$$

Let $\mathcal{A}(\Omega)$ denote the space of holomorphic functions in Ω that are continuous up to the boundary. For the next definition, we identify the elements of $\mathcal{A}(\Omega)$ with their boundary values.

Definition 4.3.2. The Hardy HF-space of Ω, $\mathrm{HF}^2(\Omega)$, is defined to be the closure of $\mathcal{A}(\Omega)$ in $L^2(\partial\Omega, \widetilde{\sigma}_\Omega)$.

Remark 4.3.3. $\mathrm{EHF}^2(\Omega)$ is a normed vector space and $\mathrm{HF}^2(\Omega)$ is a Hilbert space. Both these spaces contain $A(\Omega)$, and

$$||f||^2_{\mathrm{EHF}^2} = \int_{\partial\Omega} |f|^2 d\widetilde{\sigma}_\Omega = ||f||^2_{\mathrm{HF}^2} \quad \forall f \in \mathcal{A}(\Omega).$$

We present a scenario in which these spaces are comparable and produce a reproducing kernel with desirable transformation properties under biholomorphisms.
Claim. Suppose $\Omega \subset \mathbb{C}^d$ is a bounded domain (such that $\widetilde{\sigma}_\Omega$ exists), for which

1. there is a $c > 0$ so that $\ell_M(|f|^2) \leq c||f||^2_{\mathrm{EHF}^2}$ for all $M > 0$ and $f \in \mathrm{EHF}^2(\Omega)$.

2. for any compact $K \subset \Omega$, there is a $K > 0$ so that

$$\sup\{|f(z)| : z \in K\} \leq C_K ||f||_{\mathrm{EHF}^2}, \ \forall f \in \mathrm{EHF}^2(\Omega).$$

Then,

(i) $\mathrm{EHF}^2(\Omega)$ is a Banach space.

(ii) There is a unique $S_\Omega : \Omega \times \partial\Omega \to \mathbb{C}$ such that

- $\overline{S_\Omega(z, \cdot)} \in \mathrm{HF}^2(\Omega)$ for all $z \in \Omega$.
- $f(z) = \int_{\partial\Omega} f(w) S_\Omega(z, w) d\widetilde{\sigma}_\Omega(w)$ for all $f \in \mathrm{HF}^2(\Omega)$.

(iii) Under the hypotheses of Proposition 4.2.5, and the assumption that there is a well-defined branch of $(\det \mathrm{J}_{\mathbb{C}} F(z))^{\frac{\delta-1}{\delta}}$ in $\mathcal{A}(\Omega_1)$, we have that

$$(\det \mathrm{J}_{\mathbb{C}} F(z))^{\frac{\delta-1}{\delta}} S_{\Omega_2}(F(z), F(w)) \overline{(\det \mathrm{J}_{\mathbb{C}} F(w))}^{\frac{\delta-1}{\delta}} = S_{\Omega_1}(z, w), \quad (z, w) \in \Omega \times \overline{\Omega}.$$

Proof. Let $\{f_j\}$ be a Cauchy sequence in $\mathrm{EHF}^2(\Omega)$. By condition 2., $\{f_j\}$ is a Cauchy sequence in the uniform metric on compact sets in Ω. Hence, it converges uniformly on compacts to some $f \in \mathcal{O}(\Omega)$. In particular, for any fixed $M < M'$, $\ell_{M,M'}(|f - f_j|^2) \to 0$ as $j \to \infty$. Also, due to the boundedness of Cauchy sequences, there is an $L > 0$ so that $\ell_M(|f_j|^2)^{\frac{1}{2}} \leq \sqrt{c}||f_j||_{\mathrm{EHF}^2} \leq L$, for all j. Therefore,

$$(\ell_{M,M'}(|f|^2))^{\frac{1}{2}} \leq (\ell_{M,M'}(|f - f_j|^2))^{\frac{1}{2}} + L,$$

and $f \in \mathrm{EHF}^2(\Omega)$ with $||f||_{\mathrm{EHF}^2} \leq L$.

Now, suppose there is a subsequence $\{f_{j_k}\}_{k \in \mathbb{N}_+}$ such that $||f - f_{j_k}||_{\mathrm{EHF}^2} \geq 2\eta > 0$ for all k. Let l be large enough so that $||f_{j_l} - f_{j_k}||_{\mathrm{EHF}^2} < \eta/2$ for all $k > l$. We may choose an $M > 0$ so that $\ell_{M,M'}(|f - f_{j_l}|^2) > \eta^2$ for some $M' > 0$. Therefore, we get

$$\ell_{M,M'}(|f - f_{j_k}|^2)^{\frac{1}{2}} \geq \ell_{M,M'}(|f - f_{j_l}|^2)^{\frac{1}{2}} - ||f_{j_l} - f_{j_k}||_{\mathrm{EHF}^2} > \eta/2.$$

But, the left hand side goes to 0 as $k \to \infty$, thus proving that $\lim_{j\to\infty} ||f - f_j||_{\mathrm{EHF}^2} = 0$.

As observed in remark 4.3.3, there is a densely defined bounded operator from HF^2 to EHF^2, which is, in fact, identity on the dense subset $\mathcal{A}(\Omega)$. Thus, we get an isometry Φ from $\mathrm{HF}^2(\Omega)$ to $\overline{\mathrm{EHF}^2(\Omega)}$ — the latter being $\mathrm{EHF}^2(\Omega)$, as shown in the first part.

We now refer to the final part of Section 2.3 to observe that Φ^{-1} plays the role of P on the space $E = \mathrm{HF}^2(\Omega)$, thus yielding parts (ii) and (iii) (with $S_\Omega := S_{\widetilde{\sigma}_\Omega}$). $\square$

BIBLIOGRAPHY

[1] V. I. Arnold. *Mathematical Methods of Classical Mechanics*, volume 60. Springer, 1989.

[2] N. Aronszajn. Theory of reproducing kernels. *Trans. Amer. Math. Soc.*, pages 337–404, 1950.

[3] F. Aurenhammer. Power diagrams: properties, algorithms and applications. *SIAM J. Comput.*, 16:78–96, 1987.

[4] D. Barrett. A floating body approach to Fefferman's hypersurface measure. *Math. Scand.*, 98:69–80, 2006.

[5] D. Barrett and C. Hammond. Remarks on variational problems for Fefferman's measure. *arXiv preprint arXiv:1109.5882*, 2011.

[6] D. Barrett and L. Lee. On the Szegő metric. *J. Geom. Anal.*, 24(1):104–117, 2014.

[7] W. Blaschke. *Vorlesungen Über Differentialgeometrie II: Affine Differentialgeometrie*. Springer, Berlin, 1923.

[8] L. Boutet de Monvel and J. Sjöstrand. Sur la singularité des noyaux de Bergman et de Szegö. *Astérisque*, 34-35:123–164, 1976.

[9] L. Capogna, D. Danielli, S. D. Pauls, and J. T. Tyson. *An Introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem*. Birkhäuser, 2007.

[10] J. P. D'Angelo. A note on the Bergman kernel. *Duke Math. J.*, 45(2):259–265, 1978.

[11] C. Fefferman. The Bergman kernel and biholomorphic mappings of pseudoconvex domains. *Invent. Math.*, 26(1):1–65, 1974.

[12] C. Fefferman. Parabolic invariant theory in complex analysis. *Adv. in Math.*, 31:131–262, 1979.

[13] P. M. Gruber. Asymptotic estimates for best and stepwise approximation of convex bodies ii. *Forum Math.*, 5:521–538, 1993.

[14] P. M. Gruber and J. M. Wills. *Handbook of convex geometry*, volume A. North Holland, Amsterdam, 1993.

[15] C. Hammond. Variational problems for Fefferman hypersurface measure and volume-preserving CR invariants. *J. Geom. Anal.*, 21(2):372–408, 2011.

[16] G. Henkin and J. Leiterer. *Theory of functions on complex manifolds*. Springer, 1983.

[17] K. Hirachi. Transformation law for the Szegő projectors on CR manifolds. *Osaka J. Math.*, 27(2):301–308, 1990.

[18] L. Hörmander. L^2 estimates and existence theorems for the operator. *Acta Math.*, 113(1):89–152, 1965.

[19] L. Hörmander. *An Introduction to Complex Analysis in Several Variables*. North-Holland, Amsterdam, 1990.

[20] K. Böröczky Jr. Approximation of general smooth convex bodies. *Adv. in Math.*, 153:325–341, 2000.

[21] K. Böröczky Jr. and M. Ludwig. Approximation of convex bodies and a momentum lemma for power diagrams. *Monatsh. Math.*, 127(2):101–110, 1999.

[22] S. G. Krantz and H. R. Parks. *Geometric Integration Theory*. Springer, 2008.

[23] K. Leichtweiß. *Affine Geometry of Convex Bodies*. Johann Ambrosius Barth, Heidelberg, 1998.

[24] M. Ludwig. Asymptotic approximation of smooth convex bodies by general polytopes. *Mathematika*, 46(1):103–125, 1999.

[25] R. M. Range. *Holomorphic Functions and Integral Representations in Several Complex Variables*. Springer, New York, 1986.

[26] W. Rudin. *Function Theory in Polydiscs*. Number 41 in Mathmatics lecture note series. WA Benjamin, 1969.

[27] C. Schütt. The convex floating body and polyhedral approximation. *Israel J. Math.*, 73(1):65–77, 1991.

[28] C. Schütt and E. Werner. The convex floating body. *Math. Scand.*, 66:275–290, 1990.

[29] E. M. Stein. *Boundary Behavior of Holomorphic Functions of Several Complex Variables*. Princeton University Press Princeton, 1972.

[30] A. W. Van der Vaart. *Asymptotic Statistics*, volume 3. Cambridge university press, 2000.