ON CONNECTIONS BETWEEN INVARIANTS OF SINGULARITIES IN ZERO AND POSITIVE CHARACTERISTIC

by

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A mis padres por siempre estar ahí, a mi hermano por los muchos años, a mis amigos por las noches compartidas, a mis maestros por ayudarme a ver, y a Rui por hacer el tiempo ingrávido.

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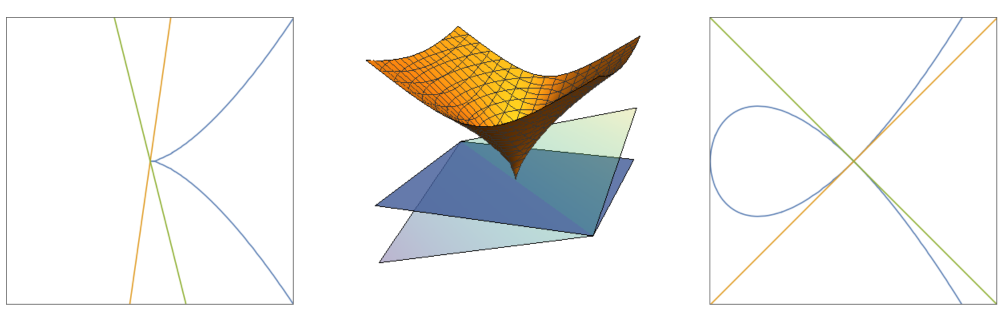
CHAPTER I

Introduction

The main goal of this thesis is to study some invariants associated to singularities in positive characteristic and compare them with their counterparts in characteristic zero. This work contains results obtained in [P1́3, NBP13]. Of these, [NBP13] is in collaboration with Luis Núñez-Betancourt. Singularities play an important role as they appear naturally in almost every area of mathematics, including commutative algebra, algebraic geometry, number theory, representation theory, analysis and topology.

Singularities

From the geometric point of view, a singularity corresponds to a point where a given geometric object, for example a manifold or an algebraic variety, has an unexpected tangent space. For instance, if we consider the following hypersurfaces



$y^2 = x^3$ $\qquad$ $z^3 - (y^2 + 3x^2) = 0$ $\qquad$ $y^2 = x^3 + x^2$

Figure I.0.0.1: Examples of Singularities we can see that in each case there is a special point where the tangent space is larger than expected.

Singularities can also be detected algebraically. For example, if $f(\boldsymbol{x})$ is a polynomial in n variables with real coefficients defining a hypersurface X in $\mathbb{R}^n$, then the singularities of X are given by those points $\boldsymbol{a} \in X$ where the partial derivatives $\frac{\partial f}{\partial x_i}(\boldsymbol{a})$ are zero for all i. For instance, the hypersurface in the middle of the above figure corresponds to $f = z^3 - (y^2 + 3x^2)$, which has partial derivatives $f_x = -6x$, $f_y = -2y$, and $f_z = 3z^2$. We conclude that the only singular point is the origin, which corresponds with our geometric intuition.

Classifying singularities has been an object of intense study in both zero and positive characteristic. In characteristic zero many invariants can be described in terms of resolutions of singularities and they are related to the minimal model program. Furthermore, many results in this setting can be approached analytically. However, in the recent decades, it has become apparent that in order to study singularities in characteristic zero, one can also reduce to positive characteristic and use Frobenius techniques to investigate singularities.

Singularities via Frobenius

Let R be a domain of positive characteristic p. The Frobenius map $F : R \to R$ takes an element r to r^p, therefore its image is the subring R^p consisting of all the p-th powers of elements in R. This induces on R a structure of R^p-module. It is a consequence of a theorem of Kunz [Kun69] that, under mild conditions, R is not singular if and only if R is a locally free R^p-module. This remarkable result tells us that we can detect singularities via the action of Frobenius. Therefore we can define different families of singularities by specifying "how close" R is to a locally free R^p-module. There are many kinds of singularities obtained by imposing restrictions on the action of Frobenius and one key feature is that they parallel the classes of singularities that have been studied in characteristic zero. The following diagram shows the relation among them and how they compare with the singularities in characteristic zero:

We do not give the definitions of these classes of singularities, which are somewhat technical, and we refer instead to Chapter II for the definitions. We also refer to [BFS13, ST12, TW14, SZ15] for surveys on F-singularities and to [KM98] for an introduction to singularities in characteristic zero.

$$
\begin{tikzcd}
\text{regular} \arrow[r, Rightarrow] & \text{strongly } F\text{-regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & F\text{-rational} \arrow[l, Rightarrow, bend right, "\text{+quasi-Gorenstein}"'] \arrow[d, Rightarrow] \\
& F\text{-pure} \arrow[r, Rightarrow] & F\text{-injective} \arrow[l, Rightarrow, bend left, "\text{+quasi-Gorenstein}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\text{regular} \arrow[r, Rightarrow] & \text{Log Terminal} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Rational} \arrow[l, Rightarrow, bend right, "\text{+Gorenstein}"'] \arrow[d, Rightarrow] \\
& \text{Log Canonical} \arrow[r, Rightarrow] & \text{Du Bois} \arrow[l, Rightarrow, bend left, "\text{+Gorenstein+Normal}"]
\end{tikzcd}
$$

Figure I.0.0.2: Families of Singularities

Multiplier and test ideals

All the invariants that we study in this thesis are related to the notion of multiplier ideals in characteristic zero, and to the notion of test ideal in positive characteristic.

Multiplier ideals have been intensively studied over the last two decades, as they play an important role in birational geometry, see for example [Laz04]. Given a smooth complex variety X and a nonzero ideal sheaf $\mathfrak{a}$, one can define for any parameter $c > 0$ an ideal $\mathcal{J}(\mathfrak{a}^c)$, called multiplier ideal. This ideal is described via a log resolution $\pi : X' \to X$ of the pair $(X, \mathfrak{a})$, i.e. a proper birational map, with X' smooth, and such that $\mathfrak{a}\mathcal{O}_{X'} = \mathcal{O}_{X'}(-E)$, where E is a simple normal crossing divisor. Then,

$$\mathcal{J}(\mathfrak{a}^c) := \pi_*\mathcal{O}(K_{X'/X} - \lfloor cE \rfloor), \tag{I.0.0.1}$$

where $K_{X'/X}$ is the relative canonical divisor.

Mixed multiplier ideals extend the previous definition to the case of several ideals: for nonzero ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_\mathfrak{n}$ and positive numbers $c_1, \ldots, c_n$ we take a log resolution for the pair $(X, \mathfrak{a}_1 \cdots \mathfrak{a}_n)$ and set the mixed multiplier ideal to be

$$\mathcal{J}(\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_n^{c_n}) := \pi_*\mathcal{O}(K_{X'/X} - \lfloor c_1E_1 + \ldots + c_nE_n \rfloor),$$

where $\mathcal{O}_{X'}(-E_i) = \mathfrak{a}_i\mathcal{O}_{X'}$.

Test ideals were introduced by Hara and Yoshida in [HY03] as an analogue of multiplier ideals in positive characteristic and are a generalization of the ones defined by Hochster and Huneke [HH90]. One question that was studied since [HY03] is which properties of multiplier ideals have analogues for test ideals. For example, for multiplier ideals the *jumping numbers* of $\mathfrak{a}$ are defined as the positive real numbers c such that $\mathcal{J}(\mathfrak{a}^c) \neq \mathcal{J}(\mathfrak{a}^{c-\epsilon})$ for every $\epsilon > 0$ (cf. [ELSV04]). It is easy to see from the definition that for each $\mathfrak{a}$ these numbers are discrete and rational. Thus it was expected that this was the case also in positive characteristic. Blickle, Mustaţă and Smith proved discreteness and rationality of the analogous positive characteristic invariants in [BMS08], but the proof was more involved.

The Invariants

Constancy regions

We study the dependence of mixed test ideals on parameters, and show that the emerging picture is quite different from that in the case of mixed multiplier ideals in characteristic zero.

In the mixed multiplier ideal setting we consider the map

$$\begin{aligned} \Psi\colon\ \mathbb{R}^n_{\geq 0} &\to \{\text{ Ideal sheaves on } X \ \} \\ (\lambda_1,\ldots,\lambda_n) &\mapsto \mathcal{J}(\mathfrak{a}_1^{\lambda_1}\cdots\mathfrak{a}_n^{\lambda_n}). \end{aligned}$$

A nonempty fiber of this map is called a *constancy region for the mixed multiplier ideals* of $\mathfrak{a}_1,\ldots,\mathfrak{a}_n$. For example, when $n=1$, these regions consist of intervals $[0,\alpha_1),[\alpha_1,\alpha_2),\ldots$, where the α_i are the jumping numbers of the ideal. The most important of these numbers is the smallest one α_1, the *log canonical threshold* of the ideal. For all n, it is known that the constancy regions are finite unions of polytopes.

In positive characteristic, the mixed test ideal $\tau(\mathfrak{a}_1^{\lambda_1}\cdots\mathfrak{a}_n^{\lambda_n})$ plays a role analogous to that of the mixed multiplier ideal (see Chapter II or [BMS08, HY03] for details), and we can define the *constancy regions for mixed test ideals* in a similar way. Blickle, Mustaţă, and Smith [BMS08] showed that, for $n=1$, the same picture holds, in this setting the α_i are called the *F-jumping numbers* and α_1 is the *F-pure threshold*. At the end of the paper, the authors asked whether the constancy regions in positive characteristic should also consist of finite unions of rational polytopes.

In chapter III we prove that this is not the case, but we can still get a nice decomposition. This decomposition depends on a p-fractal function, that is, a function $\varphi:\mathbb{R}^n_{\geq 0}\to\mathbb{N}$ satisfying the following property. If we restrict φ to a bounded domain D, then the vector space generated by the functions $\phi_{b,e}(t_1,\ldots,t_n)=\varphi((t_1+b_1)/p^e,\ldots,(t_n+b_n)/p^e)$ with b_i integers and $((t_1+b_1)/p^e,\ldots,(t_n+b_n)/p^e)\in D$, is finite dimensional (Definition III.2.1). Explicitly, we show:

Theorem (Theorem III.2.6)**.** For an F-finite, regular ring R essentially of finite type over a finite field of positive characteristic and non zero ideals $\mathfrak{a}_1,\ldots,\mathfrak{a}_n$ of R, there is a p-fractal function $\varphi:\mathbb{R}^n_{\geq 0}\to\mathbb{N}$ such that

$$\tau(\mathfrak{a}_1^{c_1}...\mathfrak{a}_n^{c_n})=\tau(\mathfrak{a}_1^{d_1}...\mathfrak{a}_n^{d_n})\Longleftrightarrow\varphi(c_1,\ldots,c_n)=\varphi(d_1,\ldots,d_n),$$

and therefore the constancy regions are of the form $\varphi^{-1}(i)$ for $i \in \mathbb{N}$.

Roughly speaking, this shows that each constancy region has a p-fractal structure that, as we see in the examples in Section III.3, can be intricate.

F-Jumping ideals and the modules M_α

For a polynomial f in $\mathbb{C}[x_1, \ldots, x_n]$, the Bernstein-Sato polynomial of f is defined as the nomic polynomial $b_f(s) \in \mathbb{C}[s]$ of minimal degree satisfying a relation of the form

$$P \cdot f^{s+1} = b_f(s) f^s$$

where P is a differential operator in $\mathbb{C}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n, s]$.

The existence of such polynomial was proved by Bernstein [Ber72]. The Bernstein-Sato polynomial carries important information about the singularities of f and it is related with many other invariants [Kol95, Bud05]. For example, the smallest jumping number of f is equal to the negative of the largest root of $b_f(s)$.

Let $D = \mathbb{C}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$, be the ring of differential operators over the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. When studying the Bernstein-Sato polynomial Kashiwara, introduced for any complex number γ the holonomic D-module $\mathcal{M}_f(\gamma) = D \cdot f^\gamma$. Note that when $\gamma, \gamma - 1, \gamma - 2, \ldots$ are not roots of the Bernstein-Sato polynomial, $\mathcal{M}_f(\gamma) \cong R_f$ as an R-module.

Theorem I.0.1 (Kashiwara, [HTT08] Corollary 6.25)**.** *If $\gamma, \gamma - 1, \gamma - 2, \ldots$ are not roots of the Bernstein-Sato polynomial of f then the D-module $\mathcal{M}_f(\gamma)$ is a simple D-module.*

It is worth mentioning that the multiplier ideals can be recovered from the theory of D-modules [BS05].

For a regular ring R of positive characteristic Lyubeznik introduced the concept of F^e-modules [Lyu97], see Section II.2.3 for the definition. We use the theory of F^e-modules to prove an analogue of Theorem I.0.1 in positive characteristic, but we do more, we characterize the simplicity depending on the parameter being an F-jumping number. We briefly describe this next.

Let R be a regular ring of positive characteristic. Blickle, Mustaţă, and Smith introduced certain F^e-modules M_α to study the discreteness and rationality of the F-jumping numbers of hypersurfaces [BMS09]. Let f be an element in R and $\alpha = \frac{r}{p^e - 1}$ a rational number without p in the denominator. The F^e-module M_α is an R-module isomorphic to R_f, together with a twisted action of the Frobenius operator F^e that depends on α. Explicitly, if we write $M_\alpha = R_f \cdot e_\alpha$, then

$$F^e\left(\frac{g}{f^\ell} \cdot e_\alpha\right) = \frac{g^{p^e}}{f^{p^e\ell+r}} \cdot e_\alpha.$$

Note that this action suggests that e_α behaves formally like $f^{-\alpha}$. In joint work with Núñez-Betancourt, we use these modules to define two families of ideals: the F-Jumping ideals $\mathcal{J}_F(f^\alpha)$ and the F-Jacobian ideals $J_F(f)$. The former ones are used to detect when α is an F-jumping number:

Theorem (Theorem IV.0.25)**.** Let R, f and α as before. The following are equivalent:

i. α is not an F-jumping number.

ii. M_α is a simple F-module.

iii. M_α is a simple D_R-module, where D_R is the ring of differential operators on R.

iv. $\mathcal{J}_F(f^\alpha) = R$.

We can also recover the test ideals from the modules M_α, explicitly:

Proposition (Proposition IV.0.22)**.** The test ideal $\tau(f^\alpha)$ is a minimal root for the F^e-modules M_α.

F-Jacobian ideals and the intersection homology modules

Suppose that k is a perfect field and f is a polynomial in $R = k[x_1, \ldots, x_n]$. The Jacobian ideal of f is defined as

$$\mathrm{Jac}\,(f) = \left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

This ideal defines the singular locus of f. Hence R/fR is regular if and only if $\mathrm{Jac}(f) = R$. Another important property of Jacobian ideals is a Leibniz rule, that is

$$\mathrm{Jac}\,(fg) \subseteq f\mathrm{Jac}\,(g) + g\mathrm{Jac}\,(f)\,.$$

The equality in the previous containment holds only in specific cases [Fab13, Proposition 8] and it is used to study transversality of singular varieties [Fab13, FA12].

In order to define the F-Jacobian ideals, we use Blickle's [Bli04] intersection homology D_R-module $\mathcal{L}(R/fR, R)$. This D_R-module is the sum of all the simple D_R-submodules of R_f/R. The F-Jacobian ideal of f, denoted by $J_F(f)$, is defined as the ideal in R that contains f and satisfies

$$J_F(f)\frac{1}{f} = \mathcal{L}(R/fR, R) \cap R\frac{1}{f}$$

in R_f/R. This ideal behaves similarly to the Jacobian ideal. For example, while the Jacobian ideal detects regularity, the F-Jacobian ideal detects F-regularity. More explicitly, if R/fR is F-regular, then $J_F(f) = R$ (Corollary V.2.11). In fact, if R/fR is F-pure, then R/fR is F-regular if and only if $J_F(f) = R$ (Corollary V.2.13). We also show that, like the Jacobian ideal, the F-Jacobian ideal satisfies a Leibniz rule

$$J_F(fg) = fJ_F(g) + gJ_F(f)$$

for relatively prime elements $f, g \in R$ (Proposition V.1.14). This is a key point, since it allows us to study interactions among different hypersurfaces.

The F-Jacobian ideals behave well with respect to p^e-th powers: $J_F(f^{p^e}) = J_F(f)^{[p^e]}$ (Proposition V.3.1). This property is essential in several proofs and contrasts with how the Jacobian ideal changes with p^e-th powers:

$$\mathrm{Jac}(f^{p^e}) = f^{p^e}R \neq \left(f^{p^e}, \left(\frac{\partial f}{\partial x_1}\right)^{p^e}, \ldots, \left(\frac{\partial f}{\partial x_n}\right)^{p^e} \right) = \mathrm{Jac}(f)^{[p^e]}.$$

The F-Jacobian ideal can be computed from the test ideal in certain cases and they are strongly related (see Proposition IV.0.11). However, they differ in general (see Example V.5.3). Moreover, the F-Jacobian ideal can be defined for elements f such that R/fR is not reduced and satisfies properties that the test ideal does not (eg. Propositions V.1.14 and V.3.1). In Section V.4, we show how the F-Jacobian arises in other contexts related with Cartier algebras, F-ideals, or $R\{F\}$-modules.

Outline

The first part of chapter II contains background in characteristic zero. This section is not necessary for any of the results in the thesis, but we include it as motivation for the positive characteristic results. The second part of chapter II contains all the necessary background in positive characteristic that we need in the later chapters.

Chapter III studies the constancy regions for mixed test ideals. We start by defining some sets associated to mixed test ideals in Section III.1. We give the structural result for constancy regions in positive characteristic in Section III.2. We finish this chapter with some examples and counterexamples in Section III.3.

In Chapter IV we introduce the F^e-modules M_α and we use these modules to define the F-jumping ideals and to give a characterization of the F-jumping numbers. In this chapter we also prove several properties of F-jumping ideals. In particular, we show that they commute with localization and completion. Furthermore, we show that they can be used to give an algorithm for determining when α is an F-jumping number, see algorithm IV.0.26.

In Chapter V we discuss the intersection homology modules $\mathcal{L}(R/fR, R)$. Even though the existence of these modules was first shown by Blickle [Bli04], we give a different proof of their existence in Section V.1. We use the intersection homology modules to define the F-Jacobian ideals. In Section V.2 we relate the F-Jacobian ideals with the usual test ideal and F-regularity. We study the behavior of the F-Jacobian ideals under different extensions in Section V.3. We finish Chapter V with several examples, see Section V.5.

CHAPTER II

Background

This chapter is divided in two parts. The first section contains several known results related to singularities in characteristic zero. Even though we do not need any of these results in this thesis, we present them so the reader can compare with the results we obtain in positive characteristic. The second section contains all the necessary background in positive characteristic.

II.1 Characteristic zero

In this section we recall some relevant results about singularities in characteristic zero. We first fix some terminology. A *variety* is an integral scheme X over a field k. We assume throughout this section that the field k is the field of complex numbers $\mathbb{C}$.

There are many tools involved in the study of singularities in characteristic zero. One classic approach is to start with a possibly singular variety Z and embed it in a smooth ambient space X. This leads to the concept of pairs.

Definition II.1.1. A *pair* (X, Z) consists of a smooth variety X and a formal sum $Z = c_1 Z_1 + \ldots + c_n Z_n$, where the c_i are real numbers and Z_i are subschemes of X. If $I_1, \ldots, I_n$ are the sheaves of ideals defining $Z_1, \ldots, Z_n$, we also denote (X, Z) by $(X, I_1^{c_1} \cdots I_n^{c_n})$.

Let X be a smooth variety over the complex numbers $\mathbb{C}$ and $\mathfrak{a} \subseteq \mathcal{O}_X$ an ideal sheaf. Recall that a (Weil) divisor $D = \sum r_i D_i$ has *simple normal crossing* support if each of its irreducible components D_i is smooth, and if locally one has coordinates $x_1, \ldots, x_n$ such that $\mathrm{Supp}(D) = \sum D_i$ is defined by a monomial in the x_i.

II.1.1 Resolution of Singularities

One of the keystone concepts in the study of singularities in characteristic zero is that of resolution of singularities.

Definition II.1.2. A *log resolution* of a nonzero ideal $\mathfrak{a} \subseteq \mathcal{O}_X$ is a proper birational map $\pi : Y \to X$ whose exceptional locus is a divisor E such that

1. Y is non-singular.

2. $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$ with F an effective divisor.

3. $F + E$ has simple normal crossing support.

The existence of a log resolution of singularities for any nonzero sheaf of ideals in a variety over a characteristic zero field was shown by Hironaka [Hir64]. Furthermore, Hironaka shows that a log resolution of singularities can be obtained by successively blowing up smooth subvarieties.

Example II.1.3. Let $X = \mathbb{A}^2 = \operatorname{Spec}(\mathbb{C}[x, y])$ and $f(x, y) = x^2 - y^3$. The singular locus of f is given by

$$\frac{\partial f}{\partial x} = 2x = 0 \text{ and } \frac{\partial f}{\partial y} = -3y^2 = 0,$$

hence f is only singular at the origin. A log resolution of f is obtained by a sequence of three blow-ups. We study in detail the first blow-up and leave to the reader the computation of the other two. The first blow-up X' can be described as the union of two charts, given in local coordinates by $\mathbb{C}[x, y/x]$ and $\mathbb{C}[x/y, y]$. The pullback of f is then $x^2(1 - (y/x)^3x)$ in the first chart and $y^2\left((x/y)^2 - y\right)$ in the second one. Note that in the first chart the pullback of f consists of two disjoint smooth divisors. By renaming the variables $u = x/y$ and $v = y$ we see that in the second chart the pullback is given by the union of the divisors $v^2 = 0$ and $u^2 - v = 0$, and, even though both divisors are smooth, they are not in simple normal crossing position. It will be necessary to blow-up twice more in order to get the simple normal crossing condition. Figure II.1.3.1 shows this process, with E_i denoting the exceptional divisor of the i-th blowup.

Example II.1.4. Let $D \subseteq \mathbb{A}^n$ be a hypersurface defined by a homogenous polynomial f such that $G = \operatorname{Proj}\left(k[x_1, \ldots, x_n]/fk[x_1, \ldots, x_n]\right)$ is smooth. This is equivalent to saying that D is the affine cone over a smooth divisor of $\mathbb{P}^{n-1}$. The origin o is the

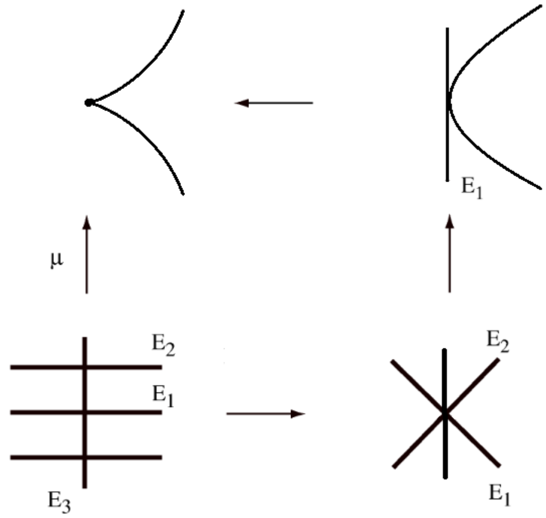


Figure II.1.3.1: A resolution for $x^2 - y^3$

only singular point of D, and a log resolution of singularities for D is obtained by blowing-up at o. Moreover, in this case the intersection of the exceptional divisor and of the proper transform of D is isomorphic to G.

II.1.2 Multiplier ideals

Multiplier ideals have been intensively studied over the last two decades, as they play an important role in birational geometry. In this section we recall the definition and basic properties of these ideals. See [Laz04] for a more thorough treatment. For $\pi : Y \to X$ a morphism between smooth varieties we set $K_{Y/X} := K_Y - \pi^* K_X = \operatorname{div}(\det(\operatorname{Jac}(\pi)))$. The following result is particularly useful when doing computations:

Proposition II.1.5. *[Har77, Exercise II.8.5] Suppose that Y is obtained as the blow-up of X along a smooth subvariety Z. If the codimension of Z in X is $r \geq 2$, then $K_{Y/X} = (r-1)E$, where E is the exceptional divisor.*

Before we give the definition of multiplier ideal, we need some extra notation: Let $D = \sum a_i D_i$ be a $\mathbb{Q}$-linear combination of distinct prime divisors D_i, by $\lfloor D \rfloor$ we denote the divisor $\sum \lfloor a_i \rfloor D_i$.

Definition II.1.6. Let $\alpha_1, \ldots, \alpha_n$ be nonnegative real numbers, $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ nonzero sheaves of ideals on X, and X' be a log resolution for the pair $(X, \mathfrak{a}_1 \cdots \mathfrak{a}_n)$. The *mixed multiplier ideal* of $\mathfrak{a}_1 \ldots \mathfrak{a}_n$ with parameters $\alpha_1, \ldots, \alpha_n$ is defined as

$$\mathcal{J}(\mathfrak{a}_1^{\alpha_1} \cdots \mathfrak{a}_n^{\alpha_n}) := \pi_* \mathcal{O}(K_{X'/X} - \lfloor \alpha_1 E_1 + \ldots + \alpha_n E_n \rfloor),$$

where $\mathcal{O}_{X'}(-E_i) = \mathfrak{a}_i\mathcal{O}_{X'}$.

It can be shown that this definition does not depend on the log resolution $\pi : X' \to X$. In the case $n = 1$ these ideals are just called multiplier ideals and when the ideals are principal and define divisors $D_1, \ldots, D_n$ we also denote this ideal by $\mathcal{J}(\alpha_1 D_1 + \ldots + \alpha_n D_n)$.

Let D be the divisor associated to $f \in \Gamma(X, \mathcal{O}_X)$. The multiplier ideals $\mathcal{J}(\alpha D)$ satisfy:

- If $\alpha < \alpha'$ then $\mathcal{J}(\alpha D) \supseteq \mathcal{J}(\alpha' D)$.
- For any $\alpha \in \mathbb{R}_{\geq 0}$, there exists $\epsilon > 0$ such that $\mathcal{J}(\alpha D) = \mathcal{J}(\alpha' D)$ for every $\alpha' \in [\alpha, \alpha + \epsilon)$.

The numbers α for which $\mathcal{J}(\alpha D) \neq \mathcal{J}((\alpha - \epsilon)D)$ for all $\epsilon > 0$ are the *jumping numbers* of D. The most important jumping number is the smallest one; this is *the log canonical threshold* of f.

Example II.1.7. If $f(x, y) = x^2 - y^3$, we can compute the multiplier ideal of f and the jumping numbers via the log resolution in Example II.1.3. If $\mu : Y \to X$ is this log resolution, then $f \cdot \mathcal{O}_Y = \mathcal{O}_Y\left(-6E_3 - 3E_2 - 2E_1 - \tilde{D}\right)$. By Proposition II.1.5, the relative canonical divisor is $K_{Y/X} = 4E_3 + 2E_2 + E_1$, hence

$$\mathcal{J}(f^\alpha) = \mu_*\mathcal{O}_Y\left(4E_3 + 2E_2 + E_1 - \lfloor 6\alpha E_3 + 3\alpha E_2 + 2\alpha E_1 + \alpha\tilde{D}\rfloor\right) \tag{II.1.7.1}$$

$$= \mu_*\mathcal{O}_Y\left(\lceil 4 - 6\alpha\rceil E_3 + \lceil 2 - 3\alpha\rceil E_2 + \lceil 1 - 2\alpha\rceil E_1 - \lfloor\alpha\tilde{D}\rfloor\right) \tag{II.1.7.2}$$

$$= \begin{cases} f^{\lfloor\alpha\rfloor} & \text{if } \alpha - \lfloor\alpha\rfloor < 5/6, \\ f^{\lfloor\alpha\rfloor}(x, y) & 5/6 \leq \alpha - \lfloor\alpha\rfloor < 1. \end{cases} \tag{II.1.7.3}$$

It follows that the jumping numbers of f are $\{5/6, 1, 1 + 5/6, 2, 2 + 5/6, \ldots\}$.

II.1.3 The constancy regions

Let X be a smooth variety over the complex numbers, and $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ nonzero ideal sheaves on X. The definition of mixed multiplier ideals allows us to define a map

$$\Psi : \mathbb{R}^n_{>0} \longrightarrow \{\text{ Ideal sheaves of } \mathcal{O}_X\}$$

$$(\alpha_1, \ldots, \alpha_n) \longmapsto \mathcal{J}(\mathfrak{a}_1^{\alpha_1} \cdots \mathfrak{a}_n^{\alpha_n})$$

Definition II.1.8. The fibers of the map Ψ are called *the constancy regions* for the mixed multiplier ideals $\mathcal{J}(\mathfrak{a}_1^{\alpha_1}\cdots\mathfrak{a}_n^{\alpha_n})$.

The following proposition follows immediately from the definition of mixed multiplier ideal.

Proposition II.1.9. *A constancy regions for a collection of ideal sheaves $\mathfrak{a}_1,\ldots,\mathfrak{a}_n$ consist of finite unions of rational polytopes with non-overlapping interiors.*

Clearly any pair of constancy regions are disjoint and the union of the constancy regions is the region $\alpha_1,\ldots,\alpha_n \geq 0$. Besides the trivial cases (when the ideals are either 0 or R) there are infinitely many constancy regions. But, as the next proposition shows, they have a periodic behavior.

Proposition II.1.10. *[Laz04, Skoda's Theorem, Theorem 11.1.1] Let $\mathfrak{a}_1,\ldots,\mathfrak{a}_n$ be ideal sheaves on a non-singular variety X. Assume further that $\mathfrak{a}_1$ is generated by s elements, then for all $l \geq s$*

$$\mathcal{J}(\mathfrak{a}_1^{l}\mathfrak{a}_2^{\alpha_2}\cdots\mathfrak{a}_n^{\alpha_n}) = \mathfrak{a}_1\cdot\mathcal{J}(\mathfrak{a}_1^{l-1}\mathfrak{a}_2^{\alpha_2}\cdots\mathfrak{a}_n^{\alpha_n}).$$

From Proposition II.1.10 it follows the above-mentioned periodicity of the constancy regions.

Corollary II.1.11. *There is an integer N such that for all integers $l_1,\ldots,l_n > N$ and any constancy region A, the intersection $A\cap[l_1,l_1+1)\times\cdots\times[l_n,l_n+1)$ is a translation of the intersection of a constancy region B with $[l_1-1,l_1)\times\cdots\times[l_n-1,l_n)$.*

Example II.1.12. The constancy regions for $f(x,y)=x^2-y^3$ are given by intervals of the form $[i,i+5/6)$ and $[i+5/6,i+1)$ for $i\geq 0$.

Example II.1.13. Let $f_1 = xy$ and $f_2 = x+y$ and D_1, D_2 the corresponding divisors in $\mathbb{A}^2$. A log resolution of singularities for $xy(x+y)$ is obtained by blowing-up at the origin. Let $\pi : Y \to X$ be such blow-up. We have $K_{Y/X} = E$, where E is the exceptional divisor. By Proposition II.1.10, it is enough to compute the constancy regions for α and β less than 1. We have

$$\mathcal{J}(f_1^{\lambda_1}f_2^{\lambda_2}) = \pi_*\mathcal{O}_Y(E-\lfloor\lambda_1\pi^*D_1+\lambda_2\pi^*D_2\rfloor) = \pi_*(\lfloor 1-\{2\lambda_1\}-\{\lambda_2\}\rfloor E))$$

$$= \begin{cases} R & \text{if } 2\lambda_1+\lambda_2<2 \\ (x,y) & \text{if } 2\leq 2\lambda_1+\lambda_2<3. \end{cases}$$

Hence there are only two constancy regions inside $[0,1)\times[0,1)$, which are separated by the line $2\lambda_1+\lambda_2=2$. The constancy regions are shown in Figure II.1.13.1:

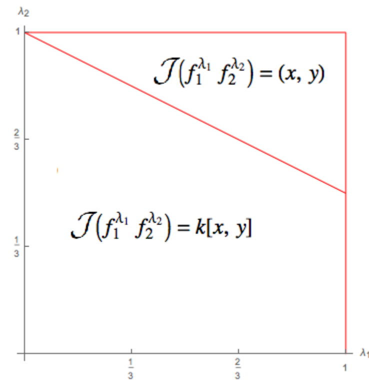


Figure II.1.13.1: Constancy regions for $xy, x+y$

II.1.4 D_R-modules associated to subvarieties

Many important invariants associated to singularities come from the D_R-module theory. This theory is quite rich and involved and we refer the reader to [HTT08] for a complete treatment. In this subsection we limit ourselves to giving the basic definitions and properties of D_R-modules. We also introduce two important D_R-modules that carry information about the singularities of a regular function.

Throughout this subsection we set $R=\mathbb{C}[x_1,\dots,x_n]$ the polynomial ring over the complex numbers and $X=\mathbb{A}^n=\operatorname{Spec}(R)$.

Definition II.1.14. The ring of differential operators is the non-commutative ring $D_R:=\mathbb{C}[x_1,\dots,x_n,\delta_1,\dots,\delta_n]$ where the ∂_i satisfy the relations

$$[\partial_i,x_j]=\delta_{ij} \qquad\qquad [\partial_i,\partial_j]=0.$$

Note that R itself has a natural structure of D_R-module with $\partial_i\cdot f=\frac{\partial f}{\partial x_i}$. A module over the ring D_R is called a D_R-module. We are interested in two families of D_R-modules associated to subvarieties of the smooth variety X.

II.1.4.1 The D_R-modules M_α

For $f\in\mathbb{C}[x_1,\dots,x_n]$ nonzero, consider the ring $D_R[s]$ obtained by adding to D_R a variable s commuting with the elements of D_R. We consider the $D_R[s]$-module $R_f\cdot f^s$

where f^s is a formal symbol and the ∂_i act by

$$\partial_i f^s = \frac{s}{f}\,(\partial_i f)\, f^s$$

The Bernstein-Sato polynomial $b_f(s)$ associated to f is the unique monic polynomial of minimal degree satisfying a relation

$$P(x_1,\ldots,x_n,\partial_1,\ldots,\partial_n,s)f^{s+1} = b_f(s)f^s,$$

for some $P \in D_R[s]$.

The Bernstein-Sato polynomial has been broadly studied and there still are many open questions in relation to it. The roots of this polynomial are always negative rational numbers [Kas76]. Furthermore, if λ is a jumping number less than 1 then $-\lambda$ is a root of $b_f(s)$ [ELSV04].

Given any complex number α, we can specialize the D_R-module $R_f \cdot f^s$ by making $s = \alpha$. We set

$$\mathcal{M}_f(\alpha) = D \cdot f^\alpha.$$

Proposition II.1.15. *Let $f \in R$ be a nonzero polynomial. If $\alpha, \alpha-1, \alpha-2, \ldots$ are not roots of the Bernstein-Sato polynomial $b_f(s)$ then $\mathcal{M}_f(\alpha) \cong R_f$ as an R-module.*

Proof. Clearly $\mathcal{M}_f(\alpha) \subseteq R_f \cdot f^\alpha$. Let $P(s) \in D_R[s]$ be the differential operator such that $P(s) \cdot f^{s+1} = b_f(s)f^s$. We will show by induction on i that $f^{\alpha-i}$ is in $\mathcal{M}_f(\alpha)$ for all $i \geq 0$. The case $i = 0$ is given by the definition of $\mathcal{M}_f(\alpha)$. Assume that $f^{\alpha-i} \in \mathcal{M}_f(\alpha)$, from the relation $P(\alpha - i) \cdot f^{\alpha-i} = b_f(\alpha-i)f^{\alpha-(i+1)}$ and the fact that $\alpha - i$ is not a root of the Bernstein-Sato polynomial $b_f(s)$, it follows that $f^{\alpha-(i+1)} \in \mathcal{M}_f(\alpha)$. The result is now immediate. $\square$

Theorem II.1.16 (Kashiwara, [HTT08] Corollary 6.25)**.** *If $\alpha, \alpha-1, \alpha-2, \ldots$ are not roots of the Bernstein-Sato polynomial, then the D_R-module $\mathcal{M}_f(\alpha)$ is a simple D_R-module.*

It is an open question whether the converse holds. More precisely, if $\mathcal{M}_f(\alpha) \cong R_f \cdot f^\alpha$ is a simple D_R-module, does it follow that α is not a root of the Bernstein-Sato polynomial?

II.1.4.2 The Brylinski-Kashiwara intersection homology D_R-modules

Suppose that $I = (f_1, \ldots, f_l)$ is an ideal of $R = \mathbb{C}[x_1,\ldots,x_n]$ defining a subvariety $Y \subseteq X = \mathbb{A}^n$ of dimension d. We recall that the local cohomology modules $H^i_I(R)$

are defined as the cohomology modules of the Čech complex

$$0 \longrightarrow R \longrightarrow \oplus R_{f_i} \longrightarrow \oplus R_{f_i f_j} \longrightarrow \ldots \longrightarrow R_{f_1 f_2 \cdots f_l} \longrightarrow 0.$$

As the localization of a D_R-module has a natural structure of D_R-module, the previous complex can be seen as a complex of D_R-modules. Hence the R-modules $H^i_I(R)$ are also D_R-modules. The Brylinski-Kashiwara intersection homology D_R-module, denoted $\mathcal{L}(Y, X)$, is defined as the unique simple D_R-submodule of $H^{n-d}_I(R)$. The existence of the D_R-module $\mathcal{L}(Y, X)$ is nontrivial and can be found in [Bjö84]. This modules can also be obtained via the Riemann-Hilbert correspondence from the pushforward of the intersection homology complex IC_Y to X.

II.2 Positive characteristic

This section gives the necessary background in positive characteristic. We start by covering test ideals and some classical classes of singularities in positive characteristic. In the second section we follow [BMS08] to give a description of the test ideals in the regular case. In the last section we go over the F-module theory introduced by Lyubeznik [Lyu97].

II.2.1 Test ideals and singularities via Frobenius

Test ideals were originally defined by Hoschter and Huneke [HH90] and later Hara and Yoshida extended this definition to include pairs [HY03]. In this subsection we recall their definition. We finish the subsection with several definitions of classes of singularities obtained by the action of Frobenius. There are several excellent surveys in this topic, see for example [BFS13, ST12, TW14, SZ15].

For every $e \geq 1$ we denote by $R^{(e)}$ the $R-R$-bimodule that as abelian group is isomorphic to R, with the usual left R-module structure, while the right one is given by the Frobenius action $F^e : R \to R$. Given an inclusion of R-modules $N \subseteq M$, by tensoring with $R^{(e)}$ we obtain a map of (left) R-modules

$$R^{(e)} \otimes N \to R^{(e)} \otimes M.$$

We denote the image of this map by $N^{[p^e]}_M$. Note for example that if I is an ideal of R then $I^{[p^e]}_R = I^{[p^e]} = \left(i^{p^e} : i \in I\right) R$.

Definition II.2.1 (Hara-Yoshida [HY03], Hochster-Huneke [HH90])**.** Given a positive real number c, *the* $\mathfrak{a}^c$*-tight closure of* N *in* M is the R-submodule $N_M^{*\mathfrak{a}^c}$ of M consisting of all elements $m \in M$ for which there exists $u \in R$, not in any minimal prime, such that

$$\text{image}\left(u\mathfrak{a}^{\lceil cp^e \rceil} \otimes m\right) \in N_M^{[p^e]}$$

for all $e \gg 0$. When $N = I$ is an ideal of $M = R$ we denote by I^* the (R^1)-tight closure of I in R.

For a maximal ideal $m \subseteq R$ we denote by $E_R(R/m)$ the injective hull of the residue field R/m. We set

$$E := \bigoplus_{m \in \text{maxSpec}(R)} E_R(R/m).$$

Definition II.2.2 (Hara-Yoshida [HY03])**.** The $\mathfrak{a}^c$-*test ideal* $\tau(\mathfrak{a}^c)$ is defined as

$$\tau(\mathfrak{a}^c) = \bigcap_{M \subseteq E} \text{Ann}_R(0_M^{*\mathfrak{a}^c})$$

where the intersection runs over all finitely generated submodules of E.

Remark II.2.3. We can recover the classical (finitistic) test ideal of Hochster and Huneke (defined in [HH90]) by taking $\mathfrak{a} = R$ and c to be any positive number. We denote this ideal by $\tau(R)$.

These definitions can easily be extended to the case of several ideal $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ and positive real numbers $c_1, \ldots, c_n$. In this case we call the corresponding ideal *the mixed test ideal associated to* $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ and we denote it by $\tau(\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_n^{c_n})$, see II.2.14 for a precise definition in the regular case.

We can use tight closure theory to define important families of singularities as follows.

Definition II.2.4. [HH90, FW89a] Let R be a reduced ring of characteristic $p > 0$.

a. We say that R is *weakly F-regular* if every ideal I in R is tightly closed, that is if $I^* = I$. We say that R is *F-regular* if every localization of R is weakly F-regular.

b. A local ring (R, m) is said to be *F-rational* if every parameter ideal is tightly closed. When R is not local, we say that R is F-rational if the local ring R_p is F-rational for every maximal ideal p of R.

Another family of singularities in positive characteristic appears naturally as a consequence of the following theorem of Kunz.

Theorem II.2.5 (Kunz's Theorem [Kun69])**.** *Given a local ring R, the following are equivalent:*

a. The ring R is regular.

b. The Frobenius map $F : R \to R$ is a flat morphism.

c. R is a free R^{p^e}-module of rank $p^e \dim(R)$ for some $e > 0$.

c. R is a free R^{p^e}-module of rank $p^e \dim(R)$ for every $e > 0$.

This remarkable result tells us that we can detect singularities via the action of Frobenius. Therefore we can define different families of singularities by specifying "how close" R is to a locally free R^p-module. We borrow notation from the scheme setting and denote by $F^e_* R$ the R-module whose underlying group structure is R and whose R-module structure is given by the Frobenius action $F^e : R \to R$. A ring R is *F-finite* if $F_* R$ is a finite R-module.

Definition II.2.6. [HH89, HR76] Let R be an F-finite ring of prime characteristic p. We say that a ring R is *F-pure* if the Frobenius map $F : R \to F_* R$ splits as an R-module homomorphism. We say that R is *strongly F-regular* if for every $c \in R^o$, there exists some $e \in \mathbb{N}$ such that the map

$$\begin{array}{c} cF^e : R \xrightarrow{F^e} F^e_* R \xrightarrow{\times F^e_* c} F^e_* R \\ \\ x \longmapsto F^e_*(x^{p^e}) \longmapsto F^e_*(cx^{p^e}) \end{array}$$

splits as an R-module homomorphism.

II.2.2 Test ideals in the regular case

When R is a regular F-finite ring, Manuel Blickle, Mircea Mustaţă and Karen Smith [BMS08] gave an alternative description of test ideals. Here we recall this construction following loc. cit. and therefore throughout this subsection we will assume that R is a regular F-finite ring.

Recall that for an ideal $I \subseteq R$ we denote by $I^{[p^e]}$ the ideal generated by the p^e-powers of elements in I.

Lemma II.2.7. *Given $u \in R$, we have $u^{p^e} \in I^{[p^e]}$ if and only if $u \in I$.*

Proof. The question is local so we may assume that R is local. From the exact sequence

$$0 \to I \to R \to R/I \to 0$$

and the flatness of Frobenius in the regular case we have that the sequence

$$0 \longrightarrow R^{(e)} \otimes I \longrightarrow R^{(e)} \otimes R \longrightarrow R^{(e)} \otimes (R/I) \longrightarrow 0$$

is exact. But as $R^{(e)} \otimes I \cong I^{[p^e]}$ and $R^{(e)} \otimes R \cong R$ it follows that $R^{(e)} \otimes (R/I) \cong R/I^{[p^e]}$. Hence if $u^{p^e} \in I^{[p^e]}$ it follows that $1 \otimes u = 0$ in $R^{(e)} \otimes (R/I)$, which in turn implies $u = 0$ in R/I. The result follows. □

Given an ideal I in R, we denote by $I^{[1/p^e]}$ the smallest ideal $\mathfrak{J}$ such that $I \subseteq \mathfrak{J}^{[p^e]}$. The existence of a smallest such ideal is a consequence of the flatness of the Frobenius map in the regular case. We record some basic properties of the ideals $I^{[1/p^e]}$.

Proposition II.2.8. *[BMS08, Lemma 2.4] Let $\mathfrak{a}, \mathfrak{b}$ be ideals in R. If e and e' be positive integers, then the following statements hold.*

(a) If $\mathfrak{a} \subseteq \mathfrak{b}$, then $\mathfrak{a}^{[1/p^e]} \subseteq \mathfrak{b}^{[1/p^e]}$.

(b) $(\mathfrak{a} \cdot \mathfrak{b})^{[1/p^e]} \subseteq \mathfrak{a}^{[1/p^e]} \cdot \mathfrak{b}^{[1/p^e]}$.

(c) $\mathfrak{a}^{[1/p^e]} \subseteq \left(\mathfrak{a}^{p^{e'}}\right)^{[1/p^{e+e'}]}$.

The following proposition gives an explicit description of $I^{[1/p^e]}$ when R is free over R^{p^e}. This holds, for example, if R is a polynomial ring or a local ring.

Proposition II.2.9. *[BMS08, Proposition 2.5] Suppose that R is free over R^q, for $q = p^e$, and let $e_1, \ldots, e_N$ be a basis of R over R^q. If $h_1, \ldots, h_n$ are generators of an ideal I of R, and if for every $i = 1, \ldots, n$ we write*

$$h_i = \sum_{j=1}^{N} a_{i,j}^q e_j$$

with $a_{i,j} \in R$, then

$$I^{[1/p^e]} = (a_{i,j} | i \leq n \text{ and } j \leq N).$$

In the regular case we have the following description of the test ideals:

Proposition II.2.10. *[BMS08, Proposition 2.22] Given a positive number c and a nonzero ideal $\mathfrak{a}$, the* generalized test ideal of $\mathfrak{a}$ with exponent c *can be described as*

$$\tau(\mathfrak{a}^c) = \bigcup_{e>0} (\mathfrak{a}^{\lceil cp^e \rceil})^{[1/p^e]},$$

where $\lceil c \rceil$ stands for the smallest integer $\geq c$.

The ideals in the above union form an increasing chain of ideals; therefore, as R is Noetherian, they stabilize. Hence for e large enough $\tau(\mathfrak{a}^c) = (\mathfrak{a}^{\lceil cp^e \rceil})^{[1/p^e]}$. In the principal ideal case we can say more.

Proposition II.2.11. *[BMS09, Lemma 2.1] If $\lambda = \frac{m}{p^e}$ for some positive integer m, then $\tau(f^\lambda) = (f^m)^{[1/p^e]}$.*

It can be shown that as the parameter c varies over the reals, only countably many different test ideals appear; moreover, we have:

Theorem II.2.12. *[BMS08, Proposition 2.14] For every nonzero ideal $\mathfrak{a}$ and every non-negative number c, there exists $\epsilon > 0$ such that $\tau(\mathfrak{a}^c) = \tau(\mathfrak{a}^{c'})$ for $c \leq c' < c + \epsilon$.*

Definition II.2.13. A positive real number c is an *F-jumping exponent* of $\mathfrak{a}$ if $\tau(\mathfrak{a}^c) \neq \tau(\mathfrak{a}^{c-\epsilon})$ for all $\epsilon > 0$.

The F-jumping exponents of an ideal $\mathfrak{a}$ form a discrete set of rational numbers, that is, there are no accumulation points of this set. In fact, they form a sequence with limit infinity (see [BMS08, Theorem 3.1]).

As in the case of one ideal, one can define the mixed test ideal of several ideals as follows.

Definition II.2.14. Given nonzero ideals $\mathfrak{a}_1, ..., \mathfrak{a}_n$ of R and non-negative real numbers $c_1, ..., c_n$, we define the *mixed generalized test ideal with exponents* $c_1, \ldots, c_n$ as:

$$\tau(\mathfrak{a}_1^{c_1} \cdots a_n^{c_n}) = \bigcup_{e>0} (\mathfrak{a}_1^{\lceil c_1 p^e \rceil} \cdots \mathfrak{a}_n^{\lceil c_n p^e \rceil})^{[1/p^e]}.$$

As in the case of $\tau(\mathfrak{a}^c)$, we have $\tau(\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_n^{c_n}) = (\mathfrak{a}_1^{\lceil c_1 p^e \rceil} \cdots \mathfrak{a}_n^{\lceil c_n p^e \rceil})^{[1/p^e]}$ for all e large enough.

Theorem II.2.15. *Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be nonzero ideals in the polynomial ring $R = k[x_1, \ldots, x_r]$, and let $c_1 = r_1/p^s, \ldots, c_n = r_n/p^s$ be such that $r_1, \ldots, r_n$ are natural numbers. If each $\mathfrak{a}_i$ can be generated by polynomials of degree at most d, then the* *ideal* $\tau(\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_n^{c_n})$ can be generated by polynomials of degree at most $\lfloor d(c_1+\ldots+c_n)\rfloor$. Here $\lfloor r\rfloor$ stands for the biggest integer $\leq r$.

Proof. We argue as in [BMS08, Proposition 3.2], where the result was proven for the case of one ideal. We know that R is free over R^{p^e} with basis

$$\{\beta_j x_1^{\alpha_1} \cdots x_r^{\alpha_r} | 0 \leq \alpha_i < p^e \text{ and } \beta_j \text{ part of a basis for } k \text{ over } k^{p^e}\}.$$

The ideal $\mathfrak{a}_1^{\lceil p^e c_1\rceil} \cdots \mathfrak{a}_n^{\lceil p^e c_n\rceil}$ can be generated by polynomials of degree at most $d\lceil p^e c_1\rceil + \ldots + d\lceil p^e c_n\rceil$. Hence taking $e > s$ large enough by Proposition II.2.9 the ideal

$$\tau(\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_n^{c_n}) = (\mathfrak{a}_1^{\lceil p^e c_1\rceil} \cdots \mathfrak{a}_n^{\lceil p^e c_n\rceil})^{[1/p^e]}$$

is generated by polynomials of degree at most $(d\lceil p^e c_1\rceil + \ldots + d\lceil p^e c_n\rceil)/p^e = (dp^{e-s}r_1 + \ldots + dp^{e-s}r_n)/p^e = d(r_1 + ... + r_n)$. $\square$

II.2.3 D-modules and F-modules

II.2.3.1 D-modules

In Section II.1.4 we defined a D-module as a module over the Weyl algebra $\mathbb{C}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$. In general, for rings that do not contain a field of characteristic zero, the ring of differential operators is more complicated. For example, in the case that R is the polynomial ring $\mathbb{F}_p[x_1, \ldots, x_n]$ it is not true that D_R is generated by the partial differentials $\partial_1, \ldots, \partial_n$. Instead, we need a more general definition.

Let R be a commutative ring and A be a subring of R. The elements of $\mathrm{Hom}_A(R, R)$ induced by multiplication by elements in R are called A-linear differential operators of order zero. We say that $\theta \in \mathrm{Hom}_A(R, R)$ is an *A-linear differential operator* of order less than or equal to $k+1$, if for every $r \in R$ the element $[\theta, r] = \theta \cdot r - r \cdot \theta$ is an A-linear differential operator of order less than or equal to k. The subset of $\mathrm{Hom}_A(R, R)$ consisting of all A-linear differential operators forms a ring that we denote by $D(R, A)$. If $A = \mathbb{Z}/p\mathbb{Z}$, we write D_R for $D(R, \mathbb{Z}/p\mathbb{Z})$, or just D if R is clear from the context.

$D(R, A)$-modules behave well under localization, that is if M is a $D(R, A)$-module and $W \subseteq R$ is a multiplicative system, then $W^{-1}M$ acquires a natural structure of $D(R, A)$-module such that the localization map $M \to W^{-1}M$ is a morphism of $D(R, A)$-modules. Furthermore $W^{-1}M$ is also a $D(W^{-1}R, A)$-module.

A nonzero $D(R, A)$-module M is *simple* if the only $D(R, A)$-submodules of M are the trivial ones, that is 0 and M. Assume R is a reduced F-finite ring of positive characteristic p. For any multiplicative system $W \subset R$ and any simple D_R-module M, the D_R-module $W^{-1}M$ is either zero or a simple $D_{W^{-1}R}$-module. Therefore, for every D_R-module of finite length N we have $\text{length}_{D_{W^{-1}R}} W^{-1}N \leq \text{length}_{D_R} N$.

Note that D is a subring of $\text{Hom}_{\mathbb{Z}/p\mathbb{Z}}(R, R)$, hence R has a natural structure of D-module. We can use D-modules to measure F-singularities.

Proposition II.2.16. *[Smi95a, Theorem 2.2] If R is an F-finite domain of positive characteristic $p > 0$, then R is a strongly F-regular ring if and only if R is F-pure and a simple D_R-module*

In addition, we have:

Proposition II.2.17. *[Yek92] If R is an F-finite domain of positive characteristic $p > 0$, then*

$$D_R = \bigcup_{e \in \mathbb{N}} \text{Hom}_{R^{p^e}}(R, R).$$

We denote $\text{Hom}_{R^{p^e}}(R, R)$ by $D_R^{(e)}$. We finish this subsection with a proposition that will be frequently use in Chapter IV.

Proposition II.2.18. *[ÀMBL05, Proposition 3.1] Let R be an F-finite regular ring of positive characteristic p. If f in a nonzero element of R, then*

$$D_R^{(e)} \cdot f = \left((f)^{[1/p^e]}\right)^{[p^e]},$$

for all $e > 0$.

II.2.3.2 F-modules

The theory of F-modules was introduced by Lyubeznik [Lyu97] to study finiteness properties of local cohomology modules. In this section we recall the basic definitions and properties. Throughout this section we assume that R is a regular ring of positive characteristic $p > 0$.

Every morphism of rings $\varphi : R \to S$ defines a functor from R-modules to S-modules φ^*, defined by $\varphi^* M = S \otimes_R M$. If $S = R$ and φ is the Frobenius morphism, we denote by FM the R-module $\varphi^* M = R^{(e)} \otimes M$. This is the Frobenius functor introduced by Peskine and Szpiro [PS73]. If R is regular, the flatness of Frobenius implies that F is an exact functor. We denote the e-th iterated Frobenius functor by F^e.

Example II.2.19. If M is the cokernel of a matrix $(r_{i,j})$, then F^eM is the cokernel of the matrix $(r_{i,j}^{p^e})$. In particular, if $I \subset R$ is an ideal, then $F^e(R/I) = R/I^{[p^e]}$.

An F^e-*module* is a pair $(\mathcal{M}, \phi)$ consisting of an R-module $\mathcal{M}$ and an isomorphism of R-modules $\phi : \mathcal{M} \to F^e\mathcal{M}$. The isomorphism ϕ is called the structural isomorphism. We note that an F^e-module has a natural structure as an $F^{\ell \cdot e}$-module for every $\ell \in \mathbb{N}$ given by the composition

$$\mathcal{M} \xrightarrow{\nu} F^e\mathcal{M} \xrightarrow{F^e\nu} F^{2e}\mathcal{M} \ldots \xrightarrow{F^{(\ell-1)}\nu} F^{\ell \cdot e}\mathcal{M}.$$

Let $R[F^e]$ be the skew-ring generated by the symbol F^e and the relation $F^e r = r^{p^e}F^e$ for any $r \in R$. Any F^e-module has a structure of $R[F^e]$-module. Indeed, if M is an F^e-module with structural morphism $\phi : M \to F^eM = R^{(e)} \otimes M$ then the action of F^e is given by:

$$\begin{aligned} F^e : M &\longrightarrow M \\ m &\longmapsto F^e(m) = \phi^{-1}(1 \otimes m). \end{aligned}$$

One way to produce F^e-modules is by using generating morphisms. More explicitly, let M be an R-module and $\beta : M \to F^eM$ be a morphism of R-modules. We consider

$$\mathcal{M} = \varinjlim(M \xrightarrow{\beta} F^eM \xrightarrow{F\beta} F^{2e}M \xrightarrow{F^2\beta} \ldots)$$

and note that the Frobenius functor F^e commutes with direct limits; therefore

$$F^e\mathcal{M} = \varinjlim(F^eM \xrightarrow{F\beta} F^{2e}M \xrightarrow{F^2\beta} F^{3e}M \xrightarrow{F^3\beta} \ldots) \cong \mathcal{M}.$$

This gives $\mathcal{M}$ a structure of an F^e-module. In this case we say that $\mathcal{M}$ is *generated by* β, or that β is a *generating morphism for* $\mathcal{M}$. If β is an injective map, then M injects into $\mathcal{M}$. In this case β is called a *root morphism* for $\mathcal{M}$. If β is understood from the context, we only say that M is a *root* for $\mathcal{M}$. If there is a root M which is finitely generated as an R-module, then $\mathcal{M}$ is called an F^e-*finite* F^e-*module*

Example II.2.20. R has a natural structure of an F^e-module for all e. Indeed, this follows from the fact that $F^eR = R$; hence, the structure morphism $\nu : R \to R$ is the identity map.

The next example plays an important role in Chapter IV.

Example II.2.21 ([BMS09, Pag. 6653])**.** For every element $f \in R$ and $r, e \in \mathbb{N}$, we take $\alpha = \frac{r}{p^e - 1}$ and define M_α as the F^e-finite F^e-module that is generated by

$$R \xrightarrow{f^r} F^e R = R.$$

If we choose another representation $\alpha = \frac{r'}{p^{e'} - 1}$ we obtain similarly an $F^{e'}$-module. However, these two are canonically isomorphic as $F^{ee'}$-modules.

We say that $\phi : \mathcal{M} \to \mathcal{N}$ is a morphism of F^e-modules if the following diagram commutes:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\phi"] \arrow[d, "\nu_{\mathcal{M}}"] & \mathcal{N} \arrow[d, "\nu_{\mathcal{N}}"] \\ F\mathcal{M} \arrow[r, "F\phi"] & F\mathcal{N}. \end{tikzcd}$$

The F^e-modules form an Abelian category and the F^e-finite F^e-modules form a full Abelian subcategory. Moreover, if $\mathcal{M}$ is F^e-finite, then $\mathcal{M}_f$ is also F^e-finite. In addition, if R is a local ring, then every F^e-finite F^e-module has a minimal root [Bli04, Lyu97].

Example II.2.22. The localization map $R \to R_f$ is a morphism of F-modules for every $f \in R$. Moreover, the cokernel of the localization map $R \to R_f$ is an F-finite F-module for every $f \in R$. Indeed, $R_f/R = H^1_f(R)$ is generated by $R/fR \xrightarrow{f^{p-1}} F(R/fR) = R/f^p R$.

We recall that every F^e-submodule $M \subset R_f/R$ is a D-module [Lyu97, Example 5.2]. We end this section by stating two important structural properties of F^e-modules.

Theorem II.2.23 ([BB11, Bli04, Lyu97, Theorem 5.13])**.** *If R is a regular F-finite ring of positive characteristic, then every F^e-finite F^e-module over R has finite length in the category of F^e-modules.*

Corollary II.2.24. *If R is a regular F-finite ring of positive characteristic, then every F^e-finite F^e-module over R has finite length in the category of D-modules.*

Proof. By Theorem II.2.23, R satisfies the hypotheses of [Lyu97, Theorem 5.6], which states that every simple F^e-module is a finite direct sum of simple D-modules. Then M has finite length as a D-module. $\square$

CHAPTER III

Constancy Regions in Positive characteristic

In this chapter we prove the first main result. The first section introduces some sets associated to mixed test ideals. In the second section we use these sets to describe the structure of the constancy regions in positive characteristic. We conclude this chapter with an example exhibiting the complexity of the constancy regions. In particular, the example shows that a constancy region in positive characteristic does not need to be a finite unions of rational polytopes.

III.1 Some sets associated to mixed test ideals

In this section we introduce the definitions needed for our study of mixed test ideals and derive some basic properties. Throughout this chapter R denotes a regular ring essentially of finite type over an F-finite field k of positive characteristic.

In order to simplify notation we denote $\mathfrak{a}_1^{c_1}...\mathfrak{a}_n^{c_n}$ by $\mathfrak{a}^{\boldsymbol{c}}$, where $\mathfrak{a} = (\mathfrak{a}_1, ..., \mathfrak{a}_n)$, $\boldsymbol{c} = (c_1, ..., c_n) \in \mathbb{R}^n_{\geq 0}$. We similarly denote the vector $(\lceil r_1 \rceil, ..., \lceil r_n \rceil)$ by $\lceil \boldsymbol{r} \rceil$, where $\boldsymbol{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n_{\geq 0}$.

Definition III.1.1. Given nonzero ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$, and I in R, we define

$$V^I(\mathfrak{a}, p^e) = \left\{ \frac{1}{p^e}\boldsymbol{c} = \left(\frac{c_1}{p^e}, \ldots, \frac{c_n}{p^e}\right) \in \frac{1}{p^e}\mathbb{Z}^n_{\geq 0} | \ \mathfrak{a}^{\boldsymbol{c}} \not\subseteq I^{[p^e]} \right\}$$

and

$$B^I(\mathfrak{a}, p^e) = \bigcup [0, l_1] \times \ldots . \times [0, l_n] \subset \mathbb{R}^n,$$

where the union runs over all $(l_1, \ldots, l_n) \in V^I(\mathfrak{a}, p^e)$.

From this definition it follows that if $e' \geq e$, then $V^I(\mathfrak{a}, p^e) \subseteq V^I(\mathfrak{a}, p^{e'})$ and $B^I(\mathfrak{a}, p^e) \subseteq B^I(\mathfrak{a}, p^{e'})$. Indeed, if $\mathfrak{a}^{\boldsymbol{c}} \not\subseteq I^{[p^e]}$, then there is an element $f \in \mathfrak{a}^{\boldsymbol{c}}$ with

$f \notin I^{[p^e]}$, hence $f^{p^{e'-e}} \in \mathfrak{a}^{p^{e'-e}\boldsymbol{c}}$ and, by the flatness of the Frobenius morphism, we get $f^{p^{e'-e}} \notin I^{[p^{e'}]}$. Therefore $\mathfrak{a}^{p^{e'-e}\boldsymbol{c}} \nsubseteq I^{[p^{e'}]}$ and we get the first inclusion. The second one is then straightforward.

Definition III.1.2. Let $B^I(\mathfrak{a}) = \bigcup_{e>0} B^I(\mathfrak{a}, p^e)$ and define $\chi^I_{\mathfrak{a}} : \mathbb{R}^n \to \mathbb{N}$ to be the characteristic function of the set $B^I(\mathfrak{a})$. That is, $\chi^I_{\mathfrak{a}}(\boldsymbol{c})$ is 1 if $\boldsymbol{c}$ is in $B^I(\mathfrak{a})$ and it is 0 otherwise.

In order to study the sets $B^I(\mathfrak{a})$ it is crucial to understand how they intersect any increasing path. This motivates the following definition.

Definition III.1.3. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$, and $I \neq R$ be nonzero ideals as before and let $\boldsymbol{r} = (r_1, \ldots, r_n) \in \mathbb{Z}^n_{\geq 0}$ be such that that $\mathfrak{a}^{\boldsymbol{r}} \subseteq \mathrm{rad}(I)$. We denote

$$V^I_{\boldsymbol{r}}(\mathfrak{a}, p^e) = \max\{m \in \mathbb{Z}_{\geq 0} |\ \mathfrak{a}^{m\boldsymbol{r}} \nsubseteq I^{[p^e]}\}.$$

Remark III.1.4. While in the definition of $V^I(\mathfrak{a}, p^e)$ one does not require any relation between $\mathfrak{a}$ and I, observe that we require that $\mathfrak{a}^{\boldsymbol{r}} \subseteq \mathrm{rad}(I)$ when we consider $V^I_{\boldsymbol{r}}(\mathfrak{a}, p^e)$.

Note that if $\mathfrak{a}^{m\boldsymbol{r}} \nsubseteq I^{[p^e]}$ then $\mathfrak{a}^{pm\boldsymbol{r}} \nsubseteq I^{[p^{e+1}]}$. Therefore $pV^I_{\boldsymbol{r}}(\mathfrak{a}, p^e) \leq V^I_{\boldsymbol{r}}(\mathfrak{a}, p^{e+1})$, hence

$$\left(\frac{V^I_{\boldsymbol{r}}(\mathfrak{a}, p^e)}{p^e}\right)_{e\geq 1} \tag{III.1.4.1}$$

is a non-decreasing sequence.

Proposition III.1.5. *The sequence (III.1.4.1) is bounded, hence it has a limit.*

Proof. If $\mathfrak{a}^{\boldsymbol{r}}$ is generated by s elements, then $\mathfrak{a}^{(s(p^e-1)+1)\boldsymbol{r}} \subseteq (\mathfrak{a}^{\boldsymbol{r}})^{[p^e]}$. For l large enough such that $\mathfrak{a}^{l\boldsymbol{r}} \subseteq I$, we have $V^I_{\boldsymbol{r}}(\mathfrak{a}, p^e) \leq l(s(p^e-1)+1) - 1$ for all e. Therefore $V^I_{\boldsymbol{r}}(\mathfrak{a}, p^e)/p^e \leq ls$, thus the sequence is bounded. ☐

Definition III.1.6. We call this limit the *F-threshold of $\mathfrak{a}$ associated to I in the direction $\boldsymbol{r} = (r_1, \ldots, r_n)$*, and we denote it by $C^I_{\boldsymbol{r}}(\mathfrak{a})$.

Remark III.1.7. In the case $n = 1$ we recover the usual definition of F-threshold [MTW05], [BMS08, Section 2.5].

Lemma III.1.8. *Let $\frac{1}{p^e}\boldsymbol{b} = (\frac{b_1}{p^e}, \ldots, \frac{b_n}{p^e})$ and $\frac{1}{p^{e'}}\boldsymbol{c} = (\frac{c_1}{p^{e'}}, \ldots, \frac{c_n}{p^{e'}})$ be two elements in $\mathbb{R}^n_{\geq 0}$, such that $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{Z}^n_{\geq 0}$. If $\frac{b_i}{p^e} \leq \frac{c_i}{p^{e'}}$, for every i, and $e' \leq e$ then $(\mathfrak{a}^{\boldsymbol{c}})^{[1/p^{e'}]} \subseteq (\mathfrak{a}^{\boldsymbol{b}})^{[1/p^e]}$.*

Proof. The assertion follows as in [BMS08, Lemma 2.8]. The condition $b_i \leq c_i p^{e-e'}$ implies that $\mathfrak{a}_i^{b_i} \supseteq \mathfrak{a}_i^{c_i p^{e-e'}}$ for every i. Therefore

$$(\mathfrak{a}^{\boldsymbol{b}})^{[1/p^e]} \supseteq (\mathfrak{a}^{p^{e-e'}\boldsymbol{c}})^{[1/p^e]} \supseteq (\mathfrak{a}^{\boldsymbol{c}})^{[1/p^{e'}]},$$

where the last inclusion follows from Proposition II.2.8. $\square$

Proposition III.1.9. *Given any $\boldsymbol{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n_{\geq 0}$, there is $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n) \in \mathbb{R}^n_{>0}$ such that for every $\boldsymbol{r} = (r_1, \ldots, r_n)$ with $0 \leq r_i < \epsilon_i$, we have $\tau(\mathfrak{a}^{\boldsymbol{c}}) = \tau(\mathfrak{a}^{\boldsymbol{c}+\boldsymbol{r}})$.*

Proof. We argue as in the proof of [BMS08, Proposition 2.14]. We first show that there is a vector $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n)$, with $\epsilon_i > 0$ for all i, such that for all vectors $\boldsymbol{r} = (r_1, \ldots, r_n) \in \mathbb{Z}^n_{\geq 0}$ with $c_i \leq \frac{1}{p^e} r_i < c_i + \epsilon_i$ we have that $(\mathfrak{a}^{\boldsymbol{r}})^{[1/p^e]}$ is constant. Indeed, otherwise there are sequences $\boldsymbol{r}_m = (r_{m,1}, \ldots, r_{m,n}) \in \mathbb{Z}^n_{\geq 0}$ and $e_m \in \mathbb{Z}_{\geq 0}$ such that $\frac{1}{p^{e_m}} \boldsymbol{r}_m$ converges to $\boldsymbol{c}$, $\left(\frac{1}{p^{e_m}} r_{m,i}\right)_m$ is a decreasing sequence for every i, $e_m \leq e_{m+1}$, and $(\mathfrak{a}^{\boldsymbol{r}_m})^{[1/p^{e_m}]} \neq (\mathfrak{a}^{\boldsymbol{r}_{m+1}})^{[1/p^{e_{m+1}}]}$. It follows from Lemma III.1.8 that $(\mathfrak{a}^{\boldsymbol{r}_m})^{[1/p^{e_m}]} \subsetneq (\mathfrak{a}^{\boldsymbol{r}_{m+1}})^{[1/p^{e_{m+1}}]}$ for all m, but this contradicts the fact that R is Noetherian.

Assume now that $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n)$ is as above and let $I = (\mathfrak{a}^{\boldsymbol{r}})^{[1/p^e]}$ for all $\boldsymbol{r} = (r_1, ..., r_n) \in \mathbb{Z}^n_{\geq 0}$ with $c_i \leq \frac{1}{p^e} r_i < c_i + \epsilon_i$. We show that $I = \tau(\mathfrak{a}^{\boldsymbol{c}})$. Take e large enough such that $\tau(\mathfrak{a}^{\boldsymbol{c}}) = (\mathfrak{a}^{\lceil p^e \boldsymbol{c} \rceil})^{[1/p^e]}$ and $\frac{\lceil p^e c_i \rceil}{p^e} < c_i + \epsilon_i$ for every i. If all $p^e c_i$ are non-integers then $\frac{\lceil p^e c_i \rceil}{p^e} > c_i$ and $\tau(\mathfrak{a}^{\boldsymbol{c}}) = I$. Let us suppose that $p^e c_i$ is an integer precisely when $i = i_1, ..., i_l$. Let $\boldsymbol{d} = (d_1, \ldots, d_n)$ be the vector whose i_j coordinates are 1 and all the other are 0. As e is arbitrarily large we may also assume that $c_i < c_i + \frac{1}{p^e} d_i < c_i + \epsilon_i$ for all $i \in \{i_1, \ldots, i_l\}$, hence $I = (\mathfrak{a}^{\lceil p^e \boldsymbol{c} \rceil + \boldsymbol{d}})^{[1/p^e]} \subseteq (\mathfrak{a}^{\lceil p^e \boldsymbol{c} \rceil})^{[1/p^e]} = \tau(\mathfrak{a}^{\boldsymbol{c}})$.

The reverse inclusion follows by showing $\mathfrak{a}^{\lceil p^e \boldsymbol{c} \rceil} \subseteq I^{[p^e]}$. Let $u \in \mathfrak{a}^{\lceil p^e \boldsymbol{c} \rceil}$. If $e' > e$ and e' is large enough, then $c_i < c_i + \frac{1}{p^{e'}} < c_i + \epsilon_i$, hence $\mathfrak{a}^{\lceil p^{e'} \boldsymbol{c} \rceil + \boldsymbol{1}} \subseteq I^{[p^{e'}]}$. Here $\boldsymbol{1}$ denotes the vector whose coordinates are all 1. Thus, for v a nonzero element in $\mathfrak{a}_1 \cdots \mathfrak{a}_n$ we have

$$vu^{p^{e'-e}} \in \mathfrak{a}^{p^{e'-e} \lceil p^e \boldsymbol{c} \rceil + \boldsymbol{1}} \subseteq \mathfrak{a}^{\lceil p^{e'} \boldsymbol{c} \rceil + \boldsymbol{1}} \subseteq (I^{[p^e]})^{[p^{e'-e}]}.$$

This implies that u is in the tight closure of $I^{[p^e]}$, but as R is a regular ring, the tight closure of $I^{[p^e]}$ is equal to $I^{[p^e]}$(see [HH90]). This gives $\mathfrak{a}^{\lceil p^e \boldsymbol{c} \rceil} \subseteq I^{[p^e]}$ hence, by definition, $\tau(\mathfrak{a}^{\boldsymbol{c}}) = (\mathfrak{a}^{\lceil p^e \boldsymbol{c} \rceil})^{[1/p^e]} \subseteq I$. $\square$

Definition III.1.10. A positive real number c is called an *F-jumping number of $\mathfrak{a}$ in the direction $\boldsymbol{r} \neq \boldsymbol{0} \in \mathbb{Z}^n_{\geq 0}$*, if c is such that $\tau(\mathfrak{a}^{c\boldsymbol{r}}) \neq \tau(\mathfrak{a}^{(c-\epsilon)\boldsymbol{r}})$ for every real number $\epsilon > 0$.

Proposition III.1.11. *If $r \in \mathbb{Z}^n_{\geq 0}$ and $\lambda \in \mathbb{R}_{\geq 0}$, then*

$$\tau(\mathfrak{a}^{\lambda r_1} \cdots \mathfrak{a}^{\lambda r_n}) = \tau(\mathfrak{J}^{\lambda}),$$

where $\mathfrak{J} = \mathfrak{a}_1^{r_1} \cdots \mathfrak{a}_n^{r_n}$.

Proof. By Propostion III.1.9, we may assume $\lambda = \frac{s}{p^{e'}}$ with $s \in \mathbb{Z}_{\geq 0}$. For e sufficiently large, we have

$$\tau(\mathfrak{a}^{\lambda r_1} \cdots \mathfrak{a}^{\lambda r_n}) = (\mathfrak{a}^{\lceil \lambda r_1 p^e \rceil} \cdots \mathfrak{a}^{\lceil \lambda r_n p^e \rceil})^{[1/p^e]} = (\mathfrak{a}^{s r_1 p^{e-e'}} \cdots \mathfrak{a}^{s r_n p^{e-e'}})^{[1/p^e]}$$

$$= ((\mathfrak{a}^{r_1} \cdots \mathfrak{a}^{r_n})^{s p^{e-e'}})^{[1/p^e]} = ((\mathfrak{a}^{r_1} \cdots \mathfrak{a}^{r_n})^{\lambda p^e})^{[1/p^e]} = \tau(\mathfrak{J}^{\lambda}).$$

□

Corollary III.1.12. *The F-threshold of $\mathfrak{a}$ associated to I in the direction $\boldsymbol{r} = (r_1, \ldots, r_n)$ is equal to the F-threshold of $\mathfrak{a}_1^{r_1} \cdots \mathfrak{a}_n^{r_n}$ associated to I.*

Corollary III.1.13. *The set of F -jumping numbers of $\mathfrak{a}$ in direction $\mathbf{r}$ is equal to the set of F-jumping numbers of $\mathfrak{a}^{\mathbf{r}}$.*

Therefore [BMS08, Corollary 2.30] implies the following.

Corollary III.1.14. *The set of F-jumping numbers of $\mathfrak{a}$ in the direction $\boldsymbol{r}$ is equal to the set of F-thresholds of $\mathfrak{a}$, associated to various ideals I, in the direction $\boldsymbol{r}$.*

Given $l_1, \ldots, l_n$ positive real numbers we denote by $[\mathbf{0}, \boldsymbol{l}]$ the set $[0, l_1] \times \ldots \times [0, l_n]$.

Proposition III.1.15. *Given nonzero ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ of R, where R is a regular, F-finite ring essentially of finite type over a finite field, the set $\{\tau(\mathfrak{a}^{\boldsymbol{c}}) | \ \boldsymbol{c} \in [\mathbf{0}, \boldsymbol{l}]\}$ is finite.*

Proof. Since R is assumed to be essentially of finite type over k, arguing as in the proof of [BMS08, Theorem 3.1], one can see that the assertion for all such R follows if we know it for $R = k[x_1, \ldots x_r]$, with $r \geq 1$. We will therefore assume that we are in this case.

By Lemma III.1.9, we may assume that $\boldsymbol{c} = (\frac{\alpha_1}{p^e}, \ldots, \frac{\alpha_n}{p^e})$ with $\alpha_i \in \mathbb{Z}_{\geq 0}$ and $e \geq 1$. Let d be an upper bound for the degrees of the generators of $\mathfrak{a}_\mathfrak{i}$, for all i. By Theorem II.2.15 we have that $\tau(\mathfrak{a}^{\boldsymbol{c}})$ is generated by polynomials of degree $\leq ndL$, where $L = \max\{l_i\}$. Since k is finite, there are only finitely many sets consisting of polynomials of bounded degree and therefore only finitely many ideals $\tau(\mathfrak{a}^{\boldsymbol{c}})$ where $\boldsymbol{c} \in [\mathbf{0}, \boldsymbol{l}]$. □

Definition III.1.16. The *constancy region* for a test ideal $\tau(\mathfrak{a}^{\mathbf{c}})$ is defined as the set of points $\boldsymbol{c'} \in \mathbb{R}^n_{\geq 0}$ such that $\tau(\mathfrak{a}^{\mathbf{c}}) = \tau(\mathfrak{a}^{\mathbf{c}'})$.

Lemma III.1.17. If J is an ideal of R, then $B^J(\mathfrak{a})$ consists of the points $\mathbf{c} \in \mathbb{R}^n_{\geq 0}$ such that $\tau(\mathfrak{a}^{\mathbf{c}}) \not\subseteq J$.

Proof. Assume first that $\mathbf{c} = (\frac{\alpha_1}{p^e}, \ldots, \frac{\alpha_n}{p^e})$ with $\alpha_i \in \mathbb{N}$. Choose a representation of $\mathbf{c}$ with e large enough such that $\tau(\mathfrak{a}^{\mathbf{c}}) = (\mathfrak{a}^{\alpha})^{[1/p^e]}$. In this case we have

$$\mathbf{c} \in B^J(\mathfrak{a}) \Longleftrightarrow \mathfrak{a}^{\alpha} \not\subseteq J^{[p^e]} \Longleftrightarrow (\mathfrak{a}^{\alpha})^{[1/p^e]} \not\subseteq J \Longleftrightarrow \tau(\mathfrak{a}^{\mathbf{c}}) \not\subseteq J.$$

For the general case, let $\mathbf{c} \in B^J(\mathfrak{a})$, this implies that $\mathbf{c} \in B^J(\mathfrak{a}, p^e)$ for some e. Therefore we can find $\mathbf{r} = (\frac{\alpha_1}{p^e}, \ldots, \frac{\alpha_n}{p^e}) \in B^J(\mathfrak{a}, p^e) \subseteq B^J(\mathfrak{a})$, with $\alpha_i \in \mathbb{N}$, $\frac{\alpha_i}{p^e} \geq c_i$. By the first part this implies $\tau(\mathfrak{a}^{\mathbf{r}}) \not\subseteq J$, but as $\frac{\alpha_i}{p^e} \geq c_i$ for all i, we have that $\tau(\mathfrak{a}^{\mathbf{r}}) \subseteq \tau(\mathfrak{a}^{\mathbf{c}})$ hence $\tau(\mathfrak{a}^{\mathbf{c}}) \not\subseteq J$.

For the reverse inclusion, let $\mathbf{c} \in \mathbb{R}^n_{\geq 0}$ be such that $\tau(\mathfrak{a}^{\mathbf{c}}) \not\subseteq J$. By Proposition III.1.9 there is a point $\mathbf{r} = (\frac{\alpha_1}{p^e}, \ldots, \frac{\alpha_n}{p^e})$ with $\alpha_i \in \mathbb{N}$, $\frac{\alpha_i}{p^e} \geq c_i$ and $\tau(\mathfrak{a}^{\mathbf{c}}) = \tau(\mathfrak{a}^{\mathbf{r}})$, therefore $\tau(\mathfrak{a}^{\mathbf{r}}) \not\subseteq J$. We use the first part again and conclude $\mathbf{r} \in B^J(\mathfrak{a})$, but as $\frac{\alpha_i}{p^e} \geq c_i$ for all i, we deduce that $\mathbf{c} \in B^J(\mathfrak{a})$. □

Theorem III.1.18. *If $\mathfrak{a}_1, ..., \mathfrak{a}_n$ are all contained in a maximal ideal $\mathfrak{m}$, then for each $\boldsymbol{c} \in \mathbb{R}^n_{\geq 0}$, there exist ideals $I_1, ..., I_d$ and J such that the constancy region for the test ideal $\tau(\mathfrak{a}^{\mathbf{c}})$ is given by $\bigcap_{i=1,\ldots,d} B^{I_i}(\mathfrak{a}) \backslash B^J(\mathfrak{a})$.*

Proof. We first show that this constancy region is bounded. As $\mathfrak{a}_i \subseteq \mathfrak{m}$ for all i we have that for any $\boldsymbol{c'} \in \mathbb{R}^n_{\geq 0}$and e sufficiently large

$$\tau(\mathfrak{a}^{\mathbf{c}'}) = (\mathfrak{a}_1^{\lceil c_1' p^e \rceil} \cdots \mathfrak{a}_n^{\lceil c_n' p^e \rceil})^{[1/p^e]} \subseteq (\mathfrak{m}^{\lceil c_1' p^e \rceil + \ldots + \lceil c_n' p^e \rceil})^{[1/p^e]}$$

$$\subseteq (\mathfrak{m}^{\lceil c_1' p^e + \ldots + c_n' p^e \rceil - n})^{[1/p^e]} \subseteq \mathfrak{m}^{\lceil c_1' + \ldots + c_n' \rceil - n + 1}.$$

Since $\cap_s \mathfrak{m}^s = 0$, there is L such that $\tau(\mathfrak{a}^{\mathbf{c}}) \not\subseteq \mathfrak{m}^L$, we deduce that for any $\mathbf{c}'$ in the constancy region $\tau(\mathfrak{a}^{\mathbf{c}'}) = \tau(\mathfrak{a}^{\mathbf{c}}) \not\subseteq \mathfrak{m}^L$, hence $c_1' + \ldots + c_n' \leq L$. This implies that the constancy region for $\tau(\mathfrak{a}^{\mathbf{c}})$ is bounded.

To deduce our description consider a sufficiently large hypercube $[\mathbf{0}, \boldsymbol{l}]$ containing the constancy region for $\tau(\mathfrak{a}^{\mathbf{c}})$. By Propostion III.1.15, we know that the set $\mathcal{A} = \{\tau(\mathfrak{a}^{\mathbf{c}}) | \, \boldsymbol{c} \in [\mathbf{0}, \boldsymbol{l}]\}$ is finite. Let $I_1, \ldots, I_d$ be the ideals in $\mathcal{A}$ that are strictly contained in $\tau(\mathfrak{a}^{\mathbf{c}})$ and let $J = \tau(\mathfrak{a}^{\mathbf{c}})$. We claim that the constancy region for $\tau(\mathfrak{a}^{\mathbf{c}})$ is equal to $\bigcap_{i=1,\ldots,d} B^{I_i}(\mathfrak{a}) \backslash B^J(\mathfrak{a})$. Lemma III.1.17 implies that the set $\bigcap_{i=1,\ldots,d} B^{I_i}(\mathfrak{a}) \backslash B^J(\mathfrak{a})$ is equal to the set of all $\mathbf{r}$ such that $\tau(\mathfrak{a}^{\mathbf{r}}) \not\subseteq I_i$ for all i and $\tau(\mathfrak{a}^{\mathbf{c}}) \subseteq \tau(\mathfrak{a}^{\mathbf{r}})$, or equivalently, $\tau(\mathfrak{a}^{\mathbf{r}}) = \tau(\mathfrak{a}^{\mathbf{c}})$ by our choice of I_i. $\square$

Remark III.1.19. We can remove the condition that all ideals $\mathfrak{a}_i$ are contained in a maximal ideal and still get a similar description. Explicitly, in each hypercube $[\mathbf{0}, \boldsymbol{l}]$ the constancy region is given by $\bigcap_{i=1,\ldots,d} B^{I_i}(\mathfrak{a}) \backslash B^J(\mathfrak{a}) \cap [\mathbf{0}, \boldsymbol{l}]$, for suitable $I_1, \ldots, I_d$ and J.

We now give a version of Skoda's theorem for mixed test ideals (see [BMS08, Proposition 2.25] for the case of one ideal). This theorem allows us to describe the constancy regions in the first octant by describing only the constancy regions in a sufficiently large hypercube $[\mathbf{0}, \boldsymbol{l}] = [0, l_1] \times \ldots \times [0, l_n]$.

Theorem III.1.20. *(Skoda's Theorem) Let $\boldsymbol{e_1}, ..., \boldsymbol{e_n}$ be the standard basis for $\mathbb{R}^n$, and assume $1 \leq i \leq n$. If $\mathfrak{a}_i$ is generated by m_i elements, then for every $\boldsymbol{s} = (s_1, \ldots, s_n)$ with $s_i \geq m_i$, we have*

$$\tau(\mathfrak{a}^{\boldsymbol{s}}) = \mathfrak{a}^{\boldsymbol{e_i}} \tau(\mathfrak{a}^{\boldsymbol{s}-\boldsymbol{e_i}}).$$

Proof. We only need to prove $(\mathfrak{a}^{\lceil p^e \boldsymbol{s} \rceil})^{[1/p^e]} = \mathfrak{a}^{\boldsymbol{e_i}} (\mathfrak{a}^{\lceil p^e (\boldsymbol{s}-\boldsymbol{e_i}) \rceil})^{[1/p^e]}$ for e large enough.

Let $\boldsymbol{d} = (d_1, .., d_n)$ be a vector with integer coordinates and $d_i \geq p^e s_i$. We want to show that

$$(\mathfrak{a}^{\boldsymbol{d}})^{[1/p^e]} = \mathfrak{a}^{\boldsymbol{e_i}} (\mathfrak{a}^{\boldsymbol{d}-p^e \boldsymbol{e_i}})^{[1/p^e]},$$

from which the result follows.

Since $\mathfrak{a}^{\boldsymbol{d}-p^e\boldsymbol{e_i}} \cdot \mathfrak{a}_i^{[p^e]} \subseteq \mathfrak{a}^{\boldsymbol{d}} \subseteq ((\mathfrak{a}^{\boldsymbol{d}})^{[1/p^e]})^{[p^e]}$, we have

$$\mathfrak{a}^{\boldsymbol{d}-p^e\boldsymbol{e_i}} \subseteq \left(((\mathfrak{a}^{\boldsymbol{d}})^{[1/p^e]})^{[p^e]} : \mathfrak{a}_i^{[p^e]}\right) = \left((\mathfrak{a}^{\boldsymbol{d}})^{[1/p^e]} : \mathfrak{a}_i\right)^{[p^e]},$$

where the equality is consequence of the flatness of Frobenius. Therefore

$$(\mathfrak{a}^{\boldsymbol{d}-p^e\boldsymbol{e_i}})^{[1/p^e]} \subseteq ((\mathfrak{a}^{\boldsymbol{d}})^{[1/p^e]} : \mathfrak{a}_i),$$

that is,

$$\mathfrak{a}^{\boldsymbol{e_i}} (\mathfrak{a}^{\boldsymbol{d}-p^e\boldsymbol{e_i}})^{[1/p^e]} \subseteq (\mathfrak{a}^{\boldsymbol{d}})^{[1/p^e]}.$$

For the reverse inclusion, note that since $d_i \geq m_i(p^e - 1) + 1$, in the product of d_i of the generators of $\mathfrak{a}_i$ at least one should appear with multiplicity $\geq p^e$. Therefore $\mathfrak{a}^{\boldsymbol{d}} = \mathfrak{a}_i^{[p^e]} \cdot \mathfrak{a}^{\boldsymbol{d}-p^e\boldsymbol{e_i}}$, hence

$$\mathfrak{a}^{\boldsymbol{d}} \subseteq \mathfrak{a}_i^{[p^e]} \cdot \mathfrak{a}^{\boldsymbol{d}-p^e\boldsymbol{e_i}} \subseteq \mathfrak{a}_i^{[p^e]} \cdot \left((\mathfrak{a}^{\boldsymbol{d}-p^e\boldsymbol{e_i}})^{[1/p^e]}\right)^{[p^e]} = \left(\mathfrak{a}^{\boldsymbol{e_i}} \cdot (\mathfrak{a}^{\boldsymbol{d}-p^e\boldsymbol{e_i}})^{[1/p^e]}\right)^{[p^e]},$$

which clearly implies $(\mathfrak{a}^{\boldsymbol{d}})^{[1/p^e]} \subseteq \mathfrak{a}^{\boldsymbol{e_i}}(\mathfrak{a}^{\boldsymbol{d}-p^e\boldsymbol{e_i}})^{[1/p^e]}$. ☐

Proposition III.1.21. *If c is an F-jumping number in the direction $\boldsymbol{r} = (r_1, \ldots, r_n)$ then also cp is an F-jumping number in the direction $\boldsymbol{r}$.*

Proof. Note that $V^I_{\boldsymbol{r}}(\mathfrak{a}, p^{e+1}) = V^{I^{[p]}}_{\boldsymbol{r}}(\mathfrak{a}, p^e)$, hence $pC^I_{\boldsymbol{r}}(\mathfrak{a}) = C^{I^{[p]}}_{\boldsymbol{r}}(\mathfrak{a})$. ☐

III.2 The constancy regions

In this section we prove our main result, Theorem III.2.6 below. We begin by recalling our definition of p-fractals.

Let $\mathcal{F}$ be the algebra of functions $\phi : \mathbb{R}^n_{\geq 0} \to \mathbb{Q}$. For each $q = p^e$ and every $\boldsymbol{b} = (b_1, \ldots, b_n) \in \mathbb{Z}^n$ with $0 \leq b_i < q$ we define a family of operators $T_{q|\boldsymbol{b}} : \mathcal{F} \to \mathcal{F}$ by

$$T_{q|\boldsymbol{b}}\phi(t_1, \ldots, t_n) = \phi((t_1 + b_1)/q, \ldots, (t_n + b_n)/q).$$

Definition III.2.1. Let $\phi : [0, l]^n \to \mathbb{Q}$ be a map and let denote also by ϕ its extension by zero to $\mathbb{R}^n_{\geq 0}$. We say that ϕ is a *p-fractal* if all the $T_{q|\boldsymbol{b}}\phi$ span a finite dimensional $\mathbb{Q}$-subspace V of $\mathcal{F}$. Furthermore, we say that an arbitrary $\phi \in \mathcal{F}$ is a *p-fractal* if its restriction to each hypercube $[\boldsymbol{0}, \boldsymbol{l}]$ is a p-fractal.

Remark III.2.2. This definition is similar to the one in [MT04, Definition 2.1]. The only difference is that in [MT04, Definition 2.1] the domain of the functions is the hypercube $[0, 1] \times \ldots \times [0, 1]$.

In this section we assume that R is a regular, F-finite ring essentially of finite type over a finite field of characteristic $p > 0$, and $\mathfrak{a}_i \subseteq R$ are nonzero ideals.

Lemma III.2.3. *Let $\boldsymbol{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n_{\geq 0}$, and $\mathfrak{a}_1, \ldots \mathfrak{a}_n$ be nonzero ideals of R then $\tau(\mathfrak{a}^{\boldsymbol{c}})^{[1/p^e]} = \tau(\mathfrak{a}^{\frac{1}{p^e}\boldsymbol{c}})$.*

Proof. Taking k large enough

$$\tau(\mathfrak{a}^{\mathbf{c}})^{[1/p^e]} = \left((\mathfrak{a}_1^{\lceil c_1p^k\rceil} \cdots \mathfrak{a}_n^{\lceil c_np^k\rceil})^{[1/p^k]}\right)^{[1/p^e]}$$

and by [BMS08, Lemma 2.4] the later contains

$$(\mathfrak{a}_1^{\lceil c_1p^k\rceil} \cdots \mathfrak{a}_n^{\lceil c_np^k\rceil})^{[1/p^{k+e}]} = (\mathfrak{a}_1^{\lceil \frac{c_1}{p^e}p^{k+e}\rceil} \cdots \mathfrak{a}_n^{\lceil \frac{c_n}{p^e}p^{k+e}\rceil})^{[1/p^{k+e}]}$$

$$= \tau(\mathfrak{a}^{\frac{1}{p^e}\boldsymbol{c}}).$$

Therefore $\tau(\mathfrak{a}^{\mathbf{c}})^{[1/p^e]} \supseteq \tau(\mathfrak{a}^{\frac{1}{p^e}\boldsymbol{c}})$.

For the other inclusion note that

$$\tau(\mathfrak{a}^{\mathbf{c}}) = (\mathfrak{a}_1^{\lceil c_1 p^k \rceil} \cdots \mathfrak{a}_n^{\lceil c_n p^k \rceil})^{[1/p^k]}$$

$$= (\mathfrak{a}_1^{\lceil \frac{c_1}{p^e} p^{k+e} \rceil} \cdots \mathfrak{a}_n^{\lceil \frac{c_n}{p^e} p^{k+e} \rceil})^{[p^e/p^{k+e}]}$$

that by [BMS08, Lemma 2.4] is contained in

$$\left((\mathfrak{a}_1^{\lceil \frac{c_1}{p^e} p^{k+e} \rceil} \cdots \mathfrak{a}_n^{\lceil \frac{c_n}{p^e} p^{k+e} \rceil})^{[1/p^{k+e}]} \right)^{[p^e]} = \tau(\mathfrak{a}^{\frac{1}{p^e}\boldsymbol{c}})^{[p^e]}$$

but this is equivalent to say

$$\tau(\mathfrak{a}^{\mathbf{c}})^{[1/p^e]} \subseteq \tau(\mathfrak{a}^{\frac{1}{p^e}\boldsymbol{c}}).$$

□

Lemma III.2.4. *Let $\boldsymbol{l} = (l_1, \ldots l_n) \in \mathbb{Z}^n$ be such that l_i is the minimum number of generators of the ideal $\mathfrak{a}_\mathfrak{i}$. Let $\boldsymbol{b} \in \mathbb{Z}^n$ such that $l_i - 1 \leq b_i$. For all e, we have*

$$T_{p^e|\boldsymbol{b}} \chi_{\mathfrak{a}}^I = T_{p^0|(\boldsymbol{l}-[1])} \chi_{\mathfrak{a}}^{(I^{[p^e]}:\mathfrak{a}^{\boldsymbol{b}-\boldsymbol{l}+\mathbf{1}})},$$

where $\chi_{\mathfrak{a}}^I$ denotes the characteristic function introduced in Definition III.1.2.

Proof. We have that

$$T_{p^e|\boldsymbol{b}} \chi_{\mathfrak{a}}^I([t]) = \chi_{\mathfrak{a}}^I \left(\frac{1}{p^e}([t] + \boldsymbol{b}) \right)$$

is equal to 1 if and only if, by Lemma III.1.17, to

$$\tau(\mathfrak{a}^{\frac{1}{p^e}(\boldsymbol{t}+\boldsymbol{b})}) \not\subseteq I,$$

and by Lemma III.2.3 this is

$$\tau(\mathfrak{a}^{\boldsymbol{t}+\boldsymbol{b}})^{[1/p^e]} \not\subseteq I,$$

but the later is equivalent to

$$\tau(\mathfrak{a}^{\boldsymbol{t}+\boldsymbol{b}}) \not\subseteq I^{[p^e]}.$$

As $b_i \geq l_i - 1$ by Skoda's Theorem the previous expresion becomes

$$\mathfrak{a}^{\boldsymbol{b}-\boldsymbol{l}+\mathbf{1}} \cdot \tau(\mathfrak{a}^{\boldsymbol{t}+\boldsymbol{l}-\mathbf{1}}) \not\subseteq I^{[p^e]}$$

Wich in turn is equivalent to

$$\tau(\mathfrak{a}^{\boldsymbol{t}+\boldsymbol{l}-\mathbf{1}}) \not\subseteq (I^{[p^e]} : \mathfrak{a}^{\boldsymbol{b}-\boldsymbol{l}+\mathbf{1}})$$

but this is the case if and only if

$$T_{p^0|(\boldsymbol{l}-\mathbf{1})}\chi_{\mathfrak{a}}^{(I^{[p^e]}:\mathfrak{a}^{\boldsymbol{b}-\boldsymbol{l}+\mathbf{1}})}(\boldsymbol{t}) = \chi_{\mathfrak{a}}^{(I^{[p^e]}:\mathfrak{a}^{\boldsymbol{b}-\boldsymbol{l}+\mathbf{1}})}(\boldsymbol{t}+\boldsymbol{l}-\mathbf{1})$$

is equal to 1

Note that a point of the form $\frac{1}{p^k}\boldsymbol{r} + (\boldsymbol{l} - \mathbf{1})$ with $\boldsymbol{r} \in \mathbb{Z}^n$ is in $B^{(I^{[p^e]}:\mathfrak{a}^{b-l+1})}(\mathfrak{a})$ if and only if $\mathfrak{a}^{\boldsymbol{r}+p^k(\boldsymbol{l}-\mathbf{1})} \not\subseteq (I^{[p^e]} : \mathfrak{a}^{\boldsymbol{b}-\boldsymbol{l}+\mathbf{1}})^{[p^k]}$ if and only if $\mathfrak{a}^{\boldsymbol{r}} \cdot \mathfrak{a}^{p^k(\boldsymbol{l}-\mathbf{1})} \cdot (\mathfrak{a}^{\boldsymbol{b}-\boldsymbol{l}+\mathbf{1}})^{[p^k]} \not\subseteq I^{[p^{e+k}]}$ this by Lemma III.2.3 occurs if and only if $\mathfrak{a}^{\boldsymbol{r}+p^k\boldsymbol{b}} \not\subseteq I^{[p^{e+k}]}$, or equivalently $\frac{1}{p^{e+k}}\boldsymbol{r} + \frac{1}{p^e}\boldsymbol{b} \in B^I(\mathfrak{a})$. From this the result follows easily. □

This lemma is especially useful when the ideals are principal, as we will see in the examples of Section 5.

Lemma III.2.5. *If R is a regular, F-finite ring essentially of finite type over a finite field, then in each hypercube $[\mathbf{0}, \boldsymbol{l}]$ there are only finitely many functions $\chi_{\mathfrak{a}}^I$. That is, the set $\{\chi_{\mathfrak{a}}^I|_{[\mathbf{0},\boldsymbol{l}]}; I \subseteq R\}$ is finite.*

Proof. By Lemma III.1.17, $B^I(\mathfrak{a})$ is the set of all points $\boldsymbol{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n_{\geq 0}$ such that $\tau(\mathfrak{a}^{\boldsymbol{c}}) \not\subseteq I$, hence $B^I(\mathfrak{a})$ is a union of constancy regions. By Lemma III.1.15, we know that there are only finitely many constancy regions for bounded exponents, therefore there are only finitely many functions $\chi_{\mathfrak{a}}^I|_{[\mathbf{0},\boldsymbol{l}]}$. □

Theorem III.2.6. There is a p-fractal function $\varphi : \mathbb{R}^n_{\geq 0} \to \mathbb{N}$ for which

$$\tau(\mathfrak{a}_1^{c_1} ... \mathfrak{a}_n^{c_n}) = \tau(\mathfrak{a}_1^{d_1} ... \mathfrak{a}_n^{d_n}) \Longleftrightarrow \varphi(c_1, ..., c_n) = \varphi(d_1, ..., d_n),$$

and therefore the constancy regions are of the form $\varphi^{-1}(i)$ for some number i.

Proof. We first show that the functions $\chi_{\mathfrak{a}}^I$ are p-fractal. We want to prove that all the $T_{p^e|\boldsymbol{b}}\chi_{\mathfrak{a}}^I$ span a finite dimensional space. Lemma III.2.4 states that all but finitely many of these functions have the form $T_{p^0|(\boldsymbol{l}-\mathbf{1})}\chi_{\mathfrak{a}}^J$ for different ideals J. Lemma III.2.5 ensures that there are only finitely many of those in each hypercube $[\mathbf{0}, \boldsymbol{l}]$. From this it follows that $\chi_{\mathfrak{a}}^I$ is a p-fractal.

For $\boldsymbol{c} \in \mathbb{R}^n_{\geq 0}$, let $\eta_{\boldsymbol{c}}$ be the characteristic function associated to the constancy region $\tau(\mathfrak{a}^{\boldsymbol{c}})$. Remark III.1.19 implies that that in each hypercube $[\mathbf{0}, \boldsymbol{l}]$, $\eta_{\boldsymbol{c}}|_{[\mathbf{0},\boldsymbol{l}]} = \left(\chi_{\mathfrak{a}}^{I_1} \cdots \chi_{\mathfrak{a}}^{I_d} - \chi_{\mathfrak{a}}^{J}\right)\big|_{[\mathbf{0},\boldsymbol{l}]}$ for some ideals $I_1, \ldots, I_d$ and J, therefore $\eta_{\boldsymbol{c}}$ is p-fractal.

Clearly there are countably many constancy regions, so we can numerate them. For every i, let $\boldsymbol{c}_i = (c_{i1}, \ldots, c_{in})$ a point in the i-th constancy region, and we define

$$\varphi = \sum_{i \in \mathbb{N}} i \cdot \eta_{\boldsymbol{c}_i}.$$

This function satisfies the desired conditions. □

Corollary III.2.7. *Let R be a regular, F-finite ring essentially of finite type over a finite field. Let $\eta_{\boldsymbol{c}}$ be the characteristic function associated to the constancy region of $\tau(\mathfrak{a}^{\boldsymbol{c}})$, then $\eta_{\boldsymbol{c}}$ is a p- fractal.*

III.3 Some examples

In section III.2 we showed that the characteristic functions of the constancy regions are p-fractal functions, see Corollary III.2.7. We use this fact and Proposition II.2.9 to compute an explicit example. Throughout this section we use a subscript $*_p$ to denote that a number $*$ is written in base p. More explicitly, given integers $0 \leq a_i \leq p-1$ by $(a_r a_{r-1} \ldots a_0.a_{-1} \ldots a_{-l})_p$ we mean the rational number $\sum_{j=-l}^{r} m_j p^j$. For example, 12.1_3 denotes the number $1 \cdot 3 + 2 \cdot 3^0 + 1 \cdot 3^{-1} = \frac{16}{3}$ written in base 3. One of the main tools for computing examples is the following theorem:

Theorem III.3.1. *(Lucas' Theorem [Luc78]) Fix non-negative integers $m \geq n \in \mathbb{N}$ and a prime number p. If we write m and n in their base p expansions: $m = (m_r m_{r-1} \ldots m_1 m_0)_p = \sum_{j=0}^{r} m_j p^j$ and $n = (n_r n_{r-1} \ldots n_1 n_0)_p = \sum_{j=0}^{r} n_j p^j$, then*

$$\binom{m}{n} \equiv \binom{m_0}{n_0} \cdot \binom{m_1}{n_1} \cdots \binom{m_r}{n_r} \mod p$$

where we interpret $\binom{a}{b}$ as zero if $a < b$. In particular, $\binom{m}{n}$ is nonzero mod p if and only if $m_j \geq n_j$ for all $j = 1, \ldots r$.

Remark III.3.2. In particular, if $m = p^k - 1$, then all coefficients in the expansion of $(x+y)^m$ are nonzero.

Example III.3.3. (The Devil's Staircase) Let $R = \mathbb{F}_3[x, y]$, $f_1 = x + y$, and $f_2 = xy$. We want to describe the constancy regions for the test ideals $\tau(f^{\boldsymbol{c}})$.

We first show that there are five different test ideals in the region $[0,1] \times [0,1]$. More precisely, we show that

$$\tau(f^{\boldsymbol{c}}) = \begin{cases} R \text{ or } (x,y), & \boldsymbol{c} \in [0,1) \times [0,1) \\ (x+y), & \boldsymbol{c} \in \{1\} \times [0,1) \\ (xy), & \boldsymbol{c} \in [0,1) \times \{1\} \\ (xy(x+y)), & \boldsymbol{c} = (1,1). \end{cases}$$

We want to compute the test ideal at $(\frac{1}{3}, \frac{2}{3})$. By Proposition III.1.11, we have

$$\tau(f^{(0.1_3, 0.2_3)}) = \tau((f_1 \cdot f_2^2)^{\frac{1}{3}}).$$

By Proposition II.2.11, we have

$$\tau((f_1 \cdot f_2^2)^{\frac{1}{3}}) = ((x+y)(xy)^2)^{[\frac{1}{3}]} = (x^3y^2 + x^2y^3)^{[\frac{1}{3}]}.$$

Finally, Proposition II.2.9 gives

$$(x^3y^2 + x^2y^3)^{[\frac{1}{3}]} = (x,y),$$

and therefore

$$\tau(f_1^{c_1} \cdot f_2^{c_2}) \subseteq (x,y) \text{ if } c_1 \geq 1/3 \text{ and } c_2 \geq 2/3.$$

In particular, the test ideal associated to the points $(1 - \frac{1}{3^k}, 1 - \frac{1}{3^k})$ is contained in (x,y). Now

$$\begin{aligned} \tau(f^{(1-\frac{1}{3^k}, 1-\frac{1}{3^k})}) &= ((x+y)^{3^k-1}(xy)^{3^k-1})^{[\frac{1}{3^k}]} \\ &= ((x^2y + xy^2)^{3^k-1})^{[\frac{1}{3^k}]}. \end{aligned}$$

Since the terms $x^{2(3^k-1)}y^{3^k-1}$ and $x^{3^k-1}y^{2(3^k-1)}$ appear in the expansion of $(x^2y + xy^2)^{3^k-1}$ with nonzero coefficient, we conclude that $\tau(f^{(1-\frac{1}{3^k}, 1-\frac{1}{3^k})}) \supseteq (x,y)$. Therefore

$$\tau(f^{(1-\frac{1}{3^k}, 1-\frac{1}{3^k})}) = (x,y).$$

Thus there are only two test ideals in the region $[0,1) \times [0,1)$, namely R and (x,y). Clearly $\tau(f^{(1,0)}) = (x+y)$, and by Skoda's Theorem

$$\tau(f^{(1, 1-\frac{1}{3^k})}) = f_1 \cdot \tau(f^{(0, 1-\frac{1}{3^k})})$$

$$= (x+y)\cdot((xy)^{3^k-1})^{[\frac{1}{3^k}]}$$

$$= (x+y),$$

hence the only test ideal in the region $\{1\}\times[0,1)$ is $(x+y)$.

In a similar way, $\tau(f^{(0,1)}) = (xy)$ and

$$\tau(f^{(1-\frac{1}{3^k},1)}) = f_2\cdot\tau(f^{(1-\frac{1}{3^k},0)})$$

$$= (xy)\cdot((x+y)^{3^k-1})^{[\frac{1}{3^k}]}$$

$$= (xy).$$

Thus (xy) is the only test ideal that appears in the region$[0,1)\times\{1\}$.

Finally, note that the test ideal at $(1,1)$ is

$$\tau(f^{(1,1)}) = ((x+y)xy).$$

We now show that $(\frac{1}{3},\frac{2}{3})$ is a point in the boundary of $B^{(x,y)}(f)$ and then use the p-fractal structure to sketch the constancy regions.

For every k

$$\tau(f^{(\frac{1}{3}-\frac{1}{3^k},\frac{2}{3}-\frac{1}{3^k})})$$
$$= ((x+y)^{3^{k-1}-1}(xy)^{2\cdot 3^{k-1}-1})^{[\frac{1}{3^k}]}.$$

But in the expansion of $(x+y)^{3^{k-1}-1}$ every term appears with nonzero coefficient, see Remark III.3.2. In particular, the term $(xy)^{\frac{3^{k-1}-1}{2}}(xy)^{2\cdot 3^{k-1}-1}$ appears with non-zero coefficient when expanding the product $(x+y)^{3^{k-1}-1}(xy)^{2\cdot 3^{k-1}-1}$. Since the degrees in x and y of this monomial are smaller than 3^k, by Proposition II.2.9 we conclude that $\tau(f^{(\frac{1}{3}-\frac{1}{3^k},\frac{2}{3}-\frac{1}{3^k})}) = R$. Thus

$$\chi_f^{(x,y)}\left(\frac{1}{3},\frac{2}{3}\right) = 0$$

and

$$\chi_f^{(x,y)}([0,\frac{1}{3})\times[0,\frac{2}{3})) = 1.$$

The later shows that the point $(\frac{1}{3},\frac{2}{3})$ is in the boundary of constancy regions for R and (x,y). We can use the p-fractal structure to find more points in this boundary. The idea is to break the region $[0,1]\times[0,1]$ into squares of length $1/3$ and find which of these must contain a boundary point. Then we apply the p-fractal structure to these squares to find the points.

For the points $(0,\frac{2}{3})$, $(\frac{2}{3},\frac{1}{3})$, $(\frac{1}{3},1)$, and $(1,\frac{2}{3})$ we have:

$$\tau(f^{(0,\frac{2}{3})}) = ((xy)^2)^{[\frac{1}{3}]} = R,$$

$$\tau(f^{(\frac{2}{3},\frac{1}{3})}) = ((x+y)^2xy)^{[\frac{1}{3}]} = (x^3y - x^2y^2 + xy^3)^{[\frac{1}{3}]} = R$$

and

$$\tau(f^{(\frac{1}{3},1)}) = ((x+y)(xy)^3)^{[\frac{1}{3}]} = (xy) \subset (x,y),$$

$$\tau(f^{(1,\frac{2}{3})}) = ((x+y)^3x^2y^2)^{[\frac{1}{3}]} = (x+y) \subset (x,y).$$

Therefore there should be boundary points in the squares $[0,1/3)\times[2/3,1)$ and $[2/3,1)\times[0,1/3)$. It is easy to check that there are no boundary points in all the other squares. From this and Lemma III.2.4 we know that $T_{3|(0,2)}\chi_f^{(x,y)} = T_{3|(2,1)}\chi_f^{(x,y)} = \chi_f^{(x,y)}$, since $\chi_f^{(x,y)}$ is the only characteristic function that is non-constant in $[0,1)\times[0,1)$. Moreover,

$$\chi_f^{(x,y)}(0.01_3, 0.22_3) = \chi_f^{(x,y)}(0_3 + 0.01_3, 0.2_3 + 0.02_3)$$

$$= \chi_f^{(x,y)}\left(\frac{0_3 + 0.1_3}{3}, \frac{2_3 + 0.2_3}{3}\right) = T_{3|(0,2)}\chi_f^{(x,y)}(0.1_3, 0.2_3)$$

$$= \chi_f^{(x,y)}(0.1_3, 0.2_3) = \chi_f^{(x,y)}\left(\frac{1}{3},\frac{2}{3}\right) = 0.$$

Similarly, we have

$$\chi_f^{(x,y)}(0.21_3, 0.12_3) = 0$$

and

$$\chi_f^{(x,y)}([0,0.01_3)\times[0,0.22_3)) = \chi_f^{(x,y)}([0,0.21_3)\times[0,0.12_3)) = 1.$$

That is, the points $(0.01_3, 0.22_3)$ and $(0.21_3, 0.12_3)$ are also in the boundary. We can repeat the proccess by subdividing the squares $[0,1/3)\times[2/3,1)$ and $[2/3,1)\times[0,1/3)$ into smaller squares of length $1/9$ and obtain more points of the boundary. This process can be summarized as follows. Let A be the set of points obtained from $(0.1_3, 0.2_3)$ by successively applying the operations

$$(0.a_1\ldots a_n1_3, 0.b_1\ldots b_n2_3) \mapsto \begin{cases} (0.a_1\ldots a_n01_3, 0.b_1\ldots b_n22_3) \\ (0.a_1\ldots a_n21, 0.b_1\ldots b_n12_3). \end{cases}$$

We have

$$\chi_f^{(x,y)}(\boldsymbol{p}) = 0$$

and

$$\chi_f^{(x,y)}([\boldsymbol{0}, \boldsymbol{p})) = 1$$

for all $\boldsymbol{p} \in A$. That is, the points of A are points in the boundary. We can now sketch the regions of constancy in $[0,1] \times [0,1]$:

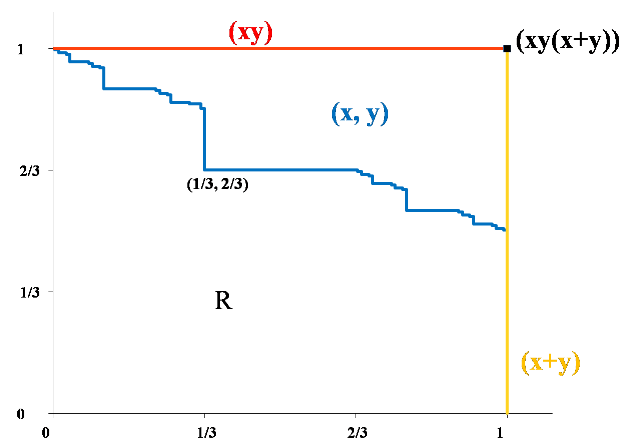


Figure III.3.3.1: The constancy regions for $f_1 = xy$ and $f_2 = (x+y)$ in $[0,1] \times [0,1]$

Using Skoda's theorem, we can describe the whole diagram of test ideals.

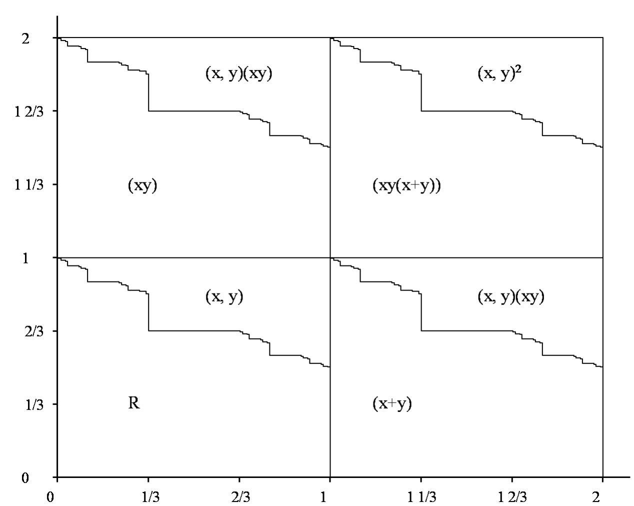


Figure III.3.3.2: The constancy regions for $f_1 = xy$ and $f_2 = (x+y)$

Remark III.3.4. We chose the name Devil's Staircase for this example because of the resemblance to the Devil's Staircases or Cantor functions that appear in the basic courses of analysis.

Example III.3.5. It can be similarly shown that for any characteristic p the same polynomials give a staircase that has infinitely many steps. Indeed,

$$\tau(f^{(\frac{1}{p^k},1-\frac{1}{p^k})}) = ((x+y)(xy)^{p^k-1})^{[\frac{1}{p^k}]} = (x,y)$$

but

$$\tau(f^{(\frac{2}{p^k},1-\frac{2}{p^k})}) = ((x+y)^2(xy)^{p^k-2})^{[\frac{1}{p^k}]} = R.$$

Which forces an infinite step situation as in the previous example. Therefore we can not expect that there are characteristics for which the region given by the test ideals will be the same as the one given by the multiplier ideals, see Example II.1.13.

CHAPTER IV

The Modules M_α and the F-jumping ideals

In this chapter we define the F-jumping ideals and give some basic properties. In particular, we relate them with the generalized test ideals, F-jumping numbers, and the modules M_α.

Throughout this chapter R denotes an F-finite regular domain of characteristic $p > 0$ and α denotes a rational number whose denominator is not divisible by p.

As the denominator of α is not divisible by p we can write $\alpha = \frac{r}{p^e - 1}$ for some $e > 0$. An easy, but crucial consequence of this is that

$$(p^{e(\ell-1)} + \ldots + p^e + 1)r + \alpha = p^{e\ell}\alpha$$

for every $\ell \in \mathbb{N}$.

Lemma IV.0.6. *If f be an element in R, then $\tau(f^{p\lambda}) \subset \tau(f^\lambda)^{[p]}$.*

Proof. We have that $\frac{\lceil pp^j\lambda \rceil}{p^j} \le \frac{p\lceil p^j\lambda \rceil}{p^j}$ and $\lim\limits_{j\to\infty} \frac{p\lceil p^j\lambda \rceil}{p^j} = p\lambda$. Then,

$$\tau(f^{p\lambda}) = \bigcup_{j\in\mathbb{N}} \left(f^{p\lceil p^j\lambda\rceil}\right)^{[1/p^j]} = \bigcup_{j\in\mathbb{N}} \left(f^{\lceil p^j\lambda\rceil}\right)^{[1/p^{j-1}]} \subset \bigcup_{j\in\mathbb{N}} \left(\left(f^{\lceil p^j\lambda\rceil}\right)^{[1/p^j]}\right)^{[p]} = \tau(f^\lambda)^{[p]}$$

by the properties of the ideals $I^{[1/p^j]}$ [BMS08, Lemma 2.4]. □

Recall from Example II.2.21 that

$$M_\alpha = \lim_{\rightarrow} R \xrightarrow{f^r} R \xrightarrow{f^{p^e r}} F(R) = R \xrightarrow{f^{p^{2e} r}} \ldots.$$

Hence, as an R_f-module, M_α is free of rank one. Let e_α denote the generator obtained as the image of $1 \in F^0R = R$ in the previous direct limit. We think of e_α as $1/f^\alpha$.

Note that M_α carries a natural structure of a D_R-module, which does not depend of the presentation of α [BMS09, Remark 2.4]. Given $P \in D_R^{(se)}$ and $c \in R$, it is shown in loc. cit. that

$$P \cdot \left(\frac{c}{f^m} \cdot e_\alpha \right) = \frac{P(cf^{m(p^{es}-1)+r(1+p^e+\ldots+p^{e(s-1)})})}{f^{mp^{es}+r(1+p^e+\ldots p^{e(s-1)})}} \cdot e_\alpha.$$

When $m = 0$, the previous expression is equal to

$$P \cdot (c \cdot e_\alpha) = \frac{P(cf^{r(1+p^e+\ldots p^{e(s-1)})})}{f^{r(1+p^e+\ldots+p^{e(s-1)})}} \cdot e_\alpha.$$

We also recall that e_α generates M_α as a D_R-module [BMS09, Theorem 2.11].

We now are ready to introduce the F-jumping ideals in terms of the D_R-module structure of M_α.

Definition IV.0.7. Let $N_\alpha = D_R f^{\lceil \alpha \rceil} \cdot e_\alpha$, which is the smallest D_R-submodule of M_α containing $f^{\lceil \alpha \rceil} e_\alpha$. We define the *$F$-jumping ideal associated to f and α* as the ideal $\mathfrak{J}_{F_R}(f^\alpha)$ of R such that $\mathfrak{J}_{F_R}(f^\alpha) \cdot e_\alpha = N_\alpha \cap R \cdot e_\alpha$. Whenever the ring is clear from the context, we simply write $\mathfrak{J}_F(f^\alpha)$.

Lemma IV.0.8. *Given any ideal $I \subseteq R$, the D_R-module generated by Ie_α is*

$$D_R \cdot Ie_\alpha = \bigcup_{s \geq 0} \left(D_R^{(es)}(f^{r(1+\ldots+p^{e(s-1)})} I) / f^{r(1+\ldots+p^{e(s-1)})} \right) \cdot e_\alpha.$$

The intersection of this D_R-module with Re_α is equal to Je_α, where

$$J = \bigcup_{s \geq 0} \left(D_R^{(es)}(f^{r(1+\ldots+p^{e(s-1)})} I) : f^{r(1+\ldots+p^{e(s-1)})} \right).$$

In particular,

$$\mathfrak{J}_{F_R}(f^\alpha) = \bigcup_{s \geq 0} \left(D_R^{(es)}(f^{r(1+\ldots+p^{e(s-1)})} f^{\lceil \alpha \rceil}) : f^{r(1+\ldots+p^{e(s-1)})} \right).$$

Proof. This follows from the description of the action of the differential operators on M_α given above. Indeed,

$$D_R \cdot Ie_\alpha = \bigcup_{s \geq 0} D_R^{(e)} Ie_\alpha$$

$$= \bigcup_{s \geq 0} \frac{D_R^{(e)}(If^{r(1+p^e+\ldots+p^{e(s-1)})})}{f^{r(1+p^e+\ldots+p^{e(s-1)})}} e_\alpha.$$

Which gives the first part. To get the second statement, note that

$$\frac{D_R^{(e)}(If^{r(1+p^e+\ldots+p^{e(s-1)})})}{f^{r(1+p^e+\ldots+p^{e(s-1)})}} e_\alpha \cap Re_\alpha = \left(D_R^{(es)}(f^{r(1+\ldots+p^{e(s-1)})}I) : f^{r(1+\ldots+p^{e(s-1)})} \right) e_\alpha.$$

□

For test ideals, if $\alpha < \beta$ then $\tau(f^\alpha) \supseteq \tau(f^\beta)$. The same statement is not true for the F-jumping ideals, but we can give a relation if $\beta = \alpha + l$ for $l \in \mathbb{N}$. Before stating this result, we need a lemma.

Lemma IV.0.9. *The morphism $M_\alpha \to M_{\alpha+1}$ defined by sending $e_\alpha \mapsto fe_{\alpha+1}$ is an isomorphism of F^e-modules (as well as of D_R-modules). In particular, $N_\alpha \cong N_{\alpha+1} = D_R f^{\lceil \alpha+1 \rceil} \cdot e_{\alpha+1}$.*

Proof. Consider the following commutative diagram of R-modules

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
R \arrow[r, "f^r"] \arrow[d, "f"] & R \arrow[r, "f^{rp^e}"] \arrow[d, "f^{p^e}"] & R \arrow[r, "f^{rp^{2e}}"] \arrow[d, "f^{p^{2e}}"] & \ldots \\
R \arrow[r, "f^{r+p^e-1}"] \arrow[d] & R \arrow[r, "f^{(r+p^e-1)p^e}"] \arrow[d] & R \arrow[r, "f^{(r+p^e-1)p^{2e}}"] \arrow[d] & \ldots \\
R/f \arrow[r, "f^{r+p^e-1}"] \arrow[d] & R/f^{p^e} \arrow[r, "f^{(r+p^e-1)p^e}"] \arrow[d] & R/f^{p^{2e}} \arrow[r, "f^{(r+p^e-1)p^{2e}}"] \arrow[d] & \ldots \\
0 & 0 & 0 &
\end{tikzcd}$$

We note that $R/f \overset{f^{r+p^e-1}}{\to} R/f^{p^e}$ is the zero map because $r \geq 1$ (recall $\alpha > 0$). By taking direct limits, we obtain that the induced morphism $M_\alpha \to M_{\alpha+1}$ is an isomorphism of F-modules, hence of D_R-modules. □

Proposition IV.0.10. *If $\ell \in \mathbb{N}$, then $f^\ell \mathfrak{J}_F(f^\alpha) \subset \mathfrak{J}_F(f^{\alpha+\ell})$.*

Proof. Let $\phi : M_\alpha \to M_{\alpha+\ell}$ be the morphism of D_R-modules given by $e_\alpha \mapsto f^\ell e_{\alpha+\ell}$. We know that ϕ is an isomorphism by Lemma IV.0.9. We obtain

$$f^\ell \mathfrak{J}_F(f^\alpha) e_{\alpha+\ell} = \mathfrak{J}_F(f^\alpha) f^\ell e_{\alpha+\ell} = \phi(\mathfrak{J}_F(f^\alpha) e_\alpha) = \phi(N_\alpha \cap Re_\alpha)$$

$$= \phi(N_\alpha) \cap \phi(Re_\alpha) = N_{\alpha+\ell} \cap Rf^\ell e_{\alpha+\ell} \subset N_{\alpha+\ell} \cap Re_{\alpha+\ell} = \mathfrak{J}_F(f^{\alpha+\ell})e_{\alpha+\ell}.$$

□

We now define inductively a sequence of ideals $\mathcal{I}^j_R(f^\alpha)$ associated to f^α. These ideals will help us relate the test ideals to the F-jumping ideals.

Definition IV.0.11. Given a representation of $\alpha = \frac{r}{p^e-1}$. We set $\mathcal{I}^1_R(f^\alpha) = \tau(f^\alpha)$. Given $\mathcal{I}^j_R(f^\alpha)$, we let

$$\mathcal{I}^{j+1}_R(f^\alpha) = \left(\left(\mathcal{I}^j_R(f^\alpha)\right)^{[p^e]} : f^r \right).$$

If it is clear in which ring we are taking these ideals, we simply write $\mathcal{I}^j(f^\alpha)$.

We collect some basic properties of this sequence of ideals in the following proposition.

Proposition IV.0.12. *If $\mathcal{I}^j_R(f^\alpha)$ is the sequence of ideals associated to $f \in R$ and $\alpha = \frac{r}{p^e-1}$, then for every positive integer j:*

(i) $\mathcal{I}^j_R(f^\alpha) \subset \mathcal{I}^{j+1}_R(f^\alpha)$, hence $\mathcal{I}^n_R(f^\alpha) = \mathcal{I}^{n+j}_R(f^\alpha)$ for some n sufficiently large.

(ii) $f^r\mathcal{I}^{j+1}_R(f^\alpha) \subset \mathcal{I}^j_R(f^\alpha)^{[p^e]}$.

(iii) For all $j > 1$ there is an equality $\mathcal{I}^j_R(f^\alpha) = (\tau(f^\alpha)^{[p^{je}]} : f^{r(1+p^e+\ldots+p^{e(j-1)})})$.

Proof. We note that

$$f^r\mathcal{I}^1(f^\alpha) = f^r\tau(f^\alpha) = \tau(f^{r+\alpha}) = \tau(f^{p^e\alpha}) \subset \tau(f^\alpha)^{[p^e]} = \mathcal{I}^1(f^\alpha)^{[p^e]}.$$

The second equality follow from Skoda's Theorem (Proposition III.1.20) and the last containment from Lemma IV.0.6. As a consequence, $\mathcal{I}^1(f^\alpha) \subset \mathcal{I}^2(f^\alpha)$ and $f^r\mathcal{I}^2(f^\alpha) \subset \mathcal{I}^1(f^\alpha)^{[p^e]}$. The assertions in (i) and (ii) follow by induction. Part (iii) follows in the case $j = 2$ by definition. Suppose that we know the result for $j-1$, then

$$\mathcal{I}^j_R(f^\alpha) = \left(\left(\mathcal{I}^{j-1}_R(f^\alpha)\right)^{[p^e]} : f^r \right) = \left(\left(\tau(f^\alpha)^{[p^{e(j-1)}]} : f^{r(1+p^e+\ldots+p^{e(j-2)})}\right)^{[p^e]} : f^r \right),$$

which, by the flatness of Frobenius and the properties of colon ideals in the regular case, is equal to

$$\left(\tau(f^\alpha)^{[p^{ej}]} : f^{r(1+p^e+\ldots+p^{e(j-1)})}\right)$$

□

As far as we know the flag of ideals associated to f and α may depend on the presentation of α. But, as the next proposition shows, the union of these ideals does not depend on the presentation of α.

Proposition IV.0.13. *$\mathfrak{J}_F(f^\alpha) = \bigcup_j \mathcal{I}^j(f^\alpha) = \mathcal{I}^n(f^\alpha)$ for $n \gg 0$.*

Proof. By the definition of $\tau(f^\alpha)$ and Proposition II.2.18, we have that, for a positive integer $n \in \mathbb{N}$, $\tau(f^\alpha)^{[p^{se}]} = \left(\left(f^{\lceil p^e\alpha\rceil}\right)^{[1/p^{se}]}\right)^{[p^{se}]} = D_R^{(es)}(f^{\lceil p^{se}\alpha\rceil})$ for $s \geq n$. Then, by Lemma IV.0.8,

$$\begin{aligned}
\mathfrak{J}_F(f^\alpha)e_\alpha &= N_\alpha \cap Re_\alpha = D_R f^{\lceil\alpha\rceil}\cdot e_\alpha \cap Re_\alpha \\
&= \bigcup_{s\geq 0}\left(D_R^{(es)}(f^{r(1+\ldots+p^{e(s-1)})}f^{\lceil\alpha\rceil}) : f^{r(1+\ldots+p^{e(s-1)})}\right)e_\alpha \\
&= \bigcup_{s\geq n}\left(D_R^{(es)}(f^{r(1+\ldots+p^{e(s-1)})}f^{\lceil\alpha\rceil}) : f^{r(1+\ldots+p^{e(s-1)})}\right)e_\alpha \\
&= \bigcup_{s\geq n}\left(D_R^{(es)}(f^{\lceil r(1+\ldots+p^{e(s-1)})+\alpha\rceil}) : f^{r(1+\ldots+p^{e(s-1)})}\right)e_\alpha \\
&= \bigcup_{s\geq n}\left(D_R^{(es)}(f^{\lceil p^{es}\alpha\rceil}) : f^{r(1+\ldots+p^{e(s-1)})}\right)e_\alpha \\
&= \bigcup_{s\geq n}\left(\tau(f^\alpha)^{[p^{se}]} : f^{r(1+\ldots+p^{e(s-1)})}\right)e_\alpha = \bigcup_{s\geq n}\mathcal{I}^s(f^\alpha)e_\alpha.
\end{aligned}$$

□

The F-jumping ideals behave well with respect to localization and completion.

Proposition IV.0.14. *If $W \subset R$ is a multiplicative system in R, then $\mathfrak{J}_{F_{W^{-1}R}}(f^\alpha) = W^{-1}\mathfrak{J}_F(f^\alpha)$.*

Proof. Choose n such that $\mathfrak{J}_{F_R}(f^\alpha) = \mathcal{I}^n_R(f^\alpha)$. By Proposition IV.0.12(iii) we have

$$\mathfrak{J}_{F_R}(f^\alpha) = (\tau(f^\alpha)^{[p^{je}]} : f^{r(1+p^e+\ldots+p^{e(j-1)})})$$

for all $j > n$. We know that test ideals commute with localization [BMS08, Proposition 2.13]. Since colon ideals and Frobenius powers of ideals commute with localization as well, the result follows from Proposition IV.0.13. □

Proposition IV.0.15. *If R is a local ring and $\widehat{R}$ is the completion of R with respect to the maximal ideal, then $\mathfrak{J}_{F_{\widehat{R}}}(f^\alpha) = \mathfrak{J}_{F_R}(f^\alpha)\widehat{R}$.*

Proof. This proof is analogous to the proof of Proposition IV.0.14. □

Notation IV.0.16. Every F^e-submodule of M_α is determined by an ideal $I \subset R$ such that we have an induced map $I \xrightarrow{f^r} I^{[p^e]}$, that is $f^r I \subset I^{[p^e]}$, [Lyu97, Corollary 2.6]. The F^e-submodule of M_α *generated by* I is given by

$$N_I = \lim_{\rightarrow} \left(I \xrightarrow{f^r} I^{[p^e]} \xrightarrow{f^{p^e r}} I^{[p^{2e}]} \xrightarrow{f^{p^{2e} r}} \dots \right).$$

Lemma IV.0.17. *If $I, J \subset R$ are ideals such that $f^r I \subset I^{[p^e]}$, $f^r J \subset J^{[p^e]}$, and $I \subset J$, then the F^e-submodule of M_α generated by I is equal to the one generated by J if and only if there exists $\ell \in \mathbb{N}$ such that $f^{r(1+\ldots+p^{e(\ell-1)})} J \subset I^{[p^{e\ell}]}$.*

Proof. Let N_I and N_J be the F^e-submodules of M_α generated by I and J respectively. In this case $J/I \xrightarrow{f^r} J^{[p^e]}/I^{[p^e]}$ generates the F^e-module N_J/N_I. Since J/I is a finitely generated R-module, $N_J = N_I$ if and only if there exists ℓ such that

$$J/I \xrightarrow{f^{r(1+\ldots+p^{e(\ell-1)})}} J^{[p^{e\ell}]}/I^{[p^{e\ell}]}$$

is the zero morphism. Therefore, $N_I = N_J$ if and only if there exists $\ell \in \mathbb{N}$ such that $f^{r(1+\ldots+p^{e(\ell-1)})} J \subset I^{[p^e]}$. $\square$

Proposition IV.0.18. *N_α is an F^e-submodule of M_α. Moreover, the morphisms $\mathfrak{J}_F(f^\alpha) \xrightarrow{f^r} \mathfrak{J}_F(f^\alpha)^{[p^e]}$ and $\tau(f^\alpha) \xrightarrow{f^r} \tau(f^\alpha)^{[p^e]}$ generate N_α as an F^e-module.*

Proof. By the definition of $\tau(f^\alpha)$ and Proposition II.2.18, there is an integer ℓ such that $\tau(f^\alpha)^{[p^{se}]} = \left(\left(f^{\lceil p^{se}\alpha \rceil} \right)^{[1/p^{se}]} \right)^{[p^{se}]} = D_R^{(es)}(f^{\lceil p^{es}\alpha \rceil})$ for all integers $s \geq \ell$. Then,

$$\begin{aligned} N_\alpha = D_R \cdot f^{\lceil \alpha \rceil} e_\alpha = \bigcup_{s \geq 0} D_R^{(es)} \cdot f^{\lceil \alpha \rceil} e_\alpha = \bigcup_{s \geq \ell} D_R^{(es)} \cdot f^{\lceil \alpha \rceil} e_\alpha \\ = \bigcup_{s \geq \ell} \left(D^{(es)}(f^{r(1+\ldots+p^{e(s-1)})} f^{\lceil \alpha \rceil}) / f^{r(1+\ldots+p^{e(s-1)})} \right) \cdot e_\alpha \\ = \bigcup_{s \geq \ell} \left(D_R^{(es)}(f^{\lceil p^{es}\alpha \rceil}) / f^{r(1+\ldots+p^{e(s-1)})} \right) \cdot e_\alpha \\ = \bigcup_{s \geq \ell} \left(\tau(f^\alpha)^{[p^{se}]} / f^{r(1+\ldots+p^{e(s-1)})} \right) \cdot e_\alpha. \end{aligned}$$

By Lemma IV.0.6, we have $f^r \tau(f^\alpha) = \tau(f^{r+\alpha}) = \tau(f^{p^e \alpha}) \subset \tau(f^\alpha)^{[p^e]}$. Hence $\tau(f^\alpha) \xrightarrow{f^r} \tau(f^\alpha)^{[p^e]}$ is a root for the F^e-submodule of M_α

$$\bigcup_s \left(\tau(f^\alpha)^{[p^{se}]} / f^{r(1+\ldots+p^{e(s-1)})} \right) \cdot e_\alpha.$$

We conclude that N_α is the F^e-submodule of M_α generated by the morphism $\tau(f^\alpha) \xrightarrow{f^r} \tau(f^\alpha)^{[p^e]}$.

For the second part, we note that $f^r\mathcal{I}^s(f^\alpha) \subset \mathcal{I}^s(f^\alpha)^{[p^e]}$ for every s by Proposition IV.0.12 (i) and (ii). As a consequence, we deduce that $\mathcal{I}^s(f^\alpha)$ also generates an F^e-submodule of M_α. By the same proposition $f^r\mathcal{I}^{s+1}(f^\alpha) \subset \mathcal{I}^s(f^\alpha)^{[p^e]}$. Therefore the F^e-submodules generated by $\mathcal{I}^s(f^\alpha)$ are all the same by Lemma IV.0.17. As $\mathfrak{I}_F(f^\alpha) = \mathcal{I}^s(f^\alpha)$ for sufficiently large s, the result follows. ☐

Lemma IV.0.19. *If $I \subset R$ is a nonzero ideal such that $I \subseteq (I^{[p^e]} : f^r)$, then $f \in \sqrt{I}$.*

Proof. The hypothesis is equivalent to $f^r I \subseteq I^{[p^e]}$. Let P be a prime ideal of R. If $f \notin P$ then $IR_P \subseteq (IR_P)^{[p^e]}$. Therefore, by Nakayama's Lemma, we have $IR_P = R_P$. Hence if a prime ideal P contains I, then it must contain f as well. The result follows since $\sqrt{I}$ is the intersection of all prime ideals containing I.

☐

Proposition IV.0.20. *For any nonzero F^e-submodule N of M_α, we have $N_\alpha \subset N$. In particular, N_α is a simple F^e-module.*

Proof. Since any two R-modules intersect nontrivially in R_f, we deduce that there is a minimal simple F^e-submodule N. If I is the ideal such that $Re_\alpha \bigcap N = Ie_\alpha$, we have $I = (I^{[p^e]} : f^r)$ [Lyu97, Corollary 2.6]. Moreover, $N = N_I$ (see Notation IV.0.16). Then $f^n \in I$ for some $n \in \mathbb{N}$ by Lemma IV.0.19.

Choose ℓ large enough such that $\tau(f^\alpha)^{[p^{se}]} = \left(\left(f^{\lceil p^{se}\alpha\rceil}\right)^{[1/p^{se}]}\right)^{[p^{se}]} = D_R^{(es)}(f^{\lceil p^{se}\alpha\rceil})$ and $\tau(f^{\frac{n}{p^{se}}+\alpha}) = \tau(f^\alpha)$ for all $s \geq \ell$ (cf. Proposition II.2.18). We have

$$\begin{aligned}
D_R \cdot f^n e_\alpha &= \bigcup_{s\geq 0} D_R^{(es)} \cdot f^n e_\alpha = \bigcup_{s\geq \ell} D_R^{(es)} \cdot f^n e_\alpha \\
&= \bigcup_{s\geq \ell} \left(D_R^{(es)}(f^{r(1+\ldots+p^{e(s-1)})} f^n)/f^{r(1+\ldots+p^{e(s-1)})}\right) \cdot e_\alpha \\
&= \bigcup_{s\geq \ell} \left(D_R^{(es)}(f^{n+r(1+\ldots+p^{e(s-1)})})/f^{r(1+\ldots+p^{e(s-1)})}\right) \cdot e_\alpha \\
&= \bigcup_{s\geq \ell} \left(\left(\left(f^{n+r(1+\ldots+p^{e(s-1)})}\right)^{[1/p^{se}]}\right)^{[p^{es}]} / f^{r(1+\ldots+p^{e(s-1)})}\right) \cdot e_\alpha \\
&= \bigcup_{s\geq \ell} \left(\tau(f^{\frac{n}{p^{es}}+\frac{r(1+\ldots+p^{e(s-1)})}{p^{es}}})^{[p^{es}]} / f^{r(1+\ldots+p^{e(s-1)})}\right) \cdot e_\alpha \\
&\supset \bigcup_{s\geq 0} \left(\tau(f^{\frac{n}{p^{es}}+\alpha})^{[p^{es}]} / f^{r(1+\ldots+p^{e(s-1)})}\right) \cdot e_\alpha
\end{aligned}$$

$$\supset \bigcup_{s\geq \ell} \left(\tau(f^{\alpha})^{[p^{es}]}/f^{r(1+\ldots+p^{e(s-1)})}\right)\cdot e_{\alpha} = D_R \cdot f^{\lceil \alpha \rceil} e_{\alpha} = N_{\alpha}.$$

Since $D_R \cdot f^n e_{\alpha} \subseteq N$, we conclude $N_{\alpha} \subseteq N$. $\square$

Corollary IV.0.21. *N_{α} is a simple D_R-module.*

Proof. Since N_{α} is a simple F^e-module, it is a direct sum of simple D_R-modules by Corollary II.2.24. Using the fact that any two R-modules in R_f intersect nontrivially, N_{α} must be a simple D_R-module. $\square$

Proposition IV.0.22. *If $I \subset \tau(f^{\alpha})$ is a nonzero ideal such that $I \subset (I^{[p^e]} : f^r)$, then $I = \tau(f^{\alpha})$. In particular, $\tau(f^{\alpha})e_{\alpha}$ is a minimal root for N_{α}.*

Proof. The flatness of Frobenius and the properties of colon ideals in the regular case imply that

$$(I^{[p^{se}]} : f^{r(1+p^e+\ldots+p^{e(s-1)})}) = \left(\left(I^{[p^{(s-1)e}]} : f^{r(1+p^e+\ldots+p^{e(s-2)})}\right)^{[p^e]} : f^r\right).$$

Hence, by a trivial induction, we deduce that $I \subset (I^{[p^{se}]} : f^{r(1+p^e+\ldots+p^{e(s-1)})})$ for every $s \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that $f^n \in I$ by Lemma IV.0.19, and we can choose it such that $n > \alpha$. Hence $f^{n+\frac{p^{es}-1}{p^e-1}r} \in I^{[p^{es}]}$ which implies

$$\left(f^{n+\frac{p^{es}-1}{p^e-1}r}\right)^{[1/p^{es}]} \subset I.$$

Note that $(p^e-1)n > r$. Thus,

$$\frac{1}{p^{es}}\left(n+\frac{p^{es}-1}{p^e-1}r\right) = \frac{(p^e-1)n+(p^{es}-1)r}{p^{es}(p^e-1)} = \frac{((p^e-1)n-r}{p^{es}(p^e-1)} + \frac{r}{p^e-1} > \frac{r}{p^e-1},$$

and

$$\lim_{s\to\infty} \frac{(p^e-1)n+(p^{es}-1)r}{p^{es}(p^e-1)} = \frac{r}{p^e-1} = \alpha.$$

Then, by Theorem II.2.12 and Proposition II.2.11, we have that for ℓ sufficiently large, $\tau(f^{\alpha}) = \left(f^{n+\frac{p^{e\ell}-1}{p^e-1}r}\right)^{[1/p^{e\ell}]} \subset I \subset \tau(f^{\alpha})$. Therefore $I = \tau(f^{\alpha})$.

We now prove that $\tau(f^{\alpha})$ is a minimal root. First, we note that Proposition IV.0.18 guaranties that $\tau(f^{\alpha})$ is indeed a root for N_{α}. If I is any other root for N_{α}, then it must satisfy $f^r I \subseteq I^{[p^e]}$, hence $I \subset (I^{[p^e]} : f^r)$. Therefore, by the first part, $\tau(f^{\alpha}) \subset I$. The result follows. $\square$

Using ideas analogous to the ones used in the previous proof, we recover a result previously obtained by Blickle [Bli08, Note before Proposition 3.5]

Proposition IV.0.23. *Let $\beta = \frac{a}{b} \in \mathbb{Q}_{>0}$. If $\alpha > \beta$, then $f^r\tau(f^\beta) \subset \tau(f^\beta)^{[p^e]}$ and $\tau(f^\beta)$ generates M_α as F^e-module.*

Proof. Since $\frac{r}{p^e-1} > \frac{a}{b}$, we have that $br + a > p^e a$. Then

$$f^r\tau(f^\beta) = \tau(f^{\beta+r}) = \tau(f^{\frac{a+br}{b}}) \subset \tau(f^{\frac{p^e a}{b}}) = \tau(f^{p^e\beta}) \subset \tau(f^\beta)^{[p^e]},$$

where the last containment follows from Lemma IV.0.6. Thus $\tau(f^\beta)$ generates an F-submodule of M_α.

Clearly $\lim\limits_{\ell\to\infty} \frac{r(1+\ldots+p^{e(\ell-1)})}{p^{e\ell}} = \alpha$, so we can pick $\ell \in \mathbb{N}$ such that $r(1+\ldots+p^{e(\ell-1)}) > p^{e\ell}\beta$. With this choice of ℓ, we have by Proposition II.2.11

$$f^{r(1+\ldots+p^{e(\ell-1)})}R \subset \left(\left(f^{r(1+\ldots+p^{e(\ell-1)})}\right)^{[1/p^{el}]}\right)^{[p^{el}]} = \tau\left(f^{\frac{r(1+\ldots+p^{e(\ell-1)})}{p^{e\ell}}}\right)^{[p^{el}]}.$$

Then $\tau\left(f^{\frac{r(1+\ldots+p^{e(\ell-1)})}{p^{e\ell}}}\right)$ generates M_α as an F^e-module by Lemma IV.0.17. In view of $\tau(f^{\frac{r(p^{e(\ell-1)}+\ldots1)}{p^{e\ell}}}) \subset \tau(f^\beta)$, we deduce that $\tau(f^\beta)$ also generates M_α as an F^e-module. ∎

Remark IV.0.24. The result in [Bli08] shows more; the test ideal $\tau(f^{\alpha-\epsilon})$ is a minimal root for the F^e-module M_α.

Theorem IV.0.25. *Let R be an F–finite regular domain and $f \in R$ be a nonzero element. Let $\alpha \in \mathbb{Q}$ be a rational number with no positive power of p in the denominator. Suppose that $(p^e-1)\alpha \in \mathbb{N}$. The following are equivalent:*

(i) *α is not an F-jumping number;*

(ii) *$N_\alpha = M_\alpha$;*

(iii) *M_α is a simple D_R-module;*

(iv) *M_α is a simple F^e-module (with the structure induced by the representation for α);*

(v) *$\mathfrak{J}_F(f^\alpha) = R$.*

Proof. By definition, $\alpha = \frac{r}{p^e-1}$ is not an F-jumping number if and only if $\tau(f^\alpha) = \tau(f^{\alpha-\epsilon})$. By Proposition IV.0.22 and Remark IV.0.24 we know that these ideals are the miminal roots of the F^e-modules M_α and N_α. Therefore the equality $\tau(f^\alpha) = \tau(f^{\alpha-\epsilon})$ happens if and only if $N_\alpha = M_\alpha$, which is equivalent to $\mathfrak{J}_F(f^\alpha) = R$. The rest follows the fact that N_α is a the unique simple F^e-submodule/D_R-submodule of M_α, see Proposition IV.0.20 and Corollary IV.0.21. □

By Theorem IV.0.25 there is an algorithm to decided whether a number of the form $\frac{r}{p^e-1}$ is an F-jumping number for $f \in R$, if there is an algorithm to compute $\mathfrak{J}_F(f^\alpha)$. We describe this algorithm next:

Algorithm IV.0.26.
Input: $\alpha = \frac{r}{p^e-1}$ and $f \in R$.
Output: R if α is not F-jumping number for f and a proper ideal otherwise.
Process: Compute $\tau(f^\alpha)$[1];
Take $J_1 = \tau(f^\alpha)$;
Compute $J_{n+1} = (J_n^{[p^e]} : f^r)$ until there is an ℓ such that $J_\ell = J_{\ell+1}$;
Return: $\mathfrak{J}_F(f^\alpha) = J_\ell$.

Example IV.0.27. Let $R = \mathbb{F}_{13}[x, y]$ and $f = x^2 + y^3$. Therefore:

(i) If $\alpha = \frac{11}{12}$, $\tau(f^\alpha) = (x, y)R$, $J_1 = (x, y)R$, and $J_2 = ((x^{13}, y^{13}) : f^{11}) = R$ because f^{11} is equal to $x^{22} - 2x^{20}y^3 + 3x^{18}y^6 - 4x^{16}y^9 + 5x^{14}y^{12} - 6x^{12}y^{15} - 6x^{10}y^{18} + 5x^8y^{21} - 4x^6y^{24} + 3x^4y^{27} - 2x^2y^{30} + y^{33}$. Therefore $\frac{11}{12}$ is not an F-jumping number.

(ii) If $\alpha = \frac{10}{12}$, $\tau(f^\alpha) = (x, y)R$, $J_1 = (x, y)R$, and $J_2 = ((x^{13}, y^{13}) : f^{10}) = (x, y)$ because f^{10} is equal to $x^{20} - 3x^{18}y^3 + 6x^{16}y^6 + 3x^{14}y^9 + 2x^{12}y^{12} + 5x^{10}y^{15} + 2x^8y^{18} + 3x^6y^{21} + 6x^4y^{24} - 3x^2y^{27} + y^{30}$. Therefore $\frac{10}{12}$ is an F-jumping number.

Proposition IV.0.28. *Let R be an F-finite regular domain and f a nonzero element of R. If α is a rational number such that p does not divide its denominator, then*

$$\sqrt{\mathfrak{J}_F(f^\alpha)} = \sqrt{(\tau(f^\alpha) : \tau(f^{\alpha-\epsilon}))}.$$

Proof. We consider a prime ideal $Q \subset R$. We have $\mathfrak{J}_{F_{R_Q}}(f^\alpha) = \mathfrak{J}_{F_R}(f^\alpha)R_Q = R_Q$ if and only if α is not an F-jumping number for f in R_Q. This is equivalent to $(\tau(f^\alpha R_Q) : \tau(f^{\alpha-\epsilon}R_Q)) = R_Q$. We conclude that $\operatorname{Supp}_R R/\mathfrak{J}_F(f^\alpha) = \operatorname{Supp}_R R/(\tau(f^\alpha) : \tau(f^{\alpha-\epsilon}))$, hence $\sqrt{\mathfrak{J}_F(f^\alpha)} = \sqrt{(\tau(f^\alpha) : \tau(f^{\alpha-\epsilon}))}$. □

[1] There are algorithms for computing the test ideal $\tau(f^\alpha)$, some of which have been implemented for *Macaulay 2* by Daniel Hernández, Moty Katzman, Sara Malec, Karl Schwede, Pedro Teixeira and Emily Witt.

Remark IV.0.29. In general, $\mathfrak{J}_F(f^\alpha)$ is not equal to $(\tau(f^\alpha) : \tau(f^\beta))$. For example, let $R = \mathbb{F}_7[x,y]$, $f = x^3y^2$, and $\alpha = \frac{4}{6}$. In this case, $\beta = \frac{3}{6}$ is the biggest F-jumping number smaller than α. We have $\tau(f^\beta) = xyR$ and $\tau(f^\alpha) = x^2yR$, hence $(x^2yR : xyR) = xR$. However, $\mathfrak{J}_F(f^\alpha) = x^2$.

For test ideals, we have $\tau(f^{p\alpha}) \subset \tau(f^\alpha)^{[p]}$. We finish this section by showing that for F-jumping ideals a stronger result is true. In the process we use the sequence of ideals from Definition IV.0.11.

Lemma IV.0.30. *If f and α are as above, then $\mathcal{I}^j(f^{p^e\alpha}) = \mathcal{I}^{j-1}(f^\alpha)^{[p^e]}$ for $j \geq 2$.*

Proof. We will prove that $\mathcal{I}^j(f^{p^e\alpha}) = \mathcal{I}^{j-1}(f^\alpha)^{[p^e]}$ for $j \geq 2$ by induction on j. If $j = 2$, then

$$\mathcal{I}^2(f^{p^e\alpha}) = \mathcal{I}^2(f^{\alpha+r}) = (\tau(f^{\alpha+r})^{[p^e]} : f^{p^e r})$$

by the flatness of Frobenius in the regular case, the later expression is equal to

$$= (\tau(f^{\alpha+r}) : f^r)^{[p^e]} = (f^r\tau(f^\alpha) : f^r)^{[p^e]} = \tau(f^\alpha)^{[p^e]} = \mathcal{I}^1(f^\alpha)^{[p^e]}.$$

If the assertion is true for j, then,

$$\mathcal{I}^{j+1}(f^{p^e\alpha}) = (\mathcal{I}^j(f^{p^e\alpha})^{[p^e]} : f^{p^e r}) = (\mathcal{I}^j(f^{p^e\alpha}) : f^r)^{[p^e]}$$

$$= (\mathcal{I}^{j-1}(f^\alpha)^{[p^e]} : f^r)^{[p^e]} = \mathcal{I}^j(f^\alpha)^{[p^e]}.$$

□

Proposition IV.0.31. *Let R be an F-finite regular domain and f a nonzero element of R. If α is a rational number such that p does not divide its denominator, then $\mathfrak{J}_F(f^{p\alpha}) = \mathfrak{J}_F(f^\alpha)^{[p]}$.*

Proof. We first note that $\mathfrak{J}_F(f^{p^e\alpha}) = \mathfrak{J}_F(f^\alpha)^{[p^e]}$ because

$$\mathfrak{J}_F(f^{p^e\alpha}) = \bigcup_j \mathcal{I}^j(f^{p^e\alpha}) = \bigcup_j \mathcal{I}^j(f^\alpha)^{[p^e]} = \mathfrak{J}_F(f^\alpha)^{[p^e]},$$

here the first equality follows from Proposition IV.0.13, and the second one from Lemma IV.0.30. In addition, from Proposition IV.0.13 and the definition of $\mathcal{I}^j(f^{p^e\alpha})$, we have that for n large enough

$$\mathfrak{J}_F(f^\alpha)^{[p]} = \mathcal{I}^n(f^\alpha)^{[p]} = \left(\mathcal{I}^n(f^\alpha)^{[p^e]} : f^r\right)^{[p]} = (\mathfrak{J}_F(f^\alpha)^{[p^{e+1}]} : f^{pr}).$$

Hence, $\mathfrak{J}_F(f^\alpha)^{[p]}$ defines the F^e-submodule of $M_{p\alpha}$ generated by the morphism

$$\mathfrak{J}_F(f^\alpha)^{[p]} \xrightarrow{f^{pr}} \mathfrak{J}_F(f^\alpha)^{[p^{e+1}]}.$$

Since $\mathfrak{J}_F(f^{p\alpha})$ defines the unique simple F-submodule of $M_{p\alpha}$ by Propositions IV.0.18 and IV.0.20, $\mathfrak{J}_F(f^{p\alpha}) \subset \mathfrak{J}_F(f^\alpha)^{[p]}$. Combining these two observations we have that

$$\mathfrak{J}_F(f^{p^e\alpha}) \subset \mathfrak{J}_F(f^{p^{e-1}\alpha})^{[p]} \subset \ldots \subset \mathfrak{J}_F(f^{p\alpha})^{[p^{e-1}]} \subset \mathfrak{J}_F(f^\alpha)^{[p^e]} = \mathfrak{J}_F(f^{p^e\alpha}).$$

We conclude that all the containments are equalities. Since the Frobenius map is faithfully flat, we deduce that $\mathfrak{J}_F(f^{p\alpha}) = \mathfrak{J}_F(f^\alpha)^{[p]}$. $\square$

CHAPTER V

The intersection homology modules and the F-Jacobian ideals

The purpose of this chapter is two-fold: First, to extend the the notion of the intersection homology modules for hypersurfaces, introduced by Blickle [Bli04] in the local case, to the case of an arbitrary regular F-finite ring R of positive characteristic. Second, to introduced the F-Jacobian ideal of an element $f \in R$.

Notation V.0.32. Throughout this chapter R denotes an F-finite regular domain of characteristic $p > 0$. For $f \in R$, let $\sigma : R/fR \to R_f/R$ be the R-linear morphism given by $\sigma([a]) = [a/f]$. Since σ is injective, we identify R/fR with its image in R_f/R. For example, by $R/fR \subset R_f/R$ we mean $\sigma(R/fR) \subseteq R_f/R$.

V.1 Definition and first properties

In this section we define the F-Jacobian ideal and deduce some of its properties. We start with a few preliminary lemmas.

Lemma V.1.1. *Let $f \in R$ be an element and $\pi : R \to R/fR$ be the quotient morphism. If*

$$\mathcal{N} = \{N \subset R_f/R \mid N \text{ is an } F\text{-submodule}\} \text{ and } \mathcal{I} = \{I \subset R \mid f \in I, (I^{[p]} : f^{p-1}) = I\},$$

then there is a bijective correspondence between $\mathcal{N}$ and $\mathcal{I}$, given by sending N to $I_N = \pi^{-1}(N \cap R/fR)$. Its inverse is defined by sending the ideal $I \in \mathcal{I}$ to the F-module N_I generated by

$$I/fR \xrightarrow{f^{p-1}} F(I/fR) = I^{[p]}/f^p R.$$

Proof. Since $\phi : R/fR \xrightarrow{f^{p-1}} R/f^pR$ is a root for R_f/R, its F-submodules are in bijective correspondence with ideals $J \subset R/fR$ such that $\phi^{-1}(F(J)) = J$ [Lyu97, Corollary 2.6]. If $J = I/fR$ for an ideal $I \subset R$, then $F(J) = I^{[p]}/f^pR$. Therefore,

$$\phi^{-1}(I^{[p]}/f^p) = \{h \in R/fR \mid f^{p-1}h \in I^{[p]}/f^p\}$$

$$= \{h \in R \mid f^{p-1}h \in I^{[p]}\}/fR = (I^{[p]} : f^{p-1})/fR$$

and the result follows. □

Lemma V.1.2. *Suppose that R is a UFD. If $f \in R$ is an irreducible element, then $N \cap M \neq 0$ for any nonzero R-submodules $M, N \subset R_f/R$.*

Proof. Let $a/f^\beta \in M \setminus \{0\}$ and $b/f^\gamma \in N \setminus \{0\}$, where $\beta, \gamma \geq 1$. Since R is a UFD and f is irreducible, we may assume that $\gcd(a, f) = \gcd(b, f) = 1$. Then, $\gcd(ab, f) = 1$, hence $ab/f \neq 0$ in R_f/R. We have $ab/f = bf^{\beta-1}(a/f^\beta) = af^{\gamma-1}(b/f^\gamma) \neq 0$, and we conclude that $ab/f \in N \cap M$. □

Lemma V.1.3. *Suppose that the ring R is a UFD. If $f \in R$ is an irreducible element, then there is a unique simple D_R-submodule in R_f/R. In particular, the result holds if R is a local ring.*

Proof. Since R_f/R is a D_R-module of finite length [Lyu97, Example 5.2], there exists a simple D_R-submodule $M \subset R_f/R$. Let N be any simple D_R-submodule of R_f/R. Since $M \cap N \neq 0$ by Lemma V.1.2 and M is a simple D_R-module, $M = M \cap N$. Likewise, $N = M \cap N$. Therefore we have a unique nonzero simple D_R-submodule of R_f/R. □

Remark V.1.4. Let $I \subset R$ be an equidimensional ideal of codimension 1. Let $Q_1, \ldots, Q_\ell$ be the minimal primes of I. By applying the Mayer-Vietories sequence [BS98, Chapter 3], we obtain, for all $2 \leq i \leq \ell$, the exact sequence

$$H^1_{Q_1 \cap \ldots \cap Q_{i-1} + Q_i}(R) \to H^1_{Q_1 \cap \ldots \cap Q_{i-1}}(R) \oplus H^1_{Q_i}(R) \to H^1_{Q_1 \cap \ldots \cap Q_i}(R).$$

Furthermore, since $\operatorname{depth}(Q_1 \cap \ldots \cap Q_{\ell-1} + Q_\ell) = 2$, it follows that $H^1_{Q_1 \cap \ldots \cap Q_{\ell-1} + Q_\ell}(R) = 0$. Therefore we have injective morphisms

$$H^1_{Q_1 \cap \ldots \cap Q_{i-1}}(R) \oplus H^1_{Q_i}(R) \to H^1_{Q_1 \cap \ldots \cap Q_i}(R).$$

We deduce that there is an injective map

$$\eta : H^1_{Q_1}(R) \oplus \ldots \oplus H^1_{Q_\ell}(R) \to H^1_{Q_1 \cap \ldots \cap Q_\ell}(R) = H^1_I(R).$$

Where the equality follows from the fact that $Q_1, \ldots, Q_\ell$ are the minimal primes of I.

Let $\mathfrak{m} \subset R$ be any maximal ideal of R. Since $R_\mathfrak{m}$ is a UFD, there exist elements $g_i, f \in R_\mathfrak{m}$ such that $Q_i R_\mathfrak{m} = g_i R_\mathfrak{m}$, $IR_\mathfrak{m} = fR_\mathfrak{m}$ and $f = g_1 \cdots g_\ell$. In this case we have $H^1_{Q_i}(R_\mathfrak{m}) \cong (R_\mathfrak{m})_{g_i}/R_\mathfrak{m}$, $H^1_I(R_\mathfrak{m}) \cong (R_\mathfrak{m})_f/R_\mathfrak{m}$, and the map $H^1_{Q_i}(R_\mathfrak{m}) \to H^1_I(R_\mathfrak{m})$ is induced by the localization map $(R_\mathfrak{m})_{g_i} \to (R_\mathfrak{m})_f$. As a consequence, we deduce that the map η is a morphism of F-modules and that $H^1_{Q_i}(R) \cong H^0_{Q_i} H^1_I(R)$.

Propositions V.1.5 and V.1.8 are extensions of [Bli04, Theorem 4.1]. We point out that the proofs presented in this manuscript use neither étale invariance nor Kashiwara equivalence.

Proposition V.1.5. *If Q is a prime ideal of height 1, then $H^1_Q(R)$ has a unique simple D_R-submodule, which is denoted by $\mathcal{L}(R/Q, R)$.*

Proof. Recall that $H^1_Q(R)$ has finite length as an F-module by Theorem II.2.23. Therefore $H^1_Q(R)$ has finite length as a D_R-module by Corollary II.2.24. Suppose that M, N are simple D_R-submodules of $H^1_Q(R)$. We pick a maximal ideal $m \subset R$. If $Q \not\subseteq \mathfrak{m}$, then $M_\mathfrak{m} = N_\mathfrak{m} = 0$. So we can assume that $Q \subset m$. Note that $QR_\mathfrak{m}$ is a principal ideal in $R_\mathfrak{m}$ because every regular local ring is a UFD. Therefore we can find $g \in R_\mathfrak{m}$ such that $gR_\mathfrak{m} = QR_\mathfrak{m}$. We note that $\operatorname{Ass}_R H^1_Q(R) = \{Q\}$. Therefore $\operatorname{Ass}_R N = \{Q\}$ because $N \neq 0$ and $\operatorname{Ass}_R N \subset \operatorname{Ass}_R H^1_Q(R)$. As a consequence, $N_\mathfrak{m} \neq 0$. Furthermore, $\operatorname{length}_{D_{R_\mathfrak{m}}} N_\mathfrak{m} = 1$ because the length as a D_R-module cannot increase under localization. Therefore, $N_\mathfrak{m}$ is a simple $D_{R_\mathfrak{m}}$-submodule of $H^1_g(R_\mathfrak{m})$. Similarly, $M_\mathfrak{m}$ is a simple $D_{R_\mathfrak{m}}$-module of $H^1_g(R_\mathfrak{m}) \cong (R_\mathfrak{m})_g/R_\mathfrak{m}$. We note that g is an irreducible element because $(g) = QR_\mathfrak{m}$ is a prime ideal. Therefore $M_\mathfrak{m} = N_\mathfrak{m}$ by Lemma V.1.3. Since this holds for every maximal ideal $\mathfrak{m}$, we obtain the assertion in the proposition. $\square$

In [Bli04] Blickle defined the intersection homology D_R-modules for a regular local ring of positive characteristic. We now present an extension of this notion to the case of any regular ring of positive characteristic.

Definition V.1.6 (cf. [Bli04])**.** Let $I \subset R$ be an equidimensional ideal of codimension 1, and $Q_1, \ldots Q_\ell \in R$ be the minimal primes of I. We define the intersection homology

D_R-module of I by

$$\mathcal{L}(R/I, R) = \sum_{i=1}^{\ell} \mathcal{L}(R/Q_i, R),$$

where this sum is taken in $H^1_I(R)$, given that $H^1_{Q_1}(R) \cong H^0_{Q_1} H^1_I(R)$ by Remark V.1.4.

Remark V.1.7 (see [Bli04, Corollary 4.2])**.** Let $I \subset R$ be am equidimensional ideal of codimension 1. Let N be a simple D_R-submodule of $H^1_I(R)$ and Q an associated prime of N. Since $H^0_Q(N)$ is a nonzero D_R-submodule of N, we have $N = H^0_Q(N)$. We deduce that N is a simple D_R-submodule of $H^0_Q H^1_I(R)$; therefore $N = \mathcal{L}(R/Q, R)$ by Remark V.1.4 and Proposition V.1.5. We conclude that $\mathcal{L}(R/I, R)$ is the direct sum of all simple D_R-submodules of $H^1_I(R)$ via the inclusion in Remark V.1.4.

Proposition V.1.8. *If $I \subset R$ is an equidimensional prime ideal of codimension 1, then $\mathcal{L}(R/I, R)$ is an F-submodule of $H^1_I(R)$.*

Proof. Since the sum of F-submodules is again an F-submodule, by Remark V.1.7 it suffices to prove the claim for prime ideals. Let Q be a prime ideal of height 1 and let N denote the intersection homology module $\mathcal{L}(R/Q, R)$.

Let $H^1_Q(R) \xrightarrow{\theta} FH^1_Q(R)$ be the structure morphism as an F-module. Let $\mathfrak{m} \subset R$ be a maximal ideal containing Q. Since $R_\mathfrak{m}$ is a UFD, there exists $g \in R_\mathfrak{m}$ such that $QR_\mathfrak{m} = gR_\mathfrak{m}$. By a similar argument to the one given in the proof of Proposition V.1.5, we conclude that $N_\mathfrak{m}$ is the unique simple D_R-submodule of $H^1_g(R_\mathfrak{m})$. That is $N_\mathfrak{m}$ is equal to the intersection homology of $R_\mathfrak{m}/gR_\mathfrak{m}$, namely $\mathcal{L}(R_\mathfrak{m}/gR_\mathfrak{m}, R_\mathfrak{m})$. Then, $N_\mathfrak{m} = \mathcal{L}(R_\mathfrak{m}/gR_\mathfrak{m}, R_\mathfrak{m})$ is an F-submodule of $H^1_g(R_\mathfrak{m})$ [Bli04, Theorem 4.3].

We have $\theta_\mathfrak{m}(N_\mathfrak{m}) = FN_\mathfrak{m}$ for every maximal ideal because N is locally an F-submodule of $H^1_Q(R)$. Then, $\theta(N) = FN$ and N is an F-submodule of $H^1_Q(R)$. $\square$

Remark V.1.9. Let $Q \subset R$ be a prime of height 1. Since $\mathcal{L}(R/Q, R)$ is a simple D_R-module and an F-finite F-module, we deduce that it is a simple F-module by Corollary II.2.24.

Remark V.1.10. Suppose that R is a UFD. As a consequence of Remark V.1.7, we deduce that if $f = f_1 \cdots f_\ell$, with the $f_1, \ldots, f_\ell$ are relatively prime, then

$$\mathcal{L}(R/fR, R) = \mathcal{L}(R/f_1R, R) \oplus \ldots \oplus \mathcal{L}(R/f_\ell R, R) \subseteq R_f/R.$$

Indeed, this follows from the fact that, for all elements $g, h \in R$ such that $\gcd(g, h) = 1$, the intersection of R_g and R_h, as R-submodules of R_{gh}, is R.

We are ready to introduce the main object studied in this section.

Definition V.1.11. Let $f \in R$ be a nonzero element, $Q_1, \dots Q_\ell \subseteq R$ the minimal primes of R/fR, and $\pi : R \to R/fR$ the quotient morphism. We denote the pullback of $\mathcal{L}(R/fR, R) \cap R/fR$ to R by $J_F(f)$ and we call it the *F-Jacobian ideal of f*. That is

$$J_F(f) = \{a \in R | \, [a/f] \in \mathcal{L}(R/fR, R)\} \, .$$

Proposition V.1.12. *Suppose that R is a UFD. If $g \in R$ is an irreducible element and $f = g^n$ for some integer $n \geq 1$, then there exists a unique ideal $I \subset R$ such that:*

(i) $f \in I$,

(ii) $I \neq fR$,

(iii) $(I^{[p]} : f^{p-1}) = I$, *and*

(iv) I *is contained in any other ideal satisfying* (i), (ii), and (iii).

Furthermore, in this case $I = J_F(f)$

Proof. We note that $R_f/R = R_g/R$ and therefore $\mathcal{L}(R/fR, R) = \mathcal{L}(R/gR, R)$. As g is an irreducible element of R, Lemma V.1.2 and Remark V.1.9 imply that $\mathcal{L}(R/fR, R)$ is a simple F-module. Let I be the ideal corresponding to $\mathcal{L}(R/fR, R)$ given in Lemma V.1.3 under the bijection in Lemma V.1.1. Therefore, from Lemma V.1.1, we deduce that I satisfies (i)-(iv) and that $I = J_F(f)$. □

Remark V.1.13. Suppose that R is a UFD. If $f \in R$ is an irreducible element, then:

(i) $\mathcal{L}(R/fR, R) = \mathcal{L}(R/f^nR, R)$ for every $n \in \mathbb{N}$ because $R_{f^n}/R = R_f/R$.

(ii) $J_F(f)$ is the minimal element of the family of ideals I containing properly fR such that $(I : f^{p-1}) = I$ by Proposition V.1.12.

(iii) $J_F(f)$ is not the usual Jacobian ideal of f. For example, if $R = \mathbb{F}_3[x, y, z, w]$ and $f = xy + zw$, the Jacobian ideal of f is $\mathfrak{m} = (x, y, z, w)R$. However, $\mathfrak{m} \neq (\mathfrak{m}^{[3]} : f^2) = (x^2, y^2, z^2, w^2)$.

(iv) $J_F(f) = R$ if and only if R_f/R is a simple F-module by the proof of Proposition V.1.12.

(v) R_f/R is a simple D_R-module if and only if $J_F(f) = R$. Indeed, both conditions are equivalent to saying that $\mathcal{L}(R/fR, R) = R_f/R$

We now study how the F-Jacobian ideal interacts with different elements. In particular, we start by proving the Leibniz Rule for F-Jacobian ideals.

Proposition V.1.14. *Suppose that R is a UFD. If $f, g \in R$ are relatively prime elements, then $J_F(fg) = fJ_F(g) + gJ_F(f)$. Moreover, we have $fJ_F(g) \cap gJ_F(f) = fgR$.*

Proof. We consider R_f/R and R_g/R as a F-submodules of R_{fg}/R, where the inclusion is induced by the localization maps $\iota_f : R_f \to R_{fg}$ and $\iota_g : R_g \to R_{fg}$.

Let $\pi : R \to R/fgR$ and $\rho : R \to R/fR$ be the quotient morphisms. The limit of the vertical morphism in the commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
R/fR \arrow[r, "f^{p-1}"] \arrow[d, "g"] & R/f^pR \arrow[r, "f^{p^2-p}"] \arrow[d, "g^p"] & R/f^{p^2}R \arrow[r, "f^{p^3-p^2}"] \arrow[d, "g^{p^2}"] & \ldots \\
R/fgR \arrow[r, "(fg)^{p-1}"] & R/f^pg^pR \arrow[r, "(fg)^{p^2-p}"] & R/(fg)^{p^2}R \arrow[r, "(fg)^{p^3-p^2}"] & \ldots
\end{tikzcd}$$

is the morphism $R_f/R \to R_{fg}/R$ induced by ι_f. Moreover, the correspondence in Lemma V.1.1 gives the commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
J_F(f)/fR \arrow[r, "f^{p-1}"] \arrow[d, "g"] & J_F(f)^{[p]}/f^pR \arrow[r, "f^{p^2-p}"] \arrow[d, "g^p"] & J_F(f)^{[p^2]}/f^{p^2}R \arrow[r] \arrow[d, "g^{p^2}"] & \ldots \\
gJ_F(f)/fgR \arrow[r, "(fg)^{p-1}"] & g^pJ_F(f)^{[p]}/f^pg^pR \arrow[r, "(fg)^{p^2-p}"] & g^{p^2}J_F(f)^{[p^2]}/(fg)^{p^2}R \arrow[r] & \ldots
\end{tikzcd}$$

such that the limit of the vertical maps give the isomorphism of F-modules, $\tilde{\iota}_f$: $\mathcal{L}(R/fR, R) \to \tilde{\iota}_f(\mathcal{L}(R/fR, R))$. We have

$$gJ_F(f) = \pi^{-1}(\mathcal{L}(R/fR, R) \cap R/fgR) \subset \pi^{-1}(\mathcal{L}(R/fgR, R) \cap R/fgR) = J_F(fg).$$

In addition, $gJ_F(f)$ is the ideal that corresponds to $\tilde{\iota}_f(\mathcal{L}(R/fR, R)) \subseteq R_{fg}/R$ via the bijection in Lemma V.1.1. Likewise, $fJ_F(g) \subset J_F(fg)$ and it corresponds, via Lemma V.1.1, to $\tilde{\iota}_g(\mathcal{L}(R/gR, R))$ as an F-submodule of R_{fg}/R. Therefore we can conclude that $fJ_F(g) + gJ_F(f) \subset J_F(fg)$.

We claim that

$$\left((f^pJ_F(g)^{[p]} + g^pJ_F(f)^{[p]}) : f^{p-1}g^{p-1}\right) = fJ_F(g) + gJ_F(f).$$

Consider first $h \in (f^p J_F(g)^{[p]} + g^p J_F(f)^{[p]} : f^{p-1}g^{p-1})$. Then $f^{p-1}g^{p-1}h = f^p v + g^p w$ for some $v \in (J_F(g))^{[p]}$ and $w \in J_F(g)^{[p]}$. Since f and g are relatively prime, f^{p-1} divides w and g^{p-1} divides v. Thus, there exist $a, b \in R$ such that $v = g^{p-1}a$ and $w = g^{p-1}b$. Then, $a \in (J_F(g)^{[p]} : g^{p-1}) = J_F(g)$ and $b \in (J_F(f)^{[p]} : f^{p-1}) = J_F(f)$. Since $f^{p-1}g^{p-1}h = f^p v + g^p w = f^p g^{p-1}a + g^p f^{p-1}b$, we deduce that $h = fa + gb \in fJ_F(g) + gJ_F(f)$.

For the reverse inclusion, it is straightforward to check that

$$fJ_F(g) + gJ_F(f) \subset \left((f^p J_F(g)^{[p]} + g^p J_F(f)^{[p]}) : f^{p-1}g^{p-1}\right).$$

Therefore $fJ_F(g) + gJ_F(f)$ generates an F-submodule of R_{fg}/R. If N is such F-submodule, then N contains both $\mathcal{L}(R/fR, R)$ and $\mathcal{L}(R/gR, R)$, we conclude that $\tilde{\iota}_f\left(\mathcal{L}(R/fR, R)\right) \oplus \tilde{\iota}_g\left(\mathcal{L}(R/gR, R)\right) \subset N$. Therefore, $J_F(fg) \subset fJ_F(g) + gJ_F(f)$ and the first statement of the proposition follows.

The last part of the proposition is an immediate consequence of the fact that $\gcd(f, g) = 1$, $f \in J_F(f)$ and $g \in J_F(g)$. □

Proposition V.1.15. *If $m, n \in \mathbb{N}$ are such that $m < n$, then $f^{n-m}J_F(f^m) \subset J_F(f^n) \subset J_F(f^m)$.*

Proof. From the definition of F-Jacobian ideal we have

$$\begin{aligned} f^{n-m}J_F(f^m) &= \left\{a \in R \mid f^{n-m} \cdot [a/f^m] \in \mathcal{L}(R/fR, R)\right\} \\ &\subset \{a \in R \mid [a/f^n] \in \mathcal{L}(R/fR, R)\} \\ &= J_F(f^n). \end{aligned}$$

The second assertion can be proved in a similar way.

□

Remark V.1.16. The first inclusion in Proposition V.1.15 can be strict strict. Indeed, let $R = \mathbb{F}_p[x]$ and $f = x$. In this case, R_f/R is a simple F-module. Therefore, $J_F(x^m) = R$ for every $m \geq 1$ and $f^{n-m}J_F(f^m) \neq J_F(f^n)$ for every $n > m$.

Corollary V.1.17. *If $f, g \in R$ are such that f divides g, then, $J_F(g) \subset J_F(f)$.*

Proof. It suffices to prove the statement for local rings, hence we may assume that R is a UFD. In this case, the claim follows from Propositions V.1.14 and V.1.15. □

V.2 Relations with test ideals and F-regularity

In this section we give some further properties of the F-Jacobian ideal. In particular, we relate it to the study of singularities in positive characteristic.

Notation V.2.1. If $f \in R$ is an element such that R/fR is a reduced ring, we denote by τ_f the pullback of the test ideal $\tau(R/fR)$ of R/fR to R.

Proposition V.2.2. *Suppose that R is a UFD. If $f \in R$ is an irreducible element, then*

$$J_F(f) = \bigcap_{\gcd(a,f)=1} \left(\bigcup_{e\in\mathbb{N}} \left(\left((f^{p^e-1}a)^{[1/p^e]}, f \right)^{[p^e]} : f^{p^e-1} \right) \right).$$

Proof. For $0 \neq a/f^n \in \mathcal{L}(R/fR, R)$, such that $\gcd(a,f) = 1$, we know that $0 \neq a/f \in \mathcal{L}(R/fR, R)$, since R is a UFD and f is an irreducible element. Therefore the simplicity of $\mathcal{L}(R/fR, R)$ implies $D_R \cdot a/f = \mathcal{L}(R/fR, R)$. We can conclude that $\mathcal{L}(R/fR, R)$ is the intersection of all nonzero cyclic D_R-submodules generated by elements $a/f \in R_f/R$. Hence,

$$J_F(f)/f = \bigcap_{\gcd(a,f)=1} ((D_R \cdot a/f) \cap R \cdot 1/f) = \bigcap_{\gcd(a,f)=1} \left(\bigcup_{e\in\mathbb{N}} \left(D_R^{(e)} \cdot a/f \cap R \cdot 1/f \right) \right).$$

Note that $b \in J_F(f)$ iff $b/f \in \mathcal{L}(R/fR, R) = \bigcap_{\gcd(a,f)=1} \bigcup_{e\in\mathbb{N}} \left(D^{(e)} \cdot a/f \right)$. In this case, for every $a \in R$ such that $\gcd(a,f) = 1$, there exists $e \in \mathbb{N}$ such that $b/f \in D_R^{(e)} \cdot a/f$. Thus, there exists $\phi \in D_R^{(e)}$ such that $\phi(a/f) = 1/f^{p^e}\phi(f^{p^e-1}a) = b/f$ in R_f/R. Then, there exists an element $r \in R$ such that $\phi(a/f) = 1/f^{p^e}\phi(f^{p^e-1}a) = b/f + r$ in R_f. We deduce that, $f^{p^e-1}b = \phi(f^{p^e-1}a) - f^{p^e}r$ and $b \in (I : f^{p^e-1})$ for $I = D_R^{(e)} \cdot (f^{p^e-1}a) + f^{p^e}R$. Since $((f^{p^e-1}a)^{[1/p^e]})^{[p^e]} = D_R^{(e)} \left(f^{p^e-1}a\right)$ by Proposition II.2.18, we conclude that $b \in \bigcap_{\gcd(a,f)=1} \left(\bigcup_{e\in\mathbb{N}} \left(\left((f^{p^e-1}a)^{[1/p^e]}, f \right)^{[p^e]} : f^{p^e-1} \right) \right)$.

On the other hand, if

$$b \in \bigcap_{\gcd(a,f)=1} \bigcup_{e\in\mathbb{N}} \left(\left((f^{p^e-1}a)^{[1/p^e]}, f \right)^{[p^e]} : f^{p^e-1} \right),$$

then for every $a \in R$ such that $\gcd(a,f) = 1$, there exist $e \in \mathbb{N}, \phi \in D_R^{(e)}$ and $r \in R$ such that $f^{p^e-1}b = \phi(f^{p^e-1}a) + f^{p^e}r$ because $\left((f^{p^e-1}a)^{[1/p^e]} \right)^{[p^e]} = D_R^{(e)}(f^{p^e-1}a)$. Therefore, after dividing by f^{p^e}, we have that $b/f \in D_R^{(e)} \cdot a/f$ in R_f/R, hence, $b \in J_F(f)$ □

The following proposition shows that the F-Jacobian ideal commutes with localization.

Proposition V.2.3. *If $f \in R$ is nonzero and $W \subset R$ is a multiplicative system, then $J_{F_{W^{-1}R}}(f) = W^{-1}J_{F_R}(f)$.*

Proof. Let $Q_1, \ldots, Q_\ell$ be the minimal primes of fR. We note that for every i, $W^{-1}\mathcal{L}(R/Q_iR, R)$ is zero if $Q_i \cap W \neq \varnothing$ or a simple $D_{W^{-1}R}$-submodule of $H^1_f(W^{-1}R)$ if $Q_i \cap W = \varnothing$. Hence, if $Q_i \cap W = \varnothing$, then $W^{-1}\mathcal{L}(R/Q_i, R)$ is the intersection homology of $W^{-1}(R/Q_i)$ by Proposition V.1.5. As a consequence, $W^{-1}\mathcal{L}(R/fR, R)$ is the intersection homology of $W^{-1}R/fW^{-1}R$. Then,

$$\begin{aligned} J_{F_{W^{-1}R}}(f)/fW^{-1}R &= W^{-1}R/fW^{-1}R \cap \mathcal{L}(W^{-1}R/fW^{-1}R, W^{-1}R) \\ &= W^{-1}(R/fR \cap \mathcal{L}(R/fR, R)) = W^{-1}(J_{F_R}(f)/fR), \end{aligned}$$

and the result follows. $\square$

We now give a theorem that relates the F-Jacobian ideal with F-regularity. We point out that since R/fR is a Gorenstein ring this theorem is a consequence of a result of Blickle [Bli04, Corollary 4.10]. However, our proof is different from the one given there.

Theorem V.2.4. *Let $f \in R$ be nonzero, such that R/fR is an F-pure ring. If $J_F(f) = R$, then R/fR is strongly F-regular.*

Proof. We may assume that $(R, \mathfrak{m}, K)$ is local and $f \in \mathfrak{m}$, because F-purity and F-regularity are local properties for R/fR and the F-Jacobian ideal commutes with localization by Proposition V.2.3. We recall that every F-pure ring is reduced [Fed87]. Note that if $f = gh$ for g, h non unit relative primes, then Proposition V.1.14 implies $J_F(f) \subset (g, h)$. Hence $J_F(f) = R$ implies that $f = g^n$ for some irreducible g in R and some $n > 0$, but since R/fR is reduced we have $n = 1$ and therefore f is irreducible. Due to the fact that $J_F(f) = R$, we obtain that for every a such that $\gcd(a, f) = 1$ there exists an $e \in \mathbb{N}$ such that $R = \left(\left((f^{p^e-1}a)^{[1/p^e]}, f\right)^{[p^e]} : f^{p^e-1}\right)$ by Lemma V.2.2. Then, $f^{p^e-1} \in \left((f^{p^e-1}a)^{[1/p^e]}, f\right)^{[p^e]}$. Since $f^{p^e-1} \not\in \mathfrak{m}^{[p^e]}$ for every $e \in \mathbb{N}$ by Fedder's criterion [Fed87], $R = (f^{p^e-1}a)^{[1/p^e]}$; otherwise, $((f^{p^e-1}a)^{[1/p^e]}, f) \subset \mathfrak{m}$. Therefore, there exists a morphism $\phi \in \operatorname{Hom}_{R^{p^e}}(R, R^{p^e})$ such that $\phi(f^{p^e-1}a) = 1$. Let $\varphi : R/fR \to R/fR$ be the morphism defined by $\varphi([x]) = [\phi(f^{p^e-1}x)]$. This corresponds to an R-linear map $\varphi' : R^{1/p^e} \to R$ that sends $[a]^{1/p^e} \to [1]$. Hence, R/fR is strongly F-regular. $\square$

Remark V.2.5. $J_F(f) = R$ does not imply that R/fR is F-pure. For example, let $R = \mathbb{F}_2[x]$ and $f = x^2$. In this case, $J_F(f) = R$ and R/fR is not even reduced.

Remark V.2.6. Blickle showed that the intersection homology module of f is has the test ideal of R/fR as minimal root [Bli04, Proposition 4.5]. As a consequence, we have $f^{p-1}\tau_f \subset \tau_f^{[p]}$.

We now turn to the relations between $J_f(f)$ and τ_f.

Lemma V.2.7. *Suppose that R/fR is reduced and let $I^j(f) = (\tau_f^{[p^{j-1}]} : f^{p^{j-1}-1})$ for $j > 0$. Then $I^j(f) \subset I^{j+1}(f)$ and $I^{j+1}(f) = (I^j(f)^{[p]} : f^{p-1})$. Moreover, $I^j(f)/fR \overset{f^{p-1}}{\to} I^j(f)^{[p]}/f^pR$ generates $\mathcal{L}(R/fR, R)$ as an F-module.*

Proof. We note that $I^1(f) = \tau_f$ and this is a root for $\mathcal{L}(R/fR, R)$ by Remark V.2.6. In addition, $f^{p-1}I^1(f) = f^{p-1}\tau_f \subset \tau_f^{[p]} = I^1(f)^{[p]}$. Thus, $I^1(f) \subset I^2(f)$ and $f^{p-1}I^2(f) \subset I_1(f)^{[p]}$. Moreover, $I^2(f)/fR$ is also a root for $\mathcal{L}(R/fR, R)$ because $I^2(f)/I^1(f)$ is the kernel of the map $R/I^1(f) \overset{f^{p-1}}{\to} R/I^1(f)^{[p]}$. Inductively, we obtain that $I^j(f) \subset I^{j+1}(f)$, $I^{j+1}(f) = (I^j(f)^{[p]} : f^{p-1})$ and that $I^j(f)/fR$ is a root for $\mathcal{L}(R/fR, R)$ for every $j \in \mathbb{N}$. The result follows. □

The following proposition allows us to compute the F-Jacobian ideal from the classical test ideal of R/fR.

Proposition V.2.8. *If f is nonzero and R/fR is reduced, then $J_F(f) = \bigcup_{j\geq 1} I^j(f)$ and*

$$J_F(f)/fR \overset{f^{p-1}}{\to} J_F(f)^{[p]}/f^pR$$

generates $\mathcal{L}(R/fR, R)$ as an F-module.

Proof. By the Noetherian property, the flag of ideals

$$I^1(f) \subset I^2(f) \subset I^3(f) \subset \dots$$

eventually stabilizes. Let k be such that $I^k(f) = I^j(f)$ for $j \geq k$. Then, $I^k(f) = I^{k+1}(f) = (I^k(f)^{[p]} : f^{p-1})$ by Lemma V.2.7. Thus, $I^k(f)$ is an ideal satisfying the conditions in Proposition V.1.12, hence $J_f(F) \subset I^k(f)$. Since $I^k(f)/fR \overset{f^{p-1}}{\to} I^k(f)^{[p]}/f^pR$ generates $\mathcal{L}(R/fR, R)$, we conclude that $I^k(f)/fR \subset \mathcal{L}(R/fR, R) \cap R/fR = J_F(f)/fR$. Therefore we have,

$$J_F(f) = I^k(f) = \bigcup_{j\geq 1} I^j(f).$$

□

Remark V.2.9. In general, we do not have $\tau_f = J_F(f)$. For example let $R = K[x]$, where K is any perfect field of characteristic $p > 0$ and let $f = x^2$. In this case, $\tau_f = xR \neq R = J_F(f)$. Example V.5.3 below shows another situation when $\tau_f \neq J_F(f)$.

Corollary V.2.10. *Suppose that R is an UFD and that it is a $\mathbb{Z}^h$-graded ring. If $f \in R$ is a nonzero homogeneous element, then $J_F(f)$ is a homogeneous ideal.*

Proof. It suffices to prove that $\mathcal{L}(R/fR, R)$ is a $\mathbb{Z}^h$-graded submodule of R_f/R. We may assume that R/fR is reduced since $\sqrt{f}$ is principal and $\mathcal{L}(R/\sqrt{fR}, R) = \mathcal{L}(R/fR, R)$. It is known that $\tau(R/fR)$ is a homogeneous ideal, see [HH94b, Theorem 4.2]. This implies that $I^j_R(f)$ is a homogeneous ideal for every j. Therefore, $J_F(f)$ is homogeneous and $\mathcal{L}(R/fR, R)$ is a $\mathbb{Z}^n$-graded submodule of R_f/R. □

Corollary V.2.11. *If $f \in R$ is nonzero, such that R/fR is reduced, then*

$$V(J_F(f)) \subset \{Q \in \mathrm{Spec}(R) \mid R_Q/fR_Q \textit{ is not } F\textit{-regular}\}.$$

Moreover, if R/fR is an F-pure ring, then

$$V(J_F(f)) = \{Q \in \mathrm{Spec}(R) \mid R_Q/fR_Q \textit{ is not } F\textit{-regular}\}.$$

Proof. For every prime ideal $Q \in V(J_F(f))$, $J_F(f)R_Q = J_{F_{R_Q}}(f) \neq R_Q$.We have $\tau_f R_Q \subset J_{F_{R_Q}}(f) \subset QR_Q$. Therefore, R_Q is not F-regular.

Now, we suppose that R/fR is F-pure. For every prime ideal $Q \subset R$ such that R_Q/fR_Q is not F-regular, we have $J_{F_{R_Q}}(f) \neq R_Q$ by Theorem V.2.4. Therefore $Q \in V(J_F(f))$. □

Lemma V.2.12. *Let $f \in R$ be a nonzero element. If R/fR is F-pure, then $R/J_{F_R}(f)$ is F-pure.*

Proof. We may assume that R is local with maximal ideal $\mathfrak{m}$. Since R/fR is F-pure, we have $f^{p-1} \notin \mathfrak{m}^{[p]}$ by Fedder's Criterion [Fed87]. We know that $f^{p-1} \in (J_F(f)^{[p]} : J_F(f))$, hence $(J_F(f)^{[p]} : J_F(f)) \not\subset \mathfrak{m}^{[p]}$. Therefore, $R/J_F(f)$ is F-pure by Fedder's Criterion loc. cit. □

Corollary V.2.13. *Let $f \in R$. If R/fR is an F-pure ring, then $J_F(f) = \tau_f$.*

Proof. We have $\sqrt{J_F(f)} = \sqrt{\tau_f}$ by Corollary V.2.11 because

$$\{Q \in \mathrm{Spec}(R) \mid R_Q/fR_Q \text{ is not } F\text{-regular}\} = V(\tau(R/fR))$$

in this case. Since $R/J_F(f)$ is F-pure by Lemma V.2.12, $J_F(f)$ is a radical ideal. In addition, $\tau(R/fR)$ is a radical ideal [FW89b, Proposition 2.5], hence τ_f is a radical ideal. Therefore $J_F(f) = \tau_f$. $\square$

V.3 Behavior under extensions

In this subsection we study the behavior of F-Jacobian ideals under completion and other flat extensions.

Proposition V.3.1. *If $f \in R$ is nonzero, then $J_{F_{R^{1/p}}}(f) = J_{F_R}(f)R^{1/p}$. Equivalently, $J_{F_R}(f^p) = J_{F_R}(f)^{[p]}$.*

Proof. By Proposition V.2.3, since taking p-roots commutes with localization, it suffices to prove the statement for regular local rings. Therefore we may assume that R is a UFD. By Proposition V.1.14, we may assume that $f = g^n$, where g is irreducible. Let h denote the length of R_f/R in the category of F-modules. Let $G : R^{1/p} \to R$ be the isomorphism defined by $r \to r^p$. Under the isomorphism G, $R_f^{1/p}/R^{1/p}$ corresponds to R_{f^p}/R. Therefore the length of $R_f^{1/p}/R^{1/p}$ in the category of $F_{R^{1/p}}$-modules is h. Let $0 = M_0 \subset \ldots \subset M_h = R_f/R$ be a chain of F_R-submodules of R_f/R such that M_{i+1}/M_i is a simple F_R-module. Let $fR = J_0 \subset \ldots \subset J_h = R$ be the corresponding chain of ideals under the bijection given in Lemma V.1.1. By Proposition V.1.3 and since $f = g^n$ and g is irreducible, $M_1 = \mathcal{L}(R/fR, R)$ and $J_1 = J_{F_R}(f)$. We note that $(J_i^{[p]}R^{1/p} : f^{p-1}R^{1/p}) = J_iR^{1/p}$ and $J_iR^{1/p} \neq J_{i+1}R^{1/p}$ because $R^{1/p}$ is a faithfully flat R-algebra.

Then, we have a strictly ascending chain of ideals

$$fR^{1/p} = J_0R^{1/p} \subset \ldots \subset J_hR^{1/p} = R^{1/p}$$

that corresponds to a strictly ascending chain of $F_{R^{1/p}}$-submodules of $R_f^{1/p}/R^{1/p}$. Since $f = \left(g^{1/p}\right)^p$, $g^{1/p}$ is irreducible and the length of $R_f^{1/p}/R^{1/p}$ is h, we have

$$J_{F_R}(f)R^{1/p} = J_1R^{1/p} = J_{F_{R^{1/p}}}(f).$$

We conclude that

$$J_{F_R}(f)^{[p]} = G(J_{F_R}(f)R^{1/p}) = G(J_{F_{R^{1/p}}}(f)) = J_{F_R}(f^p).$$

$\square$

Proposition V.3.2. *If $R \to S$ is a flat morphism of regular F-finite domains, then $J_{F_S}(f) \subset J_{F_R}(f)S$.*

Proof. By Proposition V.2.3, since flatness is a local property, it suffices to prove our claim for local rings. By Proposition V.1.14 we may assume that $f = g^\beta$, where g is an irreducible element in R. Since S is flat, $(J_{F_R}(f)^{[p]}S : f^{p-1}) = J_{F_R}(f)S$. Let M denote the F_S-submodule of S_f/S given by $J_{F_R}(f)S$ under the correspondence in Lemma V.1.1. If f is a unit in S, then $J_F(f)S = S$ and the result is immediate. We may assume that f is not a unit in S. Since $J_F(f) \neq fR$, we can pick $a \in J_F(f) \setminus fR$. Then, $a = bg^\gamma$ for some $0 \leq \gamma < \beta$ and $b \in R$ such that $\gcd(b, g) = 1$. In this case, $R/g \xrightarrow{b} R/g$ is injective, hence $S/gS \xrightarrow{b} S/gS$ is also injective. Thus, $\gcd(b, g) = 1$ in S. Hence, b/g is not zero in S_g/S. Moreover, $b/g = g^{\beta-\gamma-1}a/f \in M$ and it is not zero. Let $g_1, \ldots, g_\ell \in S$ be irreducible relatively prime elements such that $g = g_1^{\beta_1} \cdots g_\ell^{\beta_\ell}$. We have $b/g_i = h_i b/g \in S_{g_i}/S \cap M \setminus \{0\}$, where $h_i = g_1^{\beta_1} \cdots g_i^{\beta_i - 1} \cdots g_1^{\beta_\ell}$. In particular, as $\mathcal{L}(S/g_iS, S)$ is the unique simple F-submodule of S_{g_i}/S and the intersection $S_{g_i}/S \cap N$ is nontrivial we have that $\mathcal{L}(S/g_iS, S) \subset S_{g_i}/S \cap M \subset M$ for every i. Hence we deduce $\mathcal{L}(S/fS, S) \subset M$, which implies $J_{F_S}(f) \subset J_{F_R}(f)S$. □

Proposition V.3.3. *Suppose that R is a local ring. If $f \in R$ is nonzero, then $J_{F_{\widehat{R}}}(f) = J_{F_R}(f)\widehat{R}$, where $\widehat{R}$ denotes the completion of R with respect to the maximal ideal.*

Proof. We have $\mathcal{L}(\widehat{R}/f\widehat{R}, \widehat{R}) = \mathcal{L}(R/fR, R) \otimes_R \widehat{R}$ [Bli04, Theorem 4.6], hence

$$J_{F_{\widehat{R}}} = \left(\widehat{R}/f\widehat{R}\right) \cap \mathcal{L}(\widehat{R}/f\widehat{R}, \widehat{R}) = ((R/fR) \cap \mathcal{L}(R/fR, R)) \otimes_R \widehat{R} = J_{F_R}(f)\widehat{R}.$$

□

Proposition V.3.4. *Suppose that $(R, \mathfrak{m}, K)$ is local. Let (S, η, L) denote a regular F-finite ring. If $R \to S$ is a flat local morphism such that the closed fiber $S/\mathfrak{m}S$ is regular and L/K is separable, then $J_{F_S}(f) = J_{F_R}(f)S$.*

Proof. We can assume without loss of generality that R/fR is reduced because the intersection homology depends only on the local cohomology $H^1_f(R)$. We have $J_{F_{\widehat{R}}}(f) = J_{F_R}(f)\widehat{R}$ and $J_{F_{\widehat{S}}}(f) = J_{F_S}(f)\widehat{S}$ by Proposition V.3.3. In addition, the induced morphism $\widehat{R} \to \widehat{S}$ is still a flat local morphism. Since $J_{F_S}(f) \subset J_{F_R}(f)S$ and $J_{F_{\widehat{S}}}(f) \subset J_{F_{\widehat{R}}}(f)\widehat{S}$ by Proposition V.3.2, $J_{F_{\widehat{R}}}(f)\widehat{S}/J_{F_{\widehat{S}}}(f) = (J_{F_R}(f)S/J_{F_S}(f)) \otimes_S \widehat{S}$. Therefore, we can assume that R and S are complete. In this case, we know that S/fS is reduced and $\tau(R/fR)S = \tau(S/fS)$ [HH94a, Theorem 7.2]. Therefore, $I^j_S(f) = I^j_R(f)S$ and $J_{F_S}(f) = J_{F_R}(f)S$ by Proposition V.2.8. □

We now focus on F-Jacobian ideals in polynomial rings over a field. In particular, we study how the F-Jacobian ideal behaves under field extensions.

Lemma V.3.5. *Let $R = K[x_1, \ldots, x_n]$, where K is a perfect field. If $K \to L$ is a finite algebraic field extension, $S = L[x_1 \ldots, x_n]$, and $R \to S$ is the map induced by the extension, then $J_{F_S}(f) = J_{F_R}(f)S$.*

Proof. Note that L is F-finite since the extension is finite, therefore so is S. By V.1.14, we can assume that $f = g^\beta$, where g is an irreducible element in R. By Proposition V.3.2, it suffices to show that $J_{F_R}(R)S \subset J_{F_S}(S)$.

There is an inclusion $\phi : R_f/R \to S_f/S$, which is induced by $R \to S$. We take $M = \mathcal{L}(S/fS, S) \cap R_f/R$. We claim that M is a D_R-submodule of R_f/R. Since K is perfect, we have

$$D_R = D(R, \mathbb{Z}/p\mathbb{Z}) = \bigcup_{e\in\mathbb{N}} \mathrm{Hom}_{R^{p^e}}(R, R) = D(R, K) = R[\frac{1}{t!}\frac{\partial^t}{\partial x_i^t}].$$

We note that $D_R = D(R, K) \subset D(S, K) \subset D_S$, and that $\phi(\frac{1}{t!}\frac{\partial^t}{\partial x_i^t}v) = \frac{1}{t!}\frac{\partial^t}{\partial x_i^t}\phi(v)$ for every $v \in R_f/R$. As a consequence, $\frac{\partial^t}{\partial x_i^t}v \in M$ for every $v \in M$. Therefore, M is a D_R-module.

Let $I = M \cap R/fR$. We note that

$$I = \mathcal{L}(S/fS, S) \cap R/fR = (J_{F_S}(f)/fS) \cap R/fR$$

and that S/fS is an integral extension of R/fR because L is an algebraic extension of K. Let $r \in J_{F_S}(f)/fS$ not zero, and $a_j \in R/fR$ such that $a_0 \neq 0$ and $r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0 = 0$ in S/fS. Then, $r(a_{n-1}r^{n-1} + \ldots + a_1) = -a_0$, and so $a_0 \in I = (J_{F_S}(f)/fS) \cap R/fR$, and then $M \neq 0$. Therefore, $\mathcal{L}(R/fR, R) \subset M$ and so $J_F(f)/f \subset I$. Let $\pi : R \to R/fR$ be the quotient morphism. Then,

$$J_{F_R}(f) \subset \pi^{-1}(I) = J_{F_S}(f) \cap R, \text{ and } J_{F_R}(f)S \subset (J_{F_S}(f) \cap R)S \subset J_{F_S}(f).$$

□

Lemma V.3.6. *Let $R = K[x_1, \ldots, x_n]$, where K is an F-finite field. Let $L = K^{1/p}$, $S = L[x_1 \ldots, x_n]$, and $R \to S$ the map induced by the extension $K \to L$. Then $J_{F_S}(f) = J_{F_R}(f)S$.*

Proof. We have that $R \subset S \subset R^{1/p}$. Then, by Proposition V.3.2,

$$J_{F_{R^{1/p}}}(f) \subset J_{F_S}(f)R^{1/p} \subset (J_{F_R}(f)S)R^{1/p} = J_{F_R}(f)R^{1/p}.$$

Since $J_{F_{R^{1/p}}}(f) = J_{F_R}(f)R^{1/p}$ by Proposition V.3.1,

$$0 = J_{F_S}(f)R^{1/p}/(J_{F_R}(f)S)R^{1/p} = (J_{F_S}(f)/J_{F_R}(f)S) \otimes_S R^{1/p}.$$

Therefore, $J_{F_S}(f) = J_{F_R}(f)S$ because $R^{1/p}$ is a faithfully flat S-algebra. $\square$

Lemma V.3.7. *Let $R = K[x_1, \dots, x_n]$, where K is an F-finite field. Let L be the perfect closure of K, $S = L[x_1 \dots, x_n]$, and $R \to S$ the map induced by the extension $K \to L$. Then $J_{F_S}(f) = J_{F_R}(f)S$.*

Proof. We may assume that $f = g^n$ for an irreducible $g \in R$ by Proposition V.1.14. Let $S^e = K^{1/p^e}[x_1, \dots, x_n]$. Let $h_1, \dots, h_\ell$ denote a set of generators for $J_{F_S}(f)$. In this case, $(J_{F_S}(f)^{[p]} : f^{p-1}) = J_{F_S}(f)$. Then there exist $c_{i,j} \in S$ such that $f^{p-1}h_j = \sum c_{i,j}h_j^p$. Since $S = \bigcup_e S^e$, there exists m such that $c_{i,j}, h_j \in S^m$. Let $I \subset S^m$ be the ideal generated by $h_1, \dots, h_\ell$. We note that $IS = J_{F_S}(f)$; moreover, $J_{F_S}(f) \cap S^m = I$ because $S^e \to S$ splits for every $e \in \mathbb{N}$.

We claim that $(I^{[p]} : f^{p-1}) = I$. We have that $f^{p-1}h_\ell \in I^{[p]}$ by our choice of m and so $I \subset (I^{[p]} : f^{p-1})$. If $g \in (I^{[p]} : f^{p-1})$, then $f^{p-1}g \in I^{[p]}S \subset J_{F_S}(f)^{[p]}$ and $g \in J_{F_S}(f) \cap S^m = I$.

As in the proof of Lemma V.3.5, $(J_{F_S}(f)/fS) \cap (S^m/fS^m) \neq 0$ and then $J_{F_S}(f) \cap S^m = I \neq fS$. Therefore, $J_{F_{S^m}}(f) \subset I$ by Proposition V.1.12. Hence,

$$J_{F_{S^m}}(f)S \subset IS = J_{F_S}(f) \subset J_{F_{S^m}}(f)S,$$

and the result follows because

$$J_{F_R}(f)S = (J_{F_R}(f)S^m)S = J_{F_{S^m}}(f)S.$$

$\square$

Theorem V.3.8. *Let $R = K[x_1, \dots, x_n]$, where K is an F-finite field. Let L be an algebraic extension of K, $S = L[x_1 \dots, x_n]$, and $R \to S$ the map induced by the extension $K \to L$. Then $J_{F_S}(f) = J_{F_R}(f)S$.*

Proof. By Proposition V.3.2, it suffices to show $J_{F_R}(f)S \subset J_{F_S}(f)$. Let K^* and L^* denote the perfect closure of K and L respectively. Let $R^* = K^*[x_1, \dots, x_n]$ and $S^* = L^*[x_1, \dots, x_n]$. Then,

$$(J_{F_R}(f)S)S^* = J_{F_R}(f)S^* = (J_{F_R}(f)R^*)S^* = J_{F_{R^*}}(f)S^* = J_{F_{S^*}}(f) = J_{F_S}(f)S^*$$

by Lemma V.3.5 and V.3.7. Therefore,

$$(J_{F_R}(f)S/J_{F_S}(f)) \otimes_S S^* = (J_{F_R}(f)S)S^*/(J_{F_S}(f))S^* = 0.$$

Hence $J_{F_R}(f)S/J_{F_S}(f) = 0$ because S^* is a faithfully flat S-algebra. □

Example V.3.9. We can use the previous theorems to compute examples of F-Jacobian ideals. Let $R = \mathbb{F}_3[x, y]$, and $f = x^2 + y^2$ and $\mathfrak{m} = (x, y)$. We have that $(\mathfrak{m}^{[p]} : f^{p-1}) = \mathfrak{m}$. Then, $J_{F_R}(f) \subset \mathfrak{m}$. Let $L = \mathbb{F}_3[i]$ the extension of $\mathbb{F}_3$ by $\sqrt{-1}$, $S = L[x, y]$ and $\phi : R \to S$ be the inclusion given by the field extension. Then, $J_{F_S}(f) = (x, y)S$ by Proposition V.1.14 because $x^2 + y^2 = (x + iy)(x - iy)$. Since ϕ is a flat extension, $J_{F_S}(f) \subset J_{F_R}(f)S$. Then, $\mathfrak{m} = R \cap J_{F_S}(f) \subset R \cap J_{F_R}(f)S \subset J_{F_R}(f)$ Hence, $J_F(f) = \mathfrak{m}$.

V.4 Relation to $R\{F\}$-modules and Cartier modules

In this section, we discuss two different settings in which F-Jacobian ideals appear. These connections arise naturally from the relation of test ideals with Cartier modules and $R\{F\}$ modules. We refer to [BB11, Bli13, BB11, ST12] for details about Cartier modules and test ideals, and to [LS01, Sha07, Smi97, Smi95b] for $R\{F\}$-modules and test ideals.

Suppose that $(R, \mathfrak{m}, K)$ is an F-finite reduced local Gorenstein ring. In this case, $\mathrm{Hom}_R(R^{1/p^e}, R)$ has a structure of an R^{1/p^e}-module given by precomposition of maps: if $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$, then $r^{1/p^e} \cdot \phi$ is defined by the rule $r^{1/p^e} \cdot \phi(x^{1/p^e}) = \phi(r^{1/p^e} x^{1/p^e})$. As an R^{1/p^e}-module, $\mathrm{Hom}_R(R^{1/p^e}, R)$ is isomorphic to R^{1/p^e} [Fed87, Lemma 1.6], and we pick a generator Φ. We say that an ideal $I \subset R$ is a *Cartier ideal*, or *an ideal compatible with* Φ, if $\Phi(I^{1/p^e}) \subset I$. We point out that this is not the standard definition in the general theory of Cartier modules, and the one given here requires that the ring is Gorenstein.

Suppose that $(R, \mathfrak{m}, K)$ is an F-finite regular local ring. Let $f \in \mathfrak{m}$ and $\overline{R} = R/fR$. Let $\Phi \in \mathrm{Hom}_R(R^{1/p}, R)$ be a generator of this module over $R^{1/p}$. We point out that $\Phi^{-1}(I) = IR^{1/p}$ for every ideal $I \subset R$ because $R^{1/p}$ is a free R-module. Then, the map, $\overline{\Phi}_f : \overline{R}^{1/p} \to \overline{R}$, defined by $\overline{\Phi}(x^{1/p}) = \phi(f^{\frac{p-1}{p}} x^{\frac{1}{p}})$ is a generator for $\overline{R}^{1/p}$ [Fed87, Lemma 1.6 and Corollaries].

Proposition V.4.1. *If $\overline{R} = R/fR$ is reduced, then $J_F(f)\overline{R}$ is a Cartier ideal of $\overline{R}$. Furthermore, $J_F(f)$ is the largest Cartier idea $I \subset \overline{R}$ such that $\overline{\Phi}^j(I^{1/p^j}) \subset \tau(\overline{R})$ for some $j \in \mathbb{N}$.*

Proof. Since $f^{p-1}J_F(f) \subset (J_F(f))^{[p]}$, we have that $f^{\frac{p-1}{p}}(J_F(f)\overline{R})^{1/p} \subset J_F(f)\overline{R}^{1/p}$. Hence, $\overline{\Phi}(J_F(f)\overline{R}^{1/p}) \subset J_F(f)R$. Therefore, $J_F(f)$ is a Cartier ideal.

For now prove the second statement, we have that $\overline{\Phi}^{-1}(J) = (J\overline{R}^{1/p} :_{\overline{R}^{1/p}} f^{\frac{p-1}{p}})$ for any ideal $J \subset \overline{R}$. If J' is the lift of J to R, then $(J\overline{R}^{1/p} :_{\overline{R}^{1/p}} f^{\frac{p-1}{p}}) = (J'R^{1/p} :_{R^{1/p}} f^{\frac{p-1}{p}})\overline{R}^{1/p}$ because $f \in J'$. Combining this with Proposition V.2.8, we obtain that $J_F(f) = \overline{\Phi}^{-j}\left(\tau(\overline{R})\right)$ for $j \gg 0$. The result follows.

□

Note that the proposition shows more, it shows that the ideals $\tau(\overline{R})$ and $J_F(f)$ are equal up to nilpotency of Cartier modules. By the duality between Cartier modules and $\overline{R}\{F\}$-modules [BB11, Proposition 5.2], we have that $\operatorname{Ann}_{H^{\dim(\overline{R})}_{\mathfrak{m}}(\overline{R})} \tau(\overline{R})$ and $\operatorname{Ann}_{H^{\dim(\overline{R})}_{\mathfrak{m}}(\overline{R})} J_F(f)$ are $\overline{R}\{F\}$-modules, the following is an immediate consequence of [BB11, Theorem 5.3].

Corollary V.4.2. *Up to nilpotency of $\overline{R}\{F\}$-modules, we have that $\operatorname{Ann}_{H^{\dim(\overline{R})}_{\mathfrak{m}}(\overline{R})} \tau(\overline{R})$ and $\operatorname{Ann}_{H^{\dim(\overline{R})}_{\mathfrak{m}}(\overline{R})} J_F(f)$ are equal.*

V.5 Examples

In this section we present several examples of F-Jacobian ideals. These computations are based on previous calculations of the F-pure threshold and the classical test ideal.

Proposition V.5.1. *Let $f \in R$ be an element with an isolated singularity at the maximal ideal $\mathfrak{m}$. If $R_{\mathfrak{m}}/fR_{\mathfrak{m}}$ is F-pure, then*

$$J_F(f) = \begin{cases} R & \text{if } R/fR \text{ is } F\text{-regular} \\ \mathfrak{m} & \text{otherwise} \end{cases}$$

Proof. Since R/fR has an isolated singularity at $\mathfrak{m}$, we have $J_F(f)R_P = R_P$ for every prime ideal different from m, hence $\mathfrak{m} \subset \sqrt{J_F(f)}$.

If $R_{\mathfrak{m}}/fR_{\mathfrak{m}}$ is F-regular, then R/fR is F-regular, and therefore $J_F(R) = R$ by Corollary V.2.11.

On the other hand, if $R_{\mathfrak{m}}/fR_{\mathfrak{m}}$ is not F-regular, then $J_F(R) \neq R$ by Corollary V.2.11. Therefore we have $\mathfrak{m} = \sqrt{J_F(f)}$. Since $R_{\mathfrak{m}}/fR_{\mathfrak{m}}$ is F-pure, we deduce that $R_{\mathfrak{m}}/J_F(f)R_{\mathfrak{m}}$ is F-pure by Lemma V.2.12. Therefore, $R_{\mathfrak{m}}/J_F(f)R_{\mathfrak{m}}$ is a reduced ring, hence $J_F(f) = \mathfrak{m}$. □

Example V.5.2. Let K be an F-finite field and let E be an elliptic curve over K. We choose a closed immersion of E in $\mathbb{P}^2_K$ and set $R = K[x, y, z]$, the homogeneous coordinate ring of $\mathbb{P}^2_K$. We take $f \in R$ as the cubic form defining E. We know that f has an isolated singularity at $\mathfrak{m} = (x, y, z)R$. If the elliptic curve is ordinary, then R/fR is F-pure [Har77, Proposition 4.21] [BS13, Theorem 4.1]. We know that R/fR is never an F-regular ring [HH94b, Discussion 7.3$b(b)$, Theorem 7.12]. Therefore $J_F(f) = \mathfrak{m}$ by Proposition V.5.1.

Example V.5.3. Let $R = K[x, y, z]$, where K is an F-finite field of characteristic $p > 3$ and let $f = x^3+y^3+z^3 \in R$. We consider the quotient morphism $\pi : R \to R/fR$ and the maximal ideal $\mathfrak{m} = (x, y, z)R$. We have $\tau_f = \mathfrak{m}$ [Smi95b, Example 6.3]. Therefore, $\mathfrak{m} \subset J_F(f)$ by Proposition V.2.8. It is known that R/fR is F-pure if and only if $p \equiv 1(\bmod 3)$. We have $(\mathfrak{m}^{[p]} : f^{p-1}) = \mathfrak{m}$ if $p \equiv 1(\bmod 3)$[REFERENCE], and $(\mathfrak{m}^{[p]} : f^{p-1}) = R$ if $p \equiv 1 \bmod 2$. Therefore, $J_F(f) = R$ if $p \equiv 2 \bmod 3$, and $J_F(f) = \mathfrak{m}$ if $p \equiv 1 \bmod 3$.

Example V.5.4. Let $R = K[x_1, \ldots, x_n]$, where K is an F–finite field of characteristic $p > 0$. Let $f = a_1x_1^{d_1} + \ldots + a_nx_n^{d_n}$ be such that $a_i \neq 0$ for all i. The ring R/fR has an isolated singularity at the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$. If $\frac{1}{d_1} + \ldots + \frac{1}{d_n} = 1$, then R/fR is F-pure for $p \gg 0$ [Her14, Theorem 3.1]. In addition, R is not F-regular [Gla96, Theorem 3.1] because f^{p-1} is congruent to $cx_1^{p^e-1} \cdots x_n^{p^e-1}$ module $\mathfrak{m}^{[p^e]}$ for some element $c \in K \setminus \{0\}$. Therefore $J_F(f) = R$ for $p \gg 0$ by Proposition V.5.1.

Remark V.5.5. Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring and $f \in R$ be such that R/fR is reduced. We can obtain $J_F(f)$ from $\tau(R/fR)$ by Proposition V.2.8. In the case $n > 3$, $f = x_1^d + \ldots + x_n^d$ and D_R is not divisible by the characteristic of K, there is an algorithm to compute the test ideal of R/fR [McD03]. Therefore, when $f = x_1^d + \ldots + x_n^d$ there is an algorithm to compute $J_F(f)$.

Example V.5.6. Let $R = K[x_1, \ldots, x_n]$, where K is a field of characteristic $p > 0$, and let $f = x_1^d+\ldots+x_n^d$. This example is based on computations done by McDermott [McD03, Examples 11, 12 and 13].

If $p = 2, n = 5$ and $d = 5$, $\tau_f = (x_i^2x_j)_{1\leq i,j\leq 5}$. Therefore,

$$(x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_1x_2x_3x_4x_5)R = (\tau_f^{[2]} : f)$$

and $R = (\tau_f^{[4]} : f^3)$, hence, $J_F(f) = R$.

If $p = 3, n = 4$ and $d = 7$, $\tau_f = (x_i^2x_j^2)_{1\leq i,j\leq 4}$. In this case $R = (\tau_f^{[3]} : f^2)$ and $J_F(f) = R$.

If $p = 7, n = 5$ and $d = 4$, we have $\tau_f = (x_1, \ldots, x_5)R$ and $R = (\tau_f^{[7]} : f^6)$, hence $J_F(f) = R$.

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