Learning and Beliefs in Non-Centralized Markets

by

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This dissertation is dedicated to Priscilla and Bart Tablante.

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Information Percolation and Multi-Unit Demand

The paper examines learning in a decentralized market where there is uncertainty about common values, and how agents' ability to learn affects efficiency. Uninformed traders can learn about the value by searching in the market and meeting other traders. I study whether this learning is sufficient to ensure that the trading prices are ex post individually rational and the outcome is efficient, as search frictions become small. Past research demonstrated an impossibility of learning when buyers and sellers have unit demand and supply for an indivisible good. The novelty is to consider a divisible good, so that traders may engage in multiple transactions before leaving the market. The main result is that when the good becomes increasingly divisible, prices become ex post individually rational and the outcome becomes efficient. Thus, the "granularity of trade" is identified as an important determinant of trading outcomes in decentralized markets.

1.1 Introduction

This paper examines learning in a decentralized market where there is uncertainty about a common state, and how agents' ability to learn affects efficiency. In a decentralized market, different agents may transact at different prices, and if an agent trades multiple times, she may buy or sell the same good for different prices over her trading horizon. An agent's inability to learn the state leads to trades at potentially different prices, and is the central focus of this paper.

In particular, the paper focuses on the role of multi-unit supply and demand for learning and for the payoff outcomes for agents. I study this in a model with several key features. An equal mass of buyers and sellers seek to trade a finite number of discrete goods, one at a time, over discrete periods. There is a persistent state of the world, which determines the common values and costs. A fraction of buyers and sellers are uncertain about the state. Every period, all buyers and sellers match in pairs. Each pair first negotiates to trade at one of three prices or not trade, and then separates. After separating, each agent has a small probability of exiting and receiving her payoff, and otherwise continues into the next period. After some agents exit, new buyers and sellers enter and join the old agents.

The exit rate acts as an explicit search friction that incentivizes agents to trade sooner at perhaps less favorable prices. Buyers and sellers keep track of their past trades, and regularly update a belief regarding the state of the world. In a steady state equilibrium, conditional on the state, the aggregate fractions of buyers and sellers taking different trading positions is constant, and each buyer and seller chooses a price at which they are willing to trade based on his or her trading history and belief regarding the state.

By examining steady state equilibria in which the exit rate disappears, I show that the number of units, or "granularity of trade," acts as a second source of friction. If only one of these frictions disappears, not all agents can learn the common state and there is inefficiency. If both frictions disappear simultaneously, in a sequence of steady state equilibria, agents learn the common state and an efficient allocation is achieved.

The seminal paper in the literature on learning in a search environment is Wolinsky (1990), which examines bilateral trade of an indivisible good. That paper provides a negative result regarding the possibility of learning: even as search frictions disappear, outcomes are not ex post individually rational; that is, some agents realize a lower payoff than what they would have received if they had not participated. The current paper follows the setup in Wolinsky (1990) and adds the possibility to trade more than one unit.[1] I start by showing a result analogous to Wolinsky (1990) in my model when agents can trade at most a fixed number of units.

Blouin and Serrano (2001) reproduce Wolinsky's result in a non-steady state environment, in which no new agents join the market after an initial period. Their negative result shows both a significant amount of ex post regret when agents trade, and that the information asymmetry causes a loss of efficiency. In both of these papers agents take a binary action. Each of the two actions correspond to one of the

[1] Two more differences are that Wolinsky (1990) assumes discounting instead of an exit rate and has a different assumption on the relation of prices and values. The changes are motivated by the fact that I consider trade of multiple units.

two possible states of the world, and the limited action sets are useful technically to simplify strategies.

One way of understanding these negative results is to view information as a contagion. If information can only be transmitted when informed agents agree to trade at an unfavorable price, information cannot diffuse to all the agents, in a market in which agents trade only once. I show that in a market in which agents trade multiple times, agents can spread the information to more agents. This possibility of learning is a fundamental difference between markets in which agents transact only once, and markets in which agents transact multiple times.

To compare the results in this paper to earlier results, I look at several benchmarks. Both earlier papers examine whether or not agents learn the true state; in this paper I call it information percolation, since only through many meetings and trades can agents learn. Wolinsky shows an impossibility result by demonstrating that many trades happen which are not ex post individually rational: if there were no uncertainty, some agents would not have agreed to these trades. Blouin and Serrano also use this strategy, and show a similar result. In this spirit I look at the fraction of trades happening at the "wrong" prices, calling it ex post regret. Finally, I look at the fraction of gains to trade realized, since as Blouin and Serrano point out, if information does not percolate, or trades take place at a wrong price, this does not imply that welfare improving trades are not happening.

Golosov et al. (2014), also revisit the settings of Wolinsky (1990) and Blouin and Serrano (2001). As in the current paper, the good is divisible. However, offers are not restricted to be on a grid. The main friction is a possibility that the market ends in every period, similar to Blouin and Serrano (2001). However, they are interested in a different question, and study a different limit. Blouin and Serrano (2001) study the expected payoffs and prices as the probability of the game ending in any given period becomes small. Golosov et al. (2014) fix the ending probability, and instead study the allocation in period T, conditional on the game not ending before period T. They show that the allocation in period T becomes approximately ex post efficient as T becomes large. For a fixed ending probability, the expected payoffs and prices will of course generally not be interim efficient or ex post individually rational.

A significant difference in the nature of learning in the current paper from the learning that takes place in Golosov et al. (2014) is that learning here is passive, while in Golosov et al. (2014) learning is an active process. In the current paper, agents learn only from the actions of their match partners, and update their beliefs using a simple process. In Golosov et al. (2014), agents must carefully select small offers at revealing prices, and then learn based on their match partner's response. This is due to the differences in bargaining.

Duffie and Manso (2007), Duffie et al. (2014), and Duffie et al. (2010) also study information percolation in a decentralized market. In these papers, information transmission is mechanical: when two agents meet, each learns all the information that the other agent had ever learned. While the goal is similar, in the current paper learning is strategic.

When agents can meet in larger groups, learning becomes easier. In an auction environment, Pesendorfer and Swinkels (1997) find a positive result when there is a large number of goods and traders. In a search model with private values, Lauermann et al. (2011) provide a positive result when buyers search over a sequence of periods, meeting many to one with sellers to bargain until trading or exiting.

1.2 Model

1.2.1 Setup

The market consists of a continuum of buyers and a continuum of sellers of equal mass, interested in trading a homogeneous good. A persistent binary state of nature determines at the same time seller costs and buyer values. There are two states of nature $W = \{H, L\}$. One of the states $w \in W$ is termed high (H). In this state, each buyer has high values v^H when entering and all sellers have high costs c^H. The other state is called low (L), for low buyer values v^L and low seller costs c^L. For simplicity, the states are equally likely, although this has no impact on the results. To ensure gains to trade in both states, $v^H > c^H$ and $v^L > c^L$. To preclude a pooling equilibrium where all agents trade at the same price regardless of the state, I assume $c^H > v^L$.

There are two types of buyers and sellers: informed and uninformed, who differ only in their initial belief about the state. Informed agents know the state while uninformed agents have a common prior that the state is high of 1/2. All buyers start with zero units, and sellers produce a unit and incur a cost for that unit upon agreeing to trade. Agents are interested in trading a finite number of times. In the high state, buyers have the high value v^H for the first k units ($k \in \mathbb{N} \setminus \{0\}$) purchased, and the lower value v^L for any subsequent goods, while sellers have costs c^H for every good. In the low state, buyers have value v^L for all units, while sellers each face low costs c^L for the first k units sold and have higher costs c^H for any subsequent units. This value and cost structure is depicted in Figure 1.1.

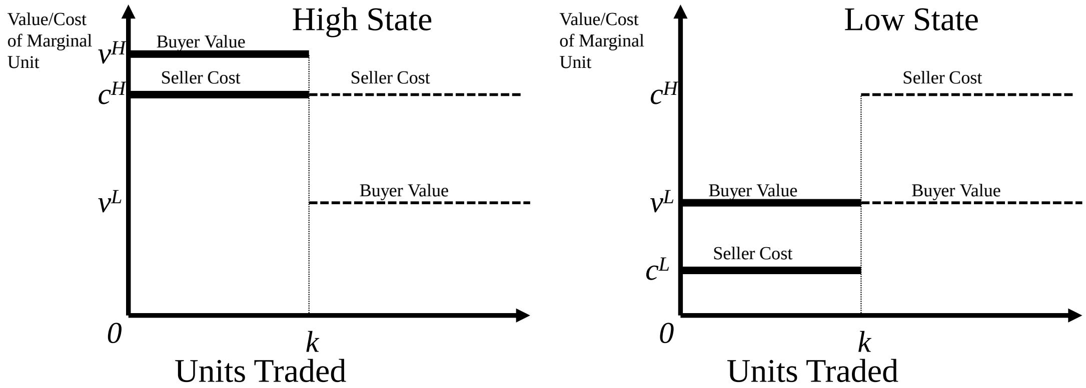

Figure 1.1: In both the high and low states, there are gains to trade for the first k units. After trading k or more units, a buyer has only low values and a seller has only high costs, so there are no gains to trade.

The purpose of having gains to trade, $v > c$, for k units, is to ensure that agents are interested in trading multiple times. While these k trades can be understood as multiple goods, they can also be interpreted as fractions of a single good: as k increases, a good becomes "less granular" and increasingly divisible. The assumption of multi-unit supply and demand is further discussed in Section 1.4.3.2.

While the value and cost structure for the first k units in Figure 1.1 is standard, the decrease [increase] in buyer values [seller costs] in the high [low] state after trading k units is not standard. The value and cost structure should be understood as follows: for a buyer and seller that have traded fewer than k units, $v > c$ in both states. After trading k units, a buyer [seller] has low values v^L [high cost c^H], regardless of the state. Then there are no gains to trade between a buyer and seller who have traded more than k times. Having the value [cost] of a buyer [seller] who has traded k or more times be v^L [c^H] has the effect that agents' behavior after trading k times is non-revealing.

Agents may trade in the market until exiting, at which point an agent realizes her payoff. Uninformed agents learn the state only when they exit, and until then update a belief based on the actions of their match partners. There are three prices at which trade may occur, p^H, p^M, and p^L. For each unit traded at a price p, a buyer receives a payoff of $(v - p)$, and similarly a seller receives $(p - c)$.

The payoffs to buyers and sellers then depend on the number of goods traded and the prices at which trade took place. Let y^H, y^M, y^L $(\in \mathbb{N})$ be the number of items a buyer has bought at the high, medium, and low prices respectively, $\mathbf{y} = (y^H, y^M, y^L)$, and $\overline{y} = y^H + y^M + y^L$. Similarly, let $\mathbf{z}$ represent the items a seller has sold at different prices. Then **y** or **z** is the set of past trades of an agent. In the high state a buyer's payoff on exiting is:

$$\min(\overline{y}, k)\, v_H + \max(\overline{y} - k, 0)\, v_L - y^H p^H - y^M p^M - y^L p^L.$$

The first component of the above value represents the high value the buyer receives for the first k units. The second term is the low value for any units after the first k. The last three terms represent the payments the buyer has made in purchasing goods, paying either a high, medium or low price for each good.

A seller's payoff on exiting in the high state is:

$$z^H p^H + z^M p^M + z^L p^L - \overline{z} c^H = z^H \left(p^H - c^H\right) + z^M \left(p^M - c^H\right) + z^L \left(p^L - c^H\right).$$

In the low state a buyer's payoff on exiting is:

$$\overline{y} v^L - y^H p^H - y^M p^M - y^L p^L = y^H \left(v^L - p^H\right) + y^M \left(v^L - p^M\right) + y^L \left(v^L - p^L\right),$$

and a seller's payoff is:

$$z^H p^H + z^M p^M + z^L p^L - \min(\overline{z}, k)\, c_L - \max(\overline{z} - k, 0)\, c_H.$$

Finally, each buyer and seller maximizes his or her expected payoff.

Trade takes place over discrete periods without beginning or end. The timing for each period is as follows: first buyers and sellers are randomly matched, then each member of each pair simultaneously makes an offer, and if offers overlap in a pair then they trade. After matching and potentially trading, a small fraction of agents exit, and finally new agents enter.

In every period, first each buyer is randomly matched to a seller in a one-to-one matching. Buyers and sellers simultaneously choose actions from a binary set $A = \{T, S\}$, choosing to be tough (T) or soft (S). A buyer playing tough requests a low price, and a seller playing tough requests a high price. A buyer playing soft offers to pay a high price, and a seller playing soft offers to sell for a low price.

In a match, if both the buyer and seller play tough, no trade is made. If at least one agent in a match plays soft, exactly one unit is traded. If the buyer chooses soft and the seller plays tough, a unit is traded at a high price p^H. If the seller plays soft and the buyer plays tough, the unit is sold at a low price p^L. Finally, if both the buyer and seller in a match play soft, a unit is traded at a medium price p^M.

I make the assumption that

$$v^H > p^H = c^H > p^M > v^L = p^L > c^L. \tag{1.1}$$

If the state is high, buyers can make k profitable trades by playing soft, and if the state is low, sellers can make k profitable trades by playing soft. Setting $p^H = c^H$ and $p^L = v^L$ corresponds to the rational expectations equilibrium prices or the Walrasian prices under complete information.

After matching and potentially trading, an agent exits and realizes her payoff with probability $1 - \delta \in (0, 1)$, and, with the remaining probability δ, remains in the market next period. This has the effect of exponential discounting, and on the aggregate level ensures a constant mass of agents. Note that agents do not exit as a result of trading, and may not opt to exit. This is a non-standard assumption and is discussed further in Section 1.4.3.3. After a fraction $(1 - \delta)$ of agents exit, an equal mass of buyers and sellers enter the market, and this mass is normalized to 1. Because exit and entry are both exogenous, the total number of buyers and sellers in the market is always the same and equal to $1 + \delta + \delta^2 + ... = \frac{1}{1-\delta}$.

A fraction $\alpha \in (0, 1)$ of the mass 1 of entering buyers and sellers are informed, and the remaining $(1 - \alpha)$ are uninformed. An agent can see all her past actions, and all the actions her past match partners have chosen. For example, a buyer who has been in the market for ten complete periods knows his past ten offers, and one action from each of the ten sellers he met. Uninformed agents have a belief that the state is high, $\theta \in \Theta$, $\Theta = [0, 1]$, which they update based on the actions of their match partners.

A game is parameterized by an 8-tuple consisting of $(\delta, k, \alpha, v^H, v^L, p^M, c^H, c^L)$. The market parameters are (δ, k, α), and describe the exit rate, number of units, and fraction of informed agents. The preference parameters are $(v^H, v^L, p^M, c^H, c^L)$ and describe values, costs, and prices.

1.2.2 Steady State Statistics

In a period of a game, there are several statistics of interest. The probability that a buyer encounters a seller playing soft and the probability that a seller encounters a buyer playing soft are of particular importance. In a period in the low state, let τ^{SL} be the fraction of sellers playing tough and τ^{BL} be the fraction of buyers playing tough. In a period in the high state, let τ^{SH} be the fraction of sellers playing tough, and τ^{BH} be the fraction of buyers playing tough. The proportions τ^{SH} [τ^{BL}] may be thought of as the proportion of sellers [buyers] playing tough "correctly," because a seller [buyer] playing soft in the high [low] state would necessarily lower her payoff. If

$\tau^{SH} = 1$, then no sellers play soft in the high state to trade at the wrong price. The proportions τ^{SL} and τ^{BH} represent agents playing tough when playing soft can yield a positive payoff. It is tempting to say that τ^{SL} is the proportion of sellers playing tough "incorrectly" in the low state, but sellers who have already traded k units in the low state should play tough. These proportions will be useful in discussing other steady state properties.

Also of interest are the characteristics of the populations of buyers and sellers. To understand these, I define two mass functions $\mathbf{B} : W \times \mathbb{N}^3 \times \Theta \times \mathbb{N} \to \mathbb{R}$ for buyers and $\mathbf{S} : W \times \mathbb{N}^3 \times \Theta \times \mathbb{N} \to \mathbb{R}$ for sellers. These functions map the state, past trades, a belief that the state is high, and an age (the number of complete periods an agent has been in the market) to population masses in a period. These populations are measured after exit and entry but before matching and trading. For example, in every game and in every period in the high state, $\mathbf{B}(H, (0, 0, 0), \frac{1}{2}, 0) = 1 - \alpha$ and $\mathbf{B}(H, (0, 0, 0), 1, 0) = \alpha$. In words, in the high state the number of buyers who have bought 0 goods, have a belief that the state is high of $\frac{1}{2}$, and have been in the market for 0 complete periods, is $1 - \alpha$.

While there may exist non-stationary equilibria, in which τ^{SL} or $\mathbf{B}$, for example, are not constant, I restrict attention to steady state (stationary) equilibria, and therefore assume $(\tau^{SL}, \tau^{SH}, \tau^{BH}, \tau^{BL}, \mathbf{B}, \mathbf{S})$ are all constant in every period of an equilibrium. This will be discussed extensively in the equilibrium definition in Section 1.2.5.

1.2.3 Strategies

An agent starts having traded zero units, and with a belief in the high state of $\frac{1}{2}$ if she is uninformed, and 0 or 1 if she is informed, depending on the state. While trading, an agent observes the actions she and her match partner make, and recalls the actions she and her past match partners have made in every period in which she has been in the market. Strategies could be described as functions mapping histories to actions, but instead, I restrict strategies to map past trades completed and current beliefs to actions.This restriction implies that a buyer, with a given belief and set of past trades, who encounters one seller playing soft and one seller playing tough, and during these two meetings plays tough twice, should behave the same after these two periods, regardless of the order of the sellers. The purpose of this restriction is to simplify analysis, allowing us to focus on the beliefs that agents hold and the trades they have made. Because I will look only at steady state equilibria (discussed in Section 1.2.5), and allow for mixed strategies, this restriction is only to simplify notation and does not affect the results. Strategies are then defined as:

$$\sigma^B : \mathbb{N}^3 \times \Theta \to [0,1] \quad \text{and} \quad \sigma^S : \mathbb{N}^3 \times \Theta \to [0,1],$$

mapping the number of past trades completed and belief in the state to a probability of playing soft. For example, if $\sigma^B((1,0,2),0.8) = 0.75$, then a buyer who has bought one unit at a high price, zero units at a medium price, and two units at a low price, and believes the state to be high with probability 0.8 will play soft with probability 0.75.

Recall the payoffs:

$$v^H > p^H = c^H > p^M > v^L = p^L > c^L.$$

These strategies and payoffs yield the following two observations:

1. Any strategy in which an informed buyer in the low state plays soft with positive probability, that is $\sigma^B(\mathbf{y},0) > 0$, or an informed seller in the high state accepts a low price, that is $\sigma^S(\mathbf{z},1) > 0$, is strictly dominated by $\sigma^B(\mathbf{y},0) = 0$ and $\sigma^S(\mathbf{z},1) = 0$.

2. Any strategy involving playing soft after having made at least k trades, for example $\sigma^B((0,0,k),\theta) > 0$, is strictly dominated by $\sigma^B((0,0,k),\theta) = 0$.

1.2.4 Value functions

I define value functions that map to an agent's expected payoff. These are used to understand interim payoffs and characterize strategies in an equilibrium. These value functions are defined ex-interim. To discuss value functions and agents' beliefs, I start by defining belief updating functions. As $\theta \in [0,1]$, let $\beta^B : A \times \Theta \to \Theta$ be the belief of a buyer who had belief θ then encountered a seller playing tough or soft. The belief updating function for sellers is analogously defined.

I first define cumulative value functions for use in understanding the game generally and payoffs specifically. I then define another set of value functions, forward looking value functions, which are used later to partially characterize equilibrium. The main difference between the two value functions is the inclusion of trades already made. The cumulative value function of a buyer $V^B : \mathbb{N}^3 \times \Theta \to \mathbb{R}$ maps the past trades and belief to a total expected payoff. It is defined just before choosing an action, and is recursively defined using the expected value functions for playing tough or soft, $\overline{V}^B : A \times \mathbb{N}^3 \times \Theta \to \mathbb{R}$ as:

$$V^B(\mathbf{y},\theta) = \max\left\{\overline{V}^B(T,\mathbf{y},\theta), \overline{V}^B(S,\mathbf{y},\theta)\right\},$$

with the expected value function for playing tough defined as:

$$
\begin{aligned}
&\overline{V}^B(T,\mathbf{y},\theta) = \qquad (1.2)\\
&\quad (1-\delta)\Big[[\theta\tau^{SH}+(1-\theta)\tau^{SL}][\beta^B(T,\theta)\left(v^H\min(\overline{y},k)+v^L\max(\overline{y}-k,0)\right)+\left(1-\beta^B(T,\theta)\right)v^L\overline{y}]\\
&\quad +[\theta(1-\tau^{SH})+(1-\theta)(1-\tau^{SL})][\beta^B(S,\theta)\left(v^H\min(\overline{y}+1,k)+v^L\max(\overline{y}+1-k,0)\right)\\
&\qquad +\left(1-\beta^B(S,\theta)\right)v^L(\overline{y}+1)-p^L]-(y^hp^H+y^Mp^M+y^Lp^L)\Big]\\
&\quad +\delta\Big[[\theta\tau^{SH}+(1-\theta)\tau^{SL}]V^B(\mathbf{y},\beta^B(T,\theta))\\
&\qquad +[\theta(1-\tau^{SH})+(1-\theta)(1-\tau^{SL})]V^B((y^h,y^M,y^L+1),\beta^B(S,\theta))\Big].
\end{aligned}
$$

Going through this five-line sum, with probability $(1-\delta)$, in the first three lines the buyer will exit later this period and realizes his payoff for trades made this period or before. In the first line, with perceived probability $\theta\tau^{SH}+(1-\theta)\tau^{SL}$ the buyer encounters a seller playing tough, and updates his beliefs accordingly. Recall θ is the buyer's belief that the state is high, τ^{SH} is the probability of encountering a seller playing tough conditional on the state being high, and τ^{SL} is the probability of encountering a seller playing tough conditional on the state being low. In the second line, with the remaining probability, the buyer encounters a seller playing soft. Encountering a seller playing tough or soft impacts the buyer's belief that the state is high through β^B, affecting his expected value for each item bought. The third line starts with buying a unit at price p^L and also includes the cost of all units bought before this period, regardless of the seller's action this period. With probability δ the buyer does not exit at the end of this period, and encounters a seller playing tough in the fourth line and soft in the fifth line, and receives a continuation payoff for each of these possibilities.

Similarly, the expected value function for a buyer playing soft is:

$$
\begin{aligned}
&\overline{V}^B(S,\mathbf{y},\theta) = \qquad (1.3)\\
&\quad (1-\delta)\Big[[\theta\tau^{SH}+(1-\theta)\tau^{SL}]\\
&\qquad \cdot[\beta^B(T,\theta)\left(v^H\min(\overline{y}+1,k)+v^L\max(\overline{y}+1-k,0)\right)+\left(1-\beta^B(T,\theta)\right)v^L(\overline{y}+1)-p^H]\\
&\quad +[\theta(1-\tau^{SH})+(1-\theta)(1-\tau^{SL})]\\
&\qquad \cdot[\beta^B(S,\theta)\left(v^H\min(\overline{y}+1,k)+v^L\max(\overline{y}+1-k,0)\right)+\left(1-\beta^B(S,\theta)\right)v^L(\overline{y}+1)-p^M]\\
&\quad -(y^hp^H+y^Mp^M+y^Lp^L)\Big]\\
&\quad +\delta\Big[[\theta\tau^{SH}+(1-\theta)\tau^{SL}]V^B(y^h+1,y^M,y^L,\beta^B(T,\theta))\\
&\qquad +[\theta(1-\tau^{SH})+(1-\theta)(1-\tau^{SL})]V^B(y^h,y^M+1,y^L,\beta^B(S,\theta))\Big].
\end{aligned}
$$

The above payoff functions will largely be used to characterize expected payoffs of agents.

1.2.4.1 Forward Looking Value Functions

Since trade only happens in one direction, trades already made cannot be altered. As another way of looking at agents' expected payoffs, I introduce forward looking value functions, which ignore the value of already made trades and focus on the benefit of future participation in the market. A buyer who has traded $\overline{y}$ times has, if $\overline{y} < k$, $\overline{y} - k$ remaining potentially profitable trades to make, or no opportunity to benefit from additional trade if $\overline{y} \geq k$. The buyer's forward looking value functions for $\overline{y} < k$ are: $v^B(k-\overline{y},\theta)$, $\overline{v}^B(T,k-\overline{y},\theta)$, and $\overline{v}^B(S,k-\overline{y},\theta)$:

$$v^B(k-\overline{y},\theta) = \max\{\overline{v}^B(T,k-\overline{y},\theta), \overline{v}^B(S,k-\overline{y},\theta)\}, \tag{1.4}$$

$$\begin{aligned}\overline{v}^B(T,k-\overline{y},\theta) = & \\ & (\theta\tau^{SH} + (1-\theta)\tau^{SL})\delta v^B(k-\overline{y},\beta^B(T,\theta)) \\ & + (\theta(1-\tau^{SH}) + (1-\theta)(1-\tau^{SL})) \\ & \quad \cdot \left(\beta^B(S,\theta)(v^H - p^L) + (1-\beta^B(S,\theta))(v^L - p^L) + \delta v^B(k-\overline{y}-1,\beta^B(S,\theta))\right),\end{aligned} \tag{1.5}$$

and

$$\begin{aligned}\overline{v}^B(S,k-\overline{y},\theta) = & \\ & (\theta\tau^{SH} + (1-\theta)\tau^{SL}) \\ & \quad \cdot \left(\beta^B(T,\theta)(v^H - p^H) + (1-\beta^B(T,\theta))(v^L - p^H) + \delta v^B(k-\overline{y}-1,\beta^B(T,\theta))\right) \\ & (\theta(1-\tau^{SH}) + (1-\theta)(1-\tau^{SL})) \\ & \quad \cdot \left(\beta^B(S,\theta)(v^H - p^M) + (1-\beta^B(S,\theta))(v^L - p^M) + \delta v^B(k-\overline{y}-1,\beta^B(S,\theta))\right).\end{aligned} \tag{1.6}$$

For $\overline{y} \geq k$, and for any θ, additional trades at a low price do not change utility, while trades at a medium or high price decrease utility, and so $v^B(k-\overline{y},\theta) = \overline{v}^B(T,k-\overline{y},\theta) = 0 > \overline{v}^B(S,k-\overline{y},\theta)$.

The difference between the cumulative value functions V and the forward looking value functions v is the value of trades already made. Because beliefs, and therefore the value for trades already made, follow a martingale, given $\mathbf{y}$,

$$V^B(\mathbf{y},\theta) = \theta\left(v^H \min(\overline{y},k) + v^L \max(\overline{y}-k,0)\right) + (1-\theta)\overline{y}v^L - (y^h p^H + y^M p^M + y^L p^L) + v^B(j,\theta).$$

1.2.5 Equilibrium Definition

The purpose of this research is to understand information percolation and efficiency in steady-state equilibria. I restrict attention to role-symmetric equilibria, so that all buyers and all sellers follow the same mixed strategy. The restriction to symmetric strategies is without loss of generality due to the continuum of agents and the allowance of mixed strategies.

A steady state perfect Bayesian equilibrium, hereafter equilibrium, consists of four main parts: populations, aggregate behaviors, strategies, and belief functions. A steady state is a 10-tuple

$$(\mathbf{B}, \mathbf{S}, \tau^{SL}, \tau^{SH}, \tau^{BH}, \tau^{BL}, \sigma^S, \sigma^B, \beta^S, \beta^B)$$

that satisfies two conditions: consistency of beliefs and stationarity of populations, and an equilibrium is a steady state that additionally satisfies optimality.

1. CONSISTENCY OF BELIEFS: Beliefs follow Bayes rule given $(\tau^{SL}, \tau^{SH}, \tau^{BH}, \tau^{BL})$, that is:

$$\beta^B(T, \theta) = \frac{\theta\tau^{SH}}{\theta\tau^{SH} + (1-\theta)\tau^{SL}}, \text{ and } \beta^B(S, \theta) = \frac{\theta(1-\tau^{SH})}{\theta(1-\tau^{SH}) + (1-\theta)(1-\tau^{SL})}. \quad (1.7)$$

Note that belief updates are invertible. Off equilibrium beliefs are assigned passively. Note that the only time an agent can identify an out of equilibrium action is if $\tau^{SL} = \tau^{SH} = 1$ or if $\tau^{BH} = \tau^{BL} = 1$, and a buyer or seller never expects to see his or her match partners playing soft. In this case I define $\beta^B(S, \frac{1}{2}) = \frac{1}{2}$, or $\beta^S(S, \frac{1}{2}) = \frac{1}{2}$.

2. STATIONARITY: the populations of buyers $\mathbf{B}$ and sellers $\mathbf{S}$ are constant, and as a result of this, the aggregate behavior proportions $\tau^{SL}, \tau^{BH}, \tau^{SH}, \tau^{BL}$ are also constant. For example, for a given belief θ, set of past trades $\mathbf{y}$, and time in the market n, with $n, y^H, y^M, y^L > 0$, and $\beta^B(T, \theta') = \theta$, and $\beta^B(S, \theta'') = \theta$:

$$\mathbf{B}(H, \mathbf{y}, \theta, n) = \quad (1.8)$$

$$\delta\Big(\tau^{SH}\Big(\sigma^B((y^H-1,y^M,y^L),\theta')\mathbf{B}(H,(y^H-1,y^M,y^L),\theta',n-1)$$
$$+(1-\sigma^B(\mathbf{y},\theta'))\mathbf{B}(H,(y^H,y^M,y^L),\theta',n-1)\Big)$$
$$+(1-\tau^{SH})\Big(\sigma^B((y^H,y^M-1,y^L),\theta'')\mathbf{B}(H,(y^H,y^M-1,y^L),\theta'',n-1)$$
$$+(1-\sigma^B((y^H,y^M,y^L-1),\theta''))\mathbf{B}(H,(y^H,y^M,y^L-1),\theta'',n-1)\Big)\Big).$$

This equation states that the mass of buyers in the high state with a given belief θ and with a set of past trades $\mathbf{y}$ is equal to the mass of buyers that will survive next period (δ) playing soft (σ^B) and playing hard ($1-\sigma^B$) and encountering a seller playing tough (τ^{SH}) or soft ($1-\tau^{SH}$) to reach the set of completed trades $\mathbf{y}$ and belief θ.

As an aside, from the exogenous exit rate and (1.8), with some abuse of notation I derive the total population masses:

$$\sum_\theta\sum_{\mathbf{z}}\sum_n \mathbf{S}(H,\mathbf{z},\theta,n)=\sum_\theta\sum_{\mathbf{y}}\sum_n \mathbf{B}(H,\mathbf{y},\theta,n)=\frac{1}{1-\delta}. \tag{1.9}$$

From (1.9) and the exit rate, the mass of exiting uninformed buyers or sellers is $(1-\alpha)$.

The second part of stationarity holds aggregate behavior constant. In order for the fractions of agents playing tough to be constant, the fractions must follow from the masses of agents and strategies, for example:

$$\tau^{BH}=\left(\sum_\theta\sum_{\mathbf{y}}\sum_n \mathbf{B}(H,\mathbf{y},\theta,n)\left(1-\sigma^B(\mathbf{y},\theta)\right)\right)/\left(\frac{1}{1-\delta}\right). \tag{1.10}$$

This is the total mass of buyers weighted by their probability of playing tough, divided by the total number of buyers, all in the high state.

3. OPTIMALITY: Strategies σ^B and σ^S are optimal given $(\tau^{SL},\tau^{SH},\tau^{BH},\tau^{BL})$, that is if
$$\overline{V}^B(T,\mathbf{y},\theta)>\overline{V}^B(S,\mathbf{y},\theta)\text{ then }\sigma^B(\mathbf{y},\theta)=0,$$
and if
$$\overline{V}^B(T,\mathbf{y},\theta)<\overline{V}^B(S,\mathbf{y},\theta)\text{ then }\sigma^B(\mathbf{y},\theta)=1.$$

The analog for seller strategies must also hold. Note that because θ can take on countably many values in a steady state, the optimality property requires agents to maximize their payoffs for beliefs that are unattainable.

1.2.5.1 Properties of Value Functions in a Steady State

The following are properties of value functions in any steady state. They are stated for buyers, but the analogous properties for sellers also hold. The properties are stated without proof, and are easy to verify.

(V1) For any $j > 0$, $v^B(j,1) > v^B(j-1,1)$, and for $j' > 1$, $v^B(j',1) - v^B(j'-1,1) < v^B(j'-1,1) - v^B(j'-2,1)$.

(V2) For any $\theta > 0$, and $j > 0$, if $\tau^{SL} \neq \tau^{SH}$, then $v^B(j,\theta) > v^B(j-1,\theta)$.

(V3) For any $\theta' > \theta > 0$, and $j > 0$, if $\tau^{SL} \neq \tau^{SH}$, then $v^B(j,\theta') > v^B(j,\theta)$.

(V4) For any $\theta > 0$, $j > 1$, and if $\tau^{SL} \neq \tau^{SH}$, then $v^B(j,\theta) - v^B(j-1,\theta) < v^B(j-1,\theta) - v^B(j-2,\theta)$.

(V5) For any $\theta' > \theta > 0$, and $j > 1$, if $\tau^{SL} \neq \tau^{SH}$, then $v^B(j,\theta') - v^B(j-1,\theta') > v^B(j,\theta) - v^B(j-1,\theta)$.

The idea of the above properties is as follows. If $\tau^{SL} = \tau^{SH}$ then the actions of sellers do not affect an uninformed buyer's belief, requiring $\tau^{SL} \neq \tau^{SH}$ prevents this situation. The first and second properties state that the total value of future trade is increasing in the number of goods left to trade. Due to the discount factor and the fact that beliefs of informed agents will not change, (V1) includes concavity. The strict inequality is because either a buyer has a chance to encounter a seller playing soft in the high state, or no sellers play soft in the high state, so encountering a large number of sellers playing tough pushes a buyer's belief to 1, at which point playing soft increases his payoff. The third property (V3) states that future payoffs are increasing with beliefs.

The fourth and fifth properties look at the value of the marginal good, and both use the non-zero discounting. The fourth property (V4) states that the marginal value of a good decreases with j, while the fifth property (V5) states that the marginal value of a good is increases with the belief.

1.2.5.2 Key Properties of an Equilibrium

The following definitions formalize efficiency and ex post regret in an equilibrium.

Definition 1 (Ex post regret)**.** *The fraction of potential trades that take place at prices that lower an agent's utility in state w of an equilibrium is x^w. In the high state, this is the fraction of possible trades that take place at p^M or p^L, so:*

$$x^H = \left(\sum_{\theta} \sum_{\mathbf{z}} \sum_{n} \mathbf{S}(H, \mathbf{z}, \theta, n) \frac{(z^L + z^M)}{k} \right) \Big/ \left(\frac{1}{1-\delta} \right). \tag{1.11}$$

Combining over states, then $x = \max(x^H, x^L)$ is **ex post regret**.

In (1.11), x^H is the mass of trades at the "wrong" price in the high state. It is each mass of sellers multiplied by the fraction that trade at the wrong price (playing soft). The denominator is one measure of the number of possible trades. In the spirit of earlier work, this is the measure of trades made that are not ex post individually rational.

Definition 2 (Gains to trade realized)**.** *The proportion of gains to trade realized in state w is d^w. In the high state this is:*

$$d^H = \left(\sum_{\theta} \sum_{\mathbf{y}} \sum_{n} \mathbf{B}(H, \mathbf{y}, \theta, n) \frac{\min(\overline{y}, k)}{k} \right) \Big/ \left(\frac{1}{1-\delta} \right). \tag{1.12}$$

To combine over states, $d = \max(d^H, d^L)$ is the fraction of **gains to trade realized**.

In (1.12), in the high state, buyers who exit having traded at least k times have realized all the gains from trade, while buyers who exit having traded fewer than k times have unrealized gains to trade (and decrease d^H). The measure of gains to trade used is the fraction of trades that 'should' be made that are made. Furthermore, if in the high state, almost all exiting buyers (almost all buyers in the market) have traded approximately k units, then d^H is close to 1.[2]

1.2.5.3 Properties in the Limit

The above two definitions serve as benchmarks to understand how agents gain and lose utility from participation in the market. To understand how agents select strategies, we must look at how beliefs evolve, and so define another belief function. Recall that β^B maps an action and a prior belief to an updated belief. Let $\tilde{\beta}^B : W \times \mathbb{N}$ be a function mapping the state and number of periods in the market to a random

[2]By measuring agents at the beginning of a period instead of after agents have traded, we include new agents who have not had a chance to trade. As δ grows large, the impact of those agents disappears.

variable for beliefs of an uninformed buyer in an equilibrium. For example, $\tilde{\beta}^B(H,1)$ is a random variable representing an uninformed buyer's belief after meeting one seller in the high state, and $Pr\{\tilde{\beta}^B(H,1) = \beta^B(T,\frac{1}{2})\} = \tau^{SH}$, and $Pr\{\tilde{\beta}^B(H,1) = \beta^B(S,\frac{1}{2})\} = 1-\tau^{SH}$.

For any game parameterized by $(\delta, k, \alpha, v^H, v^L, p^M, c^H, c^L)$ and a corresponding equilibrium $(\mathbf{B}, \mathbf{S}, \tau^{SL}, \tau^{SH}, \tau^{BH}, \tau^{BL}, \sigma^S, \sigma^B, \beta^S, \beta^B)$, if ex post regret in the high state is zero ($x^H = 0$), then in the high state, each seller makes no trades at low or medium prices ($z^L = 0$ and $z^M = 0$), so no seller can ever play soft if she believes the high state is possible (at any belief $\theta > 0$). It follows that if $x^H = 0$, then uninformed sellers will not play soft, so not all the gains to trade can be realized in the low state ($d^L < 1$). As a result, it is impossible in any equilibrium of any game for $x = 0$ and $d = 1$. For this reason I look at sequences of games and corresponding equilibria, so that in the limit information can percolate and the outcome can be efficient. The next two definitions provide benchmarks in the limit.

Definition 3 (Information percolation)**. Information percolates** *in a sequence of equilibria* $\{(\mathbf{B}_i, \mathbf{S}_i, \tau_i^{SL}, \tau_i^{SH}, \tau_i^{BH}, \tau_i^{BL}, \sigma_i^S, \sigma_i^B, \beta_i^S, \beta_i^B)\}_{i=1}^{\infty}$ *of a sequence of games parameterized by*
$\{(\delta_i, k_i, \alpha, v^H, v^L, p^M, c^H, c^L)\}_{i=1}^{\infty}$ *if there exists a corresponding sequence* $\{n_i\}$ *such that:*

1. $\lim_{i\to\infty} (\delta_i)^{n_i} = 1,$

2. $\operatorname{plim}_{i\to\infty} \tilde{\beta}_i^B(L, n_i) = 0,$

3. $\operatorname{plim}_{i\to\infty} \tilde{\beta}_i^B(H, n_i) = 1,$

4. $\operatorname{plim}_{i\to\infty} \tilde{\beta}_i^S(L, n_i) = 0,$ *and*

5. $\operatorname{plim}_{i\to\infty} \tilde{\beta}_i^S(H, n_i) = 1.$

In other words, if information percolates in a sequence of equilibria, then far along that sequence almost all agents learn the state before exiting.

Definition 4 (Efficiency)**.** *A sequence of equilibria*
$(\mathbf{B}_i, \mathbf{S}_i, \tau_i^{SL}, \tau_i^{SH}, \tau_i^{BH}, \tau_i^{BL}, \sigma_i^S, \sigma_i^B, \beta_i^S, \beta_i^B)\}_{i=1}^{\infty}$ *corresponding to a sequence of games parameterized by* $\{(\delta_i, k_i, \alpha, v^H, v^L, p^M, c^H, c^L)\}_{i=1}^{\infty}$ *is* **efficient** *if* $\lim_{i\to\infty} d_i = 1$ *and* $\lim_{i\to\infty} x_i = 0$.

The efficiency concept used is closest to unconstrained Pareto efficiency in a period. The necessity of having all gains to trade realized in an efficient outcome is straightforward. The reason that efficiency requires ex post regret to approach zero is that if it were greater than zero, for example if a significant fraction of buyers played soft in the low state, then some sellers in the low state who had already traded k units would sell a good at a high price, hurting the buyer but having no impact on the seller.

With these two definitions I can analyze games, equilibria, and sequences of these, characterizing behavior, information percolation, and payoffs for agents.

1.3 Partial Characterization of Steady States and Equilibria

In this section I find properties of steady states, in order to work towards finding conditions under which information percolates and equilibria are efficient. I start with two single crossing properties of optimal strategies in a steady state, then I establish the equivalence of information percolation and efficiency, and continue with a few lemmas to characterize strategies in a steady state. I then show equilibrium existence.

1.3.1 Characterization of Steady States

The first two lemmas will characterize steady states. These characterizations will be useful to show existence of an equilibrium, and later efficiency and information percolation results.

Lemma 1 (Strategies involve threshold beliefs)**.** *In any steady state, consider any* $\mathbf{y}, \mathbf{z}, \theta$*. If* σ^B *is optimal, and for some* $\theta, \mathbf{y}$*,* $\sigma^B(\mathbf{y}, \theta) > 0$ *then for any* $\theta' > \theta$*,* $\sigma^B(\mathbf{y}, \theta') = 1$*. For sellers, if* σ^S *is optimal, and for some* $\theta, \mathbf{z}$*,* $\sigma^S(\mathbf{z}, \theta) > 0$ *then for any* $\theta' < \theta$*,* $\sigma^S(\mathbf{z}, \theta') = 1$*.*

This lemma states that a buyer who would play soft with a given belief and set of past trades would play soft if he had the same past trades and a higher belief. This is proven by showing that for any j with $j \leq k$ and $\theta \in \Theta$, $\overline{v}^B(S, j, \theta) - \overline{v}^B(T, j, \theta)$ is strictly increasing in θ. Note that I make use of the optimality condition, which is slightly stronger than necessary for a perfect Bayesian equilibrium, to simplify notation for beliefs that could never be reached. The proof is relegated to the Appendix.

This next lemma states that in a steady state, for a strategy to be optimal, if an agent with some belief θ plays soft after having traded $\overline{y}$ ($\overline{z}$) times, she will play soft if she has the same belief but has traded fewer than $\overline{y}$ ($\overline{z}$) times. This lemma will be used to rule out certain steady states and equilibria, and bound the number of times an agent plays soft in an equilibrium. The proof is found in the Appendix.

Lemma 2 (Agents play soft first)**.** *Consider any steady state in which* $\tau^{SL} \neq \tau^{SH}$*. If* σ^B *is optimal, and for some* $\mathbf{y}$ *and* θ*,* $\sigma^B(\mathbf{y}, \theta) > 0$*, then for all* $\mathbf{y}'$ *such that* $\overline{y}' < \overline{y}$*,* $\sigma^B(\mathbf{y}', \theta) = 1$*. Similarly, consider any steady state in which* $\tau^{BH} \neq \tau^{BL}$*. If* σ^S *is optimal, if for some some* θ *and* $\mathbf{z}$*, then* $\sigma^S(\mathbf{z}, \theta) > 0$*, then for all* $\mathbf{z}'$ *such that* $\overline{z}' < \overline{z}$*,* $\sigma^S(\mathbf{z}', \theta) = 1$*.*

The parallel properties also hold if $\tau^{SL} = \tau^{SH}$ *or* $\tau^{BH} = \tau^{BL}$ *and* $\theta \in \{0, 1\}$*), in the case in which an agent is informed.*

It is worth noting that the property does apply if $\tau^{SH} = \tau^{SL} = 1$. In this case, if the parameters are such that $\frac{1}{2}(v^H - p^H) + \frac{1}{2}(v^L - p^H) = 0$, an uninformed buyer will never learn, and is always indifferent between playing soft and playing tough.

Theorem 1 (Equilibrium existence)**.** *For any game parameterized by* $(\delta, k, \alpha, v^H, v^L, p^M, c^H, c^L)$ *there is an equilibrium* $(\mathbf{B}, \mathbf{S}, \tau^{SL}, \tau^{SH}, \tau^{BH}, \tau^{BL}, \sigma^S, \sigma^B, \beta^S, \beta^B)$ *of that game.*

To prove the existence of an equilibrium, I construct an isomorphic game and construct a fixed point of that game using strategies and steady state statistics. I then show that the fixed point exactly corresponds to an equilibrium of the original game. The proof is found in the Appendix.

Lemma 3 (Equivalence of efficiency and information percolation)**.** *Consider any sequence of games* $\{(\delta_i, k_i, \alpha, v^H, v^L, p^M, c^H, c^L)\}_{i=1}^{\infty}$ *and corresponding equilibria indexed* i*,*
with $\lim_{i \to \infty} (\delta_i)^{k_i} = 1$*. In this sequence of equilibria, information percolates if and only if the sequence is efficient.*

The idea of this lemma is as follows. If an uninformed agent can learn the state almost surely, then in the state where she cannot earn a positive payoff, she will not attain a significantly negative payoff by trading badly. If no agent will make a significant number of bad trades, then no agent has an incentive to wait for trades at very positive terms that would hurt her match partner. Consequently each agent will trade at the "correct" prices and realize positive payoffs. Conversely, if an agent has traded correctly, then she must have learned the state. The proof follows exactly this idea, and is found in the Appendix.

1.3.2 Characterization of Equilibrium Strategies

The next couple of lemmas will characterize strategies in an equilibrium. These characterizations will be useful to show efficiency and information percolation results.

The first of these next lemmas states that uninformed buyers play soft more in the high state, and uninformed sellers play soft more in the low state. This is important for bounding posterior beliefs, and combined with Lemma 1, causes agents who have met more tough match partners to be more likely to play soft than agents who have met more soft match partners.

Lemma 4 (Uninformed do not trade 'badly')**.** *In any equilibrium,*

$$\left(\sum_{0<\theta<1}\sum_{\mathbf{y}}\sum_{n}\mathbf{B}(H,\mathbf{y},\theta,n)\sigma^B(\mathbf{y},\theta)\right) \geq \left(\sum_{0<\theta<1}\sum_{\mathbf{y}}\sum_{n}\mathbf{B}(L,\mathbf{y},\theta,n)\sigma^B(\mathbf{y},\theta)\right), \quad (1.13)$$

and

$$\left(\sum_{0<\theta<1}\sum_{\mathbf{y}}\sum_{n}\mathbf{S}(H,\mathbf{y},\theta,n)\sigma^S(\mathbf{y},\theta)\right) \leq \left(\sum_{0<\theta<1}\sum_{\mathbf{y}}\sum_{n}\mathbf{S}(L,\mathbf{y},\theta,n)\sigma^S(\mathbf{y},\theta)\right). \quad (1.14)$$

The proof introduces and looks at the histories of uninformed buyers. An uninformed buyer's history is the sequence of actions taken by the buyer and his match partner. The proof is split into three cases: encountering a seller playing tough can either raise, lower, or leave unchanged the belief of a buyer. The proof roughly shows that if a buyer's belief in the high state is greater, he is more likely to play soft.

Using Lemma 4 and strictly dominated strategies, I bound τ in the next lemma. Recall that τ^{SH} and τ^{BL} are the fraction of sellers and buyers playing tough in the high and low states respectively (playing tough correctly), and τ^{SL} and τ^{BH} are the fraction of sellers and buyers playing tough in the low and high states respectively.

Lemma 5 (Bounds on playing tough)**.** *In any equilibrium, the following inequalities hold:*

1. $\tau^{BL} \geq \tau^{BH} \geq \delta^k$.

2. $\tau^{SH} \geq \tau^{SL} \geq \delta^k$.

3. $\tau^{BL} \geq 1-(1-\alpha)(1-\delta^k)$.

4. $\tau^{SH} \geq 1-(1-\alpha)(1-\delta^k)$.

Proof. From Lemma 4 and the observation that informed buyers never play soft in the low state, and informed sellers never play soft in the high state, $\tau^{BL} \geq \tau^{BH}$ and $\tau^{SH} \geq \tau^{SL}$. The second inequalities in (1) and (2) come from the observation that an agent plays soft no more than k times, so the total mass of agents playing soft is bounded above by $(1 + \delta + ... + \delta^k)$, and the fraction of all agents playing soft is bounded above by $(1 - \delta)$.

To show (3), recall that in the low state only uninformed buyers would play soft, so the total mass of buyers playing soft in the low state is bounded above by $(1 - \alpha)(1 + \delta + ... + \delta^k)$. Then, the fraction of buyers playing tough in the low state is $\tau^{BL} \geq 1 - (1 - \alpha)(1 - \delta^k)$. (4) is proven analogously. □

1.4 Results

1.4.1 Negative Result

In this section, I show that for a fixed k, uninformed agents cannot learn the state with certainty before trading, and as a result of this, the market cannot be efficient.

Theorem 2 (No percolation)**.** *For any sequence of games indexed i and parameterized by $\{(\delta_i, k, \alpha, v^H, v^L, p^M, c^H, c^L)\}_{i=1}^{\infty}$, with $\delta_i \to 1$, and corresponding equilibria for which the limits of d_i and x_i exist, at least one of the following must be true:*

1. $\lim_{i \to \infty} d_i < 1$.
2. $\lim_{i \to \infty} x_i > 0$.

This result states that even as the main friction disappears, either not all the gains from trade will be realized ($d < 1$), or a significant fraction of agents will trade at the wrong prices and have ex post regret ($x > 0$). This result is analogous to the main results of Wolinsky (1990) and Blouin and Serrano (2001). The proof is similar to the proofs in those works, but made simpler here by the additional possibility of meeting agents who have already traded k times who will always play tough.

The main intuition of the proof regards the length of time an uninformed agent waits before playing soft for the first time. I show that either the length of time is too large, and the agent risks exiting before trading, even as the main friction disappears, or the length of time is too short, and the agent will almost certainly receive the same signal in both states before trading for the first time, and so will almost certainly trade at the wrong price in one of the states.

Proof of Theorem 2.

Step 0: Notation.

Let $\overline{\beta}^B : \mathbb{N} \times \mathbb{N} \to \Theta$ be a function in an equilibrium mapping from the total number of sellers a buyer has met, and the number of times an uninformed buyer has encountered a seller playing soft to a current belief. This is recursively defined: $\overline{\beta}^B(0,0) = \frac{1}{2}$, and for $n, u > 0$, $\overline{\beta}^B(n,u) = \beta^B(T, \overline{\beta}^B(n-1,u)) = \beta^B(S, \overline{\beta}^B(n-1,u-1))$.

I then define $m-1$ as the minimum number of periods an uninformed buyer waits before playing soft with positive probability in an equilibrium. For example, $m = 1$ if an uninformed buyer waits no periods and plays soft in the first period with positive probability, and $m = \infty$ if no uninformed buyer ever plays soft. Then m is defined:

$$m = \inf_{\hat{m},u,\mathbf{y}} \{\hat{m} | \sigma^B(\mathbf{y}, \overline{\beta}^B(\hat{m},u)) > 0\}.$$

The proof proceeds in two cases. First, the case in which there is a subsequence of equilibria with $\tau_i^{SL} = \tau_i^{SH}$, and second, the case in which, for i sufficiently high, $\tau_i^{SL} < \tau_i^{SH}$. From Lemma 5, in each equilibrium, $\tau_i^{SL} \leq \tau_i^{SH}$ so these cases are exhaustive. In each of these cases, I will show that either $\lim_{i\to\infty} d_i < 1$ or $\lim_{i\to\infty} x_i > 0$.

Step 1: If there is a subsequence of equilibria $\{i'\}$ with $\tau_{i'}^{SH} = \tau_{i'}^{SL}$, then $\lim_{i'\to\infty} d_{i'} < 1$ or $\lim_{i'\to\infty} x_{i'} > 0$.

If in an equilibrium i', $\tau_{i'}^{SH} = \tau_{i'}^{SL}$, then uninformed buyers cannot learn, so for any i' in the subsequence, $\beta_{i'}^B(T, \frac{1}{2}) = \beta_{i'}^B(S, \frac{1}{2}) = \frac{1}{2}$. Then from [1.8], for all $\mathbf{y}$ and n:

$$\mathbf{B}_{i'}(L, \mathbf{y}, \frac{1}{2}, n) = \mathbf{B}_{i'}(H, \mathbf{y}, \frac{1}{2}, n).$$

If $\lim_{i'\to\infty} d_{i'}^H = 1$ then almost all uninformed buyers play soft almost $k_{i'}$ times in both states before exiting, and $\lim_{i'\to\infty} x_{i'}^L > 0$.

Step 2: If, for i sufficiently high $\tau_i^{SL} < \tau_i^{SH}$, then $\lim_{i'\to\infty} d_{i'} < 1$ or $\lim_{i'\to\infty} x_{i'} > 0$.

I now restrict attention to i sufficiently high. In this case, for $\theta \in (0,1)$, from [1.7], $\beta_i^B(T,\theta) > \theta$. Consider the sequence of $\{m_i\}$. When m_i is finite, a buyer who has played tough and only encountered sellers playing tough for m_i periods, must strictly prefer playing soft in the next period, from Lemma 1 and Lemma 2. Then there are two subcases.

Step 2, subcase 1: There is a subsequence of equilibria indexed i' where $m_{i'}$ is infinite or $\lim_{i'\to\infty} (\delta_{i'})^{m_i'} < 1$.

Along the subsequence $\{i'\}$, for i' sufficiently large, from the definition of m, at least $(1 - \lim_{i' \to \infty} (\delta_{i'})^{m'_i})/2 > 0$ of all uninformed buyers will exit before ever playing soft in both states. If $d_{i'} \to 1$, then for i' sufficiently large, $d_{i'} > 1 - (1 - \lim_{i' \to \infty} (\delta_{i'})^{m'_i})/4$, and in the high state, at least half of these buyers who do not play soft in the high state must encounter at least one seller playing soft (to realize their gains from trade), so at least $(1 - \lim_{i' \to \infty} (\delta_{i'})^{m'_i})/4 > 0$ of all sellers will play soft at least once, and $\lim_{i' \to \infty} x_{i'} > 0$.

Step 2, subcase 2: $\lim_{i \to \infty} (\delta_i)^{m_i} = 1$.

In this subcase, almost all uninformed buyers survive m_i periods after entering, and so, for some histories, play soft. Consider a buyer's meetings and actions over these m_i periods. If a buyer only plays tough and only encounters sellers playing tough, from Lemma 2 and Lemma 1, he will play soft in his m^{th} period. From Lemma 5, $\tau_i^{SL} \geq (\delta_i)^k$, so the proportion of buyers who encounter only sellers playing tough for m_i periods in the high state is:

$$(\tau_i^{SL})^{m_i} \geq ((\delta_i)^k)^{m_i} = ((\delta_i)^{m_i})^k. \tag{1.15}$$

If $(\delta_i)^{m_i} \to 1$ then $(f_i^S)^{m_i} \to 1$, and in an equilibrium in the low state at least $(\delta_i)^{m_i}(\tau_i^{SL})^{m_i} \to 1$ of uninformed buyers will survive m_i periods encountering sellers only playing tough, and play soft at least once. If all uninformed buyers play soft at least once in the low state, $\lim_{i \to \infty} x_i \geq (1 - \alpha)/k$. Thus in both cases, $\tau_i^{SL} < \tau_i^{SH}$ or $\tau_i^{SL} = \tau_i^{SH}$, either $\lim_{i \to \infty} x_i > 0$ or $\lim_{i \to \infty} d_i < 1$.

□

1.4.1.1 Relationship to Prior Work

This impossibility result is related to the results of Wolinsky (1990) and Blouin and Serrano (2001), both of which show that some trades will happen at the wrong price. There are three significant differences between those earlier models and the model presented here. The first two differences are in exiting and discounting. In those models, agents only exit through trade, and discount future payoffs explicitly. Here exit is exogenous (to prevent buyers and sellers from becoming unbalanced), and is the source of discounting. Exogenous exit also introduces an additional friction not seen in the earlier models; as agents leave less often it becomes more difficult to meet an agent who has not traded k times. This additional friction makes the negative proof in my model easier, as seen in [1.15], where buyers have difficulty meeting sellers with units left to trade during their lifetime. As a result, this proof would not work in earlier models, although the intuition is the same.

On the other hand, an assumption on the values, costs, and likelihoods of states called *no fear* by Blouin and Serrano is no longer required. This assumption is present in both of those earlier models and guarantees that uninformed buyers or sellers would be willing to play soft without learning. Finally, by allowing for multi-unit trade, this result is in one aspect more general than earlier results. A discussion comparing the assumptions is found in Section 1.4.3.1.

1.4.2 Positive Result

If instead of holding k fixed, buyers and sellers make a 'large' number of trades, information percolation and efficiency can be achieved. In this section, I add a restriction on payoffs, show how informed agents playing soft provides a signal from which the uninformed can learn, show that this signal is informative enough, and finally show that information percolation and efficiency will be achieved. Lastly I remove the assumption on payoffs and show that information percolation is still possible, although not guaranteed.

The following assumption on the shape of the payoffs, and implicitly the prices, limits incentives to playing tough and hoping to meet an uninformed agent incorrectly playing soft, when there is a guaranteed payoff to playing soft.

Assumption 1 (Restriction on payoffs)**.**

$$v^H - p^L < (v^H - p^H)e^{1/(1-\alpha)} \text{ and } p^H - c^L < (p^L - c^L)e^{1/(1-\alpha)}. \tag{1.16}$$

Recall that $p^H = c^H$ and $v^L = p^L$. The necessity of this assumption is discussed in Section 1.4.3.1.

1.4.2.1 Informed Agents Reveal Information:

Lemma 6 (Informed do not always misrepresent the state)**.** *Let* $(\alpha, v^H, v^L, c^H, c^L)$ *satisfy Assumption 1. For any sequence of games parameterized by* $\{(\delta_i, k_i, \alpha, v^H, v^L, p^M, c^H, c^L)\}_{i=1}^{\infty}$ *with* $(\delta_i)^{k_i} \to 1$ *and* $k_i \to \infty$*, and corresponding equilibria indexed* i*, there exists a proportion* $s \in (0, 1)$ *such that for* i *sufficiently large, for all* $\mathbf{y_i}$ *such that* $\overline{y}_i < sk_i$ $\sigma_i^B(\mathbf{y}_i, 1) = 1$ *and for any* $\mathbf{z_i}$ *such that* $\overline{z}_i < sk_i$*,* $\sigma_i^S(\mathbf{z}_i, 0) = 1$

The purpose of this lemma is to lower bound the informativeness of match partners' actions for an uninformed agent trying to learn the state. Because a constant fraction α of the agents in an economy is informed, and a significant fraction of agents in any equilibrium and state will play tough, this fraction of informed playing soft in one state and not the other can be sufficient for learning. Although the fraction of agents playing soft in a period approaches zero as frictions disappear, a later lemma will show that the informativeness of this fraction is sufficient for positive results.

The proof strategy is to bound the payoff of an informed seller in the low state to playing tough forever. If a seller has traded fewer than k times and knows the state is low, she can guarantee a positive payoff by playing soft, or can hope to wait for an uninformed buyer playing soft. As δ^k grows, the likelihood a seller can make a significant mass of trades at a high price in the low state becomes upper bounded, because she can only make these trades with uninformed buyers who have traded fewer than k times. This upper bound is used to show that a seller will make a positive fraction of her trades in the low state at a low price.

Proof. I will show that informed sellers in the low state play soft for at least $\lfloor sk_i \rfloor$ periods.

Step 1: Notation.

Let $1-s' = (1-\alpha)\ln[\frac{p^H-c^L}{p^L-c^L}]$, so $\frac{p^H-c^L}{p^L-c^L} = [e^{1/(1-\alpha)}]^{(1-s')}$. This will be the maximum fraction of trades for which an informed agent plays tough. Pick $\kappa > 0$ to satisfy $(1+\kappa)(1-s') < 1$, and let $s = s' - \kappa + \kappa s'$, so $(1-s) = (1-s')(1+\kappa)$.

Consider an informed seller who has traded $(\lfloor sk_i \rfloor - 1)$ times. From Lemma 2, she may do one of the following:

1. Play soft for the next trade, then play tough indefinitely for expected value $V1$.

2. Play tough for the remaining trades for expected value $V2$.

To show that the informed seller will not do (2) for i sufficiently large, I will compare the expected payoffs of V1 and V2. The payoff to (1) is: $V1 = \tau^{BL}(p^L - c^L) + (1-\tau^{BL})(p^M - c^L) + \delta_i X_i$, where X_i is the payoff to playing tough indefinitely for the remaining $\lceil (1-s)k_i \rceil$ trades, and the payoff to (2) is $V2 = X_i + \rho_i \gamma_i$, X_i for the first $\lceil (1-s)k_i \rceil$ trades, plus γ_i, the expected payoff for the last trade, conditional on making those trades, with ρ_i as the probability of making those trades.

Step 2: $\limsup_{i\to\infty} \rho_i \le \left(\frac{p^L - c^L}{p^H - c^L}\right)^{1+\kappa}$**.**

The probability of a buyer making one trade in the high state then surviving while playing tough is $\frac{\delta_i(1-g_i^B)}{(1-\delta_i)+\delta_i(1-\tau_i^{BL})}$, so the probability ρ_i of not exiting while waiting for those trades $\lceil (1-s)k_i \rceil$ to complete is:

$$\rho_i = \left[\frac{\delta_i(1-\tau_i^{BL})}{(1-\delta_i)+\delta_i(1-\tau_i^{BL})}\right]^{\lceil (1-s)k_i \rceil} \le \left[\frac{(1-\alpha)(1-(\delta_i)^{k_i})}{(1-\delta_i)+(1-\alpha)(1-(\delta_i)^{k_i})}\right]^{\lceil (1-s)k_i \rceil} \tag{1.17}$$

because $(1-\tau_i^{BL}) \leq (1-\alpha)(1-(\delta_i)^{k_i})$ from Lemma 5. Simplifying the right hand side of 1.17, this equals:

$$\left[\frac{(1-\alpha)(1+\delta_i+...+(\delta_i)^{k_i-1})}{1+(1-\alpha)(1+\delta_i+...+(\delta_i)^{k_i-1})}\right]^{\lceil(1-s)k_i\rceil} \leq \left[\frac{(1-\alpha)k_i}{1+(1-\alpha)k_i}\right]^{(1-s)k_i}. \tag{1.18}$$

This is rewritten, and then along the limit of $k_i \to \infty$, this is an exponential:

$$\left[1-\frac{1}{1+(1-\alpha)k_i}\right]^{(1-s)k_i} \to \left[1-\frac{1}{(1-\alpha)k_i}\right]^{k_i(1-s)} \to e^{-\frac{1-s}{1-\alpha}}. \tag{1.19}$$

At this point the definition of $(1-s)$ is used, so that probability becomes:

$$e^{-\frac{(1-s')(1+\kappa)}{1-\alpha}} = e^{(1+\kappa)\ln\left(\frac{p^L-c^L}{p^H-c^L}\right)} = \left(\frac{p^L-c^L}{p^H-c^L}\right)^{1+\kappa}. \tag{1.20}$$

Thus $\limsup_{i\to\infty} \rho_i \leq \left(\frac{p^L-c^L}{p^H-c^L}\right)^{1+\kappa}$.

Step 3: $V1 > V2$.

I can then compare the difference in payoffs between (1) and (2):

$$V1-V2 \geq \tau^{BL}(p^L-c^L)+(1-\tau^{BL})(p^M-c^L)+\delta_i X_i\text{-}[X_i+\rho_i\gamma_i] \geq p^L-c^L+\delta_i X_i\text{-}[X_i+\rho_i\gamma_i].$$

Along the limit of $(\delta_i)^{k_i} \to 1$, and noting that $X_i \leq (p^H-c^L)(1+\delta_i+...+(\delta_i)^{\lceil sk_i\rceil})$ and $\gamma_i \leq p^H-c^L$, we can bound the difference:

$$V1-V2 \geq p^L-c^L-(1-\delta_i)(1+\delta_i+...+(\delta_i)^{k_i-1})(p^H-c^L)-(p^H-c^L)\rho_i$$

$$= p^L-c^L-(p^H-c^L)\rho_i-(1-(\delta_i)^{k_i})(p^H-c^L). \tag{1.21}$$

Using $\lim(\delta_i)^{k_i}=1$, and $\lim sup\rho \leq \left(\frac{p^L-c^L}{p^H-c^H}\right)^{(1+\kappa)}$, the right hand side of 1.21 is lower bounded:

$$\to (p^L-c^L)\left[1-\left(\frac{p^L-c^L}{p^H-c^L}\right)^{\kappa}\right] > 0. \tag{1.22}$$

From 1.22, the payoff to playing tough forever $(V2)$ must be strictly below the payoff to playing soft now then tough forever $(V1)$ for i sufficiently high. Thus for i sufficiently large an informed seller will play soft after having traded $\lfloor sk_i - 1\rfloor$ units, so by Lemma 2 an informed seller in the low state will play soft for the first $\lfloor sk_i\rfloor$ periods. The proof for buyers is identical.

□

1.4.2.2 Agents Learn

The next lemma is the last step necessary to show that information percolates. Recall that $\tilde{\beta}^B : W \times \mathbb{N} \to R$ is a function mapping the state and number of periods in the market to a random variable for beliefs in an equilibrium. For example, $Pr\{\tilde{\beta}^B(H,1) = \beta^B(T,\frac{1}{2})\} = \tau^{SH}$, and $Pr\{\tilde{\beta}^B(H,1) = \beta^B(S,\frac{1}{2})\} = 1-\tau^{SH}$.

Lemma 7 (Agents learn)**.** *Consider any sequence of games* $\{(\delta_i, k_i, \alpha, v^H, v^L, p^M, c^H, c^L)\}_{i=1}^{\infty}$ *and corresponding equilibria indexed* i *with* $\lim_{i\to\infty}(\delta_i)^{k_i} = 1$*. Suppose* $\exists s \in (0,1]$ *such that informed buyers play soft in the high state and informed sellers play soft in the low state immediately on entering for the first* $\lfloor sk_i \rfloor$ *periods.*

Then there exists a sequence $\{n_i\}_{i=1}^{\infty}$ *such that* $\lim_{i\to\infty}(\delta_i)^{n_i} = 1$ *and for* $n_i' \geq n_i$*:*

1. $\operatorname*{plim}_{i\to\infty} \tilde{\beta}_i^B(L, n_i') = 0,$
2. $\operatorname*{plim}_{i\to\infty} \tilde{\beta}_i^B(H, n_i') = 1,$
3. $\operatorname*{plim}_{i\to\infty} \tilde{\beta}_i^S(L, n_i') = 0,$ *and*
4. $\operatorname*{plim}_{i\to\infty} \tilde{\beta}_i^S(H, n_i') = 1.$

In other words, information percolates.

The strategy of the proof is to use Lemma 6 which states that some informed agents will play soft, and show that the number of times they play soft is enough, far along the sequence of equilibria, for the uninformed agents to learn the state. It is worthwhile to note the likelihood of encountering any agent playing soft in a period converges to zero, but the number of softs a buyer expects to encounter over his life grows to infinity with k.

Proof. Let

$$n_i = \frac{\ln(1 - \frac{1}{\sqrt{k_i}})}{\ln(\delta_i)}. \tag{1.23}$$

This will be the length of the learning period. From (1.23), $(\delta_i)^{n_i} = (1 - \frac{1}{\sqrt{k_i}}) \to 1$, and buyers will survive the learning phase.

Step 1: For i sufficiently large, $\exists b \in (0,1)$ such that $\tau^{SH} - \tau^{SL} < b(1-(\delta_i)^{k_i}$.

In the low state, from Lemma 6, an informed seller plays soft for at least $\lfloor sk_i \rfloor$ periods, and from $\mathbf{S}$ and σ^S, there is a mass of $\alpha(1 + \delta_i + ... + \delta_i^{sk_i - 1})$ sellers playing soft that would not play soft in the high state. The probability of meeting one of these sellers in any period is then:

$$\alpha(1+\delta_i+...+(\delta_i)^{sk_i-1})/\left(\frac{1}{1-\delta_i}\right)=\alpha(1-(\delta_i)^{sk_i}). \tag{1.24}$$

Using Lemma 4, which states that the fraction of uninformed sellers playing soft is more in the low state than in the high state, and combining those uninformed sellers and the informed sellers from (1.24):

$$\tau_i^{SL}\leq\tau_i^{SH}-\alpha(1-(\delta_i)^{sk_i}). \tag{1.25}$$

From 1.25, $\exists b>0$, such that this difference is lower bounded: $\alpha(1-(\delta_i)^{sk_i})>b(1-(\delta_i)_i^k)$, so we can bound the fraction of sellers playing soft correctly in the low state:

$$1-(\delta_i)^{k_i}\geq 1-\tau_i^{SL}\geq 1-\tau_i^{SH}+b(1-(\delta_i)^{k_i})>0.$$

This demonstrates the difference between sellers playing soft in the low state $(1-\tau_i^{SL})$ and the high state $(1-\tau_i^{SH})$. This difference approaches zero, but will be large enough for buyers to learn the state.

Step 2: For any $n_i'\geq n_i'$, $\text{plim}\,\beta_i^B(L,n_i')\to 0$ and $\text{plim}\,\beta_i^B(H,n_i')\to 1$.

Over these n_i' periods let us consider how many times a buyer will encounter a seller playing soft, conditional on the state being low. In expectation this is $n_i'(1-\tau_i^{SL})$. The number of times a buyer expects to encounter a seller playing soft, conditional on the state being high, is $n_i'(1-\tau_i^{SH})$. Comparing these two, a buyer takes n_i' draws from a Bernoulli distribution, with success probability $r_i^L=(1-\tau_i^{SL})$ if the state is low, and success probability $r_i^H=(1-\tau_i^{SH})$ if the state is high. If the buyer can determine the state based on these draws, the buyer will learn the state.

Let R_i^w be two random variables, one for each state $w\in\{H,L\}$, which is the number of softs a buyer sees during the learning phase of n_i' periods in state w. For a state, the random variable has mean $E[R_i^w]=n_i'r_i^w$ and variance $(\sigma_i^w)^2=n_i'r_i^w(1-r_i^w)$. Consider the deviation from the mean. Using Chebyshev's inequality this is:

$$P\left(|R_i^w-n_i'r_i^w|<n_i'\frac{r_i^H-r_i^L}{4}\right)\geq 1-\frac{1}{\kappa_i^2},$$

with

$$\kappa_i\sigma_i^w=n_i'\frac{r_i^L-r_i^H}{4}\ \text{so}\ \frac{1}{\kappa_i^2}=\frac{4(\sigma_i^w)^2}{(r_i^L-r_i^H)^2}.$$

Substituting, the inequality becomes:

$$P\left(|R_i^w - n_i' r_i^w| < n_i' \frac{r_i^H - r_i^L}{4}\right) \geq 1 - 16 \frac{r_i^w(1-r_i^w)}{n_i'(r_i^H - r_i^L)^2}.$$

Because $r_i^w \leq (1-(\delta_i)^{k_i})$, and $(r_i^L - r_i^H) \geq b(1-(\delta_i)^{k_i})$:

$$1 - 16 \frac{r_i^w(1-r_i^w)}{n_i'(r_i^H - r_i^L)^2} \geq 1 - 16 \frac{(1-(\delta_i)^{k_i})}{n_i' b^2 (1-(\delta_i)^{k_i})^2}$$

$$= 1 - 16 \frac{\ln \delta_i}{\ln(1-\frac{1}{\sqrt{k_i}}) b^2 (1-(\delta_i)^{k_i})}.$$

Along the limit of $i \to \infty$:

$$= 1 - \frac{16}{b^2} \lim_{i\to\infty} \left[\frac{\ln \delta_i}{1-\delta_i}\right] \lim_{i\to\infty} \left[\frac{1}{\ln(1-\frac{1}{\sqrt{k_i}})(1+\delta_i+...+(\delta_i)^{k_i-1})}\right]. \tag{1.26}$$

The second limit can be evaluated using $k_i \to \infty$ and $(\delta_i)^{k_i} \to 1$ to become

$$\lim_{i\to\infty} \frac{k_i^{-1}}{\ln(1-\frac{1}{\sqrt{k_1}})} = \lim_{i\to\infty} \frac{-(k_i)^{-2}}{(-\frac{1}{2}(k_i)^{-3/2})(\frac{1}{1-\frac{1}{\sqrt{k_i}}})} = 0.$$

So

$$\lim_{i\to\infty} P\left(|R_i^w - n_i' r_i^w| < n_i \frac{r_i^H - r_i^L}{4}\right) = 1.$$

Thus, conditional on the state and making n_i' observations, R_i^w will be at most $n_i' \frac{r_i^H - r_i^L}{4}$ from $n_i' r_i^w$. Since these are two disjoint ranges, conditional on surviving n_i' periods,
$\tilde{\beta}_i^B(H, n_i') \xrightarrow{p} 1$ and $\tilde{\beta}_i^B(L, n_i') \xrightarrow{p} 0$.

Combining this with the probability of surviving n_i periods, $\underset{i\to\infty}{\text{plim}} (\delta_i)^{n_i} \tilde{\beta}_i^B(H, n_i) = 1$ and $\underset{i\to\infty}{\text{plim}} (\delta_i)^{n_i}(1 - \tilde{\beta}_i^B(n_i, L)) = 1$. The proof for the seller is identical.

□

While learning and efficiency can occur, there is no guarantee that they will result in a sequence of equilibria with disappearing frictions. The next theorem shows that Assumption 1 is sufficient to guarantee an outcome in which agents learn is the only outcome as frictions disappear.

Theorem 3 (Positive Result). *Let $(\alpha, v^H, v^L, c^H, c^L)$ satisfy Assumption 1. For any* *sequence of games parameterized by* $\{(\delta_i, k_i, \alpha, v^H, v^L, p^M, c^H, c^L)\}_{i=1}^{\infty}$ *with* $(\delta_i)^{k_i} \to 1$ *and* $k_i \to \infty$*, and corresponding equilibria indexed* i*, the following are true:*

1. *Information percolates in the sequence of equilibria.*

2. *The sequence of equilibria is efficient.*

Proof. Information percolation follows directly from Lemmas 6 and 7, and efficiency follows from Lemma 3.

□

The importance of multi-unit supply and demand is visible in several places. Most importantly, trading a large number of units enables learning as seen in the proof of Lemma 7 in (1.26). In this equation a fixed k would prevent beliefs from converging to 0 or 1, so the high state and low state would never be perfectly distinguishable.

This result shows that eliminating frictions, with an additional assumption, makes learning efficiency not only possible but also the guaranteed outcome. In the next theorem I relax Assumption 1 to show that if frictions disappear, learning and efficiency are still possible although not guaranteed.

Theorem 4 (Positive Result Exists Generally)**.** *For any sequence of games parameterized by* $\{(\delta_i, k_i, \alpha, v^H, v^L, p^M, c^H, c^L)\}_{i=1}^{\infty}$ *with* $(\delta_i)^{k_i} \to 1$ *and* $k_i \to \infty$*, there is a corresponding sequence of equilibria indexed* i *such that the following are true:*

1. *Information percolates in the sequence of equilibria.*

2. *The sequence of equilibria is efficient.*

Proof. The proof is as follows. Pick an $s \in (0, 1)$. I define a new game with the only difference being that all informed buyers [sellers] in the high [low] state must play soft sk_i times immediately after entering. I then apply the existence proof to construct a steady state and equilibrium of this game, still restricting buyers and sellers for their first sk_i trades. As in the proof of Theorem 3, because informed agents play soft for a significant fraction of their trades, information will percolate and the sequence of equilibria is efficient. It is left to show that for i sufficiently large, equilibria in the constrained game are equilibria in the base game. To do this let $\rho_i = \left(\frac{\delta_i(1-\tau_i^{SH})}{(1-\delta_i)+\delta_i(1-\tau_i^{SH})}\right)^{(1-s)k_i}$. Notice that ρ_i is the probability that a buyer will survive to make $(1-s)k_i$ trades at a low price in the high state. Because the sequence is efficient, ρ_i will necessarily approach zero. At this point I focus on high values of

i and relax the restriction on the games and equilibria for low i. The proof proceeds identically to the proof of Lemma 6 and the remainder of Theorem 3.

□

1.4.3 Discussion of Assumptions

In this section I discuss the choice of assumptions used in the paper. I start by looking at the assumption on the payoffs, then discuss multi-unit supply and demand, and finally exogenous exit and the prices. Assumption 1 is sufficient to guarantee information percolation. The following is an example of the necessity of this assumption.

1.4.3.1 A Degenerate Equilibrium and the Shape of Payoffs

To demonstrate the importance of Assumption 1, consider the following game parameters:

$$(v^H = 40, p^H = 39 = c^H, v^L = 30 = p^L, c^L = 1, \alpha = \frac{1}{2}).$$

In this case $\frac{v^H - p^L}{v^H - p^H} = 10 > e^{1/(1-\alpha)} \approx 7.4$.

Consider $\delta = 0.99, k = 8$ and the following strategies: $\forall \mathbf{y}$ and $\theta, \sigma^B(\mathbf{y}, \theta) = 0$, and $\sigma^S(\mathbf{z}, \theta) = 1$ for $\theta \in [0, \frac{29}{38}], \overline{z} < k$. In this case $\tau^{BH} = \tau^{BL} = 1, \tau^{SL} = \delta^k, \tau^{SH} = 1 - \frac{1}{2}(1 - \delta^k)$, and the sellers' strategies are clearly optimal. From this, $\tau^{SH} \approx 0.9614$, and the probability that a buyer can make all his trades at p^L in the high state is 0.2458. In the high state, since each item bought at the low price yields a payoff of 10, relative to a payoff at the high price of 1, all buyers will always play tough, and buyers' strategies are optimal.

In the low state, the outcome is Pareto efficient (subject to the exit rate) and there is no ex post regret ($x^L = 0$), as the sellers each play soft k times. But, in the high state, the outcome is inefficient, and there is significant ex post regret, as all trades are made at the wrong price. Assumption 1 prevents this scenario by reducing the incentive that informed buyers in the high state have to play tough.

The parametrized example is one in which $\frac{v^H - p^L}{v^H - p^H} \geq e^{\frac{1}{1-\alpha}}$ and $\frac{1}{2}(p^L - c^L) + \frac{1}{2}(p^L - c^H) \geq 0$. In this game there exists an equilibrium where uninformed sellers play soft, and buyers always play tough. On the other hand, if

$$\frac{1}{2}(p^L - c^L) + \frac{1}{2}(p^L - c^H) < 0 \text{ and } \frac{v^H - p^L}{v^H - p^H} \geq e^{\frac{1}{1-\alpha}}, \tag{1.27}$$

then uninformed sellers are unwilling to trade at their priors, and some informed buyers must play soft when the state is H, allowing uninformed sellers to update their beliefs appropriately. If informed buyers play soft in the high state, some sellers will learn the state, reducing buyers' payoffs to playing tough. As it is unclear exactly what the necessary relationships are to guarantee percolation and efficiency, these conditions warrant additional study.

While I use an assumption on the shape of the payoffs, I can forgo the *no fear* assumption present in Blouin and Serrano (2001) and Wolinsky (1990), which roughly states that some uninformed agents do not need to learn to want to trade. In some ways this assumption is restrictive, as the world is replete with markets where agents learn before participating, and would not participate without learning. An example of this would be a game partially parameterized by $v^H = 12, c^H = 11, v^L = 2, c^L = 1$. In this case the results of earlier work do not apply, while the model presented in this paper requires $\alpha \geq 0.566$ for the positive result to be the unique outcome in an almost frictionless game.

Through Assumption 1, we see that not only are informed agents the source of information and learning for uninformed agents, but the informed agents can also act as a significant source of friction which discourages other informed agents from trying to trade at prices at which an informed agent would not trade.

1.4.3.2 Multi-unit Supply and Demand

The assumption of multi-unit demand is critical to the positive information percolation and efficiency results. The assumption has two features worth noting. First, agents can only trade zero or one units in a transaction. Second, marginal values and costs are flat with a single jump. More general assumptions allow agents to trade multiple goods in a single transaction, and give agents more flexible utility functions. Allowing agents to trade multiple units in a single transaction gives agents the opportunity to make all their trades at once, behaving like a single-unit consumer. Allowing them to trade more than one unit at once, up to a limit, creates a more complex model. In a model in which agents simultaneously decide the maximum units to trade at the same time that they simultaneously pick prices, the possibility result in Theorem 4 may still exist, although the uniqueness of the outcome in 3 would not.

1.4.3.3 Exogenous Exit

Under exogenous exit, agents do not have a choice of when to exit. If the assumption were changed, and buyers and sellers could choose to exit, or exit when they have finished trading, then either the mass of buyers can differ from the mass of sellers, or feasibility will be violated. If the mass of buyers differs from the mass of sellers, then during matching some agents would be without a partner, raising complications. If feasibility is violated, the total gains from trade available in a game would be a function of the equilibrium. Either of these properties is highly undesirable, and for these reasons I use the assumption of exogenous exit.

1.5 Concluding Remarks

In this paper I have presented a framework for understanding information percolation when buyers and sellers want to trade multiple goods. I first compared my model to earlier models of dynamic matching and bargaining, and demonstrated a no-percolation result analogous to the negative results of Wolinsky (1990) and Blouin and Serrano (2001). Then, with an additional assumption to eliminate trivial equilibria in which beliefs could never change, I showed that as the frictions of exit and granularity of trades disappear, information would fully percolate in all equilibria, yielding the efficiency and no ex post regret that would be found in an analogous market with complete information. I also demonstrated that this outcome could realize without the additional assumption on payoffs if frictions disappeared.

In order to show these results, several new assumptions were required. Allowing buyers and sellers to trade multiple times introduced complications absent from earlier work. In particular, using the standard assumption that agents exit if they have completed trading may yield a different number of buyers and sellers even if their entry rates are equal. Future work may further address the question of what conditions can lead to or preclude information percolation. The evidence presented in this paper suggests that if meetings are bilateral, and information is only transmitted through prices, multiple transactions are required for information to fully percolate.

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1.6 Appendix

1.6.1 Proof of Lemma 1: Strategies Involve Threshold Beliefs

Proof. Recall the forward looking value functions v^B and $\overline{v}^B$, which describe the value of future participation in the market. For $\theta \in \Theta$ and $j < k$, the value a buyer expects for playing soft is described by:

$$\begin{aligned}\overline{v}^B(S,j,\theta) =& v^L + \theta(v^H - v^L)\\ &- \tau^{SL}p^H - (1-\tau^{SL})p^M - \theta(\tau^{SH} - \tau^{SL})p^H + \theta(\tau^{SH} - \tau^{SL})p^M\\ &+ \delta\tau^{SL}v^B(j-1,\beta^B(T,\theta)) + \delta\theta(\tau^{SH} - \tau^{SL})v^B(j-1,\beta^B(T,\theta))\\ &+ \delta(1-\tau^{SL})v^B(j-1,\beta^B(S,\theta)) - \theta(\tau^{SH} - \tau^{SL})v^B(j-1,\beta^B(S,\theta)).\end{aligned}$$

The value to playing tough is:

$$\begin{aligned}\overline{v}^B(T,j,\theta) =& \theta(1-\tau^{SH})(v^H - p^L)\\ &+ \delta(\theta\tau^{SH} + (1-\theta)\tau^{SL})v^B(j,\beta^B(T,\theta))\\ &+ \delta(\theta(1-\tau^{SH}) + (1-\theta)(1-\tau^{SL})v^B(j-1,\beta^B(S,\theta)).\end{aligned}$$

From the value functions, the difference in values is:

$$\begin{aligned}\overline{v}^B(S,j,\theta)-\overline{v}^B(T,j,\theta) =&\theta\tau^{SH}(v^H-v^L)+v^L \\ &-\tau^{SL}p^H-(1-\tau^{SL})p^M-\theta(\tau^{SH}-\tau^{SL})(p^H-p^M) \\ &-\delta(\tau^{SH}\theta+(1-\theta)\tau^{SL})\big(v^B(y,\beta^B(T,\theta))-v^B(y-1,\beta^B(T,\theta))\big).\end{aligned} \tag{1.28}$$

Step 1: If $\tau^{SL}>\tau^{SH}$ then $\overline{v}^B(S,j,\theta)-\overline{v}^B(T,j,\theta)$ is increasing in θ.

With some abuse of notation, I will use several derivatives. These derivatives exist almost everywhere, except at points where $\overline{v}^B(S,j,\theta)=\overline{v}^B(T,j,\theta)$. Furthermore, even if the derivatives do not exist, the functions are still continuous.
The derivative of (1.28) is:

$$\begin{aligned}\frac{d(\overline{v}^B(S,\overline{j},\theta)-\overline{v}^B(T,\overline{j},\theta))}{d\theta} =& \\ &\tau^{SH}(v^H-v^L)-(\tau^{SH}-\tau^{SL})(p^H-p^M) \\ &-\delta(\tau^{SH}\theta+(1-\theta)\tau^{SL})\frac{d\beta^B(T,\theta)}{d\theta}\left(\frac{d(v^B(j,\beta^B(T,\theta))-v^B(j-1,\beta^B(T,\theta)))}{d\beta^B(T,\theta)}\right) \\ &-\delta(\tau^{SH}-\tau^{SL})(v^B(j,\beta^B(T,\theta))-v^B(j-1,\beta^B(T,\theta))).\end{aligned}$$

From consistency of beliefs in (1.7), $\frac{d\beta^B(T,\theta)}{d\theta}=\frac{\tau^{SH}\tau^{SL}}{(\theta\tau^{SH}+(1-\theta)\tau^{SL})^2}$. Because $\tau^{SH}<\tau^{SL}$, $\frac{\tau^{SH}\tau^{SL}}{(\theta\tau^{SH}+(1-\theta)\tau^{SL})^2}<\frac{\tau^{SH}}{(\theta\tau^{SH}+(1-\theta)\tau^{SL})}$ for $\theta\in(0,1)$.

From Property (V5), the value of the marginal good can be upper bounded:

$$\left(\frac{d(v^B(j,\beta^B(T,\theta))-v^B(j-1,\beta^B(T,\theta)))}{d\beta^B(T,\theta)}\right)\le(v^H-v^L).$$

Then a new lower bound on the derivative of the difference in values is:

$$\begin{aligned}\frac{d(\overline{v}^B(S,j,\theta)-\overline{v}^B(T,j,\theta))}{d\theta} \ge& \\ &\tau^{SH}(v^H-v^L)-(\tau^{SH}-\tau^{SL})(p^H-p^M) \\ &-\delta(\tau^{SH}\theta+(1-\theta)\tau^{SL})\frac{\tau^{SH}}{(\theta\tau^{SH}+(1-\theta)\tau^{SL})}(v^H-v^L) \\ &+\delta(\tau^{SL}-\tau^{SH})(v^B(j,\beta^B(T,\theta))-v^B(j-1,\beta^B(T,\theta))). \\ =\;& (1-\delta)\left(\tau^{SH}(v^H-v^L)\right)+(\tau^{SL}-\tau^{SH})(p^H-p^M) \\ &+\delta(\tau^{SL}-\tau^{SH})(v^B(j,\beta^B(T,\theta))-v^B(j-1,\beta^B(T,\theta))). \\ >\;& 0.\end{aligned}$$

Because the derivative is strictly greater than zero, $\overline{v}^B(S,j,\theta)-\overline{v}^B(T,j,\theta)$ is increasing in θ.

Step 2: If $\tau^{SL}<\tau^{SH}$ then $\overline{v}^B(S,j,\theta)-\overline{v}^B(T,j,\theta)$ is increasing in θ by induction.

Let $\overline{j} = arg \min_{\overline{y},\theta}\{\sigma^B(\mathbf{y},\theta) > 0\}$, so $\overline{j}$ is the fewest number of goods a buyer has where he, for some belief, plays soft. For example, if $\overline{j} = 2$, then a buyer who has traded $k-1$ times will not play soft for any belief, but for some $\theta, \overline{v}^B(S,\overline{j},\theta) \geq \overline{v}^B(T,\overline{j},\theta)$. This is the base case of the induction.

If $\overline{j} = 1$, then whenever buyers have one unit left to buy, they may play soft for some belief. If $\overline{j} > 1$, after trading in the base case and having $\overline{j}-1$ units left, they will play tough forever. Using the martingale property of beliefs, the remaining value on having $\overline{j}-1$ units left to trade or fewer is:

$$v^B(j,\theta) = \theta\left(v^H - p^L + \delta v^B(j-1,\theta)\right)\frac{1-\tau^{SH}}{1-\delta\tau^{SH}} \quad \text{for } 0 < j < \overline{j}.$$

With $v^B(0,\theta) = 0$, I unravel the recursion:

$$v^B(\overline{j}-1,\theta) = \theta\sum_{i=1}^{\overline{j}-1}(\delta)^{i-1}\left(\frac{(1-\tau^{SH})}{1-\delta\tau^{SH}}\right)^i(v^H - p^L).$$

When an agent has $\overline{j}$ units left to trade and a belief θ, he must decide how many times t to play tough before playing soft, $t \in \{\infty, 0, 1, ...\}$. Let $vm : \mathbb{N}\times\mathbb{N}\times\Theta \to \mathbb{R}$ be the value function of a buyer mapping a number j units left to buy, a number t periods to wait before playing soft, and a belief θ to a payoff.

Then the payoff to waiting t times is:

$$\begin{aligned} vm(\overline{j},t,\theta) =& \theta\Bigg[((\tau^{SH}\delta)^t)\left(v^H - \tau^{SH}p^H - (1-\tau^{SH})p^L + \delta\sum_{i=1}^{\overline{j}-1}(\delta)^{i-1}\left(\frac{(1-\tau^{SH})}{1-\delta\tau^{SH}}\right)^i(v^H-p^L)\right) \\ &+ (1-\tau^{SH})(1+\tau^{SH}\delta + ... + (\tau^{SH}\delta)^{t-1})\left(v^H - p^L + \delta\sum_{i=1}^{\overline{j}-1}(\delta)^{i-1}\left(\frac{(1-\tau^{SH})}{1-\delta\tau^{SH}}\right)^i(v^H-p^L)\right)\Bigg] \\ &+ (1-\theta)((\tau^{SL}\delta)^t)(v^L - \tau^{SL}p^H - (1-\tau^{SL})p^L). \end{aligned} \tag{1.29}$$

Then $v(\overline{j},\theta) = \max_{t\in\mathbb{N}} vm(\overline{j},t,\theta)$, since an agent will optimally wait to play soft.

By inspection, for each t, $vm(\overline{j},t,\theta)$ is linear in θ, so $\frac{dvm(\overline{j},t,\theta)}{d\theta}$ exists and is constant for every t. Moreover, for $\theta = 0$, playing soft is strictly dominated, and

$$vm(\overline{j},0,0) < vm(\overline{j},1,0) < ... < vm(\overline{j},\infty,0).$$

Finally, by the value function property (V1), since for some θ, $\overline{v}^B(S,\overline{j},\theta) > \overline{v}^B(T,\overline{j},\theta)$, replacing $\overline{v}^B(T,\overline{j},\theta)$ with any $vm(\overline{j},t,\theta)$, with $t \geq 1$, we find $\frac{d\overline{v}^B(S,\overline{j},\theta)}{d\theta} > \frac{d\overline{v}^B(T,\overline{j},\theta)}{d\theta}$.

Since $\frac{d\overline{v}^B(S,\overline{j},\theta)}{d\theta} > \frac{d\overline{v}^B(T,\overline{j},\theta)}{d\theta}$, $\frac{d\overline{v}^B(S,\overline{j},\theta)}{d\theta} - \frac{d\overline{v}^B(T,\overline{j},\theta)}{d\theta} > 0$, and:

$$
\begin{aligned}
\frac{d(\overline{v}^B(S,\overline{j},\theta) - \overline{v}^B(T,\overline{j},\theta))}{d\theta} = & \\
& \tau^{SH}(v^H - v^L) - (\tau^{SH} - \tau^{SL})(p^H - p^M) \\
& - \delta(\tau^{SH}\theta + (1-\theta)f)\frac{d\beta^B(T,\theta)}{d\theta}\left(\frac{d(v^B(\overline{j},\beta^B(T,\theta)) - v^B(\overline{j}-1,\beta^B(T,\theta)))}{d\beta^B(T,\theta)}\right) \\
& - \delta(\tau^{SH} - \tau^{SL})(v^B(\overline{j},\beta^B(T,\theta)) - v^B(\overline{j}-1,\beta^B(T,\theta))) \\
& > 0.
\end{aligned} \tag{1.30}
$$

The Inductive Step:

Recall from Property (V4) that for any θ, for $j' > j > 0$, if $\tau^{SL} \neq \tau^{SH}$, then:

$$v^B(j,\theta) - v^B(j-1,\theta) > v^B(j',\theta) - v^B(j'-1,\theta),$$

and Property (V5), which is restated, if $\beta^B(T,\theta) > \theta$, then:

$$v(j,\beta^B(T,\theta)) - v(j,\theta) > v(j',\beta^B(T,\theta)) - v(j',\theta).$$

Using these two properties, and since $\tau^{SH} > \tau^{SL}$, $\beta^B(T,\theta) > \theta$, and for any $j > \overline{j}$,

$$
\begin{aligned}
\frac{d(\overline{v}^B(S,j,\theta) - \overline{v}^B(T,j,\theta))}{d\theta} = & \\
& \tau^{SH}(v^H - v^L) - (\tau^{SH} - \tau^{SL})(p^H - p^M) \\
& - \delta(\tau^{SH}\theta + (1-\theta)\tau^{SL})\frac{d\beta^B(T,\theta)}{d\theta}\left(\frac{d(v^B(j,\beta^B(T,\theta)) - v^B(j-1,\beta^B(T,\theta)))}{d\beta^B(T,\theta)}\right) \\
& - \delta(\tau^{SH} - \tau^{SL})(v^B(j,\beta^B(T,\theta)) - v^B(j-1,\beta^B(T,\theta))) \\
& >
\end{aligned} \tag{1.31}
$$

$$
\begin{aligned}
& \tau^{SH}(v^H - v^L) - (\tau^{SH} - \tau^{SL})(p^H - p^M) \\
& - \delta(\tau^{SH}\theta + (1-\theta)\tau^{SL})\frac{d\beta^B(T,\theta)}{d\theta}\left(\frac{d(v^B(j-1,\beta^B(T,\theta)) - v^B(j-2,\beta^B(T,\theta)))}{d\beta^B(T,\theta)}\right) \\
& - \delta(\tau^{SH} - \tau^{SL})(v^B(j-1,\beta^B(T,\theta)) - v^B(j-2,\beta^B(T,\theta))) \\
& > 0
\end{aligned} \tag{1.32}
$$

then

$$\tau^{SH}(v^H - v^L) - (\tau^{SH} - \tau^{SL})(p^H - p^M) \tag{1.33}$$
$$> \delta(\tau^{SH}\theta + (1-\theta)\tau^{SL})\frac{d\beta^B(T,\theta)}{d\theta}\left(\frac{d(v^B(j,\beta^B(T,\theta)) - v^B(j-1,\beta^B(T,\theta)))}{d\beta^B(T,\theta)}\right)$$
$$+ \delta(\tau^{SH} - \tau^{SL})(v^B(j,\beta^B(T,\theta)) - v^B(j-1,\beta^B(T,\theta)))$$
$$> 0.$$

Although this difference in derivatives may be set valued, it is always positive whenever a buyer might play soft for some belief and set past of trades.

In both the case in which $\tau^{SL} > \tau^{SH}$ and the case in which $\tau^{SL} < \tau^{SH}$ the claim is true. If $\tau^{SL} = \tau^{SH}$, then $\beta^B(T,\theta) = \theta$, and the proof is trivial. The proof for sellers is symmetric and omitted.

□

1.6.2 Proof of Lemma 2: Agents Play Soft First

Proof. For a buyer, recall the forward looking value functions, which describe the value of future participation in the market. For $\theta \in \Theta$ and $j < k$, the value for playing soft is described by:

$$\begin{aligned}\overline{v}^B(S,j,\theta) =& v^L + \theta(v^H - v^L)\\ &- \tau^{SL}p^H - (1-\tau^{SL})p^M - \theta(\tau^{SH} - \tau^{SL})p^H + \theta(\tau^{SH} - \tau^{SL})p^M\\ &+ \delta\tau^{SL}v^B(j-1,\beta^B(T,\theta)) + \delta\theta(\tau^{SH} - \tau^{SL})v^B(j-1,\beta^B(T,\theta))\\ &+ \delta(1-\tau^{SL})v^B(j-1,\beta^B(S,\theta)) - \theta(\tau^{SH} - \tau^{SL})v^B(j-1,\beta^B(S,\theta)),\end{aligned}$$

and the value to playing tough is:

$$\begin{aligned}\overline{v}^B(T,j,\theta) =& \theta(1-\tau^{SH})(v^H - p^L)\\ &+ \delta(\theta\tau^{SH} + (1-\theta)\tau^{SL})v^B(j,\beta^B(T,\theta))\\ &+ \delta(\theta(1-\tau^{SH}) + (1-\theta)(1-\tau^{SL})v^B(j-1,\beta^B(S,\theta)).\end{aligned}$$

And the difference in values is:

$$\begin{aligned}\overline{v}^B(S,j,\theta) - \overline{v}^B(T,j,\theta) =& \theta\tau^{SH}(v^H - v^L) + v^L\\ &- \tau^{SL}p^H - (1-\tau^{SL})p^M - \theta(\tau^{SH} - \tau^{SL})(p^H - p^M)\\ &- \delta(\tau^{SH}\theta + (1-\theta)\tau^{SL})\big(v^B(j,\beta^B(T,\theta)) - v^B(j-1,\beta^B(T,\theta))\big).\end{aligned} \tag{1.34}$$

Only the third line of the right hand side of (1.34) depends on the number of units left to trade, and for $j' > j$ from (V4), if $\tau^{SL} \neq \tau^{SH}$ then $\overline{v}^B(S, j', \theta) - \overline{v}^B(T, j', \theta) > \overline{v}^B(S, j, \theta) - \overline{v}^B(T, j, \theta)$. For an informed buyer (a buyer for whom $\theta \in \{0, 1\}$), Property (V1) applies. The proof for sellers is identical and omitted.

□

1.6.3 Proof of Equilibrium Existence

Step 1: The isomorphic game and strategies.

The strategies for the buyer and seller are broken into 6 strategies of interest. Let $\mathbb{K} = \{0, 1, ..., k-1\}$. Let $m \in \mathbb{K}$ represent either the number of softs played by a trading partner, or the number of times traded. If an agent has traded k or more times, he or she will play tough. For this reason I restrict agents to doing this, and only examine the strategic considerations of agents who have traded fewer than k times. A strategy $\tilde{\sigma} : \mathbb{K} \times \mathbb{K} \to [0, 1]$ maps the number of softs seen and the total number of units traded to the likelihood of playing soft before exiting, conditional on only encountering trading partners playing tough. Let Ω be the space of strategies of this type. $\tilde{\sigma} \in \{\tilde{\sigma}^{BH}, \tilde{\sigma}^{BL}, \tilde{\sigma}^{UB}, \tilde{\sigma}^{SH}, \tilde{\sigma}^{SL}, \tilde{\sigma}^{US}\}$, for an informed buyer in the high state, an informed buyer in the low state, an uninformed buyer, and also three for the sellers.

A strategy dictates the one or two periods in which an agent plays soft. For example, if $\delta = 0.9$ and $\tilde{\sigma}^{UB}(0, 0) = 0.81$, then an uninformed buyer who, after entry, encounters two sellers playing tough in sequence, will play soft in the third period. If the buyer encounters one seller playing soft, and has traded two units, then $\tilde{\sigma}^{UB}(2, 1)$ tells us when the buyer will play soft to trade a third unit, if ever. If $\tilde{\sigma}^{UB}(0, 0) = 0.80$, then a buyer will play soft after the second period with probability p such that $p0.81 + (1-p)0.729 = 0.80$. Then the buyer's likelihood of exiting before playing soft is 0.80. For any $\overline{y}, u$, given $\sigma(\overline{y}, u)$, I denote the probability of playing soft in a period n as $p = \min(\max(\frac{\sigma - \delta^{n+1}}{\delta^t - \delta^{n+1}}, 0), 1)$.

To simplify notation let $\sigma : \{BH, BL, UB, SH, SL, US\} \times \mathbb{N}^3 \to [0, 1]$ map an agent's history to a likelihood of playing soft, so $\sigma(BH, \overline{y}, u, n) = \min\left(\max\left(\frac{\sigma^{BH}(\overline{y}, u) - \delta^{n+1}}{\delta^n - \delta^{n+1}}, 0\right), 1\right)$. Let $\Sigma = [0, 1]^{6k^2}$ be the set of all strategy profiles.

Let the steady state statistics $(\tau^{SL}, \tau^{SH}, \tau^{BH}, \tau^{BL})$ take the same meaning as earlier, and let $\Gamma = [0, 1]^4$ be the set of all possible steady state statistics. I use the forward looking value functions v^B and v^S as before, where v^B is defined from $\overline{v}^B$ using τ^{SH} and τ^{SL} and v^S is defined from $\overline{v}^S$ using τ^{BH} and τ^{BL}.

The remainder of the proof is structured as follows: I construct a fixed point using strategy profiles and steady state statistics. Let $F_1 : \Gamma \to 2^{\Sigma}$ map a strategy profile and steady state statistics to a new set of strategy profiles, and let $F_2 : \Gamma \times \Sigma \to \Gamma$ map to a new set of steady state statistics. Let $F = (F_1, F_2)$, so $F : \Gamma \times \Sigma \to 2^{\Gamma} \times \Sigma$ constitutes a self-mapping correspondence. As the domain of (F_1, F_2) is compact, I then prove that F_1 and F_2 satisfy the closed graph property, and that for $\tau \in \Gamma$ and $\sigma \in \Sigma$, $F_1(\tau)$ and $F_2(\sigma)$ are non-empty and convex valued. Finally I prove that a fixed point of (F_1, F_2) constitute an equilibrium of the original game.

Step 1, substep 1: F_1 definition.

As in the negative proof, let $\overline{\beta}^S, \overline{\beta}^B : \mathbb{N} \times \mathbb{N}$ map an age and number of softs seen to a belief, and that these functions are conditional on some $\tau \in \Gamma$. Given $\sigma \in \Sigma$ and $\tau \in \Gamma$,

$$F_1(\sigma, \tau) = (\tilde{\sigma}^{BH}, \tilde{\sigma}^{BL}, \tilde{\sigma}^{UB}, \tilde{\sigma}^{SH}, \tilde{\sigma}^{SL}, \tilde{\sigma}^{US}). \tag{1.35}$$

Let $F_1^{UB}(\tau)$ represent $\tilde{\sigma}^{UB}$. To construct $F_1^{UB}(\tau)(\overline{y}, u)$ for any $\overline{y}, u$, let J be the set of all values $n \geq$ that satisfy the following two inequalities:

$$\overline{v}^B \left(T, k - \overline{y}, \overline{\beta}^B(n, u)\right) \leq \overline{v}^B \left(S, k - \overline{y}, \overline{\beta}^B(n, u)\right) \tag{1.36}$$

$$\text{and if } \; n \geq 1, \overline{v}^B \left(T, k - \overline{y}, \overline{\beta}^B(n-1, u)\right) \geq \overline{v}^B \left(S, k - \overline{y}, \overline{\beta}^B(n-1, u)\right) \tag{1.37}$$

So J is the set of elements such that it is profitable to play soft, but in earlier periods it was better to play tough. If J is empty, then let $F_1^{UB}(\overline{y}, m) = 0$, otherwise, let

$$F_1^{UB}(\overline{y}, m) = [\delta^{\min(J)}, \delta^{\sup(J)}]. \tag{1.38}$$

Step 1, substep 2: $F_1(\tau)$ is nonempty and convex valued, and F_1 has the closed graph property.

By construction, for any $\tau \in \Gamma$, $F_1(\tau)$ is non-empty and convex valued. To show the closed graph property, take any pair of sequences $\tau_i \to \overline{\tau}$ and $q_i \to \overline{q}$ such that $q_i \in F_1^{UB}(\tau_i)$. Recall that for any $\overline{y}, m$, $\overline{v}_i^B(T, k - \overline{y}, m)$ and $\overline{v}_i^B(T, k - \overline{y}, m)$, which are defined in (1.4-1.6), are continuous in τ_i.

I will show the closed graph property by contradiction. Suppose that $\overline{q} > \max(F_1(\overline{\tau}))$, so according to $\overline{q}$ the buyer would wait at most j periods and the buyer following $\max(F_1(\overline{\tau}))$ would wait at least j' periods, $j' > j$. So for $n = j$, (1.36) does not hold at τ. But for τ_i near τ, (1.36) will hold as $q_i \in F_1^{UB}(\tau_i)$. This contradicts the continuity of $\overline{v}^B$.

Next suppose $\overline{q} < \min(F_1^{UB}(\overline{\tau}))$. Again consider j' corresponding to $\min(F_1^{UB}(\overline{\tau}))$ and j corresponding to $\overline{q}$. Under τ then, at j', (1.36) and (1.37) are satisfied, but at j they are not. Again this is a violation of the continuity of $\overline{v}^B$. Since neither $\overline{q} > \max(F_1(\overline{\tau}))$ nor $\overline{q} < \min(F_1(\overline{\tau}))$, $\overline{q} \in F_1(\overline{\tau})$.

Step 2, substep 1: F_2 definition, continuity and existence.

I next construct two functions: conditional buyer and seller mass functions $\tilde{\mathbf{B}} : \{0, 1, \frac{1}{2}\} \times W \times \Sigma \times \Gamma \times \mathbb{N} \times \mathbb{N}^2 \to \mathbb{R}$ and $\tilde{\mathbf{S}} : \{0, 1, \frac{1}{2}\} \times W \times \Sigma \times \Gamma \times \mathbb{N} \times \mathbb{N}^2 \to \mathbb{R}$ which map to a mass of agents. The inputs to $\tilde{B}$ and $\tilde{S}$ are: an initial belief, the state of nature w, strategy profile and steady state statistics, age n, and trading histories which are characterized by total number of goods traded y and a total number of trading partners playing soft, m.

Note that these mass functions are similar to the earlier mass functions, but take τ as an input, and so are not linked, compared to the earlier mass functions in which τ was determined endogenously. Then $\tilde{\mathbf{B}}$ is defined for the high state as follows:

$$\begin{aligned}\tilde{\mathbf{B}}(\theta, H, \sigma, \tau, n, y, m) = \quad & \delta\tau^{SH}\tilde{\mathbf{B}}(\theta, H, \tau^{SH}, n-1, y, m)(1 - \sigma(BH, y, m, n-1)) + \\ & \delta\tau^{SH}\tilde{\mathbf{B}}(\theta, H, \tau^{SH}, n-1, y-1, m)\sigma(BH, y, m, n-1) \\ & + \delta(1 - \tau^{SH})\tilde{\mathbf{B}}(\theta, H, \tau^{SH}, n-1, y-1, m-1)\end{aligned} \tag{1.39}$$

with bases cases of $\tilde{\mathbf{B}}(\frac{1}{2}, H, \sigma, \tau, 0, 0, 0) = \tilde{\mathbf{B}}(\frac{1}{2}, L, \sigma, \tau, 0, 0, 0) = (1-\alpha)$, $\tilde{\mathbf{B}}(1, H, \sigma, \tau, 0, 0, 0) = \tilde{\mathbf{B}}(0, L, \sigma, \tau, 0, 0, 0) = \alpha$, and $\tilde{\mathbf{B}}(1, L, \sigma, \tau, 0, 0, 0) = \tilde{\mathbf{B}}(0, H, \sigma, \tau, 0, 0, 0) = 0$. Note that the last line of (1.39) omits the buyer's action; once the buyer encounters a seller playing soft the price (and so buyer's action) is irrelevant. $\tilde{\mathbf{S}}$ is analogously defined.

Based on those mass functions, I define $F_2 : \Sigma \times \Gamma \to \tau$ which calculates the fractions of agents in a given state playing tough. In the high state the fraction of buyers playing tough is:

$$\begin{aligned}F_2^{BH}(\sigma, \tau) = \quad & (1-\delta)\sum_n\sum_y\sum_m \tilde{\mathbf{B}}(\frac{1}{2}, H, \tau^{SH}, n, y, m)(1 - \sigma(UB, y, m, n)) \\ & +(1-\delta)\sum_n\sum_y\sum_m \tilde{\mathbf{B}}(1, H, \tau^{SH}, n, y, m)(1 - \sigma(BH, y, m, n)).\end{aligned} \tag{1.40}$$

$F_2^{BL}, F_2^{SH}, F_2^{SL}$ are analogously defined. The existence of $F_2(\tau, \sigma)$ and the continuity of F_2 come trivially from the definition.

Step 3: (F_1, F_2) has a fixed point.

Because the function (F_1, F_2) has the closed graph property and $(F_1, F_2)(\tau, \sigma)$ is non-empty and convex, by Kakutani's fixed point theorem it has a fixed point. From this fixed point we can recover the 10-tuple constituting a steady state, this step is simple and omitted. Note that in this steady state, $\tau^{BH} \leq \tau^{BL}$ and $\tau^{SH} \geq \tau^{SL}$.

Step 4: The fixed point is an equilibrium of the original game.

By construction, the steady state satisfies consistency of beliefs and stationarity. Finally we must check optimality. Recall that we restricted the buyers and sellers in this game to waiting a number of periods before playing soft. As beliefs move monotonically in the number of toughs encountered, from Lemma 1 (the single crossing property for beliefs), if an agent does not play soft immediately but plays soft only after encountering some number of toughs, he or she would be willing to play soft after encountering more toughs. Thus the restriction we placed on constructing σ is non-binding, and in constructing the steady state, σ^B and σ^S are optimal.

1.6.4 Proof of Lemma 3: Equivalence of Efficiency and Information Percolation

Proof.

Recall that x_i^w is the proportion of trades taking place at the 'wrong' price in a state w and equilibrium i, and d_i^w is the fraction of gains to trade realized. Let $\mathbf{x^H} \equiv \limsup_{i\to\infty} x_i^H$, $\mathbf{d^H} \equiv \liminf_{i\to\infty} d_i^H$, with $\mathbf{x^L}$ and $\mathbf{d^L}$ analogously defined.

Step 1: Information percolation implies $x^L = 0$.

Suppose information percolates and $\mathbf{x^L} > 0$, and consider the subsequence $\{i'\}$ where $\lim_{i'\to\infty} x_{i'}^L = \mathbf{x^L}$.

In any equilibrium with index i', let $VP_{i'}$ be the average normalized payoff of an uninformed buyer in the high state, and $VN_{i'}$ be the average normalized payoff of an uninformed buyer in the low state. Because we are interested in the payoff of agents when exiting (after matching and bargaining), we cannot use the mass functions directly, which measure populations before matching and bargaining. So we use uninformed buyers at least one period old, of which there are a mass of $\left(\frac{\delta_i(1-\alpha)}{1-\delta_i}\right)$ from (1.9) to look at the payoffs of the uninformed buyers in both states:

$$VP_{i'} = \frac{\left(\sum_{0<\theta<1}\sum_{\mathbf{y}}\sum_{n>0}\mathbf{B}_{i'}(H,\mathbf{y},\theta,n)(\min(k_{i'},\overline{y})v^H + \max(0,\overline{y}-k_{i'})v^L - y^Hp^H - y^Mp^M - y^Lp^L)\right)}{k_{i'}\left(\frac{\delta_{i'}(1-\alpha)}{1-\delta_{i'}}\right)},$$

and

$$VN_{i'} = \left(\sum_{0<\theta<1}\sum_{\mathbf{y}}\sum_{n>0}\mathbf{B}_{i'}(L,\mathbf{y},\theta,n)(\overline{y}v^L - y^Hp^H - y^Mp^M - y^Lp^L)\right) / \left(k_{i'}\frac{\delta_{i'}(1-\alpha)}{1-\delta_{i'}}\right).$$

Then an uninformed buyer on entering in equilibrium i' expects to receive:

$$\frac{V_{i'}^B((0,0,0),\frac{1}{2})}{k} = \frac{VP_{i'}}{2} + \frac{VN_{i'}}{2}.$$

Note that $VN_{i'}$ is bounded below by $(v^L - p^H)$ and for i' sufficiently large above by $\mathbf{x}^L(v^L - p^M)$. VP_i is bounded below by 0 and above by $(v^H - p^L)$. Given these bounds there must be a convergent subsequence of $VP_{i'}$ and $VN_{i'}$. Throw out all elements of this subsequence preventing it from converging, and let VN and VP be the limits.

Let $\overline{\delta} \in (0,1)$ and $\epsilon > 0$ satisfy:

$$\overline{\delta}((1-\overline{\delta})VN + \overline{\delta}^2\mathbf{d^H}VP) - \epsilon > VN + VP,$$

which exists for $\overline{\delta}$ close to 1 and ϵ close to 0 because $VN < 0$ and $VP > 0$.

From information percolation let $\epsilon_1, \epsilon_2 = (1-\overline{\delta})$, so for $i'' > \overline{i'}$, $\delta_{i''} > \overline{\delta}$, $Pr\{\hat{\beta}^B(L, n_{i'}) < (1-\overline{\delta})\} > \overline{\delta}$ and $Pr\{\hat{\beta}^B(H, n_{i'}) > \overline{\delta}\} > \overline{\delta}$.

Take $\{n_{i'}\}$ such that from information percolation, for i' sufficiently large, $(\delta_{i'})^{n_{i'}} > \overline{\delta}$, $Pr\{\tilde{\beta}^B_{i'}(H, n_{i'}) > \overline{\delta}\} > \overline{\delta}$, and $Pr\{\tilde{\beta}^B_{i'}(L, n_{i'}) < (1-\overline{\delta})\} > \overline{\delta}$. I then construct a sequence of strategies $\hat{\sigma}^B_{i'}$ which yields a payoff in expectation exceeding $\overline{\delta}((1-\overline{\delta})VN_{i'} + \overline{\delta}^2 VP_{i'})$, by shifting all soft actions until after $n_{i'}$ periods, and not trading if after $n_{i'}$ periods, his belief θ that the state is high is below $\overline{\delta}$. So $\hat{V}_i^B((0,0,0),\frac{1}{2}) = \overline{\delta}((1-\overline{\delta})\frac{VP_i}{2} + \overline{\delta}\frac{VN_i}{2}) > V_i^B((0,0,0),\frac{1}{2})$, and $\sigma^B_{i'}$ is not optimal, so the assumption $\mathbf{x^L} > 0$ must be false.

Step 2: Information percolation and $x^L = 0$ implies $d^H = 1$.

A buyer can make six types of trades: buyer 'correct' softs at a high price in the high state, buyer 'incorrect' softs at a high price in the low state, seller 'correct' softs at low price in the low state, seller 'incorrect' softs resulting in purchases at a medium price in the high state, and buyer 'incorrect' softs resulting in a medium price in the low state. Combining these for both informed and uninformed buyers, to bound the expected payoff of an uninformed buyer, we get:

$$\begin{aligned}&(1-\alpha)V_i^B((0,0,0),\frac{1}{2}) + \alpha\frac{V_i^B((0,0,0),0) + V_i^B((0,0,0),1)}{2} \\ &\leq k_i x_i^L(v^L - p^M)/2 + k_i d_i^H(v^H - p^H)/2 + k_i x_i^H(p^H - p^L)/2.\end{aligned} \tag{1.41}$$

Because an informed buyer can imitate an uninformed, an uninformed's payoff is weakly less than the average buyer's payoff, so

$$V_i^B((0,0,0),\frac{1}{2}) \leq k_i x_i^L(v^L - p^M) + k_i d_i^H(v^H - p^H) + k_i x_i^H(p^H - p^L). \tag{1.42}$$

Using information percolation, for any $\overline{\delta} < 1$, for i sufficiently large, the payoff of an uniformed buyer is at least

$$V_i^B((0,0,0),\frac{1}{2}) \geq (1-\overline{\delta})k_i(v^L - p^H)/2 + \overline{\delta}k_i(v^H - p^H)/2, \tag{1.43}$$

since an uninformed buyer can wait n_i periods to learn the state with probability $\overline{\delta}$ and survive to play soft k_i times. Combining (1.42) and (1.43), given $\overline{\delta}$ and for i sufficiently high:

$$\begin{aligned} & x_i^L(v^L - p^M)/2 + d_i^H(v^H - p^H)/2 + x_i^H(p^H - p^L)/2 \\ \geq & \frac{1}{k_i}V_i^B((0,0,0),\frac{1}{2}) \geq (1-\overline{\delta})(v^L - p^H)/2 + \overline{\delta}(v^H - p^H)/2. \end{aligned} \tag{1.44}$$

Since $x_i \to 0$, the upper bound of (1.44) becomes $d_i^H(v^H - p^H)/2$. The lower bound also approaches $(v^H - p^H)/2$, so $d_i^H \to 1$.

Step 3: Efficiency implies information percolation.

Suppose $\lim_{i\to\infty} x_i = 0$ and $\lim_{i\to\infty} d_i = 1$, so the sequence is efficient. Let $\overline{\beta}_i^B : \{T,S\}\times\Theta \to \Theta$ be the belief of an uninformed buyer given a set of past trades and time in the market, so $\overline{\beta}_i^B(\mathbf{y}, n)$ is the belief of an uninformed buyer who has met n trading partners and has made the trades $\mathbf{y}$. This is recursively defined using the earlier belief update function:

$$\overline{\beta}_i^B((y^H,y^M,y^L),n) = \beta_i^B(T,\overline{\beta}_i^B((y^H,y^M,y^L),n{-}1)) = \beta_i^B(T,\overline{\beta}_i^B((y^H{-}1,y^M,y^L),n{-}1))$$

and

$$\overline{\beta}_i^B((y^H,y^M,y^L),n) = \beta_i^B(S,\overline{\beta}_i^B((y^H,y^M{-}1,y^L),n{-}1)) = \beta_i^B(S,\overline{\beta}_i^B((y^H,y^M,y^L{-}1),n{-}1))$$

with the base case at the prior: $\overline{\beta}^B((0,0,0),0) = \frac{1}{2}$.

Take any $\epsilon_1 > 0$. Consider an equilibrium, and let n_i be the minimum number of periods after which at least $(1-\epsilon_1)$ of the uninformed buyers have played soft for at least $\frac{k_i}{2}$ periods in the high state. From $d_i \to 1$ and the definition of d in (1.12), $(\delta_i)^{n_i} \to 1$. Let ϵ_i^w be the fraction of uninformed buyers who have played soft at least $\frac{k_i}{2}$ times in state w after n_i periods. From $x_i \to 0$, $\lim_{i\to\infty} \epsilon_i^L \to 0$, and from the definition of ϵ_1, the limit of agents who have played soft at least $\frac{k_i}{2}$ times in the high state converges to $(1-\epsilon)$. Because beliefs are consistent, $\operatorname{plim} \tilde{\beta}_i(H, n_i) = 1$ and $\operatorname{plim} \tilde{\beta}_i(L, n_i) = 0$. Thus information percolates.

□

1.6.5 Proof of Lemma 4: Uninformed do not Trade 'Badly'

Recall that the lemma states

$$\left(\sum_{0<\theta<1} \sum_{\mathbf{y}} \sum_{n} \mathbf{B}(H, \mathbf{y}, \theta, n)\sigma^B(\mathbf{y}, \theta)\right) \geq \left(\sum_{0<\theta<1} \sum_{\mathbf{y}} \sum_{n} \mathbf{B}(L, \mathbf{y}, \theta, n)\sigma^B(\mathbf{y}, \theta)\right),$$

and

$$\left(\sum_{0<\theta<1} \sum_{\mathbf{y}} \sum_{n} \mathbf{S}(H, \mathbf{y}, \theta, n)\sigma^S(\mathbf{y}, \theta)\right) \leq \left(\sum_{0<\theta<1} \sum_{\mathbf{y}} \sum_{n} \mathbf{S}(L, \mathbf{y}, \theta, n)\sigma^S(\mathbf{y}, \theta)\right).$$

Proof. To show how the mass of uninformed buyers playing soft differs between the states, I will look at histories of these buyers.

Consider enumerating the number of periods buyers have been in the market, and the actions of their trading partners over those periods. For the remainder of this proof I will call the ordered list of seller actions a buyer has encountered **histories**, $h \in \mathbb{H}$. If a buyer has been in the market for n periods then there 2^n possible histories. For example, for $n = 2$ there are four histories, which I will denote $\{(TT), (TS), (ST), (SS)\}$. Within and after each of these histories a buyer may play soft a different number of times; we will focus on these number of softs.

Let $e : \mathbb{H} \to \{T, S\}$ be a function giving the last element of a history. Let $\underline{h}$ be the subhistory of h with the last period removed. Let $\beta_H : \mathbb{H} \to \mathbb{R}$ be the belief function for uninformed buyers mapping histories to beliefs. This is defined recursively with $\beta_H(\emptyset) = \frac{1}{2}$. If $h \in \mathbb{H}/\{\emptyset\}$ and $e(h) = T$, then $\beta_H(h) = \beta(T, \beta_H(\underline{h}))$, and if h is a history and $e(h) = S$, then $\beta_H(h) = \beta(S, \beta_H(\underline{h}))$. The purpose of this belief function is to use histories in equilibrium strategies.

Let $\lambda : \mathbb{H} \times \mathbb{N}^3 \to \mathbb{R}$ be a mass function that tells the mass of uninformed buyers in a state with a given history that have a given set of past trades at the end of a period (after entry), conditional on that history. It is recursively defined, with $\lambda(\emptyset, (0, 0, 0)) = 1$. If h is a history and $e(h) = T$, and $\mathbf{y}$ consists of $y^M, y^L, y^H > 0$, then:

$$\lambda(h,\mathbf{y}) = \lambda(\underline{h},(y^H-1,y^M,y^L))\sigma^B((y^H-1,y^M,y^L),\beta_H(\underline{h})) \\ +\lambda(\underline{h},\mathbf{y})(1-\sigma^B(\mathbf{y},\beta_H(\underline{h}))), \tag{1.45}$$

and if $e(H) = S$ then

$$\lambda(h,\mathbf{y}) = \lambda(\underline{h},(y^H,y^M-1,y^L))\sigma^B((y^H,y^M-1,y^L),\beta_H(\underline{h})) \\ +\lambda(\underline{h},(y^H,y^M,y^L-1))(1-\sigma^B((y^H,y^M,y^L-1),\beta_H(\underline{h}))). \tag{1.46}$$

Then by construction $\sum_{\mathbf{y}} \lambda(h,\mathbf{y}) = 1$.

Let $\tilde{h} : \mathbb{H} \to 2^{\mathbb{H}}$ be a function that gives all subhistories of a history, for example $\tilde{h}(TTS) = \{\emptyset,(T),(TT),(TTS)\}$.

Let $s : 2^{\mathbb{H}} \to \mathbb{R}$ be a function in an equilibrium mapping sets of histories to the average number of softs a buyer will have played over the set. This is defined:

$$s(H) = \frac{\sum_{h' \in H} \sum_{h'' \in \tilde{h}(h')} \sum_{\mathbf{y} \in \mathbb{N}^3} \lambda(h'',\mathbf{y})\sigma^B(\mathbf{y},\beta_H(h''))}{|H|}. \tag{1.47}$$

I will use s for sets of histories in which each history is equally likely. Let H_n^u be the set of histories of length n that have u softs (and $n-u$ toughs). Moving from histories containing 0 softs to histories containing n softs is a sequence of sets of histories: $(H_n^0, ..., H_n^n)$. Let $\rho : W \times \mathbb{N} \times \mathbb{N}$ be a function in an equilibrium that maps from state, number of periods and number of softs to the probability that a buyer encounters that many softs over that many periods.

For example, the probability of a history of n periods having u softs in the high state, $\rho(w,n,u)$ is defined:

$$\rho(H,n,u) = \frac{n!}{u!(n-u)!}(\tau^{SH})^{(n-u)}(1-\tau^{SH})^u, \tag{1.48}$$

and

$$\rho(L,n,u) = \frac{n!}{u!(n-u)!}(\tau^{SL})^{(n-u)}(1-\tau^{SL})^u. \tag{1.49}$$

The mass of buyers playing soft in state w is $\left(\sum_{0<\theta<1} \sum_{\mathbf{y}} \sum_{n} \mathbf{B}(w,\mathbf{y},\theta,n)\sigma^B(\mathbf{y},\theta)\right)$.

The mass of softs that uninformed buyers of age n have played is then $\delta^n \sum_{u=0}^{n} s(H_u^n)\rho(w,n,u)$.

The mass played by uninformed buyers of age n in prior periods is then $\delta^n \sum_{u=0}^{n-1} s(H_u^{n-1})\rho(w, n-1, u)$. Combining these two over all periods yields:

$$\sum_{0<\theta<1} \sum_{\mathbf{y}} \sum_{n} \mathbf{B}(w, \mathbf{y}, \theta, n)\sigma^B(\mathbf{y}, \theta) = (1-\alpha) \sum_{n=0}^{\infty} \sum_{u=0}^{n} \rho(w, n, u) s(H_u^n)[\delta^n - \delta^{n+1}]. \tag{1.50}$$

From (1.50) and (1.13), Lemma 4 is then satisfied for buyers if for all n:

$$\sum_{u \leq n} \rho(L, n, u) s(H_u^n) \leq \sum_{u \leq n} \rho(H, u, n) s(H_u^n). \tag{1.51}$$

The proof proceeds in three cases. First the case in which $\tau^{SH} = \tau^{SL}$, and prices are uninformative, then the case in which $\tau^{SH} > \tau^{SL}$, and encountering a seller playing tough increases a buyer's belief that the state is high, and finally $\tau^{SH} < \tau^{SL}$, where only encountering a seller playing soft will increase a buyer's belief that the state is high. In all three cases I will show (1.51) holds true, in the latter two by showing both ρ and $s(H_u^n)$ move in the same direction to show (1.51).

Case 1: If $\tau^{SH} = \tau^{SL}$, then (1.51) is satisfied.

In this case, from (1.8), the steady state masses of uninformed buyers in the high and low states are identical, so the fraction of uninformed buyers playing soft in the high and low states is identical.

Case 2: If $\tau^{SH} > \tau^{SL}$, then (1.51) is satisfied.

This case consists of two steps, showing first that the probabilities ρ move monotonically, and more softs are expected in the low state, then that seeing more toughs leads the buyer play more softs. Intuitively, toughs are more commin in the high state and buyers will play soft more if they encounter more toughs.

Case 2, step 1: for any $n > 0$, for all $u \in \{1, ..., n\}$, $\frac{\rho(L,n,u)}{\rho(L,n,u-1)} > \frac{\rho(H,n,u)}{\rho(H,n,u-1)}$.

This follows directly from the definition of ρ and $\tau^{SH} > \tau^{SL}$. This is the first half of showing (1.51).

Case 2, step 2: If $\tau^{SH} > \tau^{SL}$, then for each $u \in \{1, ..., n\}, s(H_n^u) < s(H_n^{u-1})$.

To show this I will pair each history h with $(u-1)$ softs ($h \in H_n^{u-1}$) to a history h' with u softs ($h' \in h_n^u$), and show that the average[3] number of softs a buyer has played after h is always weakly greater than average number of softs a buyer has played after h'. Because the cardinalities of H_n^{u-1} and H_n^u may differ, the pairing is defined as

[3]If we restricted attention to pure strategies then this average would be unnecessary.

follows. For each $h \in H_n^{u-1}$ pair it with $h' \in H_n^u$ such that h and h' differ in exactly one period and $s(\{h'\})$ is maximized. From all of these paired histories h' construct a set $H_n^{u'}$. Because of the construction of $H_n^{u'}$, $s(H_n^{u'}) > s(H_n^u)$[4].

To compare $s(H_n^{u'})$ and $s(H_n^{u-1})$, take any two matched histories, h' with u and h with $u-1$ softs that differ in the $\bar{t}^{th}$ position, and a deterministic buyer[5]. Let h_t and h'_t be the subhistories after t periods of those histories. Let $\mathbf{y_t}$ and $\mathbf{y'_t}$ be the sets of past trades of the buyer after h_t and h'_t. For $t \leq \bar{t}$, we know $\mathbf{y_t} = \mathbf{y'_t}$ and $\beta_H(h_t) = \beta_H(h'_t)$. Let q_t and q'_t be the number of softs the buyer has played after h_t and h'_t respectively.

Case 2, step 2, subcase 1: The buyer plays soft in period $\bar{t}$.

In this case, $\overline{y}_{\bar{t}} = \overline{y}'_{\bar{t}}$. Suppose in some period t, $q_t = q'_t$, and $\sigma^B(\mathbf{y}', \beta_H(h')) = 1$. Then since $\overline{y}_{\bar{t}} = \overline{y}'_{\bar{t}}$, and $\beta_H(h_t) > \beta_H(h'_t)$, from Lemma 1, $\sigma^B(\mathbf{y}, \beta_H(h)) = 1$. Thus, since $q_{\bar{t}} = q'_{\bar{t}}$, and q'_t never increases beyond q_t, $q_t \geq q'_t$. Finally $q_n = s(h_n)$ and $q'_n = s(h'_n)$.

Case 2, step 2, subcase 2: The buyer plays tough in period $\bar{t}$.

In this case the buyer encountering a seller playing tough in period $\bar{t}$ trades one fewer unit than the buyer encountering a seller playing soft. Again $q_{\bar{t}} = q'_{\bar{t}}$, but $\overline{y}'_{\bar{t}} > \overline{y}_{\bar{t}}$. Suppose in some period t, $u_t = u'_t$, $\overline{y}'_t > \overline{y}_t$, and $\sigma^B(\mathbf{y'_t}, \beta_H(h')) = 1$. Then since $\overline{y}'_t > \overline{y}_t$, and $\beta_H(h_t) > \beta_H(h'_t)$, from Lemma 1 and Lemma 2, $\sigma^B(\mathbf{y_t}, \beta_H(h_t)) = 1$. Thus again q'_t cannot increase beyond q_t.

Since for every matched history, buyers encountering sellers playing soft $u-1$ times will play soft at least as many times as buyers encountering sellers playing soft u times, $s(H_n^u) > s(H_n^{u'})$.

Case 3: If $\tau^{SH} > \tau^{SL}$, then (1.51) is satisfied.

I will prove this case in a way similar to the last case, and use Lemma 2 which states that if $\sigma^B(\mathbf{y}, \theta) > 0$, then for $\mathbf{y}'$ such that $\overline{y}' < \overline{y}$, $\sigma^B(\mathbf{y}', \theta) = 0$. Note that in the high state encountering sellers playing soft is relatively more likely than in the low state, so

$$\forall u \in \{1, ..., n\}, \rho(L, n, u) - \rho(L, n, u-1) > \rho(H, n, u) - \rho(H, n, u-1). \qquad (1.52)$$

I pair histories exactly the same as I did for the case when $\tau^{SL} < \tau^{SH}$. Let us again

[4]There is an issue of replacement in a set. With some abuse of notation, allow $H_n^{u'}$ to allow multiples of the same element.

[5]We will compare the actions of two buyers, when they randomize, we would like them to always randomize in the same way, as if using perfectly correlated random numbers. With some abuse of notation, I suppose $\sigma_H^B \in \{0, 1\}$. If $\sigma_H^B \in (0, 1)$, I consider this as two cases separately.

consider a buyer who encounters two histories that differ only in the $\bar{t}^{th}$ position. Now the buyer that encounters a seller playing soft has a higher belief. As in the earlier case, before the $\bar{t}^{th}$ trade, buyers are identical. Now we must consider the buyer's action in the $\bar{t}^{th}$ period.

Case 3, subcase 1: The buyer plays soft in the $\bar{t}^{th}$ period.

If the buyer plays soft, then the buyer who encounters a seller playing tough has the same number of units to trade as the buyer meeting a seller playing soft. In this case the logic from the earlier case applies directly as well.

Case 3, subcase 2: The buyer plays tough in the $\bar{t}^{th}$ period.

In this case one buyer has a higher belief, and the other buyer has more units left to trade. I will show that because both buyers played tough in the $\bar{t}^{th}$ period, the effect of having a higher belief leads that buyer to play soft weakly more than the buyer with more units left to trade.

Case 3, subcase 2, step 1: If $\tau^{SH} < \tau^{SL}$, then a buyer only plays soft after he encounters a seller playing soft or if he played soft in the prior period.

If a buyer plays T and encounters a T, his belief drops, so this follows directly from Lemma 1. Then any sequence of softs played by the buyer with the higher belief will be also played by the buyer with the higher belief, but earlier. If this sequence overlaps with another sequence, the result is postponed/increased.

Case 3, subcase 2, step 2: If $\tau^{SH} < \tau^{SL}$, then for each $u \in \{1, ..., n\}, \bar{s}(H_n^u) > \bar{s}(H_n^{u-1})$.

The buyer encountering a seller playing tough has more units to trade and a lower belief than the buyer encountering a seller playing soft. Again h and h' differ in the $\bar{t}^{th}$ position, with $e(h_{\bar{t}}) = T$ and $e(h'_{\bar{t}}) = S$.

Let $\eta = \{\eta_j | e(h_{\eta_j}) = S, \bar{t} + j > \bar{t}\}$, so η_1 is the first time after $\bar{t}$ that the buyers encounter a seller playing soft. Furthermore, add $\eta_0 = \bar{t}$ to η.

Let $\xi = \{\xi_1, ..., \xi_{|\eta|}\}$, with $\xi_j = \max\{m | 0 \leq m \leq \eta_{j+1} - \eta_j)$ and $\sigma^B(\mathbf{y}_{\eta_j+m}, \beta_H(h_{\eta_j+m})) = 1\}$. This set contains the lengths of periods after encountering a seller playing S that the buyer with the lower belief plays soft for, until the buyer either stops playing soft or encounters another seller playing soft. Note that $\xi_0 = 0$, as the buyer with the lower belief does not play soft after period $\bar{t}$..

Let ν be the set analogously defined for the buyer with the higher belief as ξ, without cutting off at the next encounter of soft, so $\nu_j = \max\{m | 0 \leq m$ and for all$m' \leq m, \sigma^B(\mathbf{y}'_{\eta_j+m'}, \beta_H(h'_{\eta_j+m'})) = 1\}$. Note that ν_0 may be greater than 0.

Claim:

$$\forall t, 0 \leq t < |\eta| - 1, \quad \nu_{\eta_t} + s(h'_{\eta_t}) \geq \xi_{\eta_{t+1}} + s(h_{\eta_{t+1}}). \tag{1.53}$$

This claim follows directly from Lemmas 1 and 2, on the left the buyer with the greater belief will play soft more than the buyer with the lesser belief on the right, as he has seen more softs. Finally $s(h_n) = \xi_{|\eta|} + s(h_\eta)$, $s(h_n) \leq s(h'_n)$. This combined with the probabilities earlier satisfies (1.51).

$\square$

Chapter 2

Equilibria in a Market with a Front Runner

A front runner uses information about incoming market orders to make risk free profits. The direct effect of a front runner is the diversion of surplus away from traders. In addition, a front runner's existence may affect the bidding strategies of traders who become less willing to reveal trading surplus through their positions. This paper introduces a new model of front running in order to study how front running distorts equilibrium bidding and the implications for overall welfare. The front runner reduces total surplus if traders' values are distributed uniformly and bidding strategies are linear. In general, for all distributions, the front runner prevents the constrained efficient outcome from being an equilibrium.

2.1 Introduction

Front running describes one of several activities in which a market participant uses information about incoming market orders to gain a financial advantage. The goal of this paper is to investigate the impact of a front runner who observes the orders of a buyer and seller, and can place his own orders before the buyer's or seller's reach the market. Holding the orders of the buyer and seller constant, a front runner who extracts maximal surplus will reduce the payoffs of the buyer and seller. This reduction in payoffs exactly equals the front runner's profits, so total surplus is unchanged.

However, the existence of a market front runner will change the behavior of agents, and a different equilibrium will ensue. In this paper I examine a buyer and seller who may trade in a double auction and have common knowledge about the existence of a front runner. Holding the strategy of the seller fixed, if a front runner enters the market, then the buyer will bid lower. Bidding lower will increase his payoff conditional on trading, while decreasing his probability of trading. The seller will similarly raise her bid to increase her payoff conditional on trading.

If the seller increases her bid, the buyer will want to increase his bid to increase his probability of trading. There are then two effects, a primary effect in which agents reduce their chance at trading to increase their payoff from trading, and a secondary effect in which agents react to their trading partner changing their bid. The question we need to ask then is which of these two effects dominates, and how surplus is ultimately affected.

In order to provide intuition I start with an example, specifying the buyer and seller value distributions, as well as the equilibrium. In this example, I find that the first effect dominates, and the front runner reduces total surplus. The reason for this is that when a front runner enters the market, a buyer with a high value and a seller with a low value shade their bid most, while sellers or buyers near the threshold of trading cannot significantly shade their bids. Then the buyer with a very low value must raise his bid to still be able to trade, while the buyer with a very high value does not need to raise his bid. In this way a front runner might flatten the range of bids that the buyer and seller submit, and reduce the surplus generated by a market.

To understand the impact of a front runner generally, we must compare the set of equilibria with and without a front runner. I find that we cannot draw any conclusions from the comparison between these two sets without selecting a pair of equilibria to compare. We would like to compare the equilibria with maximal surplus, so we start by considering the maximum surplus available under any mechanism. I find that the front runner prevents a constrained efficient outcome. The intuition is that the front runner takes surplus from the most profitable trades, which eliminates some less profitable but still surplus-increasing trades. However, the double auction without a front runner is unlikely to have a Bayesian Nash equilibrium that implements the constrained efficient outcome. Thus we cannot compare the maximal surplus equilibria with and without a front runner and attain an unambiguous answer. To compare a particular pair of equilibria with and without a front runner, I characterize monotonic equilibria as a function of the likelihood that a front runner exists. I use this characterization to compare equilibria with and without a front runner that are close, but find again an ambiguous answer.

Thus we must restrict attention to some distributions and equilibrium selection rule to make stronger predictions regarding the impact of a front runner. I restrict attention to linear equilibria and the uniform value distribution. These are the same restrictions as the ones in which the earlier example lives. Into this environment I introduce a market maker, who provides an outside option to the buyer and seller. The market maker lists a price at which he offers to buy a single unit, and a higher price at which he offers to sell a single unit. The buyer and seller can participate in the double auction by submitting bids between the market makers' two prices, but they will trade only if they both do this and the buyer bids a higher price than the seller. If there is a front runner and the buyer trades with the seller, the front runner affects the prices at which they trade. I show that even with the market maker, under the uniform distribution and linear strategies, the front runner strictly reduces surplus.

The paper is structured as follows. In the remainder of Section 2.1, I discuss other work which examines either double auctions or front runners. In Section 2.2, I first present the model and discuss its assumptions, then in Section 2.3 I introduce the main market measures and present some general results. The general results provide a method to construct comparable equilibria with and without a front runner, and also compare the double auction with a front runner to a mechanism that yields the second best outcome. Section 2.5 focuses on the uniform case to compare equilibria with and without a front runner. This section provides stronger results regarding total surplus, showing that a front runner always reduces total surplus. Finally I conclude in Section 2.6.

2.1.1 Related Literature

An understanding of how front runners affect equilibrium strategies can help explain some impacts of high frequency trading. Using infrastructure advantages, high frequency traders can very quickly see the orders in different markets. They can then identify arbitrage opportunities created by different markets and act fast enough to engage in latency arbitrage and make risk-free profits. In this way a high frequency trader engaging in latency arbitrage acts as a market front runner. If, instead of allowing a high frequency trader to take these arbitrages, the surplus created by agents listing different prices is distributed to the agents ultimately trading, the two markets can act as a single market without a front runner.[1] Wah and Wellman (2013) study how total surplus is affected by a latency arbitrageur in a dynamic model, but this is not an equilibrium analysis; in this paper I study how surplus is a one period equilibrium model.

[1]For a description of the role of high frequency traders as market front runners on fragmented markets, see Jones (2013).

Although high frequency trading has made behaving as a front runner easier, front runners are not a recent innovation. For an introduction to front runners, and a series of models addressing their impacts, see Pagano and Roell (1992) or Danthine and Moresi (1998). In their work, they consider the strategies of background traders as exogenous, and focus on the impact of front runners on the volume of background trade.

To endogenize the strategy of the buyer and seller, I consider a modification of the bargaining problem analyzed by Chatterjee and Samuelson (1983): the double auction. In a double auction with one buyer and one seller, the buyer and seller simultaneously submit bids. If the buyer's bid exceeds the seller's, the two agents trade at a price between the two bids, with $k \in [0, 1]$ weight placed on the buyer's bid. In this work I fix $k = \frac{1}{2}$. I then add an outside option for the agents in the form of a market maker.

Myerson and Satterthwaite (1983) discuss the surplus generated from bilateral trade like a double auction, showing the impossibility of an ex post efficient allocation. They discuss the surplus that can be generated by an incentive compatible and individually rational mechanism (e.g., a double auction), and characterize equilibria where this surplus is maximized. They also discuss how a broker (e.g., a market front runner) who extracts or inserts money will affect total surplus.

A central problem of double auction analysis is the multiplicity of equilibria. Equilibria in discontinuous strategies are thoroughly discussed by Leininger et al. (1989), and continuous strategy equilibria by Satterthwaite and Williams (1989). While double auctions do not always maximize ex ante surplus over bargaining games, linear strategy equilibria, if they exist, are shown by Satterthwaite and Williams (1989) to be interim efficient[2] and achieve almost all the surplus available from trade. A general proof of the conditions necessary for efficiency is elusive.

Because a market with and without a front runner will have two different sets of equilibria, to compare the outcomes we must compare the sets. I propose two ways to compare the equilibria between the two markets. First, I look at the uniform distribution and linear strategies. Then I use a characterization of continuous equilibria to compare equilibria between the two markets using a common feature of the two equilibria.

Making assumptions regarding the distribution of values and the strategies in the equilibrium allows us to directly calculate surplus, trading probabilities, and also examine the impact of outside options. The uniform distribution has been used in

[2] This result holds under common assumptions, which appear in Section 2.2.2.

several studies. Under the uniform distribution, linear strategies have been shown to maximize surplus. Leininger et al. (1989) discuss linear strategies in depth. They look at uniform value distributions, but also several other distributions, and find that the equilibrium in linear strategies is very close to realizing all the gains from trade. To quote "[efficiency] varies little over a wide range of priors" Leininger et al. (1989, Page 97). For this reason I use linear strategies to construct equilibria in markets with and without front runners. In this environment I find that the front runner reduces total surplus.

Linear strategies are also supported by experimental evidence. Radner and Schotter (1989) show in an experimental setting linear equilibria are often selected and realize most of the available surplus.

2.2 Model

The model consists of a buyer and a seller with independent private values v_B and v_S in $[0, 1]$ for a single good. The buyer's valuation is drawn from the cumulative distribution function F_b and the seller's value from F_s. Both densities f_S and f_B are positive on $(0, 1)$. Also in the market may be a market maker who offers to buy and sell at two prices, a Bid and Ask, with $Bid, Ask \in [0, 1]$, and $Ask > Bid$. The prices are exogenous and observed by the buyer and seller, and the payoffs of the market maker are neither calculated nor included in total surplus. If $Bid = 0$ and $Ask = 1$, then I will say that the market maker does not exist. A market front runner exists with probability α. The buyer and seller have common knowledge of the market maker's prices and α. The buyer and seller observe their private values, then simultaneously submit bids to the market, the buyer submitting b and the seller submitting s.

Trade is as follows. If $b \geq Ask$ then the buyer buys from the market maker, and receives a payoff $v_B - Ask$. If $s \leq Bid$ then the seller trades with the market maker, and receives a payoff of $Bid - v_S$. If $Ask > b > s > Bid$, then the buyer and seller do not trade with the market maker, but trade with each other. In this case, if there is a market front runner, then both the buyer and seller trade at their bid and offer, and receive payoffs of $v_B - b$ and $s - v_S$ respectively. If there is no front runner, then they trade at a price of $\frac{b+s}{2}$. The buyer receives a payoff of $v_B - \frac{b+s}{2}$, and the seller receives a payoff of $\frac{b+s}{2} - v_S$. If either the buyer or the seller does not trade, he or she receives a payoff of zero.

Total surplus is the ex ante measure of gains to trade realized. Total surplus consists of the payoffs the buyer and seller receive, as well as the surplus the market front runner, if he exists, receives. The buyer and seller are risk neutral and maximize expected profits. I study markets without a market maker in section 2.4.1 and with a market maker in section 2.5. A market with a market maker $\mathbb{M}$ is parameterized by $(Ask, Bid, \alpha, F_B, F_S)$, and a market without a market maker is parameterized by $(1, 0, \alpha, F_B, F_S)$.

2.2.1 Discussion of Model

Several features of the model are worthy of additional discussion. The market maker serves two purposes, both as a robustness check and as a representative of the outside options traders have in the real world. In this sense the market maker reflects either an explicit market maker, or the current best bid and best offer in an order book. The market maker makes no profits due to competition. The exogeneity of the market maker is because an asset's underlying value distribution, and so the market maker's position, is independent of the existence of a front runner.

Of particular interest is how the front runner can come between the buyer and seller. There are two natural interpretations, both matching the model presented here. The first interpretation of the front runner is as an unscrupulous broker, who is an agent who receives the orders of his client, then trades on his own account to extract some of the surplus generated by his client's orders. The second and more recent interpretation is a high frequency trader who is able to engage in latency arbitrage. Latency arbitrage is when the prices of a single stock in different markets (or highly correlated stocks in the same market) differ for a very small period of time, and a high frequency trader takes advantage of the price discrepancies. These opportunities exist because fragmented markets may communicate more slowly than individual traders. While this is not front running, it has the same economics as front running, and can be considered as if front running. In either of these cases a market front runner can extract surplus between a single buyer and seller.

If an unscrupulous broker decides to reform, it is clear that the buyer or seller will extract additional surplus. In the second case, if fragmented markets communicate faster than a high frequency trader can act, then orders submitted to different markets will trade with each other, and more surplus will be received by the buyer or seller.

The likelihood of the front runner existing can be interpreted in several ways. The first is that a front runner exists with some probability, and if he exists he intercedes on every trade. The second, and more natural interpretation is that one or many front runners exist, and intercede with some probability on any trade.

2.2.2 Restriction on Distributions

It is difficult to analyze the equilibria of double auctions for arbitrary distributions. For the remainder of the paper I restrict the distribution of values. Let R and T be the inverse hazard rates:

$$R(v_S) = F_S(v_S)/f_s(v_S), \tag{2.1}$$

$$T(v_B) = (F_B(v_B) - 1)/f_B(v_B). \tag{2.2}$$

Furthermore, let c_S and c_B be the virtual reservation values:

$$c_S(v_S) = v_S + R(v_S), \tag{2.3}$$

$$c_B(v_B) = v_B + T(v_B). \tag{2.4}$$

These are functions that appear in the first-order conditions of the buyer and seller. Consistent with earlier work, I assume that on $(0, 1)$, R and T are C^1, $R > 0$, $T < 0$, and c_S and c_B are increasing.

2.3 Equilibrium

Recall that the buyer and seller have a common belief that a market front runner exists, and common knowledge of the Bid and Ask. Given the Bid, Ask, and belief α, a strategy for a buyer $B : [0, 1] \to [0, 1]$ maps his value v_B to a bid b. Similarly given the Bid, Ask, and α, a strategy for a seller $S : [0, 1] \to [0, 1]$ maps her value v_S to an offer s.

Given the *Bid*, *Ask*, and α, a Bayesian Nash equilibrium consists of the pair (B, S) such that each agent's strategy maximizes his expected payoff conditional on the other agent's strategy.

Given S, let $P_B : [0, 1] \to [0, 1]$ be a function mapping a bid b to the ex-interim probability that a buyer trades with the seller, and let $E_B : [0, 1] \times [0, 1] \to [0, 1]$ be a function mapping the buyer's value v_B and a bid b to the buyer's expected payoff conditional on trading with the seller. Given B, P_S and E_S are defined analogously: P_S maps the seller's offer s to a probability of trade, and E_S maps the offer and value to an expected payoff conditional on trading.

2.3.1 Additional Definitions

I next provide several definitions to aid our understanding of equilibria.

Definition 5 (Regular Equilibrium)**.** *A* **regular equilibrium** *is a pair of strategies* (B, S) *that constitute a Bayesian Nash equilibrium, such that:*

1. *both* B *and* S *are strictly increasing except where* $B(v_B) \geq Ask$ *and* $S(v_S) \leq Bid$*,*

2. B *is differentiable where* $P_B(B(v_B)) > 0$ *and* S *is differentiable where* $P_S(S(v_S)) > 0$*,*

3. *for any* v*, if* $P_B(B(v)) = 0$ *then* $B(v) = v$*, and if* $P_S(S(v)) = 0$ *then* $S(v) = v$*.*

The purpose of using regular equilibria is two-fold. First differentiable equilibria are easier to analyze. Secondly, we can rule out equilibria like the no-trade equilibrium in which $B(v_B) = 0$ and $S(v_S) = 1$, by forcing agents who would never trade to bid their values.

Because strategies in a regular equilibrium are strictly increasing, we can define inverse functions of the strategies, which map bids to values. Let $\sigma_B : [0, 1] \to [0, 1]$ and $\sigma_S : [0, 1] \to [0, 1]$ be the inverse strategies, so $\sigma_B(b) = v_B$ if $B(v_B) = b$, and $\sigma_S(s) = v_S$ if $S(v_S) = s$.

Definition 6 (Total Surplus)**.** *Given a market* $\mathbb{M} = (Ask, Bid, \alpha, F_B, F_S)$ *and an equilibrium* (B, S) *of that market, the expected* **total surplus** $TS(B, S)$ *of the equilibrium is:*

$$TS(B,S) = \int_{v_B=0}^{1} \int_{v_S=0}^{1} (v_B - v_S) p_{DA}(v_B, v_S) f_S(v_S) f_B(v_B) dv_S dv_B \tag{2.5}$$
$$+ \int_{v_S=0}^{S^{-1}(Bid)} (Bid - v_S) f_S(v_S) dv_S + \int_{v_B=B^{-1}(Ask)}^{1} (v_B - Ask) f_B(v_B) dv_B.$$

Because the front runner extracts the surplus between the buyer's bid and the seller's offer, he neither creates nor destroys surplus, so the expected surplus of a market depends only on the strategies of the buyer and seller.

Definition 7 (Trading Boundary)**.** *Suppose* (B, S) *is a regular equilibrium of a market. Let* $TB = \{(v_B, b, v_S) | B(v_B) = b = S(v_S)\}$*. Then* TB *is the trading boundary.*

The trading boundary defines the values of the buyer and seller which trade. The trading boundary can be used to compare the surplus of two equilibria.

Lemma 8 (Greater trading boundary yields greater surplus)**.** *Consider two markets* $\mathbb{M} = (1, 0, \alpha, F_B, F_S)$ $\mathbb{M}' = (1, 0, \alpha', F_B, F_S)$*. Let* (B, S) *be a regular equilibrium of*

$\mathbb{M}$ *and* (B', S') *be a regular equilibrium of* $\mathbb{M}'$. *Suppose* TB *is the trading boundary corresponding to* (B, S) *and* TB' *is the trading boundary corresponding to* (B', S').

If for each $(v_B, v_S) \in TB$, *there is a* $(v_B, v'_S) \in TB'$ *such that* $v'_S \leq v_S$, *and for some* $(v_B, v_S) \in TB$ *there is a* $(v_B, v'_S) \in TB'$ *such that* $v'_S < v_S$, *then* $TS(B, S) > TS(B', S')$.

The proof is easy and omitted. Note that if one equilibrium yields a greater surplus than another, the trading boundary corresponding to the greater surplus need not include the other trading boundary.

2.3.1.1 Example: Uniform Distribution and Linear Strategies

In order to better understand the results, I present two equilibria of two markets without market makers, one with a front runner ($\alpha = 1$), and one without ($\alpha = 0$), in which the buyer and seller value distributions are uniform. I present the total surplus and trading boundaries for this example, and will later use one or both of these equilibria to demonstrate results regarding trading ranges and total surplus.

Without a market maker, the linear strategy equilibrium discussed by Chatterjee and Samuelson (1983) is an equilibrium here. This is:

$$B(v_B) = \frac{2}{3}v_B + \frac{1}{12}, \text{ and}$$
$$S(v_S) = \frac{2}{3}v_S + \frac{1}{4}.$$

The total surplus is $\frac{9}{64}$, and the trading boundary is defined by $\{(v_B, v_S) | v_B = v_S + \frac{1}{4}\}$.

With a front runner, the linear strategy equilibrium is:

$$B(v_B) = \frac{1}{2}v_B + \frac{1}{6}, \text{ and}$$
$$S(v_S) = \frac{1}{2}v_S + \frac{1}{3}.$$

The total surplus is $\frac{9}{64}$ and the trading boundary is defined by $\{(v_B, v_S) | v_B = v_S + \frac{1}{3}\}$. The front runner receives a profit of $\frac{2}{81}$.[3] Note that this trading boundary is within the trading boundary from the equilibrium without a front runner. Next, several results without a market maker are provided.

[3]While neither of the above is a regular equilibrium, they can easily be adjusted to regular equilibria if the low-value buyers and high-value sellers, those outside the trading ranges, bid their values.

2.4 Results without a Market Maker

We would like to compare the surplus generated by the double auction without a front runner to the surplus generated by the double auction with a front runner. To do this we first must select a pair of equilibria. Because an equilibrium with no trade always exists with or without a front runner, some equilibrium with a front runner will always yield greater surplus than some equilibrium without a front runner, and vice versa. A starting point then is to examine the best, or surplus maximizing, equilibrium with or without a front runner. Although we cannot compare these directly, we can compare each to an equilibrium of a mechanism which generates the maximum surplus.

2.4.1 Mechanisms

In order to examine the efficiency of an equilibrium in a market with a front runner, we must consider the broader space of mechanisms. Let $p : [0,1]^2 \to [0,1]$ be a trading rule, mapping buyer and seller reported values to a likelihood of trade, and let $x : [0,1]^2 \to [0,1]$ be the transfer rule. A mechanism is defined by a trading rule and a transfer rule. Conditional on x then, the seller with value v_S and reported type v'_S has an interim expected payoff of $U_S(v_S, v'_S, p, x) = \int_{v_B=0}^{1} (x(v_B, v'_S) - v_S p(v_B, v'_S))\, dv_B$, and the buyer with type v_B who reports type v'_B has an interim expected payoff of $U_B(v_B, v'_B, p, x) = \int_{v_S=0}^{1} (v_B p(v'_B, v_S) - x(v'_B, v_S))\, dv_S$.

A mechanism (p, x) is incentive compatible if given a v_S, for all v'_S, $U_S(v_S, v_S, p, x) \geq U_S(v_S, v'_S, p, x)$. And a mechanism is individually rational if for all v_S, $U_S(v_S, v_S, p, x) \geq 0$.

The purpose of this additional notation is to compare the double auction to other mechanisms through the revelation principle. In particular I will compare the maximal surplus generated by the double auction with the maximal surplus generated by any incentive compatible and individually rational mechanism in Theorem 5.

Given B and S, let $p_{DA} : [0,1]^2 \to \{0,1\}$ be the trading rule for the double auction, so

$$p_{DA}(v_B, v_S) = \begin{cases} 1 & : Bid > B(v_B) \geq S(v_S) > Ask \\ 0 & : \text{otherwise.} \end{cases} \tag{2.6}$$

Let

$$\Gamma(p) = \int_{v_B=0}^{1} \int_{v_S=0}^{1} (c_B(v_B) - c_S(v_S)) p(v_B, v_S) f_S(v_S) f_B(v_B) dv_S dv_B. \tag{2.7}$$

This function provides access to mechanism design results from Myerson and Satterthwaite (1983) and Satterthwaite and Williams (1989) regarding the efficiency and total surplus generated by a mechanism.

Theorem 5 (A front runner precludes maximal total surplus)**.** *Consider a market $\mathbb{M} = (1, 0, \alpha, F_B, F_S)$ with $\alpha > 0$, so that there is no market maker, and a front runner acts with some positive probability. Then for any equilibrium (B, S) of $\mathbb{M}$, there is another mechanism that is incentive compatible and individually rational that yields a greater total surplus.*

The outline of the proof is as follows. First we use the equilibrium with the front runner to construct a mechanism using Myerson and Satterthwaite (1983). Then we can use necessary conditions from Satterthwaite and Williams (1989, Theorem 2.2) to show that that this equilibrium is not ex ante efficient. Finally, since it is not ex ante efficient there must be another mechanism yielding greater surplus.

Proof. Take an equilibrium with a front runner (B, S).

Then the p_{DA} corresponding to the double auction with the front runner is:

$$p_{DA}(v_B, v_S) = \begin{cases} 1 & : B(v_B) \geq S(v_S) \\ 0 & : B(v_B) < S(v_S) \end{cases} \tag{2.8}$$

And the buyer's interim expected payoff when having value v_B and playing as value v_B' is:

$$P_B(B(v_B'))E_B(v_B, B(v_B')) = \int_{v_S=0}^{1} p_{DA}(v_B', v_s)(v_B - B(v_B'))f(v_S)dv_S. \tag{2.9}$$

I use p_{DA} to construct a transfer rule $\overline{x}$ as follows:

$$\begin{aligned} \overline{x}(v_B, v_S) = \quad & p_{DA}(v_B, v_S)\left(\tfrac{B(v_B)+S(v_S)}{2}\right) \\ & + \int_{vs=0}^{1} p_{DA}(v_B, vs)\left(\frac{B(v_B) - S(vs)}{2}\right) f_S(vs)dvs \quad - \int_{vb=0}^{1} p_{DA}(vb, v_S)\left(\frac{B(vb) - S(v_S)}{2}\right) f_B(vb)dvb \end{aligned} \tag{2.10}$$

From [2.8-2.10], a buyer with value v_B under the new transfer rule has interim expected value:

$$\begin{aligned} U_B(v_B, v_B', p_{DA}, \overline{x}) = & \int_{vs=0}^{1} p_{DA}(v_B', v_S)v_B f_S(v_S)dv_S - \int_{vs=0}^{1} p_{DA}(v_B', v_S)\left(\frac{B(v_B') + S(v_S)}{2}\right)f_S(v_S)dv_S \\ - \int_{v_S=0}^{1}\int_{vs=0}^{1} & p_{DA}(v_B', vs)\left(\frac{B(v_B') - S(vs)}{2}\right) f_S(vs)f_S(v_S)dvsdv_S + \int_{v_S=0}^{1}\int_{vb=0}^{1} p_{DA}(vb, v_S)\left(\frac{B(vb) - S(v_S)}{2}\right) f_B(vb)f_S(v_S)dvbdv_S \end{aligned} \tag{2.11}$$

Simplifying [2.11], the buyer's interim payoff is:

$$U_B(v_B, v_B', p_{DA}, \overline{x}) = \int_{v_S=0}^{1} p_{DA}(v_B', v_s)(v_B - B(v_B'))f(v_S)dv_S \tag{2.12}$$
$$+ \int_{vb=0}^{1} \int_{vs=0}^{1} p_{DA}(vb, v_S) \left(\frac{B(vb) - S(v_S)}{2} \right) f_B(vb) f_S(v_S) dvb dv_S.$$

Note that the difference between [2.9] and [2.12], the interim expected payoff of a buyer in the double auction compared to in the constructed mechanism, is half the front runner's expected profits. Because the front runner's profits are never negative, $(p_{DA}, \overline{x})$ constitute an incentive compatible and individually rational mechanism.

There are then two cases: the front runner has expected profits of zero, or the front runner has expected profits greater than zero.

Case 1: The front runner's expected profits exceed zero.

In this case because $\Gamma(p_{DA})$ equals the front runner's profits, from $\Gamma(p_{DA}) = P_B(0) + P_S(1)$ and from Satterthwaite and Williams (1989, Theorem 2.2) the double auction with a front runner must not be ex ante efficient.

Case 2: The front runner's expected profits are zero.

Because strategies must be weakly increasing, for no $v_B < 1$ and $v_S > 0$, can $B(v_B) > S(v_S)$. From Satterthwaite and Williams (1989, Theorem 2.2), if (p, x) is ex ante efficient then there must be scalars s, t, such that:

$$p_{DA}(v_B, v_S) = \begin{cases} 1 & : v_B + sT(v_B) \geq v_S + rR(v_S) \\ 0 & : v_B + sT(v_B) < v_S + rR(v_S). \end{cases} \tag{2.13}$$

But recall that $T(1) = 0$ and $R(0) = 0$, and $R > 0$ and $T < 0$. So p_{DA} cannot satisfy [2.13], and the equilibrium with a front runner must not be ex ante efficient.

□

From Myerson and Satterthwaite (1983, Theorem 2), a mechanism that maximizes expected gains from trade and also satisfies individual rationality and incentive compatibility must exist. However as Satterthwaite and Williams (1989, Theorem 5.1) show, a surplus-maximizing equilibrium of the $\frac{1}{2}$-double auction need not maximize total surplus over the set of mechanisms. While this result does speak to the surplus of a double auction with a front runner, it cannot directly compare the equilibria in a double auction with a front runner to the equilibria without a front runner.

2.4.2 Close Equilibria

Although we cannot compare the maximal surplus of the double auction with a front runner to that without a front runner, we may be able to compare equilibria between the two sets that are similar, if two equilibria can be classified as similar. In order to make such a comparison I characterize equilibria in the next theorem.

Theorem 6 (Linked differential equations)**.** *Given a market parameterized by* $(1, 0, \alpha, F_B, F_S)$ *(without a market maker), all regular equilibria can be characterized by two differential equations. For* $v_B > S(0)$ *and* $v_S < B(1)$*:*

$$\sigma'_S(b) = \left(\frac{1+\alpha}{2}\right) R(v_S)/(v_B - b), \text{ and} \tag{2.14}$$

$$\sigma'_B(b) = \left(\frac{1+\alpha}{2}\right) T(v_B)/(v_S - b). \tag{2.15}$$

Recall that $v_B = \sigma_B(b)$ *and* $v_S = \sigma_S(b)$*.*

Furthermore, if B *and* S *are strictly increasing differentiable functions satisfying [2.14-2.15] for* $v_B > S(0)$ *and* $v_S < B(1)$*, and if* $B(v_B) = v_B$ *for* $v_B < S(0)$ *and* $S(v_S) = v_S$ *for* $v_S > B(1)$*, then* (B, S) *constitutes an equilibrium. Finally, for any* v_S, v_B, b*, such that* $0 < v_S < b < v_B < 1$*, an equilibrium exists such that* $S(v_S) = b = B(v_B)$*.*

Proof. The proof is identical to the differential equations derived in Satterthwaite and Williams (1989, Theorem 3.2). □

For markets without a market maker, we can use the differential equations to create two similar equilibria with different likelihoods of a front runner. From these two equilibria we then calculate the trading boundary and total surplus, allowing us to compare the outcomes with and without a front runner. It is worth noting that Theorem 6 does not generate closed form equilibria, so equilibria must be constructed numerically.

Comparing equilibria requires us to select both an equilibrium with a front runner and an equilibrium without a front runner. To make these selections we start with any regular equilibrium, with some probability of a front runner. We pick a point on the interior of that trading boundary of that equilibrium, which is $v_B \in (0, 1)$, $v_S \in (0, 1)$ such that $B(v_B) = S(v_S)$. From this point on the trading boundary of an equilibrium without a front runner, we can construct an equilibrium with a different probability of a front runner using Theorem 6. The idea of this mapping is that for some pair of values, v_B and v_S, the buyer and seller submit the same bids, and trade with the same likelihoods. Buyer and seller values close to this point are also submitting bids that are similar between the two equilibria, so we will call these two equilibria close.

Generally I find that increasing the likelihood of the front runner causes strategies to become steeper, and the range of prices at which trade can occur will decrease. This is consistent with the front runner causing strategies to become flatter. However for some distributions and start points this is not true, as in Figure 2.5 in the Appendix.

The trading boundary is much more difficult to interpret. Even for the same pair of value distributions, the starting point on the trading boundary and the price can determine whether increasing the likelihood of the front runner moves the trading boundary inside or outside of the original trading boundary. As a result, the impact on surplus is highly ambiguous using this method. This ambiguity highlights the importance of equilibrium selection.

2.5 Uniform Distribution and Linear Strategies

By applying Theorem 6 numerically, one can demonstrate that introducing a front runner can actually increase total surplus. If we know the distribution of buyer and seller values and apply a selection rule, then we can calculate the equilibria and total surplus directly. For the remainder of the paper I assume the uniform distribution, that is $F_B(v_B) = v_B$ and $F_S(v_S) = v_S$. Note that from Chatterjee and Samuelson (1983) we know that the double auction yields the second best surplus, and so from Theorem 5 the linear equilibrium without a front runner yields a greater total surplus than any equilibrium with a front runner.

From the starting point of the linear equilibrium, which yields the second best outcome without a front runner, I will reintroduce the market maker. I then define a linear strategy with a threshold which will allow us to compare the outcomes with and without a front runner directly.

Definition 8 (Linear threshold strategy)**.** *A* **linear threshold strategy** *is one of the following:*

1. *A strategy B parameterized by $\beta_0, \beta_1, \overline{v_B}$, with $\overline{v_B} \in (0,1)$, such that $B(v_B) = \beta_0 + \beta_1 v_B$ for $v_B < \overline{v_B}$ and $B(v_B) = Ask$ for $v_B \geq \overline{v_B}$.*

2. *A strategy S parameterized by $\eta_0, \eta_1, \overline{v_S}$, with $\overline{v_S} \in (0,1)$, such that $S(v_S) = \eta_0 + \eta_1 v_S$ for $v_S < \overline{v_S}$ and $S(v_S) = Bid$ for $v_S \geq \overline{v_S}$.*

	Eqm strategies without a front runner:			Eqm strategies with a front runner:
β_0^0	$\frac{1}{54}(-4+9(Bid+Ask)+2\sqrt{4-9(Ask-Bid)})$		β_0^1	$\frac{1}{32}(-3+8(Bid+Ask)+\sqrt{9-16(Ask-Bid)})$
β_1^0	$\frac{2}{3}$		β_1^1	$\frac{1}{2}$
η_0^0	$\frac{1}{54}(4+9(Bid+Ask)-2\sqrt{4-9(Ask-Bid)})$		η_0^1	$\frac{1}{32}(3+8(Bid+Ask)-\sqrt{9-16(Ask-Bid)})$
η_1^0	$\frac{2}{3}$		η_1^1	$\frac{1}{2}$
$\overline{v_B^0}$	$\frac{Bid+Ask}{2}+\frac{4}{9}-\sqrt{\frac{16}{81}-\frac{4}{9}(Ask-Bid)}$		$\overline{v_B^1}$	$\frac{1}{16}(9+8(Ask+Bid)-3\sqrt{9-16(Ask-Bid)})$
$\overline{v_S^0}$	$\frac{Bid+Ask}{2}-\frac{4}{9}+\sqrt{\frac{16}{81}-\frac{4}{9}(Ask-Bid)}$		$\overline{v_S^1}$	$\frac{1}{16}(-9+8(Bid+Ask)+3\sqrt{9-16(Ask-Bid)})$

Figure 2.1: Strategies of linear threshold equilibria without a front runner on the left and with a front runner on the right.

Definition 9 (Linear threshold equilibria)**.** *An equilibrium of a market* $\mathbb{M}$ (B, S, α) *is a* **linear threshold equilibrium** *if both* B *and* S *are linear threshold strategies.*

Note that for some (Bid, Ask, α), a linear threshold equilibrium does not exist. For example, if $Bid = 0$ then there cannot exist a strategy S such that for some $\overline{v_S}$ it is optimal for a seller with value $v_S < \overline{v_S}$ to bid Bid.

2.5.1 A Deterministic Front Runner

First I solve for the two linear threshold equilibria with $\alpha \in \{0, 1\}$ as a function of Ask and Bid. These are found in Figure 2.1. Note that the equilibrium strategies described in Figure 2.1 do not constitute an equilibrium if $Bid = 0$ and $Ask = 1$; in this case no buyer or seller would ever trade with the market maker. This is reflected by invalid strategy parameters (e.g., $\overline{v_B} > 1$), and there are no linear threshold equilibria. The next section discusses this issue.

2.5.1.1 Equilibrium Existence

As discussed earlier, if $Bid = 0$ and $Ask = 1$, a linear threshold equilibrium does not exist. The space of parameters for which linear threshold equilibria exist is:

$$\begin{aligned} &\left(Ask \leq \frac{2}{3} \text{ and } 4\sqrt{16-18Ask}+9(Ask+Bid) > 16\right), \\ \text{or } &\left(\frac{4}{3}\sqrt{2-2Ask}+Ask+Bid > 2 \text{ and } Ask > \frac{7}{9}\right), \\ \text{or } &\left(Ask > \frac{2}{3} \text{ and } Ask < \frac{7}{9} \text{ and } Ask \leq \frac{4}{9}+Bid\right). \end{aligned} \tag{2.16}$$

The space of equilibrium existence is displayed in Figure 2.2 and is found in the Appendix.

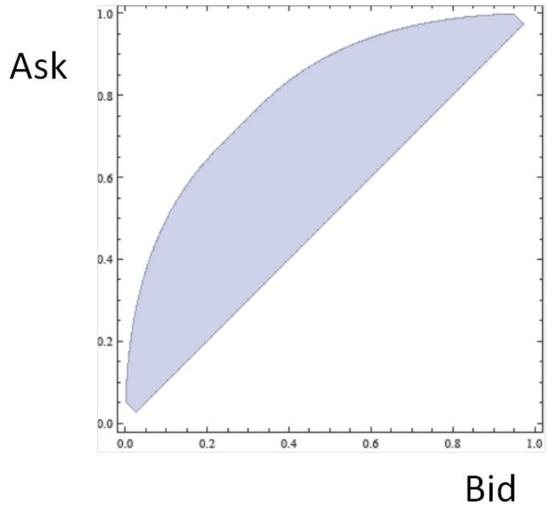

Figure 2.2: Parameter space for linear threshold equilibrium existence.

2.5.2 Uniqueness of Linear Threshold Equilibria

Lemma 9 (Uniqueness of linear threshold equilibria)**.** *If Bid and Ask satisfy the inequalities in (2.16), then there exists a unique linear threshold equilibrium for* $\alpha = 0$ *and a unique linear threshold equilibrium for* $\alpha = 1$*.*

For either $\alpha = 1$ or $\alpha = 0$, the proof follows directly from the derivation of the equilibrium strategies in the appendix. Equilibrium existence comes automatically. If the seller follows a linear threshold strategy, then there is a unique best response from the buyer's first-order condition. The two first order-conditions yield a system of equations with two solutions. One of the solutions corresponds to a case in which either $\overline{v_S} < 0$ or $\overline{v_B} > 1$ or both, which is not an equilibrium, so there is at most one equilibrium.

2.5.3 Total surplus

As discussed earlier, surplus consists of the payoffs the buyer and seller receive, as well as the surplus the market front runner, if he exists, receives. The market maker is assumed to make no profits. In calculating total surplus, recall that the buyer's and seller's private values are distributed uniformly in $[0, 1]$.

Theorem 7 (A Market Front Runner Decreases Total Surplus)**.** *For any* (Bid, Ask) *satisfying [2.16],* $TS^0 > TS^1$*, the total surplus of the linear threshold equilibrium with* *a front runner is strictly less than the total surplus of the linear threshold equilibrium without a front runner.*

The proof comes directly from (2.44) in the appendix, for no market maker parameters Bid and Ask is $TS^0 - TS^1$ negative.

2.5.4 An Example of an Equilibrium with and without a Front Runner

For example, if $Bid = 0.3$ and $Ask = 0.7$, the equilibrium strategies with and without a front runner are as follows:

$$B^0(v_B) = 0.12 + \frac{2}{3}v_B \quad \overline{v_B^0} = 0.804 \tag{2.17}$$

$$S^0(v_S) = 0.22 + \frac{2}{3}v_S \quad \overline{v_S^0} = 0.196 \tag{2.18}$$

$$B^1(v_B) = 0.21 + \frac{1}{2}v_B \quad \overline{v_B^1} = 0.766 \tag{2.19}$$

$$S^1(v_S) = 0.29 + \frac{1}{2}v_S \quad \overline{v_S^1} = 0.234 \tag{2.20}$$

The resulting trades for different values of v_B and v_S are depicted in Figure 2.3. As can be seen in that figure, the region of surplus from trade is strictly greater when there is not a front runner. Note that a buyer with a low value bids more in the equilibrium with a front runner than the equilibrium without a front runner. This is because he needs to increase his probability of trade, since in the equilibrium without a front runner there are sellers making much lower offers. For higher values, buyers in an equilibrium with a front runner bid lower in order to maximize their profits. This demonstrates the condensing effect a front runner has on the bids.

2.5.5 Probabilistic Front Runner

In this section we examine the equilibria when there is a front runner with probability $\alpha \in (0, 1)$. The derivation of the strategies in Figure 2.4 is found in the appendix.

After finding the strategies, we find the parameter values Bid and Ask for which a linear threshold equilibrium exists for any α, and find the range to be identical to the range for the deterministic front runner. We then apply strategies from Figure 2.4 to find total surplus in the appendix.

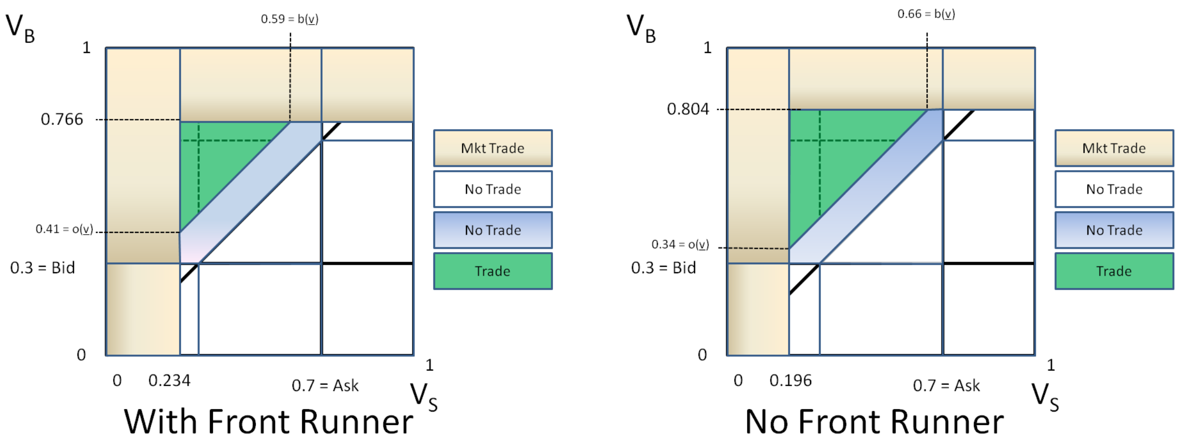

Figure 2.3: A graph of the trade that takes place for different buyer and seller values when there is and is not a front runner. The regions of no trade, trade with the market maker, and trade between the agents are labeled. The difficulty of making generalizations about the linear threshold equilibrium is apparent.

β_0	$\frac{(\alpha+1)\left(2\left(\sqrt{(\alpha+2)^2\left(-(\alpha+3)^2\text{Ask}+(\alpha+3)^2\text{Bid}+(\alpha+2)^2\right)}+9\text{Ask}+9\text{Bid}-4\right)+\alpha(\alpha((\alpha+8)\text{Ask}+(\alpha+8)\text{Bid}-2)+21\text{Ask}+21\text{Bid}-8)\right)}{2(\alpha+2)(\alpha+3)^3}$
β_1	$\frac{2}{3+\alpha}$
η_0	$\frac{(\alpha+1)\left(\alpha(\alpha((\alpha+8)\text{Ask}+(\alpha+8)\text{Bid}+2)+21\text{Ask}+21\text{Bid}+8)-2\left(\sqrt{(\alpha+2)^2\left(-(\alpha+3)^2\text{Ask}+(\alpha+3)^2\text{Bid}+(\alpha+2)^2\right)}-9\text{Ask}-9\text{Bid}-4\right)\right)}{2(\alpha+2)(\alpha+3)^3}$
η_1	$\frac{2}{3+\alpha}$
$\overline{v_S}$	$\frac{\sqrt{(\alpha+2)^2\left(-(\alpha+3)^2\text{Ask}+(\alpha+3)^2\text{Bid}+(\alpha+2)^2\right)}}{(\alpha+3)^2}+\frac{2}{\alpha+3}-\frac{1}{(\alpha+3)^2}+\frac{1}{2}(\text{Ask}+\text{Bid}-2)$
$\overline{v_B}$	$-\frac{\sqrt{(\alpha+2)^2\left(-(\alpha+3)^2\text{Ask}+(\alpha+3)^2\text{Bid}+(\alpha+2)^2\right)}}{(\alpha+3)^2}-\frac{2}{\alpha+3}+\frac{1}{(\alpha+3)^2}+\frac{1}{2}(\text{Ask}+\text{Bid}+2)$

Figure 2.4: Equilibrium linear threshold strategies with a probabilistic front runner. These match the strategies in Figure 2.1 for $\alpha \in \{0, 1\}$.

Theorem 8. *For Bid, Ask such that a linear threshold equilibrium exists for $\alpha \in (0,1)$, $\frac{dTS^\alpha}{d\alpha} < 0$.*

The proof follows directly from the fact that in the range where equilibria exist, the value of $\frac{dTS^\alpha}{d\alpha}$

is always strictly negative, except for $\alpha = 0, \frac{2}{9} < Bid < \frac{1}{3}$, and $Ask = \frac{4}{9} + Bid$ at which point $\frac{dTS^\alpha}{d\alpha}$ does not exist.

2.6 Conclusion

A front runner is an agent who has knowledge about the stream of orders coming into a market, and can act on this knowledge to make profit. Past research involving front runners has not allowed agents to strategically respond to the existence of a front runner. In this work I use an equilibrium model to examine how agents' beliefs in a front runner's existence preclude the second best outcome, and in the linear uniform case unambiguously reduce total surplus. It appears that the existence of the front runner affects the market by dampening the bids which traders submit. This affects the probabilities of trade for agents with extreme values in a clear way; for example a buyer with a low value can no longer trade, and a buyer with a high value is less likely to trade. The impact on a buyer or seller with a value near the middle of the trading range is ambiguous.

This research shows that a front runner unambiguously decreases surplus when buyer and seller values are distributed uniformly and the linear strategy equilibrium is selected (with or without a market maker). However this result does not generalize for a different pair of value distributions or a different equilibrium selection rule. Furthermore, there is no obvious comparison between the two sets of equilibria. For a given pair of value distributions, there is always an equilibrium with a front runner that yields a greater surplus than some equilibrium without a front runner; the reverse is also true. This remains true even if we restrict attention to regular equilibria. Also, because it is difficult to find a surplus maximizing equilibrium in this framework, we cannot generally draw conclusions regarding the comparison between the surplus maximizing equilibria with and without a front runner. Finally, while Section 2.4.2 introduced a way to compare equilibria with and without a front runner that are similar, using this approach we also find that introducing the front runner can increase or decrease surplus, depending on the equilibria being selected.

Although the results do suggest that a front runner will decrease total surplus, either when selecting the linear equilibrium or the surplus maximizing equilibrium, they are far from conclusive. In ongoing research, numerical analysis has shown that for some finite support distributions a regular equilibrium[4] with a market front runner can yield a greater surplus than any regular equilibrium without a front runner.

Although the linear equilibria in the uniform case suggests that regulations should restrict agents from behaving as front runners, more research can be done to determine more general conditions under which total surplus is reduced or improved. As a result, it is still too early to understand exactly what impact front running has, and how regulation would affect market outcomes. Future work may impose additional equilibrium selection rules for a more clear answer.

[4] By regular equilibrium I mean that strategies are weakly increasing, and, if an agent has no opportunity for profitable trade, then she bids her value.

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2.7 Appendix

2.7.1 Calculation of Optimal Linear Strategies

I first solve for the values of $\overline{v_B}^{\alpha}, \beta_1^{\alpha}, \beta_0^{\alpha}, \overline{v_S}^{\alpha}, \eta_1^{\alpha}, \eta_0^{\alpha}$, where $\alpha = 1$, and there is a front runner, then $\alpha = 0$, when a front runner does not exist. To solve for those values I use the first-order conditions.

2.7.1.1 Buyer Payoff Maximization

The expected value of a buyer who submits a bid $b < Ask$ is:

$$P_B^{\alpha}(b)E_B^{\alpha}(v_B, b). \tag{2.21}$$

Recall that P_B^{α} gives the probability that a buyer trades with the seller, and E_B^{α} is the expected value of a buyer conditional on trading. Given $S^{\alpha}(v_S) = \eta_1^{\alpha} v_S + \eta_1^{\alpha}$,

$$P_B^{\alpha}(b) = Pr\{b > o > Bid\} = \frac{b - \eta_0^{\alpha}}{\eta_1^{\alpha}} - \overline{v_S^{\alpha}}, \tag{2.22}$$

and

$$E_B^{\alpha}(v_B, b) = \alpha(v_B - b) + (1 - \alpha)\left(v_B - \left(\frac{b + \frac{b + \eta_1^{\alpha}\overline{v_S^{\alpha}} + \eta_0^{\alpha}}{2}}{2}\right)\right). \tag{2.23}$$

From Equations [2.22] and [2.23], the expected payoff of a buyer who submits a bid $b < Ask$ is:

$$P_B^{\alpha}(b)E_B^{\alpha}(v_B, b) = \left(\frac{b - \eta_0^{\alpha}}{\eta_1^{\alpha}} - \overline{v_S^{\alpha}}\right)\left(\alpha(v_B - b) + (1 - \alpha)\left(v_B - \left(\frac{b + \frac{b + \eta_1^{\alpha}\overline{v_S^{\alpha}} + \eta_0^{\alpha}}{2}}{2}\right)\right)\right). \tag{2.24}$$

Differentiating [2.24] yields a first-order condition for $\alpha = 1$ of:

$$b = B^1(v_B) = \frac{v_B}{2} + \frac{\eta_0^1 + \overline{v^1_{S1}}\eta_1^1}{2}, \tag{2.25}$$

and when there is not a front runner ($\alpha = 0$), differentiating 2.24 yields a first-order condition of:

$$b = B^0(v_B) = \frac{2v_B}{3} + \frac{\eta_0^0 + \overline{v_S^0}\eta_1^0}{3}. \tag{2.26}$$

At the threshold, when $v_B = \overline{v_B^\alpha}$, the buyer is indifferent between buying from the market maker and submitting a bid to potentially trade with the seller, so:

$$\overline{v_B^\alpha} - Ask = \left(\frac{B^\alpha(\overline{v_B^\alpha}) - \eta_0^\alpha}{\eta_1^\alpha} - \overline{v_S^\alpha}\right)\left(\alpha(\overline{v_B^\alpha} - B^\alpha(\overline{v_B^\alpha})) + (1-\alpha)\left(\overline{v_B^\alpha} - \left(\frac{B^\alpha(\overline{v_B^\alpha}) + \frac{B^\alpha(\overline{v_B^\alpha}) + \eta_1\overline{v_S} + \eta_0}{2}}{2}\right)\right)\right). \tag{2.27}$$

2.7.1.2 Seller expected value maximization

Setting up the same optimality for the seller's strategy yields an expected payoff for submitting an offer above Bid of:

$$P_S^\alpha E_S^\alpha(v_S, o) = \left(\overline{v_B^\alpha} - \frac{o - \beta_0^\alpha}{\beta_1^\alpha}\right)\left(v_S - \alpha o - (1-\alpha)\left(\frac{o + \frac{o + \beta_1^\alpha\overline{v_B^\alpha} + \beta_0^\alpha}{2}}{2}\right)\right) \tag{2.28}$$

which gives a first-order condition of:

$$o = S^1(v_S) = \frac{v_S}{2} + \frac{\beta_0^1 + \overline{v_B^1}\beta_1^1}{2}, \tag{2.29}$$

and

$$o = S^0(v_S) = \frac{2v_S}{3} + \frac{\beta_0^0 + \overline{v_B^1}\beta_1^0}{3}. \tag{2.30}$$

The seller's threshold condition is:

$$Bid - \overline{v_S^\alpha} = \left(\overline{v_B^\alpha} - \frac{S^\alpha(\overline{v_S^\alpha}) - \beta_0^\alpha}{\beta_1^\alpha}\right)\left(\overline{v_S^\alpha} - \alpha S^\alpha(\overline{v_S^\alpha}) - (1-\alpha)\left(\frac{S^\alpha(\overline{v_S^\alpha}) + \frac{S^\alpha(\overline{v_S^\alpha}) + \beta_1^\alpha\overline{v_B^\alpha} + \beta_0^\alpha}{2}}{2}\right)\right) \tag{2.31}$$

2.7.1.3 Equilibrium strategies

Using Equations [2.25], [2.26], [2.27], [2.29], [2.30], and [2.31], $\overline{v_B}, \beta_1, \beta_0, \overline{v_S}, \eta_1, \eta_0$ are identified as functions of (Bid, Ask, α). These are stated in Figure 2.1. It is easy to verify then that the linear strategies with cutoff types described above constitute an equilibrium.

2.7.2 Space of Equilibrium Existence

With a market front runner, a linear threshold equilibrium exists if $\overline{v_B^1}, \overline{v_S^1} \in (0,1)$. This is true if:

$$0 < Bid \leq \frac{2}{9} \text{ and } Bid < Ask < \frac{3\sqrt{Bid}}{\sqrt{2}} - Bid, \tag{2.32}$$

or if

$$\frac{2}{9} < Bid < 1 \text{ and } Bid < Ask < \frac{3}{4}\sqrt{1+8Bid} - \frac{1}{4}(1+4Bid). \tag{2.33}$$

Without a market front runner, a linear threshold equilibrium exists if $\overline{v_B^0}, \overline{v_S^0} \in (0,1)$. This holds true if:

$$0 < Bid \leq \frac{2}{9} \text{ and } Bid < Ask < \frac{4\sqrt{2Bid}}{3} - Bid, \tag{2.34}$$

or if

$$\frac{2}{9} < Bid < \frac{1}{3} \text{ and } Bid < Ask \leq \frac{1}{9}(4+9Bid), \tag{2.35}$$

or if

$$\frac{1}{3} \leq Bid < 1 \text{ and } Bid < Ask < \frac{1}{9}(2-9Bid) + \frac{4\sqrt{18Bid-2}}{9}. \tag{2.36}$$

The intersection of those two ranges characterizes the parameter space for which both equilibria exist:

$$\begin{gathered} Ask \leq \frac{2}{3} \text{ and } 4\sqrt{16-18Ask} + 9(Ask+Bid) > 16 \\ \text{or } \frac{4}{3}\sqrt{2-2Ask} + Ask + Bid > 2 \text{ and } Ask > \frac{7}{9} \\ \text{or } Ask > \frac{2}{3} \text{ and } Ask < \frac{7}{9} \text{ and } Ask \leq \frac{4}{9} + Bid. \end{gathered} \tag{2.37}$$

An image of the space described by [2.16] is provided in Figure 2.2. The inequalities in [2.16] will be used later to discuss the equilibria.

Recall the buyer and seller expected value from trade between the spread in [2.24] and [2.28], and their threshold conditions in equations [2.27] and [2.31]. Using [2.27] and [2.31], and the first-order conditions from [2.24] and [2.28].

2.7.3 Total Surplus

The expected contribution to total surplus of the buyer trading with the market maker is:

$$\int_{\overline{v_B}}^{1} (v_B - Ask) dv_B = |_{v_B=1}^{v_B=\overline{v_B}} (\frac{v_B^2}{2} - v_B Ask) = (\frac{1}{2} - Ask) - (\frac{\overline{v_B}^2}{2} - \overline{v_B} Ask), \quad (2.38)$$

and the analog for the seller is:

$$\int_{0}^{\overline{v_S}} (Bid - v_S) dv_S = |_{v_S=0}^{v_S=\overline{v_S}} (v_S \cdot Bid - \frac{v_S^2}{2}) = (\overline{v_S} \cdot Bid - \frac{\overline{v_S}^2}{2}). \quad (2.39)$$

The expected contribution to total surplus from trade between the buyer and seller is:

$$\int_{v_B=\eta_1\overline{v_S}+\eta_0}^{v_B=\overline{v_B}} \int_{v_S=\overline{v_S}}^{v_S=(\beta_1 v_B+\beta_0-\eta_0)/\eta_1} (v_B - v_S) dv_B dv_S. \quad (2.40)$$

Combining [2.38], [2.39], and [2.40], we find a total surplus of:

$$\begin{aligned} TS = \quad & \tfrac{1}{6}\Bigg(6\text{Ask}\overline{v_B} - 6\text{Ask} + \quad (2.41) \\ & \tfrac{1}{\beta_1^2}\Bigg((\eta_0 + \eta_1\overline{v_S} - \overline{v_B})\big(3\beta_0^2 + 3\beta_0(\beta_1 - 1)(\eta_0 + \eta_1\overline{v_S} + \overline{v_B}) + \overline{v_S} \\ & (3\beta_1^2(\eta_0 + \overline{v_B}) - 2\beta_1\eta_1(2\eta_0 + \overline{v_B}) + \eta_1(2\eta_0 + \overline{v_B})) \\ & + \overline{v_S}^2 \left(3\beta_1^2(\eta_1 - 1) - 2\beta_1\eta_1^2 + \eta_1^2\right) - (2\beta_1 - 1) \\ & (\eta_0^2 + \eta_0\overline{v_B} + \overline{v_B}^2)\big)\Bigg) + 6\text{Bid}\overline{v_S} - 3\overline{v_B}^2 - 3\overline{v_S}^2 + 3\Bigg). \end{aligned}$$

Substituting in the strategies from Figure 2.1 into [2.41], the expected total surplus in a market with a market front runner is:

$$TS^1 = \frac{1}{32}\left(16 + 8Ask^2 + Ask\left(-23 + 16Bid - 3\sqrt{9 - 16Ask + 16Bid}\right) + Bid\left(-9 + 8Bid + 3\sqrt{9 - 16Ask + 16Bid}\right)\right). \quad (2.42)$$

The same total surplus in a market without a front runner is:

$$\begin{aligned} TS^0 = \frac{1}{1944}\Big(& 486Ask^2 + 9Bid\left(-42 + 54Bid - 23\sqrt{4 - 9(Ask - Bid)}\right) + 4\left(251 + 4\sqrt{4 - 9(Ask - Bid)}\right) \quad (2.43) \\ & + 9Ask\left(-174 + 108Bid + 23\sqrt{4 - 9(Ask - Bid)}\right)\Big). \end{aligned}$$

The difference in total surplus, [2.43] – [2.42] is:

$$TS^0 - TS^1 = \frac{1}{7776}\left(64\left(2 + \sqrt{4 - 9Ask + 9Bid}\right) + 9(Ask - Bid)(-75 + 92\sqrt{4 - 9(Ask - Bid)} + 81\sqrt{9 - 16(Ask - Bid)})\right) \quad (2.44)$$

2.7.4 Total Surplus with a Probablistic Front Runner

$$
\begin{aligned}
TS^{\alpha} = {} & \frac{1}{24}\Bigg(\alpha^2(3\text{Ask} - 3\text{Bid} - 4) + \\
& \frac{\left((\alpha+3)^2(\alpha(\alpha(\alpha(\alpha+8)+18)+48)+69)\text{Bid}+4(\alpha-1)(\alpha(\alpha(\alpha+9)+15)+3)(\alpha+2)^2\right)\sqrt{(\alpha+2)^2\left(-(\alpha+3)^2\text{Ask}+(\alpha+3)^2\text{Bid}+(\alpha+2)^2\right)}}{(\alpha+2)(\alpha+3)^5} \\
& +2(\alpha+2)\left(3(\alpha+3)^5\text{Bid}^2 + 3(\alpha(\alpha(5\alpha+13)-5)-21)(\alpha+3)^2\text{Bid} + 2(\alpha(\alpha(\alpha(\alpha(12\alpha+97)+344)+762)+1104)+753)\right) \\
& +\alpha(3\text{Ask} - 3\text{Bid} + 4) \\
& +\text{Ask}\left(\left(-\frac{48}{(\alpha+3)^3} + \frac{3}{\alpha+2} - 1\right)\sqrt{(\alpha+2)^2\left(-(\alpha+3)^2\text{Ask} + (\alpha+3)^2\text{Bid} + (\alpha+2)^2\right)} - \frac{6(\alpha(\alpha(9\alpha+49)+103)+87)}{(\alpha+3)^3} + 12\text{Bid}\right) + 6\text{Ask}^2\Bigg).
\end{aligned}
\tag{2.45}
$$

Taking the derivative of total surplus with respect to α yields:

$$
\begin{aligned}
\frac{dTS^{\alpha}}{d\alpha} = {} & \frac{1}{24(\alpha+3)^6\sqrt{(\alpha+2)^2\left(-(\alpha+3)^2\text{Ask}+(\alpha+3)^2\text{Bid}+(\alpha+2)^2\right)}} \\
& (\alpha+1)\Big(2\alpha^8(\text{Ask}-\text{Bid}-4)(\text{Ask}-\text{Bid}-1) + \alpha^7\left(46\text{Ask}^2 - \text{Ask}(92\text{Bid}+219) + \text{Bid}(46\text{Bid}+219) + 164\right) \\
& +\alpha^6\left(462\text{Ask}^2 - 3\text{Ask}(308\text{Bid} - 2\kappa + 691) + 3\text{Bid}(154\text{Bid} - 2\kappa + 691) - 8\kappa + 1436\right) \\
& +\alpha^5\left(2598\text{Ask}^2 + \text{Ask}(-5196\text{Bid} + 105\kappa - 10658) + \text{Bid}(2598\text{Bid} - 105\kappa + 10658) - 132\kappa + 6604\right) \\
& +\alpha^4\left(8778\text{Ask}^2 - 33\text{Ask}(532\text{Bid} - 23\kappa + 956) + 33\text{Bid}(266\text{Bid} - 23\kappa + 956) - 876\kappa + 16876\right) \\
& +\alpha^3\left(9\left(-3956\text{AskBid} + \text{Ask}(1978\text{Ask} - 5851) + 1978\text{Bid}^2\right) + 2694\kappa(\text{Ask}-\text{Bid}) + 52659\text{Bid} - 2572\kappa + 23104\right) \\
& +3\alpha^2\left(6750\text{Ask}^2 + \text{Ask}(-13500\text{Bid} + 1512\kappa - 14399) + 6750\text{Bid}^2 - 4(378\text{Bid}+257)\kappa + 14399\text{Bid} + 4384\right) \\
& +3\alpha\left(3510\text{Ask}^2 - 3\text{Ask}(2340\text{Bid} - 297\kappa + 880) + 3\text{Bid}(1170\text{Bid} - 297\kappa + 880) - 32(5\kappa+26)\right) \\
& +12(549\text{Ask} - 549\text{Bid} + 92(\kappa-4)) + 81(\text{Ask}-\text{Bid})(12\text{Ask} - 12\text{Bid} - 5\kappa)\Big),
\end{aligned}
\tag{2.46}
$$

where $\kappa = \sqrt{(2+\alpha)^2\left((2+\alpha)^2 - (3+\alpha)^2 Ask + (3+\alpha)^2 Bid\right)}$.

2.7.5 Numerically Generated Equilibria

Using theorem 6 we compare equilibria with and without a front runner using a common starting point. Under the uniform distribution these are depicted in Figure 2.5.

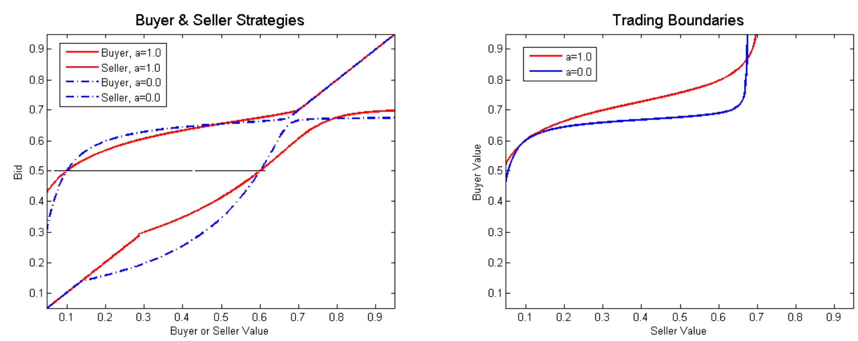

Figure 2.5: In the case where both the buyer and seller have values distributed uniformly, we calculate equilibria with ($\alpha = 1$) and without ($\alpha = 0$) a front runner. We find that strategies without a front runner are generally steeper, except for high buyer and seller values. This causes trades to happen with a front runner that would not happen without a front runner.

Chapter 3

Efficiency in a Non-Stationary Decentralized Market

with Stephan Lauermann

The paper studies the possibility of information aggregation and efficiency in a non-stationary, dynamic matching and bargaining model with strictly positive, non-vanishing frictions. The allocation that results from decentralized trade is shown to approach over time an efficient allocation if the good is divisible but not necessarily if the good is indivisible.

3.1 Introduction

The paper examines trading in a decentralized market with a fixed set of buyers and sellers who are differentially informed about the value and cost of the traded good. For this, we follow the Blouin and Serrano (2001) model of a matching and bargaining game with pairwise meetings, no new entry, and discounting. They show that the equilibrium allocation of their model is not the ex-ante efficient, rational expectation allocation even as discounting is removed, assuming that the good is indivisible and traders can take only one of two bargaining postures. Roughly speaking, this negative result arises because buyers may try to hold out too long insisting on low prices while sellers may try to hold out too long insisting on high prices for the market to clear sufficiently quickly to enable efficient trade.

Golosov et al. (2014) (hereafter GLT) study a variation of the Blouin and Serrano model in which (i) the good is perfectly divisible, (ii) traders have strictly decreasing marginal utility for the good, (iii) bargaining postures (prices) are not restricted, and (iv) instead of discounting there is a probability that the market stops in any period. In this variation, they study the long run limit allocation that results conditional on the event that the market continues to operate in every period and never stops. GLT show that this long-run allocation is an efficient allocation.

Our aim is clarify the relation between these models and their trading outcomes. To this end, we introduce a model that can nest critical elements from both setups. Specifically, we modify Blouin and Serrano (2001) by replacing discounting by a fixed ending probability and assume that the good is divisible into k parts, with $k = 1$ corresponding roughly to their setting and $k \to \infty$ roughly to the GLT setting, albeit with constant marginal valuations. We keep the restriction to two bargaining postures. Similar to GLT, we restrict the class of equilibria that we consider (while they impose a symmetry assumption, we impose a monotonicity condition). We study the properties of the ex-post allocation in our variation, that is, the benchmark suggested by GLT. We show that for $k = 1$ (indivisible good), the long-run allocation is not necessarily efficient. For $k \to \infty$, however, the long-run allocation is shown to be always efficient.

Our analysis suggests that the critical ingredient of the positive result by GLT is the assumption that the good is divisible but not the assumption that prices can be freely adjusted. In particular, the restriction to two bargaining postures in Blouin and Serrano- building on the earlier paper by Wolinsky (1990)- may not be as significant as previously thought.

Our analysis draws heavily on insights developed in previous work by Tablante (2015). Tablante studies a steady-state model similar to Wolinsky (1990), examining the properties of the allocation among entering traders as frictions vanish, akin to the ex-ante benchmark by Blouin and Serrano and identical to Wolinsky's own benchmark. He shows that the divisibility of the good is critical, replicating Wolinsky's negative result when the good is indivisible but showing that the allocation among entering traders converges to the efficient, rational expectations equilibrium as frictions are removed. The current note complements that paper by studying GLT's benchmark of the long-run allocation with trading among a fixed set of buyers and sellers without new entry.

3.2 Model

The market consists of a continuum of buyers and a continuum of sellers seeking to trade a single homogeneous good. A binary state of nature determines both seller costs and buyer values for all agents. There are two states of nature $W = \{H, L\}$. The state H represents the high state and corresponds to high buyer values and high seller costs, and the other state (L) for the low state represents low buyer values and low seller costs.

The game starts in the first period, period 0. The probability of the high state is $\frac{1}{2}$ and the low state is $\frac{1}{2}$. At the start of period 0 there is a continuum of buyers and a continuum of sellers, each with a mass of 1. The common state of nature is decided, and a fraction α of both buyers and sellers know the state. These agents are called informed, and the remaining $1-\alpha$ of agents are uninformed, but have a common prior that the state is high of $\frac{1}{2}$.

The good is broken into $k \in \mathbb{N}$ pieces and buyers and sellers have a constant value and cost for each of the k units. Trade takes place over discrete periods, $t \in \{0, 1, 2, 3, ...\}$ starting from an initial period $t = 0$. The timing for each period is as follows: first buyers and sellers are randomly matched, then each member of each pair simultaneously makes an offer, and if offers overlap in a pair then they trade. Finally, the game ends with probability $\delta \in (0, 1)$. This risk of ending serves as a source of friction which encourages agents to trade.

After matching, buyers and sellers simultaneously choose one of two actions $A = \{T, S\}$. An agent who has completed trading automatically plays tough. A buyer choosing T plays *tough*, and offers to pay no more than a low price p^L for a unit. Conversely, a seller choosing T offers to sell a unit for no less than a high price p^H. A buyer choosing S plays *soft* and offers to pay a high price p^H, and similarly, a seller playing S offers to sell a unit for a low price p^L.

In a pair, if both the buyer and seller play tough, they do not trade. If at least one of the two plays soft, and the other has not completed trading, a unit is traded at a price $p \in \{p^L, p^M, p^H\}$, with $p^L < p^M < p^H$. If the buyer plays soft and the seller tough, then $p = p^H$. If the seller plays soft and the buyer tough, then $p = p^L$. If both agents in a pair play soft then $p = p^M$. If one plays soft, and the other has completed trading, then there is no trade, and the agent playing soft observes a very tough (VT).

To inform their actions, agents know their personal trading histories and starting beliefs. An element of a trading history for time t is $e^t \in \{(T, T), (T, S), (S, T), (S, S), (S, VT), (VT, S)\}$ where the first element represents the action the agent took, and the second element is his trading partner's action in that period. The history of an agent at the end of period t is denoted by $h^t \in \mathbb{H}^t$, where $h^t = (e^0, e^1, e^2, ..., e^t)$, and $\mathbb{H}^t$ is the set of all histories of length t. Let $r(h)$ be the number of times an agent has traded, and $r^p(h)$ be the number of times an agent has traded at a price p. If $r(h) = k$, an agent can no longer trade.

To ensure gains from trade and eliminate a pooling equilibrium price, we assume $v^H > p^H \geq c^H > p^M > v^L \geq p^L > c^L$. In the high state, when the game ends, a buyer with history h receives a payoff of:

$$r(h)v^H - \left(r^H(h)p^H + r^M(h)p^M + r^L(h)p^L\right) \tag{3.1}$$

and the payoff of a buyer in the low state is:

$$r(h)v^L - \left(r^H(h)p^H + r^M(h)p^M + r^L(h)p^L\right). \tag{3.2}$$

The payoff a seller receives in the high state when the game ends is:

$$r^H(h)p^H + r^M(h)p^M + r^L(h)p^L - r(h)c^H \tag{3.3}$$

and the payoff of a seller in the low state is:

$$r^H(h)p^H + r^M(h)p^M + r^L(h)p^L - r(h)c^L. \tag{3.4}$$

3.3 Strategies and Equilibrium

In this section we discuss the interrelated components of an equilibrium.

3.3.1 Strategies

A strategy $\sigma : \mathbb{H} \times \Sigma \to [0,1]$ maps a history and initial belief to a probability of playing soft. There are six strategies of interest, $\sigma^{UB}, \sigma^{US}, \sigma^{BH}, \sigma^{BL}, \sigma^{SH}, \sigma^{SL}$, for the strategies of uninformed buyers, uninformed sellers, and informed buyers [sellers] in the high [low] state. For notation we assume that all agents of the same type follow the same strategy. Because we allow for mixed strategies, this is without loss of generality.

In an equilibrium, mass functions $m : \mathbb{H} \to \mathbb{R}$ tell how many agents of a given type have a given trading history. There are eight mass functions, one for each informed or uninformed buyer or seller, in the high or low state. Then $m^{UBH}(h^t)$ is the mass of uninformed buyers in the high state having history h^t.

3.3.2 Equilibrium

A game is parameterized by $(v^H, v^L, c^H, c^L, p^H, p^M, p^L, \alpha, k, \delta)$. A Perfect Bayesian Nash equilibrium of a game, hereafter equilibrium, is a set of strategies such that agents maximize expected payoffs.

A proof of equilibrium existence has so far been elusive. The central challenge is the involvement of the very tough action. In Wolinsky (1990) and Blouin and Serrano (2001), a strategy can be represented by a single integer or ∞ representing a waiting time. In Tablante (2015), in an isomorphic game, a strategy can be represented by a finite set of integers or ∞. In the current model, an agent can encounter either a tough or a very tough while not trading. These two signals may move an agent's belief in opposing directions and make equilibrium existence much more difficult to show.

3.3.3 Efficiency

There are two questions we seek to address in the model: if the game continues arbitrarily long, do all the potential trades take place, and do all agents learn the state. To answer these questions we define a few benchmarks.

Definition 10 (Ex Post Efficiency)**.** *A sequence of equilibria indexed i is* **ex post efficient** *if:*

1. $\lim_{i\to\infty}\lim_{t\to\infty}\sum_{h^t\in\mathbb{H}^t}\left(\frac{m_i^{UBH}(h^t)r(h^t)+m_i^{IBH}(h^t)r(h^t)}{k_i}\right)=1,$ *and*
2. $\lim_{i\to\infty}\lim_{t\to\infty}\sum_{h^t\in\mathbb{H}^t}\left(\frac{m_i^{UBL}(h^t)r(h^t)+m_i^{IBL}(h^t)r(h^t)}{k_i}\right)=1.$

Ex post efficiency states that all buyers in the high and low states have traded all their units, which implies that all sellers have traded as well.

3.4 Results

This section will demonstrate how information asymmetry can lead to inefficiency, before showing that divisibility of trade eliminates that inefficiency. The next lemma will show that the informed are not the source of inefficiency, and will be useful for later proofs.

3.4.1 Partial Equilibrium Characterization

Lemma 10 (Informed Trade)**.** *In any equilibrium, the following are true:*

1. $\lim_{t\to\infty}\sum_{h^{\underline{t}}\in\mathbb{H}}\frac{m^{IBH}(h^{\underline{t}})r(h^{\underline{t}})}{\alpha k}=1.$

2. $\lim_{t\to\infty} \sum_{h^{\underline{t}}\in\mathbb{H}} \frac{m^{ISL}(h^{\underline{t}})r(h^{\underline{t}})}{\alpha k} = 1.$

Lemma 10 states that in any equilibrium there is some $\underline{t}$ after which almost all of the units of informed buyers [sellers] will have been bought [sold] in the high [low] state. The idea is that there is some period after which uninformed sellers [buyers] will no longer be making a significant fraction of mistakes. Because no more mistakes are made, and due to the discount rate, informed buyers [sellers] will try to trade all their units by playing soft.

It is worth noting that a decreasing fraction of informed agents may be unable to trade due to the possibility of always encountering match partners who have completed trading.

Proof. Let $\overline{s} = \lim_{t\to\infty} \sum_{h^t\in\mathbb{H}^t} \frac{m^{UBL}(h^t)\left(r^M(h^t)+r^H(h^t)\right)}{k}$. This is the fraction of trades uninformed buyers do incorrectly in the low state. Because agents can trade a maximum of k times, $\overline{s}$ is bounded.

Let $\overline{s}^{Lt} = \overline{s} - \sum_{h^t\in\mathbb{H}^t} \frac{m^{UBL}(h^t)\left(r^M(h^t)+r^H(h^t)\right)}{k}$. This is the fraction of trades that have yet to be executed by period t in the low state. By construction, $\lim_{t\to\infty} s^{Lt} = 0$.

Let $C(h)$ denote the continuation value after h. Let d^{Lt} be the probability of encountering a buyer who has completed his trades in period t and so plays very tough. Let s^{Lt} be the probability of encountering a buyer playing soft in a period t. We will compare the value of playing tough and soft.

Let vs be the value of playing soft after some history h, then:

$$vs = s^{Lt}(p^M - c^L + \delta C(h,(S,S))) + (1-d^{Lt}-s^{Lt})(v^H - p^H + \delta C(h,(S,T))) + d^{Lt}\delta C(h,(S,VT)),$$

and the value to playing tough vt is:

$$vt = s^{Lt}(p^H - c^L + \delta C(h,(S,S))) + d^{Lt}\delta C(h,(T,VT)) + (1-d^{Lt}-s^{Lt})\delta(C(h,(T,T))).$$

Because an informed agent's belief does not change, all that matters for her continuation payoff is the number of units left to trade, so $C(h,(T,VT)) = C(h,(S,VT))$. Because $\lim_{t\to\infty} s^{Lt} = 0$, the difference $(vs - vt)$ becomes:

$$\lim_{t\to\infty}(vs - vt) = (1-d^{Lt})(v^H - p^H + \delta C(h,(S,T))) - (1-d^{Lt})\delta C(h,(T,T)) \quad (3.5)$$

The value of a single unit is upper bounded: $C(h,(T,T))-C(h,(S,T)) \leq \overline{s}^{Lt}(v^H - p^L) + (v^H - p^H)$, so that (3.5) is bounded again:

$$\lim_{t\to\infty}(vs - vt) \geq (1 - d^{Lt})(v^H - p^H - \delta(v^H - p^H)) > 0. \tag{3.6}$$

Thus there is some period t after which all informed sellers play soft in the low state. □

Definition 11 (Trade is on Average Ex Post Rational)**.** *In an equilibrium* **trade is on average ex post rational** *if in every period t, the following two are true:*

- $\sum_{h^t \in \mathbb{H}} m^{UBH}(h^t)\sigma^{UB}(h^t) \geq \sum_{h^t \in \mathbb{H}} m^{UBL}(h^t)\sigma^{UB}(h^t)$.
- $\sum_{h^t \in \mathbb{H}} m^{USL}(h^t)\sigma^{US}(h^t) \geq \sum_{h^t \in \mathbb{H}} m^{USH}(h^t)\sigma^{US}(h^t)$.

This is a monotonicity condition which states that the uninformed play soft more correctly than incorrectly, and is analogous to Tablante (2015, Lemma 4). So far the proof that trade is on average ex post rational holds in every equilibrium has been elusive. There are several ways the proof of this might work. One approach would be to try to prove a stronger property. The first step would be to take two histories of the same length h^t and h^t_1 such that $r(h^t) < r(h^t_1)$ and the buyer's belief that the state is higher after h^t than h^t_1, and show that if $\sigma^{UB}(h^t) > 0$ then $\sigma^{UB}(h^t_1) = 1$. The second step would show that encountering a trading partner playing soft always decreases a buyer's belief that the state is high. Both of these steps are confounded by the non-stationary nature of the game. Furthermore, while the first step is intuitive, it is possible that an agent may want to play soft in order to encounter a match partner playing 'very tough' and gain information from that. The second step is made more difficult by the fact that uninformed buyers with a lower belief might be more likely to play soft in a later period than an earlier period.

3.4.2 Granular Trade Precludes Efficiency

If the good is indivisible, then for some parameter settings not all trades will take place. The reason for this is that learning is more difficult when there are fewer trades from which to learn.

Example 1. *Consider the following parameter setting.*

v^H	99		v^L	3
p^H	98		p^L	2
c^H	97		c^L	1
p^M	50		α	0.1
k	1		δ	0.9

Notice that this game is symmetric for buyers and sellers. One equilibrium is as follows: all informed buyers in the high state and informed sellers in the low state play soft in the first period, and uninformed buyers and sellers play tough in every period. Thus trade only happens in the first period, and not all the gains from trade are achieved.

To show that this is an equilibrium we must consider $\sigma^{UB}, \sigma^{US}, \sigma^{BH}, \sigma^{BL}, \sigma^{SH}, \sigma^{SL}$. Clearly the latter four are optimal, but consider σ^{UB} and σ^{US}. After a first period history of (T, S), an agent is done trading, and so plays tough in subsequent periods. After a history of $(T, T), (T, T), ...$ a buyer's [seller's] belief that the state is high [low] is $\frac{0.5}{0.5+0.5\cdot 0.9} \approx 0.53$. Given this belief and the knowledge that no trading partner will play soft, the expected payoff of playing soft is clearly negative. Thus $\sigma^{UB}(\cdot) = 0$ and $\sigma^{US}(\cdot) = 0$ are optimal.

To show that this is the only equilibrium, suppose that another equilibrium exists in which an uninformed agent plays soft in some period. Take the earliest period t in which an uninformed agent (for notation, a buyer) plays soft. Prior to that only informed agents may have played soft, and the likelihood of having encountered one is at most 0.1. The posterior belief that the state is high based on not having encountered a seller playing soft is then at most 0.53. Let s^H be the probability of encountering a seller playing soft in period t if the state is high, and let s^L be the probability of encountering a seller playing soft in period t if the state is low. If the buyer encounters a seller playing very tough, his belief does not change and he faces an isomorphic problem with a new s^H and s^L, so this possibility is omitted. Because the buyer plays soft, the following two inequalities are true:

$$0.53(99 - 50s^H - 98(1 - s^H)) + 0.47(3 - 50s^L - 98(1 - s^L)) \geq 0, \qquad (3.7)$$

and

$$0.53(99-50s^H-98(1-s^H))+0.47(3-50s^L-98(1-s^L)) \geq 0.53s^H(99-2)+0.47s^L(3-2). \qquad (3.8)$$

The first inequality states that the payoff to playing soft exceeds zero, and the second inequality states that the payoff to playing soft exceeds the payoff to playing tough. It is straightforward that (3.7) and (3.8) cannot be satisfied simultaneously, thus no equilibrium in which an uninformed agent plays soft can exist. Since no uninformed will play soft in any equilibrium, informed will play soft in the first period.

3.4.3 Good Divisibility Promotes Trading

If the good is made increasingly divisible, then the problem of an insufficient signal for learning is overcome, and it is possible for all the trade to take place. This is because uninformed agents can learn the true state to take advantage of profitable trades.

Proposition 1. *Any sequence of equilibria indexed i, in which $k_i \to \infty$ and trade is on average ex post rational, is ex post efficient.*

The proof proceeds by contradiction and assumes a subsequence of inefficient equilibria. First we show that informed must play soft for a significant fraction of periods. As k grows large, after many periods uninformed agents will be able to learn from these trades. Finally, because uninformed learn, they will complete all of their trades as well.

Proof. Take any sequence of equilibria indexed i in which trade is on average ex post rational and $k_i \to \infty$. Suppose for some $\epsilon > 0$ there is a subsequence of equilibria indexed i' such that $\forall i', \forall t, \sum_{h^t \in \mathbb{H}^t} \frac{m_{i'}^{USL}(h^t) r(h^t)}{(1-\alpha)k_{i'}} < 1 - \epsilon$.We will restrict attention to this subsequence and show that if such a subsequence exists, uninformed can learn and so trade.

Step 1: $\forall i'$, $\lim_{t\to\infty} \sum_{h^t \in \mathbb{H}^t} \frac{m_{i'}^{ISL}(h^t)\left(r^M(h^t) + r^L(h^t)\right)}{\alpha k_{i'}} \geq \epsilon$.

The first step states that informed sellers play soft for a significant fraction of trades in each equilibrium. This is shown by contradiction. From the continuum of agents, uninformed and informed sellers will encounter buyers playing soft in equal proportions. Thus in any equilibrium and for any t:

$$\sum_{h^t \in \mathbb{H}^t} \frac{m^{USL}(h^t)(r(h^t))}{(1-\alpha)k} \geq \sum_{h^t \in \mathbb{H}^t} \frac{m^{ISL}(h^t)(r^H(h^t))}{\alpha k}. \tag{3.9}$$

The left hand side of (3.9) is the fraction of trades uninformed sellers have made, while the right hand side of (3.9) is the fraction of trades informed sellers have made from buyers playing soft by some period in the low state.

Because the left side of (3.9) is less than $1-\epsilon$, and $\lim_{t\to\infty}\sum_{h^t\in\mathbb{H}^t}\frac{m^{ISL}(h^t)r(h^t)}{\alpha k}=1$ from Lemma 10, $\lim_{t\to\infty}\sum_{h^t\in\mathbb{H}^t}\frac{m_{i'}^{ISL}(h^t)\left(r^M(h^t)+r^L(h^t)\right)}{\alpha k_{i'}}\geq\epsilon$. This provides a lower bound on the signal uninformed buyers can use to learn.

Step 2: Let $\tilde{h}_i^{Ht}$ and $\tilde{h}_i^{Lt}$ be a random variable for the history of an uninformed buyer in the high and low states after t periods in equilibrium i. Let $\beta_i(h)$ be an uninformed buyer's belief after h. Then there is a sequence $\{\underline{t}_i\}$ such that for any sequence $\{t_i\}$ satisfying $t_i>\underline{t}_i$, $\operatorname{plim}_{i\to\infty}\beta_i(\tilde{h}_i^{Ht_i})=1$.

Let $\bar{t}_i$ be the earliest period such that $\sum_{h^{\bar{t}_i}\in\mathbb{H}^{\bar{t}_i}}\frac{m_i^{SL}(h^{\bar{t}_i})(r^L(h^{\bar{t}_i})+r^M(h^{\bar{t}_i}))}{\alpha k_i}\geq\frac{\epsilon}{2}$, which must exist from Step 1 and Lemma 10.

In an equilibrium, let

$$f^L(\bar{t})=\sum_{t=0}^{\bar{t}}\sum_{h^t\in\mathbb{H}^t}\frac{m^{USL}(h^t)\sigma^{US}(h^t)+m^{SL}(h^t)\sigma^{SL}(h^t)}{k},$$

and

$$f^H(\bar{t})=\sum_{t=0}^{\bar{t}}\sum_{h^t\in\mathbb{H}^t}\frac{m^{USH}(h^t)\sigma^{US}(h^t)+m^{SH}(h^t)\sigma^{SH}(h^t)}{k}.$$

These are the mass of softs that have been played in the high and low states up to period t. Recall that because playing soft for a seller is a dominated strategy, $\sigma^{SH}(\cdot)=0$. From Step 1, there is an $\underline{t}$ such that for $\bar{t}>\underline{t}$, informed sellers have traded at a low or medium price at least $\frac{\epsilon}{2}$ times and so have played soft at least $\frac{\epsilon}{2}$ times. From this and the fact that trade is on average ex post rational, $f^L(\bar{t})\geq f^H(\bar{t})+\frac{\epsilon}{2}$. We will now focus on uninformed buyers. Let $w(h)$ be the number of times a buyer encounters a seller playing soft in history h.

Take a sequence $\{t_i\}$ such that $t_i>\underline{t}_i$. Consider the sequence of random variables $w(\tilde{h}_i^{Lt_i})$ and $w(\tilde{h}_i^{Ht_i})$. That is, consider the number of times an uninformed buyer encounters sellers playing soft. In an equilibrium, $w(\tilde{h}_i^{Lt_i})$ and $w(\tilde{h}_i^{Ht_i})$ each follow the Poisson binomial distribution, because in each there are t_i independent draws from t_i Bernoulli distributions. Furthermore, the mean of $\frac{w(\tilde{h}_i^{Lt_i})}{k_i}$ is $f_i^L(t_i)$ which satisfies $0\leq f_i^H(t_i)<f_i^H(t_i)+\frac{\epsilon}{2}\leq f_i^L(t_i)\leq 1$, and the variance is:

$$\sum_{t=0}^{\bar{t}} \sum_{h^t \in \mathbb{H}^t} \frac{\left(m_i^{USL}(h^t)\sigma_i^{US}(h^t) + m_i^{SL}(h^t)\sigma_i^{SL}(h^t)\right)\left(1 - m_i^{USL}(h^t)\sigma_i^{US}(h^t) + m_i^{SL}(h^t)\sigma_i^{SL}(h^t)\right)}{k_i^2}$$

which approaches 0 as $i \to \infty$, so $\operatorname*{plim}_{i\to\infty}\left(\frac{w(\tilde{h}_i^{Lt_i})}{k_i} - f_i^L(t_i)\right) = 0$. Similarly the mean of $\frac{w(\tilde{h}_i^{Ht_i})}{k_i}$ is $f_i^{Ht_i}$ and the variance of $\frac{w(h_i^{Ht_i})}{k_i}$ converges to 0 so that $\operatorname*{plim}_{i\to\infty}\left(\frac{w(\tilde{h}_i^{Ht_i})}{k_i} - f_i^H(t_i)\right) = 0$.

Finally, because $\lim_{i\to\infty} Pr\{\frac{w(\tilde{h}_i^{Ht_i})}{k_i} > f_i^L(t_i) + \frac{\epsilon}{2}\} = 0$ and $\lim_{i\to\infty} Pr\{\frac{w(\tilde{h}_i^{Ht_i})}{k_i} < f_i^L(t_i) + \frac{\epsilon}{2}\} = 1$, $\operatorname*{plim}_{i\to\infty} \beta_i(\tilde{h}_i^{Ht_i}) = 1$.

Step 3: There is a sequence $\{\bar{t}_i\}$ such that:

$$\lim_{i\to\infty}\left(\sum_{h^t\in\mathbb{H}^t|t>\bar{t}_i, r(h^t)<k_i} m_i^{UBH}(h^t)\sigma_i^{UB}(h^t) + \sum_{h^t\in\mathbb{H}^t|t>\bar{t}_i, r(h^t)=k_i} m_i^{UBH}(h^t)\right) = 1 - \alpha.$$

This states that at some point almost all the uninformed will either have completed trading or be playing soft in every period. This will complete the proof of the proposition and is proven by examining the payoff of playing soft relative to the payoff of paying tough.

In some equilibrium i and some period t, let s_i^{Ht} and s_i^{Lt} be the probabilities of encountering a seller playing soft in the high and low states, $\overline{s}_i^{Ht}$ and $\overline{s}_i^{Lt}$ be the remaining mass of softs in the high and low states, and d_i^{Ht} and d_i^{Lt} be the probabilities of encountering a seller playing very tough in the high and low states. Let $C_i(h)$ be the continuation payoff after a history h.

For some history h, if $r(h) < k$ then the payoff to playing tough is:

$$\begin{aligned} vt = {} & \beta_i(h)\left(s_i^{Ht}(v^H - p^L + \delta C_i(h,(T,S))) + (1 - s_i^{Ht})\delta C_i(h,(T,T))\right) \\ & + (1-\beta_i(h))\left(s_i^{Lt}(v^L - p^L + \delta C_i(h,(T,S))) + (1 - s_i^{Lt})\delta C_i(h,(T,T))\right) \end{aligned}$$

and the payoff to playing soft is:

$$\begin{aligned} vs = {} & \beta_i(h)\left(s_i^{Ht}(v^H - p^M + \delta C_i(h,(S,S))) + (1 - s_i^{Ht} - d_i^{Ht})(v^H - p^H + \delta C_i(h,(S,T))) + d_i^{Ht}\delta C_i(h,(S,VT))\right) \\ & + (1-\beta_i(h))\left(s_i^{Lt}(v^L - p^M + \delta C_i(h,(S,S))) + (1 - s_i^{Lt} - d_i^{Lt})(v^L - p^H + \delta C_i(h,(S,S))) + d_i^{Lt}\delta C_i(h,(S,VT))\right). \end{aligned}$$

We suppress the arguments (h, i, t) of vt and vs for clarity. For $t_i > \underline{t}_i$, $\operatorname{plim}_{i\to\infty} \beta_i(\tilde{h}_i^{Ht_i}) = 1$. As $t \to \infty$, $s^H \to 0$, $s^L \to 0$ so that the difference in payoffs becomes:

$$\lim_{i\to\infty}\lim_{t\to\infty}(vs-vt) = (1-d_i^{Ht})(v^H-p^H+\delta C_i(h,(S,T)))+d_i^{Ht}\delta C_i(h,(S,VT))-\delta C_i(h,(T,T)). \tag{3.10}$$

Because both $\operatorname{plim}\beta_i(\tilde{h}_i^{Ht_i},(T,T)) = 1$ and $\operatorname{plim}\beta_i(\tilde{h}_i^{Ht_i},(S,VT)) = 1$, $\operatorname{plim}\Big(C_i(\tilde{h}_i^{Ht_i},(T,T)) - C_i(\tilde{h}_i^{Ht_i},(S,VT))\Big) = 0$[1]. Finally because

$$C_i(h,(T,T))-C_i(h,(S,T)) < \beta_i(h)(\overline{s}_i^{Ht}\frac{\delta}{1-\delta}(v^H-p^L)+(v^H-p^H))+(1-\beta_i(h))\overline{s}_i^{Lt}\frac{\delta}{1-\delta}(v^L-p^L)$$

which converges to $v^H - p^H$ as $\beta_i(h) \to 1$ and $\overline{s}_i^{Ht} \to 0$, we can bound $(vs - vt)$ to find that:

$$\operatorname*{plim}_{i\to\infty,t\to\infty}(vs-vt) \geq (1-d_i^{Ht})(1-\delta)(v^H-p^H) > 0. \tag{3.11}$$

Thus, as $\beta_i(\tilde{h}^{Ht_i})$ converges in probability to 1 and $t \to \infty$ so that the other limits hold, for almost all buyers in the high state either $vs - vt > 0$ or they have completed trading.

The proof is symmetric for sellers in the low state, so that the sequence of equilibria is ex post efficient.

□

Divisibility is important for efficiency because it allows the beliefs of uninformed agents to approach the truth arbitrarily close.

3.5 Conclusion

In this research we have studied information aggregation and efficiency in a decentralized market, and identified weaker conditions are necessary for learning and efficiency than the conditions found in past research. In particular we have provided evidence that the indivisibility of trade is the significant friction which can preclude learning and efficiency, and that flexible prices are not necessary. Furthermore, our research demonstrates that learning can be completely achieved as a passive process, unlike past research which requires agents carefully select bargaining postures in order to learn. While learning may be faster if agents carefully craft offers to investigate

[1] Trivially, C is continuous in the belief. C is bounded between 0 and $\frac{1}{1-\delta}(v^H - p^L)$ and so far any k, h, and h' such that $r(h) = r(h')$, as $\beta_i(h) - \beta_i(h') \to 0$ then $C(h) - C(h') \to 0$.

the state, this research finds that agents can learn by behaving passively and insisting on favorable prices while observing the actions of their match partners.

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