

Multi-Duality in Minimal Surface-Type Problems

By David Yang Gao and Wei H. Yang

The multi-duality of the nonlinear variational problem $\inf J(u, \Lambda u)$ is studied for minimal surfaces-type problems. By using the method developed by Gao and Strang [1], the Fenchel–Rockafellar’s duality theory is generalized to the problems with affine operator Λ . Two dual variational principles are established for nonparametric surfaces with constant mean curvature. We show that for the same primal problem, there may exist different dual problems. The primal problem may or may not possess a solution, whereas each dual problem possesses a unique solution. An evolutionary method for solving the nonlinear optimal-shape design problem is presented with numerical results.

1. Introduction

We are interested in the dual variational problems associated with minimal surface-type equations. Let Ω be a bounded open set of \mathbf{R}^2 , with boundary $\Gamma = \partial\Omega$; we seek a function u in a feasible set \mathcal{U} which satisfies

$$P(u) = \inf_{v \in \mathcal{U}} \int_{\Omega} \sqrt{\alpha + |\nabla v|^2} \, d\Omega. \quad (1)$$

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For different given feasible sets \mathcal{U} and the parameters $\alpha \in \mathbf{R}$, problem (1) yields various minimal surface-type problems:

1.1. *H-surface problems*

Let $\alpha = 1$ and

$$\mathcal{U} = \left\{ u \in W_0^{1,1}(\Omega) \mid \int_{\Omega} u \, d\Omega = \text{const.} \right\},$$

then problem (1) is to find a surface, which is the graph of $u(x, y)$ bounded by the domain Ω , such that volume $\int_{\Omega} u \, d\Omega$ is a given constant. The Euler–Lagrange equation associated with this problem can be given by

$$\nabla \cdot (\nabla u / (1 + |\nabla u|^2)^{1/2}) + \lambda = 0, \quad (2)$$

in which $\lambda \in \mathbf{R}$ is the Lagrange multiplier, depending on the constraint of the constant volume $\int_{\Omega} u \, d\Omega = \text{const.}$ Equation (2) shows that the mean curvature of the surface is a constant over the domain Ω . We call this problem simply the H-surface problem. As primal problems, surfaces with given mean curvature have been studied by many authors (see Miranda [2], Giusti [3], Brakke [4], Huisken [5], and Oliker [6], etc.). In this paper, we study this problem's multi-duality.

1.2. *Surface with obstacle*

If we let

$$\mathcal{U} = \left\{ u \in W_0^{1,1}(\Omega) \mid u(x) \geq \psi(x) \quad \forall x \in \Omega \right\},$$

where the real-valued function ψ is a prescribed concave obstacle function. In this case, problem (1) is equivalent to a variational inequality. Such a problem has been studied by Kinderlehrer [7, 8], Nitsche [9], and Stampacchia [10]). The dual problem for von Karman plate has been studied by Yau and Gao [11]. Their result shows that for a geometrical nonlinear problem, the dual problem is equivalent to a coupled quadratic optimization.

1.3. *Plastic limit analysis*

We consider u as the velocity field in three-dimensional space, $\mathbf{u} = \{u_i\}$ ($i = 1, 2, 3$), and $|\nabla \mathbf{u}|$ is a properly defined norm function of ∇u (see [12]). Let \mathcal{U} be the kinematically admissible space

$$\mathcal{U} = \left\{ \mathbf{u} \in BD(\Omega) \mid \int_{\Gamma_t} \mathbf{f} \mathbf{u} \, d\Gamma = 1, \quad \mathbf{u} = 0 \text{ on } \Gamma_u \right\},$$

where $BD(\Omega)$ is the space of bounded deformation vector functions introduced by Strang and Temam [13]; $\Gamma_t \cup \Gamma_u = \partial\Omega$; $\mathbf{f} : \Gamma \rightarrow \mathbf{R}^3$ is the unit surface

traction. Then problem (1) provides the limit loading factor for rigid-perfect plasticity ($\alpha = 0$):

$$\lambda = \inf_{v \in \mathcal{U}} \int_{\Omega} |\nabla v| \, d\Omega. \quad (3)$$

For the problem of steady plastic flow in pipe, problem (3) can be written as

$$\lambda = \inf_{v \in \mathcal{U}} \int_{\Omega} \sqrt{\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2} \, d\Omega, \quad (4)$$

which gives the collapse pressure factor λ for plastic flow. This is indeed a minimal surface-type problem with $\alpha = 0$. As the primal problem, the variational solution of (3) or (4) gives only the upper bound approach for the limit loading factor λ . However, its dual problem provides the lower bound, which is important for engineering designs. Duality theory for such problems has been studied by, among others, Temam [14], Kohn and Temam [15], Temam and Strang [16], Cesari and Yang [17], Yang [18, 19], Gao [20, 21], and Gao and Strang [22].

The classical minimal surface problems have been studied for more than a century. It is known that the primal problem (1) may or may not possess a solution u . Its dual problem(s), on the other hand, possess a unique solution \mathbf{p}^* (see [23, 24]). The extremality conditions link \mathbf{p}^* to the solution u of the primal problem if it exists. When the primal problem possesses no solution, the dual problems allow us to define the generalized solutions.

Generally speaking, the primal functional for minimal surface-type problems can be written as $P(u) = J(u, \Lambda u)$. If Λ is continuous linear homogeneous operator, the dual variational problem can easily be established by using the well-known Fenchel–Rockafellar theory of duality. For nonparametric minimal hypersurfaces, the dual problem has been studied extensively (see [23]). For the general parametric surface problems, the operator Λ is usually quadratic. In this case, the Fenchel–Rockafellar theory cannot be applied to construct the dual variational problems. By introducing a so-called complementary gap function, a new duality theory for nonlinear operator Λ has been developed by Gao and Strang [1]. Applications of this theory have been given to large deformation elastoplastic analysis [22, 25], finite plastic dynamics [26], and obstacle problems for von Karman equations [11].

The results of our study of the multi-duality in minimal surfaces-type problems show that by choosing variant geometric maps Λ , there may exist different dual problems for the same primal variational problem. If Λ is an affine operator, then there also exists a complementary gap function between the dual variational functional and the conjugate functional of J obtained by using the Legendre–Fenchel transformation. This gap function plays an important role in dual analysis. Our idea and method were motivated by studying the H-surface problem; it turns out they can be applied to other minimal surface-type problems.

In Section 2 we briefly describe the concept of the Riemannian geometry of general minimal surfaces. Following the notation introduced by Ekeland and Temam [23] and Gao and Strang [1], in Section 3 we give a generalized geometric operator, which maps the configuration space to different metric spaces, and we illustrate the duality relations between the dual variables and operators. In Section 4, the dual variational principle for nonparametric H-surface is discussed. This theory has been generalized in Section 5 by using a different geometrical operator. Where we proved that there exists at least two different dual problems for the same primal variational problem. In Section 6, an evolutionary method for solving a nonlinear constrained optimization problem is presented.

2. H-surface and primal problem

Let \mathcal{M} be a two-dimensional Riemannian surface in \mathbf{R}^3 with a nonempty boundary $\partial\mathcal{M}$. We denote by $\{\Omega, X^\alpha\}$ ($\alpha = 1, 2$) the local chart in \mathcal{M} . Then the surface \mathcal{M} can be represented parametrically by the equations

$$x^i = x^i(X^\alpha), \quad i = 1, 2, 3. \quad (5)$$

It is assumed that the map $x(X^\alpha)$ is of class C^2 , which induces a Riemannian metric $\mathbf{C} = \{C_{\alpha\beta}(X)\}$ on \mathcal{M} :

$$\mathbf{C} = \mathbf{D}^t \mathbf{D}, \quad C_{\alpha\beta} = \sum_{i=1}^3 D_\alpha^i D_\beta^i = \langle x_\alpha, x_\beta \rangle, \quad (6)$$

where $\mathbf{D} = \{D_\alpha^i\} = \{\frac{\partial}{\partial X^\alpha} x\}$ is a two-point tensor, and the superscript t denotes transpose of a vector or a tensor. The inverse of \mathbf{C} is denoted by $C^{\alpha\beta} = (C_{\alpha\beta})^{-1}$.

Following the notation given in Ekeland and Temam [23], we designate two real vector spaces, \mathcal{V} and \mathcal{V}^* , placed in duality by a bilinear pairing denoted by $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V}^* \rightarrow \mathbf{R}$. For example, if $\mathcal{V} \subset \mathbf{R}^3$, we simply have

$$\langle \mathbf{v}, \mathbf{w} \rangle = v^i w_i = \delta_{ij} v^i w^j = v_i w_i \quad \forall \mathbf{v} \in \mathcal{V}, \quad \mathbf{w} \in \mathcal{V}^*. \quad (7)$$

If $\mathcal{V} \subset T_X \mathcal{M}$ (the tangent space of the surface \mathcal{M}) $\subset \mathbf{R}^2$,

$$\langle \mathbf{A}, \mathbf{B} \rangle = A^\alpha B_\alpha = C_{\alpha\beta} A^\alpha B^\beta \quad \forall (\mathbf{A}, \mathbf{B}) \in \mathcal{V} \times \mathcal{V}^*, \quad (8)$$

where the usual summation convention is assumed. In continuum mechanics, $C_{\alpha\beta}$ is called the Green strain tensor.

Let L^p and $W^{p,q}$ denote the usual Lebesgue and Sobolev spaces. Domain and range may be specified as $L^p(\Omega; \mathbf{R}^3)$. Here we assume that $\Omega \subset \mathbf{R}^2$ is an open

set of class C^2 and that ϕ_o is a regular function defined over $\Gamma = \partial\Omega$. Let \mathcal{C} be the space of configurations

$$\mathcal{C} := \{x \in W^{1,1}(\Omega; \mathbf{R}^3) \mid x(X) = \phi_o \quad \forall X \in \partial\Gamma\}. \quad (9)$$

For any given Riemannian surface: $x \in \mathcal{C}$, $\mathbf{C} : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ is positive definite:

$$\det \mathbf{C} \geq 0. \quad (10)$$

The total area of the surface \mathcal{M} in this metric is denoted by $P : \mathcal{C} \rightarrow \mathbf{R}$:

$$P(x) := \int_{\mathcal{M}} d\mathcal{M} = \int_{\Omega} \sqrt{\det \mathbf{C}(x(X))} d\Omega \quad \forall x \in \mathcal{C}. \quad (11)$$

The mean curvature for the parametric surface \mathcal{M} is given by

$$H(x) = G^{\alpha\beta} h_{\alpha\beta}(x) = G^{\alpha\beta} \left\langle \frac{\partial^2 x}{\partial X^\alpha \partial X^\beta}, \nu \right\rangle$$

where $\nu \in \mathbf{R}^3$ is the unit normal vector of the surface \mathcal{M} . Then the primal problem for minimizing the parametric surface with constant mean curvature is stated below:

$$\inf_{x \in \mathcal{C}} P(x) \quad s.t. \quad H(x(X)) = const. \quad \forall X \in \Omega. \quad (12)$$

For a nonparametric surface, $x(X^\alpha)$ takes the form

$$x(X^\alpha) = \{X^1, X^2, u(X^1, X^2)\}. \quad (13)$$

Here $u : \Omega \rightarrow \mathbf{R}$ is the graph of the mapping $x : \Omega \rightarrow \mathcal{M}$. Substituting (13) into (6), we have

$$\begin{aligned} \mathbf{D} &= \mathbf{J} + \mathbf{J}_3 \otimes \nabla u, & \text{or } D_\alpha^i &= \delta_\alpha^i + \delta_3^i u_\alpha; \\ \mathbf{C} &= \mathbf{I} + (\nabla u)(\nabla u)^t & \text{or } C_{\alpha\beta} &= \delta_{\alpha\beta} + u_\alpha u_\beta, \end{aligned} \quad (14)$$

where $\mathbf{J} = \{\delta_\alpha^i\}$ is a 3×2 identity and $\mathbf{J}_3 = \{\delta_3^i\} = (0, 0, 1)^t \in \mathbf{R}^3$ is a vector so that

$$\mathbf{J}_3 \otimes \nabla u = \{\delta_3^i u_\alpha\} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ u_x & u_y \end{bmatrix}.$$

It is easy to verify that

$$\det \mathbf{C} = 1 + |\nabla u|^2 > 0 \quad \forall u \in W^{1,1}(\Omega; \mathbf{R}). \quad (15)$$

For a nonparametric surface, the total area of the surface \mathcal{M} can be written as

$$P(u) = \int_{\Omega} \sqrt{\det \mathbf{C}(u)} \, d\Omega = \int_{\Omega} \sqrt{1 + |\nabla u|^2} \, d\Omega. \quad (16)$$

In this case, the primal problem (12) becomes

$$\inf_{v \in W_0^{1,1}(\Omega)} P(v), \quad \text{s.t.} \int_{\Omega} v \, d\Omega = \text{const.} \quad (17)$$

This is a nonlinear minimization problem constrained by a constant volume.

Introducing the Lagrange multiplier $\lambda \in \mathbf{R}$ to relax the constant volume constraint, we get

$$L(v, \lambda) = \int_{\Omega} \sqrt{1 + |\nabla v|^2} \, d\Omega + \lambda \left[\int_{\Omega} u \, d\Omega - \text{const.} \right]. \quad (18)$$

So we have the saddle point problem:

$$\inf_{v \in W_0^{1,1}(\Omega)} \sup_{\lambda \in \mathbf{R}} L(v, \lambda). \quad (19)$$

The associated Euler–Lagrange system is

$$\begin{cases} A(u) - \lambda = 0, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad \int_{\Omega} u \, d\Omega = \text{const.} \quad (20)$$

The existence and uniqueness of the solutions for problem (17) or (20) depend on the constant λ as well as the boundary of the domain. The following lemma shows the convexity of the functional $P(v)$:

LEMMA 1. *For any given nonparametrical surface $v \in W_0^{1,1}(\Omega) : \Omega \rightarrow \mathbf{R}$, the functional $P : W_0^{1,1}(\Omega) \rightarrow \mathbf{R}$ is convex.*

Proof: Let $\epsilon = \nabla v$. It is easy to find that the functional $P(v)$ is convex if and only if the function $w(\epsilon) = \sqrt{1 + \epsilon \epsilon^t}$ is convex $\forall \epsilon(X) \in L^1(\Omega, \mathbf{R}^2)$. The second order directional derivative of w at ϵ in the direction e is given by

$$\delta^2 w(\epsilon; e, e) = e_{\alpha} H_{\epsilon}^{\alpha\beta} e_{\beta},$$

where $H_{\epsilon}^{\alpha\beta}$ is the Hessian of $w(\epsilon)$ defined by

$$H_{\epsilon}^{\alpha\beta} = \frac{\partial^2 w(\epsilon)}{\partial \epsilon_{\alpha} \partial \epsilon_{\beta}} = \frac{(1 + \epsilon \epsilon^t) \delta^{\alpha\beta} - \epsilon^{\alpha} \epsilon^{\beta}}{(1 + \epsilon \epsilon^t)^{3/2}}.$$

By the Cauchy–Schwartz inequality, we have

$$e_\alpha e^\alpha \epsilon_\beta \epsilon^\beta - (e_\alpha \epsilon^\alpha)^2 \geq 0 \quad \forall e \in \mathbf{R}^2.$$

So for any given $\epsilon \in L^1(\Omega, \mathbf{R}^2)$, we have

$$\delta^2 w(\epsilon; e, e) \geq \frac{e_\alpha e^\alpha}{(1 + e_\alpha e^\alpha)^{3/2}} \geq 0 \quad \forall e \in L^1(\Omega, \mathbf{R}^2),$$

which shows that $w(\epsilon)$ is convex. Actually, $w : L^1(\Omega, \mathbf{R}^2) \rightarrow \mathbf{R}$ is strictly convex because $\delta^2 w$ is strictly positive for any given non-zero vector $e \in L^1(\Omega, \mathbf{R}^2)$. ■

Now our interest is in constructing the dual problems of (17).

3. Geometrical mappings and dual variables

In the duality theory of Fenchel–Rockafellar, the dual variational problem is based on the linear geometrical operator: $\Lambda \in \mathcal{L}(\mathcal{V}, \mathcal{Y})$ (see [23]). In this paper, we show that the same primal problem admits different dual variational problems by introducing different geometrical operators.

Using the notation similar to those introduced by Ekeland and Temam [23], we set

$$\begin{aligned} \mathcal{V} &= W^{1,1}(\Omega; \mathbf{R}), & \mathcal{Y}_1 &= L^1(\Omega; \mathbf{R}^2), & \mathcal{Y}_1^* &= L^\infty(\Omega; \mathbf{R}^2), \\ \mathcal{V}^* &= W^{1,1}(\Omega; \mathbf{R})^* = \text{the dual of } W^{1,1}(\Omega; \mathbf{R}); \\ \mathcal{Y}_2 &= \{\mathbf{p} \in L^1(\Omega; \mathbf{R}^3 \times \mathbf{R}^2) \mid \det(\mathbf{p}' \mathbf{p})(X) > 0 \quad \forall X \in \Omega \text{ a.e.}\} \\ \mathcal{Y}_2^* &= \{\mathbf{p}^* \in L^\infty(\Omega; \mathbf{R}^2 \times \mathbf{R}^3) \mid \det(\mathbf{p}^{*t} \mathbf{p}^*)(X) > 0 \quad \forall X \in \Omega \text{ a.e.}\}. \end{aligned}$$

By definition, we let $\epsilon : \mathcal{V} \rightarrow \mathbf{R}^2$:

$$\epsilon := \nabla u. \tag{21}$$

So we obtain

$$\mathbf{D} = \mathbf{J} + \mathbf{J}_3 \otimes \epsilon. \tag{22}$$

So far we have two kinds of geometrical variables: \mathbf{D} and ϵ , which depend linearly on the function $u \in \mathcal{V}$. Let $\mathbf{p} = \{p_s\} \in \mathcal{Y}$ with $p_1 = \epsilon$ for $s = 1$, $p_2 = \mathbf{D}$ for $s = 2$. We may introduce the geometric operator $\Lambda = \{\Lambda^s\} : \mathcal{V} \rightarrow \mathcal{Y}$ defined as

$$\Lambda u = \begin{cases} \nabla u & s = 1 \\ \mathbf{J} + \mathbf{J}_3 \otimes \nabla u & s = 2. \end{cases} \tag{23}$$

We see here that for $s = 2$, Λ has a nonhomogeneous term \mathbf{J} . The geometrical relations between the space \mathcal{V} and \mathcal{Y} can be written in a compact form,

$$\mathbf{p}(u) = \Lambda u = \begin{cases} \epsilon & s = 1 \\ \mathbf{D} & s = 2 \end{cases} \quad (24)$$

The Jacobian $w(\mathbf{p}) : \mathcal{Y} \rightarrow \mathbf{R}$ can be denoted by

$$w(\mathbf{p}) = \begin{cases} \sqrt{1 + |\mathbf{p}|^2} = \sqrt{1 + \epsilon\epsilon^t} & \forall \mathbf{p} \in \mathcal{Y}_1 \\ \sqrt{\det(\mathbf{p}^t \mathbf{p})} = \sqrt{\det(\mathbf{D}^t \mathbf{D})} & \forall \mathbf{p} \in \mathcal{Y}_2. \end{cases} \quad (25)$$

The conjugate variable $\mathbf{p}^* \in \mathcal{Y}^*$ of $\mathbf{p} \in \mathcal{Y}$ becomes

$$\begin{aligned} \mathbf{p}^* &= \frac{\partial w(\mathbf{p})}{\partial \mathbf{p}} \\ \Rightarrow \begin{cases} \epsilon^* = \frac{\partial w(\epsilon)}{\partial \epsilon} = \epsilon / \sqrt{1 + \epsilon\epsilon^t} & \text{for } s = 1 \\ \mathbf{D}^* = \frac{\partial w(\mathbf{D})}{\partial \mathbf{D}} = \sqrt{\det(\mathbf{D}^t \mathbf{D})} (\mathbf{D}^t \mathbf{D})^{-1} \mathbf{D}^t & \text{for } s = 2 \end{cases} \end{aligned} \quad (26)$$

Substituting equation (23) into (26), we have

$$\mathbf{D}^* = \{D_i^{*\alpha}\} = \frac{1}{\sqrt{1 + |\nabla u|^2}} \begin{bmatrix} 1 + u_y^2 & -u_x u_y & u_x \\ -u_x u_y & 1 + u_x^2 & u_y \end{bmatrix} \quad (27)$$

From Gao and Strang [1], for any given geometric mapping $\Lambda : \mathcal{V} \rightarrow \mathcal{Y}$, we have the following decomposition law:

$$\Lambda = \Lambda_t + \Lambda_n. \quad (28)$$

Where Λ_t is the Gâteaux derivative of $\mathbf{p}(u) = \Lambda u$ at $u \in \mathcal{V}$,

$$\Lambda_t v = \lim_{t \rightarrow +0} \frac{\mathbf{p}(u + tv) - \mathbf{p}(u)}{t}.$$

Λ_n is the complementary operator of Λ_t . In this paper, for $s = 1$ we have $\Lambda_t = \Lambda$. But for $s = 2$, the Gâteaux derivative of the geometric mapping $\Lambda : \mathcal{V} \rightarrow \mathcal{Y}_2$ and its complementary operator should be

$$\Lambda_t v = \mathbf{J}_3 \otimes \nabla v, \quad \Lambda_n v = \mathbf{J}.$$

By using the Gauss–Green law, the conjugate operator of Λ_t satisfies the relation,

$$\langle \Lambda_t u, \mathbf{p}^* \rangle = \langle u, \Lambda_t^* \mathbf{p}^* \rangle. \quad (29)$$

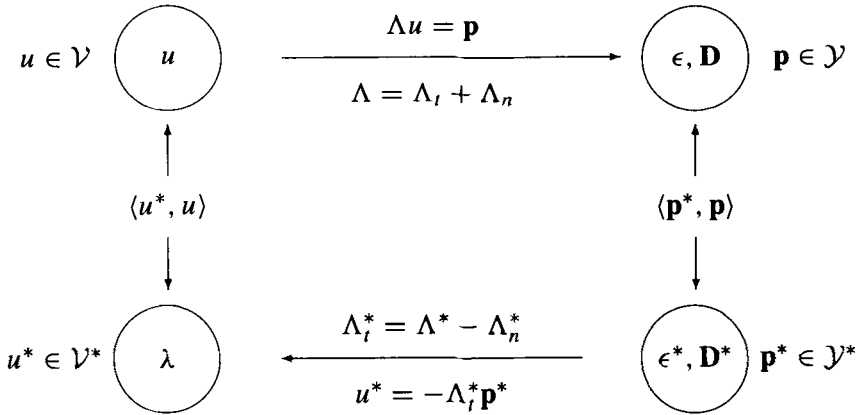


Figure 1. Dual variables and spaces.

For $s = 1$, $\Lambda_t^* = \Lambda^* : \mathcal{Y}_1^* \rightarrow \mathcal{V}^*$ is simply defined by

$$\Lambda^* \epsilon^* = \begin{cases} -\nabla \cdot \epsilon^* & \text{in } \Omega, \\ \mathbf{n} \cdot \epsilon^* & \text{on } \Gamma, \end{cases} \quad (30)$$

where $\mathbf{n} \in \mathbf{R}^2$ is the unit normal vector on $\partial\Omega$. For $s = 2$, $\Lambda_t^* : \mathcal{Y}_2^* \rightarrow \mathcal{V}^*$ should be

$$\Lambda_t^* \mathbf{D}^* = \begin{cases} -\nabla \cdot (\mathbf{D}^* \mathbf{J}_3) = -\nabla_\alpha D_3^{*\alpha} & \text{in } \Omega, \\ \mathbf{n} \cdot (\mathbf{D}^* \mathbf{J}_3) = n_\alpha D_3^{*\alpha} & \text{on } \Gamma. \end{cases} \quad (31)$$

It is easy to find that $\Lambda_t^* : \mathcal{Y}^* \rightarrow \mathcal{V}^*$ is the balance (equilibrium) operator. For a given geometrical measure $\mathbf{p} = \Lambda u \in \mathcal{Y}$, the equilibrium equation for nonparametric surfaces can be written as

$$-\Lambda_t^* \mathbf{p}^*(\Lambda u(X)) = A(u) = \lambda \quad \forall X \in \bar{\Omega}. \quad (32)$$

The relations of these spaces and operators are presented in Figure 1.

To relax the constant volume constraint in problem (17), we introduce a feasible subset $\mathcal{U} \subset \mathcal{V}$,

$$\mathcal{U} := \{v \in \mathcal{V} \mid v(X) = 0 \quad \forall X \in \Gamma, \int_\Omega v \, d\Omega = \text{const.}\}, \quad (33)$$

and define the following general functional $F : \mathcal{V} \rightarrow \bar{\mathbf{R}} := \mathbf{R} \cup \{+\infty\}$,

$$F(v) := \Psi_{\mathcal{U}}(v) = \begin{cases} 0 & \text{if } v \in \mathcal{U} \\ +\infty & \text{otherwise} \end{cases} \quad (34)$$

where $\Psi_{\mathcal{U}}$ is the indicator function of the set \mathcal{U} . It is obvious that $\mathcal{U} = \text{dom}F(v) := \{v \in \mathcal{V} \mid F(v) < +\infty\}$. Furthermore, for any given $\mathbf{p} \in \mathcal{Y}$, we define the functional $W : \mathcal{Y} \rightarrow \mathbf{R}$:

$$W(\mathbf{p}) = \int_{\Omega} w(\mathbf{p}) \, d\Omega \quad \forall \mathbf{p} \in \mathcal{E}. \quad (35)$$

Then the total area $P(u)$ of the surface can be written as

$$P(u) = J(u, \Lambda u) = W(\Lambda u) + F(u), \quad (36)$$

where the functional J takes $\mathcal{U} \times \mathcal{Y}$ into $\bar{\mathbf{R}}$. So the primal problem (17) now takes the following generalized form

$$\inf_{v \in \mathcal{V}} J(v, \Lambda v). \quad (37)$$

Its Euler–Lagrange equation is

$$0 \in \Lambda_i^* \partial W(\Lambda u) + \partial F(u),$$

which is equivalent to equation (20).

4. The first dual variational principle

For the nonparametric surface, the geometric mapping $\Lambda : \mathcal{V} \rightarrow \mathcal{Y}$ given in (23) is a linear operator, so using the Fenchel–Rockafellar theory of duality we can construct the dual H-surface problem. First, we consider the geometrical mapping $\epsilon = \Lambda u = \nabla u$. In this case, the conjugate function $J^*(-\Lambda^* \epsilon^*, \epsilon^*)$ is given by the following lemma:

LEMMA 2. *If $\text{dom}F(v) \neq \emptyset$, then there exists a $\lambda \in \mathbf{R}$ such that the conjugate function $J^* : \mathcal{V}^* \times \mathcal{Y}^* \rightarrow \bar{\mathbf{R}}$ is*

$$J^*(-\Lambda^* \epsilon^*, \epsilon^*) = W^*(\epsilon^*) + F(-\Lambda^* \epsilon^*), \quad (38)$$

where $W^* : \mathcal{Y}^* \rightarrow \bar{\mathbf{R}}$ is the conjugate function of $W(\epsilon)$:

$$W^*(\epsilon^*) = \begin{cases} -\int_{\Omega} \sqrt{1 - |\epsilon^*|^2} \, d\Omega & \text{if } |\epsilon^*|^2 \leq 1 \\ +\infty & \text{if } |\epsilon^*|^2 > 1, \end{cases} \quad (39)$$

and $F^* : \mathcal{V}^* \rightarrow \bar{\mathbf{R}}$ is the conjugate function of $F(u)$:

$$F^*(-\Lambda^* \epsilon^*) = \begin{cases} \lambda c & \text{if } \Lambda^* \epsilon^* - \lambda = 0 \\ +\infty & \text{otherwise} \end{cases} \quad (40)$$

where c is the constant from $c = \int_{\Omega} u \, d\Omega$.

The proof of this Lemma is straightforward using the Legendre–Fenchel transformation (see, e.g., [23]).

For any given $\lambda \in \mathbf{R}$, we define the so-called statically admissible space $\mathcal{U}^* \subset \mathcal{V}^*$:

$$\mathcal{U}^* := \{v^* \in \mathcal{V}^* \mid v^*(X) = \lambda \quad \forall X \in \Omega \text{ a.e.}\}. \quad (41)$$

So the conjugate function $F^*(-\Lambda^* \epsilon^*)$ can be written as

$$F^*(-\Lambda^* \epsilon^*) = \lambda c + \Psi_{\mathcal{U}^*}(-\Lambda^* \epsilon^*)$$

where $\Psi_{\mathcal{U}^*} : \mathcal{V}^* \rightarrow \bar{\mathbf{R}}$ is the indicator of the subset \mathcal{U}^* :

$$\Psi_{\mathcal{U}^*}(v^*) = \begin{cases} 0 & \text{if } v^* \in \mathcal{U}^* \\ +\infty & \text{otherwise} \end{cases}$$

Furthermore, we define a closed convex subset $\mathcal{K}_1 \subset \mathcal{Y}_1^*$:

$$\mathcal{K}_1 := \{\epsilon^* \in \mathcal{Y}^* \mid |\epsilon^*(X)| \leq 1 \quad \forall X \in \Omega \text{ a.e.}\}, \quad (42)$$

and write the functional $W^* : \mathcal{Y}_1^* \rightarrow \bar{\mathbf{R}}$ as

$$W^*(\epsilon^*) = - \int_{\Omega} \sqrt{1 - |\epsilon^*|^2} \, d\Omega + \Psi_{\mathcal{K}_1}(\epsilon^*), \quad (43)$$

where $\Psi_{\mathcal{K}_1}$ is the indicator of the subset $\mathcal{K}_1 \subset \mathcal{Y}_1^*$. Over the statistically admissible space \mathcal{U}^* , we write

$$P^*(\epsilon^*) = -J^*(-\Lambda^* \epsilon^*, \epsilon^*) = \int_{\Omega} \sqrt{1 - |\epsilon^*|^2} \, d\Omega - \Psi_{\mathcal{K}_1}(\epsilon^*) - \lambda c \quad (44)$$

and define the following problem:

$$P^*(\epsilon^*) = \sup_{\substack{\tau^* \in \mathcal{Y}_1^* \\ -\Lambda^* \tau^* \in \mathcal{U}^*}} P^*(\tau^*). \quad (45)$$

THEOREM 1. *There exists a $\lambda \in \mathbf{R}$ such that the primal problem (17) admits problem (45) as its dual and*

$$\inf P = \sup P^*. \quad (46)$$

Furthermore, the primal problem (17) may or may not possess a solution, whereas the dual problem (45) possesses a unique solution $\epsilon^(\lambda) \in \mathcal{Y}^*$ for the given $\lambda \in \mathbf{R}$.*

Proof: Suppose that $u \in \mathcal{V}$ is a solution of the primal problem (17). For a given $\epsilon^* \in \mathcal{Y}^*$, the Lagrangian $L : \mathcal{V} \times \mathcal{Y}_1^* \rightarrow \bar{\mathbf{R}}$ of the primal problem (17) is defined by

$$-L(u, \epsilon^*) = \sup_{\epsilon \in \mathcal{Y}} \{ \langle \epsilon - \Lambda u, \epsilon^* \rangle - J(u, \epsilon) \}.$$

We prove that the Lagrange multiplier ϵ^* is the solution of problem (45). Actually, for a given $(u, \epsilon^*) \in \mathcal{U} \times \mathcal{Y}^*$,

$$\begin{aligned} L(u, \epsilon^*) &= \langle u, \Lambda^* \epsilon^* \rangle - \sup_{\epsilon \in \mathcal{Y}} \{ \langle \epsilon, \epsilon^* \rangle - W(\epsilon) \} \\ &= \langle u, \Lambda^* \epsilon^* \rangle - W^*(\epsilon^*). \end{aligned}$$

For a given $\epsilon^* \in \mathcal{Y}^*$ such that $-\Lambda^* \epsilon^* \in \mathcal{U}^*$, we have

$$L(u, \epsilon^*) = -\lambda c - W^*(\epsilon^*) = \sup P^*(\tau^*),$$

since $u \in \mathcal{U}$ is the solution of the primal problem (17). This shows that the problem (17) and (45) are mutually dual.

Meanwhile

$$\begin{aligned} \inf_{v \in \mathcal{V}} P(v) &= W(\Lambda u) = \langle \epsilon^*, \Lambda u \rangle - W^*(\epsilon^*) \\ &= \langle \Lambda^* \epsilon^*, u \rangle - W^*(\epsilon^*) = \langle -\lambda, u \rangle - W^*(\epsilon^*) \\ &= -\lambda \int_{\Omega} u \, d\Omega - W^*(\epsilon^*) = P^*(\epsilon^*) \\ &= \sup_{\tau^* \in \mathcal{U}^*} P^*(\tau^*). \end{aligned}$$

Since $P^* : \mathcal{Y}^* \rightarrow \mathbf{R}$ is strictly concave, upper semicontinuous (u.s.c.), if the convex subset $\mathcal{K}_1 \neq \emptyset$, then for a given $\lambda \in \mathbf{R}$, the dual problem (45) has a unique solution. ■

Remark: While the primal problem (17) is similar to the upper bound theorem in the theory of plastic limit analysis, the dual problem (45) corresponds to the generalized lower bound theorem (see [20]). The physical meaning of the dual variable ϵ^* can be considered as the surface tension. The constraint $|\epsilon^*| \leq 1$ corresponds to the yield condition in plastic limit analysis. If $|\epsilon^*| > 1$, the surface is broken. Using the property of the “generalized complementary energy” $W^* : \mathcal{Y}^* \rightarrow \bar{\mathbf{R}}$, various approximate methods can be derived to solve the dual problem (see [21]).

In the closed convex subset $\mathcal{K}_1 \subset \mathcal{Y}_1^*$ the functional P^* (46) can be written as

$$P^*(\epsilon^*) = \int_{\Omega} \sqrt{1 - |\epsilon^*|^2} \, d\Omega - \lambda c. \quad (47)$$

The dual problem (45) has a degenerated form:

THEOREM 2. *For a given $\lambda \in \mathcal{V}^*$, the dual problem*

$$\sup_{\substack{\epsilon^* \in \mathcal{K}_1 \\ -\Lambda^* \epsilon^* = \lambda}} \left\{ \int_{\Omega} \sqrt{1 - |\epsilon^*(X)|^2} \, d\Omega - \lambda c \right\} \quad (48)$$

possesses a unique solution.

This theorem can easily be proved by considering the concavity of P^* over the closed convex subset \mathcal{K}_1 . Problem (48) is a constrained extremum problem defined on a closed convex subset $\mathcal{K}_1 \in \mathcal{Y}_1^*$ and hyperplane \mathcal{U}^* . In the theory of plastic limit analysis, this theorem is similar to the classical lower bound theorem.

5. The second dual variational principle

Our aim here is to show that for the same primal problem (17), we may have different dual problems. For the given geometric relation (22), we look instead at (21), for the second dual problem of the nonparametric H-surface. In this case, the dual variables of $u \in \mathcal{V}$ and $\mathbf{D} \in \mathcal{Y}_2$ are $-\Lambda_t^* \mathbf{D}^* \in \mathcal{V}^*$ and $\mathbf{D}^* \in \mathcal{Y}_2^*$, respectively. The conjugate function of $J(v, \mathbf{D})$ should be $J^*(-\Lambda_t^* \mathbf{D}^*, \mathbf{D}^*)$ as explained by the following lemma:

LEMMA 3. *If $\text{dom}F(v)$ is not empty, then there exists a $\lambda \in \mathbf{R}$ such that*

$$J_2^*(-\Lambda_t^* \mathbf{D}^*, \mathbf{D}^*) = W^*(\mathbf{D}^*) + \lambda c + \Psi_{\mathcal{U}^*}(-\Lambda_t^* \mathbf{D}^*), \quad (49)$$

where $W^* : \mathcal{Y}_2^* \rightarrow \mathbf{R}$ is a convex, l.s.c. function defined by

$$W^*(\mathbf{D}^*) = \int_{\Omega} \sqrt{\det(\mathbf{D}^* \mathbf{D}^{*t})} \, d\Omega + \Psi_{\mathcal{Y}_2^*}(\mathbf{D}^*). \quad (50)$$

Proof: By the Legendre–Fenchel transformation we have

$$\begin{aligned} J_2^*(-\Lambda_t^* \mathbf{D}^*, \mathbf{D}^*) &= \sup_v \sup_{\mathbf{D}} \{ \langle -\Lambda_t^* \mathbf{D}^*, v \rangle + \langle \mathbf{D}^*, \mathbf{D} \rangle - J(v, \mathbf{D}) \} \\ &= W^*(\mathbf{D}^*) + F_2^*(-\Lambda_t^* \mathbf{D}^*) \end{aligned}$$

where

$$W^*(\mathbf{D}^*) = \sup_{\mathbf{D} \in \mathcal{Y}_2} \{ \langle \mathbf{D}^*, \mathbf{D} \rangle - \int_{\Omega} \sqrt{\det(\mathbf{D}^t \mathbf{D})} \, d\Omega \}.$$

Since $W(\mathbf{D}) : \mathcal{Y}_2 \rightarrow \mathbf{R}$ is convex, the critical point

$$\mathbf{D} = \frac{1}{\sqrt{\det(\mathbf{D}^* \mathbf{D}^{*t})}} \mathbf{D}^{*t} (\mathbf{D}^t \mathbf{D})$$

maximizes the above optimal problem and gives the result (50)

On the other hand, the dual variable of $v \in \mathcal{V}$ is $\Lambda_t^* \mathbf{D}^* \in \mathcal{V}^*$. We produce

$$\begin{aligned} F_2^*(-\Lambda_t^* \mathbf{D}^*) &= \sup_{v \in \mathcal{V}} \{ \langle -\Lambda_t^* \mathbf{D}^*, v \rangle - F(v) \} \\ &= \lambda c + \sup_{v \in \mathcal{U}} \{ \langle -\Lambda_t^* \mathbf{D}^* - \lambda, v \rangle \} \\ &= \lambda c + \Psi_{\mathcal{U}^*}(-\Lambda_t^* \mathbf{D}^*) \end{aligned} \tag{51}$$

to prove Lemma 2. ■

It is obvious that $J_2^* : \mathcal{U}^* \times \mathcal{Y}_2^* \rightarrow \bar{\mathbf{R}}$ is convex. Unfortunately, for the geometric operator $\Lambda : \mathcal{V} \rightarrow \mathcal{Y}_2$ with nonhomogeneous term \mathbf{J} , $J_2^*(-\Lambda_t^* \mathbf{D}^*, \mathbf{D}^*)$ is not the dual variational functional of the primal functional $J(v, \Lambda v)$ because its Euler-Lagrangian is not equivalent to the governing equations of the H-surface problem. The difference between the conjugate functional J_2^* and the dual functional of J is the so-called complementary gap function defined by (see [1])

$$\begin{aligned} G^*(-\Lambda_n^* \mathbf{D}^*) &= \langle -\Lambda_n^* \mathbf{D}^*, u \rangle = -\langle \mathbf{D}^*, \Lambda_n u \rangle \\ &= - \int_{\Omega} \text{tr}(\mathbf{D}^* \mathbf{J}) \, d\Omega = - \int_{\Omega} D_i^{*\alpha} \delta_{\alpha}^i \, d\Omega. \end{aligned} \tag{52}$$

Here we define the functional $P_2^* : \mathcal{Y}_2^* \rightarrow \mathbf{R} \cup \{-\infty\}$:

$$P_2^*(\mathbf{D}^*) = -W^*(\mathbf{D}^*) - F_2^*(-\Lambda_t^* \mathbf{D}^*) - G^*(-\Lambda_n^* \mathbf{D}^*). \tag{53}$$

Then the second dual problem of H-surface problem (17) should be

$$\sup_{\mathbf{D}^* \in \mathcal{Y}_2^*} P_2^*(\mathbf{D}^*). \tag{54}$$

THEOREM 3. *Problem (17) and problem (54) are mutually dual, and*

$$\inf P = \sup P_2^*. \tag{55}$$

If $\text{dom} W^(\mathbf{D}^*) \cap \text{dom} F_2^*(-\Lambda_t^* \mathbf{D}^*) \neq \emptyset$, then the problem (54) possesses a unique solution.*

Proof: We produce

$$\begin{aligned}
 \inf P &= W(\Lambda u) = \langle \Lambda u, \mathbf{D}^* \rangle - W^*(\mathbf{D}^*) \\
 &= \langle \Lambda_n u, \mathbf{D}^* \rangle + \langle u, \Lambda_t^* \mathbf{D}^* \rangle - W^*(\mathbf{D}^*) \\
 &= \int_{\Omega} \text{tr}(\mathbf{D}^* \mathbf{J}) \, d\Omega - \lambda c - W^*(\mathbf{D}^*) \\
 &= \sup P_2^*
 \end{aligned}$$

as the required duality relation. Since $P_2^* : \mathcal{Y}_2^* \rightarrow \bar{\mathbf{R}}$ is strictly concave, u.s.c. functional, then over the convex subset \mathcal{Y}_2^* , the problem possesses a unique solution. ■

For any given $\mathbf{D}^* \in \mathcal{Y}_2^*$, if $-\Lambda_t^* \mathbf{D}^* \in \mathcal{U}^*$, the functional P_2^* can be written as

$$P_2^*(-\Lambda^* \mathbf{D}) = \int_{\Omega} \text{tr}(\mathbf{D}^* \mathbf{J}) \, d\Omega - \lambda c - \int_{\Omega} \sqrt{\det(\mathbf{D}^* \mathbf{D}^{*t})} \, d\Omega. \quad (56)$$

So the second dual problem (54) has the following degenerated form:

$$\sup_{\substack{\mathbf{D}^* \in \mathcal{Y}_2^* \\ -\Lambda_t^* \mathbf{D}^* = \lambda}} \left\{ \int_{\Omega} \text{tr}(\mathbf{D}^* \mathbf{J}) \, d\Omega - \lambda c - \int_{\Omega} \sqrt{\det(\mathbf{D}^* \mathbf{D}^{*t})} \, d\Omega \right\}. \quad (57)$$

By now we have proved that for the same primal problem (17), there exists at least two different dual problems, whose significance is as follows. Since the primal functional P is strictly convex, if the primal problem possesses a solution u , the dual solutions for problem (45) and (54) are equivalent and we have

$$P(u) = P_1^*(\epsilon^*(u)) = P_2^*(\mathbf{D}^*(u)).$$

When the primal problem possesses no solution, the dual problems (45) and (54) define two different generalized solutions. From the point of view of the optimization theory, the first dual problem (45) is a nonlinear optimal problem with the nonlinear inequality constraint $|\epsilon^*| \leq 1$ and the equilibrium constraint $-\Lambda^* \epsilon^* = \lambda$. The numerical solutions for this kind of problems are very difficult to obtain, whereas the second dual problem (54) is constrained by only one linear equilibrium condition. It is easy to get the approximate solution by using discrete methods. Physically speaking, the first dual problem is equivalent to the lower bound theory in plastic limit analysis (see [2]). The second dual problem, on the other hand, is similar to the complementary energy principle in nonlinear elasticity. By the way, the second dual problem given here can easily be generated to study parametrical surfaces.

6. Evolutionary method for computing H-surfaces

An evolutionary method for solving the H-surface problem was first studied by Temam [24] and Lichnewsky and Temam [27] for minimal hypersurface problems. Here we give only the applications to the H-surface problem. We assume that the graph $u : \Omega \times [0, T] \rightarrow \mathbf{R}$ is time dependent. The evolution equations

$$\begin{cases} u_t = A(u(X, t)) - \lambda & \text{in } \Omega \times]0, T[, \\ u = 0 & \text{on } \partial\Omega \times [0, T] \\ u(X, 0) = u_0(X) & \text{in } \Omega \end{cases} \quad (58)$$

associated with the primal problem (17) are treated as an initial-boundary value problem. In this evolutionary problem we have two unknowns, $u(X, t)$ and $\lambda(t)$, but only one equation. From the constraint $\int_{\Omega} u \, d\Omega = V$ (V is a given constant), we have

$$\frac{d}{dt} \int_{\Omega} u \, d\Omega = \int_{\Omega} u_t \, d\Omega = \int_{\Omega} [A(u(X, t)) - \lambda] \, d\Omega = 0.$$

This gives the equation for the Lagrange multiplier:

$$\lambda(t) = \frac{\int_{\Omega} A(u(X, t)) \, d\Omega}{\text{area}(\Omega)}.$$

But our numerical experiments show that this formulation yields an unstable iteration. From the weak form of the constraint $\int_{\Omega} u[A(u(X, t)) - \lambda] \, d\Omega$, a weighted formulation

$$\lambda(t) = \frac{\int_{\Omega} u A(u(X, t)) \, d\Omega}{V}, \quad (59)$$

which is equally valid, produces stable solutions. We solve this evolutionary equation for $\Omega = \{(x, y) \in \mathbf{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$ and discretize this initial-boundary value problem by a regular three-dimensional net, which is generated from a regular net in the region Ω with the space mesh size h and the time mesh size k . Then for a given $\lambda(t_j)$, we look for approximate solution $u_{\mu, \nu}^j$ of the solutions $u(x, y, t)$ at the grid point (x_{μ}, y_{ν}, t_j) . Here we use the standard finite difference method to compute the quantities

$$u_x(x_{\mu}, y_{\nu}, t_j) = \frac{u_{\mu+1, \nu}^j - u_{\mu-1, \nu}^j}{2h}, \quad u_y(x_{\mu}, y_{\nu}, t_j) = \frac{u_{\mu, \nu+1}^j - u_{\mu, \nu-1}^j}{2h},$$

$$u_{xx}(x_{\mu}, y_{\nu}, t_j) = \frac{u_{\mu+1, \nu}^j - 2u_{\mu, \nu}^j + u_{\mu-1, \nu}^j}{h^2},$$

$$u_{yy}(x_{\mu}, y_{\nu}, t_j) = \frac{u_{\mu, \nu+1}^j - 2u_{\mu, \nu}^j + u_{\mu, \nu-1}^j}{h^2},$$

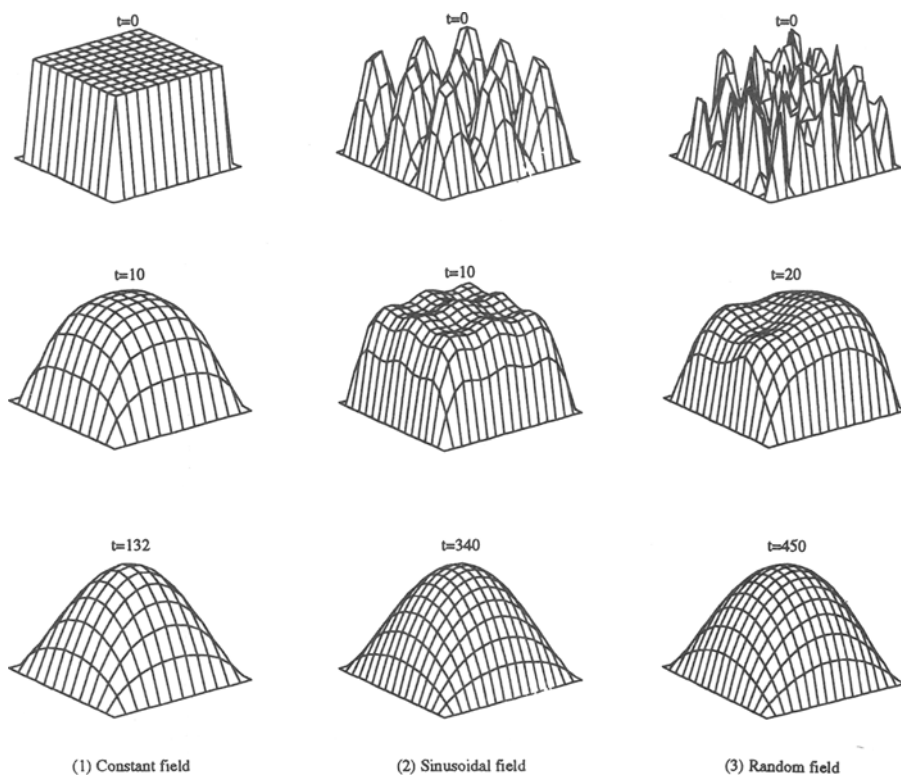


Figure 2. Numerical results for evolutionary solutions.

$$\begin{aligned}
 u_{xy}(x_\mu, y_\nu, t_j) &= \frac{u_{\mu+1, \nu+1}^j - u_{\mu-1, \nu+1}^j - u_{\mu+1, \nu-1}^j + u_{\mu-1, \nu-1}^j}{h^2} \\
 &= u_{yx}(x_\mu, y_\nu, t_j), \\
 u_t &= \frac{u_{\mu, \nu}^{j+1} - u_{\mu, \nu}^j}{h}.
 \end{aligned}$$

By an explicit method, we use the algorithm:

- (i) Given initial condition $u(x, y, t_0)$, constant V , Lagrange multiplier $\lambda(t_0)$, and a previously given precision $\omega > 0$. Let $j = 0$.
- (ii) Determine $u(t_{k+1})$ at every regular grid point $P(x_\mu, y_\nu, t_j)$ in the domain Ω by

$$u_{\mu, \nu}^{j+1} = u_{\mu, \nu}^j + k[A(u_{\mu, \nu}^j) - \lambda(t_j)]. \quad (60)$$

- (iii) Find $\lambda(t_{j+1})$ by

$$\lambda(t_{j+1}) = \frac{\int_{\Omega} u_{\mu, \nu}^{j+1} A(u_{\mu, \nu}^{j+1}) d\Omega}{V}. \quad (61)$$

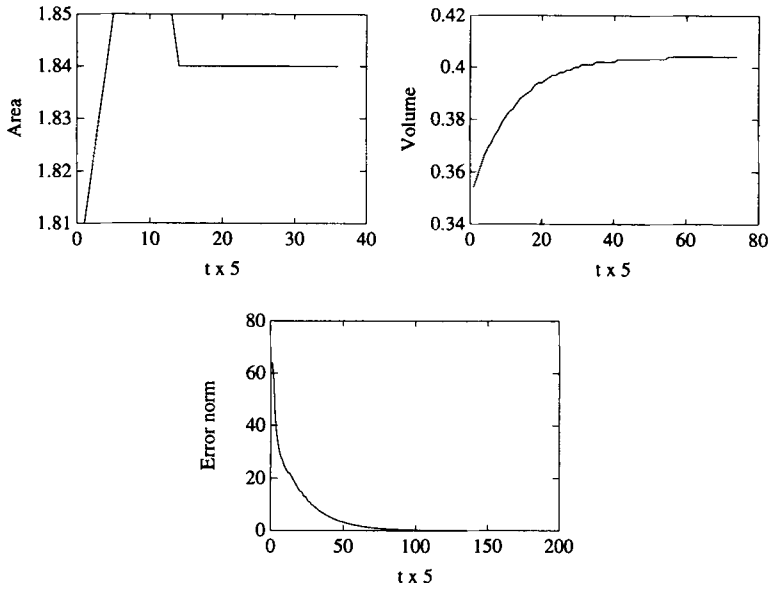


Figure 3. Convergence of evolution solutions for a random initial field.

- (iv) If $\|A(u_{\mu,v}^{j+1}) - \lambda\| \leq \omega$ or $|\int_{\Omega} u_{\mu,v}^{j+1} d\Omega - V| \leq \omega$, u^{j+1} is the desirable numerical solution. Otherwise, by setting $u^{j+1} = u^j$, $\lambda(t_{j+1}) = \lambda(t_j)$, let $j = j + 1$ then go to step (ii).

For three different initial data u_0 —(1) a constant field; (2) a sinusoidal field; (3) a random field—our evolutionary algorithm produced the identical final minimal surface as shown in Figure 2. The scale $u(x, y)$ is amplified. The convergence is shown in Figure 3 for $\Omega = \{(x, y) \in \mathbf{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$, where the area $P(u)$ of the surface, volume $\int_{\Omega} u d\Omega$, and error norm $\|u_t\|$ are plotted against the time variable t .

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