ON TOROIDAL WAVE FUNCTIONS

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Introduction. A method was developed by Weston [1] for solving the Helmholtz equation for a class of non-separable coordinate systems. This method was applied to the Toroidal coordinate system [2] and a complete set of solutions were obtained with represented radiation from a ring source. In this paper, there will be derived from these wave functions, simpler wave functions which are more useful in practice. Besides having simpler asymptotic values in the radiation or far field, their asymptotic values for thin rings hold down to smaller wavelengths.

1. Toroidal Coordinates. The relation between toroidal and cartesian coordinate systems is given by Magnus and Oberhettinger [3],

$$x = \frac{d \sinh \zeta \cos \theta}{\cosh \zeta - \cos \eta}, \qquad y = \frac{d \sinh \zeta \sin \theta}{\cosh \zeta - \cos \eta}, \qquad z = \frac{d \sin \eta}{\cosh \zeta - \cos \eta} \tag{1.1}$$

Domains of the coordinates are $0 \leqq \eta \leqq 2\pi$, $0 \leqq \phi \leqq 2\pi$, and $0 \leqq \zeta \leqq \infty$, where $\zeta = \zeta_0$ defines the torus

$$z^2 + (\rho - d \coth \zeta_0)^2 = d^2 \operatorname{csch}^2 \zeta_0$$

and $\eta = \eta_0$ defines the sphere

$$(z - d \cot \eta_0)^2 + \rho^2 = d^2 \csc \eta_0^2$$

where $\rho = (x^2 + y^2)^{1/2}$.

The metric coefficients are given by the following relations

$$h_\zeta = h_\eta = \frac{d}{\cosh \zeta - \cos \eta}, \qquad h_\phi = \frac{d \sinh \zeta}{\cosh \zeta - \cos \eta}. \tag{1.2}$$

In order to facilitate the analysis, the variable s will be used instead of ζ, where the two are related by the equation

$$\cosh \zeta = s. \tag{1.3}$$

The variable s has more physical significance. If $s = s_0$ defines a toroidal surface formed by rotating a circle of radius b about a line in the plane of the circle and at a distance a from its center, then we have the relation $s_0 = a/b$. For the case of a thin torus or wire we will define a and b as the loop and wire radii respectively.

The parameter d in the system of equations (1.1) and (1.2) is related to the radii a and b defined above for a given torus, by

$$d = (a^2 - b^2)^{1/2}.$$

Hence we see that d represents the radius a of the limiting torus $s = \infty$.

2. Toroidal Wave Functions. In Weston [2], solutions of the Helmholtz equation

$$\nabla^2\psi + k^2\psi = 0 \tag{2.1}$$

where $k = 2\pi/\lambda$, were derived in toroidal coordinates. These solutions

$$e^{im\phi}V_{m+2l}^m(s, \eta) \quad \text{and} \quad e^{im\phi}W_{m+2l+1}^m(s, \eta)$$

with ($l = 0, 1, 2, \cdots$), possessed a ring singularity (i.e., singular at $s = \infty$) and satisfied the radiation condition. These wave functions were defined in terms of auxiliary functions S_n^m, T_n^m by the relation

$$V_n^m = S_n^m + i(-1)^{n+1}S_{-n-1}^m, \qquad W_m^m = T_n^m + i(-1)^{n+1}T_{-n-1}^m. \tag{2.2}$$

These are given explicitly in terms of the following series representation

$$S_n^m = c_n^m \sum_{p=0}^{\infty} \sum_{r=0} a_p^r(m/2 + n/2 + \tfrac{1}{2})_r (m/2 - n/2)_r P_{p+(n+m)/2}^{-m-r}(s) \tag{2.3}$$

$$T_n^m = c_n^m 2^{-1/2} \sin\eta(s - \cos\eta)^{-1/2} \sum_{p=0}^{\infty} \sum_{r=0} a_p^r(m/2 + n/2 + 1)_r (m/2 - n/2 + \tfrac{1}{2})_r P_{p+(n+m-1)/2}^{-m-r}(s) \tag{2.4}$$

where

$$c_n^m = (n - m + 1)_{2m}(kd)^n(-1)^m\pi^{1/2}2^{-(m+n+2)/2}$$

$$a_p^r = \frac{(kd)^{2p}(-1)^p(s^2 - 1)^{r/2}(s - \cos\eta)^{-p-r-(n+m)/2}}{(p)!2^p\Gamma(n + \frac{3}{2} + p)(r)!}$$

where $P_\nu^m(s)$ is the associated Legendre function.

The asymptotic values for large r, i.e., in the radiation field, where (r, θ, ϕ) represents the spherical polar coordinates are

$$V_{m+2l}^m \sim (-i)^{m+2l+1} \frac{e^{ikr}}{kr} R_{m+2l}^m(\cos\theta) \tag{2.5}$$

$$W_{m+2l+1}^m \sim (-i)^{m+2l+2} \frac{e^{ikr}}{kr} R_{m+2l+1}^m(\cos\theta) \tag{2.6}$$

where

$$R_{m+2l}^m(\cos\theta) = (2l + 1)_{2m}(-kd)^{-m} \sum_{r=0}^{l} \frac{(-l)_r(m + l + \frac{1}{2})_r}{(r)!} \left(\frac{2\sin\theta}{kd}\right)^r J_{m+r}(kd\sin\theta) \tag{2.7}$$

$$R_{m+2l+1}^m(\cos\theta) = (2l + 2)_{2m}(-kd)^{-m}\cos\theta \sum_{r=0}^{l} \frac{(-l)_r(m + l + \frac{3}{2})_r}{(r)!} \left(\frac{2\sin\theta}{kd}\right)^r J_{m+r}(kd\sin\theta) \tag{2.8}$$

They have the following asymptotic values for s approaching ∞.

$$V_{m+2l}^{m} \sim i(-1)^{m+l+1}(2l+1)_{2m}\Gamma(m+2l+\tfrac{1}{2})(2/\pi)^{1/2} \cdot (kd)^{-m-2l-1}s^{1/2}P_{l-1/2}^{-m}(s)\cos l\eta \qquad l \neq 0 \tag{2.9}$$

$$V_{m}^{m} \sim i(-1)^{m+1}(1)_{2m}\Gamma(m+\tfrac{1}{2})(2\pi)^{-1/2}(kd)^{-m-1}s^{1/2}P_{-1/2}^{-m}(s)$$

$$W_{m+2l+1}^{m} \sim i(-1)^{m+l+1}(2l+2)_{2m}\,\Gamma(m+2l+\tfrac{3}{2})(2\pi)^{-1/2} \cdot (kd)^{-m-2l-2}s^{1/2}P_{l+1/2}^{-m}(s)\sin(l+1)\eta \tag{2.10}$$

However, for these asymptotic expressions for large s we must place the wavelength restriction $s > (kd)^2$.

If we express the Helmholtz equation in Toroidal coordinates and approximate it in the region $s \gg 1$, it can be shown [6], that there exist solutions which have the above asymptotic values (2.9), (2.10) and (2.11), but which hold for a greater range of wavelength for a particular s, i.e. they hold for $s > kd$ or for thin wires $\lambda > 2\pi b$.

In solving the Helmholtz equation in Toroidal coordinates for solutions satisfying the radiation condition [2], the following condition was used. Since a ring source of radius d approaches a point shource when d approaches zero, the Toroidal wave functions were required to be identical to the spherical polar wave functions

$$e^{im\phi}h_n^{(1)}(kr)P_n^m(\cos\theta)$$

in the limit of vanishing d. This is a necessary condition that the wave functions satisfy the radiation condition.

However, this places too great a restriction on the wave functions. We expect that we can obtain simpler wave functions such that their asymptotic values for large s are similar to that of Eqns. (2.9), (2.10), and (3.11) but hold for a greater range in wavelength.

Rather than solving the differential equations for the wave function with less restrictive conditions in the applying the radiation condition, we can make use of the fact that any simpler set of wave functions must be a linear combination of the Toroidal wave functions $e^{im\phi}V_{m+2l}^{m}$ and $e^{im\phi}W_{m+2l+1}^{m}$.

3. Toroidal Wave Functions Simplified. In the series expansion for $S_{-m-2l-1}^{m}(s,\eta)$ replace the Legendre functions by their integral representation

$$P_v^{-m}(s) = \frac{\Gamma(v-m+1)}{2\pi\Gamma(v+1)}\int_{-\pi}^{\pi} e^{imt}[s+(s^2-1)^{\frac{1}{2}}\cos t]^v\,dt \tag{3.1}$$

It can be shown that we may invert the order of summation and integration. The integrand can be summed to give an expression involving spherical Bessel functions of argument x (defined below) and their derivations. After some analysis just involving the integrand, we obtain the following

$$S^m_{-m-2l-1}(s,\eta) \tag{3.2}$$
$$= \frac{(2m+2l)!(kd)^{-m}}{(2l)!2\pi}(-1)^{m+l}\sum_{p=0}^{l}\frac{(-l)_p(m+l+\frac{1}{2})_p}{(p)!}\left(\frac{2}{kd}\right)^p(-1)^p\mathcal{S}^m_{-p-1}$$

where

$$\mathcal{S}^m_{-l-1} = \int_{-\pi}^{\pi} e^{imt}z^l j_{-l-1}(x)\,dt \tag{3.3}$$

$$x = kd2^{\frac{1}{2}}\left[\frac{s+\sqrt{s^2-1}\cos t}{s-\cos\eta}\right]^{\frac{1}{2}} \tag{3.4}$$

$$z = \frac{(e^{it}\sqrt{s^2-1}+s-\cos\eta)}{2^{\frac{1}{2}}(s-\cos\eta)^{\frac{1}{2}}(s+\sqrt{s^2-1}\cos t)^{\frac{1}{2}}} \tag{3.5}$$

and $j_l(x)$ is the spherical Bessel function of order l. In a similar manner it can be shown that $S^m_{m+2l}(S,\eta)$ can be expressed in the form

$$S^m_{m+2l}(s,\eta) = \frac{(2m+2l)!(kd)^{-m}}{(2l)!2\pi}(-1)^l\sum_{p=0}^{l}\frac{(-l)_p(m+l+\frac{1}{2})_p}{(p)!}\left(\frac{2}{kd}\right)^p\mathcal{S}^m_p \tag{3.6}$$

where

$$\mathcal{S}^m_l = \int_{-\pi}^{\pi} e^{imt}z^l j_l(x)\,dt \tag{3.7}$$

Thus we see that the original wave functions $V^m_{m+2l}(s,\eta)$ can be decomposed into the new set $\mathcal{V}^m_l(s,\eta)(l = 0, 1, 2, 3, \cdots)$ where

$$\mathcal{V}^m_l(s,\eta) = \mathcal{S}^m_l + i(-1)^{l+1}\mathcal{S}^m_{-l-1} \tag{3.8}$$

$$= \int_{-\pi}^{\pi} e^{imt}z^l h^{(1)}_l(x)\,dt. \tag{3.9}$$

Also it can be shown that the original wave functions $W^m_{m+2l+1}(s,\eta)$ can be decomposed into the set $\mathcal{W}^m_l(s,\eta)(l = 0, 1, 2, \cdots)$ where

$$\mathcal{W}^m_l = \mathfrak{J}^m_l + i(-1)^{l+2}\mathfrak{J}^m_{-l-1} \tag{3.10}$$

$$= \frac{\sin\eta}{(s-\cos\eta)}\int_{-\pi}^{\pi} e^{imt}z^l x^{-1}h^{(1)}_{l+1}(x)\,dt \tag{3.11}$$

and

$$\mathfrak{J}^m_l = \frac{\sin\eta}{(s-\cos\eta)}\int_{-\pi}^{\pi} e^{imt}z^l x^{-1}j_{l+1}(x)\,dt \tag{3.12}$$

The relationship between $W^m_{m+2l+1}(s,\eta)$ and $\mathcal{W}^m(s,\eta)$ is given by

$$W^m_{m+2l+1}(s,\eta) \tag{3.13}$$
$$= \frac{(2m+2l+1)!(kd)^{-m+1}}{(2l+1)!2\pi}(-1)^l\sum_{p=0}^{l}\frac{(-l)_p(m+l+3/2)_p}{(p)!}\left(\frac{2}{kd}\right)^p\mathcal{W}^m_p$$

Their asymptotic values in the far field can be easily obtained from the integral expressions (3.9) and (3.11). Using the relations

$$x \sim kr + kd \cos t \sin \theta + O\left(\frac{1}{r}\right), \qquad z \sim e^{it} \sin \theta + O\left(\frac{1}{r}\right)$$

we obtain

$$\mathcal{V}_l^m(s, \eta) \underset{R\to\infty}{\sim} i^{m-1} 2\pi \frac{e^{ikr}}{kr} (\sin \theta)^l J_{m+l}(kd \sin \theta) \tag{3.14}$$

$$\mathcal{W}_l^m(s, \eta) \underset{R\to\infty}{\sim} \frac{-i^m 2\pi}{(kd)} \frac{e^{ikr}}{kr} \cos \theta \, (\sin \theta)^l J_{m+l}(kd \sin \theta) \tag{3.15}$$

4. Asymptotic Values of $\mathcal{V}_l^m$ and $\mathcal{W}_l^m$ for s approaching ∞. We want to consider the asymptotic values of $\mathcal{V}_l^m$ and $\mathcal{W}_l^m$ for large s. In this paper we will restrict ourselves to the frequency range $s > kd$ or $\lambda > 2\pi b$. Now it can be shown that for large s

$$| z | \leqq 1 \tag{4.1}$$

and x has the bounds

$$kd2^{1/2} \left[\frac{s - \sqrt{s^2 - 1}}{s - \cos \eta}\right]^{1/2} \leqq x \leqq kd2^{1/2} \left[\frac{s + \sqrt{s^2 - 1}}{s - \cos \eta}\right]^{1/2}$$

for t in the interval $-\pi \leqq t \leqq \pi$.

Hence, using the integral representation Eqn. (3.9) for $\mathcal{V}_l^m$ we have

$$\begin{aligned}\mathcal{V}_l^m(s, \eta) &= \frac{(-i)(2l)!}{2^l(l)!} \int_{-\pi}^{\pi} e^{imt} z^l x^{-l-1} e^{ix}\, dt + (-i)^{l+1} \\ &\quad \cdot \sum_{r=0}^{l-1} \frac{(l+1)_r (-l)_r}{(r)!(2i)^r} \int_{-\pi}^{\pi} e^{imt} z^l x^{-r-1} e^{ix}\, dt \\ &= \frac{-i(2l)!}{2^l(l)!} \int_{-\pi}^{\pi} e^{imt} z^l x^{-l-1}\, dt + R(s, \eta)\end{aligned} \tag{4.2}$$

The remainder $R(s, \eta)$ comprises of terms of the form

$$\int_{-\pi}^{\pi} e^{imt} z^l x^{-r-1} e^{ix}\, dt \qquad r = 0, 1, 2, \cdots, l - 1 \tag{4.3}$$

and

$$\int_{-\pi}^{\pi} e^{imt} z^l x^{-l-1} (e^{ix} - 1)\, dt \tag{4.4}$$

However

$$\left| \int_{-\pi}^{\pi} e^{imt} z^l e^{ix} x^{-r-1}\, dt \right| \leqq \int_{-\pi}^{\pi} x^{-r-1}\, dt \tag{4.5}$$

and

$$\left| \int_{-\pi}^{\pi} e^{imt} z^l x^{-l-1} (e^{ix} - 1)\, dt \right| \leqq \int_{-\pi}^{\pi} x^{-l}\, dt \tag{4.6}$$

But

$$\begin{aligned} \int_{-\pi}^{\pi} x^{-r-1}\, dt &= \frac{(s - \cos\eta)^{(r+1)/2}}{(kd)^{r+1} 2^{(r+1)/2}} \int_{-\pi}^{\pi} (s + \sqrt{s^2 - 1}\cos t)^{-(r+1)/2}\, dt \\ &= \frac{2\pi(s - \cos\eta)^{(r+1)/2}}{(kd)^{r+1} 2^{(r+1)/2}} P_{-(r+1)/2}(s) \\ &\underset{s>1}{\sim} \frac{\pi^{1/2}\Gamma(r/2) s^r}{(kd)^{r+1}\Gamma(r/2 + \frac{1}{2})}, \qquad r \neq 0 \end{aligned} \tag{4.7}$$

Hence we see that for large s and $s > kd$, $R(s, \eta)$ is of the order of $s^{l-1}/(kd)^l$. Thus it follows that

$$\mathfrak{V}_l^m \sim \frac{-i(2l)!}{2^l(l)!} \int_{-\pi}^{\pi} e^{imt} z^l x^{-l-1}\, dt + O\left(\frac{s^{l-1}}{(kd)^l}\right), \quad l \neq 0 \tag{4.8}$$

We will evaluate the remaining integral in (4.8) exactly. Expand z^l as a power series to obtain

$$\begin{aligned} \int_{-\pi}^{\pi} e^{imt} z^l x^{-l-1}\, dt = \frac{(kd)^{-l-1}}{2^{l+1/2}} \sum_{n=0}^{l} \frac{(-l)_n(-1)^n(s^2 - 1)^{n/2}}{(n)!(s - \cos\eta)^{n-l-1/2}} \\ \cdot \int_{-\pi}^{\pi} e^{i(m+n)t}(s + \sqrt{s^2 - 1}\cos t)^{-l-1/2}\, dt \end{aligned} \tag{4.9}$$

and using Eqn. (3.1) the expression (4.9) becomes equal to

$$\frac{(kd)^{-l-1}}{2^{l+1/2}} \sum_{n=0}^{l} \frac{(-l)_n(-1)^n(s^2 - 1)^{n/2}}{(n)!(s - \cos\eta)^{n-l-1/2}} \frac{2\pi\Gamma(-l + \frac{1}{2})}{\Gamma(-l + \frac{1}{2} - m - n)} P_{l-1/2}^{-m-n}(s) \tag{4.10}$$

which can be written in the form

$$\frac{(l + \frac{1}{2})_m (kd)^{-l-1}\pi(-1)^m}{2^{l-1/2}} \cdot \left\{ (s - \cos\eta)^{l+1/2} \sum_{n=0}^{l} \frac{(-l)_n(m + l + \frac{1}{2})_n}{(n)!(s - \cos\eta)^n} (s^2 - 1)^{n/2} P_{l-1/2}^{-m-n}(s) \right\}$$

The expression inside the brackets is identically equal to the following [2]

$$(s - \cos\eta)^{1/2} \sum_{n=0}^{l} \frac{\epsilon_n(2l)!(-1)^n}{(l - n)!(l + n)!2^{l-1}} P_{n-1/2}^{-m}(s) \cos n\eta \tag{4.11}$$

where

$$\epsilon_n = 1 \text{ when } n = 1, 2, 3, \cdots$$

$$\epsilon_n = \tfrac{1}{2} \text{ when } n = 0.$$

Hence from (4.11), (4.9) and (4.8) we have

$$\mathcal{V}_l^m \sim \frac{(-1)^{l+m+1} i (kd)^{-l-1} \pi (2l)!(l+\frac{1}{2})_m}{2^{3l-3/2}(l)!} \cdot \sqrt{s-\cos\eta}\, P_{l-1/2}^{-m}(s) \cos l\eta + O(s^{l-1}) + O\left(\frac{s^{l-1}}{(kd)^l}\right), \quad l \neq 0 \tag{4.12}$$

In a similar manner it can be shown that the asymptotic expression for $\mathcal{W}_l^m$ is the following

$$\mathcal{W}_l^m \sim \frac{(-1)^{l+m+1} i (kd)^{-l-3} \pi (2l+1)!(l+\frac{3}{2})_m}{(l)!2^{3l+1/2}} \cdot \sqrt{s-\cos\eta}\, P_{l+1/2}^{-m}(s) \sin(l+1)\eta \tag{4.13}$$

We will consider the case $l = 0$ for $\mathcal{V}_0^m$ in more detail.

$$\mathcal{V}_0^m = (-1)\int_{-\pi}^{\pi} e^{imt+ix} x^{-1}\, dt = -i\int_{-\pi}^{\pi} e^{imt} x^{-1}\, dt - i\int_{-\pi}^{\pi} e^{imt} x^{-1}(e^{ix}-1)\, dt \tag{4.14}$$

$$= \frac{-i\pi 2^{1/2}\sqrt{s-\cos\eta}}{(\frac{1}{2})_m (kd)} P_{-1/2}^m(s) - i\int_{-\pi}^{\pi} e^{imt} x^{-1}(e^{ix}-1)\, dt \tag{4.15}$$

for $s \gg 1$ and $s > kd$ we have that

$$x = kd2\left(\frac{s}{s-\cos\eta}\right)^{1/2} \cos\frac{t}{2} + O\left(\frac{kd}{s}\right) \tag{4.16}$$

Hence the integral in (4.15) is approximately equal to

$$\beta^{-1}\int_{-\pi}^{\pi} e^{imt}\int_0^{\beta} e^{iy\cos t/2}\, dy\, dt + O\left(\frac{kd}{s}\right)$$

where $\beta = 2kd\,[(s)/(s-\cos\eta)]^{1/2}$. We can interchange the order of integration and then set $t = 2T - \pi$, to obtain

$$2\beta^{-1}(-1)^m \int_0^{\beta} dy \int_0^{\pi} e^{i2mT+iy\sin T}\, dt$$

which equals

$$2\beta^{-1}\pi(-1)^m \int_0^{\beta} \{J_{2m}(y) - iE_{2m(y)}\}\, dy \tag{4.17}$$

where $E_{2m}(y)$ is Weber's function. Thus the asymptotic value of $\mathcal{V}_0^m$ is

$$\mathcal{V}_0^m \sim \frac{-i\pi(s-\cos\eta)^{1/2} P_{-1/2}^m(s)}{2^{-1/2}(\frac{1}{2})_m (kd)} + 2\beta^{-1}(-1)^m \pi \cdot \int_0^{\beta} \{J_{2m}(y) - iE_{2m}(y)\}\, dy + O\left(\frac{kd}{s}\right) \tag{4.18}$$

5. Conclusion. The basic toroidal wave functions

$$\mathcal{S}_l^m = \int_{-\pi}^{\pi} e^{imt} z^l j_l(x)\, dt, \qquad \mathcal{S}_{-l-1}^m = \int_{-\pi}^{\pi} e^{imt} z^l j_{-l-1}(x)\, dt,$$

$$\mathfrak{T}_l^m = \frac{\sin \eta}{s - \cos \eta} \int_{-\pi}^{\pi} e^{imt} z^l x^{-1} j_{l+1}(x)\, dt,$$

$$\mathfrak{T}_{-l-1}^m = \frac{\sin \eta}{s - \cos \eta} \int_{-\pi}^{\pi} e^{imt} z^l x^{-1} j_{-l-2}(x)\, dt$$

which can be coupled to represent outgoing radiation from a ring source by the following

$$\mathcal{V}_l^m = \mathcal{S}_l^m + i(-1)^{l+1} \mathcal{S}_{-l-1}^m, \qquad \mathcal{W}_l^m = \mathfrak{T}_l^m + i(-1)^{l+2} \mathfrak{T}_{-l-1}^m$$

are more useful and simpler than the ones developed previously [2]. Besides having a simpler integral representation, and asymptotic values in the far field, they are orthogonal asymptotically as s approaches ∞ (i.e., for a thin ring) for a greater range of wavelengths than the wave functions [2].

Because of their asymptotic values for their rings, these wave functions will be extremely useful in problems involving radiation from thin rings. The previous set of Toroidal wave functions [2] have already been used to obtain the scattered field for an electromagnetic plane wave incident on a perfectly conducting thin ring [5].

Electromagnetic and acoustic radiation problems for thick rings or toroids are theoretically possible to solve, since the above set of wave functions form a complete set. However, the practical difficulty is at present in the matching of boundary conditions, since the wave functions are not orthogonal for any except thin rings.

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