RADIO SCIENCE, Vol. 2 (New Series), No. 2, February 1967

Present Views on Electrodynamics of Moving Media[1]

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(Received July 18, 1966; revised August 6, 1966)

The recent formulations on electrodynamics of moving media proposed by some authors are reviewed. It is pointed out that all these apparently new formulations can be uncovered in the earlier work of Minkowski and Born. It is recommended that this classical work should be thoroughly covered in the curriculum of educational institutions offering graduate courses in electromagnetic field theory.

1. Introduction

The recent controversy, if indeed it is a controversy, on the electrodynamics of moving media, began with the publication of a neoteric book on electromagnetism (Fano, Chu, and Adler, 1960). In that book, Minkowski's theory was severely criticized both for alleged internal imperfections and for inconsistency with experimental evidence. Later, in a series of two papers (Tai, 1964a,b), the author of this paper showed that Minkowski's theory not only gives the complete, correct answer but also comprises all the essential results of Panofsky and Phillips, and Boffi as well as that of Fano, Chu, and Adler. It appears, however, that some misunderstandings remain, as witnessed by a number of recent communications (Brown, 1964; Penfield, 1964, 1965; Szablya, 1965; Penfield and Haus, 1965; Tai, 1965). Since most of the authors are present at this meeting, we now have a wonderful opportunity to hear each viewpoint expressed in detail.

As one of the panel members, I will take advantage of this occasion to present a more detailed analysis of the various formulations based upon the earlier work of Minkowski and Born. It is hoped that such an analysis will reveal more clearly the intimate relations between the various formulations and will enhance my affirmation that the different formulations are but different manifestations of Minkowski's classical theory.

2. Maxwell-Minkowski Equations and the Transformation of the Field Vectors

As reviewed in detail by Sommerfeld (1952), Maxwell's equations are invarient in all inertial systems. The three independent field equations are

$$\nabla x\bar{E}=-\frac{\partial \bar{B}}{\partial t} \qquad (1)$$

$$\nabla x\bar{H}=\bar{J}+\frac{\partial \bar{D}}{\partial t} \qquad (2)$$

$$\nabla\cdot\bar{J}=-\frac{\partial \rho}{\partial t} \qquad (3)$$

where

$\bar{E}$, $\bar{D}$ = electric field vectors

$\bar{H}$, $\bar{B}$ = magnetic field vectors

$\bar{J}$, ρ = free-current and free-charge densities.

For convenience, we shall designate (1) to (3) as Maxwell's equations in the "indefinite" form as long as the constitutive relations between the field quantities are unknown or unspecified. They become "definite" when the constitutive relations are known.

3. The Relativistic Transformation of Field Vectors

When two inertial systems are moving uniformly with respect to each other in the z direction, as shown in figure 1, the field variables in these two systems transform according to the relations:

$$\bar{E}'=\bar{\bar{\gamma}}\cdot(\bar{E}+\bar{v}x\bar{B}) \qquad (4)$$

$$\bar{B}'=\bar{\bar{\gamma}}\cdot\left(\bar{B}-\frac{1}{c^2}\bar{v}x\bar{E}\right) \qquad (5)$$

$$\bar{H}'=\bar{\bar{\gamma}}\cdot(\bar{H}-\bar{v}x\bar{D}) \qquad (6)$$

$$\bar{D}'=\bar{\bar{\gamma}}\cdot\left(\bar{D}+\frac{1}{c^2}\bar{v}x\bar{H}\right) \qquad (7)$$

[1] The material contained here is the same as that presented at the 1966 IEEE International Convention.

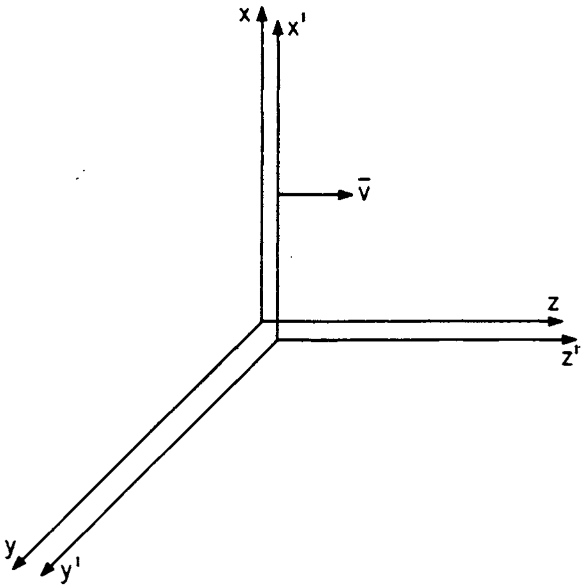

FIGURE 1.

$$\bar{J}' = \frac{1}{\sqrt{1-\beta^2}}\,\bar{\bar{\gamma}}^{-1}\cdot(\bar{J}-\rho\bar{v}) \tag{8}$$

$$\rho' = \frac{1}{\sqrt{1-\beta^2}}\left(\rho - \frac{1}{c^2}\bar{v}\cdot\bar{J}\right) \tag{9}$$

where $c = (\mu_0\epsilon_0)^{-1/2}$

$\beta = v/c$

$$\bar{\bar{\gamma}} = \begin{bmatrix} (1-\beta^2)^{-1/2} & 0 & 0 \\ 0 & (1-\beta^2)^{-1/2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $\bar{\bar{\gamma}}^{-1}$ denotes the reciprocal of $\bar{\bar{\gamma}}$. If one introduces two material field vectors $\bar{P}$ and $\bar{M}$ such that

$$\bar{D} = \epsilon_0\bar{E} + \bar{P} \tag{10}$$

$$\bar{B} = \mu_0(\bar{H}+\bar{M}) \tag{11}$$

and similarly,

$$\bar{D}' = \epsilon_0\bar{E}' + \bar{P}' \tag{12}$$

$$\bar{B}' = \mu_0(\bar{H}'+\bar{M}') \tag{13}$$

then by substituting (10) through (13) into (4) through (7), one finds that the material field vectors transform according to the following relations:

$$\bar{P}' = \bar{\bar{\gamma}}\cdot\left(\bar{P} - \frac{1}{c^2}\bar{v}x\bar{M}\right) \tag{14}$$

$$\bar{M}' = \bar{\bar{\gamma}}\cdot(\bar{M}+\bar{v}x\bar{P}). \tag{15}$$

4. Various Indefinite Forms of Maxwell's Equations

To derive the various indefinite forms, we first write the Maxwell's equations in the EPHM form, i.e.,

$$\nabla x\bar{E} = -\frac{\partial}{\partial t}\mu_0(\bar{H}+\bar{M}) \tag{16}$$

$$\nabla x\bar{H} = \bar{J} + \frac{\partial}{\partial t}(\epsilon_0\bar{E}+\bar{P}). \tag{17}$$

Since the continuity equation relating $\bar{J}$ and ρ is the same for all indefinite forms, we do not have to consider it again. By solving (14) and (15) for $\bar{P}$ and $\bar{M}$ and substituting them into (16) and (17), one obtains

$$\nabla x\bar{E} = -\frac{\partial}{\partial t}\mu_0[\bar{H}+\bar{\bar{\gamma}}\cdot(\bar{M}'-\bar{v}x\bar{P}')] \tag{18}$$

$$\nabla x\bar{H} = \bar{J} + \frac{\partial}{\partial t}\left[\epsilon_0\bar{E}+\bar{\bar{\gamma}}\cdot\left(\bar{P}'+\frac{1}{c^2}\bar{v}x\bar{M}'\right)\right]. \tag{19}$$

Equations (18) and (19) had previously been derived by Born (1910). Now, let us define

$$\bar{\bar{\gamma}}\cdot\bar{P}' = \bar{P}_c \tag{20}$$

$$\bar{\bar{\gamma}}\cdot\bar{M}' = \bar{M}_c \tag{21}$$

$$\epsilon_0\bar{E} + \frac{1}{c^2}\bar{\bar{\gamma}}\cdot(\bar{v}x\bar{M}') = \epsilon_0\bar{E}_c \tag{22}$$

$$\bar{H} - \bar{\bar{\gamma}}\cdot(\bar{v}x\bar{P}') = \bar{H}_c \tag{23}$$

then (18) and (19) can be written in the form

$$\nabla x(\bar{E}_c - \mu_0\bar{v}x\bar{M}_c) = -\frac{\partial}{\partial t}\mu_0(\bar{H}_c+\bar{M}_c) \tag{24}$$

$$\nabla x(\bar{H}_c + \bar{v}x\bar{P}_c) = \bar{J} + \frac{\partial}{\partial t}(\epsilon_0\bar{E}_c+\bar{P}_c). \tag{25}$$

Equations (24) and (25) are the same as the ones obtained by Fano, Chu, and Adler (1960) using a kinematic method. They can also be derived by means of the method of motional flux (Tai, 1964a). The present derivation, however, shows more clearly the role played by the field variables in the EPHMv formulation. We can trace the origin of these quantities in terms of the conventional quantities which define them, namely (20) through (23).

If we write (18) in the form

$$\nabla x\overline{E} = -\frac{\partial \overline{B}}{\partial t}, \tag{26}$$

(19) can be changed into

$$\nabla x\left(\frac{\overline{B}}{\mu_0} - \overline{M}_c + \overline{v}x\overline{P}_c\right) = \overline{J} + \frac{\partial}{\partial t}\left(\epsilon_0\overline{E} + \overline{P}_c + \frac{1}{c^2}\overline{v}x\overline{M}_c\right). \tag{27}$$

Equations (26) and (27) correspond to the EPBMv form. They were first described by Panofsky and Phillips (1950) for the special case when $\overline{M}_c = 0$.[2] It should be pointed out that the material vectors $\overline{M}_c$ and $\overline{P}_c$ appearing in the EPHMv form and in the EPBMv form are the same and are related to the material vectors defined in the primed system by (20) and (21). Table 1 summarizes the various indefinite forms.

TABLE 1. *Some Indefinite Forms of Maxwell's Equations*

Form	Equivalent Quantities			
EDBH	$\overline{E}$	$\overline{D}$	$\overline{B}$	$\overline{H}$
EPHM	$\overline{E}$	$\epsilon_0\overline{E} + \overline{P}$	$\mu_0(\overline{H} + \overline{M})$	$\overline{H}$
EPBM	$\overline{E}$	$\epsilon_0\overline{E} + \overline{P}$	$\overline{B}$	$\frac{\overline{B}}{\mu_0} - \overline{M}$
EPHMv	$\overline{E}_c - \mu_0\overline{v}x\overline{M}_c$	$\epsilon_0\overline{E}_c + \overline{P}_c$	$\mu_0(\overline{H}_c + \overline{M}_c)$	$\overline{H}_c + \overline{v}x\overline{P}_c$
EPBMv	$\overline{E}$	$\epsilon_0\overline{E} + \overline{P}_c + \frac{1}{c^2}\overline{v}x\overline{M}_c$	$\overline{B}$	$\frac{\overline{B}}{\mu_0} - \overline{M}_c + \overline{v}x\overline{P}_c$
Remarks:	$\overline{P}_c = \overline{\overline{\gamma}}\cdot\overline{P}' = \overline{\overline{\gamma}}\cdot\overline{\overline{\gamma}}\cdot(\overline{P} - \frac{1}{c^2}\overline{v}x\overline{M})$			
	$\overline{M}_c = \overline{\overline{\gamma}}\cdot\overline{M}' = \overline{\overline{\gamma}}\cdot\overline{\overline{\gamma}}\cdot(\overline{M} + \overline{v}x\overline{P})$			

The EPBM form was used by Boffi (1957) in his study of electrodynamics of moving media.

Before we discuss the constitutive relations we would like to call attention to the apparently different expressions for the Lorentz force. From table 1, one can easily show that

$$\overline{E} + \overline{v}x\overline{B} = \overline{E}_c + \mu_0\overline{v}x\overline{H}_c \tag{28}$$

and

$$\overline{H} - \overline{v}x\overline{D} = \overline{H}_c - \epsilon_0\overline{v}x\overline{E}_c. \tag{29}$$

In the work of Fano, Chu, and Adler (1960), $\overline{E}_c + \mu_0\overline{v}x\overline{H}_c$ and $\overline{H}_c - \epsilon_0 vx\overline{E}_c$ are postulated as the forces exerted respectively on a unit electric charge and a unit "magnetic pole" in order to give a vivid "physical" interpretation of the EPHMv formulation. It is clear from the present derivation that such postulates are not necessary. In fact, our present knowledge indicates that the science of electromagnetism is just the science of electricity. Electric charges and electric currents are, so far, known to be the sole agencies responsible for the electromagnetic field. Except, perhaps, for historical reasons, electricity can be taught without magnetism. Those who do not share this point of view should refer to Bitter (1959), one of the renowned authorities in magnetism, saying, "In this chapter and next we shall describe in detail the information available about magnetic poles—and in the end we shall conclude that there is no such thing."

5. Constitutive Relations for Uniformly Moving Isotropic Media

Minkowski, recognizing the invariant property of Maxwell's equations for material media, was the first to apply the special theory of relativity to determine the constitutive relations for a uniformly moving isotropic medium, provided its properties at rest are known. If one identifies the primed system as being at rest with respect to an isotropic, lossless medium, then

$$\overline{D}' = \epsilon'\overline{E}' \tag{30}$$

$$\overline{B}' = \mu'\overline{H}'. \tag{31}$$

By substituting these relations into (4) through (6) and simplifying the results, one finds

$$\overline{D} + \frac{1}{c^2}\overline{v}x\overline{H} = \epsilon'(\overline{E} + \overline{v}x\overline{B}) \tag{32}$$

$$\overline{B} - \frac{1}{c^2}\overline{v}x\overline{E} = \mu'(\overline{H} - \overline{v}x\overline{D}). \tag{33}$$

By solving for $\overline{B}$ and $\overline{D}$ in terms of $\overline{E}$ and $\overline{H}$ from (32) and (33), one obtains

$$\overline{D} = \epsilon'\overline{\overline{\alpha}}\cdot\overline{E} + \overline{\Omega}x\overline{H} \tag{34}$$

$$\overline{B} = \mu'\overline{\overline{\alpha}}\cdot\overline{H} - \overline{\Omega}x\overline{E} \tag{35}$$

where $\overline{\Omega} = \frac{(n^2-1)\beta}{(1-n^2\beta^2)c}\hat{z}$

$$\beta = v/c, \qquad n = \left(\frac{\mu'\epsilon'}{\mu_0\epsilon_0}\right)^{1/2}$$

$$\overline{\overline{\alpha}} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad a = \frac{1-\beta^2}{1-n^2\beta^2}.$$

[2] Dr. Robert C. Costen called the author's attention to the fact that the EPBMv form was first described by H. A. Lorentz.

By means of these relations, the constitutive relations for the field vectors defined in other indefinite forms can easily be derived. In the first place, we have

$$\begin{aligned}\overline{P}' &= (\epsilon' - \epsilon_0)\overline{E}' \\ &= (\epsilon' - \epsilon_0)\overline{\overline{\gamma}} \cdot (\overline{E} + \overline{v}x\overline{B}) \\ &= (\epsilon' - \epsilon_0)\overline{\overline{\gamma}} \cdot \overline{\overline{\alpha}} \cdot (\overline{E} + \mu'\overline{v}x\overline{H}) \end{aligned} \quad (36)$$

and

$$\begin{aligned}\overline{M}' &= (\mu' - \mu_0)\overline{H} \\ &= (\mu' - \mu_0)\overline{\overline{\gamma}} \cdot (\overline{H} - \overline{v}x\overline{D}) \\ &= (\mu' - \mu_0)\overline{\overline{\gamma}} \cdot \overline{\overline{\alpha}} \cdot (\overline{H} - \epsilon'\overline{v}x\overline{E}). \end{aligned} \quad (37)$$

The constitutive relations for the material vectors contained in the EPHMv formulation, for example, are given by

$$\begin{aligned}\overline{P}_c &= \overline{\overline{\gamma}} \cdot \overline{P}' = (\epsilon' - \epsilon_0)\overline{\overline{\gamma}} \cdot \overline{\overline{\gamma}} \cdot (\overline{E} + \overline{v}x\overline{B}) \\ &= (\epsilon' - \epsilon_0)\overline{\overline{\gamma}} \cdot \overline{\overline{\gamma}} \cdot (\overline{E}_c + \mu_0\overline{v}x\overline{H}_c) \end{aligned} \quad (38)$$

and

$$\begin{aligned}\overline{M}_c &= \overline{\overline{\gamma}} \cdot \overline{M}' = (\mu' - \mu_0)\overline{\overline{\gamma}} \cdot \overline{\overline{\gamma}} \cdot (\overline{H} - \overline{v}x\overline{D}) \\ &= (\mu' - \mu_0)\overline{\overline{\gamma}} \cdot \overline{\overline{\gamma}} \cdot (\overline{H}_c - \epsilon_0\overline{v}x\overline{E}_c). \end{aligned} \quad (39)$$

Equations (38) and (39) had previously been given by Fano, Chu, and Adler (1960), based upon a rather complicated scheme as discussed in their four-dimensional formulations. Finally, we would like to remark that in view of (6), (8), (28) and (29), we have

$$\overline{E}_c + \mu_0\overline{v}x\overline{H}_c = \overline{\overline{\gamma}}^{-1} \cdot \overline{E}' \quad (40)$$

$$\overline{H}_c - \epsilon_0\overline{v}x\overline{E}_c = \overline{\overline{\gamma}}^{-1} \cdot \overline{H}'. \quad (41)$$

By solving (40) and (41) for $\overline{E}_c$ and $\overline{H}_c$ in terms of $\overline{E}'$ and $\overline{H}'$, we obtain

$$\overline{E}_c = \overline{\overline{\gamma}} \cdot (\overline{E}' - \mu_0\overline{v}x\overline{H}') \quad (42)$$

$$\overline{H}_c = \overline{\overline{\gamma}} \cdot (\overline{H}' + \epsilon_0\overline{v}x\overline{E}'). \quad (43)$$

Equations (42) and (43) together with (20) and (21) reveal more clearly the real nature of the EPHMv formulation from the point of view of Minkowski's theory. These four relations, of course, are independent of the constitutive relations of the material medium.

6. Conclusion

A detailed presentation of various indefinite forms of Maxwell's equations for moving media has been given. Minkowski's theory and Born's equations were used to coordinate the different formulations. In particular, it has been shown that the magnetic pole model is not necessary to establish the EPHMv formulation. Furthermore, the constitutive relations in various formulations can also be obtained directly from Minkowski's theory without an independent relativistic investigation. Since the classical work of Minkowski and Born contains an enormous resource of information, it is recommended that a complete coverage of Minkowski's theory be included in the curriculum of educational institutions offering graduate courses in electromagnetic field theory.

The author is most grateful to Professor Dale Grimes for several valuable discussions.

The research reported in this paper was supported in part by the Langley Research Center of NASA under Grant Number NGR-23-005-107.

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