Technical Note:
Capacity Expansion and Cost Efficiency Improvement in the Warehouse Problem

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Abstract

The warehouse problem with deterministic production cost, selling prices, and demand was introduced in the 1950's and there is a renewed interest recently due to its applications in energy storage and arbitrage. In this paper we consider two extensions of the warehouse problem, and develop efficient computational algorithms for finding their optimal solutions. First, we consider a model where the firm can invest in capacity expansion projects for the warehouse while simultaneously making production and sales decisions in each period. We show that this problem can be solved with a computational complexity that is linear in the product of the length of the planning horizon and the number of capacity expansion projects. We then consider a problem in which the firm can invest to improve production cost efficiency while simultaneously making production and sales decisions in each period. The resulting optimization problem is non-convex with integer decision variables. We show that under some mild conditions on the cost data, the problem can be solved in linear computational time.

1 Introduction

Cahn [3] introduces the deterministic warehouse problem as follows: "Given a warehouse with fixed capacity and an initial stock of a certain product, which is subject to known seasonal price and cost variations, what is the optimal pattern of purchasing (or production), storage and sales?" This problem has a finite planning horizon, and a procurement and a sales quantity are determined for each period, with costs and revenues proportional to the chosen volumes. As a linear programming (LP) with very special structure, Charnes and Cooper [4] showed that it can be solved with a linear time algorithm. Under the optimal policy, in each period only one of four possible actions is taken, a property already shown by Bellman [2]:

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(i) produce and sell nothing;

(ii) produce nothing and sell all available inventory;

(iii) produce up to the available capacity and sell nothing;

(iv) produce up to the available capacity and sell all available inventory.

Moreover, the action choice for each period is independent of the prevailing capacity volumes.

In this paper we address two generalizations of the warehouse problem. In the first generalization, we allow the decision maker to expand the capacity in any period by implementing one of n distinct expansion projects. Any given project may be implemented in at most one period and the expansion costs are time and project dependent. We exploit the fact that the optimal operational policy is independent of the capacity values, to project the problem into one in which only capacity expansion decisions need to be made, and show that this problem can be solved in an amount of time that is linear in the product of the number of periods and the number of projects. We then consider the cost efficiency improvement problem in which the firm can invest to reduce production costs while simultaneously making purchasing and sales decisions. This gives rise to a non-convex optimization problem with integer decision variables. We focus on a special case of this problem and present an efficient algorithm to solve it. These models can be useful in investment related optimization problems in energy storage and arbitrage. For example, the warehouse capacity expansion problem may be used for scheduling the installation of additional storage units, while the warehouse cost efficiency improvement problem could be used to determine the measures that should be taken to attain high storage efficiency.

This work is motivated by the first author's industrial experience in petrochemical plants. In these production facilities the firm is always searching for ways to improve profits and teams of consultants are usually hired to study options to improve productivity. The outcome of these studies is a list of project proposals, each with different impacts on the facility's profits, that range from zero or low cost operational improvements (such as equipment load management) to major capital intensive projects (e.g., installing steam turbine generators to produce electricity from the plant's excess heat). Timing of the project execution is critical as early execution can have longer lasting benefits, while delaying the execution could also have merit because the economic factors (such as construction raw material and production interruption costs) may become favorable in later time periods. The majority of these project opportunities are categorized as either *capacity debottlenecking* or *cost efficiency* projects. The firm needs to determine which projects to implement and when to implement them, while maintaining an optimal operations schedule.

The warehouse problem can also be considered as an Economic Lot Sizing (ELS) problem with pricing and inventory bounds. The ELS problem with pricing, first introduced by Thomas [12], considers an ELS problem in which a facility simultaneously chooses its production level and selling price in every time period, where the demand in a period is a known decreasing function of the selling price. Subsequent formulations, such as Geunes et al. [6], use the demand and production levels as decision variables, with the revenue given as a function of the demand. Note that since the demand is a decreasing function of selling price, the pricing decision can be transformed to demand or sales quantity decision, and the ELS problem reduces to the warehouse problem when the demand is linearly decreasing in price. Although this ELS formulation captures most of the warehouse problem's dynamics, the ELS with pricing literature does not impose warehouse capacity limits. The warehouse capacity limits are accounted for using inventory bound constraints as studied in Hwang and van den Heuvel [7] and Hwang et al. [8], among others, in which no pricing decisions are considered.

Recent years have shown a renewed research interest in the warehouse problem. An energy storage element (e.g., underground gas storage or electric batteries) can be modeled as a warehouse, and the warehouse problem lends itself to the analysis of price arbitrage and commodity trading as well as operational policies for fixed storage capacities (see for example [5], [9], [11], [13], and [14]). Several extensions of the warehouse model have been studied to address these applications. For example, Rempala [10] introduces a production rate capacity limitation, Secomandi [11] applies rate capacities on both production and sales, and Lai et al. [9] study the case when prices in different periods are dependent.

The classic warehouse problem is introduced in the following section together with its optimal solution. Section 3 studies the capacity expansion while Section 4 studies cost efficiency improvement in the warehouse problem. The paper concludes with a discussion in Section 5.

2 Warehouse Production and Sales Planning

In this section we introduce a slightly generalized version of the original warehouse problem solved by Charnes and Cooper [4], and derive some simple results; these preliminary analyses will be used in the subsequent sections on capacity expansion and cost efficiency improvement problems.

In the classic warehouse problem, the firm determines its production quantities x_t and the sales quantities y_t of a single product in every time period over a planning horizon of T periods. The sales in time t, y_t, take place at the beginning of the time period at a known price of p_t, and the production in time t, x_t, is completed by the end of the period at a known cost of c_t, so products made in period t can be sold starting from period $t+1$, i.e., the production lead time is one period. Both the selling prices p_t and product costs c_t are known in advance. Unsold products can be stored in the warehouse and sold in later periods. The objective of the firm is to maximize its total profit over the planning horizon.

To formulate the warehouse problem, we denote by I_t the warehouse's inventory at the end of period t. The system dynamics can be written as $I_t = I_{t-1} + x_t - y_t$ with $y_t \leq I_{t-1}$. The initial inventory level I_0 is known. There is no limit on the production and sales rates, but the warehouse has a known storage capacity $B_t > 0$ in time t which is non-decreasing over time, thus $I_t \leq B_t$ for $t = 0, 1, \ldots, T$. Note that the original warehouse problem has a single capacity B for all time periods, but we need this generalization for our model in §3 where the B_t's become decision variables.

The formulation above can be simplified by noticing

$$I_t = I_0 + \sum_{\tau=1}^{t}(x_\tau - y_\tau), \quad t = 1, \ldots, T, \tag{1}$$

which allows us to eliminate the decision variables $I_t, t = 1, \ldots, T$. This substitution would also reduce the number of constraints by T because there would be no need to specify $I_t = I_{t-1} + x_t - y_t$. This problem can be formulated as the linear programming **WH** below.

$$\textbf{WH:} \qquad \max \quad \sum_{t=1}^{T}(p_t y_t - c_t x_t) \tag{2}$$

$$\text{s. t.} \quad \sum_{\tau=1}^{t} y_\tau - \sum_{\tau=1}^{t-1} x_\tau \leq I_0, \quad t = 1, \ldots, T, \tag{3}$$

$$\sum_{\tau=1}^{t}(x_\tau - y_\tau) \leq B_t - I_0, \quad t = 1, \ldots, T, \tag{4}$$

$$x_t, y_t \geq 0, \quad t = 1, \ldots, T. \tag{5}$$

The **WH** problem has linear objective function and linear constraints, hence the KKT conditions are both necessary and sufficient for optimality. We associate the dual variables λ_t and μ_t for $t = 1, \ldots, T$ to the sets of constraints (3) and (4) respectively, then the Lagrangian for this problem is

$$\begin{aligned} \mathcal{L}^{\mathbf{WH}}(\mathbf{x}, \mathbf{y}; \lambda, \mu) \;=\; & \sum_{t=1}^{T}(p_t y_t - c_t x_t) + \sum_{t=1}^{T} \lambda_t \left(I_0 + \sum_{\tau=1}^{t-1} x_\tau - \sum_{\tau=1}^{t} y_\tau \right) \\ & + \sum_{t=1}^{T} \mu_t \left(B_t - I_0 - \sum_{\tau=1}^{t} x_\tau + \sum_{\tau=1}^{t} y_\tau \right). \end{aligned} \tag{6}$$

The first order optimality conditions are obtained by taking the derivative of the Lagrangian with respect to each variable then imposing complementary slackness. We will denote the partial derivatives of $\mathcal{L}^{\mathbf{WH}}$ with respect to x_t, y_t, λ_t, and μ_t by $\mathcal{L}^{\mathbf{WH}}_{x_t}$, $\mathcal{L}^{\mathbf{WH}}_{y_t}$, $\mathcal{L}^{\mathbf{WH}}_{\lambda_t}$, and $\mathcal{L}^{\mathbf{WH}}_{\mu_t}$. We use I_t in (1) to simplify notation. The KKT optimality conditions are

$$x_t \geq 0; \quad \mathcal{L}^{\mathbf{WH}}_{x_t} = -c_t + \sum_{\tau=t+1}^{T} \lambda_\tau - \sum_{\tau=t}^{T} \mu_\tau \leq 0; \quad x_t \cdot \mathcal{L}^{\mathbf{WH}}_{x_t} = 0, \quad t = 1, \ldots, T \tag{7}$$

$$y_t \geq 0; \quad \mathcal{L}^{\mathbf{WH}}_{y_t} = p_t - \sum_{\tau=t}^{T} \lambda_\tau + \sum_{\tau=t}^{T} \mu_\tau \leq 0; \quad y_t \cdot \mathcal{L}^{\mathbf{WH}}_{y_t} = 0, \quad t = 1, \ldots, T \tag{8}$$

$$\lambda_t \geq 0; \quad \mathcal{L}^{\mathbf{WH}}_{\lambda_t} = I_{t-1} - y_t \geq 0; \quad \lambda_t \cdot \mathcal{L}^{\mathbf{WH}}_{\lambda_t} = 0, \quad t = 1, \ldots, T \tag{9}$$

$$\mu_t \geq 0; \quad \mathcal{L}^{\mathbf{WH}}_{\mu_t} = B_t - I_{t-1} + y_t - x_t \geq 0; \quad \mu_t \cdot \mathcal{L}^{\mathbf{WH}}_{\mu_t} = 0, \quad t = 1, \ldots, T. \tag{10}$$

Each of the formulas (7)-(10) has two inequalities and one equality that must be satisfied to ensure optimality. The two inequalities achieve primal and dual feasibility, respectively. The complementary slackness requires that at least one of these inequalities holds with equality, which is guaranteed by the third condition. We will solve this problem by developing an algorithm that satisfies these optimality conditions next.

The variables λ_t are duals to the constraints $y_t \leq I_{t-1}$, so λ_t has an economic interpretation of the marginal benefit due to increasing inventory in period $t-1$, which is desirable when we want to sell more products in period t. The μ_t variables are duals to $I_t \leq B_t$, so μ_t can be interpreted as the marginal benefit due to increasing capacity in period t, which can be used to produce and store more products x_t. Based on this observation, the values of λ_t and μ_t for $t = 1, \ldots, T$ can be used to determine x_t and y_t for $t = 1, \ldots, T$. We obtain the optimal solution to **WH** by first finding the optimal λ_t and μ_t values, and then use them to calculate the optimal production and sales quantities x_t and y_t. To simplify notation, we will denote $\overline{\mu}_t = \sum_{\tau=t}^{T} \mu_\tau$ and $\overline{\lambda}_t = \sum_{\tau=t}^{T} \lambda_\tau$. Then they can be obtained recursively by, $\overline{\lambda}_{T+1} = \overline{\mu}_{T+1} = 0$, and for $t \leq T$,

$$\overline{\mu}_t = \max\{\overline{\lambda}_{t+1} - c_t, \overline{\mu}_{t+1}\}, \qquad \overline{\lambda}_t = \max\{\overline{\mu}_t + p_t, \overline{\lambda}_{t+1}\}.$$

The above analysis immediately extends the algorithm in [4] for the case with constant capacity $B_t \equiv B$, $t = 1, \ldots, T$. In the rest of this paper, we use the notation $a^+ = \max\{a, 0\}$ and the indicator function $\mathbf{1}_{\{A\}} = 1$ if event A is true and 0 if A is false.

Algorithm 1 (Optimal Production and Sales Schedule)**.**

> ***Step 1:*** *Start from the last time period* $t = T$ *and recursively calculate* $\mu_t = \left(\bar{\lambda}_{t+1} - \bar{\mu}_{t+1} - c_t\right)^+$ *then* $\lambda_t = \left(\bar{\mu}_t - \bar{\lambda}_{t+1} + p_t\right)^+$ *down to the first period* $t = 1$.
>
> ***Step 2:*** *Start from the first period* $t = 1$ *and recursively calculate* $y_t = I_{t-1} \cdot \mathbf{1}_{\{\lambda_t > 0\}}$ *then* $x_t = (B_t - I_{t-1} + y_t) \cdot \mathbf{1}_{\{\mu_t > 0\}}$ *up to the last period* $t = T$.

This algorithm has linear computational complexity $\mathcal{O}(T)$. It is straightforward to verify its optimality by checking that the KKT conditions of (7)-(10) are satisfied for a solution produced by this algorithm, implying that the algorithm produces the optimal solution.

Algorithm 1 gives a policy with the following 4 actions based on the values of λ_t and μ_t:

(i) (Sell all inventory, produce to capacity): when $\lambda_t > 0$ and $\mu_t > 0$;

(ii) (Sell all inventory, do not produce): when $\lambda_t > 0$ and $\mu_t = 0$;

(iii) (Do not sell, produce to capacity): when $\lambda_t = 0$ and $\mu_t > 0$; and

(iv) (Do not sell, do not produce): when $\lambda_t = 0$ and $\mu_t = 0$.

This policy is in line with Bellman's characterization of the four decision options. Using this policy, the firm can make an optimal decision knowing only λ_t and μ_t for every period $t = 1, \dots, T$. An important observation to make is that the marginal sales and production benefits λ_t and μ_t are independent of the warehouse's capacities B_t and the inventory I_t for all periods $t = 1, \dots, T$ because they are calculated independently from B_t and I_t in Step 1 of Algorithm 1. In essence, this implies that the λ_t and μ_t values only depend on the production costs c_t and sales prices p_t, so modifying the capacities for the different periods does not change the λ_t and μ_t values. This important observation will be used in solving the capacity expansion problem in §3.

A special case of problem **WH** is when production costs are non-increasing over time, i.e., c_t is non-increasing in t. It may seem beneficial in this case to withhold from production unless it is profitable to sell the entire production quantity in the immediate following period. In that scenario, the plant would naturally produce if the sales price in the following period is greater than the production cost in the current period, and refrain from production otherwise. Indeed, the following result shows that if the problem has no starting inventory and the costs are non-increasing over time, then Algorithm 1 gives a simple solution under which whatever is produced is sold in the following period.

Proposition 1. *If $I_0 = 0$ and costs are non-increasing over time, i.e., $c_1 \geq c_2 \geq \cdots \geq c_T$, an optimal solution to **WH** is*

$$x_t = I_t = y_{t+1} = \mathbf{1}_{\{p_{t+1} > c_t\}} B_t, \qquad t < T. \tag{11}$$

Proof. This result can be proved by verifying that the given solution satisfies the KKT condition. Here we apply a simpler argument, suggested by a referee.

Suppose that, in an optimal solution the firm produces a unit in period $t-2$ and sells it in period t for some period $t>2$. Then this unit occupies the warehouse capacity in period $t-1$. Now, delay the production of this unit to period $t-1$ (this can be done because the demand for period $t-1$ is the same for both cases). By $c_{t-2} \geq c_{t-1}$, the profit margin for the sale of this unit in period t is at least as high, but it frees up an additional unit of warehouse capacity for period $t-2$. Continuing this argument we either contradict the optimality of the assumed optimal solution, or we obtain an alternative optimal solution that produces only when it sells in the following period. Furthermore, if it is profitable to produce in a period t and sell it at the beginning of the next period $t+1$ (implying $c_t \leq p_{t+1}$), then it is clearly beneficial to produce as much as possible in period t, i.e., produce the capacity level $x_t = B_t$; and otherwise, the firm should produce nothing in period t. This shows that (11) is an optimal solution. □

3 Capacity Expansion Investment Problem

Consider the warehouse problem of the previous section, but now the firm can invest in projects to increase the warehouse capacity while scheduling production and sales. Specifically, suppose the firm can choose from N project options that have different expansion increments and costs. An investment in project $n \in \{1,\ldots,N\}$ in time $t \in \{1,\ldots,T\}$ costs g_{nt} and expands the capacity by b_n units. This capacity expansion becomes effective in period t and lasts to the end of the time horizon, so we assume that the project execution duration is negligible (but the results can be easily extended to include project execution lead times). The firm can execute a project at most once over the planning horizon and cannot have partial project execution decisions, and it needs to decide whether or not to execute each project, and if so in which time period. The firm's objective remains to maximize its total profit.

To formulate the mathematical programming problem, we use the binary decision variable $z_{nt} \in \{0,1\}$ for $n=1,\ldots,T$ and $t=1,\ldots,N$ to choose between project options, so $z_{nt}=1$ if project n is executed in time t and 0 otherwise. The constraints $\sum_{t=1}^{T} z_{nt} \leq 1$ for $n=1,\ldots,N$ ensure that each project is executed at most once over the planning horizon. The warehouse capacity in time t becomes $B_t = B_{t-1} + \sum_{n=1}^{N} b_n z_{nt}$. Similar to (1) in the previous section, we can eliminate decision variables B_t by expressing it in terms of the initial capacity B_0 and the investment decisions as

$$B_t = B_0 + \sum_{n=1}^{N} b_n \sum_{\tau=1}^{t} z_{n\tau}, \quad t=1,\ldots,T. \tag{12}$$

We relax the integer constraints of the z_{nt} variables and express this problem as the following linear program **WH-B**:

$$\textbf{WH-B:} \qquad \max \quad \sum_{t=1}^{T}(p_t y_t - c_t x_t - \sum_{n=1}^{N} g_{nt} z_{nt}) \tag{13}$$

$$\text{s. t.} \quad (3), (4), (5)$$

$$\sum_{t=1}^{T} z_{nt} \leq 1, \quad n = 1, \ldots, N \tag{14}$$

$$z_{nt} \geq 0, \quad n = 1, \ldots, N, t = 1, \ldots, T, \tag{15}$$

where B_t in (4) is defined by (12). Notice that we do not need to specify the constraint $z_{nt} \leq 1$ in **WH-B** because it is implied by (14) and (15). Although z_{nt}'s integrality constraints are ignored in this formulation, we will show in Theorem 2 that there exists an integer optimal solution to this LP relaxation.

As in **WH**, the optimization problem **WH-B** has a linear objective function and a set of linear constraints, which means that a solution that satisfies the first order KKT optimality conditions is optimal. We will use the same dual variables λ_t and μ_t, $t = 1, \ldots, T$ for constraints (3) and (4) defined in the previous section. We will also use the dual variables α_n, $n = 1, \ldots, N$ for the constraints (14), where α_n can be interpreted as the marginal benefit from executing project n, i.e., $\alpha_n = 0$ if the project is not implemented and should be implemented whenever $\alpha_n > 0$. The Lagrangian of problem **WH-B** is

$$\mathcal{L}^{\textbf{WH-B}}(\mathbf{x}, \mathbf{y}, \mathbf{z}; \lambda, \mu, \alpha) = \mathcal{L}^{\textbf{WH}}(\mathbf{x}, \mathbf{y}, \mathbf{z}; \lambda, \mu) - \sum_{t=1}^{T}\sum_{n=1}^{N} g_{nt} z_{nt} + \sum_{n=1}^{N} \alpha_n \left(1 - \sum_{t=1}^{T} z_{nt}\right). \tag{16}$$

Notice that we include the index $\mathbf{z}$ in $\mathcal{L}^{\textbf{WH}}$ because the B_t variables in problem **WH** were fixed, but we assume here that B_t depends on $z_{n\tau}, n = 1, \ldots, N, \tau = 1, \ldots, t$, as given by (12). Equivalently, the last term in (6) can be written as

$$\sum_{t=1}^{T} \mu_t \left(B_0 + \sum_{n=1}^{N} b_n \sum_{\tau=1}^{t} z_{n\tau} - I_0 + \sum_{\tau=1}^{t}(y_\tau - x_\tau)\right).$$

Because none of the x_t, y_t, λ_t, and μ_t variables appear outside $\mathcal{L}^{\textbf{WH}}$ in (16), the KKT conditions (7)-(10) apply to problem **WH-B**. Note that the z_{nt} variables appear in the B_t terms in (10). Furthermore, if we let $\mathcal{L}^{\textbf{WH-B}}_{z_{nt}}$ and $\mathcal{L}^{\textbf{WH-B}}_{\alpha_n}$ be $\mathcal{L}^{\textbf{WH-B}}$'s partial derivatives with respect to z_{nt} and α_n, then the following KKT conditions must also hold in an optimal solution to **WH-B**:

$$z_{nt} \geq 0; \quad \mathcal{L}^{\textbf{WH-B}}_{z_{nt}} = -g_{nt} + b_n \textstyle\sum_{\tau=t}^{T} \mu_\tau - \alpha_n \leq 0; \quad z_{nt} \cdot \mathcal{L}^{\textbf{WH-B}}_{z_{nt}} = 0, \quad t = 1, \ldots, T, n = 1, \ldots, N \tag{17}$$

$$\alpha_n \geq 0; \qquad \mathcal{L}^{\textbf{WH-B}}_{\alpha_n} = 1 - \textstyle\sum_{t=1}^{T} z_{nt} \geq 0; \qquad \alpha_n \cdot \mathcal{L}^{\textbf{WH-B}}_{\alpha_n} = 0, \quad n = 1, \ldots, N. \tag{18}$$

To obtain an optimal solution, we need to ensure that all the inequalities hold in the conditions (7)-(10), (17), and (18) and that at least one inequality holds with equality for each condition. We will present an algorithm that attains such a solution.

Notice that the KKT conditions (17) and (18) only depend on z_{nt}, α_n, and μ_t, and can be satisfied independently of x_t, y_t, and λ_t. Therefore, if we can find the optimal μ_t values from Step 1 of Algorithm 1 then pass them to another algorithm that finds z_{nt} and α_n for which the KKT conditions (17) and (18) hold, then the relaxed capacity expansion problem **WH-B** would be solved. Algorithm 2 below does just that.

Algorithm 2 (Optimal Capacity Expansion Algorithm)**.**

Step 1: *Run Step 1 of Algorithm 1 to get* λ_t *and* μ_t *for* $t = 1, \ldots, T$.

Step 2: *Start with* $z_{nt} = 0$ *for* $n = 1, \ldots, N$ *and* $t = 1, \ldots, T$. *Then, for* $n = 1, \ldots, N$, *set* $t_n = \mathrm{argmax}_t \{b_n\bar{\mu}_t - g_{nt}\}$ *(ties can be resolved arbitrarily),* $\alpha_n = [b_n\bar{\mu}_{t_n} - g_{nt_n}]^+$, *and* $z_{nt_n} = \mathbf{1}_{\{\alpha_n > 0\}}$.

Step 3: *Perform Step 2 of Algorithm 1 to get the production and sales schedules* x_t *and* y_t *for* $t = 1, \ldots, T$.

The following theorem establishes the optimality of the algorithm.

Theorem 2. *Algorithm 2 gives an optimal capacity expansion and implementation solution in time* $\mathcal{O}(NT)$.

Proof. Notice that all the z_{nt} and α_n values from Algorithm 2 are nonnegative, all z_{nt} assignments are binary, at most a single z_{nt} is 1, and since t_n is selected as the time period with the largest $b_n\bar{\mu}_t - g_{nt}$ among all time periods, we have $\alpha_n \geq b_n\bar{\mu}_t - g_{nt}$ for $n = 1, \ldots, N$ and $t = 1, \ldots, T$. Therefore, primal and dual feasibility hold for (17) and (18) and we need only to demonstrate complementary slackness to show optimality.

For a given n, the complementary slackness for (17) holds for $t \neq t_n$ since $z_{nt} = 0$ for $t \neq t_n$, so we need only to verify that complementary slackness holds for (17) in time period t_n. Now consider a project n for which $b_n\bar{\mu}_{t_n} \leq g_{nt_n}$. In this case $\alpha_n = (b_n\bar{\mu}_{t_n} - g_{nt_n})^+ = 0$ and $z_{nt_n} = \mathbf{1}_{\{\alpha_n>0\}} = 0$, so complementary slackness holds for (17) and (18). If on the other hand $b_n\bar{\mu}_{t_n} > g_{nt_n}$, then $\alpha_n = b_n\bar{\mu}_{t_n} - g_{nt_n} > 0$ and complementary slackness would hold for (17), and $z_{nt_n} = \mathbf{1}_{\{\alpha_n>0\}} = 1$, hence complementary slackness would also hold for (18) since $\sum_{t=1}^{T} z_{nt} = z_{nt_n} = 1$. Finally, since the z_{nt} values are binary, it follows that the algorithm also gives an optimal solution to the original problem.

To find the computational complexity of the algorithm, consider a project option n. The largest term $b_n\bar{\mu}_t - g_{nt}$ can be found by evaluating every term for $t = 1, \cdots, T$, which can be done in $\mathcal{O}(T)$. Since we have N project options, the algorithm runs in $\mathcal{O}(NT)$. □

4 Cost Efficiency Improvement Problem

In this problem, we are given a set of warehouse capacities $B_1, \ldots, B_T$, a set of prices in every period $p_1, \ldots, p_T$, a starting inventory level I_0, and a starting production cost c_0. Given M cost efficiency improvement projects, the firm decides on the projects to implement and their execution time periods to maximize its total profit. A project $m \in \{1, \ldots, M\}$ executed in time $t \in \{1, \ldots, T\}$ costs q_{mt} and leads to a constant unit production cost decrement k_m, which becomes effective immediately, so the production cost in time t is $c_t = c_{t-1} - k_m$. Therefore, if all cost efficiency improvements projects are implemented then the production cost would be reduced to $c_0 - \sum_{m=1}^{M} k_m$. A project may be executed at most once over the planning horizon and a selected project must be fully executed (i.e., partial project implementations are not allowed). Naturally, the production cost will still be positive even after implementing all projects, thus we assume $c_0 > \sum_{m=1}^{M} k_m$. We further assume in this section that c_t are non-increasing over time. As in the capacity investment problem, we assume without loss of generality that the project execution duration is negligible.

As with the capacity expansion problem, we let the binary variable $w_{mt} \in \{0, 1\}$ be 1 if project m is implemented in period t and 0 otherwise, $m \in \{1, \ldots, M\}$ and $t \in \{1, \ldots, T\}$. Following the simplification for I_t and B_t in (1) and (12), we express the cost in time t in terms of c_0 and the investment decisions as

$$c_t = c_0 - \sum_{m=1}^{M} k_m \sum_{\tau=1}^{t} w_{m\tau}, \quad t = 1, \ldots, T. \tag{19}$$

To ensure that projects are never executed more than once, we need to include the constraint $\sum_{t=1}^{T} w_{mt} \leq 1$, $m = 1, \ldots, M$. The objective function for this problem is

$$\sum_{t=1}^{T} \left(p_t y_t - c_t x_t - \sum_{m=1}^{M} q_{mt} w_{mt} \right).$$

Observe that the optimal operational policy given in §2 depends only on λ_t and μ_t, that are independent of the starting inventory level. In particular, when a sale is made under this policy, all inventory is sold, which means that the starting inventory level may only affect the first production period τ, i.e., the smallest $\tau \in \{1, \ldots, T\}$ with $\mu_\tau > 0$, which produces to raise the inventory level to the warehouse capacity B_τ. We will therefore assume without loss of generality that $I_0 = 0$ for this problem, which also implies that $y_1 = 0$. Moreover, given that the production costs are non-increasing over time, we can apply Proposition 1 to conclude that

$$y_{t+1} = x_t = I_t = \mathbf{1}_{\{p_{t+1} > c_t\}} B_t, \qquad \text{for } t = 1, \dots, T-1. \tag{20}$$

We further simplify our formulation by noticing that it is never economical to produce or invest in the last period, i.e., $x_T = w_{nT} = 0$ for all n. Substituting (20) into the objective function, we obtain the cost efficiency improvement problem as the following mathematical program:

$$\textbf{WH-C}: \qquad \max \sum_{t=1}^{T-1} \left((p_{t+1} - c_t)^+ B_t - \sum_{m=1}^{M} q_{mt} w_{mt} \right) \tag{21}$$

$$\text{s. t.} \sum_{t=1}^{T-1} w_{mt} \leq 1, \quad m = 1, \dots, M, \tag{22}$$

$$w_{mt} \in \{0, 1\}, \quad m = 1, \dots, M. \tag{23}$$

The decision variables in this problem are w_{mt} for $m = 1, \dots, M$ and $t = 1, \dots, T-1$. Note that the objective function (21) is non-concave because the costs c_t depend on the w_{mt} variables as given by (19), hence this problem is difficult in general, and we do not have a polynomial time algorithm for finding its global optimal solution. In Al-Gwaiz et al. [1], we developed an $\mathcal{O}(M^2T^2)$ algorithm that obtains a local optimal solution. In this note we focus on the special case that the selling prices are not lower than the initial production cost, for which we can find the global optimal solution in polynomial time.

When $p_{t+1} \geq c_0$, we have $p_{t+1} \geq c_t$ and the objective function for the mathematical program above is simplified to

$$\begin{aligned}
& \sum_{t=1}^{T-1} \left((p_{t+1} - c_t) B_t - \sum_{m=1}^{M} q_{mt} w_{mt} \right) \\
= & \sum_{t=1}^{T-1} (p_{t+1} - c_0) B_t + \sum_{t=1}^{T-1} \sum_{m=1}^{M} k_m \sum_{\tau=1}^{t} w_{m\tau} B_t - \sum_{t=1}^{T-1} \sum_{m=1}^{M} q_{mt} w_{mt} \\
= & \sum_{t=1}^{T-1} (p_{t+1} - c_0) B_t + \sum_{t=1}^{T-1} \sum_{m=1}^{M} \left(k_m \sum_{\tau=t}^{T-1} B_\tau - q_{mt} \right) w_{mt}.
\end{aligned}$$

Since the first term is a constant, the optimization problem is equivalent to maximizing the second term, subjecting to constraints (22) and (23).

This simplified optimization problem is easy to solve, and it can be decomposed into M optimization subproblems, one for each cost efficiency project m:

$$\begin{aligned} \max \quad & \sum_{t=1}^{T-1}\left(k_m \sum_{\tau=t}^{T-1} B_\tau - q_{mt}\right) w_{mt} \\ \text{s. t.} \quad & \sum_{t=1}^{T-1} w_{mt} \leq 1, \\ & w_{mt} \in \{0,1\}. \end{aligned}$$

The optimal solution for the optimization problem above is easy to find: For cost efficiency project $m = 1, \ldots, M$, if $k_m \sum_{\tau=t}^{T-1} B_\tau - q_{mt} \leq 0$ for all $t = 1, \ldots, T-1$, then do not execute this project; otherwise, execute project m in period

$$t_m^* = \underset{t=1,\ldots,T-1}{\operatorname{argmax}} \left\{ k_m \sum_{\tau=t}^{T-1} B_\tau - q_{mt} \right\}.$$

Standard method can be applied to find t_m^* in $\mathcal{O}(T)$.

Proposition 3. *If $p_t \geq c_0$ for all t, i.e., the selling price in each period is no less than the initial production cost, then the cost efficiency improvement problem can be solved in polynomial computational time $\mathcal{O}(MT)$.*

5 Conclusion

In this note, we have considered the capacity expansion and cost efficiency improvement extensions of the classic warehouse problem and developed efficient computational algorithms to solve them. Other extensions are possible. For example, holding costs were not included in our models, but they can be easily incorporated. Since accounting for linear holding cost would introduce constant multiples to the inventory terms, it can be transformed into our problem with no holding costs but modified purchasing costs and selling prices. Another extension could be to include lead times for investments, which is also trivial because, given the deterministic nature of our problem, we can shift the investment times back by the lead time periods. A project to invest to boost sales prices would be similar in spirit to the cost efficiency improvement problem because it introduces the same nonlinear positive term in the objective function but on the y_t variables instead of x_t. Similarly, the cost efficiency improvement problem can also be used for a problem with holding cost reduction projects. More complicated options can be considered for future research, such as options that simultaneously increase capacity and reduce costs. Finally, we have assumed that all project options are independent in the sense that the cost of one project is not correlated with the cost of another one. It will be interesting to study the case where the cost of one project influences the cost of another.

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References

[1] M. Al-Gwaiz, X. Chao, and E. H. Romeijn. Capacity expansion and cost efficiency improvement in the warehouse problem. *Working Paper, Department of Industrial and Operations Engineering, University of Michigan, Ann Arbor, MI*, 2015.

[2] R. Bellman. On the theory of dynamic programming – a warehousing problem. *Management Science*, 2(3):272–275, 1956.

[3] A. S. Cahn. The warehouse problem. *Bulletin of the American Mathematical Society*, 54:1073, 1948.

[4] A. Charnes and W.W. Cooper. Generalizations of the warehousing model. *Operations Research*, 6(4):131–172, 1955.

[5] A. Faghih, M. Roozbehani, and M. A. Dahleh. Optimal utilization of storage and the induced price elasticity of demand in the presence of ramp constraints. In *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*, pages 842–847. IEEE, 2011.

[6] J. Geunes, E. Romeijn, and K. Taaffe. Requirements planning with pricing and order selection flexibility. *Operations Research*, 54(2):394–401, 2006.

[7] H.-C. Hwang and W. van den Heuvel. Improved algorithms for a lot-sizing problem with inventory bounds and backlogging. *Naval Research Logistics*, 59(3-4):244–253, 2012.

[8] H.-C. Hwang, W. van den Heuvel, and A. P. M. Wagelmans. The economic lot-sizing problem with lost sales and bounded inventory. *IIE Transactions*, 45(8):912–924, 2013.

[9] G. Lai, F. Margot, and N. Secomandi. An approximate dynamic programming approach to benchmark practice-based heuristics for natural gas storage valuation. *Operations Research*, 58(3):564–582, 2010.

[10] R. Rempala. *System Modelling and Optimization*, volume 197 of *Lecture Notes in Control and Information Sciences*, chapter Optimal strategy in a trading problem with stochastic prices, pages 560–566. Springer, Berlin, 1994.

[11] N. Secomandi. Optimal commodity trading with a capacitated storage asset. *Management Science*, 56(3):449–467, 2010.

[12] J. Thomas. Price-production decisions with deterministic demand. *Management Science*, 16 (11):747–750, 1970.

[13] O. Wu, D. Wang, and Z. Qin. Seasonal energy storage operations with limited flexibility: The price-adjusted rolling intrinsic policy. *Manufacturing & Service Operations Management*, 14 (3):455–471, 2012.

[14] Y. Zhou, A. Scheller-Wolf, N. Secomandi, and S. Smith. Managing wind-based electricity generation with storage and transmission capacity. *Available at SSRN 1962414*, 2011.