# How Theory-Building Research on Instruction can Support Instructional Improvement: Toward a Modeling Perspective in Secondary Geometry

Patricio Herbst[1]
*University of Michigan, USA*
<pgherbst@umich.edu>

How can basic research on mathematics instruction contribute to instructional improvement? In our research on the practical rationality of geometry teaching we describe existing instruction and examine how existing instruction responds to perturbations. In this talk I consider the proposal that geometry instruction could be improved by infusing it with activities where students use representations of figures to model their experiences with shape and space and I show how our basic research on high school geometry instruction informs the implementing and monitoring of such modeling perspective. I argue that for mathematics education research on instruction to contribute to improvements that teachers can use in their daily work our theories of teaching need to be mathematics-specific.

## Introduction

What is the role that research in mathematics education can play to support efforts toward instructional improvement? This seems like an impossibly general question and one that admits of many answers. I use it to point to an important need in our field's portfolio of activity: The need to develop theories of mathematics teaching that are mathematics-specific. I will explain what I mean by that and how such mathematics-specific theories of mathematics teaching can be instrumental in designing and studying regimes that can improve instruction and students' outcomes.

For over a decade I have been working in an area that Herbst & Chazan (2003, 2011, 2012) have named the practical rationality of mathematics teaching. With that expression we name an effort to provide the means to describe and understand the work of teaching mathematics in school classrooms. Our effort has included the development of constructs and methods for the study of the work of the teacher in mathematics instruction. We've carried out this effort in the context of researching the teaching of proof in geometry and the teaching of equation and word-problem solving algebra. The locations of our research have been more than contexts, though; they have served to highlight the value of attending to the specifics of mathematical work in theory-building research on mathematics teaching. I will be using examples from our work in secondary geometry to illustrate how a mathematics-specific theory and research on instruction can support the conception of instructional improvements and applied research on such improvement.

I make three points: generic theories of teaching are insufficient to support instructional improvement, the constructs afforded by practical rationality permit the development of theories of teaching that are mathematics-specific, and the case of improving the teaching of geometry from a modelling perspective illustrates how theories

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of teaching built with practical rationality constructs can support the work of instructional improvement.

### *Mathematics educators and instructional improvement: Where our efforts have been*

Our scholarly community has historically contributed to improving the practice of mathematics education. For example, researchers whose scholarship involved understanding students' opportunity to learn, their mathematical thinking, and their development have also been involved in the more practical work of developing standards, curricula, and assessments. In all of this work of research and development, our community has shown keen attention to the mathematics at stake. When members of our community have studied the learning of mathematics they have been more interested in *what* students learn and *how they use it* in mathematical problem solving than in *how much* they learn or in the general psychological characteristics of their thinking.

Our community's efforts to improve practice through the development of curricula and assessments have likewise been informed by the history and epistemology of mathematical practice. Our community has attended carefully to mathematical issues in our contributions to understanding students' learning and those have also shown in the development of curriculum materials and assessments. Current approaches to the improvement on the curriculum from a learning trajectories perspective (see Clements & Sarama, 2014) illustrate how this subject specificity in our studies of learning can help the work of teachers by giving them ways of understanding where their students are in their learning of specific ideas and tasks that can help the students grow. This is certainly one important way in which we have contributed to instructional improvement. As I elaborate below, the work of teaching has not been addressed from a mathematics-specific perspective in theory and research, but it could benefit from such perspective.

Our community's focus on students' thinking and learning brought important attention to the role of the teacher creating learning environments and observing students' work. The work of teaching includes such tasks and others that could be described in equally generic ways. I argue that such generic descriptions are however not sufficiently helpful to support instructional improvement: We need to attend to the role of subject matter in how we describe and understand the work of mathematics teaching (see Romberg & Carpenter, 1986; Chazan, Herbst, & Clark, 2016). I argue that a look at the work of teaching anchored in what is being taught and learned can be the basis for a key, unique contribution our community alone can make to instructional improvement. This is a contribution that we cannot expect from scholars who look at the act of teaching with the general perspectives of research on teaching and teacher education.

### *How mathematics educators became interested in teaching*

While mathematics educators' interest in the work of teaching in instruction is more recent than our interest in students' thinking and learning, interest in the work of teaching has been part of our community for a couple of decades. For centuries philosophers of education have had an interest in understanding the relationship between teacher and student as people shaping each other; in these philosophical approaches to teaching the content of studies has been seen as part of the context, a means to an end that might include social reproduction, individual actualization, or perhaps social justice. But these ideas seem to have been less of a part of our mathematics education community's concern when our field emerged in the crucible of the new math movement in the 50s and 60s. Those days seemed to have seen the teacher as someone who would deliver the riches of an improved curriculum to the student; with the design of the materials as the key to the quality of the students' experience (Remillard, 2005). Back in the day of the new math, the teacher was seen as somebody who had to be controlled so that the message of the curriculum could arrive to the children without blemishes (Clarke, Clarke, & Sullivan, 1996).

After the failure of the new math reform, later reforms placed interest not only on curriculum development but also on teacher development. The so-called "failure of the new math" has often been identified as a moment when those who work on reform realized the need to attend to teacher education and professional development and to the work that teachers do implementing those reforms. The Standards-based reforms placed a lot of attention to the quality of the tasks in which students would engage, capitalizing on research on students' thinking and problem solving. The notion that the success of reform would need teacher development led to efforts to educate teachers to propose worthwhile tasks to students, stay out of the way while students worked on those tasks, and manage classroom discourse in which students shared their thinking (NCTM, 1991). As Smith (1996) noted, the discourse addressed to teachers, while clear about when not to intervene, was patently unclear about what teachers could actively do to teach, to the point that it created a dilemma concerning teachers' sense of efficacy.

### *Are generic tasks of teaching a sufficient basis for a theory of mathematics teaching?*

The professional development offered to teachers attempted to enable teachers to support reform by developing capacity for some generic tasks of teaching such as proposing worthwhile tasks, noticing student thinking, and managing classroom discourse. These are generic tasks of teaching as they are, at least in principle, capable to be combined with many mathematical ideas that could be studied in a classroom. They clearly can be used to describe and assess at a coarse level of granularity what teachers do in mathematics classes (e.g., Silver, Mesa, Morris, Star, & Benken, 2009); they can serve that purpose just as much as words like *lecture* and *recitation* do. It is also possible to consider refinements of those tasks that increase granularity while maintaining their genericity. For example, the task of managing classroom discourse can and has been unpacked into sets of discursive moves that a teacher can make, for example to follow up on a student's response (e.g., confirm, press, revoice; see Brodie, 2011). Those generic descriptors for the work of teaching are economical in that they reduce the burden of knowledge of the observer--any act of mathematics teaching is describable by combining a generic task of teaching and an item of mathematical knowledge, which acts as object or argument of the task of teaching. That approach is clearly possible to use in the study of teaching by researchers and by novices. Those generic descriptors for the work of teaching can and have been useful to make observations across lessons with different content (e.g., see Stein, Grover, & Henningsen, 1996) as well as to organize the practice of teaching for initial teacher education (e.g., see Ghousseini & Herbst, 2016). But their capacity to support instructional improvement is limited, fundamentally because of two concerns that together amount to the need for addressing the mathematical specificity of the work of teaching.

The first concern is epistemological. Generic tasks of teaching create an impression of homogeneity of the practice of teaching mathematics across the different subject matter to be taught--they maintain epistemological considerations in a black box. Yet, epistemological considerations matter at the time of deciding whether one such task of teaching makes sense. For example, it is relatively easy for me to envision engaging students in a task to get them to define a mathematical concept; but I have a hard time thinking of engaging them in a task if my goal is to introduce a convention or a name; telling students directly what the convention or the name is seems more efficient and less manipulative to me. Likewise, it is relatively easy for me to consider asking "why would you say that?" when a student makes a mathematical assertion but not when a student reminds the class what the name of a concept is. In other words, the adequacy of application of generic tasks of teaching to mathematical objects, requires some epistemological mediation, some analysis of the nature of the mathematical objects being addressed, but those objects are rather unspecified in those generic tasks of teaching. Even the apparently subject-specific task of *correcting an error* requires the mathematical sophistication of the observer or of the student of teaching to decide whether an error has been made and how important it is to correct it (Hill & Grossman, 2013).

The second concern is practical and has to do with the extent to which such generic tasks of teaching really permit us to think of incremental improvement as a path toward reform. The rhetoric of reform and the critique of existing practices that goes with it may need to rely on stark contrasts as presented in slogans like "teacher-centered" versus "student-centered" instruction or generic practices such as "lecture" versus "discussion" (see Scheffler, 1960, pp. 36-46). But practicing teachers are in the predicament of having to do something new while they continue to do what they are expected to: Every change happens over a surface of background practices that remain constant. Thus they need to be able to capitalize on what they know how to do while they handle orientations to change and try changing specific practices; in other words, improvement needs to be incremental. A theory of mathematics teaching that could support incremental improvements over existing practices could also enable practicing teachers to maintain some degree of control of the consequences of such improvement. But while generic slogans like *reform teaching* and *student-centered instruction* may be useful to provide value orientations, we need theories of mathematics teaching that can anchor incremental ways of improving instruction in existing practices.

That is where the generic character of some of those tasks of teaching becomes a liability: Practicing teachers are never engaging students in a general task nor attending to what students are thinking or doing in generic terms; rather they engage students in doing particular tasks with specific instructional goals and they attend to what students are thinking or doing inasmuch as that gives them information about specific knowledge and skills they expect students might or might not have. In other words, the work of teaching mathematics is situated in the practice of teaching the specific content that features in a given course of studies. In particular, the range of options that a teacher has on what to do (for example what discursive move to use at a given moment in time) depends not only on what are the available discursive moves that a generic theory of teaching (such as, a theory of classroom discourse) makes available in general, but also on what their costs and benefits are in the situation in which the teacher may choose to use them. Those costs and benefits may vary depending on the situation. For example, to ask "why would you say that?" to a high school geometry student who is doing a proof and just wrote that two triangles are congruent, and to ask the same question, "why would you say that?," to a 4th grade student who just wrote a digit in one of the partial products of a multidigit multiplication are not really equivalent questions, even when one could code both as *press for explanation*: the question in the first situation is merely a counterpoint for what the student is expected to do next to a statement in a proof (Herbst, Chen, Weiss, & González, 2009) while the question in the second situation, in spite of being possibly productive, is likely to be experienced as an interruption (Lampert, 1986).

These two concerns, epistemological and practical, invite us to inquire what else might be possible. What would a theory of teaching look like that pays attention to the mathematical work of the teacher in practice? In our work on the practical rationality of mathematics teaching, we have been deliberately seeking to understand whether and how desirable practices might be anchored in existing practices. We are interested in finding out whether desirable practices can be seen as viable from the perspective of practitioners. To inspect such viability, we make the hypothesis that the subject and the classroom situation matter or that existing practice matters; and we try to anchor possible improvements in the teaching of specific ideas in descriptive accounts of existing, specific instructional practices. In an effort to sketch how our approach supports thinking of instructional improvement, I describe in the following sections how constructs of contract, situation, norm, and obligations help us get inside the work of the teacher in instruction and illustrate them with a case in geometry instruction.

## Instruction

By mathematics *instruction* I refer to the interactions among students, the teacher, and the mathematical content designated for students to learn, that occur inside environments such as classrooms, schools, and educational systems (Cohen, Raudenbush, & Ball, 2003). So defined, *instruction* not only adds the teacher's work to considerations of students' mathematical learning, but it also brings attention to the contexts, societal and institutional, in which such interactions exist. This definition of instruction highlights that instruction realizes three-way relationships among teacher, students, and content. The concept thus defined characterizes the role of the teacher as more than administering the curriculum to students; it highlights the possibility that the content may be altered, shaped in and through those interactions that purportedly serve to transact it. The definition also highlights that those interactions happen in broader environments that might shape them. The concept of instruction is a descriptive concept--in particular, it does not say what the role of the teacher *should* be, but it rather tries to capture what the role of the teacher is. For us the concept of instruction is the cornerstone of an approach to develop descriptive and explanatory theories of mathematics teaching (as opposed to prescriptive theories; see Silver & Herbst, 2007).

Cohen et al (2003) propose this view of instruction in response to earlier policy work that allocated causality for students' improved outcomes to having or not having resources. Cohen et al.'s (2003) view of resources is extensive--class size and teacher knowledge are two examples they provide; class size and teacher knowledge illustrate quite well that resources are more than the material resources often considered (e.g., technologies, textbooks, etc.). Cohen et al. (2003) point out, and illustrate with an analysis of class size, that despite the quantity and quality of the resources available in classrooms (i.e., despite the quality of material resources such as textbooks or technology, despite the amount of teacher knowledge or the makeup of the class), the quality of students' learning depends on the use of those resources in instruction. Cohen's et al.'s (2003) approach suggests to us that generic ideas about what teaching should be like (e.g., the notions of student-centered instruction or inquiry-based learning) are also resources; they are intellectual resources that make possible some ways of speaking and may inspire some actions.

Those intellectual resources are just as often proclaimed as capable to intervene in improving student outcomes as material resources are. Teaching has often been the object of prescription and professional development, using generic ideas like inquiry based learning and generic practices such as having students work in groups; but the relative impact of such prescriptions seems to have been low (Fullan & Miles, 1992). Examples of inquiry based lessons shown in professional development sessions get replicated in practice without necessarily making an impact in the teaching of the ideas of a course of studies, discussions may increase social engagement but at the expense of depth in the mathematical ideas discussed (Nathan & Knuth, 2003). It seems that we cannot merely talk about the impact of generic descriptions of kinds of teaching but rather we need to consider how these play out when they are used in instruction. While generic ideas and practices about teaching may be important resources to shape the disposition of teachers to work in certain ways, the use of those generic practices, like that of any other resource happens in the midst of interactions a teacher has with students and content in environments.

Cohen et al. (2003) speak of the need to characterize instructional regimes and to study their gains in students' learning. My argument that our research could contribute to instructional improvement builds on that definition of instruction: What may need to be improved is not only the content designated for students to learn or the design of the tasks in which the students could learn it, but also the work teachers do when they manage the interactions with students and content. To that end I propose that our analyses of instruction need to start not from the resource-ideas for improvement but from the study of the actual instructional practices where those ideas might be infused. Indeed, I contend that we need descriptive accounts of mathematics teaching and theories that provide us with ways of examining how the teaching that is desirable might emerge from the teaching that exists.

## A Modeling Perspective in the Teaching of Geometry

I am deeply interested in the improvement of geometry instruction in secondary schools, particularly as it regards students' induction into proof-based mathematics. My interest in it comes from two sources that betray my interest in an incremental approach to instructional improvement. First I note the contrast between, on the one hand, the ritualized proof practices that have been observed in high school geometry (Herbst et al., 2009) and, on the other hand, Lakatos's (1976) description of the methodological role that informal proof plays as a tool to find out what is true and shape definitions of mathematical ideas. Second, I have a deep appreciation for the course of studies in geometry; at least in American classrooms, the high school geometry course has maintained a place for declarative statements about mathematical concepts and for proofs when those are almost non-existent in other courses of studies. There clearly is work to do to improve geometry instruction but the high school geometry course already is and has been a beachhead for mathematical practice more so than any other institutionalized course of mathematical studies in school. Diagrams are prominent among all that there is to appreciate and to critique in the high school geometry course: Diagrams are both used to visualize ideas and criticized in making a case for proofs; and when learning to do proofs, diagrams are again used to suggest the statements that could be made and to caution students against unwarranted assumptions. What might seem like a schizoid way of treating diagrams could be improved if the geometry course was infused of a modeling perspective. I consider this an incremental idea for change. It is a change from a way of organizing the study of geometry that traditionally separated intuitive and demonstrative geometry (see Breslich,

1931) with the former concerned with the exploration and measurement of concrete diagrams and the latter concern with the derivation of logical conclusions from definitions.

What do I mean by a modeling perspective? Let's consider, as a broad definition of what mathematics education could do for students, that mathematics education teaches us to solve problems in our heads by making assumptions about states of affairs and engaging those assumptions in a propositional calculus. A modeling perspective counsel us not to expect that those states of affairs will be already formalized before we engage them in in-the-head problem solving, but rather to assume that the percepts and pre-concepts that we use to organize our experience, including, in particular, the diagrams we use in geometry, may be good enough to start reasoning with. We can treat them as if they were mathematical objects and involve them in predicting some information that can be confirmed or dispelled by experience, thus possibly inducing questions on our conceptualizations. A modeling perspective, inasmuch as it calls to produce new information without the expense of trial and error, allocates value to informal proof. And to the extent that informal proof shows limits in producing correct information, a modeling perspective allocates value to the progressive conceptualization of percepts and pre-concepts. This modeling perspective can give rise to a curriculum in which mathematical concepts and theorems from geometry are introduced by way of engagement in the prediction of information about geometric diagrams and other concrete artifacts.

A modeling perspective in geometry suggests a course that does not alienate the students' earlier experiences with shape and space (what Kuzniak, 2006, calls the workspace of natural geometry), but one where students engage in the progressive sophistication of their intellectual means to model, predict, and control geometric representations, so that they can be reliably used in making and transacting meanings (Herbst, Fujita, Halverscheid, & Weiss, in preparation). A modeling approach can be used to help students transition into what Kuzniak (2006) calls the Geometry II paradigm or the workspace of axiomatic natural geometry. A question for us as researchers is then: If this is an improvement worth moving toward, what kind of theory of mathematics teaching can help us anticipate and understand the difficulties a teacher may have to manage and identify the resources he or she could use to manage those difficulties? In the next section I use a couple of geometric examples to describe in broad strokes how the ideas of practical rationality can help.

## Practical Rationality

Practical rationality is not a theory but a set of intellectual resources that can be used to develop theories of mathematics instruction in specific courses of studies. In general it identifies personal and sociotechnical resources that might intervene in the decisions that a teacher makes in particular instructional systems. Personal resources include teacher preparation and experience, mathematical knowledge for teaching, and teacher beliefs. Sociotechnical resources include norms of didactical contracts and instructional situations, and the professional obligations of mathematics teaching. I introduce these sociotechnical resources as I describe how they can help understand and implement a modeling approach in geometry.

*Identifying difficulties with a modeling approach*

In some of my earlier empirical work, doing classroom teaching experiments in collaboration with teachers, I've asked the question of whether we could use diagrammatic or concrete representations of geometric concepts in the contexts of tasks that favor a use of informal proof with which students might participate in the mathematical construction of the concepts represented. Herbst (2003) reported one example of this modeling perspective in the context of the notion of area: At a time when 14-year-old students only knew (from early schooling) area formulas for plane figures, they were asked to rank order a set of 8 cardboard triangles according to area but using the area formula as little as possible; they were asked to justify every pairwise comparison they could make. The triangles and the task had been designed to get students to make explicit some of the properties that define the area function (i.e., inclusion, additivity) by provoking them to use the area formula to represent rather than to calculate the area of a figure. The set included two triangles (D and E; see Figure 1 below) that had been constructed so that the side of one of them was half as long as a side of the other, and the corresponding altitude of the first one was twice as long as the corresponding altitude of the other; yet the relationships between those measures may not have been preserved when those shapes were cut out from their cardboard printouts.

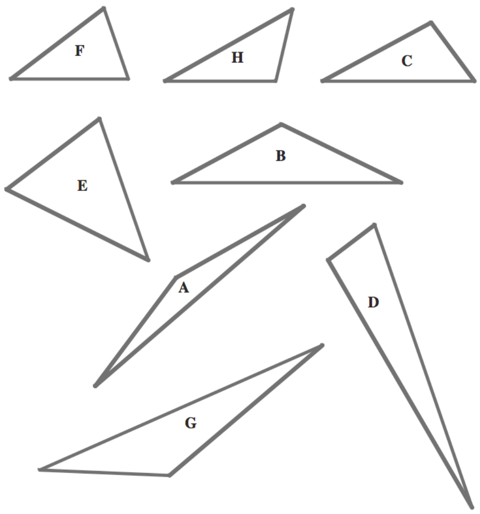



Figure 1. The 8 shapes used in the ranking triangles task.
(Reproduced from the original published in Herbst, 2003,
http://aer.sagepub.com.proxy.lib.umich.edu/content/40/1/197.short)

I bring this example first to illustrate a more general point about how the way we discuss mathematics teaching can benefit from mathematics specificity and needs to benefit from it if we really are to improve instruction. Because in spite of how much design went into the task and how hard the students and the teacher worked, the lesson did not quite work out. The ranking triangles task was implemented with students who drew on their prior knowledge and worked by themselves most of the hour, they actively discussed with each other and as a whole group and eventually came up with a ranking for the eight triangles. Yet an observer of the mathematics in that lesson could easily have the impression that the class had merely engaged in work that they had done in elementary school--measuring bases and heights and calculating—and had missed opportunities to represent relationships among areas using the area formula as an algebraic expression (those opportunities were made available at least by one student). If we had stopped our analysis of the task with its design and only looked at the work of the teacher using tools from discourse analysis, we might have missed how the moves he made to manage the evolution of the task as students worked on it actually compromised the mathematics. My analysis of the instruction attended to how the teacher experienced tensions managing at the same time students' engagement in work and the way the work embodied mathematical meanings. These tensions were indexed by three characteristics of the task. As far as the goal of the task, there was a tension between the goal expressed to students and the instructional goal sought; as far as the resources for the task, there was a tension in whether to see the cardboard triangles either as objects or as representations; and as far as operations, there was a tension in how to treat the preference not to use the area formula. It is clear that the teacher I was working with could have been better prepared to handle that complexity and that some of the preparation he could have used files into what one might call mathematical knowledge for teaching. But having or not having those resources does not obliterate the need to address sociotechnical demands, related to the responsibility for the teacher to engage with the mathematics and the student, indeed with the students' mathematical work. The hypothesis that in leading instruction the teacher had to rely on and maintain a didactical contract (Brousseau, 1997) with the students and the content at stake served to describe what happened in terms of work complexities, regardless whether those could have been mitigated or exacerbated by individual resources.

This takes me to a first bit of theory, which is the hypothesis (in the mathematical sense of assumption) that mathematics instruction is possible because a didactical contract exists that binds the teacher, the students, and the content at stake with some rights and responsibilities. Their interactions are predicated on the existence of a content, which is at stake for students to learn with the assistance of the teacher. The didactical contract makes the teacher responsible to assign work for the students on behalf of the knowledge at stake, work in which the student may develop and demonstrate their knowledge of the content at stake; the teacher is also responsible to see, in the students' work, evidence that they have acquired the content at stake. Students' engagement in mathematical work thus serves at the very least to confirm that learning has happened, but it also plays a role in learning: If the student has not learned the content, their work is unlikely to show evidence that they know it; and engaging the students in solving problems that involve the knowledge at stake has historically been at least a part of what it takes to learn the knowledge. It becomes crucial for a teacher to organize activity and the division of labor in the classroom so that the didactical contract can be complied with--that is, for the teacher to eventually observe in students' work evidence of students' knowledge. There was a fundamental ambiguity in the goal of the ranking triangles task--for the students, the goal was to rank all the triangles, but for the teacher it was to get students to formulate and use particular properties of area that would justify assertions about rankings. This ambiguity probed or stressed what I'd call a norm of the contract in which many secondary school students in the US are socialized--that everything the teacher asks students to do is deliberately chosen to aim at their learning.

The didactical contract and its norms exemplify how the study of the practical rationality of mathematics teaching aims to identify sociotechnical demands and resources for the teacher's action. One of the theory development tasks consists of identifying norms of the didactical contract that might be stressed with bids for instructional improvement in a given course of studies, such as those issued from the modeling perspective in geometry. The goal of such theoretical research is to inventory resources and how the teacher might use them to manage a negotiation in which the didactical contract may survive such stress. The theory is useful for improvement because if improvement ideas hinge on stressing or breaching norms, knowing what norms will be stressed can only help the gathering of intelligence for robust improvement. The example of the ranking triangles task, and how it relied on a fundamental ambiguity in its goal, highlights the need for a teacher to manage changes in the mathematical task attending both to students' expectations of what they are asked to do and the teacher's sense of what will best serve the instructional goal. In the next example I zoom into mathematical-work-specific versions of the didactical contract, which I call instructional situations, to illustrate how sociotechnical demands may support instruction of specific ideas.

### *Norms of instructional situations*

Instructional situations are work-specific varieties of the didactical contract, or customary ways in which labor is divided for mathematical work on particular ideas. An important emblem of an instructional situation is what might be called a *problem type* or a *canonical task*. An example of an instructional situation in American high school geometry courses is what I have called "doing proofs" (Herbst et al., 2009). Problems used in that situation have some typical characteristics--they tend to include a labeled diagram and to state the conclusion to be proved in terms of diagrammatic objects, using those labels (Herbst, Kosko, & Dimmel, 2013). Likewise those problems rely on a normative division of labor: While earlier in the students' learning of a particular idea the teacher may demonstrate to students how to do the kind of problem that mobilizes that idea, responsibility is later relinquished to students for some of the labor in those problems while other aspects may stay with the teacher. Each of those problem-types that put at stake an item of knowledge is a key part of what we call an instructional situation. Instructional situations are basically classes of mathematical work, characterized by the kind of knowledge involved as well as by the division of labor between what the teacher and the students are expected to do. Instruction in a course of studies can be described as the successive induction of the students into instructional situations, which often build on each other.

New ideas are often introduced through the teacher's demonstration of how to complete an instance of the instructional situations in which the new idea plays out. But it is conceivable, of course, that new knowledge might get introduced in different ways than by the teacher's demonstration; in particular students might get engaged in work that is novel, where their own adaptation to the demands of the task serves to bring to the fore the knowledge at stake. That was the idea with the ranking triangles task described above, and more generally with the proposition that geometric knowledge could be introduced through engaging students in modeling. Our proposition of the notion of instructional situation serves to scaffold the identification of more sociotechnical resources with which to create work contexts in which new knowledge can be introduced. I explain this with another example.

Consider the following task used by Chen and Herbst (2013). This task was designed to introduce students to the relationships among angles formed by intersecting and parallel lines, including the equivalence between the triangle sum theorem (which we knew they knew from middle school) and the parallel postulate. Students were given a diagram like the one in Figure 2 and asked to determine how many angles they would need to measure in order to know the measures of all of the angles formed by lines in the diagram. I would describe this task as an opportunity to engage in mathematical modeling: The task puts a premium on knowing before measuring, but considering that it provides no actual information, it calls for students to make hypotheses and calculate with those hypotheses to see whether they give them enough leverage to make knowledge claims; they can add or subtract hypotheses to optimize their model, and they can find out ways of calculating with those hypotheses. They can also measure to check then pretend they did not need to measure if they are convinced their calculation is sound. I want to use this task to examine how the mathematics specificity afforded by attending to instructional situations helps us describe and understand what might happen with the work of teaching that can be observed around this task.

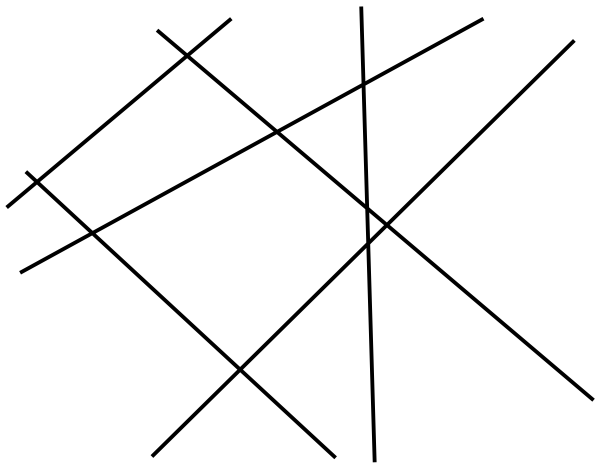


Figure 2. The intersecting lines task
(reproduced according to Herbst & Chen, 2013,
http://link.springer.com.proxy.lib.umich.edu/article/10.1007/s10649-012-9454-2/fulltext.html)

While this task is novel, there are two instructional situations that serve as anchors for this task. That is, there are two kinds of problems that students are likely to have been socialized into by the time they are in high school geometry and that can serve as background against which to inscribe this task. One is the situation of *exploration of a figure*, in which students are given a diagram and various tools (e.g., rulers, protractors) and asked to find information about the figure represented (Chazan, 1995; Herbst, 2010). The other is the situation of *geometric calculation in number* (Hsu & Silver, 2014), in which students are given some dimensions of a figure and asked to find out other dimensions. Figures 3 and 4 show what tasks that pertain to each of those situations could look like.

These two tasks have some similarities in the concepts they involve, such as line and angle. They are different in the work they call for, hence how they involve those concepts. The first one calls for knowledge of how to use a protractor to measure angles, while the second one calls for knowledge of properties of figures. They are cases of the more general instructional situations of *exploration of a figure* and *calculation of a measure* because they each rely on a different normative division of labor over knowledge. In the first case, the diagram is the mathematical object; the teacher must allow for some instruments of measurement and the students need to use the instruments to read those diagrammatic objects. In the second, the diagram represents objects; the teacher must provide sufficient numbers to use in the calculation and the students need to use them to add more information to the representation (see Herbst, 2004). Neither of these situations supports well the notion of modeling in geometry: In the first one, angles are de facto the same as their representation, while in the second one a model of the geometric figure is already imposed distinguishing the representation of an angle from its actual measure. But they both provide anchors for the task in Figure 2.

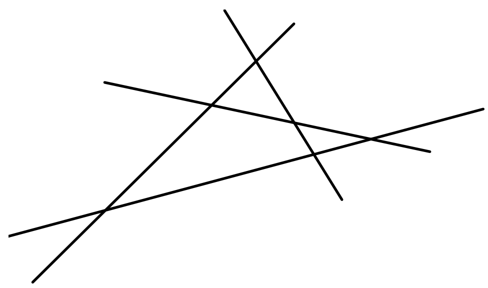


Figure 3. What are the measures of the angles in the figure?

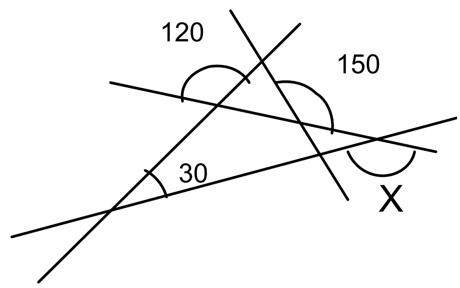



Figure 4. What is the measure of *X*?

What does it mean that these situations provide anchors for the task in Figure 2? It means that the task associated to the diagram in Figure 2 could be seen as relying on breaches or departures from the norms of the situations of exploration and of calculation, and hence the teacher could count on students' capacity to consider the resemblance between the intersecting lines tasks and tasks that belong in those situations. The teacher can pose the new task against the background of those situations. It is possible for the teacher to use that familiarity to devolve to the students responsibility for the task. For example, if students needed some help processing the question "how many angles would you need to measure…?," experience with the situation of exploration could suggest that the teacher could ask "if you were to measure all the angles, how many would you need to measure? Are these all the angles made by the lines shown in the figure?" Experience with the situation of calculation could instead suggest the teacher to ask "what would you need to know in order to calculate the measure of this angle?" In other words the instructional situations of calculation and exploration provide the teacher with sociotechnical resources that can be used to manage students' engagement in the novel, intersecting lines task.

The norms of those situations provide specific resources for the teacher to identify what may make the new task hard to manage and to know what specific moves might make the task more manageable. Specifically, because in a situation of exploration it is normative for students to use all sorts of available tools to read the diagram, the teacher could see an option to simplify the complicated question "how many angles would you need to measure to know them all?" into "how many angles are there to be measured?" In other words, the norms of instructional situations are useful not only to understand how close or how far from normative a proposed task is, but also as available resources to manage students' engagement in a task by way of making small changes to a task to facilitate students' engagement.

The instructional situations that exist in a class can thus play the interactional equivalent of collective prior knowledge in instruction, pointing in directions for the teacher to reduce complexity. This of course does not guarantee that the mathematical demands of the task will be held constant: Defaulting to one of those instructional situations could undermine the opportunity to learn. But our knowledge of those situations provides us with means to anticipate how a task might evolve and to create resources that might increase robustness: For example, while the situation of exploration is available, defaulting to it would not make this task worth its time; thus, to support the expectation that at least some measures would be calculated as opposed to measured, when we designed the task we drew lines that would clearly intersect but whose intersections are not on the page (see Figure 2); thus the angles they make can be visualized and counted, but it would take a lot of extra work to get to measure them.

I bring those examples of instructional situations and their norms to show two things. First, in terms of theory building, the building of a theory of teaching geometry calls for inventorying and relating the various instructional situations that enable mathematical work and that may articulate geometry learning in existing classrooms. Second, in terms of tools for theory building, situation and norm are key constructs that enable us to construct theories with which we can understand the practical rationality of mathematics teaching. To bring those two observations together, if much of the work of teaching includes managing students' engagement in mathematical work, situations and norms are constructs that help us describe that mathematical work. That mathematical work can be described either as familiar practices that instantiate situations, complying with norms, or as novel tasks that are novel because they depart from instructional situations by way of breaching with specific norms and therefore requiring ad hoc negotiations of the didactical contract. Instructional situations and their norms illustrate how our approach to practical rationality is mathematics-specific: The constructs themselves are general, but the way they are used in building local theories with which to understand teaching requires the researcher to use them in regard to specific mathematical conten. This content is not merely the mathematical topic that could be found in the book, but a real blend of mathematical ideas and mathematical work, as realized in potential interactions among teacher, students, and content. Inasmuch as researchers strive to describe existing practice, the constructs of contract, situation, and norm serve to describe specific patches of practice such as the teaching of particular ideas in particular courses of study. In the practical rationality approach, to understand mathematics teaching means to understand the instructional situations a teacher needs to manage (as well as the mathematical knowledge for teaching that would make a difference in how the teacher manages those situations; see Herbst & Kosko, 2014). From this perspective, it is important for researchers to develop local theories of the teaching of specific mathematical ideas in given contexts (e.g., courses of study) by accounting for the instructional situations in which those ideas are at stake, which includes identifying the norms of a situation and the natural variability of those norms in practice. In our empirical work we have been doing this by asking teachers to comment, rate, and engage in simulations that use scenarios that represent instructional situations, and in which teacher and student actions vary in the extent to which they abide by hypothesized norms (see Herbst et al., 2013).

Thus the constructs of situation and norm provide some guidance on what it means to develop mathematics-specific accounts of mathematics teaching: As researchers we should be interested in describing the set of instructional situations that span a course of studies, and for each of those situations to specify the set of norms that regulate how teacher and student divide mathematical labor. Such research is descriptive--by making such inventory one does not necessarily subscribe to it as desirable; rather, one creates groundwork or background over which designers and teachers can conceive desirable practices. That is, such work specific theories can assist the design and the study of instructional regimes that might improve student outcomes.

The background practices specified by contracts, situations, and norms can serve practice as a way to compare the costs of different tasks. Two alternative lessons for a given idea might be pondered in regard to the extent to which they take teacher and students away from instructional situations that they know how to handle. Imagine for example that instead of providing Figure 2, we only said to students--"suppose that you had 6 lines on the plane, how many angles would you need to measure in order to know them all?" Inasmuch as that task requires a formal understanding of lines, angles, and intersections, that task does not exemplify a modeling perspective; but it could also be criticized on account of how far it is from the canonical tasks of the instructional situations available. Which suggests the task might be hard to manage anyway. Possibly, individuals with significant mathematical knowledge for teaching might be able to navigate a very novel task. But a closer distance between a proposed novel task and the canonical task of an instructional situation may support a teacher's management of the evolution of the task. But why would a teacher use a novel task?

### *Professional obligations and departures from instructional situations*

Instructional situations can be seen as ways in which instructional systems satisfice (in the sense of finding a non-optimal solution to a problem) the conditions under which a teacher is expected to teach a given mathematical idea. Among the conditions under which a teacher works, there are some of institutional nature that a teacher needs to contend with—a curriculum to cover in the course of studies, a sequence of courses to feed, periods of time during which such instruction needs to happen, tests and other accountability mechanisms, etc. But the institutional obligation is only one of them. As a member of a profession, a teacher of mathematics has other obligations to meet: An obligation to the discipline of mathematics, an obligation to students as individuals, and an obligation to the class as an interpersonal space. Each of those obligations can support a variety of dispositions to act. For example the individual obligation can support actions to challenge students to do problems beyond their comfort zone as well as actions to shield students from embarrassment or frustration. The disciplinary obligation can support actions to engage students in practices of conjecturing as well as actions to confirm the correction of one's notes using reputable mathematical sources. The interpersonal obligation can support actions to ensure equitable opportunities for all students to participate and actions to maintain a peaceful atmosphere in which everybody can study and learn. The institutional obligation can support actions to move to a new topic of study as well as actions to spend time preparing for an upcoming test. The obligations are four sources of justification that Herbst and Chazan (2012; see also Chazan et al., 2016) hypothesize to be common for all teachers of mathematics, even if under each of those obligations one could find more specific dispositions that are ascribable to only some teachers of mathematics. These obligations present an array of valid, though possibly contradictory, sources of justification for teacher actions. For that reason, they can assist teachers in pondering possible deviations from the norms of instructional situations. If the instructional situations found in a course of studies are arrays of practices and norms that satisfice the aggregate set of obligations, each of the obligations on its own can justify some departures from norms. Research on the obligations is ongoing; we have been developing instruments to measure practitioners' recognition of the four obligations and the extent to which they use them in justifying departures from ordinary practice (see Herbst, Dimmel, Erickson, Ko, & Kosko, 2014). As a theoretical proposition, the notion of professional obligation, serves for us to represent the conditions in which teachers work in a more accurate way. Teachers can also use them to improve instruction.

The reforms that were proposed by the Standards movement and the US NCTM Standards' documents from 1989 and 2000 might be justified on account of the disciplinary obligation--those reforms attempted to make school mathematics more interesting, connected to the intellectual activities of mathematicians in problem solving and theory building as well as to the work of applying mathematics to the real world. Other attempts to improve instruction have emphasized the obligation of a teacher to educate students to live in a diverse society and work for a more just society. Yet other attempts to improve practice are justified on account to attend to (more of) the needs of individual students. Inasmuch as it is plausible that the four obligations represent the conditions in which teachers work, our own work promoting improvement needs to consider that all those obligations are at play when a teacher engages in work to enact such improvements.

So, how could a teacher use the obligations to support the work of instructional improvement? Let's go back to the use of the task in Figure 2. Upon presentation of the problem, the teacher may have to contend with events such as shown in Figure 5.

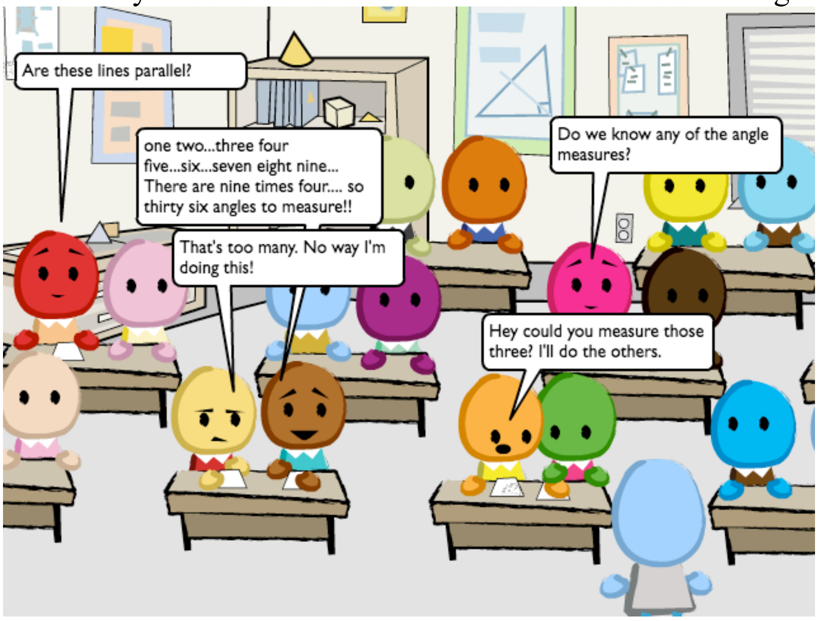
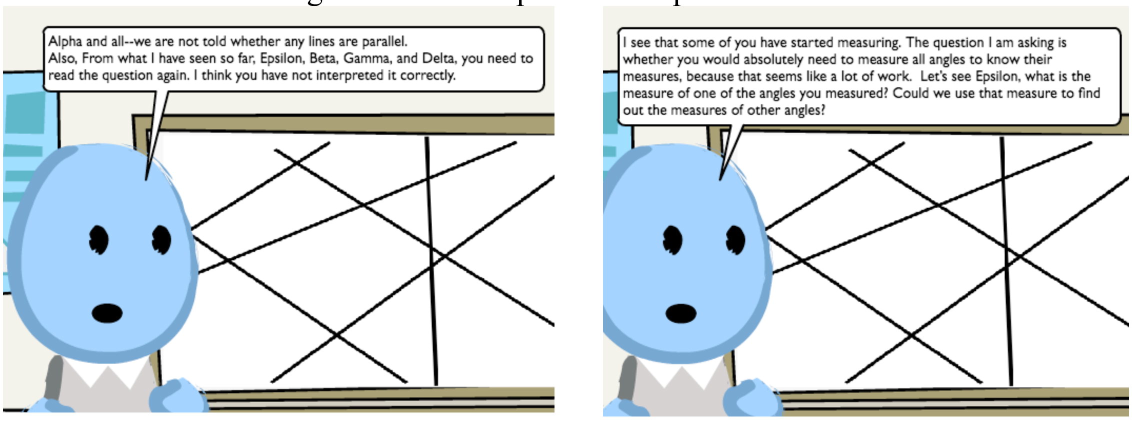



Figure 5. Possible events after the teacher present the intersecting lines task.
Graphic characters are ©2016, The Regents of the University of Michigan, used with permission.

Those events may happen at about the same time, and thus require the teacher not only to come up with possible ways of handling each of them but also to consider how to handle their simultaneity. The obligations can help us examine the space of possibilities facing the teacher. I do not contend that the present analysis of these events is exhaustive or that it provides enough for the teacher to optimize the use of time and resources. I only illustrate how the obligations can help analyze the events and look for ways to maintain the task alive. Let's note first that the two students on the right represent opportunities to default to the existing instructional situations of calculation and exploration—those students seem to know what to do, but to be rather unclear about what the work at hand is. They present a challenge that one could associate with the individual obligation—if the teacher lets Epsilon and his tablemate to continue to measure, he may increase his sense that he is on the way to completing the task; also, if the teacher leaves Delta's question about angle measures unanswered, Delta may also become apathetic. How could the teacher address these circumstances? Figures 6a and 6b present two possibilities.




Figures 6a and 6b. Two possible responses to the events in Figure 5.
(Graphic characters are ©2016, The Regents of the University of Michigan, used with permission.)

It seems that in making the first move (Figure 6a) the teacher is intent in letting the task do its job by enforcing the role he had envisioned for the students in the task, while in the second one (Figure 6b), the teacher is amenable to negotiating some of the task's implicit division of labor. While letting the task change in some way, the teacher is also being proactive, accepting what students are doing and working with it. Note that in making that move the teacher is addressing Epsilon by redirecting his drift toward exploration, the teacher is also giving a nod to the student who thought the task was too much, and providing a way for Delta to use what she knows. If Epsilon happened to say that one angle is 40 degrees, the teacher would have an opportunity to address the interpersonal obligation as well as Alpha's earlier comment about "these lines" by asking students how they could be more precise so that students can understand which angle (viz. which lines) they are referring to. The student who merely counted (some of) the angles might realize she oversimplified the task but the teacher could ask her to label the intersections as a way to keep her engaged, which would be one way of meeting the individual obligation. Then the teacher could ask Epsilon which one is the angle that he says measured 40 degrees and ask the class whether they understand what angle Epsilon is referring to. The teacher could also ask Alpha which lines are the ones he was asking whether they were parallel. The students' responses might give an opportunity for the teacher to attend to the disciplinary obligation by emphasizing the value of precision embodied in referring to angles by using three letters (viz. to lines by using two letters). To Alpha the teacher could ask back "What if the lines were parallel? How would that help?" The teacher could also see what students like Delta can do with the information that one angle is 40 degrees and ask after "what if the angle was not 40 degrees but an unknown measure *X*?" In the first move we see the teacher's question as providing an opportunity for conjecturing while in the second one as providing an opportunity for generalizing, both of which can justified on the disciplinary obligation. Thus the obligations can help a teacher identify resources and constraints in classroom happenstances, including in particular what students say and do in response to a task. The obligations can also help a teacher justify or discourage ways of addressing those happenstances. Thus the obligations can help a teacher examine what opportunities each of those events present.

## Conclusion: Working toward Instructional improvement

My argument is that we need mathematics-specific theories of mathematics teaching in order for our research to contribute to efforts at instructional improvement. I do not discount practitioner-oriented efforts that try to promote generic practices such as classroom discourse and inquiry-based learning but note that those are merely resources available in instruction and I emphasize, with Cohen et al. (2003), that the key to their effectiveness is on how resources are used in instruction. Theories of mathematics teaching that attend to the specifics of mathematical work can provide ways of describing such use. In the preceding sections I illustrate how the constructs of practical rationality, particularly the notions of contract, situation, norm, and obligation, serve to develop theories of teaching of specific mathematics that can serve to describe existing and innovative teaching, including the teaching that uses generic resources such as the notion of inquiry-based learning; the obligations can be used to describe specific moves a teacher might make when teaching particular content. But these constructs may also be used to support efforts at instructional improvement.

Morris and Hiebert (2011) have argued that lessons constitute the knowledge base of teaching and proposed that to improve instruction we need to increase and improve that knowledge base. I take it that by lessons we should mean neither solely the curriculum of the lesson, nor a lesson plan, but rather the anticipation of all that can happen in a lesson. Hence, a lesson in that knowledge base is not a linear narration of the best possible lesson but an array of possibilities forking from predictable events and conceivable decisions. I contend that a subject specific approach to mathematics instruction like what is offered by practical rationality can help us develop capacity for scoping the space of a lesson and to calculate piecemeal, incremental ways in which specific lessons might be improved. In the context of our work in technology-mediated teacher education using the LessonSketch platform (see Herbst, Chazan, Chieu, Kosko, Milewski, & Aaron, 2016) we have developed and been using the practice of StoryCircles (see Herbst & Milewski, in review). In this practice, groups of teachers interact with a facilitator to script the development of a lesson and visualize it as a storyboard, using the Depict tool in the LessonSketch platform (with which the graphics on Figures 5 and 6 were created). The approach lends itself to creating alternative branches of a lesson and to visualize a lesson not just as one story but as multiple possible stories. These story families can be contributed to a community of teachers who can use it to annotate their own experiences with the lesson for future reference. It can be noted that while such production of the many stories that could unfold for a given lesson could be done without the scaffold of a theory, a theory of mathematics teaching provides conceptual elements to predict possible stories including those that might be conceivable, even though they might stretch the conventional wisdom of what is viable to do in classrooms. An online environment where practitioners can visualize such lessons and comment on the complexities that could be exacerbated by particular decisions and actions can then support the creation of a professional knowledge base that can support instructional improvement.

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