A POSTSCRIPT TO A PAPER OF A. BAKER

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In a recent paper Baker [1] showed that if the field $\mathbf{Q}(\sqrt{-d})$, $d > 0$, has class number 2 and discriminant $-\Delta$, where $\Delta \not\equiv 3 \pmod 8$, then $d < 10^{500}$. We will show that the combination of Baker's method and an electronic computing machine can be used to find all complex quadratic fields with the above mentioned properties. The fields are the known ones, namely those given by

$$d = 5, 6, 10, 13, 15, 22, 37, 58.$$

As the famous "class number one" problem one can attack this special class number two problem from the standpoint of modular functions. The analogue of the Heegner–Stark method has been worked out by M. Kenku in his 1968 Oxford Ph.D. thesis and independently by P. Weinberger in his 1969 Berkley Ph.D. thesis.

Generally speaking we will follow the notation and terminology of [1]. However there are two small errors in [1]. The first is in the definition of A, where $2ak$ should be ak and the second is the assertion on page 101 that $\frac{1}{4}d$ is a prime; $\frac{1}{4}d$ can be twice a prime. These errors do not affect the validity of the argument.

Professor D. H. Lehmer informs us that he has checked all the fields $\mathbf{Q}(\sqrt{-d})$, $d > 0$, having class number 2 in the range $1 < d < 10^{12}$. The only such fields are the known ones, in particular the only complex quadratic fields with class number 2 and even discriminant in this range are the ones named above. We will be making use of Lehmer's result later.

The reduction step. Following the notation of [1] we have

$$|h_1 \log \alpha_1 - 16\pi\sqrt{d}/21| \leqslant 84\eta_1 (1-\eta_1)^{-2} \tag{1}$$

$$|h_2 \log \alpha_2 - 40\pi\sqrt{d}/11| \leqslant 132\eta_2 (1-\eta_2)^{-2} \tag{2}$$

where

$$\eta_1 = \exp(-\pi\sqrt{d}/84),\ \eta_2 = \exp(-\pi\sqrt{d}/132),\ h_1 = h(21d),$$
$$h_2 = h(33d),\ \alpha_1 = (5+\sqrt{21})/2,\ \alpha_2 = 23+4\sqrt{33}.$$

If $\sqrt{d} \geqslant 900$, then η_1 and η_2 are both less than $\frac{1}{2}$ and we have

$$|h_1 \log \alpha_1 - 16\pi\sqrt{d}/21| \leqslant 336\eta_1 \tag{3}$$

$$|h_2 \log \alpha_2 - 40\pi\sqrt{d}/11| \leqslant 528\eta_2. \tag{4}$$

The two upper bounds in the inequalities (3) and (4) are both less than $\exp(-\pi\sqrt{d}/200)$ provided that $\sqrt{d} \geqslant 200 \times 132 \times 6{\cdot}2691/68\pi$. This inequality is

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certainly satisfied if $\surd d \geqslant 900$. Furthermore if $\surd d \geqslant 900$ we have

$$|h_1 - 16\pi\surd d/21 . \log \alpha_1| < 10^{-6}$$
$$|h_2 - 40\pi\surd d/11 . \log \alpha_2| < 10^{-6}.$$

Thus for $\surd d \geqslant 900$ we certainly have

$$1{\cdot}52\surd d \leqslant h_1 \leqslant 1{\cdot}53\surd d \text{ and } 2{\cdot}98\surd d \leqslant h_2 \leqslant 2{\cdot}99\surd d.$$

Putting $b_1 = 105h_1$ and $b_2 = -22h_2$ we deduce from (3) and (4) that for $\surd d \geqslant 900$ we have

$$|b_1 \log \alpha_1 + b_2 \log \alpha_2| \leqslant 127 \exp(-\pi\surd d/200). \tag{5}$$

Let $H = 210\surd d$ and then $\max(|b_1|, |b_2|) \leqslant H$ and we can write (5) as

$$|b_1 \log \alpha_1 + b_2 \log \alpha_2| \leqslant \exp(4{\cdot}84419 - \pi H/42000). \tag{6}$$

We will take H so large that

$$\pi H/42000 - 4{\cdot}84419 > H/21000.$$

This inequality will be satisfied if $H > 180{,}000$ and as we are assuming for the moment that $\surd d \geqslant 900$ we certainly have $H \geqslant 189{,}000$. Thus we have

$$|b_1 \log \alpha_1 + b_2 \log \alpha_2| < \exp(-\delta H), \tag{7}$$

where $\delta^{-1} = 21{,}000$. We can now conclude with Baker that $H < 10^{250}$ and $d < 10^{500}$.

A check is now made to see if the inequality (7) has any solutions in integers b_1, b_2 when H lies in the range $180{,}000 \leqslant H \leqslant 10^{250}$. A simple way to make this check is to recall a lemma of *A–M*. Legendre, *Theorie des Nombres*, tome 1, page 147.

LEMMA. *If θ is a real number and p/q is a rational approximation to θ which satisfies the inequality $|\theta - p/q| < \frac{1}{2}q^2$, then p/q must occur as a convergent in the continued fraction expansion of θ.*

We also recall the following inequalities

$$1/(a_{n+1}+2)q_n^{\,2} < |\theta - p_n/q_n| < 1/a_{n+1}q_n^{\,2},$$

where a_n is the nth partial quotient in the continued fraction expansion of θ. Hence if we have an inequality of the form

$$|\theta - p_n/q_n| < \exp(-\delta q_n)/\beta q_n,$$

where β and δ are positive real numbers, then we must have

$$a_{n+1} > \beta . \exp(\delta q_n)/q_n - 2. \tag{9}$$

We can now write the inequality (7) in the form

$$\left|\frac{b_1}{b_2} + \frac{\log \alpha_2}{\log \alpha_1}\right| \leqslant \frac{\exp(-\delta H)}{\log \alpha_1}\frac{7}{|b_2|} \leqslant \frac{\exp(-\delta|b_2|)}{\log \alpha_1}\frac{1}{|b_2|}.$$

On taking account of Lehmer's computations we need only check values of $\surd d$ in the range $10^6 \leqslant \surd d \leqslant 10^{250}$ i.e. we can assume that

$$10^{254} > 22 \times 2{\cdot}99 \times 10^{250} \geqslant |b_2| \geqslant 22 \times 2{\cdot}98\surd d > 10^6.$$

The inequality $\exp(-\delta|b_2|)/\log\alpha_1 < \frac{1}{2}|b_2|$ is certainly satisfied for all $|b_2| > 10^6$. Consequently if b_1, b_2 is a solution of (7) with $|b_2| > 10^6$ then by Legendre's lemma b_1/b_2 must occur as a convergent in the continued fraction expansion of $\log\alpha_2/\log\alpha_1$, say p_n/q_n. Moreover, the partial quotient a_{n+1} must satisfy the inequality (9). In particular since $q_n > 10^6$ and $\exp(\delta x)/x$ increases for $x > \delta^{-1}$ the partial quotient a_{n+1} must satisfy the inequality

$$a_{n+1} > |\log\alpha_1| . \exp(1000/21) . 10^{-6} - 2 > 10^{10}.$$

We computed θ to 750 decimal places, then we developed the continued fraction expansion of θ until the convergents q_n exceeded 10^{254}. The partial quotients are given below. The largest partial quotient is 241. We conclude that there are no solutions of the inequality (7) with $|b_2|$ in the range $10^6 < |b_2| < 10^{254}$. Hence the only complex quadratic fields with class number 2 and even discriminant are the ones given above.

The computation was done on the University of Michigan's I.B.M. 360/67 calculating machine and took about 100 seconds. As a check on the machine computation the entire calculation was repeated by Mr. F. Lunnon on the Atlas I computer at the Science Research Council's Atlas Computer Laboratory, Chilton, Berkshire. taking about 30 minutes computation time. The computational routines used by Mr. Lunnon were completely different from the Michigan routines. As the two computations agree we feel quite confident that the continued fraction expansion is correct.

The Ann Arbor computation used the SRARITHMETIC multiprecision package and the Chilton computation used the ABC multilength system. At Ann Arbor the logarithms of α_1 and α_2 were computed by the Newton approximation method applied to the equation $0 = f(x) = \exp(x) - \alpha$ whilst at Chilton they were computed using Thiele's continued fraction method. In both computations the results were checked by computing $\exp(\log(\alpha))$ and then comparing the result with α. The continued fraction expansion of θ, in both computations, was found by iterating

$$a_i = [\theta],\ \theta \to 1/(\theta - a_i).$$

At each iteration the convergent p_i/q_i was tested to see if the accuracy of the original θ had been exceeded. If this test had ever been positive the computation would have been terminated.

Appendix

The continued fraction expansion of $\dfrac{\log\alpha_2}{\log\alpha_1}$ is

2, 2, 3, 1, 10	17, 2, 2, 2, 6	1, 3, 35, 2, 1
14, 8, 1, 4, 3	1, 1, 7, 3, 1	90, 1, 4, 7, 1
4, 1, 1, 47, 1	1, 1, 140, 5, 8	1, 1, 1, 3, 3
3, 1, 4, 6, 3	3, 1, 96, 241, 3	3, 1, 5, 2, 1
2, 1, 1, 2, 16	3, 4, 6, 1, 13	1, 4, 15, 2, 1
10, 2, 4, 1, 17	9, 2, 1, 1, 1	5, 1, 1, 4, 1
3, 6, 1, 16, 2	1, 8, 3, 1, 5	2, 8, 2, 6, 2
1, 7, 1, 39, 2	8, 1, 6, 1, 1	1, 1, 36, 1, 1

5, 1, 3, 1, 1	5, 26, 44, 1	1, 1, 1, 2, 3
1, 1, 1, 1, 1	10, 2, 1, 1, 3	1, 4, 2, 4, 1
1, 5, 1, 1, 2	1, 1, 1, 1, 5	1, 1, 2, 2, 23
1, 1, 2, 16, 2	2, 1, 1, 1, 1	1, 1, 1, 2, 14
70, 1, 2, 8, 1	3, 2, 7, 1, 1	15, 5, 2, 1, 1
1, 3, 5, 22, 1	6, 3, 32, 17, 5	6, 3, 2, 5, 4
6, 1, 2, 7, 2	7, 2, 7, 23, 2	5, 1, 1, 3, 3
8, 1, 3, 1, 1	8, 1, 1, 1, 5	2, 1, 1, 4, 1
1, 3, 4, 1, 24	28, 1, 1, 2, 171	2, 4, 1, 3, 1
7, 1, 9, 1, 1	5, 11, 96, 1, 4	3, 1, 4, 53, 1
6, 1, 2, 1, 1	3, 1, 2, 1, 8	9, 1, 1, 1, 1
3, 184, 4, 2, 2	1, 4, 2, 5, 3	2, 1, 23, 5, 1
2, 2, 3, 1, 1	1, 1, 10, 43, 15	1, 56, 9, 4, 1
11, 1, 1, 4, 2	1, 2, 7, 1, 1	3, 1, 76, 1, 9
1, 5, 1, 5, 1	5, 3, 1, 2, 2	1, 1, 139, 1, 2
3, 3, 51, 2, 18	4, 5, 1, 1, 9	7, 1, 4, 1, 2
2, 3, 1, 2, 1	7, 1, 5, 1, 2	3, 7, 7, 3, 21
3, 1, 2, 1, 1	1, 2, 1, 4, 1	8, 1, 1, 3, 1
1, 6, 1, 1, 2	1, 1, 2, 3, 1	6, 1, 16, 2, 2
4, 3, 4, 1, 3	1, 4, 7, 2, 10	11, 1, 1, 1, 4
2, 4, 1, 1, 3	3, 3, 31, 15, 1	1, 2, 2, 5, 7
2, 3, 4, 2, 1	6, 3, 43, 13, 5	5, 15, 1, 2, 1
8, 6, 2, 1, 1	1, 4, 1, 57, 5	2, 1, 14, 1, 2
1, 1, 3, 2, 1	6, 1, 1, 6, 3	6, 1, 2, 17, 9
1, 103, ··· .		

Reference

1. A. Baker, " A remark on the class number of quadratic fields ", *Bull. London Math. Soc.*, 1 (1969), 98–102.

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