ESTIMATION OF COEFFICIENTS OF UNIVALENT FUNCTIONS BY A TAUBERIAN REMAINDER THEOREM

PETER L. DUREN

Let S denote the class of functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots \tag{1}$$

analytic and univalent in the unit disk $|z| < 1$. One of the most penetrating results on the coefficients a_n is Hayman's theorem [**6, 7**] that

$$\lim_{n\to\infty} \frac{|a_n|}{n} = \alpha \leqslant 1, \tag{2}$$

for each $f \in S$, with equality only for the Koebe function

$$k(z) = z(1-z)^{-2} = \sum_{n=1}^{\infty} nz^n$$

or for one of its rotations. Hayman's proof begins with the elementary observation that each $f \in S$ has a direction $e^{i\theta_0}$ of maximal growth, in the sense that

$$\lim_{r\to 1} (1-r)^2 \, |f(re^{i\theta_0})| = \alpha \tag{3}$$

and the limit is 0 for every other direction. In particular, the direction $e^{i\theta_0}$ is unique if $\alpha > 0$ (see also Milin [**12**; p. 82]). The second step is the deduction of (2) from (3). Hayman's argument is relatively simple for $\alpha = 0$, but quite complicated for $\alpha > 0$.

Milin [**11, 12**] has recently simplified this latter step in the case $\alpha > 0$. His argument is essentially based upon the following result, which may be viewed as a new Tauberian theorem. Let

$$g(r) = \sum_{n=0}^{\infty} b_n r^n, \qquad b_0 = 1,$$

be a power series with complex coefficients, convergent for $-1 < r < 1$. Let

$$s_n = \sum_{k=0}^{n} b_k, \qquad \sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} s_k,$$

and

$$\log g(r) = \sum_{n=1}^{\infty} c_n r^n. \tag{4}$$

THEOREM A (Milin). *Suppose* $|g(r)| \to \alpha$ *as* $r \to 1$, *and*

$$\sum_{n=1}^{\infty} n|c_n|^2 < \infty.$$

Received 22 July, 1972.

[J. LONDON MATH. SOC. (2), 8 (1974), 279–282]

Then $|s_n| \to \alpha$ *and* $|\sigma_n| \to \alpha$ *as* $n \to \infty$.

This theorem is applied to the function

$$g(r) = \frac{(1-r)^2}{r} f(r), \tag{5}$$

where f is given by (1) and it is assumed, after a rotation, that $\theta_0 = 0$. Then a calculation gives

$$s_n = a_{n+1} - a_n \quad \text{and} \quad \sigma_{n-1} = a_n/n.$$

The Tauberian condition $\sum n\,|c_n|^2 < \infty$ is a consequence of the following theorem of Bazilevich [2, 12], for $\alpha > 0$.

THEOREM B (Bazilevich). *Let* $f \in S$, *and let*

$$\log\{f(z)/z\} = 2 \sum_{n=1}^{\infty} \gamma_n z^n. \tag{6}$$

Suppose $\alpha > 0$ *and* $\theta_0 = 0$. *Then*

$$\sum_{n=1}^{\infty} n|\gamma_n - (1/n)|^2 \leqslant \tfrac{1}{2} \log 1/\alpha.$$

One curious feature of Milin's Tauberian theorem is the assumption that $|g(r)|$, rather than $g(r)$, has a limit. An early step in Milin's proof is to show that the partial sums s_n are bounded. Hence, if $g(r) \to \alpha$, one can appeal to a classical Tauberian theorem [5; p. 154] to conclude that $\sigma_n \to \alpha$.

Under a much stronger hypothesis, Bazilevich [1, 2, 12] has estimated the rate of convergence of $|a_n/n|$ to α. He assumes that $f \in S$ maps the unit disk onto the exterior of an analytic arc. Since the arc is analytic at ∞, a Schwarz reflection shows that the square-root transform of f has the form

$$\sqrt{(f(z^2))} = \frac{\sqrt{(\alpha)}z}{1-z^2}\,[1 + B_1(1-z^2) + B_2(1-z^2)^2 + \ldots]$$

near the pole at $z = 1$. In fact, an elementary argument [1] shows that B_1 is purely imaginary. Bazilevich concludes that

$$|a_n| \leqslant \alpha n + C_1 \sqrt{(\log n)} + C_2,$$

where C_1 and C_2 depend only on α, B_1, and $\operatorname{Re} B_2$.

The purpose of the present paper is to show that under a relatively mild assumption on the behaviour of f along its ray of maximal growth, one can obtain a somewhat weaker estimate on the rate of convergence of a_n/n. The result is as follows. Observe the implicit assumption, made without loss of generality, that $\theta_0 = 0$.

THEOREM. *For some positive constants* B *and* δ, *and for some complex number* $s \neq 0$, *let* $f \in S$ *satisfy the inequality*

$$\left|\frac{(1-r)^2}{r} f(r) - s\right| \leqslant B(1-r)^{\delta}, \qquad 0 < r < 1.$$

Then

$$\left|\frac{a_n}{n} - s\right| \leqslant \frac{C}{\log n}, \qquad n = 2, 3, \ldots,$$

where C depends only on $|s|$, B, and δ.

The proof depends upon a Tauberian remainder theorem, essentially due to Freud [3] and Korevaar [9], which we now state in the notation of Theorem A. A proof of Theorem C by the Karamata–Wielandt method is also implicit in Ganelius' notes [**4**; pp. 3–6].

THEOREM C (Freud–Korevaar). *For some positive constants B and δ, suppose*

$$|g(r)-s| \leqslant B(1-r)^{\delta}, \qquad 0 < r < 1,$$

and suppose $|s_n| \leqslant M$, $\quad n = 1, 2, \ldots.$ Then

$$|\sigma_n - s| \leqslant C/\log n, \qquad n = 2, 3, \ldots,$$

where C depends only on B, M, and δ.

Proof of Theorem. Let $\alpha = |s|$, and define g by the relation (5). Since $\sigma_{n-1} = a_n/n$, we need only obtain a uniform bound for $s_n = a_{n+1} - a_n$ in terms of α and B. It is known, of course, that $\big|\,|a_{n+1}| - |a_n|\,\big|$ has an absolute bound. Our hypothesis on f implies [**12**; p. 87] that $|a_n| \to \infty$ and $\arg a_n \to \arg s$, but it seems difficult to estimate the rate of convergence in terms of α and B alone. Accordingly, we follow Milin's argument to estimate s_n.

Let c_n be defined by (4), and γ_n by (6). Then $c_n = 2(\gamma_n - 1/n)$, and Theorem **B** gives

$$\sum_{n=1}^{\infty} n|c_n|^2 \leqslant 2\log 1/\alpha. \tag{7}$$

On the other hand,

$$\sum_{n=0}^{\infty} s_n r^n = \frac{g(r)}{1-r} = \exp\left\{\sum_{n=1}^{\infty} (c_n + (1/n))\, r^n\right\}.$$

Hence, by an inequality of Lebedev and Milin [**12**; p. 51],

$$|s_n|^2 \leqslant \exp\left\{\sum_{k=1}^{n} k|c_k + (1/k)|^2 - \sum_{k=1}^{n} (1/k)\right\}$$

$$\leqslant (1/\alpha^2)\exp\left\{2\,\mathrm{Re}\sum_{k=1}^{n} c_k\right\},$$

where (7) has been used. Furthermore, by hypothesis,

$$\exp\left\{\mathrm{Re}\sum_{k=1}^{n} c_k\right\} = |g(r)|\exp\left\{\mathrm{Re}\left(\sum_{k=1}^{n} c_k - \sum_{k=1}^{\infty} c_k r^k\right)\right\}$$

$$\leqslant (\alpha + B)\exp\left\{\left|\sum_{k=1}^{n} c_k - \sum_{k=1}^{\infty} c_k r^k\right|\right\}.$$

But as in the proof of Fejér's Tauberian theorem [**10**; p. 65],

$$\left|\sum_{k=1}^{n} c_k - \sum_{k=1}^{\infty} c_k r^k\right| \leqslant \left|\sum_{k=1}^{n} c_k(1-r^k)\right| + \left|\sum_{k=n+1}^{\infty} c_k r^k\right|$$

$$\leqslant \left\{\sum_{k=1}^{n} k|c_k|^2\right\}^{\frac{1}{2}} \left\{\sum_{k=1}^{n} (1/k)\,(1-r^k)^2\right\}^{\frac{1}{2}} + \frac{1}{n}\left\{\sum_{k=n+1}^{\infty} k|c_k|^2\right\}^{\frac{1}{2}} \left\{\sum_{k=n+1}^{\infty} kr^{2k}\right\}^{\frac{1}{2}}$$

$$\leqslant 2(2\log(1/\alpha))^{\frac{1}{2}},$$

with the choice $r = 1-(1/n)$. This establishes the required bound on s_n, which completes the proof of the theorem.

One can make a similar quantitative statement when $\alpha = 0$. In this case one can proceed directly, without recourse to Tauberian remainder theorems. Since there is no longer a unique direction of maximal growth, the natural counterpart of our theorem simply estimates a_n in terms of the rate of growth of the maximum modulus of f. See, for example, Hayman [**8**; p. 392].

I wish to thank the referee for helpful criticism of the manuscript.

References

1. I. E. Bazilevich, " On the dispersion of the coefficients of univalent functions ", *Mat. Sb.*, 68 (1965), 549–560 (in Russian).
2. ———, " On a univalence criterion for regular functions and the dispersion of their coefficients ", *Mat. Sb.*, 74 (1967), 133–146 (in Russian).
3. G. Freud, " Restglied eines Tauberschen Satzes, I ", *Acta Math. Acad. Sci. Hungar.*, 2 (1951), 299–308; II, *ibid.* 3 (1952), 299–307.
4. T. Ganelius, *Tauberian Remainder Theorems* (Lecture Notes in Mathematics, No. 232, Springer-Verlag, 1971).
5. G. H. Hardy, *Divergent Series* (Oxford University Press, 1949).
6. W. K. Hayman, " The asymptotic behaviour of p-valent functions ", *Proc. London Math. Soc.*, 5 (1955), 257–284.
7. W. K. Hayman, *Multivalent Functions* (Cambridge University Press, 1958).
8. W. K. Hayman, " Coefficient problems for univalent functions and related function classes ", *J. London Math. Soc.*, 40 (1965), 385–406.
9. J. Korevaar, " Best L_1 approximation and the remainder in Littlewood's theorem ", *Nederl. Akad. Wetensch. Proc. Ser. A.* 56 = *Indagationes Math.*, 15 (1953), 281–293.
10. E. Landau, *Darstellung und Begründung einiger neurerer Ergebnisse der Funktionentheorie* (Zweiten Aufl., J. Springer, Berlin, 1929).
11. I. M. Milin, " Hayman's regularity theorem for the coefficients of univalent functions ", *Dokl. Akad. Nauk SSSR* 192 (1970), 738–741 (in Russian).
12. I. M. Milin, *Univalent Functions and Orthonormal Systems* (Izdat. " Nauka ", Moscow, 1971; in Russian).

Department of Mathematics,
University of Michigan,
Ann Arbor, Michigan 48104
U.S.A.