

# The sum of the Möbius function

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## ABSTRACT

We derive from the Riemann Hypothesis an estimate for  $M(x) = \sum_{n \leq x} \mu(n)$ . This is the first improvement of the bound that Titchmarsh established in 1927.

## 1. Introduction

Let  $M(x) = \sum_{1 \leq n \leq x} \mu(n)$ , where  $\mu(n)$  denotes the Möbius function. In [3], Littlewood proved that if the Riemann hypothesis (RH) is true then  $1/\zeta(1/2 + \varepsilon + it) \ll t^\varepsilon$  for any fixed  $\varepsilon > 0$ , and it follows by Perron's formula that

$$M(x) \ll x^{1/2+\varepsilon}. \quad (1)$$

The converse is trivial, since this estimate, by partial summation, implies that the series  $\sum_{n=1}^{\infty} \mu(n)n^{-s} = 1/\zeta(s)$  converges for  $\sigma > 1/2$ . Subsequently, Landau [2] showed, still assuming RH, that (1) is valid with  $\varepsilon \ll (\log \log \log x)/\log \log x$ , and Titchmarsh [15] improved this to  $\varepsilon \ll 1/\log \log x$ . Titchmarsh's analysis was based on the estimate

$$\zeta\left(\frac{1}{2} + \frac{1}{\log \log t + it}\right)^{-1} \ll \exp\left(\frac{c \log t}{\log \log t}\right) \quad (t \geq 4)$$

that Littlewood [4] derived from RH. The above still stands as the best known estimate of its kind but, from the work of Selberg [13], we know that a much better estimate applies for most  $t$ . In particular, Selberg (unpublished) derived asymptotic formulae for the moments

$$\int_0^T (\log |\zeta(\sigma + it)|)^{2k} dt,$$

from which it can be seen that the distribution of

$$\frac{\log |\zeta(1/2 + it)|}{\sqrt{\log \log t}},$$

for  $4 \leq t \leq T$ , tends weakly to normal distribution with mean 0 and variance  $1/2$  as  $T \rightarrow \infty$ . By utilizing Selberg's techniques, we can bound the frequency with which  $|\log \zeta(\sigma + it)|$  is large. Once our basic result is in place, it can be put to various uses. For example, in Corollary 1 below, we find that (1) holds with  $\varepsilon = (\log x)^{-22/61}$ .

**THEOREM (Assume RH).** *For any  $x \geq 2$  there is a piecewise linear contour lying in the rectangle  $1/2 < \sigma < 1$  and  $-x \leq t \leq x$  that links the bottom edge of the rectangle to the top, for which the following estimates apply:*

$$\int_{0 \leq t \leq 16} \left| \frac{x^s}{\zeta(s)} \right| |ds| \ll x^{1/2} \log \log x. \quad (2)$$

For  $16 \leq T \leq \exp((\log x)^{39/61})$ , we have

$$\int_{T \leq t \leq 2T} \left| \frac{x^s}{\zeta(s)} \right| |ds| \ll x^{1/2} T \left( \frac{e \log x}{\log T} \right)^{C \log T / \log \log T}. \quad (3)$$

For  $\exp((\log x)^{39/61}) \leq T \leq x$ , we have

$$\int_{T \leq t \leq 2T} \left| \frac{x^s}{\zeta(s)} \right| |ds| \ll x^{1/2} T (\log x)^A \exp \left( \left( \frac{\log x}{\log T} \right)^{39/22} \right). \quad (4)$$

Finally, for each real number  $t$ , where  $\exp((\log x)^{39/61}) \leq t \leq x$ , let  $\sigma(t)$  be chosen so that  $\sigma(t) + it \in \mathcal{C}$ . Then the quantity

$$\frac{x^\sigma}{|\zeta(\sigma + it)|}$$

is an increasing function of  $\sigma$  for  $\sigma(t) \leq \sigma < \infty$ .

In the above, and elsewhere, we denote by  $A$  and  $C$  effectively computable absolute constants, which may be different from one occasion to the next. The limit of our method would allow the exponent  $39/22$  to be replaced by a slightly smaller (presumably transcendental) number, but, for simplicity, we content ourselves with the above.

**COROLLARY 1** (Assume RH). For  $x \geq 2$ , we have

$$M(x) \ll x^{1/2} \exp(C(\log x)^{39/61}).$$

**COROLLARY 2** (Assume RH). There is an absolute constant  $A$  such that, if  $x \geq 2$  and  $h \leq x^{1-\delta}$ , then

$$\sum_{x < n \leq x+h} \mu(n) \ll_\delta x^{1/2} (\log x)^A.$$

The above estimates are remarkably inferior to the corresponding estimates for  $\pi(x)$ : assuming RH, we know that  $\pi(x) = \text{li } x + O(x^{1/2} \log x)$  (see [9, Theorem 13.1]). In the opposite direction it is easy to prove that  $M(x) = \Omega_\pm(x^{1/2})$ . Mertens conjectured that  $|M(x)| \leq x^{1/2}$ , but this has been disproved by Odlyzko and te Riele [11] by means of extensive numerical calculations. The finer behavior of  $M(x)$  depends in a complicated way both on the distribution of  $|\zeta'(\rho)|$  as  $\rho = 1/2 + i\gamma$  runs over the nontrivial zeros of the zeta function, and on the extent of linear independence of the imaginary parts of the  $\gamma > 0$ . Based on speculations relating to random matrix theory, Hughes, Keating, and O'Connell [1] have conjectured that

$$\sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(\rho)|} \sim \alpha T (\log T)^{1/4}, \quad (5)$$

where  $\alpha$  is a certain specified positive constant. This estimate implies (assuming RH) that

$$M(x) \ll x^{1/2} (\log x)^{5/4} \quad (6)$$

for  $x \geq 2$ . However, more should be true, and indeed Gonek (unpublished) conjectured that

$$\overline{\lim}_{x \rightarrow \infty} \frac{M(x)}{x^{1/2} (\log \log \log x)^{5/4}} \geq 0 \quad (7)$$

(see [10]). This corresponds to the conjecture of Montgomery [7] that

$$\overline{\lim}_{x \rightarrow \infty} \frac{\psi(x) - x}{x^{1/2}(\log \log \log x)^2} = \pm \frac{1}{2\pi}. \quad (8)$$

## 2. Monotonicity principles

The following lemmas capture in a succinct manner some of the main ingredients in the arguments of Landau [2] and Titchmarsh [15].

LEMMA 1 (Assume RH). *Let  $\xi(s) = (1/2)s(s-1)\zeta(s)\Gamma(s/2)\pi^{-s/2}$ . Then, for any fixed  $t$ ,*

$$\left(\sigma - \frac{1}{2}\right) \Re \frac{\xi'}{\xi}(\sigma + it) \quad (9)$$

*is a non-negative strictly increasing function of  $\sigma$  for  $1/2 \leq \sigma < \infty$ .*

By the symmetry of the functional equation  $\xi(1-s) = \xi(s)$ , it follows that expression (9) is invariant when  $\sigma$  is replaced by  $1-\sigma$ . Thus, for  $\sigma \leq 1/2$ , the expression is non-negative and strictly decreasing.

*Proof.* By Montgomery and Vaughan [9, (10.28) and (10.30)], we know that

$$\Re \frac{\xi'}{\xi}(s) = \sum_{\rho} \Re \frac{1}{s-\rho}, \quad (10)$$

where the sum is over all nontrivial zeros  $\rho = 1/2 + i\gamma$  of the zeta function. Hence expression (9) is equal to

$$\sum_{\gamma} \frac{(\sigma - 1/2)^2}{(\sigma - 1/2)^2 + (t - \gamma)^2}.$$

Since each term of this sum has the required properties, the result is immediate.  $\square$

LEMMA 2. *There is a constant  $C > 0$  such that, if  $t \geq C$ , then*

$$\left(\sigma - \frac{1}{2}\right) \left( \Re \frac{\zeta'}{\zeta}(\sigma + it) - \Re \frac{\xi'}{\xi}(\sigma + it) + \frac{1}{2} \log \frac{t}{2} \right)$$

*is a non-negative strictly increasing function of  $\sigma$  for  $1/2 \leq \sigma \leq 2$ .*

It seems likely that  $C = 2.8$  is admissible in the above. Certainly  $C = 2.7$  is not.

*Proof.* Since

$$\frac{\zeta'}{\zeta}(s) = \frac{\xi'}{\xi}(s) - \frac{1}{s-1} + \frac{1}{2} \log \pi - \frac{1}{2} \frac{\Gamma'}{\Gamma} \left( \frac{s}{2} + 1 \right),$$

we need to show that

$$\frac{1}{2} \left( \sigma - \frac{1}{2} \right) \left( \frac{2(1-\sigma)}{(\sigma-1)^2 + t^2} + \log \frac{\pi t}{2} - \Re \frac{\Gamma'}{\Gamma} \left( \frac{\sigma}{2} + 1 + \frac{it}{2} \right) \right) \quad (11)$$

is non-negative and increasing. By the Euler–MacLaurin sum formula, we find that

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O(1/|s|)$$

uniformly for  $|\arg s| \leq \pi - \delta$  and  $|s| \geq 1$  (see [9, Theorem C.1]). By applying Cauchy's formula to  $(\Gamma'/\Gamma)(s) - \log s$ , or by applying the Euler–MacLaurin formula to the expansion

$$\left(\frac{\Gamma'}{\Gamma}\right)'(s) = \sum_{n=0}^{\infty} \frac{1}{(s+n)^2},$$

we also find that

$$\left(\frac{\Gamma'}{\Gamma}\right)'(s) = \frac{1}{s} + O\left(\frac{1}{|s|^2}\right)$$

uniformly for  $|\arg s| \leq \pi - \delta$  and  $|s| \geq 1$ . By appealing to these estimates, we discover that the derivative of (11) with respect to  $\sigma$  is  $(1/2)\log \pi + O(1/t)$ , which is positive if  $t \geq C$ , and so expression (11) is strictly increasing. Since it vanishes when  $\sigma = 1/2$ , it also follows that it is non-negative.  $\square$

On summing the quantities in the two preceding lemmas, we obtain the following.

LEMMA 3 (Assume RH). *There is an absolute constant  $C > 0$  such that, if  $t \geq C$ , then*

$$\left(\sigma - \frac{1}{2}\right) \left(\Re \frac{\zeta'}{\zeta}(\sigma + it) + \frac{1}{2} \log \frac{t}{2}\right) \quad (12)$$

*is a non-negative strictly increasing function of  $\sigma$  for  $1/2 \leq \sigma \leq 2$ .*

LEMMA 4 (Assume RH). *If  $1/2 < \sigma_1 \leq \sigma_2 \leq 2$  and  $t \geq C$ , then*

$$|\zeta(\sigma_1 + it)| \geq |\zeta(\sigma_2 + it)| \left(\frac{\sigma_1 - 1/2}{\sigma_2 - 1/2}\right)^{(\sigma_2 - 1/2)(\Re(\zeta'/\zeta)(\sigma_2 + it) + (1/2)\log(t/2))}.$$

*Proof.* Let  $f(\sigma)$  denote expression (12). Thus  $f(\sigma) \leq f(\sigma_2)$  for  $1/2 \leq \sigma \leq \sigma_2$ . We divide both sides of this by  $\sigma - 1/2$  and integrate over  $\sigma_1 \leq \sigma \leq \sigma_2$ , exponentiate, and discard a factor  $(t/2)^{(\sigma_1 - \sigma_2)/2}$  to obtain the stated inequality.  $\square$

Littlewood [4] showed (assuming RH) that

$$\log \zeta(s) \ll \frac{\log \tau}{\log \log \tau}, \quad \Re \frac{\zeta'}{\zeta}(s) \ll \log \tau \quad (13)$$

for  $\sigma \geq 1/2 + 1/\log \log \tau$ , where  $\tau = |t| + 4$ . Suppose that  $4 \leq t \leq x$ , and take

$$\sigma_1 = \frac{1}{2} + \frac{\log t}{(\log x) \log \log t}, \quad \sigma_2 = \frac{1}{2} + \frac{1}{\log \log t}$$

in Lemma 4. Then, by the estimates (13), we see that

$$\frac{1}{\zeta(s)} \ll \left(\frac{e \log x}{\log t}\right)^{\frac{C \log t}{\log \log t}}$$

for  $s = \sigma_1 + it$ . Titchmarsh's estimate follows immediately by applying Perron's formula on this contour.

### 3. Large value estimates

We first establish a basic tool.

LEMMA 5. Suppose that

$$S(s) = \sum_{p \leq N} a(p)p^{-s},$$

where the  $a(p)$  are arbitrary real or complex numbers. Suppose that  $\alpha, T$ , and  $T_0$  are real numbers such that  $T \geq 2$ . For  $1 \leq r \leq R$ , let  $s_r = \sigma_r + it_r$  be points such that  $\sigma_r \geq \alpha$  and  $T_0 \leq t_r \leq T_0 + T$ . Also, suppose that these points are well spaced to the extent that  $|s_{r_1} - s_{r_2}| \geq 1/\log T$  for  $1 \leq r_1 < r_2 \leq R$ . If  $k$  is a positive integer such that  $N^k \leq T$ , then

$$\sum_{r=1}^R |S(s_r)|^{2k} \ll T(\log T)^2 k! \left( \sum_{p \leq N} |a(p)|^2 p^{-2\alpha} \right)^k.$$

*Proof.* Let  $D(s) = S(s)^k = \sum_{n \leq N^k} c_n n^{-s}$ . We show first that

$$\sum_{r=1}^R |D(s_r)|^2 \ll T(\log T)^2 \sum_n |c_n|^2 n^{-2\alpha}. \quad (14)$$

To this end, let  $(a)$  denote a disc of radius  $1/(2 \log T)$  centered at  $a$ . Then

$$|D(s)|^2 \leq \frac{4}{\pi} (\log T)^2 \iint_{(s)} |D(x + iy)|^2 dx dy$$

for any  $s$ . Since the discs  $(s_r)$  are disjoint and since they all lie in the half-strip  $\sigma \geq \alpha - 1/\log T$  and  $T_0 - 1 \leq t \leq T_0 + T + 1$ , it follows that

$$\sum_{r=1}^R |D(s_r)|^2 \ll (\log T)^2 \int_{\alpha-1/\log T}^{\infty} \int_{T_0-1}^{T_0+T+1} |D(\sigma + it)|^2 dt d\sigma.$$

By a standard mean-value theorem (see [5, Theorem 6.1; 6, 8]), we know that the inner integral above is

$$(T + O(N^k)) \sum_n |c_n|^2 n^{-2\sigma}.$$

On integrating this with respect to  $\sigma$ , we find that

$$\sum_{r=1}^R |D(s_r)|^2 \ll T(\log T)^2 \sum_n \frac{|c_n|^2}{n^{2\alpha} \log n}.$$

Note that the term  $n = 1$  does not occur in the above, since  $c_n \neq 0$  only when  $\Omega(n) = k$ . Thus  $\log n \gg 1$  for all the above  $n$ , and hence we have (14).

To complete the proof, we note that, if  $n$  has the canonical factorization  $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$  with  $\Omega(n) = \sum_i k_i = k$ , then

$$c_n = \binom{k}{k_1 \ k_2 \ \dots \ k_m} \prod_{i=1}^m a(p_i)^{k_i}.$$

Hence

$$\sum_n |c_n|^2 n^{-2\alpha} = \sum_n \binom{k}{k_1 \ k_2 \ \dots \ k_m}^2 \prod_{i=1}^m \frac{|a(p_i)|^{2k_i}}{p_i^{2k_i \alpha}}.$$

Here the multinomial coefficient is at most  $k!$  in all cases, and so the above is at most

$$k! \sum_n \binom{k}{k_1 \ k_2 \ \dots \ k_m} \prod_{i=1}^m \frac{|a(p_i)|^{2k_i}}{p_i^{2k_i \alpha}} = k! \left( \sum_{p \leq N} |a(p)|^2 p^{-2\alpha} \right)^k$$

by the multinomial theorem. The stated result now follows by combining the above with (14).  $\square$

We now use the above, and Selberg's technique, to estimate the number of times that  $|(\zeta'/\zeta)(s)|$  is large. It transpires that the exponent  $\xi$  in our Theorem 1 depends on the constants that arise in the next lemma, and so we take care to optimize the parameters. The main parameter,  $\eta$ , is left undetermined until its optimal value becomes apparent, in the next section.

**LEMMA 6** (Assume RH). *Suppose that  $0 < \eta \leq 1/2$ , that  $\varepsilon > 0$ , that  $T \geq T_0(\varepsilon)$ , and that  $\alpha \geq 1/2 + 1/\log T$ . For  $1 \leq r \leq R$ , let  $s_r = \sigma_r + it_r$  be points such that  $\sigma_r \geq \alpha$  and  $T \leq t_r \leq 2T$ , with the  $t_r$  well spaced in the sense that  $|t_{r_1} - t_{r_2}| \geq 1$  whenever  $r_1 \neq r_2$ . Finally, suppose that  $|(\zeta'/\zeta)(s_r)| \geq \eta \log T$  for  $1 \leq r \leq R$ . Then*

$$R \ll T(\log T)^3 \exp \left( -(f(\eta) - \varepsilon) \left( \alpha - \frac{1}{2} \right) (\log T) \log \left( \left( \alpha - \frac{1}{2} \right) (\log T) \right) \right),$$

where  $f(\eta) = 1/(\psi + \log(1 + 1/2\eta))$ . Here  $\psi$  is the unique real number such that  $e^{-\psi} + 1 = \psi$ .

By Newton's method it is easily found that  $\psi = 1.27846 \dots$

*Proof.* Since  $\psi > 1$ , it is clear that  $f(\eta) < 1$  for any choice of  $\eta$ . Thus the bound to be proved is worse than the trivial bound  $R \ll T$  if

$$\frac{1}{2} + \frac{1}{\log T} \leq \alpha \leq \frac{1}{2} + \frac{\log \log T}{(\log T) \log \log \log T}.$$

Hence we may assume that

$$\alpha \geq \frac{1}{2} + \frac{\log \log T}{(\log T) \log \log \log T}. \quad (15)$$

Let

$$w(n) = \begin{cases} 1 & \text{if } n \leq u, \\ \frac{\log uv/n}{\log v} & \text{if } u < n \leq uv, \\ 0 & \text{if } n > uv. \end{cases} \quad (16)$$

Then

$$\sum_n \frac{\Lambda(n)w(n)}{n^s} = \frac{-1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\zeta'}{\zeta}(s+z) \frac{u^z(v^z-1)}{z^2 \log v} dz.$$

We move the contour to the left and apply the calculus of residues to see that the above is equal to

$$-\frac{\zeta'}{\zeta}(s) + \sum_{\rho} \frac{u^{\rho-s}(1-v^{\rho-s})}{(\rho-s)^2 \log v} + \frac{u^{1-s}(v^{1-s}-1)}{(s-1)^2 \log v} + \sum_{k=1}^{\infty} \frac{u^{-2k-s}(1-v^{-2k-s})}{(2k+s)^2 \log v}, \quad (17)$$

provided that  $s \neq 1$ , and that  $\zeta(s) \neq 0$ . In the case  $u = v$ , this is Lemma 2 of Selberg [12]. For a more elaborate formula of this kind, see [14, Lemma 10]. The above is true unconditionally, but if we assume RH, then we find that the second term above has modulus at most

$$\frac{u^{1/2-\sigma}(1+v^{1/2-\sigma})}{(\sigma-1/2) \log v} \sum_{\rho} \frac{\sigma-1/2}{(\sigma-1/2)^2 + (t-\gamma)^2}.$$

Here the sum is  $\Re(\xi'/\xi)(s)$ , which by Lemma 2 is at most  $\Re(\zeta'/\zeta)(s) + (1/2)\log(t/2)$ . Thus the above is at most

$$\frac{u^{1/2-\alpha}(1+v^{1/2-\alpha})}{(\alpha-1/2)\log v} \left( \Re \frac{\zeta'}{\zeta}(s) + \frac{1}{2} \log T \right) \quad (18)$$

for  $\sigma \geq \alpha$  and  $T \leq t \leq 2T$ .

Write  $S(s) = \sum_{p \leq uv} w(p)(\log p)p^{-s}$ . We must choose  $u$  and  $v$  to be sufficiently large so as to ensure that  $|S(s_r)| \geq \delta \log T$ . We may assume that  $u \geq 2$ ,  $v \geq 2$ , and that  $uv \leq T$ . Thus the last two terms in (17) are much less than  $1/T$ . Also,

$$\sum_p \sum_{k=2}^{\infty} \frac{w(p^k) \log p}{p^{ks}} \ll \sum_p \frac{\log p}{p^{2\alpha}} \ll \frac{1}{2\alpha-1} \ll \frac{(\log T) \log \log \log T}{\log \log T} = o(\log T)$$

by (15), and so the quantity on the left has absolute value at most  $\delta \log T$  for all sufficiently large  $T$ . Hence, if  $|(\zeta'/\zeta)(s_r)| \geq \eta \log T$ , then  $|S(s_r)| \geq \delta \log T$ , provided that

$$\eta \left( 1 - \frac{u^{1/2-\alpha}(1+v^{1/2-\alpha})}{(\alpha-1/2)\log v} \right) - \frac{u^{1/2-\alpha}(1+v^{1/2-\alpha})}{(2\alpha-1)\log v} \geq 2\delta. \quad (19)$$

We want to take  $k$  in Lemma 5 as large as possible. Therefore we want the above to hold with  $uv$  as small as possible. In order to determine the optimal choice of these parameters, we find it convenient to introduce a change of variables as follows:

$$u = \exp \left( \frac{U}{\alpha-1/2} \right), \quad v = \exp \left( \frac{V}{\alpha-1/2} \right).$$

Then (19) is equivalent to the inequality

$$\frac{\eta - 2\delta}{\eta + 1/2} \geq \frac{e^{-U}(1 + e^{-V})}{V},$$

and we want  $U + V$  to be as small as possible. We take  $U$  such that the above holds with equality, since it would be wasteful to take  $U$  any larger than necessary. Then

$$U + V = \log \left( \frac{1 + e^{-V}}{V} \right) + \log \left( \frac{\eta + 1/2}{\eta - 2\delta} \right) + V.$$

This is minimized by taking  $V = \psi$ , where  $\psi$  is the unique real number such that  $1 + e^{-\psi} = \psi$ . These considerations lead us to the choice

$$u = \left( \frac{\eta + 1/2}{\eta - 2\delta} \right)^{1/(\alpha-1/2)}, \quad v = \exp \left( \frac{\psi}{\alpha-1/2} \right).$$

We take  $k = \lceil (\log T) / \log uv \rceil$  in Lemma 5. If  $\delta$  is sufficiently small as a function of  $\varepsilon$ , then

$$k \geq \left( f(\eta) - \frac{\varepsilon}{2} \right) \left( \alpha - \frac{1}{2} \right) (\log T). \quad (20)$$

Now

$$\sum_p \frac{(\log p)^2}{p^{2\alpha}} \ll \frac{1}{(2\alpha-1)^2}.$$

Indeed, with a little work it should be possible to show that the left-hand side above is strictly less than the right-hand side (that is, that the implicit constant can be taken to be 1), for all  $\alpha > 1/2$ . Since  $k! \leq k^k$ , by Lemma 5 it follows that

$$(\delta \log T)^{2k} R \ll T(\log T)^2 \left( \frac{Ck}{(2\alpha-1)^2} \right)^k.$$

Thus

$$R \ll T(\log T)^2 \left( \frac{Ck}{\delta^2(2\alpha-1)^2(\log T)^2} \right)^k \ll T(\log T)^2 \left( \frac{C}{\delta^2(2\alpha-1)\log T} \right)^k.$$

This gives the stated result, in view of (15) and (20).  $\square$

LEMMA 7 (Assume RH). For  $1 \leq r \leq R$  let  $s_r = \sigma_r + it_r$  be points such that  $\sigma_r \geq \alpha \geq 1/2 + 10(\log \log T)/\log T$ ,  $T \leq t_r \leq 2T$ ,  $|t_{r_1} - t_{r_2}| \geq 1$  when  $r_1 \neq r_2$ , and

$$|\log \zeta(s_r)| \geq (\alpha - \frac{1}{2}) \log T, \quad \Re \frac{\zeta'}{\zeta}(s_r) \leq \frac{1}{2} \log T. \quad (21)$$

Then

$$R \ll T(\log T)^3 \exp \left( -\frac{1}{2} \left( \alpha - \frac{1}{2} \right) (\log T) \log \frac{(\alpha - 1/2) \log T}{2 \log \log T} \right).$$

*Proof.* In (17) we replace  $s$  by  $s + x$ , where  $0 \leq x \leq 1$ . The second term on the right-hand side has absolute value not exceeding

$$\begin{aligned} & \frac{u^{1/2-\sigma-x}(1+v^{1/2-\sigma-x})}{\log v} \sum_{\rho} \frac{1}{(\sigma+x-1/2)^2 + (t-\gamma)^2} \\ & \leq \frac{u^{1/2-\sigma-x}(1+v^{1/2-\sigma-x})}{(\sigma-1/2)\log v} \sum_{\rho} \frac{\sigma-1/2}{(\sigma-1/2)^2 + (t-\gamma)^2}. \end{aligned}$$

Here the last sum is  $\Re(\xi'/\xi)(s)$ . Hence, by Lemma 2, the above is at most

$$\frac{u^{1/2-\sigma-x} + (uv)^{1/2-\sigma-x}}{(\sigma-1/2)\log v} \left( \Re \frac{\zeta'}{\zeta}(s) + \frac{1}{2} \log T \right)$$

for  $T \leq t \leq 2T$ . We integrate over  $0 \leq x \leq 1$  to see that

$$\left| \log \zeta(s) - \sum_n \frac{\Lambda(n)w(n)}{(\log n)n^s} \right| \leq \left( \frac{u^{1/2-\alpha}}{\log u} + \frac{(uv)^{1/2-\alpha}}{\log uv} \right) \frac{\Re(\zeta'/\zeta)(s) + (1/2)\log T}{(\alpha-1/2)\log v} + O(1).$$

We now take  $u = v = \exp(1/(\alpha-1/2))$ . Thus, if  $\Re(\zeta'/\zeta)(s) \leq (1/2)\log T$ , then the above is less than  $(9/20)(\alpha-1/2)\log T$ . Write  $S(s) = \sum_p w(p)p^{-s}$ . We note that

$$\begin{aligned} \left| \sum_p \sum_{k=2}^{\infty} \frac{w(p^k)}{kp^{ks}} \right| & \leq \frac{1}{2} \sum_{p \leq uv} \frac{1}{p} + O(1) \leq \frac{1}{2} \log \frac{2}{\alpha-1/2} + O(1) \\ & \leq \frac{1}{2} \log \log T \leq \frac{1}{20} \left( \alpha - \frac{1}{2} \right) \log T. \end{aligned}$$

Thus we see that if (21) holds then  $|S(s_r)| \geq (1/2)(\alpha-1/2)\log T$ . We apply Lemma 5 with  $k = [(1/2)(\alpha-1/2)\log T]$ . Thus we find that

$$\left( \frac{1}{2} \left( \alpha - \frac{1}{2} \right) \log T \right)^{2k} R \ll T(\log T)^2 k! \left( \sum_p \frac{w(p)^2}{p^{2\alpha}} \right)^k.$$

Now

$$\sum_p \frac{w(p)^2}{p^{2\alpha}} \leq \sum_{p \leq uv} \frac{1}{p} = \log \frac{2}{\alpha-1/2} + O(1) \leq \log \log T.$$

Hence

$$\begin{aligned} R &\ll T(\log T)^2 \left( \frac{4k \log \log T}{(\alpha - 1/2)^2 (\log T)^2} \right)^k \leq T(\log T)^2 \left( \frac{2 \log \log T}{(\alpha - 1/2) \log T} \right)^k \\ &\ll T(\log T)^3 \left( \frac{2 \log \log T}{(\alpha - 1/2) \log T} \right)^{(1/2)(\alpha - 1/2) \log T}. \end{aligned}$$

This gives the stated result.  $\square$

LEMMA 8 (Assume RH). For  $1 \leq r \leq R$  let  $s_r = \sigma_r + it_r$  be points such that  $\sigma_r \geq 1/2$ ,  $T \leq t_r \leq 2T$ , and  $|t_{r_1} - t_{r_2}| \geq 1$  when  $r_1 \neq r_2$ . Suppose also that  $V \geq 15 \log \log T$ , that  $\log |\zeta(s_r)| \leq -V$ , and that

$$\Re \frac{\zeta'}{\zeta}(\sigma + it_r) \leq \frac{1}{2} \log T \quad (22)$$

for  $\sigma_r \leq \sigma < \infty$ . Then

$$R \ll T(\log T)^3 \exp \left( -\frac{V}{3} \log \frac{V}{3 \log \log T} \right).$$

*Proof.* Write  $\sigma'_r = \sigma_r + 2V/(3 \log T)$ , and set  $s'_r = \sigma'_r + it_r$ . Then

$$V + \log |\zeta(s'_r)| \leq \log |\zeta(s'_r)| - \log |\zeta(s_r)| = \int_{\sigma_r}^{\sigma'_r} \Re \frac{\zeta'}{\zeta}(\sigma + it_r) d\sigma.$$

By the bound (22), this is at most

$$(\sigma'_r - \sigma_r) \frac{1}{2} \log T = \frac{V}{3}.$$

Hence  $\log |\zeta(s'_r)| \leq -2V/3$ . Write  $\alpha = 1/2 + 2V/(3 \log T)$ , and note that  $\sigma'_r \geq \alpha$  for all  $r$ . The stated bound now follows from Lemma 7 with the  $s_r$  replaced by the  $s'_r$ .  $\square$

#### 4. Proof of Theorem 1

First we define the contour. The contour is to be symmetric with respect to the real axis, and so it suffices to describe it in the upper half-plane. Let  $c = 39/61$ , and set  $J = [(\log x)^c / \log 2]$ . Thus  $2^J \leq \exp((\log x)^c) \leq 2^{J+1}$ . For  $0 \leq t \leq 2^J$  we proceed along a polygonal path with vertices

$$\begin{aligned} &\frac{1}{2} + \frac{2}{\log x}, \quad \frac{1}{2} + \frac{2}{\log x} + 16i, \quad \frac{1}{2} + \frac{4}{(\log 4) \log x} + 16i, \quad \frac{1}{2} + \frac{4}{(\log 4) \log x} + 32i, \\ &\frac{1}{2} + \frac{5}{(\log 5) \log x} + 32i, \end{aligned}$$

and, in general,

$$\dots, \frac{1}{2} + \frac{j}{(\log j) \log x} + 2^j i, \frac{1}{2} + \frac{j}{(\log j) \log x} + 2^{j+1} i, \frac{1}{2} + \frac{j+1}{(\log(j+1)) \log x} + 2^{j+1} i, \dots$$

until we reach the point

$$\frac{1}{2} + \frac{J}{(\log J) \log x} + 2^J i. \quad (23)$$

For  $j \geq J$  set  $T = 2^j$ . We now define the contour for  $T \leq t \leq 2T$ . For each integer  $r$ , where  $T \leq r < 2T$ , define  $\sigma_2(r)$  to be the least number such that

$$\Re \frac{\zeta'}{\zeta}(s) \leq \eta \log T$$

for all  $s$  in the half-strip  $\sigma \geq \sigma_2(r)$  and  $r \leq t \leq r+1$ . Here  $\eta$  is a positive parameter whose value will be chosen later. For the present, we assume only that  $0 < \eta < 1/2$ . Let  $1/2 + i\gamma$  be a zero of the zeta function with  $r \leq \gamma \leq r+1$ . From (10) we see that  $\Re(\xi'/\xi)(s) > \log T$  when  $s = 1/2 + 1/\log T + i\gamma$ . From Lemma 2 it then follows that  $\Re(\zeta'/\zeta)(s) > (1/2)\log T > \eta \log T$  for this same  $s$ . Thus

$$\sigma_2(r) \geq \frac{1}{2} + \frac{1}{\log T} \quad (24)$$

for all  $r$ . With  $\sigma_2(r)$  determined in this way, we write

$$\sigma_1(r) = \frac{1}{2} + \left( \sigma_2(r) - \frac{1}{2} \right) \frac{\log T}{\log x}. \quad (25)$$

To extend our contour from the point (23), we first move to the point  $\sigma_1(2^J) + 2^J i$ . After that, for each  $r \geq 2^J$ , we move from  $\sigma_1(r) + ri$  to  $\sigma_1(r) + (r+1)i$ , and from there to  $\sigma_1(r+1) + (r+1)i$ .

We now prove (2). Let  $\gamma_1 = 14.13\dots$  and  $\gamma_2 = 21.02\dots$  denote the ordinates of the first two zeros of the zeta function. Since  $\zeta'(1/2 + i\gamma_1) \neq 0$ , it follows that  $|\zeta(s)| \asymp |s - 1/2 - i\gamma_1|$  for  $s$  near  $1/2 + i\gamma_1$ . Since  $\gamma_2 > 16$ , (2) is immediate.

Next we prove (3). Suppose that  $T = 2^j$  with  $4 \leq j \leq J$ . From Littlewood's estimates (13) and Lemma 4 it is clear that

$$\frac{1}{\zeta(s)} \ll \left( \frac{e \log x}{\log T} \right)^{\frac{C \log T}{\log \log T}}$$

for  $s \in$  with  $T \leq t \leq 2T$ . Thus we have (3).

With these preliminaries completed, we initiate the proof of the main estimate, (4). Suppose that  $T = 2^j$  with  $j \geq J$ . For  $T \leq r < 2T$ , let  $t_1(r)$  be chosen such that  $|\zeta(\sigma_1(r) + it)|$  takes its minimum, for  $r \leq t \leq r+1$ , at  $t = t_1(r)$ . Set  $s_1(r) = \sigma_1(r) + it_1(r)$ , and set  $m(r) = 1/|\zeta(s_1(r))|$ . Then

$$\int_{r < t < r+1} \left| \frac{x^s}{\zeta(s)} \right| |ds| = \int_r^{r+1} \frac{x^{\sigma_1(r)}}{|\zeta(\sigma_1(r) + it)|} dt \leq x^{\sigma_1(r)} m(r). \quad (26)$$

Next we establish the last clause of Theorem 1. The logarithmic derivative of the expression in question is  $\log x - \Re(\zeta'/\zeta)(\sigma + it)$ . Suppose that  $r \leq t \leq r+1$ . By the definition of  $\sigma_2(r)$ , we know that

$$\Re \frac{\zeta'}{\zeta}(\sigma + it) \leq \eta \log T \leq \frac{1}{2} \log T \leq \log x$$

for  $\sigma \geq \sigma_2(r)$  and  $r \leq t \leq r+1$ . As for the remaining range,  $\sigma_1(r) \leq \sigma \leq \sigma_2(r)$ , we note that, by Lemma 3 we have

$$\begin{aligned} \Re \frac{\zeta'}{\zeta}(\sigma + it) &\leq \Re \frac{\zeta'}{\zeta}(\sigma + it) + \frac{1}{2} \log \frac{t}{2} \leq \frac{\sigma_2(r) - 1/2}{\sigma - 1/2} \left( \Re \frac{\zeta'}{\zeta}(\sigma_2(r) + it) + \frac{1}{2} \log \frac{t}{2} \right) \\ &\leq \frac{\sigma_2(r) - 1/2}{\sigma - 1/2} \log T \leq \frac{\sigma_2(r) - 1/2}{\sigma_1(r) - 1/2} \log T = \log x. \end{aligned}$$

Since  $x^\sigma/|\zeta(\sigma + ir)|$  is monotonically increasing for  $\sigma \geq \min(\sigma_1(r-1), \sigma_1(r))$ , and since the interval from  $\sigma_1(r-1) + ir$  to  $\sigma_1(r) + ir$  has length at most 1, it follows that

$$\int_{t=r} \left| \frac{x^s}{\zeta(s)} \right| |ds| \leq m(r-1)x^{\sigma_1(r-1)} + m(r)x^{\sigma_1(r)}.$$

On combining this with (26), we deduce that

$$\int_{T < t \leq 2T} \left| \frac{x^s}{\zeta(s)} \right| |ds| \ll \sum_{r=T}^{2T} x^{\sigma_1(r)} m(r). \quad (27)$$

From the definition of  $\sigma_1(r)$ , it is clear that  $x^{\sigma_1} = x^{1/2} T^{\sigma_2(r)-1/2}$ . Also, by Lemma 4 we see that

$$m(r) = \frac{1}{|\zeta(\sigma_1(r) + it_1(r))|} \leq \frac{(\log x / \log T)^{(\sigma_2(r)-1/2)(\eta+1/2) \log T}}{|\zeta(\sigma_2(r) + it_1(r))|}$$

since  $\Re(\zeta'/\zeta)(\sigma_2(r) + it_1(r)) \leq \eta \log T$ . On combining this with (27), we deduce that

$$\int_{T < t \leq 2T} \left| \frac{x^s}{\zeta(s)} \right| |ds| \ll x^{1/2} \sum_{r=T}^{2T} \frac{(8 \log x / \log T)^{(\sigma_2(r)-1/2)(\eta+1/2) \log T}}{|\zeta(\sigma_2(r) + it_1(r))|}. \quad (28)$$

To estimate the right-hand side above, we consider three types of  $r$ . Let  $\mathcal{I}_1$  denote the set of those  $r$ , where  $T \leq r \leq 2T$ , for which

$$\frac{1}{|\zeta(\sigma_2(r) + it_1(r))|} \leq (\log T)^{15} \exp \left( \varepsilon \left( \sigma_2(r) - \frac{1}{2} \right) (\log T) \log \left( \left( \sigma_2(r) - \frac{1}{2} \right) \log T \right) \right). \quad (29)$$

Let  $\mathcal{I}_2$  denote the set of those  $r$ , where  $T \leq r \leq 2T$ , for which

$$\sigma_2(r) \geq \frac{1}{2} + \frac{C_1 \log \log T}{(\log T) \log \log T} \quad (30)$$

and

$$\frac{1}{|\zeta(\sigma_2(r) + it_1(r))|} > \exp \left( \varepsilon \left( \sigma_2(r) - \frac{1}{2} \right) (\log T) \log \left( \left( \sigma_2(r) - \frac{1}{2} \right) \log T \right) \right). \quad (31)$$

Here  $C_1 = C_1(\varepsilon)$  is a large constant whose value will be determined later. Finally, let  $\mathcal{I}_3$  denote the set of those  $r$ , where  $T \leq r \leq 2T$ , for which

$$\sigma_2(r) \leq \frac{1}{2} + \frac{C_1 \log \log T}{(\log T) \log \log T} \quad (32)$$

and

$$\frac{1}{|\zeta(\sigma_2(r) + it_1(r))|} > (\log T)^{15}. \quad (33)$$

We note that, if (30) holds but (31) fails, then  $r \in \mathcal{I}_1$ , and that, if (32) holds but (33) fails, then  $r \in \mathcal{I}_1$ . Thus every  $r$  is in at least one of the  $\mathcal{I}_i$ .

For  $r \in \mathcal{I}_1$ , choose  $t_2(r)$ , where  $r \leq t_2(r) \leq r+1$ , such that  $\Re(\zeta'/\zeta)(\sigma_2(r) + it_2(r)) = \eta \log T$ . Among the  $r \in \mathcal{I}_1$ , consider those for which  $\alpha \leq \sigma_2(r) < \alpha + \delta$ , where  $\delta = (\log T)^{-2}$ . By Lemma 6, the contribution of these  $r$  to (28) is at most of the order of

$$x^{1/2} T (\log T)^{18} \exp(g(\alpha)),$$

where

$$g(\alpha) = \left( \alpha - \frac{1}{2} \right) (\log T) \left( \eta + \frac{1}{2} \right) \log \frac{8 \log x}{\log T} - (f(\eta) - 2\varepsilon) \left( \alpha - \frac{1}{2} \right) (\log T) \log \left( \left( \alpha - \frac{1}{2} \right) \log T \right).$$

This function assumes its maximum at

$$\alpha = \frac{1}{2} + \frac{1}{e \log T} \left( \frac{8 \log x}{\log T} \right)^{(\eta+1/2)/(f(\eta)-2\varepsilon)}.$$

The maximum value attained is

$$\frac{f(\eta) - 2\varepsilon}{e} \left( \frac{8 \log x}{\log T} \right)^{(\eta+1/2)/(f(\eta)-2\varepsilon)}.$$

This motivates us to take  $\eta$  so as to minimize the above exponent; that is, we take  $\eta$  to be the unique real number such that

$$\psi + \log \left( 1 + \frac{1}{2\eta} \right) = \frac{1}{2\eta}. \quad (34)$$

Numerically,  $\eta = 0.196570958763\dots$ ,  $f(\eta) = 0.393141917526\dots$ , and  $(\eta + 1/2)/f(\eta) = 1/2 + 1/(4\eta) = 1.771805365213\dots < 39/22$ . We also set  $\varepsilon = 10^{-4}$ , and observe that  $(\eta + 1/2)/(f(\eta) - 2\varepsilon) < 39/22$ . On summing over  $\alpha = 1/2 + 1/\log T + k\delta$ , we conclude that the total contribution of all  $r \in \mathcal{I}_1$  is at most of the order of

$$x^{1/2} T (\log T)^{20} \exp \left( \left( \frac{\log x}{\log T} \right)^{39/22} \right). \quad (35)$$

Among the  $r \in \mathcal{I}_2$ , we consider those for which  $\alpha \leq \sigma_2(r) < \alpha + \delta$  and  $V \leq -\log |\zeta(\sigma_2(r) + it_1(r))| < 2V$ , where

$$\alpha \geq \frac{1}{2} + \frac{C_1 \log \log T}{(\log T) \log \log \log T}, \quad V \geq V_0 = \varepsilon \left( \alpha - \frac{1}{2} \right) (\log T) \log \left( \left( \alpha - \frac{1}{2} \right) \log T \right).$$

We now take  $C_1 = (4/\varepsilon) \exp(9/\varepsilon)$ . This ensures that

$$\log \frac{V}{3 \log \log T} \geq \frac{9}{\varepsilon}.$$

Hence, by Lemma 8, the number of such  $r$  is at most of the order of  $T(\log T)^3 \exp(-3V/\varepsilon)$ . Since  $1/|\zeta(\sigma_2(r) + it_1(r))| \leq \exp(2V)$  for these  $r$ , on summing over  $V = V_0 2^k$  we deduce that the total contribution to (28) of those  $r \in \mathcal{I}_2$  for which  $\alpha \leq \sigma_2(r) < \alpha + \delta$  is at most of the order of  $x^{1/2} T (\log T)^3 \exp(h(\alpha))$ , where

$$h(\alpha) = \left( \alpha - \frac{1}{2} \right) (\log T) \left( \eta + \frac{1}{2} \right) \log \frac{8 \log x}{\log T} - \left( \alpha - \frac{1}{2} \right) (\log T) \log \left( \left( \alpha - \frac{1}{2} \right) \log T \right).$$

This is of the same form as the function  $g(\alpha)$  that arose in the preceding case, but with a more favorable constant. By proceeding as in the former case, we find that the total contribution to (28) of all  $r \in \mathcal{I}_2$  is at most of the order of

$$x^{1/2} T (\log T)^5 \exp \left( \frac{\log x}{\log T} \right). \quad (36)$$

Suppose that  $V \geq 15 \log \log T$ , and consider those  $r \in \mathcal{I}_3$  for which  $V \leq -\log |\zeta(\sigma_2(r) + it_1(r))| < (51/50)V$ . By Lemma 8, the sum of  $1/|\zeta(\sigma_2(r) + it_1(r))|$  over such  $r$  is at most of the order of

$$T (\log T)^3 \exp \left( \frac{51}{50} V - \frac{V}{3 \log \log T} \right).$$

On summing this over  $V = 15(51/50)^k \log \log T$ , we find that

$$\sum_{r \in \mathcal{I}_3} \frac{1}{|\zeta(\sigma_2(r) + it_1(r))|} \ll T (\log T)^{11}.$$

On the other hand, for  $r \in \mathcal{I}_3$  we have

$$\left( \frac{8 \log x}{\log T} \right)^{(\sigma_2(r) - 1/2)(\eta + 1/2) \log T} \leq \left( \frac{8 \log x}{\log T} \right)^{C_1 (\log \log T) / \log \log \log T}. \quad (37)$$

The ratio of the above with  $\exp((\log x)/\log T)^{39/22}$  is largest when  $(\log x)/\log T$  is of the form

$$C \left( \frac{\log \log T}{\log \log \log T} \right)^{22/39},$$

and therefore the right-hand side of (37) is at most of the order of

$$(\log T)^A \exp \left( \left( \frac{\log x}{\log T} \right)^{39/22} \right).$$

Hence the total contribution of all  $r \in \mathcal{R}_3$  to (28) is at most of the order of

$$x^{1/2} T (\log T)^A \exp \left( \left( \frac{\log x}{\log T} \right)^{39/22} \right).$$

On combining this with (35) and (36) in (28), we obtain (4), and the proof is complete.

### 5. Proof of the corollaries

By the truncated form of Perron's formula (see [9, Corollary 5.3]), we know that

$$M(x) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{\zeta(s)s} ds + O \left( \frac{x \log x}{T} \right),$$

where  $c = 1 + 1/\log x$ . Let  $\sigma(T)$  be chosen such that  $\sigma(T) + iT \in \mathcal{R}_3$ . By the last clause of the Theorem 1, we know that

$$\int_{\sigma(T)+iT}^{c+iT} \frac{x^s}{\zeta(s)s} ds \ll \frac{x \log x}{T}.$$

Hence

$$M(x) = \frac{1}{2\pi i} \int_{-T \leq t \leq T} \frac{x^s}{\zeta(s)s} ds + O \left( \frac{x \log x}{T} \right). \quad (38)$$

We take  $T = x^{1/2}$ , and apply the estimates (2)–(4) to obtain Corollary 1.

In proving Corollary 2, we may assume that  $h \leq x^{3/4}$ , as otherwise the stated bound follows directly from Corollary 1. From (38) we see that

$$M(x+h) - M(x) = \frac{1}{2\pi i} \int_{-T \leq t \leq T} \frac{(x+h)^s - x^s}{\zeta(s)s} ds + O \left( \frac{x \log x}{T} \right).$$

Clearly,

$$\frac{(x+h)^s - x^s}{s} \ll \begin{cases} hx^{\sigma-1} & (|t| \leq x/h), \\ x^\sigma/|t| & (|t| \geq x/h). \end{cases}$$

Hence from (2)–(4) we see that

$$M(x+h) - M(x) \ll_\delta x^{1/2} (\log x)^A.$$

This suffices.

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### References

1. C. P. HUGHES, J. P. KEATING, and N. O'CONNELL, 'Random matrix theory and the derivative of the Riemann zeta function', *Proc. R. Soc. London Ser. A Math. Phys. Eng. Sci.* 456 (2000) 2611–2627.
2. E. LANDAU, 'Über die Möbiussche Funktion', *Rend. Circ. Mat. Palermo* 48 (1924) 277–280.
3. J. E. LITTLEWOOD, 'Quelques conséquences de l'hypothèse que la fonction  $\zeta(s)$  de Riemann n'a pas de zéros dans le demi-plan  $\Re(s) > 1/2$ ', *C. R. Math. Acad. Sci. Paris* 154 (1912) 263–266.
4. J. E. LITTLEWOOD, 'On the zeros of the Riemann zeta-function', *Proc. Cambridge Philos. Soc.* 22 (1924) 295–318.

5. H. L. MONTGOMERY, *Topics in multiplicative number theory*, Lecture Notes in Mathematics 227 (Springer, Berlin, 1971).
6. H. L. MONTGOMERY, 'The analytic principle of the large sieve', *Bull. Amer. Math. Soc.* 84 (1978) 547–567.
7. H. L. MONTGOMERY, 'The zeta function and prime numbers', *Proceedings of the Queen's Number Theory Conference*, Kingston, Ontario, 1979, Queen's Papers in Pure and Applied Mathematics 54 (ed. P. Ribenboim; Queen's University, Kingston, 1980), 1–31.
8. H. L. MONTGOMERY and R. C. VAUGHAN, 'Hilbert's inequality', *J. London Math. Soc.* (2) 8 (1974) 73–82.
9. H. L. MONTGOMERY and R. C. VAUGHAN, *Multiplicative number theory I. Classical theory*, Cambridge Studies in Advanced Mathematics 97 (Cambridge University Press, Cambridge, 2007).
10. N. NG, 'The distribution of the summatory function of the Möbius function', *Proc. London Math. Soc.* (3) 89 (2004) 361–389.
11. A. M. ODLYZKO and H. J. J. TE RIELE, 'Disproof of the Mertens conjecture', *J. Reine Angew. Math.* 357 (1985) 138–160.
12. A. SELBERG, 'On the normal density of primes in small intervals', *Archiv for Matematik og Naturvidenskab* 47, Collected Papers 1 (Springer, Berlin, 1989) 160–178.
13. A. SELBERG, 'On the remainder in the formula for  $N(T)$ , the number of zeros of  $\zeta(s)$  in the strip  $0 < t < T$ ', *Avhandl. Norske Vid. Akad. Oslo I. Mat.-Naturv. Klasse* (1944), No. 1, 1–27.
14. A. SELBERG, 'Contributions to the theory of the Riemann zeta-function', *Archiv for Matematik og Naturvidenskab* 48, Collected Papers 1 (Springer, Berlin, 1989) 214–280.
15. E. C. TITCHMARSH, 'A consequence of the Riemann hypothesis', *J. London Math. Soc.* 2 (1927) 247–254.

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