This expresses $I_{k,m}(z)$ as a series of functions of the same type. It generalizes the expansion obtained in connexion with the wave functions in Coulomb fields. That expansion corresponds to (5) with $\alpha=0$. In the applications to the wave functions, the parameters k and z are purely imaginary so that it is then convenient to write

$$k=i\eta, \quad z=i\zeta, \quad \alpha=i\lambda$$

and

$$J_{\eta,m}(\zeta)=i^{-(\frac{1}{2}+m)}I_{i\eta,m}(i\zeta). \tag{6}$$

Making these substitutions in (5), we have

$$\begin{aligned}J_{\eta,m}(\zeta) &= \sum_{r=0}^{\infty}(-)^r\frac{i^{r+1}(\eta-\lambda)\Gamma(i\eta-i\lambda+\frac{1}{2}r)\zeta^{\frac{1}{2}r}}{r!\,\Gamma(i\eta-i\lambda-\frac{1}{2}r+1)}J_{\lambda,m+\frac{1}{2}r}(\zeta)\\ &= J_{\lambda,m}(\zeta)+(\eta-\lambda)\zeta^{\frac{1}{2}}J_{\lambda,m+\frac{1}{2}}(\zeta)+\tfrac{1}{2}(\eta-\lambda)^2\zeta J_{\lambda,m+1}(\zeta)\\ &\qquad +\tfrac{1}{6}(\eta-\lambda)\left((\eta-\lambda)^2+\tfrac{1}{4}\right)\zeta^{\frac{3}{2}}J_{\lambda,m+\frac{3}{2}}(\zeta)+\dots. \end{aligned}\tag{7}$$

The expansions (5) and (7) may be useful in interpolation with respect to the parameters k and η for $I_{k,m}(z)$ and $J_{\eta,m}(\zeta)$. It may be noted that, when $k=\eta=0$,

$$I_{0,m}(z)=\frac{(\pi z)^{\frac{1}{2}}}{\Gamma(\frac{1}{2}+m)}I_m(\tfrac{1}{2}z), \quad J_{0,m}(\zeta)=\frac{(\pi\zeta)^{\frac{1}{2}}}{\Gamma(\frac{1}{2}+m)}J_m(\tfrac{1}{2}\zeta)$$

so that (5) and (7) then yield expansions of the Bessel functions I_m and J_m in terms of Whittaker's functions.

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ON THE GROUP OF A GRAPH WITH RESPECT TO A SUBGRAPH*

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1. *Introduction.*

A *graph* G consists of a finite set P of *points* together with a prescribed collection L of unordered pairs of distinct points called *lines*. Two points of a graph are *adjacent* if they belong to a line of the graph. Two graphs G_1, G_2 are *isomorphic* if there is a one-one mapping of P_1 onto P_2 which preserves adjacency. An *automorphism* of a graph is a self-isomorphism.

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It is well known (König [6], p. 5) that the set of all automorphisms of a graph G forms a permutation group which acts on P. This group is called *the group of the graph* G. We also call it here the *point-group* of G. Our object is to extend this notion to the group $\Gamma(G, H)$ of G with respect to any subgraph H and study some of the consequences of this generalization. Using the powerful enumeration techniques of Pólya [7], we find in the next section a polynomial expression for the number of dissimilar occurrences of collections of copies of a given subgraph H in a graph G in terms of the cyclic structure of $\Gamma(G, H)$. As an application of the method, we obtain a formulation for the number of abstract 2-complexes with a given 1-skeleton. We conclude with some unsolved problems.

Let $\Gamma(G)$ be the group of the graph G. By the *line group* $\Gamma_1(G)$ of G we mean the permutation group which acts on the line set L of G which is induced by the elements of $\Gamma(G)$. The group $\Gamma_1(G)$ is defined in [5] and in Sabidussi [9] while Tutte [12] essentially defines $\Gamma(G)$ and $\Gamma_1(G)$ concurrently. To generalize, *the group of a graph G with respect to a subgraph H*, or more briefly *the H-group of G*, is the permutation group $\Gamma(G, H)$ which acts on all subgraphs of G isomorphic to H, induced by the automorphisms of G. This definition was anticipated in Sabidussi [9], where $M(G, H)$ is defined as the group of all injections of H into G. The *complete graph* K_p of p points is the graph in which every two distinct points are adjacent. Let us denote $\Gamma(G, K_{p+1})$ by $\Gamma_p(G)$. Then $\Gamma_0(G) = \Gamma(G)$ and $\Gamma_1(G) = \Gamma(G, K_2)$ as above.

2. *Collections of copies of a given subgraph.*

Two points of a graph G are *similar* if there is an automorphism mapping one onto the other. Similarity of two subgraphs of G is defined analogously. Given a subgraph H of G, let c_k be the number of dissimilar occurrences in G of k indistinguishable copies of H and let

$$c(G, H, x) = c_0 + c_1 x + c_2 x^2 + \dots . \tag{1}$$

We wish to derive a formula for this generating function.

We now require Pólya's Theorem [7] in the one-variable form given in [4], p. 447. Since the definitions already appear there, we include only a statement of the theorem here.

PÓLYA'S THEOREM. *The configuration counting series $F(x)$ is obtained by substituting the figure counting series $\phi(x)$ into the cycle index $Z(\Gamma)$ of the configuration group Γ. Symbolically:*

$$F(x) = Z\big(\Gamma, \phi(x)\big). \tag{2}$$

This theorem reduces the problem of finding the configuration counting series $F(x) = c(G, H, x)$ in the present context to the determination of the figure counting series and the cycle index of the configuration group.

Since any copy of H belonging to G is either absent or present in a collection of copies of H, we consider a "set of figures" having just two members, say ϕ_0 and ϕ_1, of weight 0 and 1 respectively. Hence the figure counting series is $\phi(x) = 1+x$. The configuration group is clearly $\Gamma(G, H)$. Thus an application of Equation (2) gives the following result.

THEOREM. *The generating function for the number of dissimilar occurrences in a graph G of collections of subgraphs isomorphic to H is:*

$$c(G, H, x) = Z\left(\Gamma(G, H),\ 1+x\right). \tag{3}$$

Some special cases of this result have already been obtained. One of these is the counting polynomial for all graphs with p points [4], *i.e.* the polynomial $g_p(x) = \Sigma g_{pq} x^q$ where g_{pq} is the number of non-isomorphic graphs of p points and q lines:

$$g_p(x) = Z\left(\Gamma_1(K_p),\ 1+x\right). \tag{4}$$

Another is the polynomial for the number [5] of dissimilar spanning subgraphs of G, *i.e.*, subgraphs of G containing all the points of G:

$$s_G(x) = Z\left(\Gamma_1(G),\ 1+x\right). \tag{5}$$

The exact form of $Z\left(\Gamma(G, H)\right)$ depends on the particular form of the graphs G and H. For certain graph-subgraph combinations, this cycle index has been explicitly formulated.

3. *Example: The number of symmetry types of boolean functions.*

A *boolean function* of n variables $x_1, x_2, \ldots, x_n$, also sometimes called a *switching function* in electric network theory, is a function in which the domain of each variable is 0, 1 and the range is 0, 1. Two boolean functions f_1 and f_2 are of the *same type* if f_1 can be transformed into f_2 by a permutation of the n variables followed by the complementation of a subset (possibly empty) of the variables (the complement of 0 is 1 and of 1 is 0).

The 1-skeleton or *graph of the n-cube*, denoted Q_n, may be defined as the cartesian product of n copies of K_2, or equivalently as the graph of 2^n points each of which is a binary sequence of n digits such that two points are adjacent whenever they differ in exactly one digit. It has been shown by Pólya ([8], 102) that there is a one-one correspondence between the boolean functions of n variables and the subsets of the set of points of Q_n such that two functions are of the same type if and only if their corresponding subsets of points are similar in Q_n. Hence the result of Pólya [**8**] that the counting polynomial for the symmetry types of boolean functions of n variables is given by $Z\left(\Gamma(Q_n),\ 1+x\right)$ is an application of a special case of the Theorem (3). A detailed algorithm for finding the cycle index $Z\left(\Gamma(Q_n)\right)$ has been given by Slepian [**11**].

The polynomial $Z\bigl(\Gamma(Q_n, Q_m), 1+x\bigr)$ gives for all values of k the number of dissimilar sets of k m-cubes in Q_n. The *composition* of two permutation groups A and B, introduced by Pólya [7] under the name "Gruppenkranz" and denoted $A[B]$, provides the mechanism for expressing one class of these polynomials, namely the case $m=n-1$, in closed form. This operation is also discussed in Frucht [3]. For it is easily seen (*cf.* Pólya [8], footnote 7) that $\Gamma(Q_n, Q_{n-1}) = S_n[S_2]$, the automorphism group of the hyperoctahedron. Hence the corresponding counting polynomial is

$$c(Q_n, Q_{n-1}, x) = Z(S_n[S_2], 1+x). \tag{6}$$

It is shown in Pólya [7], p. 180, that for any permutation groups A and B, $Z(A[B])$ is the composition of $Z(A)$ with $Z(B)$. This, together with the usual formula for $Z(S_n)$ gives this polynomial concisely, in the form:

$$c(Q_n, Q_{n-1}, x) = \sum_{(j)} \prod_{i=1}^{n} \frac{(1+x^i+x^{2i})^{j_i}}{i^{j_i} j_i!}, \tag{7}$$

where the sum is taken over all partitions (j) of n, *i.e.*, all n-tuples $(j) = (j_1, j_2, \ldots, j_n)$ such that

$$1j_1+2j_2+\ldots+nj_n = n. \tag{8}$$

4. *The number of* 2-*complexes with given* 1-*skeleton.*

One of the unsolved problems stated in [4] is to enumerate the non-isomorphic abstract simplicial complexes with a given number of simplexes of each dimension. An application of the theorem yields partial information by supplying an enumeration for non-isomorphic 2-complexes with prescribed 1-skeleton G. Let $c_2(G, r)$ be the number of isomorphism classes (defined in the usual way) of 2-complexes whose 1-skeleton is G, containing r ($\geqslant 0$) 2-simplexes. Let $c_2(G, x) = \sum_r c_2(G, r)x^r$. Then by the theorem, we immediately have

$$c_2(G, x) = Z\bigl(\Gamma(G, K_3), 1+x\bigr). \tag{9}$$

For example, if G is the graph with six points consisting of an equilateral triangle and the triangle determined by the midpoints of its sides, then $\Gamma(G, K_3) = S_3 \cdot S_1$ (direct product) so that

$$Z\bigl(\Gamma(G, K_3)\bigr) = \tfrac{1}{6}(f_1^4+3f_1^2f_2+2f_1f_3).$$

Thus

$$\begin{aligned} c_2(G, x) &= \tfrac{1}{6}[(1+x)^4+3(1+x)^2(1+x^2)+2(1+x)(1+x^3)] \\ &= 1+2x+2x^2+2x^3+x^4. \end{aligned}$$

Diagrams which verify this last polynomial are readily drawn.

5. *Problem.*

It was first shown, constructively, by Frucht [1] that for any given abstract group, there exists a graph whose group is isomorphic to the given group. This result was subsequently extended by Frucht [2] to cubical graphs with a given abstract group and more recently by Sabidussi [10] to graphs with given group and given graph-theoretic properties. However, none of these results is directed toward the question of which *permutation groups* belong to a graph. It is known that not all permutation groups have this property. For example any cyclic group of degree n and order n for $n > 2$ does not belong to any graph. The approach of the present paper raises similar questions. For the sequence of permutation groups $\Gamma_0(G)$, $\Gamma_1(G)$, $\Gamma_2(G)$, ... is an invariant of the graph G. To what extent is this sequence a complete set of invariants for G? Given a sequence of abstract groups A_0, A_1, A_2, ..., under what conditions does there exist a graph G such that for all n, $\Gamma_n(G)$ is abstractly isomorphic to A_n? In particular, given an abstract group A_0, which groups A_1 have the property that the pair (A_0, A_1) are isomorphic to the point-group and line-group of the same graph?

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