WEAKLY INVERTIBLE ELEMENTS IN THE SPACE OF SQUARE-SUMMABLE HOLOMORPHIC FUNCTIONS

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1. *Introduction*

1.1 Beurling [3] considered the following problem: *given a function f, holomorphic in the unit disc and belonging to the Hardy space H^2, are its polynomial multiples dense in H^2?* An immediate *necessary* condition is (denoting by $\mathbb{D}$ the open unit disc)

$$f(z) \neq 0, \qquad z \in \mathbb{D}, \tag{1.1}$$

but this does not suffice; a necessary and sufficient condition is, as Beurling showed

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|\,d\theta = \log|f(0)|. \tag{1.2}$$

Beurling called functions satisfying (1.2) *outer.*

If X denotes some topological vector space of holomorphic functions in $\mathbb{D}$, such that

(a) *X contains the constant functions,*

(b) *X is stable under multiplication by z (and hence by all polynomials),*

we may again consider, for a given f, whether its polynomial multiples are dense in X. If (as will be the case for the concrete spaces we consider) X also satisfies

(c) *the set of all polynomials is dense in X,*

then f has the aforementioned property if, and only if, (using an obvious notation for the function identically equal to 1),

$$\textit{the closure of the set of polynomial multiples of } f \textit{ contains } 1. \tag{1.3}$$

Following a terminology employed in [16], we say that an f satisfying (1.3) is *weakly invertible* (abbreviated henceforth w.i.). Weak invertibility can also be reformulated in several slightly different (equivalent) ways, which are natural in the context of certain other investigations: a reformulation in terms of *weighted polynomial approximation* may be found in [16]; also, to say that f is w.i. is equivalent to saying that it is a *cyclic vector* for the linear operator $f \to zf$ (or, equivalently, for the " forward shift " applied to its sequence of Taylor coefficients), i.e. the " orbit " of f under this operator spans a dense linear submanifold of X. In this connection see Rabindranathan [14].

For spaces X subjected to the natural further condition,

(d) *for each $\alpha \in \mathbb{D}$, evaluation at the point α is a continuous linear functional,*

the condition (1.1) is clearly necessary for f to be w.i.; in general, however, it is not sufficient, and the problem of characterizing the w.i. elements of a concretely given X, in terms of more accessible properties of the given function f, may be extremely difficult. It is noteworthy that the Beurling criterion (1.2) persists when X is any H^p space $(0 < p < \infty)$ (*cf.* Duren [5; p. 114]). The present paper continues the study of

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the case when X is B, the square-summable holomorphic functions on $\mathbb{D}$, which was begun in [16]; the results to date, although far from giving a complete picture, make it seem most unlikely that any simple metric condition like (1.2) could possibly characterize the w.i. elements of B; the main achievement of the present paper is a new *sufficient* condition for weak invertibility in B (*cf.* Theorem 2).

Although we explicitly study only the space B, we wish to emphasize that Theorem 2 (like the results in [15, 16] on which it depends) retains its validity when B is replaced by any of a large class of Banach spaces determined by norms of the type $L^p(\rho)$, where ρ is a positive measure on $\mathbb{D}$. These generalizations are completely straightforward; their inclusion here would merely complicate the notations, therefore we leave them to the reader. Another reason for concentrating on the space B is, as we shall see below, that the question of weak invertibility is in that case equivalent to a very natural (but apparently new) problem concerning *mean square approximation by bounded analytic functions on a Riemann surface*; thus, our main result is a contribution to the study of this problem, too. Finally, from the point of view of technique, the space B presents some of the challenge of the most general case, because functions in B may be of unbounded (Nevanlinna) characteristic; there exists no complete theory of their zeros, let alone a factorization theory such as we have for H^p spaces, and in general they do not possess radial boundary values. Hence the powerful and elegant methods that have been developed for the study of weakly invertible elements, invariant subspaces, etc. in H^p spaces, heavily based as they are on a complete boundary value and factorization theory, are unavailable in the present instance.

1.2 Before turning to our main result, it will be convenient to make a few more preliminary observations about our problem, establish our notations, etc.

1.2.1 B will denote the Hilbert space of holomorphic functions in $\mathbb{D}$ for which the norm

$$\|f\| = \|f\|_B = \left(\frac{1}{\pi}\int_{\mathbb{D}} |f(z)|^2\, d\sigma\right)^{1/2} \tag{1.4}$$

is finite, σ denoting areal measure on $\mathbb{D}$.

Introducing the Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

we may also write

$$\|f\|_B = \left(\sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}\right)^{1/2}. \tag{1.5}$$

We shall, without further reference, employ standard notations and terminology of H^p spaces, as in [5]. Observe, for any $f \in B$, $g \in H^\infty$, that $fg \in B$ and

$$\|fg\|_B \leqslant \|f\|_B\, \|g\|_\infty.$$

Hence, one deduces readily: *an element $f \in B$ is w.i. if and only if fH^∞ is dense in B or (equivalently) if and only if there exists a sequence $\{g_n\} \subset H^\infty$ such that*

$$\lim_{n\to\infty} \|g_n f - 1\|_B = 0. \tag{1.6}$$

1.2.2 This may be cast in yet another form, that is useful for some purposes: *An element $f \in B$ is w.i. if and only if there exists a sequence $\{g_n\} \subset H^\infty$ and a positive number M such that*

(i) $$\|g_n f\|_B \leqslant M$$

(ii) $$\lim_{n\to\infty} g_n(z) = 1/f(z), \quad z \in \mathbb{D}.$$

Indeed, the necessity of these conditions follows at once from (1.6), and the continuity of point evaluations (property (d) above) for B. If, conversely, $\{g_n\}$ and M exist satisfying (i) and (ii), there is a sequence of integers $\{n_i\}$ such that $\{g_{n_i} f\}$ converges in the weak topology of B to some element $h \in B$. Then h is in the *strong* closure of fH^∞, and moreover (ii) implies, using once again the continuity of point evaluations, that $h = 1$.

1.2.3 A somewhat less obvious reformulation of weak invertibility will now be given, which also adds greatly to the function-theoretic interest of the problem of w.i. elements in B. It is analogous to the relation between w.i. elements of H^J, and polynomial approximation to analytic functions on rectifiable Jordan curves in the metric $L^p(ds)$ (ds = arc length), due essentially to Smirnov (for an up-to-date account of these matters see **[6]**). As already remarked, in studying w.i. elements of B, we may as well restrict our attention *a priori* to functions f satisfying (1.1). Then the integrated function

$$F(z) = \int_0^z f(\zeta)\, d\zeta \tag{1.7}$$

is holomorphic in $\mathbb{D}$ with non-vanishing derivative; consequently, it is locally univalent, and maps $\mathbb{D}$ conformally onto some simply connected Riemann surface S. Because $f \in B$, F has a finite Dirichlet integral, so that S has finite area. Conversely, given any such Riemann surface S, the derivative f of any conformal map from $\mathbb{D}$ onto S is an element of B without zeros. Of special interest is the case

$$f \text{ is the derivative of a univalent function in } \mathbb{D}, \tag{1.8}$$

which is equivalent to S being a (one-sheeted) region in the complex plane.

THEOREM 1. *Let f be an element of B satisfying* (1.1), *and S the above-described associated Riemann surface. Then, f is weakly invertible if and only if the bounded analytic functions on S are dense in the square summable analytic functions on S.*

Proof. Let $L_a^p(S)$ denote the closed subspace of $L^p(S) = L^p(S; \sigma)$ (σ = areal measure on S) consisting of those functions which are holomorphic on S. Our problem is thus to show that f is w.i. if and only if $L^\infty{}_a(S)$ is dense in $L^p{}_a(S)$. In what follows we shall write $H^\infty(S)$ to denote $L_a{}^\infty(S)$.

Now consider the function F defined by (1.7), mapping $\mathbb{D}$ conformally onto S. Let G denote the function inverse to F. Observe that $G' \in L_a{}^2(S)$. Also, because of the identity

$$G'(F(z))f(z) = \frac{d}{dz} G(F(z)) = 1,$$

we have, for every $h \in H^\infty(\mathbb{D})$,

$$\pi \|hf - 1\|^2 = \int_{\mathbb{D}} |h(z)f(z) - G'(F(z))f(z)|^2 \, d\sigma.$$

Now make the change of variables $w = F(z)$. The Jacobian is $|f(z)|^2$, so the last integral equals

$$\int_S |h(G(w)) - G'(w)|^2 \, d\sigma_m.$$

Now, as h runs over all bounded analytic functions on $\mathbb{D}$, $h \circ G$ runs over all bounded analytic functions on S. We have therefore shown: *f is w.i. if and only if G' lies in the closure (in $L^2{}_a(S)$) of $H^\infty(S)$.* In particular, if this closure is *all* of $L^2{}_a(S)$, it certainly contains G', and consequently f is w.i. Thus, the " if " part of the theorem is established.

To prove the " only if " part, it remains to show that if the closure of $H^\infty(S)$ in $L^2{}_a(S)$ contains G', this closure comprises *all* of $L^2{}_a(S)$. Since this closure is stable under multiplication by elements of $H^\infty(S)$, we have therefore only to show: *the multiples of G' by elements of $H^\infty(S)$ are dense in $L^2{}_a(S)$.*

This can be seen in various ways; the most elegant is to observe that for each $\lambda \in S$ the function

$$w \to \frac{\overline{G'(\lambda)}\, G'(w)}{\left(1 - \overline{G(\lambda)}\, G(w)\right)^2} \tag{1.9}$$

is a bounded analytic function multiplied by G'. But, (1.9) is just the reproducing kernel of $L_a{}^2(S)$ for evaluation at the point λ (*cf.* Bergman [**2**]; the fact that our surface S is not schlicht imposes no changes), and these are dense in $L_a{}^2(S)$. Alternatively, the most general element of $L^2{}_a(S)$ can be written in the form $(\phi \circ G)\, G'$ where $\phi \in B$ (this is immediate, by changing variables as above), and so we need only show, given $\phi \in B$ and $\varepsilon > 0$, that there exists $h \in H^\infty(\mathbb{D})$ such that

$$\int_S |h(G(w))\, G'(w) - \phi(G(w))\, G'(w)|^2 \, d\sigma_m < \varepsilon,$$

i.e., changing variables, such that

$$\int_{\mathbb{D}} |h(z) - \phi(z)|^2 \, d\sigma_z < \varepsilon.$$

But this merely expresses that $H^\infty(\mathbb{D})$ is dense in B, and the proof is finished.

Remark. The problem of approximating elements of $L^2{}_a(S)$ by elements of $L_a{}^\infty(S)$, even when S is a schlicht region, does not seem to have been studied in the literature. Carleman proved in a 1923 paper [**4**] that if S is a *Jordan domain* the *polynomials* are already dense in $L^2{}_a(S)$; this polynomial result has been extended to more general domains (*cf.* Mergelyan [**10**], also Havin [**8**]) and the conditions for its validity are now fairly well understood. The corresponding questions for *rational* approximation were completely settled by Havin [**7**]. An interesting feature of approximation in the space $L^2{}_a(S)$ by bounded holomorphic functions, as opposed to (say) rational functions, is that the presence of *slits* is an obvious obstruction to the latter, but not to the former. Recently, in response to a query by the authors, L. I.

Hedberg was able to show that if S is any (schlicht) region of finite connectivity, $H^\infty(S)$ is indeed dense in $L^2{}_a(S)$, and in fact in all spaces $L^p{}_a(S)$, $1 \leqslant p < \infty$. The proof is outlined in a letter to one of the authors received in Dec. 1971; it is based upon techniques employed by Hedberg in [9]. In view of Theorem 1, Hedberg's result implies: *if $f \in B$ satisfies* (1.8), *it is w.i.* As a test of the efficacy of our methods, we have attempted to prove this latter proposition directly. This we have thus far been unable to do; on the other hand, our methods establish that $f \in B$ is w.i. if it is the derivative of a *finitely* valent (but not necessarily univalent) function subjected to some very mild further restrictions. This yields results on mean approximation by bounded holomorphic functions on *non-schlicht* regions, which do not seem accessible to the potential-theoretic methods employed by Hedberg.

2. *Main results*

We shall lean heavily on the following result, proved in [15]:

THEOREM A. *Let $f \in B$ satisfy* (1.1), *and for* $0 \leqslant r < 1$ *write*

$$m(r) = \min_{\theta} |f(re^{i\theta})|.$$

If, for some $\varepsilon > 0$,

$$\int_0^{2\pi} \int_0^1 |f(re^{i\theta})|^2 \, m(r)^{-\varepsilon} r \, dr \, d\theta < \infty, \tag{2.1}$$

then f is w.i. in B.

We note a special case of this theorem: *suppose, for some positive constants δ, c, we have*

$$|f(z)| \geqslant \delta(1-|z|)^c, \qquad z \in \mathbb{D}. \tag{2.2}$$

Then f is w.i. if, in addition to (2.2),

$$\int_0^{2\pi} \int_0^1 |f(re^{i\theta})|^2 (1-r)^{-\varepsilon} r \, dr \, d\theta \tag{2.3}$$

holds for some $\varepsilon > 0$. This follows at once from Theorem A.

Our main concern in this paper is the weak invertibility of functions satisfying (2.2). Observe that any f satisfying (1.8) must satisfy (2.2) (with $c = 1$) by virtue of Koebe's distortion theorems. It is a very interesting question whether (2.2) alone implies that f is w.i.; we know of no counter-example. Since (2.3) certainly holds (with $\varepsilon = \frac{1}{2}$, say) when f is *bounded*, a weak corollary of Theorem A is

COROLLARY A. *If f satisfies* (2.2), *and moreover $f(z)$ omits all values in some neighbourhood of infinity, then f is w.i. in B.*

Thus (2.2) *plus* a suitable restriction on the range of f implies that f is w.i. Our main result is on this same pattern, except that the set of values which f is required to omit (which in Corollary A has non-empty interior) is now cut down drastically.

THEOREM 2. *Let $f \in B$ satisfy* (2.2), *and suppose, moreover, $f(z)$ omits a set of values having positive logarithmic capacity. Then f is w.i. in B.*

Actually, we can prove a little more, namely

THEOREM 2′. *Let $f \in B$ satisfy* (2.2), *and suppose, moreover, for some positive integer n, $f = g^n$, where g is holomorphic in $\mathbb{D}$ and omits a set of values having positive logarithmic capacity. Then f is w.i. in B.*

Surprisingly, Theorem 2 is deducible from the *apparently* much weaker Corollary A; Theorem 2′ is a consequence of Theorem A. These deductions are based on the following lemma.

LEMMA. *Let E be a compact subset of the complex plane having connected complement and positive logarithmic capacity, and such that*

$$\operatorname{diam} E \leqslant 1/2. \tag{2.4}$$

Let V denote the (open) set of points having distance $< 1/2$ from E. Then, there exists a function ϕ, holomorphic in $\mathbb{C}\setminus E$ (not necessarily single-valued), indefinitely continuable holomorphically in $\mathbb{C}\setminus E$, and non-vanishing, such that

(i) $|\phi(w)|$ *is single-valued*, $w \in \mathbb{C}\setminus E$,

(ii) $\lim\limits_{w\to\infty} |w\phi(w)| = 1$,

(iii) $|\phi(w)|$ *is bounded on* $\mathbb{C}\setminus E$,

(iv) $|\phi(w)| \geqslant 1, \quad w \in V\setminus E$.

Proof. There exists a positive measure μ (the "equilibrium distribution", *cf.* [**19**; p. 55]) supported on E and having total mass 1, such that the function

$$u(w) = -\int \log |w-s|\, d\mu(s) \tag{2.5}$$

("conductor potential") satisfies, for some finite positive number M,

$$u(w) \leqslant M, \quad w \in \mathbb{C}\setminus E \tag{2.6}$$

(see [**19**; p. 60]). Now, u is harmonic and single-valued off E; let v denote any (in general, multiple-valued) harmonic conjugate of u, and

$$\phi = \exp(u+iv).$$

Then ϕ is holomorphic off E, and $|\phi(w)| = e^{u(w)}$ is single-valued. Moreover,

$$\log |w\phi(w)| = \int \log \frac{|w|}{|w-s|}\, d\mu(s),$$

which tends to 0 as $w \to \infty$, hence (ii) holds. Also, (2.6) implies (iii). Finally, because of (2.4), $\log |w-s| \leqslant 0$ for $s \in E$, $w \in V$; hence $u(w) \geqslant 0$, and consequently $|\phi(w)| \geqslant 1$, when $w \in V\setminus E$. Thus, (iv) holds, and the lemma is proved.

Proof of Theorem 2′. Let E be a compact set having connected complement and positive logarithmic capacity, satisfying (2.4), and disjoint from the range of g. The hypotheses clearly imply the existence of such a set. Let ϕ be the associated function constructed in the lemma. Then, by the monodromy theorem, $h = \phi \circ g$ is *single-valued*, holomorphic and bounded on $\mathbb{D}$. By virtue of the discussion in 1.2.1, it suffices to prove that $f_1 = hf$ is w.i. Now, gh is bounded in $\mathbb{D}$, since $w\phi(w)$ is bounded on $\mathbb{C}\setminus E$; therefore

$$|f_1(z)| \leqslant A|g(z)|^{n-1} = A|f(z)|^{(n-1)/n} \tag{2.7}$$

for some constant A. Also, since f satisfies (2.2), we have

$$|g(z)| \geqslant \delta_1(1-|z|)^{c_1} \tag{2.8}$$

where $\delta_1 = \delta^{1/n}$, $c_1 = c/n$. We claim now that

$$|g(z)\,h(z)| \geqslant \delta_2(1-|z|)^{c_1} \tag{2.9}$$

for some $\delta_2 > 0$. Indeed, because of property (ii) and the fact that ϕ never vanishes, $|w\phi(w)|$ has a *positive* minimum on the compact complement (relative to the *extended* plane) of V, and so $|g(z)\,h(z)|$ has a positive lower bound on the set $\{z : g(z) \notin V\}$. Therefore, it is enough to prove (2.9) for values of z such that $g(z) \in V$. But, because of (iv), $|h(z)| \geqslant 1$ for such z, and (2.9) follows from (2.8). Thus, (2.9) is established, and so $f_1 = hg^n$ also satisfies an estimate of the form

$$|f_1(z)| \geqslant \delta_3(1-|z|)^{c_3}. \tag{2.10}$$

To complete the proof we have only to observe that (2.7) implies (2.3), with f_1 in place of f and suitably small $\varepsilon > 0$. For, by (2.7), $f_1 \in L^p(\sigma)$, $p = 2n/(n-1) > 2$; therefore the product of $|f_1|^2$ and $(1-r)^{-\varepsilon}$ is integrable over $\mathbb{D}$, for suitably small $\varepsilon > 0$, by Hölder's inequality. Since also (2.10) shows that f_1 satisfies a condition of the type (2.2), Theorem A shows that f_1 is w.i., and Theorem 2′ is proved.

As an application of Theorem 2′, we can prove

THEOREM 3. *Let K be any compact connected subset of $\mathbb{C}\setminus\{0\}$, but not a single point, which does not separate 0 from ∞. Suppose f satisfies* (2.2), *and moreover $f^{-1}(K)$, as a subset of $\mathbb{D}$ in its relative topology, is a finite union of connected sets. Then f is w.i. in B.*

Proof. By hypothesis, $f^{-1}(K)$ is the union of disjoint connected sets $E_1, \ldots, E_m$. Let $n = m+1$. Slit the plane from 0 to ∞ in the complement of K. In the slit plane there are n distinct, single-valued analytic branches of the function $z = w^{1/n}$. Let $J_1, \ldots, J_n$ be the images of K under these n branches. These n sets are disjoint, compact, connected, and each contains more than one point. Hence each has positive capacity.

Let g be any holomorphic (in D) branch of $f^{1/n}$. Then z lies in $E_1 \cup \ldots \cup E_m$ if and only if $f(z) \in K$, if and only if $g(z)$ lies in $J_1 \cup \ldots \cup J_n$. Since the image of $E_1 \cup \ldots \cup E_m$ under g is the union of at most m connected sets, at least one of the sets J_i is disjoint from the range of g. The result now follows from Theorem 2′.

By virtue of Theorem 1, Theorems 2 and 3 can be reformulated as theorems on mean approximation by bounded holomorphic functions on certain Riemann surfaces. We do not trouble to formulate these explicitly; the theorems so obtained suffer from the blemish that a condition is imposed on the value distribution of the derivative of the mapping function from the unit disc onto the Riemann surface. Unfortunately almost nothing seems to be known concerning conditions of a metric character pertaining to a Riemann surface which would force restrictions on the range of values of the derivative of this mapping function (e.g. omission of a continuum, or of a set of positive capacity); this problem seems to be of interest. On the other hand, condition (2.2) can be shown to hold whenever the Riemann surface has at most finitely many sheets, i.e. the mapping function is n-valent for some finite n. In **[16]** it was shown that (2.2), *together with the hypothesis* $f \in H^1$, *imply that* f *is w.i.* Since $f \in H^1$ is equivalent to rectifiability of the boundary of the image domain under the map (1.7), we can therefore conclude: *if* S *is any simply connected (not necessarily schlicht) region, at most finitely sheeted and bounded by rectifiable arcs of finite total length, then* $H^\infty(S)$ *is dense in* $L^2{}_a(S)$.

In view of the distortion theorem for finitely valent functions, the study of weak invertibility of functions subjected *a priori* to (2.2) seems not unreasonable, but it is of course clear from Theorem A that (2.2) is not *necessary* for f to be w.i. However, a function that is of too rapid decrease along a radius vector, in particular one satisfying

$$|f(x)| \leqslant A \exp(-1/(1-x)), \qquad 0 \leqslant x < 1, \tag{2.11}$$

for some constant A, cannot be w.i. in B (or in any "reasonable" space, for that matter).

THEOREM 4. *If* $f \in B$ *satisfies* (2.11) *it is not w.i.*

Proof. Let Δ denote the open disc enclosed by the circle centred at $z = 1/2$, of radius $1/2$. Because every $f \in B$ satisfies the estimate

$$|f(z)| \leqslant \|f\|_B(1-|z|^2)^{-1},$$

a simple computation shows that the restriction of f to Δ is of class H^p (relative to Δ), for any $p < 1/2$, and the H^p norm is majorized by $\|f\|_B$. Thus, if f is w.i. in B, $f|_\Delta$ is w.i. in $H^{1/3}(\Delta)$. Therefore, $f|_\Delta$ must be outer. However, it is readily seen that if (2.11) holds, $f|_\Delta$ must have a non-trivial inner factor, indeed a factor of the type $\exp(c(z+1)/(z-1))$, $c > 0$, relativized by a change of variable from $\mathbb{D}$ to Δ. Thus f cannot be w.i. in B.

3. *Concluding remarks*

The main question left open by this paper is whether or not (2.2) alone is sufficient for f to be w.i. Until this question is decided, or if it should be decided in the negative, it is of interest to seek supplementary conditions on f, as weak as possible, which together with (2.2) imply that f is w.i. Thus (2.3) is such a condition, as is the condition on omitted values imposed on Theorem 2′. In this connection, observe that if f satisfies the hypotheses of Theorem 2′, it is necessarily of bounded (Nevanlinna) characteristic, since g is, by virtue of a well-known theorem of R. Nevanlinna [**11**; p. 213]. The same is true of f which satisfy the hypotheses of Theorem 3. Thus, a generalization of these theorems, which would also include the theorem from **[16]** referred to earlier, that (2.2) and $f \in H^1$ imply that f is w.i., would be:

(*) *If* $f \in B$ *satisfies* (2.2) *and is of bounded characteristic it is w.i.*

We have been unable to prove (*). One approach we have tried, although thus far unsuccessful, leads to interesting factorization problems, of an apparently new kind, for functions of bounded characteristic; therefore we describe briefly this approach. It is based on the following

LEMMA. *Suppose* $f_i \in L^{p_i}{}_a(\mathbb{D})$, $i = 1, 2, \dots n$ *and* f_i *is w.i. in* $L^{p_i}{}_a(\mathbb{D})$. *Then* $f = f_1 f_2 \dots f_n$ *belongings to* $L^p{}_a(\mathbb{D})$, *where* $1/p = \sum_{i=1}^{n} 1/p_i$, *and is w.i. in* $L^p{}_a(\mathbb{D})$.

Proof. That $f \in L^p{}_a(\mathbb{D})$ follows from Hölder's inequality. The rest follows by a simple induction. For, the case $n = 1$ is trivial. Now consider the case $n = 2$. By hypothesis there exists a sequence $\{\phi_j\} \subset H^\infty$ such that $\phi_j f_1 \to 1$ in L^{p_1}. Then, with $1/p = (1/p_1) + (1/p_2)$,

$$\|\phi_j f_1 f_2 - f_2\|_p \leqslant \|\phi_j f_1 - 1\|_{p_1} \|f_2\|_{p_2} \to 0$$

as $j \to \infty$. Thus, the closure of $(f_1 f_2)H^\infty$ in $L^p(\sigma)$ contains f_2. But the closure of $f_2 H^\infty$ in $L^{p_2}(\sigma)$, and *a fortiori* in $L^p(\sigma)$, contains 1. Thus $f_1 f_2$ is w.i. in $L^p{}_a(\sigma)$.

For the general case, suppose $n \geqslant 3$ and the lemma has already been proved for $n-1$ (or fewer) functions. Let $g = f_1 \dots f_{n-1}$. By the inductive hypothesis, g is w.i. in $L^r{}_a(\mathbb{D})$, $1/r = \sum_{i=1}^{n-1} 1/p_i$. Now, using the case $n = 2$ of the lemma it follows that gf_n is w.i. in $L^p{}_a(\mathbb{D})$, completing the inductive proof.

Suppose now $f \in B$ satisfies (2.2), and moreover is of bounded characteristic. This does not force f to omit any values, except of course 0. Suppose, however, we could prove the following proposition:

(**) *Let* $f \in B$ *be of bounded characteristic and satisfy* (2.2). *Then there exists a positive integer* n, *and a factorization* $f = f_1 f_2 \dots f_n$ *such that*

(i) $f_i \in L^{2n}{}_a(\mathbb{D})$, $i = 1, 2, \dots n$,

(ii) $|f_i(z)| \geqslant \delta_i(1-|z|)^{c_i}$, $i = 1, \dots n$, *where the* δ_i, c_i *are positive constants,*

(iii) *each* f_i *omits a set of values of ppsitive capacity.*

Any f admitting such a factorization is w.i. in B by the lemma, since the f_i are all w.i. in $L^{2n}{}_a(\mathbb{D})$ by Theorem 2 (or, more precisely, by Theorem 2 as generalized in an obvious way from $B = L^2{}_a(\mathbb{D})$ to $L^{2n}{}_a(\mathbb{D})$). We do not know whether (**) is true. Obviously, one could also try to work with more general factorizations whereby the f_i are in different L^{p_i} spaces.

In any case, the following special case of (*) is true (*cf.* M. Rabindranathan **[14]**).

If $f \in B$ *satisfies* (2.2) *and belongs to* H^p *for some* $p > 0$ *it is w.i.*

Rabindranathan has also observed that if $f = g/h$ is in B, where g and h are in H^∞ and g is outer, then f is weakly invertible. Indeed, g is a bounded multiple of f, and the multiples of g span 1 even in the H^2 topology.

The referee has raised the following question: if $f = FS_1/S_2$ is in B, where S_1 and S_2 are singular inner functions, and if f satisfies (2.2), must S_1 satisfy (2.2)? If so it would follow that S_1, and hence f, is weakly invertible. Unfortunately, S_1 need not satisfy (2.2). Indeed, by Theorem 2 of **[17]** the question is equivalent to the following: *if a real-valued function of bounded variation has modulus of continuity* $O(t \log 1/t)$, *must its positive variation also have this property?* A counter-example can be constructed along the lines indicated in **[18]** (see Theorems 3 and 4 on pages 271–2).

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