SUCCESSIVE COEFFICIENTS OF UNIVALENT FUNCTIONS

P. L. DUREN

Let S be the class of functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots, \ |z| < 1,$$

analytic and univalent in the unit disc. Hayman [3] proved that the difference of successive coefficients is bounded:

$$||a_{n+1}| - |a_n|| \leqslant A, \quad n = 1, 2, \dots,$$

for all $f \in S$, where A is an absolute constant. Milin [7, 8] discovered a different proof which showed that $A < 9$. Ilina [5] sharpened Milin's method and found $A < 4{\cdot}26$. Quite recently, Grinspan [2] modified Milin's approach to show that $A < 3{\cdot}61$. It is known that A cannot be reduced to 1, even for the subclass of odd functions. For the subclass of starlike functions, however, Leung [6] recently reduced the bound to 1.

The object of this note is to establish another bound which improves upon Grinspan's for a certain subclass of S. The precise statement requires some preliminary discussion.

Hayman [3] showed that for each $f \in S$, the limits

$$\alpha = \lim_{r \to 1} (1-r)^2 M_\infty(r, f) = \lim_{n \to \infty} \frac{|a_n|}{n}$$

exist, where $M_\infty(r, f)$ is the maximum of $|f(z)|$ on $|z| = r$. The number α $(0 \leqslant \alpha \leqslant 1)$ is called the *Hayman index* of f. Furthermore, f has a *direction of maximal growth* $e^{i\theta_0}$ with the property

$$\lim_{r \to 1} (1-r)^2 |f(re^{i\theta_0})| = \alpha.$$

The direction of maximal growth is unique if $\alpha > 0$.

The *logarithmic coefficients* γ_n of f are defined by

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n.$$

The following two results [8, 9, 1] give information about the logarithmic coefficients.

MILIN'S LEMMA. *For each $f \in S$,*

$$\sum_{n=1}^{N} n|\gamma_n|^2 \leqslant \sum_{n=1}^{N} \frac{1}{n} + \delta, \quad N = 1, 2, \dots,$$

where $\delta < 0{\cdot}312$.

BAZILEVICH'S THEOREM. *For each $f \in S$ with Hayman index $\alpha > 0$,*

$$\sum_{n=1}^{\infty} n \left| \gamma_n - \frac{1}{n} e^{-in\theta_0} \right|^2 \leqslant \frac{1}{2} \log \frac{1}{\alpha},$$

where $e^{i\theta_0}$ is the direction of maximal growth.

Received 11 October, 1978.

[J. LONDON MATH. SOC. (2), 19 (1979), 448–450]

The smallest possible number δ is known as Milin's constant. It is a positive number whose exact value is unknown.

Our result can now be stated.

THEOREM. *Let $f \in S$ have Hayman index $\alpha > 0$. Then*

$$\left||a_{n+1}|-|a_n|\right| \leqslant e^{\delta}\alpha^{-\frac{1}{2}} < 1\cdot 37\alpha^{-\frac{1}{2}}, \quad n = 1, 2, \ldots,$$

where δ is Milin's constant.

The proof uses an inequality due to Lebedev and Milin [8, 9, 1]. Let $\phi(z) = \sum_{n=1}^{\infty} \alpha_n z^n$ be an arbitrary power series with positive radius of convergence, and let

$$\exp\{\phi(z)\} = \sum_{n=0}^{\infty} \beta_n z^n.$$

Then

$$|\beta_n|^2 \leqslant \exp\left\{\sum_{k=1}^{n}\left(k|\alpha_k|^2 - \frac{1}{k}\right)\right\}, \quad n = 1, 2, \ldots.$$

Proof of theorem. Let ζ be a complex parameter with $|\zeta| = 1$. Then

$$\psi(z) = (1-\zeta z)\frac{f(z)}{z} = 1 + \sum_{n=1}^{\infty} (a_{n+1}-\zeta a_n)z^n$$

and

$$\log\psi(z) = \sum_{n=1}^{\infty}\left(2\gamma_n - \frac{1}{n}\zeta^n\right)z^n.$$

According to the Lebedev–Milin inequality,

$$|a_{n+1}-\zeta a_n|^2 \leqslant \exp\left\{\sum_{k=1}^{n}\left(k\left|2\gamma_k - \frac{1}{k}\zeta^k\right|^2 - \frac{1}{k}\right)\right\}$$

$$= \exp\left\{2\sum_{k=1}^{n}\left(k|\gamma_k|^2 - \frac{1}{k}\right) + 2\sum_{k=1}^{n} k\left|\gamma_k - \frac{1}{k}\zeta^k\right|^2\right\}.$$

In view of Milin's lemma and Bazilevich's theorem, it follows that

$$\left||a_{n+1}|-|a_n|\right| \leqslant |a_{n+1} - e^{-i\theta_0} a_n| \leqslant e^{\delta}\alpha^{-\frac{1}{2}}, \quad n = 1, 2, \ldots.$$

This completes the proof.

For functions $f \in S$ with large Hayman index α (more precisely, with $\alpha \geqslant 0\cdot 15$), our bound improves upon that of Grinspan.

References

1. P. L. Duren, *Univalent Functions*, Springer-Verlag, New York and Heidelberg, to appear.
2. A. Z. Grinspan, " Improved bounds for the difference of adjacent coefficients of univalent functions ", in *Some Questions in the Modern Theory of Functions*, Sib. Inst. Mat., Novosibirsk, 1976, pp. 41–45 (Russian).
3. W. K. Hayman, " The asymptotic behaviour of *p*-valent functions ", *Proc. London Math. Soc.* 5 (1955), 257–284.

4. W. K. Hayman, " On successive coefficients of univalent functions ", *J. London Math. Soc.* 38 (1963), 228–243.
5. L. P. Ilina, " On the mutual growth of neighbouring coefficients of univalent functions ", *Mat. Zametki*, 4 (1968), 715–722 = Math. Notes 4 (1968), 918–922.
6. Yuk Leung, " Successive coefficients of starlike functions ", *Bull. London Math. Soc.* 10 (1978), 193–196.
7. I. M. Milin, " Adjacent coefficients of univalent functions ", *Dokl. Akad. Nauk SSSR* 180 (1968), 1294–1297 = *Soviet Math. Dokl.* 9 (1968), 762–765.
8. I. M. Milin, *Univalent Functions and Orthonormal Systems*, Izdat. " Nauka ", Moscow, 1971; English transl., Transl. Math. Monographs, vol. 49, Amer. Math. Soc., Providence, R.I., 1977.
9. Ch. Pommerenke, *Univalent Functions*, Vandenhoeck und Ruprecht, Göttingen, 1975.

Department of Mathematics,
University of Michigan,
Ann Arbor,
Michigan 48109,
U.S.A.