ANGLES AND QUASICONFORMAL MAPPINGS†

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To J. E. LITTLEWOOD *on his 80th birthday*

1. Introduction

The class of plane quasiconformal mappings, introduced by Ahlfors (**2**) and Pfluger (**20**), has been studied rather extensively in the last ten years. In particular, there exist a surprisingly large number of equivalent definitions for this class of mappings. The definition of Ahlfors and Pfluger involves the moduli of quadrilaterals. However, these mappings can also be characterized by means of the moduli of rings ((**11**) (**22**)), by means of extremal lengths (**28**), by means of harmonic or hyperbolic measure (**13**), or by studying how they distort infinitesimal circles ((**6**) (**19**)). There are, in addition, several analytic definitions ((**3**) (**4**) (**10**) (**16**) (**21**)), as well as more qualitative definitions concerning compactness or distortion properties ((**7**) (**8**)). See also (**9**), (**23**), and (**24**).

All of the above definitions involve selecting a certain property of conformal mappings and then studying the class of all homeomorphisms which enjoy a slightly weakened form of this property. However, until very recently, no definition for this class has been given which generalizes the fact that a conformal mapping is an angle-preserving diffeomorphism. Perhaps one reason for this is that a plane quasiconformal mapping may have an exceptional set of zero measure at which it is not differentiable. Hence an angle with vertex at an exceptional point may be carried onto a pair of arcs which do not have tangents at their common end-point. In order to circumvent this difficulty, one must assign a kind of measure to each topological angle consisting of two arcs with just one end-point in common.

One can introduce such an angular measure in several ways. For example, one might use auxiliary conformal mappings to straighten out one of the sides of the topological angle; the measure could then be defined by means of a lower limit as in (2.4).‡ However, this method is a little complicated, and in § 2 we use the triangle inequality to give

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‡ A characterization for quasiconformal mappings using such an angular measure appears in (**26**).

a direct geometric definition for the measure of a topological angle. In §§ 3, 4, and 5, we consider how this angular measure is changed under various kinds of mappings. In particular, we establish in § 4 a new distortion theorem for K-quasiconformal mappings of the extended plane which is of independent interest. Then in § 6 we show how quasiconformal mappings can be characterized in terms of what they do to the measure of topological angles, and in § 7 we obtain a new theorem on conformal mappings which is similar to earlier results of Menchoff (**15**).

2. Inner measure of a topological angle

We say that two arcs γ_1 and γ_2 form a *topological angle* at a point z_0 if both γ_1 and γ_2 have z_0 as an end-point and if z_0 is the only point γ_1 and γ_2 have in common. We then define the *inner measure* $A(\gamma_1, \gamma_2)$ of this topological angle as follows:

$$A(\gamma_1, \gamma_2) = \liminf_{z_1, z_2 \to z_0} 2 \arcsin\left(\frac{|z_1 - z_2|}{|z_1 - z_0| + |z_2 - z_0|}\right), \quad z_i \in \gamma_i. \tag{2.1}$$

We see that $0 \leqslant A(\gamma_1, \gamma_2) \leqslant \pi$, that $A(\gamma_1, \gamma_2)$ does not depend upon the behaviour of γ_1 and γ_2 outside of a neighbourhood of z_0, and that

$$A(f(\gamma_1), f(\gamma_2)) = A(\gamma_1, \gamma_2) \tag{2.2}$$

when f is a similarity mapping or a reflexion in a line.

To see how this inner measure is related to the usual unsigned measure of an angle, given two distinct points z_1, $z_2 \neq z_0$, let $\theta = \theta(z_1, z_0, z_2)$ denote the radian measure of the angle at z_0 in the triangle whose vertices are z_1, z_0, z_2. Then by the law of cosines,

$$\left(\frac{|z_1 - z_2|}{|z_1 - z_0| + |z_2 - z_0|}\right)^2 = 1 - \frac{4|z_1 - z_0|\,|z_2 - z_0|}{(|z_1 - z_0| + |z_2 - z_0|)^2} (\cos \tfrac{1}{2}\theta)^2,$$

and we obtain

$$\liminf_{z_1, z_2 \to z_0} \sin \tfrac{1}{2}\theta \leqslant \sin \tfrac{1}{2} A(\gamma_1, \gamma_2) \leqslant \liminf_{\substack{z_1, z_2 \to z_0 \\ |z_1 - z_0| = |z_2 - z_0|}} \sin \tfrac{1}{2}\theta, \quad z_i \in \gamma_i. \tag{2.3}$$

We see from (2.3) that

$$A(\gamma_1, \gamma_2) = \liminf_{z_1, z_2 \to z_0} \theta(z_1, z_0, z_2)$$

provided that γ_1 or γ_2 has a tangent at z_0. In particular, if both γ_1 and γ_2 have tangents λ_1 and λ_2 at z_0, then $A(\gamma_1, \gamma_2)$ gives the radian measure of the smaller of the two angles determined by λ_1 and λ_2 at z_0.

As we mentioned earlier, one can use conformal mapping to define another kind of measure of the topological angle formed by γ_1 and γ_2. Let γ be the segment joining 0 to 1, and for $i = 1, 2$ let $w = f_i(z)$ map the complement of γ_i conformally onto the complement of γ so that z_0

corresponds to the origin. Then if we take $A^*(\gamma_1, \gamma_2)$ as the minimum of

$$(2.4)\qquad \begin{aligned} &\liminf_{w_1, w_2 \to 0} \theta(w_1, 0, w_2), \qquad w_1 \in \gamma, \quad w_2 \in f_1(\gamma_2), \\ &\liminf_{w_1, w_2 \to 0} \theta(w_1, 0, w_2), \qquad w_1 \in f_2(\gamma_1), \quad w_2 \in \gamma, \end{aligned}$$

we obtain a second kind of inner measure. It is not difficult to show that

$$A^*(\gamma_1, \gamma_2) = A(\gamma_1, \gamma_2)$$

whenever γ_1 and γ_2 have tangents at z_0.

However, there do not exist any non-trivial relations between $A(\gamma_1, \gamma_2)$ and $A^*(\gamma_1, \gamma_2)$ when γ_1 and γ_2 are arbitrary arcs. For example, if $0 < a < \infty$ and if γ_1 and γ_2 are the closures of the logarithmic spirals

$$z = e^{-(a+i)t}, \quad z = -e^{-(a+i)t}, \qquad 0 \leqslant t < \infty,$$

then it is easy to show that

$$A(\gamma_1, \gamma_2) \leqslant 2 \arcsin\left(\frac{e^{a\pi} - 1}{e^{a\pi} + 1}\right), \quad A^*(\gamma_1, \gamma_2) = \pi.$$

Since the bound for $A(\gamma_1, \gamma_2)$ tends to 0 as $a \to 0$, there can be no inequality of the form

$$(2.5)\qquad A(\gamma_1, \gamma_2) \geqslant \psi(A^*(\gamma_1, \gamma_2)),$$

where $\psi(t) > 0$ for $0 < t \leqslant \pi$, relating these two measures. Next a complicated but elementary construction in the logarithm plane yields a pair of arcs γ_1 and γ_2 which form a topological angle with

$$A(\gamma_1, \gamma_2) > 0, \quad A^*(\gamma_1, \gamma_2) = 0.$$

(See (**1**).) Hence there can be no inequality of the form

$$(2.6)\qquad A^*(\gamma_1, \gamma_2) \geqslant \psi(A(\gamma_1, \gamma_2)),$$

where $\psi(t) > 0$ for $0 < t \leqslant \pi$.

On the other hand, if we combine Theorem 3 of § 5 with known results on the behaviour of harmonic measure under quasiconformal mappings (**12**), we can show that for each K, $1 \leqslant K < \infty$, there exists a continuous increasing function $\psi_K(t) > 0$ for $0 < t \leqslant \pi$ with the following property: if γ_1 and γ_2 form a topological angle and if, for $i = 1, 2$, there exists a K-quasiconformal mapping f_i of a neighbourhood U_i of γ_i which carries γ_i onto a segment, then both (2.5) and (2.6) hold with $\psi = \psi_K$.

3. Inner measure under differentiable homeomorphisms

We study here how the inner measure of a topological angle is changed under a homeomorphism which is differentiable at the vertex of the angle. We require two preliminary results.

LEMMA 1. *Suppose that f is a homeomorphism of a neighbourhood U of the origin, that*

$$f(z) = z + o(|z|)$$

near the origin, and that γ_1 and γ_2 are two arcs in U which form a topological angle at the origin. Then $f(\gamma_1)$ and $f(\gamma_2)$ form a topological angle and

$$A(f(\gamma_1), f(\gamma_2)) = A(\gamma_1, \gamma_2).$$

Proof. Given that $0 < \varepsilon < 1$, we may choose $\delta > 0$ such that $|f(z) - z| \leqslant \varepsilon |z|$ for $|z| < \delta$. Choose z_i in γ_i so that $0 < |z_i| < \delta$, $i = 1, 2$. Then

$$\frac{|f(z_1) - f(z_2)|}{|f(z_1)| + |f(z_2)|} \leqslant \frac{|z_1 - z_2|}{|z_1| + |z_2|} + \frac{4\varepsilon}{1-\varepsilon},$$

and letting $z_1, z_2 \to 0$ yields

$$\sin \tfrac{1}{2} A(f(\gamma_1), f(\gamma_2)) \leqslant \sin \tfrac{1}{2} A(\gamma_1, \gamma_2) + \frac{4\varepsilon}{1-\varepsilon}.$$

Since ε is arbitrary, we obtain

$$A(f(\gamma_1), f(\gamma_2)) \leqslant A(\gamma_1, \gamma_2).$$

The reverse inequality follows by symmetry.

LEMMA 2. *Suppose that $D \geqslant 1$ and that*

$$g(z) = Dx + iy. \tag{3.1}$$

If $K \geqslant D$, then

$$A(g(\gamma_1), g(\gamma_2)) \geqslant \frac{1}{K} A(\gamma_1, \gamma_2) \tag{3.2}$$

for each pair of arcs γ_1 and γ_2 which form a topological angle at the origin. Conversely, if (3.2) holds for each pair of segments γ_1 and γ_2 which form an angle at the origin, then $K \geqslant D$.

Proof. Choose $z_1 = x_1 + iy_1$ in γ_1 and $z_2 = x_2 + iy_2$ in γ_2 so that $z_1, z_2 \neq 0$, and set

$$\varphi = \arcsin\left(\frac{|z_1 - z_2|}{|z_1| + |z_2|}\right), \quad \varphi' = \arcsin\left(\frac{|g(z_1) - g(z_2)|}{|g(z_1)| + |g(z_2)|}\right). \tag{3.3}$$

Then (3.1) and (3.3) yield

$$\begin{aligned}(\tan \varphi')^2 &= \frac{1}{2} \frac{D^2(x_1 - x_2)^2 + (y_1 - y_2)^2}{D^2 x_1 x_2 + y_1 y_2 + (D^2 x_1^2 + y_1^2)^{\frac{1}{2}} (D^2 x_2^2 + y_2^2)^{\frac{1}{2}}} \\ &\geqslant \frac{1}{2D^2} \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{x_1 x_2 + y_1 y_2 + (x_1^2 + y_1^2)^{\frac{1}{2}} (x_2^2 + y_2^2)^{\frac{1}{2}}} = \frac{1}{D^2} (\tan \varphi)^2.\end{aligned}$$

Hence

$$\varphi' \geqslant \arctan\left(\frac{1}{D} \tan \varphi\right) \geqslant \frac{1}{D} \varphi,$$

and we obtain

$$A(g(\gamma_1), g(\gamma_2)) = \liminf_{z_1, z_2 \to 0} 2\varphi' \geqslant \frac{1}{D} \liminf_{z_1, z_2 \to 0} 2\varphi = \frac{1}{D} A(\gamma_1, \gamma_2).$$

Thus (3.2) holds if $K \geqslant D$. Next, for $\theta > 0$ let γ_1 and γ_2 denote the segments from 0 to $e^{i\theta}$ and $e^{-i\theta}$. Then

$$A(g(\gamma_1), g(\gamma_2)) = 2 \arctan\left(\frac{1}{D} \tan\theta\right) \sim \frac{1}{D} A(\gamma_1, \gamma_2)$$

as $\theta \to 0$, and hence (3.2) implies that $K \geqslant D$.

THEOREM 1. *Suppose that f is a homeomorphism of a domain G, that f has a differential at z_0, and that*

$$\max_\theta |D_\theta f(z_0)| > 0, \tag{3.4}$$

where $D_\theta f$ denotes the directional derivative of f. If

$$\max_\theta |D_\theta f(z_0)|^2 \leqslant K |J(z_0)|, \tag{3.5}$$

where J denotes the Jacobian of f, then

$$A(f(\gamma_1), f(\gamma_2)) \geqslant \frac{1}{K} A(\gamma_1, \gamma_2) \tag{3.6}$$

for each pair of arcs γ_1 and γ_2 which form a topological angle in G at z_0. Conversely, if (3.6) *holds for each pair of segments γ_1 and γ_2 which form an angle in G at z_0, then* (3.5) *holds.*

Proof. Suppose that (3.5) holds. Then (3.4) implies that $J(z_0) \neq 0$. Hence by performing preliminary similarity mappings and reflexions, we may assume that $z_0 = f(z_0) = 0$ and that, near $z_0 = 0$,

$$f(z) = g(z) + o(|z|) = g(z) + o(|g(z)|), \tag{3.7}$$

where g is as in (3.1) and $D \geqslant 1$. Inequality (3.5) implies that $D \leqslant K$, and with Lemmas 1 and 2 we obtain

$$A(f(\gamma_1), f(\gamma_2)) = A(g(\gamma_1), g(\gamma_2)) \geqslant \frac{1}{K} A(\gamma_1, \gamma_2)$$

for each pair of arcs γ_1 and γ_2 in G which form a topological angle at the origin. This completes the proof of the first part of Theorem 1.

For the second part, suppose first that $J(z_0) \neq 0$. Then again we may assume that $z_0 = f(z_0) = 0$ and that (3.7) holds, where g is as in (3.1) and $D \geqslant 1$. Then Lemma 1 and (3.6) imply that

$$A(g(\gamma_1), g(\gamma_2)) = A(f(\gamma_1), f(\gamma_2)) \geqslant \frac{1}{K} A(\gamma_1, \gamma_2)$$

for each pair of segments γ_1 and γ_2 which form an angle in G at the origin. Hence $D \leqslant K$ by Lemma 2, and we obtain (3.5).

Finally, to complete the proof, we observe that (3.4) and (3.6) imply that $J(z_0) \neq 0$. For suppose that $J(z_0) = 0$. Then by performing preliminary similarity mappings, we may assume that $z_0 = f(z_0) = 0$ and that, near $z_0 = 0$,

$$f(z) = x + o(|z|).$$

Next, for $0 < \theta < \pi/2$ and $r > 0$, let γ_1 and γ_2 denote the segments joining 0 to $re^{i\theta}$ and $re^{-i\theta}$. Then these segments lie in G for small r, it is easy to see that

$$A(f(\gamma_1), f(\gamma_2)) = 0, \quad A(\gamma_1, \gamma_2) = 2\theta,$$

and we have a contradiction.

4. A distortion theorem for quasiconformal mappings

We establish next a distortion theorem for quasiconformal mappings of the extended plane. This will yield a sharp estimate for the change in the inner measure of a topological angle under a quasiconformal mapping.

We introduce some notation. Let $C(z_1, \ldots, z_n)$ denote the domain which consists of the extended plane minus the n points $z_1, \ldots, z_n$. Next let $z(\zeta)$ map the half-plane $\text{Im}(\zeta) > 0$ conformally onto the universal covering surface of $C(-1, 1, \infty)$, and for z in $C(-1, 1, \infty)$ set

$$\rho(z) = \frac{|\zeta'(z)|}{2 \,\text{Im}\, \zeta(z)}, \tag{4.1}$$

where $\zeta(z)$ is a local inverse of $z(\zeta)$. The function ρ is called the *hyperbolic density* for $C(-1, 1, \infty)$; it is easy to show that it does not depend upon the choice of the local inverse of $z(\zeta)$. The *hyperbolic distance* between two points z_1, z_2 of $C(-1, 1, \infty)$ is then given by

$$h(z_1, z_2) = \inf_{\gamma} \int_{\gamma} \rho(z)\,|dz|,$$

where γ is any rectifiable arc which joins z_1 and z_2 in $C(-1, 1, \infty)$.

Suppose next that f is a quasiconformal mapping of $C(-1, 1, \infty)$ onto itself. Then f can be extended to be a quasiconformal mapping of the extended plane. We say that f is *normalized* if for the extended mapping we have $f(-1) = -1$, $f(1) = 1$, and $f(\infty) = \infty$. Our distortion theorem is based upon the following fundamental result due to Teichmüller ((**27**) 29–31).

LEMMA 3. *If f is a normalized K-quasiconformal mapping of $C(-1, 1, \infty)$ onto itself, then*

$$h(z_0, f(z_0)) \leqslant \tfrac{1}{2} \log K$$

for each z_0 *in* $C(-1,1,\infty)$. *Moreover, given any pair of points* z_0, w_0 *of* $C(-1,1,\infty)$ *satisfying*

$$h(z_0, w_0) \leqslant \tfrac{1}{2}\log K,$$

there exists a normalized K*-quasiconformal mapping* f *of* $C(-1,1,\infty)$ *onto itself such that* $f(z_0) = w_0$.

In order to make use of Lemma 3, we shall establish a result which yields a lower bound for the hyperbolic distance between pairs of points in $C(-1,1,\infty)$. However, first we require some information about the density ρ. For ζ in $C(-1,0,1,\infty)$, let

$$f(\zeta) = \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right). \tag{4.2}$$

Then $f(\zeta) \in C(-1,1,\infty)$, and we set

$$u(\zeta) = \rho(f(\zeta))\,|f'(\zeta)|. \tag{4.3}$$

LEMMA 4. *For* ζ *in* $C(-1,0,1,\infty)$,

$$u(\zeta) \geqslant u(i|\zeta|). \tag{4.4}$$

Proof. (Cf. (**14**).) The function u is continuous, and is symmetric with respect to the real and imaginary axes, that is

$$u(\zeta) = u(\bar{\zeta}) = u(-\bar{\zeta}) \tag{4.5}$$

for ζ in $C(-1,0,1,\infty)$. Next, since ρ satisfies the differential equation

$$\Delta \log \rho(z) = 4\rho(z)^2$$

in $C(-1,1,\infty)$ ((**18**) 51), and since $\log|f'(\zeta)|$ is harmonic in $C(-1,0,1,\infty)$, we see that

$$\Delta \log u(\zeta) = \Delta \log \rho(f(\zeta)) = 4\rho(f(\zeta))^2 |f'(\zeta)|^2 = 4u(\zeta)^2 \tag{4.6}$$

in $C(-1,0,1,\infty)$. Now $f(\zeta) \to \infty$ as $\zeta \to 0$ or ∞, and $f(\zeta) \to 1$ as $\zeta \to 1$. Hence, if we compare the asymptotic behaviour of $f(\zeta)$ and $f'(\zeta)$ near 0, 1, ∞ with that of $\rho(z)$ near 1 and ∞ ((**18**) 246), we obtain

$$\log u(\zeta) = \begin{cases} \log \dfrac{1}{|\zeta|} - \log\left(\log \dfrac{1}{|\zeta|}\right) + \psi_0(\zeta) & \text{near } \zeta = 0, \\ \log \dfrac{1}{|\zeta-1|} - \log\left(\log \dfrac{1}{|\zeta-1|}\right) + \psi_1(\zeta) & \text{near } \zeta = 1, \\ -\log|\zeta| - \log(\log|\zeta|) + \psi_\infty(\zeta) & \text{near } \zeta = \infty, \end{cases} \tag{4.7}$$

where ψ_0, ψ_1, ψ_∞ are continuous at 0, 1, ∞ respectively.

Now choose θ_1, θ_2, r so that $0 \leqslant \theta_1 < \theta_2 \leqslant \pi/2$, $0 < r < \infty$, and $re^{i\theta_1} \neq 1$. We shall show that

$$u(re^{i\theta_1}) \geqslant u(re^{i\theta_2}). \tag{4.8}$$

For this, let $\alpha = \frac{1}{2}(\theta_2 - \theta_1)$ and $\beta = \frac{1}{2}(\theta_2 + \theta_1)$, and set

$$v(\zeta) = \log u(\zeta e^{i\alpha}) - \log u(\zeta e^{-i\alpha}). \tag{4.9}$$

From (4.7) it follows that v has limits at 0 and ∞, and from (4.5) we see that v vanishes on the real and imaginary axes. Hence we can extend v to be continuous at 0 and ∞ by defining $v(0) = v(\infty) = 0$. Let Q be the open first quadrant of the ζ-plane minus the point $e^{i\alpha}$. Then v is continuous in Q, and since, by (4.7),

$$\lim_{\zeta \to e^{i\alpha}} v(\zeta) = -\infty,$$

we conclude that v has non-positive boundary values at each point of ∂Q. Now suppose that $v(\zeta) > 0$ for some ζ in Q. Then v must have a positive maximum at some point ζ_0 of Q. From (4.6) it follows that

$$\Delta v(\zeta_0) = 4(u(\zeta_0 e^{i\alpha})^2 - u(\zeta_0 e^{-i\alpha})^2).$$

By (4.9), $v(\zeta_0) > 0$ implies that $u(\zeta_0 e^{i\alpha}) > u(\zeta_0 e^{-i\alpha})$, and since u is non-negative, this in turn implies that $\Delta v(\zeta_0) > 0$. On the other hand, since v has a maximum at ζ_0, the second-derivative condition implies that $\Delta v(\zeta_0) \leqslant 0$, and we have a contradiction. Thus $v \leqslant 0$ in Q, and (4.8) follows by setting $\zeta = re^{i\beta} \neq e^{i\alpha}$ in (4.9).

Finally, (4.4) follows from (4.8) for ζ in $C(-1, 0, 1, \infty)$, provided that $0 \leqslant \arg \zeta \leqslant \frac{1}{2}\pi$. The general case then follows from (4.5).

LEMMA 5. *If* $z_0, w_0 \in C(-1, 1, \infty)$ *then*

$$h(z_0, w_0) \geqslant h(i \cot \alpha, i \cot \beta), \tag{4.10}$$

where

$$\alpha = \arcsin\left(\frac{2}{|z_0+1|+|z_0-1|}\right), \quad \beta = \arcsin\left(\frac{2}{|w_0+1|+|w_0-1|}\right).$$

Proof. By means of a limiting argument, we may assume that z_0 and w_0 are not on the segment joining -1 to 1. Next, by symmetry, we may further assume that $\beta < \alpha$. Let f and u be as in (4.2) and (4.3). Then we choose a and b, with $1 < a < b$, so that f maps the circles $|\zeta| = a$ and $|\zeta| = b$ onto the ellipses which have foci at -1 and 1 and which pass through $i \cot \alpha$ and z_0 and through $i \cot \beta$ and w_0, respectively. Let γ be any rectifiable arc which joins z_0 and w_0 in $C(-1, 1, \infty)$. Then there exists an arc γ' in $a \leqslant |\zeta| \leqslant b$ which joins the boundary circles and for which $f(\gamma') \subseteq \gamma$. Since ρ and u are non-negative, we have, by (4.4),

$$\int_\gamma \rho(z)\,|dz| \geqslant \int_{f(\gamma')} \rho(z)\,|dz| = \int_{\gamma'} u(\zeta)\,|d\zeta| \geqslant \int_a^b u(i|\zeta|)\,d|\zeta| = \int_\delta \rho(z)\,|dz|, \tag{4.11}$$

where δ is the segment joining $i\cot\alpha$ and $i\cot\beta$. Inequality (4.10) then follows from (4.11).

LEMMA 6. *If* $0<\beta\leqslant\alpha\leqslant\frac{1}{2}\pi$, *then*

$$h(i\cot\alpha, i\cot\beta) = \tfrac{1}{2}\log\left(\frac{\mu(\sin\frac{1}{2}\beta)}{\mu(\sin\frac{1}{2}\alpha)}\right),$$

where, for $0<r<1$, $\mu(r)$ *is the modulus of the unit disk slit along the real axis from* 0 *to* r.

Proof. It is easy to see that the imaginary axis is a geodesic for the hyperbolic metric in $C(-1,1,\infty)$. Hence, by symmetry, it will be sufficient to show that

$$h(i\cot\alpha, 0) = \tfrac{1}{2}\log\frac{2}{\pi}\mu(\sin\tfrac{1}{2}\alpha). \tag{4.12}$$

Let $\nu(z)$ be the local inverse of the elliptic modular function which maps the half plane $\mathrm{Im}(z)>0$ conformally onto the curvilinear triangle $\triangle$ which lies in $\mathrm{Im}(\zeta)>0$ and is bounded by $\mathrm{Re}(\zeta)=0$, $\mathrm{Re}(\zeta)=1$, and $|\zeta-\frac{1}{2}|=\frac{1}{2}$, so that $\nu(0)=\infty$, $\nu(1)=0$, and $\nu(\infty)=1$. Next let

$$f_1(z) = \nu\left(\frac{1}{2}-\frac{1}{2}\left(\frac{z}{z-1}\right)^{\frac{1}{2}}\right),\quad f_2(z) = \frac{1+\nu(\frac{1}{2}z^{\frac{1}{2}}+\frac{1}{2})}{1-\nu(\frac{1}{2}z^{\frac{1}{2}}+\frac{1}{2})},$$

where we choose the branches of the square roots with non-negative real part. Then it is easy to verify that both f_1 and f_2 map $\mathrm{Im}(z)>0$ conformally onto a second curvilinear triangle $\triangle_0$, and that $f_1(0)=f_2(0)$, $f_1(1)=f_2(1)$, and $f_1(\infty)=f_2(\infty)$. Hence $f_1(z)=f_2(z)$ in $\mathrm{Im}(z)\geqslant 0$, and setting $z=-(\cot\alpha)^2$ yields

$$\nu((\sin\tfrac{1}{2}\alpha)^2) = \frac{1+\nu(\frac{1}{2}i\cot\alpha+\frac{1}{2})}{1-\nu(\frac{1}{2}i\cot\alpha+\frac{1}{2})}. \tag{4.13}$$

By (**12**) and by ((**5**) 437) or ((**17**) 319),

$$\frac{2}{\pi}\mu(\sin\tfrac{1}{2}\alpha) = -i\nu((\sin\tfrac{1}{2}\alpha)^2). \tag{4.14}$$

Now $\zeta(z)=\nu(\frac{1}{2}z+\frac{1}{2})$ is a local inverse of the function which maps $\mathrm{Im}(\zeta)>0$ conformally onto the universal covering surface of $C(-1,1,\infty)$. Hence $h(i\cot\alpha,0)$ is equal to the hyperbolic distance in $\mathrm{Im}(\zeta)>0$ between $\zeta(i\cot\alpha)$ and $\zeta(0)$; that is

$$h(i\cot\alpha,\ 0) = \tfrac{1}{2}\log\left(\frac{|\zeta(i\cot\alpha)-\overline{\zeta(0)}|+|\zeta(i\cot\alpha)-\zeta(0)|}{|\zeta(i\cot\alpha)-\overline{\zeta(0)}|-|\zeta(i\cot\alpha)-\zeta(0)|}\right).$$

Since $|\zeta(i\cot\alpha)|=1$ and $\zeta(0)=i$, an elementary calculation gives

$$h(i\cot\alpha, 0) = \tfrac{1}{2}\log\left(i\frac{\nu(\frac{1}{2}i\cot\alpha+\frac{1}{2})+1}{\nu(\frac{1}{2}i\cot\alpha+\frac{1}{2})-1}\right), \tag{4.15}$$

and (4.12) follows from (4.13), (4.14), and (4.15).

For $0 < r < 1$ and $K \geqslant 1$, we introduce the distortion function

$$\varphi_K(r) = \mu^{-1}(K\mu(r)), \tag{4.16}$$

where, as above, $\mu(r)$ is the modulus of the unit disk slit along the real axis from 0 to r, and where μ^{-1} is the inverse of μ. Then φ_K is continuous and strictly increasing in $0 < r < 1$, with boundary values $\varphi_K(0) = 0$ and $\varphi_K(1) = 1$. It is also easy to verify that $\varphi_K(t) \leqslant t$, and that

$$\lim_{r\to 0} \frac{\varphi_K(r)}{r^K} = 4^{1-K},$$

using known properties of $\mu(r)$ (**12**).

If we now combine Lemmas 3, 5, and 6, we obtain the following distortion theorem.

THEOREM 2. *Suppose that f is a K-quasiconformal mapping of the extended plane, and that $f(\infty) = \infty$. Then for each triple of distinct finite points z_1, z_0, z_2,*

$$\sin \tfrac{1}{2}\beta \geqslant \varphi_K(\sin \tfrac{1}{2}\alpha), \tag{4.17}$$

where φ_K is as in (4.16) *and*

$$\alpha = \arcsin\left(\frac{|z_1 - z_2|}{|z_1 - z_0| + |z_2 - z_0|}\right),$$

$$\beta = \arcsin\left(\frac{|f(z_1) - f(z_2)|}{|f(z_1) - f(z_0)| + |f(z_2) - f(z_0)|}\right).$$

This inequality is sharp.

Proof. By performing preliminary similarity transformations, we may assume that $z_1 = f(z_1) = -1$ and that $z_2 = f(z_2) = 1$. Next, since $\varphi_K(t) \leqslant t$, we may assume that $\beta \leqslant \alpha$, for otherwise (4.17) follows trivially. Now f is a normalized K-quasiconformal mapping of $C(-1, 1, \infty)$ onto itself, and hence Lemma 3 implies that

$$h(z_0, f(z_0)) \leqslant \tfrac{1}{2} \log K. \tag{4.18}$$

From Lemmas 5 and 6 we obtain

$$h(z_0, f(z_0)) \geqslant \tfrac{1}{2} \log\left(\frac{\mu(\sin \frac{1}{2}\beta)}{\mu(\sin \frac{1}{2}\alpha)}\right), \tag{4.19}$$

and (4.17) follows from (4.18) and (4.19).

To show that (4.17) is sharp, given that $0 < \alpha \leqslant \frac{1}{2}\pi$ and $1 \leqslant K < \infty$, we must find a K-quasiconformal mapping of the extended plane, and three finite points z_1, z_0, z_2, such that $f(\infty) = \infty$ and (4.17) holds with equality. Choose β, with $0 < \beta \leqslant \alpha$, so that $\sin\frac{1}{2}\beta = \varphi_K(\sin\frac{1}{2}\alpha)$, and let $z_0 = i\cot\alpha$ and $w_0 = i\cot\beta$. Then by Lemma 6,

$$h(z_0, w_0) = \tfrac{1}{2}\log\left(\frac{\mu(\sin\frac{1}{2}\beta)}{\mu(\sin\frac{1}{2}\alpha)}\right) = \tfrac{1}{2}\log K,$$

and hence, by Lemma 3, there exists a normalized K-quasiconformal mapping f of $C(-1,1,\infty)$ onto itself such that $f(z_0) = w_0$. If we set $z_1 = -1$ and $z_2 = 1$, we see that f is the desired mapping and z_1, z_0, z_2 the desired triple of points.

5. Inner measure under quasiconformal mappings

We consider next how much the inner measure of a topological angle is changed under a quasiconformal mapping.

THEOREM 3. *Suppose that f is a K-quasiconformal mapping of a domain G. Then*

$$\sin\tfrac{1}{4}A(f(\gamma_1),f(\gamma_2)) \geqslant \varphi_K(\sin\tfrac{1}{4}A(\gamma_1,\gamma_2)) \tag{5.1}$$

for each pair of arcs γ_1 and γ_2 which form a topological angle in G, where φ_K is as in (4.16). *This inequality is best possible.*

Proof. Suppose that γ_1 and γ_2 are arcs which form a topological angle at z_0 in G. By performing a preliminary similarity mapping, we may assume that $z_0 = 0$ and that G contains the unit disk $|z| < 1$. We may further assume that γ_1 and γ_2 lie in $|z| < 1$, since $A(\gamma_1,\gamma_2)$ does not depend upon the behaviour of γ_1 and γ_2 outside any neighbourhood of $z_0 = 0$.

Now let U denote the image of $|z| < 1$ under f, and let g map U conformally onto $|w| < 1$ so that $g(f(0)) = 0$. Then $g\circ f$ is a K-quasiconformal mapping of $|z| < 1$ onto $|w| < 1$ which we can extend, by reflecting in $|z| = 1$ and $|w| = 1$, to obtain a K-quasiconformal mapping h of the extended plane with $h(\infty) = \infty$. For z_1 in γ_1, z_2 in γ_2, $z_1, z_2 \neq 0$, let

$$\alpha = \arcsin\left(\frac{|z_1 - z_2|}{|z_1|+|z_2|}\right), \quad \beta = \arcsin\left(\frac{|h(z_1)-h(z_2)|}{|h(z_1)|+|h(z_2)|}\right).$$

Then by Theorem 2,

$$\begin{aligned}\sin\tfrac{1}{4}A(h(\gamma_1),h(\gamma_2)) &= \liminf_{z_1,z_2\to 0}\sin\tfrac{1}{2}\beta \\ &\geqslant \liminf_{z_1,z_2\to 0}\varphi_K(\sin\tfrac{1}{2}\alpha) = \varphi_K(\sin\tfrac{1}{4}A(\gamma_1,\gamma_2)).\end{aligned} \tag{5.2}$$

Since $h = g\circ f$ in $|z| < 1$, and since g is conformal,

$$A(h(\gamma_1),h(\gamma_2)) = A(f(\gamma_1),f(\gamma_2)) \tag{5.3}$$

by virtue of Theorem 1, and (5.1) follows from (5.2) and (5.3).

To show that (5.1) is best possible, given $\varepsilon > 0$, $0 < \alpha \leqslant \frac{1}{2}\pi$, and $1 \leqslant K < \infty$, we shall exhibit a K-quasiconformal mapping h of a domain G, and two segments γ_1 and γ_2 which form an angle in G, such that $A(\gamma_1, \gamma_2) = 2\alpha$ and

$$\sin \tfrac{1}{4}A(h(\gamma_1), h(\gamma_2)) \leqslant \varphi_K(\sin \tfrac{1}{4}A(\gamma_1, \gamma_2)) + \varepsilon. \tag{5.4}$$

The argument is similar to one given in (**14**).

First choose β, with $0 < \beta \leqslant \alpha$, so that

$$\sin \tfrac{1}{2}\beta = \varphi_K(\sin \tfrac{1}{2}\alpha). \tag{5.5}$$

Then, as in the proof of the last part of Theorem 2, we can find a normalized K-quasiconformal mapping f of $C(-1, 1, \infty)$ onto itself, and a point z_0 on the imaginary axis, such that

$$\alpha = \arcsin\left(\frac{2}{|z_0+1|+|z_0-1|}\right), \quad \beta = \arcsin\left(\frac{2}{|w_0+1|+|w_0-1|}\right), \tag{5.6}$$

where $w_0 = f(z_0)$. For $n > |z_0 - 1|$, let

$$A_n = \left\{z: \frac{1}{n} < |z - z_0| < n\right\},$$

and let g_n map $f(A_n)$ conformally onto an annulus

$$B_n = \{w: a_n < |w - w_0| < b_n\}$$

so that $g_n(1) = 1$ and so that the inner components of ∂A_n and ∂B_n correspond under $h_n = g_n \circ f$. Given a compact subset E of $C(1, w_0, \infty)$, the g_n are defined in E for $n > n(E)$ and omit the values 1, w_0, ∞ there. Hence the g_n form a normal family in $C(1, w_0, \infty)$, and there exists a subsequence $\{g_{n_k}\}$ which converges in $C(1, w_0, \infty)$ to a function g which is easily seen to be the identity mapping. Let

$$\beta_n = \arcsin\left(\frac{|h_n(-1) - h_n(1)|}{|h_n(-1) - w_0| + |h_n(1) - w_0|}\right). \tag{5.7}$$

Then since $h_n(1) = 1$ and $h_{n_k}(-1) \to -1$, we can find an N such that

$$\sin \tfrac{1}{2}\beta_N \leqslant \varphi_K(\sin \tfrac{1}{2}\alpha) + \varepsilon, \tag{5.8}$$

by (5.5) and (5.6).

Now $h = h_N$ is a K-quasiconformal mapping of A_N onto B_N under which the inner components of ∂A_N and ∂B_N correspond. Hence by reflecting successively in the pairs of circles

$$C_m = \{z: |z - z_0| = N^{-2m+1}\}, \quad D_m = \left\{w: |w - w_0| = a_N\left(\frac{a_N}{b_N}\right)^{m-1}\right\},$$

$m = 1, 2, \ldots$, we can extend h as a K-quasiconformal mapping of $0 < |z - z_0| < N$ onto $0 < |w - w_0| < b_N$. Then since z_0 is an isolated boundary point, we can further extend h to be K-quasiconformal in $|z-z_0| < N$ by setting $h(z_0) = w_0$.

Let γ_1 and γ_2 denote the segments joining z_0 to -1 and 1, respectively, and for $m = 1, 2, \ldots$, let $z_{1,m}$ and $z_{2,m}$ denote the points obtained by reflecting -1 and 1 successively in the circles $C_1, C_2, \ldots, C_m$. Then $h(z_{1,m})$ and $h(z_{2,m})$ are obtained by reflecting $h(-1)$ and $h(1)$ successively in the circles $D_1, D_2, \ldots, D_m$, and hence with (5.7) we have

$$\arcsin\left(\frac{|h(z_{1,m})-h(z_{2,m})|}{|h(z_{1,m})-h(z_0)|+|h(z_{2,m})-h(z_0)|}\right) = \beta_N \tag{5.9}$$

for $m = 1, 2, \ldots$. By (5.6), $A(\gamma_1, \gamma_2) = 2\alpha$, and since $z_{1,m}$ and $z_{2,m}$ converge to z_0 along γ_1 and γ_2 respectively, we obtain, from (5.8) and (5.9),

$$\sin \tfrac{1}{4}A(h(\gamma_1), h(\gamma_2)) \leqslant \sin \tfrac{1}{2}\beta_N \leqslant \varphi_K(\sin \tfrac{1}{4}A(\gamma_1, \gamma_2)) + \varepsilon,$$

as required.

6. Characterization of quasiconformal mappings

We shall require some properties of the linear measure of plane sets. For $d > 0$, let $\mathscr{E}$ denote any covering of a plane set E by sets E_α, where $\operatorname{dia} E_\alpha \leqslant d$, and let

$$\Lambda(E, d) = \inf \sum_\alpha \operatorname{dia} E_\alpha,$$

where the infimum is taken over all such coverings $\mathscr{E}$. Then $\Lambda(E, d)$ is non-increasing in d, and the *outer linear measure* of E is defined as

$$\Lambda(E) = \lim_{d \to 0} \Lambda(E, d). \tag{6.1}$$

This is a regular Carathéodory outer measure (**25**). Hence all Borel sets are measurable with respect to Λ.

LEMMA 7. *If F is a bounded perfect linear set, then for each $\varepsilon > 0$ there exists a $\delta > 0$ with the following property: given that $0 < t < \delta$, there exist N non-overlapping intervals I_n, with end-points in F and lengths not greater than t, such that*

$$F \subseteq \bigcup_1^N I_n \quad \textit{and} \quad Nt \leqslant \Lambda(F) + \varepsilon. \tag{6.2}$$

Proof. We may clearly assume that F lies in the positive half of the real axis λ. Then for $t > 0$ let $F(t)$ be the set of points of λ within distance t of F. Since F is compact,

$$F = \bigcap_{t>0} F(t), \quad \Lambda(F) = \lim_{t \to 0} \Lambda(F(t)),$$

and we can choose $\delta > 0$ so that

$$\Lambda(F(t)) \leqslant \Lambda(F) + \varepsilon$$

for $0 < t < \delta$. Pick any such t, and let $J_1, J_2, \ldots, J_N$ denote the intervals of the form $[(m-1)t, mt]$ which contain at least two points of F. Since F contains no isolated points, these intervals cover F in $F(t)$, and hence

$$Nt = \Lambda\left(\bigcup_1^N J_n\right) \leqslant \Lambda(F(t)) \leqslant \Lambda(F) + \varepsilon.$$

For each n, let I_n denote the smallest closed subinterval of J_n which contains the set $F \cap J_n$. It is then easy to see that the intervals I_n have all of the desired properties.

We show now how quasiconformal mappings can be characterized in terms of what they do to the inner measure of angles.

THEOREM 4. *A homeomorphism f of a domain G is K-quasiconformal, $1 \leqslant K < \infty$, if and only if it satisfies the following conditions.*

(i) *For all z_0 in G, and for all segments γ_1 and γ_2 which form an angle in G at z_0,*

$$A(f(\gamma_1), f(\gamma_2)) > 0. \tag{6.3}$$

(ii) *For almost all z_0 in G, and for all segments γ_1 and γ_2 which form an angle in G at z_0,*

$$A(f(\gamma_1), f(\gamma_2)) \geqslant \frac{1}{K} A(\gamma_1, \gamma_2). \tag{6.4}$$

Proof. Suppose that f is a K-quasiconformal mapping of G, and let E be the set of points z_0 of G at which f is differentiable with

$$0 < \max_\theta |D_\theta f(z_0)|^2 \leqslant K|J(z_0)|.$$

Then $m(G \backslash E) = 0$ by (**10**) or (**16**) and by (**3**) or (**11**). Next let γ_1 and γ_2 be any pair of segments which form an angle in G at z_0. Since $A(\gamma_1, \gamma_2) > 0$, (6.3) follows from Theorem 3. Moreover, if $z_0 \in E$, then (6.4) follows from Theorem 1, and hence f satisfies both conditions (i) and (ii).

Now suppose that f is a homeomorphism of G which satisfies these two conditions. To prove that f is a K-quasiconformal mapping, we must first show that f is ACL (absolutely continuous on lines) in G.† This will imply that f has finite partial derivatives a.e. in G, and hence, by (**10**), that f is differentiable a.e. in G. Then we must show that

$$\max_\theta |D_\theta f(z)|^2 \leqslant K|J(z)| \tag{6.5}$$

a.e. in G.

† A function f is said to be ACL in a domain G if, for each closed rectangle $R \subset G$ with sides parallel to the coordinate axes, f is absolutely continuous on almost all horizontal and vertical segments in R.

Since the ACL proof is rather long, we postpone it and consider (6.5) first. For this, let E be the set of points z_0 at which (6.4) holds, let F be the set of points at which f is differentiable, and fix z_0 in $E \cap F$. If

$$\max_{\theta} |D_\theta f(z_0)| > 0, \tag{6.6}$$

then we see from Theorem 1 that (6.5) holds for $z = z_0$. If (6.6) does not hold, then (6.5) holds trivially for $z = z_0$. Thus (6.5) holds for z in $E \cap F$, and hence a.e. in G.

We turn now to the proof that f is ACL in G. Let R be a closed rectangle which lies in G and has sides parallel to the coordinate axes. We must show that f is absolutely continuous on almost all horizontal and vertical segments in R. By performing a preliminary similarity transformation, we may assume that R is the rectangle $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant c$. Then, by symmetry, it will be sufficient to prove that $f(x+iy)$ is absolutely continuous in $0 \leqslant x \leqslant 1$ for almost all y in $0 \leqslant y \leqslant c$.

For $0 < y_0 \leqslant c$, let $I(y_0)$ denote the interval $0 \leqslant x \leqslant 1$, $y = y_0$, let $R(y_0)$ denote the rectangle $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant y_0$, and set $g(y_0) = m(f(R(y_0)))$. Since $f(R)$ is compact, $g(y)$ is finite and increasing in $0 < y \leqslant c$. Hence $g'(y_0)$ exists and is finite for almost all y_0 in $0 < y < c$. Next fix r so that $0 < r < \frac{1}{2}\rho(R, \partial G)$, where $\rho(R, \partial G)$ denotes the distance between R and ∂G, and for each z_0 in R let $\gamma_i = \gamma_i(z_0)$ be the segment joining z_0 to $z_0 + \zeta_i$, where

$$\zeta_1 = r, \quad \zeta_2 = ir, \quad \zeta_3 = -r, \quad \zeta_4 = -r + ir.$$

By condition (i), we have

$$A(f(\gamma_1), f(\gamma_2)) > 0, \quad A(f(\gamma_3), f(\gamma_4)) > 0.$$

This, in turn, implies that

$$\begin{aligned} &\limsup_{z_1, z_2 \to z_0} \frac{|f(z_1)-f(z_0)|+|f(z_2)-f(z_0)|}{|f(z_1)-f(z_2)|} < \infty, \\ &\limsup_{z_3, z_4 \to z_0} \frac{|f(z_3)-f(z_0)|+|f(z_4)-f(z_0)|}{|f(z_3)-f(z_4)|} < \infty. \end{aligned} \qquad z_i \in \gamma_i(z_0), \tag{6.7}$$

Now for each pair of integers p and q, with $p > 0$ and $0 < 1/q < r$, let $H(p,q)$ denote the set of z_0 in R such that

$$\begin{aligned} |f(z_1)-f(z_0)|+|f(z_2)-f(z_0)| &\leqslant p\,|f(z_1)-f(z_2)|, \\ |f(z_3)-f(z_0)|+|f(z_4)-f(z_0)| &\leqslant p\,|f(z_3)-f(z_4)|, \end{aligned} \tag{6.8}$$

whenever $|z_i - z_0| \leqslant 1/q$ and $z_i \in \gamma_i(z_0)$ for $i = 1, 2, 3, 4$. Then $H(p,q)$ is compact, and, by (6.7),

$$R = \bigcup_{p,q} H(p,q), \tag{6.9}$$

where the sum is taken over relevant p and q.

LEMMA 8. *Suppose that* $0 < y_0 < c$, *that* $g'(y_0)$ *exists and is finite, and that* F *is a compact set in* $I(y_0) \cap H(p,q)$. *Then*

$$\Lambda(f(F))^2 \leqslant 2pg'(y_0)\Lambda(F). \tag{6.10}$$

Proof of Lemma 8. Suppose that I is a closed subinterval of $I(y_0)$ with end-points a, b in F, where $b-a > 0$ and

$$2^{\frac{1}{2}}(b-a) \leqslant \min(1/q, c-y_0), \tag{6.11}$$

and let T be the open right triangle which has a, b, and $a+i(b-a)$ as its vertices. We say that T is the triangle *associated* with the interval I. Clearly $T \subseteq R$, by (6.11).

We shall show that

$$|f(a)-f(b)|^2 \leqslant 2pm(f(T)). \tag{6.12}$$

By performing a change of variables, we may assume that $f(a) = 0$ and $f(b) = l > 0$. Then for each u_0, $0 < u_0 < l$, the line $u = u_0$ contains an open interval which lies in $f(T)$ and has end-points w_1 in $f(I)$ and w_2 in $f(\alpha) \cup f(\beta)$, where α and β are the sides of T which join a and b to $a+i(b-a)$, respectively. Suppose that $w_2 \in f(\alpha)$, and let $z_i = f^{-1}(w_i)$ for $i = 1, 2$. Then $z_i \in \gamma_i(a)$, $|z_i - a| \leqslant 1/q$ by (6.11), and hence (6.8) yields

$$2u_0 \leqslant |w_1|+|w_2| \leqslant p|w_1-w_2|.$$

If $w_2 \in f(\beta)$, a similar argument yields

$$2(l-u_0) \leqslant |w_1-l|+|w_2-l| \leqslant p|w_1-w_2|.$$

Hence, for $0 < u_0 < l$, the line $u = u_0$ contains an open interval which lies in $f(T)$ and has length not less than $(2/p)\min(u_0, l-u_0)$. By Fubini's theorem

$$m(f(T)) \geqslant \frac{2}{p}\int_0^l \min(u, l-u)\,du = \frac{l^2}{2p},$$

from which (6.12) follows.

Since F is closed, $F = F_1 \cup F_2$, where F_1 is countable and F_2 is either perfect or empty. Obviously

$$\Lambda(F) = \Lambda(F_2), \quad \Lambda(f(F)) = \Lambda(f(F_2)),$$

and hence for the proof of (6.10) we may assume that F is a perfect set.

Fix $\varepsilon > 0$, choose the corresponding δ of Lemma 7, and fix $0 < t < \delta$ so that

$$2^{\frac{1}{2}}t \leqslant \min(1/q, c-y_0). \tag{6.13}$$

Next let $I_1, \ldots, I_N$ be the covering of F described in Lemma 7, and let T_n be the open right triangle associated with I_n. Then each pair of points a, b in $F \cap I_n$, with $b-a > 0$, bounds a closed interval I whose associated triangle T lies in T_n. Since $b-a \leqslant t$, (6.13) implies (6.11), and hence

$$|f(a)-f(b)|^2 \leqslant 2pm(f(T)) \leqslant 2pm(f(T_n))$$

by (6.12). From this it follows that

$$(\operatorname{dia} f(E_n))^2 = d_n{}^2 \leqslant 2pm(f(T_n)), \tag{6.14}$$

where $E_n = F \cap I_n$.

Let $d = \max(d_1,\dots,d_N)$. Then the sets $f(E_n)$ form a covering of $f(F)$, $\operatorname{dia} f(E_n) \leqslant d$, and hence by (6.2), (6.14), and the Schwarz inequality,

$$\Lambda(f(F),d)^2 \leqslant \left(\sum_1^N \operatorname{dia} f(E_n)\right)^2 \leqslant 2Np \sum_1^N m(f(T_n))$$

$$\leqslant 2p\,\frac{g(y_0+t)-g(y_0)}{t}\,(\Lambda(F)+\varepsilon).$$

If we now let $t \to 0$, then $d \to 0$ by the continuity of f, and we obtain

$$\Lambda(f(F))^2 \leqslant 2pg'(y_0)(\Lambda(F)+\varepsilon)$$

by (6.1). Since ε is arbitrary, this implies (6.10), and the proof of Lemma 8 is complete.

LEMMA 9. *Suppose that* $0 < y_0 < c$, *that* $g'(y_0)$ *exists and is finite, and that* E *is a subset of* $I(y_0)$ *with* $\Lambda(E) = 0$. *Then* $\Lambda(f(E)) = 0$.

Proof of Lemma 9. Suppose first that E is compact. Then $F = E \cap H(p,q)$ is compact for relevant p and q, and from (6.9) and (6.10) we conclude that

$$\Lambda(f(E)) \leqslant \sum_{p,q} \Lambda(f(E \cap H(p,q))) = 0.$$

Suppose next that E is a G_δ-Borel set. Then, since $F = I(y_0) \cap H(p,q)$ is compact,

$$\Lambda(f(E \cap H(p,q)))^2 \leqslant 2pg'(y_0)\,\Lambda(I(y_0) \cap H(p,q)) \leqslant 2pg'(y_0) < \infty$$

by (6.10). Hence, by (6.9), $f(E)$ is of Σ-finite linear measure; that is, it is the countable union of sets of finite outer linear measure. Since $f(E)$ is itself a G_δ-Borel set, Lemma 2 of (**6**) implies that

$$\Lambda(f(E)) = \sup\{\Lambda(F')\colon F' \text{ compact}, F' \subseteq f(E)\}. \tag{6.15}$$

Now let F' be any compact subset of $f(E)$, and set $F = f^{-1}(F')$. Then F is compact and $F \subseteq E$. Hence $\Lambda(F) = 0$ and $\Lambda(F') = 0$, by what was proved above. Thus $\Lambda(f(E)) = 0$ by (6.15).

Finally, in the general case, we can find a G_δ-Borel set H such that $E \subseteq H \subseteq I(y_0)$ and $\Lambda(H) = \Lambda(E) = 0$. Then $\Lambda(f(E)) \leqslant \Lambda(f(H)) = 0$, and this completes the proof of Lemma 9.

With the help of these two lemmas, we can now complete the proof of the ACL property of f as follows. For each integer $p > 0$, set

$$H(p) = \bigcup_q H(p,q),$$

where the sum is taken over relevant q. Then condition (ii) implies that $m(R\backslash H(p)) = 0$ whenever $p > \csc(\pi/8K)$. Fix such a p. Then by Fubini's theorem,

$$\Lambda(I(y_0)\backslash H(p)) = 0 \tag{6.16}$$

for almost all y_0 in $0 \leqslant y \leqslant c$. Fix $0 < y_0 < c$ so that $g'(y_0)$ exists and is finite, and so that (6.16) holds, and let E be any compact set in $I(y_0)$. Then

$$E = (E \cap H(p)) \cup (E\backslash H(p)),$$

where $\Lambda(E\backslash H(p)) = 0$ by (6.16). Hence, by Lemmas 8 and 9,

$$\begin{aligned}\Lambda(f(E))^2 = \Lambda(f(E\cap H(p)))^2 &= \lim_{q\to\infty} \Lambda(f(E\cap H(p,q)))^2 \\ &\leqslant 2pg'(y_0) \lim_{q\to\infty} \Lambda(E\cap H(p,q)) = 2pg'(y_0)\,\Lambda(E),\end{aligned} \tag{6.17}$$

and it follows that $f(x+iy_0)$ is absolutely continuous in $0 \leqslant x \leqslant 1$. Since (6.17) holds for almost all y_0 in $0 \leqslant y \leqslant c$, f has the desired ACL property, and the proof of Theorem 4 is complete.

7. Conformal mappings

We conclude with a result on conformal mappings which shows how the sufficiency part of Theorem 4 could be established under weakened hypotheses.

We say that two segments γ_1 and γ_2 form an angle at z_0 *parallel* to an angle in a triangle $\triangle$ if there exists a mapping of the form $f(z) = az+b$, where $a > 0$, which maps z_0 onto a vertex of $\triangle$ and γ_1 and γ_2 into the corresponding sides of $\triangle$.

THEOREM 5. *Suppose that $\triangle$ is a fixed triangle and that f is a sense-preserving homeomorphism of a domain G which satisfies the following conditions:*

(i) *for all z_0 in $G\backslash E$, where E is of Σ-finite linear measure, and for all segments γ_1 and γ_2 which form an angle in G at z_0 parallel to an angle in $\triangle$,*

$$A(f(\gamma_1), f(\gamma_2)) > 0; \tag{7.1}$$

(ii) *for almost all z_0 in G, and for all segments γ_1 and γ_2 which form an angle in G at z_0 parallel to an angle in $\triangle$,*

$$A(f(\gamma_1), f(\gamma_2)) \geqslant A(\gamma_1, \gamma_2). \tag{7.2}$$

Then f is a conformal mapping.

Proof. We begin by showing that f is ACL in G under the assumption that the triangle $\triangle$ has its vertices at 0, 1, i. For this, let R, $I(y)$, γ_i $H(p,q)$, and $H(p)$ be as in the proof of Theorem 4, and set

$$H = \bigcup_p H(p).$$

Then Lemma 8 holds as before, and it is easy to verify that Lemma 9 is also valid, provided that $I(y_0)\backslash H$ is countable. Since the segments γ_1, γ_2 and γ_3, γ_4 form angles which are parallel to angles in $\triangle$, condition (i) implies that $R\backslash H$ is of Σ-finite linear measure, and condition (ii) implies that $m(R\backslash H(p)) = 0$ for $p > \csc(\pi/8)$. Fix such a p. Then by a theorem due to Gross ((**25**) 279), $I(y_0)\backslash H$ is countable for almost all y_0 in $0 \leqslant y \leqslant c$. Hence for almost all y_0 in $0 \leqslant y \leqslant c$, (6.16) holds by Fubini's theorem, $E \subseteq I(y_0)$ and $\Lambda(E) = 0$ imply that $\Lambda(f(E)) = 0$, by Lemma 9, and thus $f(x+iy_0)$ is absolutely continuous in $0 \leqslant x \leqslant 1$. Since $\triangle$ is symmetric in the line $y = x$, the same argument shows that $f(x_0+iy)$ is absolutely continuous in $0 \leqslant y \leqslant c$ for almost all x_0 in $0 \leqslant x \leqslant 1$, and we conclude that f is ACL in G.

We prove next that f is conformal under the assumption that f is ACL and $\triangle$ is an arbitrary triangle. For this it is sufficient to show that

$$\max_\theta |D_\theta f(z_0)|^2 = J(z_0) \tag{7.3}$$

at each point z_0 of G where (7.2) holds and where f is differentiable with $\max_\theta |D_\theta f(z_0)| > 0$. By performing preliminary similarity transformations, we may assume that $z_0 = f(z_0) = 0$ and that, near $z_0 = 0$,

$$f(z) = Dx+iy+o(|z|),$$

where $0 \leqslant D < \infty$. We must show that $D = 1$. If $D > 0$, then by Lemma 1 we may assume that $f(z) = Dx+iy$, and (7.2) implies $D = 1$ by elementary trigonometry. If $D = 0$, it is easy to see that there exist segments γ_1 and γ_2 which form an angle parallel to an angle in $\triangle$ and for which

$$A(f(\gamma_1),f(\gamma_2)) = 0.$$

This contradicts (7.2), and hence the proof for (7.3) is complete.

Finally, to complete the proof of Theorem 5, we must show that f is ACL under the assumption that $\triangle$ is an arbitrary triangle. For this, let g be an affine mapping with dilatation K which carries the vertices of $\triangle$ onto the points 0, 1, i, and set $h = f\circ g^{-1}$. Then h satisfies conditions (i) and (ii) with $\triangle$, G, E replaced by $g(\triangle)$, $g(G)$, $g(E)$, and $A(\gamma_1,\gamma_2)$ by $(1/K)A(\gamma_1,\gamma_2)$ in (7.2). Since $g(E)$ is of Σ-finite linear measure, and since $g(\triangle)$ has its vertices at 0, 1, i, a slight modification of the above arguments shows that h is quasiconformal in $g(G)$ with maximal dilatation dependent upon K. Hence $f = h \circ g$ is quasiconformal and, *a fortiori*, ACL in G.

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