APPROXIMATION ON WILD JORDAN CURVES

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1. *Introduction*

Throughout this paper, the letter γ denotes a (positively oriented) Jordan curve in the z-plane which contains the origin in its interior. By a well-known result of Walsh [7, 8], all continuous functions on such a curve γ are uniform limits of polynomials in z and $1/z$. In other words, the integral powers z^n, $n = 0, \pm 1, \ldots$ form a spanning set for the Banach space $C(\gamma)$.

One may ask if *all* powers z^n are required for such a spanning set. If γ has finite length L, the answer is yes: by Cauchy's theorem,

$$|2\pi i| = \left| \int_\gamma \left(z^{-1} - \sum_{n \neq -1} c_n z^n \right) dz \right| \leqslant \left\| z^{-1} - \sum_{n \neq -1} c_n z^n \right\| \cdot L$$

for every finite sum $\sum_{n \neq -1} c_n z^n$, and hence the power z^{-1} has positive distance (actually equal to $2\pi/L$, *cf.* [2]) to the closed span of the other powers z^n. The same is true for every integral power z^s.

In 1957, Wermer [9] observed that for every Jordan curve γ of infinite length, (at least) one power of z is superfluous (the powers z^n, $n \neq s$, form a spanning set for $C(\gamma)$). A different proof and an extension of this result have been given by Pia Pfluger and the first author [2]. They constructed curves for which precisely $p(\geqslant 1)$ powers are superfluous, as well as a curve for which any finite set of powers can be omitted. In the present note, we obtain simple Jordan curves γ^* with the following property. *For every* (increasing) *sequence of positive integers* $\{p_k\}$ *of positive density, the set of powers*

$$\{z^n, n = \ldots, -p_2, -p_1, 0, 1, 2, \ldots\} \tag{1.1}$$

spans the space $C(\gamma^*)$. We observe further that for any such curve, $C(\gamma^*)$ must even have spanning sets (1.1) corresponding to certain sequences $\{p_k\}$ of density zero.

One is thus led to ask: Could there be a curve γ such that (1.1) is a spanning set for $C(\gamma)$, no matter how the (infinite) sequence $\{p_k\}$ is selected?

2. *Measures orthogonal to* (1.1) *as boundary values of holomorphic differentials*

Let D be the interior of our curve γ in the z-plane, Δ the open unit disc in the w-plane. We let $w = \Phi(z)$ be the 1–1 conformal map of D onto Δ, normalized so that $\Phi(0) = 0$, $\Phi'(0) > 0$. By the theorem of Carathéodory and Osgood, this map has a continuous 1–1 extension to $\bar{D}$ (onto $\bar{\Delta}$), which we also call Φ. The inverse map will be called $z = \Psi(w)$. For $0 < r \leqslant 1$, we denote the positively oriented circle $|w| = r$ by Γ_r, its image $\Psi(\Gamma_r)$ in the z-plane by γ_r (so that $\gamma_1 = \gamma$).

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LEMMA. *Let μ be a complex Borel measure on γ orthogonal to the powers z^n, $n \geqslant 0$ and z^{-p_k}, $k = 1, 2, \ldots$, where the p_k's are distinct positive integers. Then there is a (unique) holomorphic function g on D with the following properties*:

(i) *μ is a weak boundary value of dg: for every Borel set $A \subseteq \gamma$, one has*

$$\mu(A) \left(= \int_A d\mu(z) \right) = \lim_{r \uparrow 1} \int_{A_r} dg(z), \tag{2.1}$$

where A_r is the set obtained by "pulling A back to γ_r": $A_r = \Psi\{r\Phi(A)\}$.

(ii) *Near $z = 0$, $g(z)$ has the form*

$$\sum b_j z^{q_j}, \tag{2.2}$$

where $\{q_j\}$ is the set of positive integers complementary to the set $\{p_k\}$ (the powers z^{-q_j} are "missing" in (1.1)).

COROLLARIES. (i) *One has*

$$\mathrm{Var}_A\, \mu = \int_A |d\mu(z)| = \lim_{r \uparrow 1} \int_{A_r} |g'(z)|\, |dz|. \tag{2.3}$$

(ii) *If γ' is a closed subarc of an open analytic (or merely rectifiable) arc in γ, and g' has a continuous extension to $D \cup \gamma'$, then*

$$\int_{\gamma'} |d\mu(z)| = \int_{\gamma'} |g'(z)|\, |dz|. \tag{2.4}$$

Proof of the lemma. By a theorem of Walsh [6, 8], all continuous functions f on $\bar{D}$ that are holomorphic on D are uniform limits on $\bar{D}$ (hence on γ) of polynomials. Thus, the condition

$$\int_\gamma z^n\, d\mu(z) = 0,\ n = 0, 1, 2, \ldots$$

implies in particular that

$$\int_\gamma \Phi(z)^k\, d\mu(z) = 0,\ k = 0, 1, 2, \ldots .$$

We now introduce the measure $\nu = \mu \circ \Psi$ on $\Gamma_1 = \Gamma$. Then

$$0 = \int_\Gamma w^k\, d\nu(w) = \int_0^{2\pi} e^{ikt}\, d\nu(e^{it}),\ k = 0, 1, 2, \ldots :$$

the measure $\nu(e^{it})$ has all its Fourier coefficients with non-positive index equal to zero. Thus, by the theorem of F. and M. Riesz [5, *cf.* 1, 4],

$$d\nu(w) = dh(w),\ |w| = 1,$$

with $h(e^{it})$ absolutely continuous and

$$h(w) = \sum_{1}^{\infty} a_n w^n \qquad \text{for} \qquad |w| \leqslant 1;$$

the derivative $h'(w)$ being equal to the Poisson integral of $h'(e^{it})$ on Δ, one has

$$\int_{0}^{2\pi} |h'(e^{it}) - h'(re^{it})| \, dt \to 0 \qquad \text{as} \qquad r \uparrow 1. \tag{2.5}$$

Going back to D, we introduce the holomorphic function $g = h \circ \Phi$, so that $dh = dg$ at corresponding points of Δ, D. Now let A be any Borel set on γ, and $B = \Phi(A)$. Then $A_r = \Psi(rB)$, and the boundary behaviour (2.5) of h' readily implies that

$$\mu(A) = \nu(B) = \int_B h'(w) \, dw = \lim_{r \uparrow 1} \int_{rB} h'(w) dw$$

$$= \lim_{r \uparrow 1} \int_{A_r} g'(z) \, dz,$$

which is (2.1).

Again using (2.5), the condition

$$0 = \int_\gamma z^{-p_k} d\mu(z) = \int_\Gamma \Psi(w)^{-p_k} h'(w) \, dw, \; k = 1, 2, \ldots$$

shows that

$$\int_{\Gamma_r} \Psi(w)^{-p_k} h'(w) \, dw = \int_\Gamma = 0, \quad 0 < r < 1.$$

Thus

$$\int_{\gamma_r} z^{-p_k} dg(z) = 0, \quad k = 1, 2, \ldots,$$

which establishes part (ii) of the lemma.

Comments on the corollaries. Relation (2.3) is a consequence of (2.1)—or can be proved by the same method. Now let $\tilde{\gamma}$ be an open analytic arc in γ. Then the mapping function Φ has a 1–1 holomorphic extension which maps a certain open set Ω containing $\tilde{\gamma}$ onto an open set $\Phi(\Omega)$ containing $\Phi(\tilde{\gamma})$; on $\Phi(\Omega)$, the inverse Ψ is also holomorphic. It follows that for every closed subarc $w = e^{it}$, $\alpha \leqslant t \leqslant \beta$ of $\Phi(\tilde{\gamma})$,

$$\int_{\alpha}^{\beta} |\Psi'(e^{it}) - r\Psi'(re^{it})| \, dt \to 0 \quad \text{as} \quad r \uparrow 1.$$

This relation (which can be proved also if $\tilde{\gamma}$ is only rectifiable—for example, by comparing Ψ with a suitable conformal map Ψ_1 of Δ onto a subdomain D_1 of D bounded by a rectifiable Jordan curve containing $\tilde{\gamma}$, *cf.* also [**4**; p. 158]) suffices to establish (2.4):

$$\left(\int_{\gamma'} |d\mu(z)| = \right) \lim_{r \uparrow 1} \int_{\gamma_r'} |g'(z)| \, |dz| = \int_{\gamma'} |g'(z)| \, |dz|.$$

3. *Conditions on* γ *under which* (1.1) *spans* $C(\gamma)$

Let γ be a Jordan curve around 0, and μ a measure on γ orthogonal to the powers (1.1). We wish to impose conditions which force μ, or the determining holomorphic function g on D, to be zero. Matters will be arranged so that γ behaves very badly in the vicinity of a point where g must be regular. This presents no problem at all if g is a polynomial $\sum b_j z^{q_j}$ (case where finitely many negative powers z^{-q_j} are omitted, *cf.* [**2**]).

To get beyond this case, we ask that γ behave badly where it comes closest to $z = 0$. It is simplest to require that γ *contain only one point closest to the origin, the point* $z = 1$, say. A simple sufficient condition for regularity of g at $z = 1$ is then given by evenness or oddness of g: the point $z = -1$ is inside D! Oddness of g is achieved by including all the even powers z^{-2k} in (1.1) (we may omit *all* the odd powers z^{-2k-1}). More generally, the condition that $\{p_k\}$ have *positive density* (defined as $\lim k/p_k$) will guarantee that $z = 1$ be a regular point of g. Indeed, the (increasing) sequence $\{q_j\}$ complementary to $\{p_k\}$ will then have density $d < 1$. Thus by a theorem of Fabry and Pólya [3], every arc of the circle of convergence of the power series (2.2) for $g(z)$, of angular measure $> 2\pi d$, will contain a singular point. It follows that this circle must have radius greater than 1 (otherwise, there would be at least two singular points on the unit circle).

We now specify some conditions of "bad behaviour" of γ near $z = 1$.

Concrete example. Let us start with a cardioid-type curve,

$$z = \phi_0(t) = (1+4\sin\tfrac{1}{2}t)\,e^{it}, \quad 0 \leqslant t \leqslant 2\pi,$$

and superimpose an "exponential wiggle" near $z = 1$ to define γ:

$$z = \phi(t) = \begin{cases} \phi_0(t), \ 1/\log\pi \leqslant t \leqslant 2\pi, \\ \phi_0(t) + (t\sin e^{1/t})\,e^{it}, \ 0 \leqslant t \leqslant 1/\log\pi. \end{cases}$$

Suppose now that $g \neq 0$. Then g' has a zero of finite multiplicity m (which may be zero!) at $z = 1$. Thus on a small neighbourhood N of $z = 1$, it satisfies an inequality of the form

$$|g'(z)| \geqslant c\,|z-1|^m, \ c > 0.$$

We choose an analytic subarc γ' of γ in N corresponding to, say $\delta \leqslant t \leqslant \varepsilon$, where $0 < \delta < \varepsilon < 1/\log\pi$. Then

$$\int_{\gamma'} |g'(z)|\,|dz| \geqslant \int_{\delta}^{\varepsilon} c\,|\phi(t)-1|^m\,|\phi'(t)|\,dt \geqslant c\int_{\delta}^{\varepsilon} t^{m-1}\,e^{1/t}\,|\cos e^{1/t}|\,dt - O(1).$$

However, the right-hand side tends to ∞ as $\delta \downarrow 0$, contradicting the boundedness of (2.4).

An *alternative condition* of bad behaviour on γ is that there should be a neighbourhood N of $z = 1$ in which every subarc of γ has infinite length. If $g \neq 0$, the function g' would be regular and its modulus would have a positive lower bound on some such subarc. One would then appeal to (2.3) to obtain a contradiction.

4. *A spanning set* (1.1) *where* $\{p_k\}$ *has density zero*

Let γ^* be any Jordan curve around 0 with the property that (1.1) is a spanning set for $C(\gamma^*)$ whenever the sequence $\{p_k\}$ has positive density. We will show that for such a curve γ^*, there is also a spanning set (1.1) with $\{p_k\}$ of density zero.

Our final sequence $\{p_k\}$ will be of the following type. For integers

$$0 \leqslant k_1 \leqslant k_2 \leqslant \ldots \leqslant k_r \leqslant \ldots$$

which will be specified below, we define

$$\left.\begin{aligned} &p_k = k,\ 0 < k \leqslant k_1;\ p_k = 2k,\ k_1 < k \leqslant k_2; \ldots; \\ &p_k = rk,\ k_{r-1} < k \leqslant k_r; \ldots . \end{aligned}\right\} \tag{4.1}$$

To define the numbers k_r, it will be convenient to give a name, S_r say, to the closed span in $C(\gamma^*)$ of the powers

$$z^n;\ n \geqslant 0 \qquad \text{and} \qquad z^{-p_k},\ 1 \leqslant k \leqslant k_r.$$

For k_1, we choose the smallest integer $\geqslant 0$ such that z^{-1} has distance to S_1 not exceeding 1. For k_2, we next take the smallest integer $\geqslant k_1$ such that z^{-1} and z^{-2} have distance to S_2 not exceeding 1/2. Assuming that $k_1, \ldots, k_{r-1}$ have been determined, we take for k_r the smallest integer $\geqslant k_{r-1}$ such that $z^{-1}, z^{-2}, \ldots, z^{-r}$ all have distance to S_r not exceeding $1/r$. That k_r exists follows from the fact that $z^{-1}, \ldots, z^{-r}$ belong to the closed span of the powers

$$z^n,\ n \geqslant 0;\ z^{-p_k},\ 1 \leqslant k \leqslant k_{r-1};\ z^{-rk},\ k_{r-1} < k < \infty.$$

(Indeed, for this set, the opposites of the negative exponents have density $1/r$.)

The above construction leads to a set (1.1) *whose closed span* S_∞ *is all of* $C(\gamma^*)$, *while* $\{p_k\}$ *has density zero.* Indeed, the distance between an arbitrary negative power z^{-q} and S_∞ (which contains all S_r) is $\leqslant 1/r$ for all $r \geqslant q$, and hence $z^{-q} \in S_\infty$. Also, it is clear that a sequence $\{p_k\}$ of the form (4.1) must have density zero, whether k_r tends to ∞ or not.

It is easy to see, incidentally, that k_r must tend to ∞. For suppose not. Then our present set (1.1) would contain only a finite number of negative powers. However, if no power in (1.1) would have exponent $\leqslant -s$ (s integral, $\geqslant 0$), then z^{-s-1} could not be in the closed span on γ^*, or z^{-1} would be a uniform limit of polynomials on γ^*, and therefore, also on the interior D^*!

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