SUCCESSIVE COEFFICIENTS OF STARLIKE FUNCTIONS

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Let S denote the class of functions

$$f(z) = z + a_2 z^2 + \dots$$

which are analytic and univalent in the unit disk $|z| < 1$. Also, let S^* denote the subclass of S consisting of functions which map the unit disk onto domains which are starlike with respect to the origin. The purpose of this paper is to prove a conjecture made by Pommerenke [3, Problem 3.5]; namely, $\big| |a_{n+1}| - |a_n| \big| \leqslant 1$ for all $f \in S^*$.

THEOREM. *For every $f \in S^*$,*

$$\big| |a_{n+1}| - |a_n| \big| \leqslant 1, \qquad n = 1, 2, 3, \dots.$$

Equality occurs for fixed n only for the functions

$$\frac{z}{(1-\gamma z)(1-\zeta z)}$$

for some γ and ζ with $|\gamma| = |\zeta| = 1$.

COROLLARY 1. *For every $f \in S^*$,*

$$|a_n| \leqslant n, \qquad n = 2, 3, \dots.$$

COROLLARY 2. *For every odd function $f \in S^*$,*

$$|a_{2n+1}| \leqslant 1, \qquad n = 1, 2, \dots.$$

The first corollary is well known and the second is due to Privalov [5].

The proof of the theorem depends on a lemma due to MacGregor [2], concerning the class P of functions

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

analytic and of positive real part in $|z| < 1$.

LEMMA. Let $p \in P$, *and let $\lambda_n \geqslant 0$. If $q(z) = \sum_{n=1}^{\infty} \lambda_n p_n z^n$ is analytic in $|z| < 1$ and $\mathrm{Re}\,\{q(z)\} \leqslant M$ for some positive M, then $\sum_{n=1}^{\infty} \lambda_n |p_n|^2 \leqslant 2M$.*

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Proof of the lemma. Let $p_n = c_n + id_n$ and

$$u(r,\theta) = \operatorname{Re}\{p(re^{i\theta})\} = 1 + \sum_{n=1}^{\infty} (c_n \cos n\theta - d_n \sin n\theta) r^n;$$

$$v(r,\theta) = \operatorname{Re}\{q(re^{i\theta})\} = \sum_{n=1}^{\infty} \lambda_n (c_n \cos n\theta - d_n \sin n\theta) r^n.$$

$$\int_0^{2\pi} u(r,\theta)\, v(r,\theta)\, d\theta = \pi \sum_{n=1}^{\infty} \lambda_n (c_n^2 + d_n^2) r^{2n} = \pi \sum_{n=1}^{\infty} \lambda_n |p_n|^2 r^{2n}.$$

Since $u \geqslant 0$ and $v \leqslant M$,

$$\int_0^{2\pi} u(r,\theta)\, v(r,\theta)\, d\theta \leqslant M \int_0^{2\pi} u(r,\theta)\, d\theta = 2M.$$

Thus $\sum_{n=1}^{\infty} \lambda_n |p_n|^2 r^{2n} \leqslant 2M$. Letting $r \to 1$, we have the lemma.

COROLLARY. *For every $p \in P$ and every positive integer n, there is a ζ with $|\zeta| = 1$ such that*

$$\sum_{k=1}^{n} \frac{1}{k} |p_k - \zeta^k|^2 \leqslant \sum_{k=1}^{n} \frac{1}{k}.$$

Proof. Apply the lemma with

$$q(z) = \sum_{k=1}^{n} \frac{1}{k} p_k z^k.$$

This gives

$$\sum_{k=1}^{n} \frac{1}{k} |p_k - \zeta^k|^2 = \sum_{k=1}^{n} \frac{1}{k} |p_k|^2 - 2 \operatorname{Re}\{q(\zeta)\} + \sum_{k=1}^{n} \frac{1}{k}$$

$$\leqslant 2M - 2 \operatorname{Re}\{q(\zeta)\} + \sum_{k=1}^{n} \frac{1}{k}.$$

Choosing ζ with $|\zeta| = 1$ so that $\operatorname{Re}\{q(\zeta)\} = M = \operatorname{Max} \operatorname{Re}\{q(z)\}$, we obtain the result.

Proof of Theorem. If $f \in S^*$, then it is well known that

$$\frac{zf'(z)}{f(z)} = p(z)$$

for some $p \in P$. Simple integration gives

$$\log \frac{f(z)}{z} = \int_0^z \frac{p(t) - 1}{t}\, dt = \sum_{k=1}^{\infty} \frac{1}{k} p_k z^k.$$

For $|\zeta| = 1$,

$$\log\left\{(1-\zeta z)\frac{f(z)}{z}\right\} = \sum_{k=1}^{\infty} \alpha_k z^k,$$

where

$$\alpha_k = \frac{1}{k}(p_k - \zeta^k).$$

On the other hand,

$$(1-\zeta z)\frac{f(z)}{z} = \sum_{k=0}^{\infty} \beta_k z^k,$$

where

$$\beta_k = a_{k+1} - \zeta a_k.$$

Because

$$\sum_{k=0}^{\infty} \beta_k z^k = \exp\left\{\sum_{k=1}^{\infty} \alpha_k z^k\right\},$$

we may apply one of the Lebedev–Milin inequalities (see [4], Lemma 3.4) to get

$$|\beta_n|^2 \leqslant \exp\left\{\sum_{k=1}^{n} k|\alpha_k|^2 - \sum_{k=1}^{n} \frac{1}{k}\right\}, \tag{1}$$

with equality only for $\alpha_k = (\gamma^k/k)$, $k = 1, \ldots, n$, with $|\gamma| = 1$. In other words, we have

$$|a_{n+1} - \zeta a_n|^2 \leqslant \exp\left\{\sum_{k=1}^{n} \frac{1}{k}|p_k - \zeta^k|^2 - \sum_{k=1}^{n} \frac{1}{k}\right\}. \tag{2}$$

By the corollary to the lemma, we can pick some ζ with $|\zeta| = 1$ to make the exponent nonpositive. Hence $|a_{n+1} - \zeta a_n| \leqslant 1$. Because $\big|\,|a_{n+1}| - |a_n|\,\big| \leqslant |a_{n+1} - \zeta a_n|$ for all $|\zeta| = 1$, this completes the proof of the inequality.

If $\big|\,|a_{n+1}| - |a_n|\,\big| = 1$, then equality occurs in (2) for some ζ. Hence we have

$$\sum_{k=0}^{\infty} \beta_k z^k = \exp\left\{\sum_{k=1}^{n} \frac{\gamma^k}{k} z^k + O(z^{n+1})\right\}.$$

Comparing the Taylor coefficients of both sides, we get

$$\beta_k = \gamma^k \quad \text{for} \quad k = 1, 2 \ldots, n,$$

or

$$a_{k+1} = \zeta a_k + \gamma^k, \qquad k = 1, 2, \ldots, n.$$

A simple inductive argument gives

$$a_{k+1} = \frac{\gamma^{k+1} - \zeta^{k+1}}{\gamma - \zeta}, \qquad k = 1, 2, \ldots, n.$$

Hence

$$|a_{k+1}| = \left|\frac{1 - e^{i(k+1)\theta}}{1 - e^{i\theta}}\right|, \qquad e^{i\theta} = \zeta/\gamma.$$

Thus the assumption that $\big|\,|a_{n+1}|-|a_n|\,\big| = 1$ leads to

$$\big|\,|1-e^{i(n+1)\theta}|-|1-e^{in\theta}|\,\big| = |1-e^{i\theta}|.$$

In general, however,

$$\big|\,|1-e^{i(n+1)\theta}|-|1-e^{in\theta}|\,\big| \leqslant |e^{i(n+1)\theta}-e^{in\theta}| = |1-e^{i\theta}|,$$

with equality occurring only when the two vectors $1-e^{i(n+1)\theta}$ and $1-e^{in\theta}$ are collinear. Hence we have three cases to consider: (1) $\theta = 0$; (2) $1-e^{in\theta} = 0$; (3) $1-e^{i(n+1)\theta} = 0$. In the first case, when $\theta = 0$, we get $\zeta = \gamma$ and $|a_2| = |\zeta+\gamma| = 2$, therefore, $f(z)$ must be a rotation of the Koebe function $z/(1-z)^2$. In the second case, when $1-e^{in\theta} = 0$, we get $a_n = 0$, $|a_{n+1}| = 1$. Let V_{k+1}^* be the $2k$-dimensional region composed of all points $(a_2, a_3, \ldots, a_{k+1})$ with $z+\sum_{n=2}^{\infty} a_n z^n$ belonging to S^*. The second case tells us that $(a_2, \ldots, 0, a_{n+1})$ must be on the boundary of V_{n+1}^*. Otherwise we would have $(\rho a_2, \ldots, 0, \rho a_{n+1}) \in V_{n+1}^*$ for some $\rho > 1$. This is impossible because we would get $\big|\,|\rho a_n|-|\rho a_{n+1}|\,\big| = \rho > 1$. Hence by a theorem of Hummel [**1**, Theorem 1||, we get a unique $(a_2, \ldots, 0, a_{n+1}, a_{n+2})$ belonging to V_{n+2}^* with

$$a_{n+2} = \frac{\gamma^{n+2}-\zeta^{n+2}}{\gamma-\zeta}.$$

We can repeat the above argument to show that $(a_2, \ldots, 0, a_{n+1}, a_{n+2})$ is on the boundary of V_{n+2}^* and obtain, in general,

$$a_{n+j} = \frac{\gamma^{n+j}-\zeta^{n+j}}{\gamma-\zeta} \quad \text{for all positive } j.$$

Thus, our function in this case has the form $z/((1-\gamma z)(1-\zeta z))$. Similarly, we get the same result for the third case. This completes the proof of the theorem.

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References

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