UNIFORM APPROXIMATION BY HOLOMORPHIC AND HARMONIC FUNCTIONS

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ABSTRACT

Let K be a compact set in the plane. It is shown that if L is a peak set for $A(K)$, then $A(K)\,|\,L = A(L)$. It is also shown that if E is a compact subset of K with no interior such that each component of $\mathbb{C}\setminus E$ intersects $\mathbb{C}\setminus K$, then $A(K)\,|\,E$ is dense in $C(E)$. One consequence of the latter result is a characterization of the real-valued continuous functions that when adjoined to $A(K)$ generate $C(K)$.

Introduction

This paper concerns certain problems involving uniform algebras on sets in the plane. Throughout the paper, K will denote an arbitrary compact set in the plane, except where indicated otherwise. Let $A(K)$ denote the algebra of functions in $C(K)$ that are holomorphic on the interior of K, and let $R(K)$ denote the algebra of functions in $C(K)$ that can be approximated uniformly on K by rational functions with poles off K.

If A is a uniform algebra on a compact Hausdorff space X, a subset Y of X is said to be a *peak set* for A if there is a function $f \in A$ such that $f(y) = 1$ for $y \in Y$, and $|f(x)| < 1$ for $x \in X\setminus Y$. In this situation, the function f is said to *peak on* Y. A point $x \in X$ is said to be a *peak point* for A if $\{x\}$ is a peak set for A. It follows easily from Runge's theorem that if K is a compact set in the plane and L is a peak set for $R(K)$, then $R(K)\,|\,L = R(L)$. (The proof can be found in [**3**, pp. 164–165].) The main result of Section 1 is the analogous assertion for $A(K)$. Several corollaries concerning $A(K)$ are then obtained.

Wermer's maximality theorem [**6**, Theorem II.5.1] states that the disc algebra on the circle is a maximal subalgebra of $C(\partial D)$. (Here D denotes the open unit disc.) In contrast, the algebra $A(\bar{D})$ (the disc algebra on the disc) is not a maximal subalgebra of $C(\bar{D})$. In other words, there is a function f in $C(\bar{D})$ but not in $A(\bar{D})$ such that the norm-closed subalgebra $A(\bar{D})[f]$ of $C(\bar{D})$ generated by $A(\bar{D})$ and f is not equal to $C(\bar{D})$. For example, take f to be zero on a nonempty proper open subset of the disk. There is no known characterization of the functions f in $C(\bar{D})$ such that $A(\bar{D})[f] = C(\bar{D})$. A theorem of John Wermer [**9**] shows that when f is continuously differentiable on a neighborhood of $\bar{D}$, then $A(\bar{D})[f] = C(\bar{D})$ if and only if the graph of f is polynomially convex in $\mathbb{C}^2$ and $R(E) = C(E)$, where E is the zero set of $\bar{\partial} f$. It was observed by E. M. Čirka [**4**] that Wermer's technique yields a more general result. Using that more general result, Čirka obtained the following theorem.

1 THEOREM. *Let K be a compact set in the plane and suppose that every point of ∂K is a peak point for $R(K)$. Let $f \in C(K)$ be harmonic on the interior of K, but nonholomorphic on each component of the interior of K. Then, the norm-closed subalgebra of $C(K)$ generated by $R(K)$ and f is equal to $C(K)$.*

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Unaware of the above theorem, Axler and Shields [1] proved the special case where f is real-valued (under a somewhat more restrictive condition on K) using entirely different methods from those employed by Čirka. More precisely, the case they proved was the following.

2 THEOREM. *Let K be a compact set in the plane and suppose that there is a positive number d such that each component of $\mathbb{C}\backslash K$ has diameter greater than d. Let $u \in C(K)$ be real-valued and harmonic on the interior of K, but nonconstant on each component of the interior of K. Then the norm-closed subalgebra of $C(K)$ generated by $A(K)$ and u is equal to $C(K)$.*

This result led Axler and Shields to raise the following two questions.

3 QUESTION. *Does the above theorem continue to hold if K is an arbitrary compact set in the plane, that is, if the hypothesis on the components of $\mathbb{C}\backslash K$ is dropped?*

4 QUESTION. *Which compact sets K in the plane have the property that $A(K)\,|\,E$ is dense in $C(E)$ whenever E is a compact subset of K with no interior such that each component of $\mathbb{C}\backslash E$ contains a component of $\mathbb{C}\backslash K$?*

At first glance, Question 4 appears unrelated to Theorem 2. However, Axler and Shields noted that their proof for Theorem 2 works whenever K has the property indicated in Question 4. It should be observed that the condition in Question 4 that each component of $\mathbb{C}\backslash E$ contains a component of $\mathbb{C}\backslash K$ is equivalent to the condition that each component of $\mathbb{C}\backslash E$ intersects $\mathbb{C}\backslash K$.

Section 2 is devoted primarily to answering the two questions above. First an affirmative answer to Question 3 is obtained. Next it is shown that all compact sets in the plane have the property indicated in Question 4. This yields a characterization of those compact subsets E of K for which $A(K)\,|\,E$ is dense in $C(E)$. Another consequence is a characterization of the real-valued continuous functions u for which the norm-closed subalgebra of $C(K)$ generated by $A(K)$ and u is equal to $C(K)$.

In Section 3 the methods of Sections 1 and 2 are used to give new proofs of some results of Christopher Bishop [2]. For Ω an open set in the Riemann sphere, let $H^\infty(\Omega)$ denote the algebra of bounded holomorphic functions on Ω. The main result of the section asserts that if $f \in H^\infty(\Omega)$ is nonconstant on each component of Ω, then the norm-closed subalgebra of $L^\infty(\Omega)$ generated by $H^\infty(\Omega)$ and $\bar{f}$ contains $C(\bar{\Omega})$.

The paper concludes with some open questions.

Throughout the paper, m will denote planar measure (that is, Lebesgue measure on the plane), and 'a.e.' will always mean almost everywhere *with respect to planar measure*. The word 'measure' will mean regular complex Borel measure. Given a set L in the plane, ∂L will denote the boundary of L, and $\operatorname{Int}(L)$ will denote the interior of L. If f is a complex-valued function, and X is a subset of its domain, then by definition $\|f\|_X = \sup_{x \in X} |f(x)|$.

I would like to dedicate this paper to Allen Shields, whose talk in the Functional Analysis Colloquium at Berkeley on November 10, 1986 first interested me in the problems considered here. Most of the material in this paper is from my doctoral dissertation [7], and it is a pleasure to take this opportunity to thank my thesis advisor, Donald Sarason, for his valuable guidance. In addition, I would like to thank Theodore Gamelin, Walter Rudin, and especially Sheldon Axler for helpful correspondence and encouragement. I would also like to thank Christopher Bishop for sharing his preprints with me.

1. *The restriction of* $A(K)$ *to a peak set*

The following theorem is the main result of this section.

1.1 THEOREM. *If K is a compact set in the plane and L is a peak set for $A(K)$, then $A(K) \mid L = A(L)$.*

Before proving Theorem 1.1, we need some preliminaries. For K a compact set in the Riemann sphere S^2 and U an open set in S^2 contained in K, let $A(K, U)$ denote the algebra of continuous functions on K that are holomorphic on U. The following lemma, which is a generalization of [5, Lemma 1.1], will be used throughout the paper.

1.2 LEMMA. *Suppose that K is a compact set in the Riemann sphere S^2, U is an open set in S^2 contained in K and such that $\infty \notin K \setminus U$, and N is a relatively closed subset of U that does not contain ∞ and has planar measure zero. Then, $C((K\setminus U) \cup N)$ is the closed linear span of $A(K, U)$ and the functions $z \mapsto (z-z_0)^{-1}$, $z_0 \in U\setminus(N \cup \{\infty\})$.*

Lemma 1.2 can be proven by the same argument used to prove [5, Lemma 1.1]. However, a more elementary proof will be given below.

If μ is a measure on the plane with compact support, then the *Cauchy transform* $\hat{\mu}$ of μ is defined by

$$\hat{\mu}(z) = \int \frac{d\mu(w)}{w-z}$$

for all z such that the integral converges absolutely.

The following lemma, which can be proven by applying Fubini's theorem, will be used in the proof of Lemma 1.2.

1.3 LEMMA. *Let B be a Borel set in the plane and let μ be a measure on the plane with compact support. Then, $\hat{\mu} = 0$ a.e. on B if and only if μ annihilates all functions of the form*

$$f(w) = \int \frac{h(z)}{z-w}\, dm(z)$$

where h is a bounded Borel function on the plane with compact support vanishing off B.

Proof of Lemma 1.2. Suppose that μ is a measure on $(K\backslash U)\cup N$ that annihilates $A(K, U)$ and the functions $z\mapsto(z-z_0)^{-1}$, $z_0\in U\backslash(N\cup\{\infty\})$. Then, obviously $\hat{\mu}(z_0)=0$ for every $z_0\in U\backslash(N\cup\{\infty\})$. If h is a bounded Borel function on the plane with compact support such that h is zero on U, then the function f defined on K by

$$f(w)=\int\frac{h(z)}{z-w}\,dm(z)$$

is in $A(K, U)$, and hence is annihilated by μ. Consequently, Lemma 1.3 shows that $\hat{\mu}=0$ a.e. on $\mathbb{C}\backslash U$. Since N has planar measure zero, we conclude that $\hat{\mu}=0$ a.e., so μ is the zero measure.

Lemma 1.2 will not be used in its full generality. For convenience we state here two special cases that will be used. For Ω an open set in the Riemann sphere, let $A(\Omega)$ denote the algebra of continuous functions on $\bar{\Omega}$ that are holomorphic on Ω.

1.4 COROLLARY. *If Ω is an open set in the Riemann sphere and $\infty\notin\partial\Omega$, then $C(\partial\Omega)$ is the closed linear span of $A(\Omega)$ and the functions $z\mapsto(z-z_0)^{-1}$, $z_0\in\Omega\backslash\{\infty\}$.*

1.5 COROLLARY. *If K is a compact set in the plane and N is a relatively closed subset of* Int(K) *having planar measure zero, then $C(\partial K\cup N)$ is the closed linear span of $A(K)$ and the functions $z\mapsto(z-z_0)^{-1}$, $z_0\in\operatorname{Int}(K)\backslash N$.*

The next lemma is an immediate consequence of the maximum modulus principle and the definition of peak set.

1.6 LEMMA. *If K and L are as in Theorem* 1.1, *then $\partial L\subset\partial K$.*

1.7 LEMMA. *If K is a compact set in the plane and L is a peak set for $A(K)$ (that is, the hypotheses of Theorem* 1.1 *hold), then each component of $\mathbb{C}\backslash L$ intersects $\mathbb{C}\backslash K$.*

Proof. Assume, to get a contradiction, that some component U of $\mathbb{C}\backslash L$ lies entirely in K. Since L is a closed subset of the plane, each component of $\mathbb{C}\backslash L$ is open. It follows that ∂U is contained in L. Now let f in $A(K)$ be a function that peaks on L. Then, $f|\bar{U}$ is in $A(\bar{U})$ and is 1 on ∂U. Consequently, by the maximum modulus principle, f is 1 on all of $\bar{U}$. Hence, $\bar{U}$ is contained in L, contradicting our assumption that U was a component of $\mathbb{C}\backslash L$.

1.8 LEMMA. *Let K and L be as in Theorem* 1.1, *and let $M=(\overline{K\backslash L})\cap L$. Then $A(\overline{K\backslash L})\,|\,M=C(M)$.*

Proof. Let f in $A(K)$ be a function that peaks on L. Then, $f|(\overline{K\backslash L})$ is in $A(\overline{K\backslash L})$ and peaks on $\overline{K\backslash L}\cap L=M$. Thus, M is a peak set for $A(\overline{K\backslash L})$. Therefore, $A(\overline{K\backslash L})\,|\,M$ is closed in $C(M)$ [8, Lemma 12.3]. In addition, Lemma 1.7 (with $\overline{K\backslash L}$ as K and M as L) shows that each component of the complement of M intersects the complement of $\overline{K\backslash L}$.

Note that M is contained in ∂L. Hence, M is contained in ∂K (by Lemma 1.6), and consequently, M is contained in $\partial(\overline{K\backslash L})$. Hence, by applying Corollary 1.5 (with $\overline{K\backslash L}$ as K and N empty), we see that the linear span of $A(\overline{K\backslash L})\,|\,M$ and the functions $z\mapsto(z-z_0)^{-1}$, $z_0\in\operatorname{Int}(\overline{K\backslash L})$, is dense in $C(M)$. Since (as noted in the preceding paragraph) each component of the complement of M intersects the complement of $\overline{K\backslash L}$, Runge's theorem shows that each of the functions $z \mapsto (z-z_0)^{-1}$, $z_0 \in \mathrm{Int}(\overline{K\backslash L})$, can be approximated uniformly on M by rational functions with poles off $\overline{K\backslash L}$. Thus, $A(\overline{K\backslash L})\,|\,M$ is dense in $C(M)$.

Having shown that $A(\overline{K\backslash L})\,|\,M$ is both closed and dense in $C(M)$, we conclude that $A(\overline{K\backslash L})\,|\,M$ equals $C(M)$.

Proof of Theorem 1.1. Since the inclusion $A(K)\,|\,L \subset A(L)$ is obvious, it is enough to consider the reverse inclusion.

Let f in $A(L)$ be arbitrary. By the preceding lemma, there exists a function g in $A(\overline{K\backslash L})$ that agrees with f on $(\overline{K\backslash L}) \cap L$. Hence, there is a well-defined function h on K given by

$$h(z) = \begin{cases} f(z) & \text{if } z \in L \\ g(z) & \text{if } z \in \overline{K\backslash L}. \end{cases}$$

Now h is continuous on K and holomorphic on $\mathrm{Int}(K)$ (since $\partial L \subset \partial K$ by Lemma 1.6). Thus, h is in $A(K)$ and clearly $h\,|\,L = f$. Hence, $A(L) \subset A(K)\,|\,L$.

The above proof shows that Theorem 1.1 follows readily from Lemma 1.8. On the other hand, it is easy to see that Lemma 1.8 is actually a special case of the theorem. The following special case of Theorem 1.1 should also be noted.

1.9 Corollary. *If E is a peak set for $A(K)$ contained in ∂K, then $A(K)\,|\,E = C(E)$.*

The above corollary can also be proven without Theorem 1.1, using essentially the same argument as that used to prove Lemma 1.8.

If A is a uniform algebra on a compact Hausdorff space X, a subset Y of X is said to be an *interpolation set* for A if $A\,|\,Y = C(Y)$. The set Y is said to be a *peak-interpolation set* for A if for each non-zero $f \in C(Y)$ there is an $F \in A$ such that $F\,|\,Y = f$ and $|F(x)| < \|f\|_Y$ for all $x \in X\backslash Y$. It can be shown that a set is a peak-interpolation set if and only if it is simultaneously a peak set and an interpolation set [**8**, Lemma 20.1]. Thus, Corollary 1.9 can be reformulated as follows.

1.9′ Corollary. *If E is a peak set for $A(K)$ contained in ∂K, then E is a peak-interpolation set for $A(K)$.*

The next result is an immediate consequence of Theorem 1.1 and its analogue for $R(K)$.

1.10 Corollary. *If K is a compact set in the plane for which $A(K)$ and $R(K)$ coincide, and L is a peak set for this common algebra, then $A(L)$ and $R(L)$ also coincide.*

We conclude this section with one more corollary of Theorem 1.1. (For the analogue for $R(K)$ see [**3**, p. 165] and [**8**, p. 309].) First we need some definitions. Suppose A is a uniform algebra on a compact Hausdorff space X. A subset Y of X is said to be a *set of antisymmetry* for A if every function in A that is real-valued on Y is in fact constant on Y. We say that A is *antisymmetric* if every real-valued function in A is constant. (Obviously these concepts can be defined for any subalgebra of $C(X)$, not just for uniform algebras. In Section 3 we will make use of the sets of antisymmetry for an algebra of functions that does not separate points and hence is not a uniform algebra.)

1.11 COROLLARY. *Let $\{E_\alpha\}$ be the family of maximal sets of antisymmetry for $A(K)$. Then, $A(K)\,|\,E_\alpha = A(E_\alpha)$ for every α, each E_α is connected, and E_α is a singleton for all but countably many α.*

Proof. Each E_α is a peak set [6, Theorem II.13.1 and Lemma II.12.1], so the first assertion is an immediate consequence of Theorem 1.1. Since $A(K)\,|\,E_\alpha$ is antisymmetric, the second assertion follows from the first, and so does the last assertion as E_α has empty interior for all but countably many α.

2. *Two questions raised by Sheldon Axler and Allen Shields*

For $u \in C(K)$, let $A(K)[u]$ denote the norm-closed subalgebra of $C(K)$ generated by $A(K)$ and u. This section is devoted primarily to proving the following two theorems which answer the questions of Axler and Shields mentioned in the introduction.

2.1 THEOREM. *Let K be a compact set in the plane. Let $u \in C(K)$ be real-valued and harmonic on the interior of K, but nonconstant on each component of the interior of K. Then $A(K)[u] = C(K)$.*

2.2 THEOREM. *If K is a compact set in the plane, and E is a compact subset of K with no interior such that each component of $\mathbb{C}\backslash E$ intersects $\mathbb{C}\backslash K$, then $A(K)\,|\,E$ is dense in $C(E)$.*

Since, as mentioned in the introduction, the proof of Theorem 2 given by Axler and Shields works whenever K has the property established in Theorem 2.2, we see that Theorem 2.1 follows from Theorem 2.2. However, Theorem 2.1 does not require the full strength of Theorem 2.2. To illustrate this, we first given an independent proof of Theorem 2.1. The overall strategy of the proof is the same as that of the proof of Theorem 2 given by Axler and Shields. We shall make use of the following elementary fact whose proof is omitted.

2.3 LEMMA. *Let K and u be as in Theorem 2.1, and let E be a level set of u. Then, $E \cap \operatorname{Int}(K)$ has planar measure zero.*

Proof of Theorem 2.1. Let E be a maximal set of antisymmetry for $A(K)[u]$. The closure of every antisymmetric set is antisymmetric, so E is compact. Moreover, since u is a real-valued function in $A(K)[u]$, clearly u must be constant on E, so Lemma 2.3 shows that $E \cap \operatorname{Int}(K)$ has planar measure zero.

We claim that each component of $\mathbb{C}\backslash E$ intersects $\mathbb{C}\backslash K$. To prove this, let F be a component of $\mathbb{C}\backslash E$ and assume, to get a contradiction, that F is contained in K. Then, u is continuous on $\bar{F}$ and harmonic on F. Moreover, ∂F is contained in E, so u is constant on ∂F. Consequently, u is constant on F, which contradicts the hypothesis that u is nonconstant on each component of $\operatorname{Int}(K)$. Thus, each component of $\mathbb{C}\backslash E$ intersects $\mathbb{C}\backslash K$.

Since $E \cap \operatorname{Int}(K)$ has planar measure zero, Corollary 1.5 shows that the linear span of $A(K)|E$ and the functions $z \mapsto (z-z_0)^{-1}$, $z_0 \in (\operatorname{Int}(K))\backslash E$, is dense in $C(E)$. Since each component of $\mathbb{C}\backslash E$ intersects $\mathbb{C}\backslash K$, Runge's theorem shows that each of the functions $z \mapsto (z-z_0)^{-1}$, $z_0 \in (\operatorname{Int}(K))\backslash E$ can be approximated uniformly on E by rational functions with poles off K. Thus, $A(K)|E$ is dense in $C(E)$. Hence, $A(K)[u]|E$ is certainly dense in $C(E)$. Since E is a maximal set of antisymmetry for $A(K)[u]$, we know [**8**, Theorem 12.1] that $A(K)[u]|E$ is closed in $C(E)$, and so $A(K)[u]|E$ equals $C(E)$. The Bishop antisymmetric decomposition [**8**, Theorem 12.1] now implies that $A(K)[u]$ equals $C(K)$.

In view of the theorem of Čirka stated in the introduction, it should be noted that the existence of compact sets K with no interior and with $R(K) \neq C(K)$ implies that Theorem 2.1 does not remain valid if $A(K)$ is replaced by $R(K)$.

We now give the proof of Theorem 2.2.

Proof of Theorem 2.2. Let f in $C(E)$ be arbitrary and fix $\varepsilon > 0$. By Corollary 1.5, the linear span of $A(K)$ and the functions $z \mapsto (z-z_0)^{-1}$, $z_0 \in \operatorname{Int}(K)$, is dense in $C(\partial K)$, and hence dense in $C(E \cap \partial K)$. Since E has empty interior, each component of $\mathbb{C}\backslash(E \cap \partial K)$ contains a component of $\mathbb{C}\backslash E$, and hence intersects $\mathbb{C}\backslash K$. Therefore, by Runge's theorem, each of the functions $z \mapsto (z-z_0)^{-1}$, $z_0 \in \operatorname{Int}(K)$, can be approximated uniformly on $E \cap \partial K$ by rational functions with poles off K. Consequently, $A(K)|(E \cap \partial K)$ is dense in $C(E \cap \partial K)$.

So choose g in $A(K)$ such that

$$\|f-g\|_{(E \cap \partial K)} < \frac{\varepsilon}{2}.$$

By the Tietze extension theorem, there is a continuous function ϕ on E that agrees with $f-g$ on a neighborhood of $E \cap \partial K$ in E, and satisfies

$$\|\phi\|_E \leqslant \frac{\varepsilon}{2}. \tag{1}$$

Now $f-g-\phi$ is a continuous function on E that vanishes on a neighborhood of $E \cap \partial K$. For each point z lying in E but not on the boundary of K, choose an open disc Δ_z containing z and contained in K. The hypothesis that each component of $\mathbb{C}\backslash E$ intersects $\mathbb{C}\backslash K$ implies that $E \cap \bar{\Delta}_z$ has connected complement. Hence, recalling that E has empty interior, Lavrentiev's theorem [**6**, Theorem II.8.7] shows that $R(E \cap \bar{\Delta}_z) = C(E \cap \bar{\Delta}_z)$. In particular, the restriction of $f-g-\phi$ to $E \cap \bar{\Delta}_z$ is in $R(E \cap \bar{\Delta}_z)$. Since $f-g-\phi$ vanishes on a neighborhood of $E \cap \partial K$ in E, the localization theorem for rational approximation [**6**, Corollary II.10.3] now shows that $f-g-\phi$ is in $R(E)$. Therefore, since each component of $\mathbb{C}\backslash E$ intersects $\mathbb{C}\backslash K$, Runge's theorem shows that $f-g-\phi$ can be approximated uniformly (on E) by rational functions with poles off K, and hence certainly by elements of $A(K)$. So choose h in $A(K)$ such that

$$\|(f-g-\phi)-h\|_E < \frac{\varepsilon}{2}. \tag{2}$$

Combining (1) and (2) yields

$$\|f-(g+h)\|_E \leqslant \|(f-g-\phi)-h\|_E+\|\phi\|_E<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Since g and h are both in $A(K)$, this establishes the theorem.

As mentioned earlier, Theorem 2.2 enables us to give another proof of Theorem 2.1. One shows that if E is a maximal set of antisymmetry for $A(K)[u]$, then E satisfies the hypotheses in Theorem 2.2, so $A(K)\,|\,E$ is dense in $C(E)$. Consequently, $A(K)[u]\,|\,E=C(E)$, and the conclusion of Theorem 2.1 follows from the Bishop antisymmetric decomposition. Moreover, using Theorem 2.2 we obtain the following characterization of the real-valued continuous functions on K that when adjoined to $A(K)$ generate all of $C(K)$.

2.4 COROLLARY. *Suppose that K is a compact set in the plane, and $u\in C(K)$ is real-valued. Then, $A(K)[u]=C(K)$ if and only if u is nonconstant on the boundary of each open set (of the plane) contained in K.*

Proof. The proof of the 'if' part is similar to the proof of Theorem 2.1 suggested above. To prove the 'only if' part, suppose u is constant on the boundary of an open set of the plane contained in K. Then there exists a connected open set U of the plane such that u is constant on ∂U. Since the maximum modulus principle implies that $A(\bar{U})\,|\,(\partial U)$ is closed in $C(\bar{U})$, it is easy to see that

$$A(K)[u]\,|\,(\partial U)\subset\overline{A(K)\,|\,(\partial U)}\subset A(\bar{U})\,|\,(\partial U).$$

Moreover, $A(\bar{U})\,|\,(\partial U)$ is not equal to $C(\partial U)$ since the connectedness of U implies that every real-valued function in $A(\bar{U})\,|\,(\partial U)$ is constant. Consequently, $A(K)[u]\neq C(K)$.

As Christopher Bishop* has pointed out, an alternative formulation of Corollary 2.4 is as follows.

2.4′ COROLLARY. *Assume that K and u are as in Corollary 2.4. Then, $A(K)[u]=C(K)$ if and only if each level set of u has empty interior and has the property that each component of its complement contains a component of the complement of K.*

The condition on the set E in Theorem 2.2 characterizes the compact subsets of K for which $A(K)\,|\,E$ is dense in $C(E)$. More precisely, we have the following corollary.

2.5 COROLLARY. *Suppose that K is a compact set in the plane, and E is a compact subset of K. Then, $A(K)\,|\,E$ is dense in $C(E)$ if and only if E has empty interior and each component of $\mathbb{C}\setminus E$ intersects $\mathbb{C}\setminus K$.*

Proof. The 'if' statement is Theorem 2.2. To prove the 'only if' statement, suppose some component U of $\mathbb{C}\setminus E$ does not intersect $\mathbb{C}\setminus K$. Then, $\bar{U}\subset K$ and $\partial U\subset E$. Now, $A(K)\,|\,(\partial U)\subset A(\bar{U})\,|\,(\partial U)$, and $A(\bar{U})\,|\,(\partial U)$ is closed in $C(\partial U)$ (by the

* Christopher Bishop should not be confused with Errett Bishop after whom the antisymmetric decomposition and splitting lemma are named.

maximum modulus principle) and is not equal to $C(\partial U)$ (since every real-valued function in $A(\bar{U})|(\partial U)$ is constant). Thus, $A(K)|(\partial U)$ is not dense in $C(\partial U)$, so $A(K)|E$ is not dense in $C(E)$.

In the proof of Theorem 2.2, we started with a function f in $C(E)$ and approximated it by elements of $A(K)|E$. It is also possible to prove the theorem using duality. We now illustrate this approach.

2.6 LEMMA. *Let K be a compact set in the plane, and L a compact subset of K such that each component of $\mathbb{C}\backslash L$ intersects $\mathbb{C}\backslash K$. Suppose μ is a measure on L that annihilates $A(K)$. Then, $\hat{\mu} = 0$ a.e. off* $\operatorname{Int}(L)$.

In the proof we will need the following lemma which is an easy consequence of Lemma 1.3.

2.7 LEMMA. *Suppose that K is a compact set in the plane, and μ is a measure on K that annihilates $A(K)$. Then, $\hat{\mu} = 0$ a.e. off* $\operatorname{Int}(K)$.

Proof of Lemma 2.6. By the above lemma, $\hat{\mu} = 0$ a.e. off $\operatorname{Int}(K)$. The Cauchy transform of a measure is holomorphic off the closed support of the measure, so $\hat{\mu}$ is holomorphic off L. In view of the hypothesis on the components of $\mathbb{C}\backslash L$, it follows that $\hat{\mu} = 0$ off L. Thus, it suffices to show that $\hat{\mu} = 0$ a.e. on $(\partial L) \cap \operatorname{Int}(K)$.

For each $z \in (\partial L) \cap \operatorname{Int}(K)$, choose an open disc Δ_z centered at z and contained in K. Then, $L \cap \bar{\Delta}_z$ has connected complement, so by Mergelyan's theorem [**6**, Theorem II.9.1] $R(L \cap \bar{\Delta}_z) = A(L \cap \bar{\Delta}_z)$. Let Δ'_z be the disc centered at z with radius half that of Δ_z.

Now fix $w \in (\partial L) \cap \operatorname{Int}(K)$, and let $U_1 = L \cap \Delta_w$, and $U_2 = L \cap (\mathbb{C}\backslash \bar{\Delta}'_w)$. Since $\hat{\mu} = 0$ off L, we have that $\mu \perp R(L)$ [**6**, Theorem II.8.1]. Hence, by the Bishop splitting lemma [**6**, Theorem II.10.2] there exist measures μ_1 and μ_2 such that $\mu = \mu_1 + \mu_2$, $\mu_j \perp R(\bar{U}_j)$, and the closed support of μ_j is contained in U_j $(j = 1, 2)$. Now, $\mu_1 \perp R(L \cap \bar{\Delta}_w) = A(L \cap \bar{\Delta}_w)$, so by Lemma 2.7, $\hat{\mu}_1 = 0$ a.e. off $\operatorname{Int}(L \cap \bar{\Delta}_w)$. In particular, $\hat{\mu}_1 = 0$ a.e. on ∂L. Moreover, $\hat{\mu}_2 = 0$ off $\bar{U}_2$ (since $\mu_2 \perp R(\bar{U}_2)$), so $\hat{\mu}_2 = 0$ on Δ'_w. We conclude that $\hat{\mu} = \hat{\mu}_1 + \hat{\mu}_2 = 0$ a.e. on $\partial L \cap \Delta'_w$. Since there exists a countable collection $\mathscr{Z}$ of points in $(\partial L) \cap \operatorname{Int}(K)$ such that the discs Δ'_z, $z \in \mathscr{Z}$, cover $(\partial L) \cap \operatorname{Int}(K)$, it follows that $\hat{\mu} = 0$ a.e. on $(\partial L) \cap \operatorname{Int}(K)$, and the proof is complete.

Theorem 2.2 is an immediate consequence of the lemma just proven, for if L has no interior, then the conclusion of the lemma is that $\hat{\mu} = 0$ a.e. whenever μ is a measure on L that annihilates $A(K)$. Thus, the zero functional is the only linear functional on $C(L)$ that annihilates $A(K)|L$, and hence $A(K)|L$ is dense in $C(L)$.

For K a compact set in the plane and M a Borel subset of K, let $R(K, M)$ denote the closed linear span in $C(K)$ of the functions of the form

$$f(w) = \int \frac{h(z)}{z - w} \, dm(z)$$

where h is a bounded Borel function on the plane with compact support such that $h = 0$ a.e. on M. Lemma 1.3 shows that a measure μ on K annihilates $R(K, M)$ if and only if $\hat{\mu} = 0$ a.e. off M. Now we can repeat the proof of Lemma 2.6 to show that the lemma continues to hold if we replace the hypothesis that μ annihilates $A(K)$ by the hypothesis that μ annihilates $R(K, \text{Int}(K))$. Hence, we obtain the following generalization of Theorem 2.2.

2.8 THEOREM. *With K and L as in Lemma* 2.6, $R(K, \text{Int}(K))\,|\,L$ *is dense in* $R(L, \text{Int}(L))$.

To see that this result is a generalization of Theorem 2.2, note that $R(K, \text{Int}(K)) \subset A(K)$, and that if L has no interior then $R(L, \text{Int}(L)) = C(L)$.

3. *Some results of Christopher Bishop*

For h a bounded harmonic function on an open set Ω in the Riemann sphere, let $H^\infty(\Omega)[h]$ denote the norm-closed subalgebra of $L^\infty(\Omega)$ generated by $H^\infty(\Omega)$ and h. The following theorem is due to Christopher Bishop [2].

3.1 THEOREM. *Suppose that* Ω *is an open set in the Riemann sphere and that* $f \in H^\infty(\Omega)$ *is nonconstant on each component of* Ω. *Then* $C(\bar{\Omega}) \subset H^\infty(\Omega)[\bar{f}\,]$.

We shall present a proof of this result using the methods of Sections 1 and 2. As in Bishop's original proof, the following lemma will be needed.

3.2 LEMMA. *If* Ω *is an open set in the Riemann sphere and* $g \in C(\bar{\Omega})$, *then g can be approximated uniformly by a continuous function on* $\bar{\Omega}$ *that is smooth on* Ω *and holomorphic on* $\Omega \cap U$, *for some neighborhood U of* $\partial\Omega$.

Bishop's proof of this lemma is rather long and involves the notion of continuous analytic capacity. We present here a proof that is both shorter and more elementary.

Proof. Assume without loss of generality that $\infty \notin \partial\Omega$. Fix $\varepsilon > 0$. By Corollary 1.4, there exists a function h in $A(\Omega)$, points $z_1, \ldots, z_n$ in Ω, and complex numbers $a_1, \ldots, a_n$ such that

$$|g(z) - (h(z) + a_1(z - z_1)^{-1} + \ldots + a_n(z - z_n)^{-1})| < \varepsilon \quad \forall z \in \partial\Omega. \tag{3}$$

Let V be a neighborhood of $\partial\Omega$ in the plane such that (3) holds with $V \cap \bar{\Omega}$ in place of $\partial\Omega$. Let $\{\phi, \psi, \tau\}$ be a smooth partition of unity on the Riemann sphere subordinate to the cover $\{V, \Omega, S^2 \backslash \bar{\Omega}\}$ with $\text{supp}\,\phi \subset V$, $\text{supp}\,\psi \subset \Omega$, and $\text{supp}\,\tau \subset S^2 \backslash \bar{\Omega}$. Let l be a smooth function on Ω such that

$$\sup_{z \in \Omega} |g(z) - l(z)| < \varepsilon.$$

Now the function

$$z \mapsto \phi(z)(h(z) + a_1(z - z_1)^{-1} + \ldots + a_n(z - z_n)^{-1}) + \psi(z)\, l(z)$$

is a continuous function on $\bar{\Omega}$ that is smooth on Ω and holomorphic on $\Omega \cap (S^2 \backslash \text{supp}\,\psi)$, and approximates g uniformly to within ε.

We are now ready to prove Theorem 3.1. Much of the proof given below is similar to Bishop's original proof. However, Bishop's proof involves a partition of unity argument which is avoided below by using the Stone–Čech compactification so as to make an application of the Bishop antisymmetric decomposition possible.

Proof of Theorem 3.1. Assume without loss of generality that $\infty \notin \Omega$. Let X denote the Stone–Čech compactification of Ω. We will identify each bounded continuous function on Ω (or $\bar{\Omega}$) with the function it induces on X, and thus regard $H^\infty(\Omega)[\bar{f}]$ and $C(\bar{\Omega})$ as closed subalgebras of $C(X)$. By the Bishop antisymmetric decomposition [8, Theorem 12.1], it suffices to show that $H^\infty(\Omega)[\bar{f}]\,|\,E$ is dense in $C(\bar{\Omega})\,|\,E$, for each maximal set of antisymmetry E for $H^\infty(\Omega)[\bar{f}]$. Since both f and $\bar{f}$ are in $H^\infty(\Omega)[\bar{f}]$, the real and imaginary parts of f are in $H^\infty(\Omega)[\bar{f}]$. Hence, each maximal set of antisymmetry for $H^\infty(\Omega)[\bar{f}]$ is contained in a level set of f. Consequently, it suffices to show that if g is in $C(\bar{\Omega})$ and λ is in the image of f, then g can be approximated uniformly on the set $\{f=\lambda\}$ by elements of $H^\infty(\Omega)$.

So fix g in $C(\bar{\Omega})$ and λ in the image of f. By Lemma 3.2, we may assume without loss of generality that g is smooth on Ω and holomorphic on $\Omega \cap U$, for some neighborhood U of $\partial\Omega$. Then, the set $\{f=\lambda\} \cap \operatorname{supp}(\bar{\partial} g)$ is finite, so by modifying g to be constant in a neighborhood of each point of this set, we may assume that the support of $\bar{\partial} g$ is in fact disjoint from the set $\{f=\lambda\}$. Now

$$\frac{\bar{\partial} g}{f-\lambda}$$

is a smooth function on Ω with compact support. Let h be its Cauchy transform divided by $-\pi$. Explicitly,

$$h(z) = -\frac{1}{\pi}\int \frac{\bar{\partial} g(w)}{(f(w)-\lambda)(w-z)}\,dm(w).$$

Now h is a smooth function on the complex plane, and

$$\bar{\partial} h = \frac{\bar{\partial} g}{f-\lambda}$$

on Ω. Therefore, $g-h(f-\lambda)$ is a bounded holomorphic function on Ω. Thus, $g-h(f-\lambda)$ is an element of $H^\infty(\Omega)$ that obviously agrees with g on the set $\{f=\lambda\}$. This concludes the proof.

It is very easy to see that adjoining to $H^\infty(\Omega)$ the complex conjugate of a function in $H^\infty(\Omega)$ is equivalent to adjoining a real-valued bounded harmonic function on Ω having a single-valued bounded harmonic conjugate. Thus, in Theorem 3.1 we can replace the condition $f \in H^\infty(\Omega)$ by the condition that f be a real-valued bounded harmonic function on Ω having a single-valued bounded harmonic conjugate. In fact, as noted in [2], it is not necessary to assume that the conjugate is bounded. Explicitly we have the following result.

3.3 COROLLARY. *Suppose that Ω is an open set in the Riemann sphere and that u is a real-valued bounded harmonic function on Ω that is nonconstant on each component of Ω and has a single-valued harmonic conjugate. Then,* $C(\bar{\Omega}) \subset H^\infty(\Omega)[u]$.

Proof. Let u^* denote the harmonic conjugate of u, and let $F = \exp(u + iu^*)$. Then, $F \in H^\infty(\Omega)$, and Theorem 3.1 shows that $C(\bar{\Omega}) \subset H^\infty(\Omega)[\bar{F}]$. Since $\bar{F} = \exp(-(u + iu^*))\exp(2u)$, we see that $\bar{F} \in H^\infty(\Omega)[u]$, and the conclusion of the theorem follows.

The following result (also due to Bishop) can be proved by making minor changes in the proof of Theorem 3.1 given above.

3.4 THEOREM. *Suppose that Ω is an open set in the Riemann sphere and that $f \in A(\Omega)$ is nonconstant on each component of Ω. Then, $C(\bar{\Omega}) = A(\Omega)[\bar{f}]$.*

The proof of this theorem is actually easier than the argument given above for Theorem 3.1, as the Stone–Čech compactification is not needed since all the functions involved are continuous on the compact space $\bar{\Omega}$.

The reader has undoubtedly noticed the similarity between Theorems 3.4 and 2.1. In view of this similarity, one might hope to obtain an analogue of Theorem 2.1 for $A(\Omega)$ (Ω an open set in the Riemann sphere). More precisely, one might hope that if u is a real-valued continuous function on $\bar{\Omega}$ that is harmonic on Ω and nonconstant on each component of Ω, then $C(\bar{\Omega}) = A(\Omega)[u]$. However, Bishop [2] has given an example showing that this is not the case.

4. *Open questions*

4.1. Does Theorem 2.1 continue to hold if the hypothesis that u be real-valued is dropped? More precisely, if K is a compact set in the plane and f is a complex-valued continuous function on K that is harmonic on the interior of K but nonholomorphic on each component of the interior of K, does $A(K)[\bar{f}] = C(K)$? As noted in the introduction, it was shown by Čirka [4] that the answer is affirmative in the special case where each point of ∂K is a peak point for $R(K)$. For the specific case where K is the closed disc, another proof was given by Axler and Shields [1]. Some other special cases (for example, the case when $A(K)$ is Dirichlet) have been considered by Bishop [2].

4.2. If K is a compact set in the plane, and L is a compact subset of K such that each component of $\mathbb{C} \backslash L$ intersects $\mathbb{C} \backslash K$, is $A(K) \mid L$ dense in $A(L)$? An affirmative answer would obviously generalize Theorem 2.2 and would easily yield a generalization of Corollary 2.5. In view of Lemma 1.7 (and [8, Lemma 12.3]), it would also generalize Theorem 1.1.

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