ON THE DISTRIBUTION OF GENERATING FUNCTIONS

R. C. VAUGHAN AND T. D. WOOLEY

1. *Introduction*

Investigations concerning the generating function associated with the kth powers,

$$f_P(\alpha) = \sum_{1 \leqslant n \leqslant P} e(\alpha n^k), \tag{1.1}$$

originate with Hardy and Littlewood in their famous series of papers in the 1920s, 'On some problems of "Partitio Numerorum"' (see [**7**, Chapters 2 and 4]). Classical analyses of this and similar functions show that when P is large the function approaches P in size only for α in a subset of $(0, 1)$ having small measure. Moreover, although it has never been proven, there is some expectation that for 'most' α, the generating function is about $\sqrt{P}$ in magnitude. The main evidence in favour of this expectation comes from mean value estimates of the form

$$\int_0^1 |f_P(\alpha)|^s \, d\alpha \sim \Gamma(\tfrac{1}{2}s+1)\, P^{s/2}. \tag{1.2}$$

An asymptotic formula of the shape (1.2), with strong error term, is immediate from Parseval's identity when $s = 2$, and follows easily when $s = 4$ and $k > 2$ from the work of Hooley [**2**, **3**, **4**], Greaves [**1**], Skinner and Wooley [**5**] and Wooley [**9**]. On the other hand, (1.2) is false when $s > 2k$ (see [**7**, Exercise 2.4]), and when $s = 4$ and $k = 2$. However, it is believed that when $t < k$, the total number of solutions of the diophantine equation

$$x_1^k + \cdots + x_t^k = y_1^k + \cdots + y_t^k, \tag{1.3}$$

with $1 \leqslant x_j, y_j \leqslant P$ $(1 \leqslant j \leqslant t)$, is dominated by the number of solutions in which the x_i are merely a permutation of the y_j, and the truth of such a belief would imply that (1.2) holds for even integers s with $0 \leqslant s < 2k$.

The purpose of this paper is to investigate the extent to which knowledge of the kind (1.2) for an initial segment of even integer exponents s can be used to establish information concerning the general distribution of $f_P(\alpha)$, and the behaviour of the moments in (1.2) for general real s. Of particular interest is the case $s = 1$ because, although it seems hard to prove (1.2) in that case, it is relatively easy to carry out computations, and these are in strikingly close agreement with (1.2). We remark that D. Covert, at the University of Michigan, has very recently performed extensive computations in the cases $k = 2$ and $k = 3$, which provide compelling evidence in favour of the conjectured asymptotic formula (1.2) when $0 \leqslant s < 2k$.

It is conceivable that the behaviour (1.2) might even persist into the region $s \geqslant 2k$ provided that one excludes the 'peaks' of the generating functions $f_P(\alpha)$ in the

---

Received 16 December 1996.

1991 *Mathematics Subject Classification* 11L15.

First author supported by an EPSRC Senior Fellowship. Second author is a Packard Fellow, supported in part by NSF grant DMS-9622773.

neighbourhoods of rational points having relatively small denominators. Let $\eta(P)$ denote any positive decreasing function with $\eta(P) \to 0$ as $P \to \infty$, and let $\mathfrak{m}_P$ denote the set of α in $(0, 1]$ with the property that for every pair a, q with $(a, q) = 1$ and $|q\alpha - a| \leqslant \eta(P) P^{-k/2}$ one has $q > \eta(P) P^{k/2}$. This is just about the thinnest set of 'minor arcs' that is likely to be useful, yet we observe that the measure of $\mathfrak{m}_P$ approaches 1 as $P \to \infty$. If we replace the interval $(0, 1]$ in (1.2) by $\mathfrak{m}_P$, then we can hypothesise that (1.2) holds for each $s \geqslant 2k$ and this would be in line with our philosophy that the 'minor arcs' correspond to 'trivial' solutions (see [**8**] for connected remarks and results). It is not at all clear that $\mathfrak{m}_P$ is the most appropriate choice for our purposes, and quite likely a set which includes more of the peaks may be necessary in order to achieve (1.2).

In view of the general limitations to our current knowledge, it is convenient to work with a somewhat idealised situation. To this end let

$$f(\alpha) = \begin{cases} P^{-1/2}|f_P(\alpha)|, & \text{when } \alpha \in \mathfrak{m}_P, \\ 0, & \text{otherwise} \end{cases}$$

and consider the consequences of the supposition that for $t = 0, 1, \ldots, n$, one has, for some positive number δ, the asymptotic formula

$$\int_0^1 f(\alpha)^{2t}\, d\alpha = \Gamma(t+1) + O_t(P^{-\delta}). \tag{1.4}$$

As we discussed in our opening paragraph, it is expected that (1.2) should hold with a strong error term for each even integer s with $0 \leqslant s < 2k$. Consequently, if we define $f(\alpha)$ now by $f(\alpha) = P^{-1/2}|f_P(\alpha)|$ for each $\alpha \in (0, 1]$, then the relation (1.4) is expected to hold for $t = 0, 1, \ldots, k-1$. It is convenient to retain this possible ambiguity in the definition of $f(\alpha)$, such making no material difference in either our methods or our conclusions since the only properties of $f(\alpha)$ of which we make use are the assumed asymptotic formulae (1.4).

It is relatively easy to show that if (1.4) were to hold without the error term for all positive integers t, then the distribution function for $f(\alpha)$ would be $1 - e^{-\lambda^2}$. In Section 2, as our first step, we show that we can approximate to this ideal situation with a precision which depends on n and P. In order to be precise, let $\chi_{\mathscr{A}}(x)$ denote the indicator function of the set $\mathscr{A}$, and let

$$\phi(\lambda) = \int_0^1 \chi_{[0,\lambda]}(f(\alpha))\, d\alpha. \tag{1.5}$$

THEOREM 1. *Suppose that* $\lambda > 0$ *and that the asymptotic formula* (1.4) *holds for* $t = 0, 1, \ldots, n$. *Then*

$$\phi(\lambda) = 1 - e^{-\lambda^2} + O(\lambda^{1/2} e^{-\lambda^2/2} n^{-1/4} + (\lambda + \lambda^{-1})\, n^{-1/2}) + O_n((\lambda + \lambda^{-1})\, P^{-\delta}).$$

The conclusion of Theorem 1, of course, is not unexpected given that one believes that each of the summands in (1.1) is behaving almost everywhere approximately like an independent random variable on the unit circle, whence $f_P(\alpha)$ should have a Normal distribution.

The question then arises as to how well one can interpolate between the even moments in order to obtain (1.4), at least approximately, for non-integral values of t. For smaller values of t this is comparatively easy, but as t grows it becomes much harder to keep control of the situation. In Section 3 we are able to make use of Theorem 1 to conclude as follows.

THEOREM 2. *Suppose that the asymptotic formula* (1.4) *holds for* $t = 0, 1, \ldots, n$. *Then for each* s *with* $0 \leqslant s \leqslant 2n - n^{2/3}$, *we have*

$$\int_0^1 f(\alpha)^s \, d\alpha = \Gamma(\tfrac{1}{2}s+1)\,(1+O((s+1)^{1/2}\, 2^{s/2} n^{-1/4})) + O_n(P^{-\delta}).$$

Moreover, for each s *with* $0 \leqslant s \leqslant 2n$, *we have*

$$\int_0^1 f(\alpha)^s \, d\alpha = \Gamma(\tfrac{1}{2}s+1)\left(1+O\left(\frac{1}{\log(2n)}\right)\right) + O_n(P^{-\delta}).$$

With essentially no additional effort, our methods may be generalised so as to handle multidimensional exponential sums. In particular, if $k_1, \ldots, k_t$ are integers with $1 \leqslant k_t < k_{t-1} < \cdots < k_1$, then analogues of Theorems 1 and 2 may be derived for the exponential sum

$$f_{\mathbf{k}}(\boldsymbol{\alpha}) = P^{-1/2} \sum_{1 \leqslant x \leqslant P} e(\alpha_1 x^{k_1} + \cdots + \alpha_t x^{k_t}).$$

Moreover, a theorem of Steinig [**6**] implies that when r is an integer with $0 \leqslant r \leqslant t$, one has

$$\int_{(0,1]^t} |f_{\mathbf{k}}(\boldsymbol{\alpha})|^{2r} \, d\boldsymbol{\alpha} = \Gamma(r+1) + O_r(P^{-\delta}),$$

for a positive number δ. In consequence, one may establish the asymptotic formula

$$\lim_{P\to\infty} \int_{(0,1]^t} |f_{\mathbf{k}}(\boldsymbol{\alpha})| \, d\boldsymbol{\alpha} = \Gamma(3/2) + O(t^{-1/4}) = \tfrac{1}{2}\sqrt{\pi} + O(t^{-1/4})$$

unconditionally, and this might be regarded as offering evidence in favour of the conjectured asymptotic formula (1.2) in the case $s = 1$. For comparison, we remark that the best unconditional bounds known for the first moment are

$$(2^{-1/2} + o(1))\, P^{1/2} \leqslant \int_0^1 |f_P(\alpha)| \, d\alpha \leqslant P^{1/2},$$

which follow easily from known asymptotic formulae for the second and fourth moments of $f_P(\alpha)$ via an application of Hölder's inequality.

Throughout this paper, implicit constants occurring in Landau's O-notation and Vinogradov's notation $\ll$ will depend at most on those variables occurring as subscripts to the aforementioned notations.

2. *Determining the distribution function*

Our main objective in this section is the proof of Theorem 1. However, in order to motivate our proof, we first consider the following simple formal argument. We assume that the formula (1.4) holds, without error term, for all non-negative integers t. Next we note that

$$\int_0^\infty \frac{\sin(ay)}{y} \, dy = \begin{cases} \pi/2, & \text{when } a > 0, \\ 0, & \text{when } a = 0, \\ -\pi/2, & \text{when } a < 0, \end{cases}$$

and hence when x is a positive real number with $x \neq \lambda$, one has

$$\chi_{[0,\lambda]}(x) = \frac{2}{\pi}\int_0^\infty y^{-1}\sin(y\lambda)\cos(yx)\,dy.$$

Therefore, *without* justifying the manipulations,

$$\int_0^1 \chi_{[0,\lambda]}(f(\alpha))\,d\alpha = \frac{2}{\pi}\int_0^\infty y^{-1}\sin(y\lambda)\int_0^1 \cos(yf(\alpha))\,d\alpha\,dy. \tag{2.1}$$

We now make use of the power series expansion for cosine together with the formula (1.4), without error term, for all non-negative integers t. Thus, on recalling the formula

$$m! = \int_0^\infty t^m e^{-t}\,dt, \tag{2.2}$$

we conclude from (1.5) and (2.1) that

$$\begin{aligned}\phi(\lambda) &= \frac{2}{\pi}\int_0^\infty y^{-1}\sin(y\lambda)\int_0^\infty \cos(yt^{1/2})\,e^{-t}\,dt\,dy\\ &= \int_0^\infty \left(\frac{2}{\pi}\int_0^\infty y^{-1}\sin(y\lambda)\cos(yt^{1/2})\,dy\right)e^{-t}\,dt\\ &= \int_0^{\lambda^2} e^{-t}\,dt = 1-e^{-\lambda^2}.\end{aligned}$$

In order to establish Theorem 1, we need to modify the above argument in various ways. First, we must truncate the power series for cosine, in view of the incomplete nature of our input from (1.4) regarding the even moments. This, in turn, forces us to truncate the range for the dummy variable y in (2.1). In order to perform this truncation without too much pain, we apply a smoothing argument, in the sense that we replace the indicator function $\chi_{[0,\lambda]}(x)$ by $\max\{0, \lambda - x\} = \int_0^\lambda \chi_{[0,\mu]}(x)\,d\mu$. This replacement has the effect, of course, of accelerating the convergence. We then complete the argument by unsmoothing.

We begin by noting that for non-negative x,

$$\max\{0, \lambda - x\} = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin(y\lambda)}{y}\right)^2 \cos(2yx)\,dy,$$

and hence for any positive real number X, one has

$$\max\{0, \lambda - x\} = \frac{2}{\pi}\int_0^X \left(\frac{\sin(y\lambda)}{y}\right)^2 \cos(2yx)\,dy + O(X^{-1}). \tag{2.3}$$

It follows that

$$\int_0^1 \max\{0, \lambda - f(\alpha)\}\,d\alpha = \frac{2}{\pi}\int_0^X \left(\frac{\sin(y\lambda)}{y}\right)^2 \mathscr{F}(y)\,dy + O(X^{-1}), \tag{2.4}$$

where

$$\mathscr{F}(y) = \int_0^1 \cos(2yf(\alpha))\,d\alpha. \tag{2.5}$$

But by Taylor's Theorem with a remainder term, one has

$$\cos u = \sum_{j=0}^{n-1} \frac{(-1)^j u^{2j}}{(2j)!} + \theta_1(u,n)\frac{u^{2n}}{(2n)!},$$

where $|\theta_1(u,n)| \leqslant 1$. On recalling the hypothesis that (1.4) holds for $t = 0, 1, \dots, n$, we therefore deduce that

$$\mathscr{F}(y) = \sum_{j=0}^{n-1} \frac{(-1)^j (2y)^{2j} j!}{(2j)!} + \theta_2(y,n)\frac{(2y)^{2n} n!}{(2n)!} + O_n((1+y^{2n})P^{-\delta}), \tag{2.6}$$

where $|\theta_2(y,n)| \leqslant 1$. Next we observe that in view of (2.2), one has

$$\sum_{j=0}^{n-1} \frac{(-1)^j (2y)^{2j} j!}{(2j)!} = \int_0^\infty \sum_{j=0}^{n-1} \frac{(-1)^j (2yt^{1/2})^{2j}}{(2j)!} e^{-t}\, dt.$$

Thus, on applying Taylor's Theorem once again, we deduce that

$$\sum_{j=0}^{n-1} \frac{(-1)^j (2y)^{2j} j!}{(2j)!} = \int_0^\infty \left(\cos(2yt^{1/2}) + \theta_3(yt^{1/2}, n)\frac{(2yt^{1/2})^{2n}}{(2n)!}\right) e^{-t}\, dt, \tag{2.7}$$

where $|\theta_3(u,n)| \leqslant 1$. On substituting (2.7) into (2.6), we obtain

$$\mathscr{F}(y) = \int_0^\infty \cos(2yt^{1/2})\, e^{-t}\, dt + 2\theta_4(y,n)\frac{(2y)^{2n} n!}{(2n)!} + O_n((1+y^{2n})P^{-\delta}),$$

where $|\theta_4(y,n)| \leqslant 1$. We may therefore conclude from (2.4) that

$$\int_0^1 \max\{0, \lambda - f(\alpha)\}\, d\alpha = \mathscr{G}(\lambda) + E, \tag{2.8}$$

where

$$\mathscr{G}(\lambda) = \frac{2}{\pi}\int_0^X \left(\frac{\sin(y\lambda)}{y}\right)^2 \int_0^\infty \cos(2yt^{1/2})\, e^{-t}\, dt\, dy \tag{2.9}$$

and

$$E \ll \frac{(2X)^{2n-1} n!}{(2n-1)(2n)!} + X^{-1} + O_n((X + X^{2n-1})P^{-\delta}). \tag{2.10}$$

We now interchange the order of integration, this interchange being justified by absolute convergence. Thus we obtain

$$\mathscr{G}(\lambda) = \int_0^\infty \left(\frac{2}{\pi}\int_0^X \left(\frac{\sin(y\lambda)}{y}\right)^2 \cos(2yt^{1/2})\, dy\right) e^{-t}\, dt,$$

whence by (2.3) we have

$$\mathscr{G}(\lambda) = \int_0^\infty \max\{0, \lambda - t^{1/2}\}\, e^{-t}\, dt + O(X^{-1}). \tag{2.11}$$

We take $X = \frac{1}{2}((2n)!/(n-1)!)^{1/(2n)}$ in (2.8)–(2.11), and conclude that

$$\int_0^1 \max\{0, \lambda - f(\alpha)\}\, d\alpha = \int_0^\infty \max\{0, \lambda - t^{1/2}\}\, e^{-t}\, dt + O(n^{-1/2}) + O_n(P^{-\delta}). \tag{2.12}$$

The unsmoothing operation is straightforward. We write

$$\Phi(\lambda) = \int_0^1 \max\{0, \lambda - f(\alpha)\}\, d\alpha, \tag{2.13}$$

and consider the expression $\Phi(\lambda \pm \eta) - \Phi(\lambda)$. When x is non-negative and η is a real number with $0 < \eta \leqslant \lambda$, it is easily verified that

$$\max\{0, \lambda - x\} - \max\{0, \lambda \pm \eta - x\} \pm \eta \chi_{[0,\lambda]}(x) \leqslant 0.$$

It therefore follows from (2.13) and (1.5) that

$$\Phi(\lambda) - \Phi(\lambda - \eta) \leqslant \eta \phi(\lambda) \leqslant \Phi(\lambda + \eta) - \Phi(\lambda). \tag{2.14}$$

Meanwhile, from (2.12) and (2.13) one has

$$\pm(\Phi(\lambda \pm \eta) - \Phi(\lambda)) = \eta \int_0^{\lambda^2} e^{-t}\, dt \pm \int_{\lambda^2}^{(\lambda \pm \eta)^2} (\lambda \pm \eta - t^{1/2})\, e^{-t}\, dt + O(n^{-1/2}) + O_n(P^{-\delta}). \tag{2.15}$$

Thus by combining (2.14) and (2.15), we conclude that whenever $0 < \eta \leqslant \lambda$ and $\eta\lambda \ll 1$, one has

$$\phi(\lambda) = 1 - e^{-\lambda^2} + O(\eta\lambda e^{-\lambda^2} + \eta^{-1} n^{-1/2}) + O_n(\eta^{-1} P^{-\delta}). \tag{2.16}$$

We now proceed to minimise the error term

$$\mathscr{E}(n, \lambda) = \eta\lambda e^{-\lambda^2} + \eta^{-1} n^{-1/2}$$

in (2.16) subject to the constraints $0 < \eta \leqslant \lambda$ and $\eta\lambda \ll 1$. We observe that in the absence of any constraints, a good approximation to this minimum is provided by the choice

$$\eta = \lambda^{-1/2} e^{\lambda^2/2} n^{-1/4}. \tag{2.17}$$

Suppose first that $\lambda \leqslant 1$, so that $\lambda \leqslant \lambda^{-1}$. When $\lambda^{-1/2} e^{\lambda^2/2} n^{-1/4} \leqslant \lambda$, we may make the ideal choice (2.17), and hence obtain

$$\mathscr{E}(n, \lambda) \ll \lambda^{1/2} e^{-\lambda^2/2} n^{-1/4}. \tag{2.18}$$

On the other hand, when $\lambda^{-1/2} e^{\lambda^2/2} n^{-1/4} > \lambda$, the simple choice $\eta = \lambda$ yields

$$\mathscr{E}(n, \lambda) \ll \lambda^{-1} n^{-1/2}. \tag{2.19}$$

Suppose next that $\lambda > 1$, so that $\lambda^{-1} < \lambda$. Then the ideal choice (2.17) is again accessible when $\lambda^{-1/2} e^{\lambda^2/2} n^{-1/4} \leqslant \lambda^{-1}$, and we obtain (2.18) once more. Meanwhile, when $\lambda^{-1/2} e^{\lambda^2/2} n^{-1/4} > \lambda^{-1}$, the choice $\eta = \lambda^{-1}$ yields

$$\mathscr{E}(n, \lambda) \ll \lambda n^{-1/2}. \tag{2.20}$$

The proof of Theorem 1 is completed on collecting together (2.16) and (2.18)–(2.20).

3. *Interpolating between the even moments*

We now employ Theorem 1 to prove Theorem 2. In advance of the main body of our argument, we provide two estimates for $\phi(\lambda)$ which, although more trivial than the conclusion of Theorem 1, are nonetheless of greater utility in certain circumstances.

LEMMA 3.1. *Suppose that the asymptotic formula* (1.4) *holds for* $t = 0, 1, \ldots, n$. *Then when* $1 \leqslant \lambda \leqslant \sqrt{n}$ *one has*

$$1-\phi(\lambda) \ll \lambda e^{-\lambda^2} + O_\lambda(P^{-\delta}), \tag{3.1}$$

and when $\lambda \geqslant \sqrt{n}$ *one has*

$$1-\phi(\lambda) \leqslant \lambda^{-2n} n! + O_n(\lambda^{-2n} P^{-\delta}). \tag{3.2}$$

Proof. Suppose that λ is a positive number, and that m is any natural number with $m \leqslant n$. Then plainly

$$1-\phi(\lambda) = \int_0^1 \chi_{[\lambda,\infty)}(f(\alpha))\, d\alpha \leqslant \lambda^{-2m} \int_0^1 f(\alpha)^{2m}\, d\alpha.$$

Consequently, in view of the assumed asymptotic formula (1.4) in the case $t = m$,

$$1-\phi(\lambda) \leqslant \lambda^{-2m}(m! + O_m(P^{-\delta})). \tag{3.3}$$

On applying Stirling's formula to (3.3), one obtains

$$1-\phi(\lambda) \ll \exp(-2m\log\lambda + (m+\tfrac{1}{2})\log m - m) + O_m(\lambda^{-2m}P^{-\delta}),$$

and the estimate (3.1) follows when $1 \leqslant \lambda \leqslant \sqrt{n}$ on taking $m = [\lambda^2]$. Meanwhile, (3.2) is immediate from (3.3) on taking $m = n$. This completes the proof of the lemma.

Proof of Theorem 2. We may suppose without loss of generality that n is large. We next observe that

$$\int_0^1 f(\alpha)^s\, d\alpha = \int_0^\infty \lambda^s d(\phi(\lambda)) = \int_0^\infty s\lambda^{s-1}(1-\phi(\lambda))\, d\lambda. \tag{3.4}$$

It transpires that the range of integration in the latter integral may be broken up into four subintervals, the contribution of each one of which may be estimated successfully by using either Theorem 1 or Lemma 3.1. To this end, we define

$$\lambda_0 = n^{-1/6}, \quad \lambda_1 = (\tfrac{1}{2}\log n)^{1/2}, \quad \lambda_2 = n^{1/2}. \tag{3.5}$$

(i) The contribution of the interval $[0, \lambda_0]$. We have

$$\int_0^{\lambda_0} s\lambda^{s-1}(1-\phi(\lambda))\, d\lambda = \int_0^{\lambda_0} s\lambda^{s-1} e^{-\lambda^2}\, d\lambda + \mathscr{I}_1, \tag{3.6}$$

where

$$\mathscr{I}_1 = \int_0^{\lambda_0} s\lambda^{s-1}(1-e^{-\lambda^2}-\phi(\lambda))\, d\lambda.$$

Since $\phi(\lambda)$ is an increasing function with $\phi(0) = 0$, it follows that $\mathscr{I}_1$ lies between

$$\int_0^{\lambda_0} s\lambda^{s-1}(1-e^{-\lambda^2})\, d\lambda \quad \text{and} \quad \int_0^{\lambda_0} s\lambda^{s-1}(1-e^{-\lambda^2}-\phi(\lambda_0))\, d\lambda.$$

Consequently,

$$\mathscr{I}_1 \ll \frac{s}{s+2}\lambda_0^{s+2} + \lambda_0^s \phi(\lambda_0),$$

and thus we deduce from Theorem 1 that $\mathscr{I}_1$ is $O(n^{-1/3}) + O_n(P^{-\delta})$. We therefore conclude from (3.6) that

$$\int_0^{\lambda_0} s\lambda^{s-1}(1-\phi(\lambda))\, d\lambda = \int_0^{\lambda_0} s\lambda^{s-1}e^{-\lambda^2}\, d\lambda + O(n^{-1/3}) + O_n(P^{-\delta}). \tag{3.7}$$

(ii) The contribution of the interval $[\lambda_0, \lambda_1]$. We apply Theorem 1 directly to the second interval to obtain

$$\int_{\lambda_0}^{\lambda_1} s\lambda^{s-1}(1-\phi(\lambda))\,d\lambda - \int_{\lambda_0}^{\lambda_1} s\lambda^{s-1}e^{-\lambda^2}\,d\lambda \ll \mathscr{I}_2 + \mathscr{I}_3 + \mathscr{I}_4 + O_n(P^{-\delta}), \tag{3.8}$$

where

$$\mathscr{I}_2 = \int_{\lambda_0}^{\lambda_1} s\lambda^{s-1/2}e^{-\lambda^2/2}n^{-1/4}\,d\lambda, \tag{3.9}$$

$$\mathscr{I}_3 = \int_{\lambda_0}^{\lambda_1} s\lambda^{s}n^{-1/2}\,d\lambda, \tag{3.10}$$

$$\mathscr{I}_4 = \int_{\lambda_0}^{\lambda_1} s\lambda^{s-2}n^{-1/2}\,d\lambda. \tag{3.11}$$

We estimate $\mathscr{I}_2$ through the substitution $\lambda = \sqrt{(2t)}$, thereby obtaining the bound

$$\mathscr{I}_2 = n^{-1/4}\int_{\lambda_0^2/2}^{\lambda_1^2/2} s(2t)^{\frac{s}{2}-\frac{3}{4}}e^{-t}\,dt \leqslant s2^{\frac{s}{2}-\frac{3}{4}}\Gamma\left(\frac{s}{2}+\frac{1}{4}\right)n^{-1/4}. \tag{3.12}$$

On applying the same substitution, and recalling (3.5), we deduce from (3.10) that

$$\mathscr{I}_3 \leqslant n^{-1/4}\int_{\lambda_0}^{\lambda_1} s\lambda^{s}e^{-\lambda^2/2}\,d\lambda \leqslant s2^{\frac{s}{2}-\frac{1}{2}}\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)n^{-1/4}. \tag{3.13}$$

Meanwhile,

$$\begin{aligned}\mathscr{I}_4 &\leqslant n^{-1/4}\int_{1}^{\lambda_1} s\lambda^{s}e^{-\lambda^2/2}\,d\lambda + n^{-1/2}\int_{\lambda_0}^{1} s\lambda^{s-2}\,d\lambda\\ &\ll s2^{\frac{s}{2}-\frac{1}{2}}\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)n^{-1/4} + n^{-1/3}.\end{aligned} \tag{3.14}$$

On collecting together (3.8) and (3.12)–(3.14), therefore, we conclude that

$$\int_{\lambda_0}^{\lambda_1} s\lambda^{s-1}(1-\phi(\lambda))\,d\lambda = \int_{\lambda_0}^{\lambda_1} s\lambda^{s-1}e^{-\lambda^2}\,d\lambda + O\left(2^{s/2}\Gamma\left(\frac{s}{2}+\frac{3}{2}\right)n^{-1/4}\right) + O_n(P^{-\delta}). \tag{3.15}$$

(iii) The contribution of the interval $[\lambda_1, \lambda_2]$. Using estimate (3.1) of Lemma 3.1,

$$\int_{\lambda_1}^{\lambda_2} s\lambda^{s-1}(1-\phi(\lambda))\,d\lambda - \int_{\lambda_1}^{\lambda_2} s\lambda^{s-1}e^{-\lambda^2}\,d\lambda \ll \int_{\lambda_1}^{\infty} s\lambda^{s}e^{-\lambda^2}\,d\lambda + O_n(P^{-\delta}).$$

But on recalling (3.5) and making the same substitution as in case (ii), one obtains

$$\int_{\lambda_1}^{\infty} s\lambda^{s}e^{-\lambda^2}\,d\lambda \leqslant n^{-1/4}\int_{\lambda_1}^{\infty} s\lambda^{s}e^{-\lambda^2/2}\,d\lambda \leqslant s2^{\frac{s}{2}-\frac{1}{2}}\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)n^{-1/4}.$$

Thus

$$\int_{\lambda_1}^{\lambda_2} s\lambda^{s-1}(1-\phi(\lambda))\,d\lambda = \int_{\lambda_1}^{\lambda_2} s\lambda^{s-1}e^{-\lambda^2}\,d\lambda + O\left(2^{s/2}\Gamma\left(\frac{s}{2}+\frac{3}{2}\right)n^{-1/4}\right) + O_n(P^{-\delta}). \tag{3.16}$$

(iv) The contribution of the interval $[\lambda_2, \infty)$. We note first that on recalling the argument of case (iii) above,

$$\int_{\lambda_2}^{\infty} s\lambda^{s-1}e^{-\lambda^2}\,d\lambda \leqslant \int_{\lambda_1}^{\infty} s\lambda^{s-1}e^{-\lambda^2}\,d\lambda \leqslant s2^{\frac{s}{2}-\frac{1}{2}}\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)n^{-1/4}. \tag{3.17}$$

Next, on using estimate (3.2) of Lemma 3.1,

$$\int_{\lambda_2}^{\infty} s\lambda^{s-1}(1-\phi(\lambda))\,d\lambda \leqslant n!\int_{\lambda_2}^{\infty} s\lambda^{s-1-2n}\,d\lambda + O_n\left(P^{-\delta}\int_{\lambda_2}^{\infty} s\lambda^{s-1-2n}\,d\lambda\right).$$

By hypothesis we have $s \leqslant 2n - n^{2/3}$, and hence by Stirling's formula,

$$\begin{aligned}\int_{\lambda_2}^{\infty} s\lambda^{s-1}(1-\phi(\lambda))\,d\lambda &\leqslant \frac{s}{2n-s}\,n!\,n^{\frac{s}{2}-n} + O_n(P^{-\delta})\\ &\ll n\Gamma(\tfrac{1}{2}s+1)\exp\left(\mu - n + \mu\log(n/\mu)\right) + O_n(P^{-\delta}),\end{aligned}$$

where $\mu = \frac{1}{2}(s+1)$. But the function of μ in the latter exponent is an increasing function for $0 < \mu \leqslant n$, so that since $s \leqslant 2n - n^{2/3}$, we deduce that

$$\int_{\lambda_2}^{\infty} s\lambda^{s-1}(1-\phi(\lambda))\,d\lambda \ll \Gamma(\tfrac{1}{2}s+1)\,n^{-1/4} + O_n(P^{-\delta}). \tag{3.18}$$

Collecting together (3.4), (3.7), (3.15) and (3.16)–(3.18), we conclude that

$$\int_0^1 f(\alpha)^s\,d\alpha = \int_0^{\infty} s\lambda^{s-1}e^{-\lambda^2}\,d\lambda + O\left(2^{s/2}\Gamma\left(\frac{s}{2}+\frac{3}{2}\right)n^{-1/4}\right) + O_n(P^{-\delta}).$$

Moreover, by substituting $t = \lambda^2$, the integral on the right-hand side of the latter equation is readily seen to be $\Gamma(\frac{1}{2}s+1)$, and the first conclusion of Theorem 2 follows.

As an alternative, we could argue directly from (2.12) via an expression of the kind

$$\int_0^F \lambda^t \int_0^1 \max\{0, \lambda - f(\alpha)\}\,d\alpha\,d\lambda.$$

We might also hope to do better by imposing more smoothing, so that the Fourier transform used in Section 2 converges even more quickly. However, it soon becomes apparent that none of these devices gives any substantial advantage, when s is as large as $\log n$, over the argument given above.

In order to complete the proof of Theorem 2, we have merely to note that when s is a real number with $2 \leqslant s \leqslant 2n$, and m is the integer with $2m \leqslant s < 2m+2$, then by Hölder's inequality one has

$$\int_0^1 f(\alpha)^s\,d\alpha \leqslant \left(\int_0^1 f(\alpha)^{2m}\,d\alpha\right)^p \left(\int_0^1 f(\alpha)^{2m+2}\,d\alpha\right)^q,$$

where $p = m+1-s/2$ and $q = s/2 - m$, and

$$\int_0^1 f(\alpha)^{2m}\,d\alpha \leqslant \left(\int_0^1 f(\alpha)^{s}\,d\alpha\right)^{p'} \left(\int_0^1 f(\alpha)^{2m-2}\,d\alpha\right)^{q'},$$

where $p' = 2/(s-2m+2)$ and $q' = (s-2m)/(s-2m+2)$. In view of the assumed asymptotic formulae (1.4), therefore, one has

$$\Delta_s^-\,\Gamma(\tfrac{1}{2}s+1) + O_n(P^{-\delta}) \leqslant \int_0^1 f(\alpha)^s\,d\alpha \leqslant \Delta_s^+\,\Gamma(\tfrac{1}{2}s+1) + O_n(P^{-\delta}),$$

where

$$\log\Delta_s^+ = p\log\Gamma(m+1) + q\log\Gamma(m+2) - \log\Gamma(\tfrac{1}{2}s+1)$$

and

$$\log\Delta_s^- = \frac{1}{p'}\log\Gamma(m+1) - \frac{q'}{p'}\log\Gamma(m) - \log\Gamma(\tfrac{1}{2}s+1).$$

An application of Stirling's formula reveals that $\Delta_s^{\pm} = 1 + O(1/s)$, and the desired conclusion follows immediately whenever $s > \frac{1}{2}\log(2n)$. Meanwhile, the first part of Theorem 2 yields a stronger conclusion in the complementary case $s \leqslant \frac{1}{2}\log(2n)$. This completes the proof of Theorem 2.

References

1. G. Greaves, 'On the representation of a number as a sum of two fourth powers, II', *Mat. Zametki* 55 (1994) 47–58.
2. C. Hooley, 'On the numbers that are representable as the sum of two cubes', *J. Reine Angew. Math.* 314 (1980) 146–173.
3. C. Hooley, 'On another sieve method and the numbers that are a sum of two hth powers', *Proc. London Math. Soc.* (3) 43 (1981) 73–109.
4. C. Hooley, 'On another sieve method and the numbers that are a sum of two hth powers: II', *J. Reine Angew. Math.* 475 (1996) 55–75.
5. C. M. Skinner and T. D. Wooley, 'Sums of two kth powers', *J. Reine Angew. Math.* 462 (1995) 57–68.
6. J. Steinig, 'On some rules of Laguerre's, and systems of equal sums of like powers', *Rend. Mat.* (6) 4 (1971) 629–644.
7. R. C. Vaughan, *The Hardy–Littlewood method* (Cambridge University Press, 1981; 2nd edn, 1997).
8. R. C. Vaughan and T. D. Wooley, 'On a certain nonary cubic form and related equations', *Duke Math. J.* 80 (1995) 669–735.
9. T. D. Wooley, 'Sums of two cubes', *Internat. Math. Res. Notices* (1995) 181–185.

Mathematics Department
Huxley Building
Imperial College
180 Queen's Gate
London SW7 2BZ

Department of Mathematics
University of Michigan
Ann Arbor, MI 48109-1109
USA