THE BOUNDARY BEHAVIOUR OF BLOCH FUNCTIONS

S. ROHDE

1. *Introduction and results*

A function f, analytic in the unit disk $\mathbb{D}$, is a *Bloch function* if

$$\|f\|_{\mathscr{B}} = \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < \infty.$$

The (linear) space of all Bloch functions is denoted by $\mathscr{B}$. The number $\|f\|_{\mathscr{B}}$ is the Bloch semi-norm of f. Bloch functions have been extensively studied because of their close connection to univalent functions: if g is univalent in $\mathbb{D}$ then $f = \log g' \in \mathscr{B}$, and if $f \in \mathscr{B}$ with $f = c\log g'$, then g is univalent whenever c is small enough. See, for example, [1] for further information on $\mathscr{B}$.

There are Bloch functions f that have no finite radial limits, that is, the limit

$$f(\zeta) = \lim_{r\to 1} f(r\zeta) \tag{1.1}$$

exists for no point $\zeta \in \mathbb{T}$. Here $\mathbb{T}$ denotes the unit circle $\partial\mathbb{D}$. An example of such a Bloch function is the Hadamard gap series $f(z) = \sum_{n\geqslant 0} z^{2^n}$, [9, Chapter 8]. We denote by $R(f)$ the set of those points $\zeta \in \mathbb{T}$ for which the limit (1.1) exists and is finite. When studying the boundary behaviour of f, the set $R(f)$ is rather uninteresting since f behaves nicely there. We furthermore consider the set $P(f) \subset \mathbb{T}$ of those $\zeta \in \mathbb{T}$ for which the image $f(S_\zeta)$ is dense in $\mathbb{C}$, for any Stolz cone $S_\zeta \subset \mathbb{D}$ at ζ. These are the Plessner points of f and the classical theorem of Plessner [9, Chapter 6] shows that, if f is a Bloch function, then

$$\mathbb{T} = R(f) \cup P(f) \cup E(f), \tag{1.2}$$

where $E(f)$ is a set of Lebesgue measure 0. For subsets $A \subset \mathbb{T}$ we shall use the notation $|A|$ for the Lebesgue measure divided by 2π, thus $|\mathbb{T}| = 1$.

In this paper we study the behaviour of f on the set $E(f)$. Since this set is already small by Plessner's theorem, we consider the Hausdorff dimension, abbreviated to dim, of several subsets of $E(f)$. To avoid trivialities, we restrict our attention to the subclass of those functions for which $|R(f)| = 0$. With a suitable normalization this is

$$\tilde{\mathscr{B}} = \{f \in \mathscr{B} : f(0) = 0,\ \|f\|_{\mathscr{B}} = 1 \text{ and } |R(f)| = 0\}. \tag{1.3}$$

The above mentioned gap series belongs to this class as well as $\log g'$ if g is a conformal map onto a domain with a complicated boundary like the snowflake domain.

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Our main result is the following.

THEOREM 1.1. *Let $f \in \tilde{\mathscr{B}}, a > 0$ and let $\gamma: [0,1) \to \mathbb{C}, \gamma(0) = 0$ be an arbitrary continuous curve. Then there is a set $E_{\gamma,a} \subset \mathbb{T}$ with*

$$\dim E_{\gamma,a} \geqslant 1 - \frac{K}{a}, \tag{1.4}$$

so that for every $\zeta \in E_{\gamma,a}$ there is a homeomorphism $\phi_\zeta: [0,1) \to [0,1)$ such that

$$\sup_{0 \leqslant r < 1} |f(r\zeta) - \gamma(\phi_\zeta(r))| \leqslant a. \tag{1.5}$$

Here K denotes a universal constant.

Inequality (1.4) only makes sense if $a > K$, and it becomes powerful when a tends to infinity. If we denote by E_γ the set

$$E_\gamma = \bigcup_{0 < a < \infty} E_{\gamma,a} \tag{1.6}$$

then (1.4) yields $\dim E_\gamma = 1$. Roughly speaking, (1.5) says that the image of a radius follows the given curve γ (after a reparametrization) within an a-neighbourhood, and for any given curve γ there is a set E_γ of Hausdorff dimension 1 so that for every point $\zeta \in E_\gamma$ the curve $f([0,\zeta))$ follows γ (that is has bounded distance).

It is impossible to remove the reparametrization ϕ_ζ since the growth of Bloch functions has certain bounds, whereas the curve γ is arbitrary.

Next we show that Theorem 1.1 implies some well-known results in a slightly weaker form.

Choosing γ to be the positive real axis, that is $\gamma(r) = r/(1-r)$, we obtain the following corollary from (1.6).

COROLLARY 1.2. *If $f \in \tilde{\mathscr{B}}$ then*

$$\dim \{\zeta \in \mathbb{T} : \operatorname{Re} f(r\zeta) \to +\infty \text{ as } r \to 1), \sup_{0 \leqslant r < 1} |\operatorname{Im} f(r\zeta)| < \infty\} = 1.$$

A weaker statement, involving the whole class of Bloch functions, is as follows.

COROLLARY 1.3. *If $f \in \mathscr{B}$ then*

$$\dim \{\zeta \in \mathbb{T} : f(\zeta) \textit{ exists as a finite radial limit or } \operatorname{Re} f(r\zeta) \to -\infty \textit{ as } r \to 1\} = 1.$$

Using the connection to univalent functions mentioned above, Corollary 1.3 implies the following result of Anderson and Pitt [2] and (independently) Makarov [8].

THEOREM APM. *For any function g univalent in $\mathbb{D}$ we have*

$$\dim \{\zeta \in \mathbb{T} : g'(\zeta) \textit{ exists as a finite radial limit}\} = 1.$$

In fact, the results in [2] and [8] are slightly stronger since they show that the set under discussion has positive Hausdorff measure with respect to a certain weight function involving an iterated logarithm.

Considering the curve $\gamma(r) \equiv 0$ we obtain the following result of Makarov.

THEOREM M. *For any function $f \in \mathscr{B}$ we have*

$$\dim\{\zeta \in \mathbb{T}: \sup_{0 \leqslant r < 1} |f(r\zeta)| < \infty\} = 1.$$

Our proof of Theorem 1.1 is similar to Makarov's proof of the previous theorem in so far as we use the same lemma controlling the Hausdorff dimension of Cantor-like sets, Lemma 3.1 below. However, whereas Makarov considered an approximation of Bloch functions by a martingale, we approximate the level sets of a Bloch function f by chord-arc domains and compare the restrictions of f to these sets to inner functions.

Our final application of Theorem 1.1 is again to conformal mappings. Let G be a simply connected domain and $f: \mathbb{D} \to G$ be conformal. Recall that a point $f(\zeta) = w \in \partial G$ is called *well-accessible* [9, Chapter 11] if there is a Jordan arc $C \subset G$ ending at w and a constant $c > 0$ so that $\operatorname{diam} C_z \leqslant c \operatorname{dist}(z, \partial G)$ for every $z \in C$, where C_z denotes the subarc from z to w. Denote by W the set of those points $\zeta \in \mathbb{T}$ for which $f(\zeta)$ is well-accessible. It is not hard to construct a domain G so that $|W| = 0$. On the other hand, it follows from known estimates (see again [9, Chapter 11]) that $f(\zeta)$ is well-accessible if $\inf_{0 \leqslant r < 1} |f'(r\zeta)| > 0$ and $\sup_{0 \leqslant r < 1} |f'(r\zeta)| < \infty$. Hence Theorem 1.1 easily implies the following result.

COROLLARY 1.4. *For any simply connected domain we have*

$$\dim W = 1.$$

By another theorem of Makarov [7] this implies that $\dim f(W) = 1$, that is, for any simply connected domain the set of well-accessible boundary points has dimension 1.

In a forthcoming paper we shall modify the method developed in this paper to obtain results about functions in the little Bloch space $\mathscr{B}_0$ and about inner functions in $\mathscr{B}_0$. In particular we shall show that every function in $\tilde{\mathscr{B}} \cap \mathscr{B}_0$ assumes every value $w \in \mathbb{C}$ as a radial limit on a set of dimension 1, and similarly that every inner function in $\mathscr{B}_0$ assumes every $w \in \mathbb{D}$ on a set of dimension 1. This sharpens results of Hungerford [4] and Makarov [8].

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2. *Level sets of Bloch functions*

Throughout this chapter f will be a Bloch function. For points $z_0 \in \mathbb{D}$, and for $a > 0$ we consider the component $\Omega_a = \Omega_a(f, z_0)$ of $\{z \in \mathbb{D}: |f(z) - f(z_0)| < a\}$ that contains z_0. These components have also been studied in [10]. The boundary $\partial\Omega_a$ need not be rectifiable, even if f is bounded [5]. The main object of this chapter is to construct rectifiable domains $G_a \subset \mathbb{D}$ that behave like the components Ω_a, that is, they will have the property $\Omega_{a-c} \subset G_a \subset \Omega_{a+c}$.

Our construction relies on the following analog due to Pommerenke [9, Chapter 4] of a well-known result of Lehto and Virtanen.

THEOREM P. *Let $B \subset \mathbb{D}$ be a circular arc with endpoints on $\mathbb{T}$ and consider a Jordan arc $C \subset \mathbb{D}\backslash B$ with endpoints on B. Let G be the component of $\mathbb{D}\backslash(B \cup C)$ for which $\bar{G} \cap \mathbb{T} = \varnothing$. Let β be the angle between B and $\mathbb{T}$ towards G. Finally, let*

$$\alpha = \sup_{z\in G}(1-|z|^2)|f'(z)|. \tag{2.1}$$

Then

$$f(G) \subset \left\{w\in\mathbb{C}: \operatorname{dist}(w, f(C)) \leqslant \frac{e\alpha\beta}{2\sin\beta}\right\}. \tag{2.2}$$

This result implies that

$$\overline{\Omega_a(f, z_0)} \cap \mathbb{T} \neq \varnothing \tag{2.3}$$

if $f\in\mathscr{B}$, $\|f\|_{\mathscr{B}} \leqslant 1$, $z_0\in\mathbb{D}$ and $a > \frac{1}{2}e$ (see [3]). Since Bloch functions are in the MacLane-class $\mathscr{A}$ of analytic functions [6], the level curves end at points. More precisely (see [6]), we have the following result.

THEOREM ML. *If $f\in\mathscr{B}$, Ω is a component of $\{|f(z)| < a\}$ and $\partial\Omega \cap \mathbb{T} \neq \varnothing$, then the components of $\partial\Omega \cap \mathbb{D}$ are (open) Jordan arcs. The closures of these Jordan arcs intersect $\mathbb{T}$ in at most two points.*

Now we are in the position to describe our approximating domains. Let $f\in\tilde{\mathscr{B}}$, $a > \frac{1}{2}e$ and $z_0\in\mathbb{D}$. By (1.3) and (2.3) we have $\overline{\Omega_a(f, z_0)} \cap \mathbb{T} \neq \varnothing$, hence

$$\mathbb{T}\backslash\overline{\Omega_a(f, z_0)} = \bigcup_n I_n, \tag{2.4}$$

where the I_n are disjoint open arcs.

Since f is a Bloch function and $|R(f)| = 0$ ((1.2) and (1.3)) we obtain

$$\limsup_{r\to 1} |f(r\zeta)| = \infty$$

for almost all $\zeta\in\mathbb{T}$. From this we conclude that

$$|\mathbb{T} \cap \overline{\Omega_a(f, z_0)}| = 0 \tag{2.5}$$

in the following way.

After applying a Möbius transformation we may assume that $z_0 = 0$. Let

$$z\in\mathbb{T} \cap \overline{\Omega_a(f, 0)}.$$

The radius $[0, \zeta)\backslash\partial\Omega_a$ consists of line segments that are entirely in Ω_a or in $\mathbb{D}\backslash\overline{\Omega_a}$. On the line segments in Ω_a, f is bounded above by a, and an application of Theorem P, with $\beta = \frac{1}{2}\pi$, shows that $|f(z)| \leqslant a + \frac{1}{4}e\pi$ on the other segments. Hence

$$\limsup_{r\to 1} |f(r\zeta)| \leqslant a + \tfrac{1}{4}e\pi < \infty \quad (z\in\mathbb{T} \cap \overline{\Omega_a(f, 0)}) \tag{2.6}$$

and (2.5) is established.

We fix a small constant $0 < \beta < \frac{1}{2}\pi$ whose value is to be determined later. We denote by $B_n = B(I_n)$ the circular arc in $\mathbb{D}$ whose endpoints are the endpoints of I_n and which intersects $\mathbb{T}$ in the angle β. The intersecting angle is measured between B_n and I_n. The midpoint of B_n (in the obvious meaning) will be denoted by z_n or $z(I_n)$. It depends, of course, on the angle β.

Let $G_a = G_a(f, z_0)$ denote the component of $\mathbb{D}\backslash\overline{\bigcup_n B_n}$ that contains z_0. We shall use the notation $\Lambda(A)$ for the linear measure of sets $A \subset \mathbb{C}$. One of the important features of the domain G_a is that it is a chord-arc domain, that is, for points $z_1, z_2 \in \partial G_a$ we have $\Lambda(C(z_1, z_2)) \leqslant c|z_1 - z_2|$, where $C(z_1, z_2)$ denotes the shorter subarc of ∂G_a between z_1 and z_2. The constant c tends to 1 as $\beta \to 0$. We shall not use this fact explicitly, but it is the reason why our proof of Theorem 2.2 below works.

LEMMA 2.1. *Let $f \in \tilde{\mathscr{B}}$, $a > \frac{1}{2}e$ and $G_a = G_a(f, 0)$. There are constants K_1 and K_2, depending only on β, so that for any arcs I_n, B_n described above we have*

$$a - K_1 \leqslant |f(z)| < a + K_1 \quad \text{for } z \in B_n, \tag{2.7}$$

$$|f(z)| \leqslant a + K_1 \quad \text{in } G_a(f, 0) \tag{2.8}$$

and

$$|I_n| \leqslant K_2 e^{-a}. \tag{2.9}$$

Proof. Let V be the component of $\mathbb{D}\backslash\overline{\Omega_a}$ so that $V \cap \mathbb{T} = I_n$. Pick two points $z_1, z_2 \in \partial V\backslash\mathbb{T}$ near the endpoints of I_n and consider the circular arc B that contains z_1 and z_2 and intersects $\mathbb{T}$ in the angle β. Let B' denote the subarc of B with endpoints z_1, z_2. By MacLane's result (Theorem ML) we know that $C = \partial V\backslash\mathbb{T}$ is a Jordan arc and we can apply Theorem P to all those components C_k of $C\backslash B'$ for which $\overline{C_k} \subset \mathbb{D}$. We obtain from (2.2) with $\beta' = \pi - \beta$

$$f(B') \subset \left\{ w \in \mathbb{C} : \operatorname{dist}(w, f(C)) \leqslant \max\left(\frac{e\beta}{2\sin\beta}, \frac{e\beta'}{2\sin\beta'}\right)\right\}.$$

Letting z_1, z_2 tend to $\mathbb{T}$ gives (2.7).

To prove (2.8), let us first assume that f is unbounded in G_a. Then there is a curve in G_a so that $|f|$ tends to infinity along this curve and we conclude, again using Theorem P, that there is a radial segment $[0, z]$ in G_a along which $|f|$ tends to infinity. The endpoint z cannot be in $\mathbb{D}$ because of (2.7). Hence $z \in \mathbb{T} \cap \overline{\Omega_a(f, z_0)}$ and we obtain a contradiction to (2.6). Hence f is bounded in G_a. But then (2.8) follows from the maximum principle by (2.6) and (2.7).

Next we show that $|I_n| < \frac{1}{2}$ for $a > c_1$. Assume that this is not true, then there are points $z_1, z_2 \in C$ satisfying $\arg z_1 = \arg z_2 + \pi$. Applying Theorem P (with $\beta = \frac{1}{2}\pi$) to the interval $[z_1, z_2]$ we obtain $|f(z)| > a - c_1$ (with $c_1 = \frac{1}{4}e\pi$) for any $z \in [z_1, z_2]$ and choosing $z = 0$ gives the contradiction $|f(0)| > 0 = f(0)$ if $a > c_1$.

Since $\|f\|_{\mathscr{B}} = 1$ and $f(0) = 0$ we have $|f(z)| \leqslant \log(1 + |z|)/(1 - |z|)$. Since z_n is the midpoint of B_n, (2.7) yields

$$1 - |z_n| \leqslant 2 e^{K_1} e^{-a}.$$

Now (2.9) follows since $1 - |z_n| \geqslant c_2 |I_n|$ with a constant c_2 depending only on β.

For points $w \in \mathbb{D}\backslash\{0\}$ we shall denote by $p(w)$ the projection of w onto $\mathbb{T}$, that is,

$$p(w) = \frac{w}{|w|}.$$

The next result says, roughly speaking, that the map

$$p \circ f \colon (\partial G_a(f, 0), \Lambda) \longrightarrow (\mathbb{T}, |\cdot|)$$

is nearly measure preserving if $f \in \tilde{\mathscr{B}}$ (see (1.3)), β is small and a is large.

THEOREM 2.2. *Let $f \in \tilde{\mathscr{B}}$ and consider an arc $A \subset \mathbb{T}$. Let*

$$J = \{j : p(f(z_j)) \in A\}. \tag{2.10}$$

Then

$$\sum_{j \in J} |I_j| \geqslant \left(1 - \frac{2\beta}{\pi}\right)|A| \tag{2.11}$$

for $a \geqslant a_0(|A|)$.

Before proving the theorem we establish the following.

LEMMA 2.3. *Let $\phi = \phi_a : \mathbb{D} \to G_a$ with $\phi(0) = 0$ be conformal and let*

$$h(z) = \frac{1}{a + K_1} f(\phi(z)). \tag{2.12}$$

Then

$$|\{\zeta \in \mathbb{T} : p(h(\zeta)) \in A_1\}| = |A_1| + o(1) \quad \text{as } a \to \infty, \tag{2.13}$$

for any arc $A_1 \subset \mathbb{T}$. The $o(1)$ depends only on a and β.

Note that for inner functions $g : \mathbb{D} \to \mathbb{D}$ with $g(0) = 0$ we have

$$|A_1| = |g^{-1}(A_1)| = |\{\zeta \in \mathbb{T} : g(\zeta) \in A_1\}| = |\{\zeta \in \mathbb{T} : p(g(\zeta)) \in A_1\}|. \tag{2.14}$$

Now h is nearly an inner function in the sense that h is a selfmap of the unit disk with boundary values close to 1 almost everywhere. Hence (2.13) is a generalization of this result. On the other hand, (2.14) holds for arbitrary Borel sets on $\mathbb{T}$ whereas in (2.13) we make the assumption that A_1 is an arc. The proof of (2.13) follows the proof of (2.14).

Proof of Lemma 2.3. We have already noted that $|h(z)| \leqslant 1$ in $\mathbb{D}$ by (2.8). Additionally we have

$$|h(\zeta)| \geqslant \frac{a - K_1}{a + K_1} \quad \text{for a.e. } \zeta \in \mathbb{T} \tag{2.15}$$

by (2.5) and (2.7), since ∂G_a is rectifiable. Here it is important that f is in $\tilde{\mathscr{B}}$ and not merely in $\mathscr{B}$. Let

$$\varepsilon = 1 - \frac{a - K_1}{a + K_1} = \frac{2K_1}{a + K_1}$$

and

$$A_1' = \{\zeta \in \mathbb{T} : \operatorname{dist}(\zeta, A_1) \leqslant \sqrt{\varepsilon}\} \supset A_1.$$

With $B = \{\zeta \in \mathbb{T} : p(h(\zeta)) \in A_1\}$ we consider the harmonic functions

$$u_1(z) = \frac{1}{2\pi} \int_B \frac{1 - |z|^2}{|\zeta - z|^2} |d\zeta|,$$

$$u_2(z) = \frac{1}{2\pi} \int_{A_1'} \frac{1 - |h(z)|^2}{|\zeta - h(z)|^2} |d\zeta| = \tilde{u}_2(h(z)).$$

Let $\delta > 0$ be given. Standard estimates with the Poisson kernel show that for any $\varepsilon < \varepsilon_0(\delta)$ (hence for all $a \geqslant a(\delta)$) and all $w \in \mathbb{D}$ such that $p(w) \in A_1$ and $|w| > 1 - \varepsilon$, we have $\tilde{u}_2(w) \geqslant 1 - \delta$. We conclude that $u_1(\zeta) = 0$ almost everywhere on $\mathbb{T} \backslash B$ and

$u_2(\zeta) \geqslant 1-\delta$ almost everywhere on B. Hence $(u_1-u_2)(\zeta) \leqslant \delta$ almost everywhere on $\mathbb{T}$ and, since u_1-u_2 is a bounded harmonic function, the maximum principle yields $u_1(0) \leqslant u_2(0)+\delta$. With $u_1(0) = |B|$ and $u_2(0) = |A_1'|$ we obtain

$$|B| \leqslant |A_1'|+\delta \leqslant |A_1|+\delta+2\sqrt{\varepsilon}.$$

Applying the same reasoning to the complement of A_1 establishes (2.13).

Proof of Theorem 2.2. Let $A_1 \subset A$ be the concentric subarc so that

$$|A_1| = \left(1-\frac{1}{\sqrt{a}}\right)|A|. \tag{2.16}$$

With the function h defined by (2.12) and

$$B = \{\zeta \in \mathbb{T} : p(h(\zeta)) \in A_1\},$$

Lemma 2.3 gives

$$|B| = |A|+o(1) \quad \text{as } a \to \infty. \tag{2.17}$$

Set c to be a number with $0 < c < 1$ and c close to 1, whose value is to be determined later. For every arc B_j of ∂G_a we denote by $B_j' = B_j'(c)$ the concentric subarc for which $\Lambda(B_j') = c\Lambda(B_j)$.

The maximum principle shows that

$$\omega(z, B_j, G_a) \leqslant \left(1-\frac{\beta}{\pi}\right)^{-1} \omega(z, I_j, \mathbb{D})$$

in G_a, since the left-hand side is 0 on $\partial G_a \backslash B_j$ and both sides coincide on B_j. Hence

$$\omega(0, B_j, G_a) \leqslant \left(1-\frac{\beta}{\pi}\right)^{-1} |I_j|. \tag{2.18}$$

Similarly,

$$\omega(0, B_j\backslash B_j', G_a) \leqslant c_3|p(B_j\backslash B_j')| \leqslant c_4(1-c)|I_j|. \tag{2.19}$$

Here c_4 depends only on β. Now we consider

$$J_1 = \{j : B_j' \cap \phi(B) \neq \varnothing\}.$$

Then

$$\phi(B) \subset \bigcup_{j\in J_1} B_j \cup \bigcup_j (B_j\backslash B_j')$$

and (2.18) implies that

$$\begin{aligned}\sum_{j\in J_1}\left(1-\frac{\beta}{\pi}\right)^{-1}|I_j| &\geqslant \omega\left(0, \bigcup_{j\in J_1} B_j, G_a\right)\\ &\geqslant \omega(0, \phi(B), G_a) - \omega\left(0, \bigcup_j (B_j\backslash B_j'), G_a\right)\\ &\geqslant |B| - c_4(1-c).\end{aligned}$$

The last inequality follows from (2.19). Using (2.17) we see that we can choose $c = c(|A|)$ so that

$$\sum_{j\in J_1}|I_j| \geqslant \left(1-\frac{2\beta}{\pi}\right)|A|$$

for $a \geqslant a_0(|A|)$.

Thus the theorem is proven if we show that $J_1 \subset J$.

To this end, pick $j \in J_1$. Since $f \in \mathscr{B}$ and $\|f\|_{\mathscr{B}} \leqslant 1$ we have

$$|f(z)-f(z_j)| \leqslant c_5$$

for all $z \in B_j'$, where c_5 depends only on c and β, hence only on $|A|$ (and β). By (2.7) and the definition of the projection p,

$$\begin{aligned}|p(f(z))-p(f(z_j))| &= \left|\frac{f(z)-f(z_j)}{|f(z_j)|}+\left(\frac{|f(z_j)|}{|f(z)|}-1\right)\frac{f(z)}{|f(z_j)|}\right| \\ &\leqslant \frac{c_5}{a-K_1}+\frac{2K_1}{a-K_1}\frac{a+K_1}{a-K_1} \\ &\leqslant \frac{c_6(|A|)}{a}\end{aligned}$$

for sufficiently large $a > 2K_1$. By the definition of J_1 there is a $z \in B_j'$ satisfying $\zeta = \phi^{-1}(z) \in B$, and the definition of h and B gives

$$p(h(\zeta)) = p(f(z)) \in A_1.$$

Finally, (2.16) implies that

$$p(f(z_j)) \in A$$

for large a and we have shown that $j \in J$.

Theorem 2.2 was formulated only for the special case $G = G_a(f,0)$. The next result applies to all domains $G_a(f, z_0)$ and is essentially a Möbius invariant formulation of Theorem 2.2. In what follows all constants will depend (only) on β, unless otherwise stated.

COROLLARY 2.4. *Let $f \in \tilde{\mathscr{B}}$ and let $A \subset \mathbb{T}$ be an arc. Let $I \subset \mathbb{T}$ be an arc with $|I| \leqslant \frac{1}{2}$. Consider the domain $G = G_a(f, z_I)$ and the corresponding points z_j and arcs $I_j \subset \mathbb{T}$. Let*

$$J = \{j: p(f(z_j)-f(z_I)) \in A \text{ and } I_j \subset I\}, \tag{2.20}$$

where again $p(w) = w/|w|$. If $a \geqslant a_0(|A|)$, then

$$\sum_{j \in J} |I_j| \geqslant c|I|\left(\left(1-\frac{2\beta}{\pi}\right)\frac{|A|}{2}-\frac{2\beta}{\pi}\right) \tag{2.21}$$

and

$$|I_j| \leqslant \frac{1}{c} e^{-a}|I| \quad \textit{for } j \in J. \tag{2.22}$$

Proof. Let $T(z) = (z+z_I)/(1+\overline{z_I}z)$ and

$$b(z) = f(T(z))-f(T(0)) = f(T(z))-f(z_I).$$

Furthermore, let $G' = G_a(b(z),0)$. Then $G = T(G')$ and $I_j = T(I_j')$, if we denote by I_j' the arcs corresponding to G'. Standard estimates show that

$$c_7|I| \leqslant |T'(\zeta)| \leqslant c_8|I| \tag{2.23}$$

for $\zeta \in T^{-1}(I)$. Applying Lemma 2.1 to $b(z)$ we obtain from (2.9) that

$$|I_j| \leqslant c_8 K_2 e^{-a}|I|$$

and this proves (2.22). Next we consider the concentric arc $A_1 \subset A$ so that $|A_1| = \frac{1}{2}|A|$, and the set

$$J' = \{j: p(b(z_j')) \in A_1, I_j' \subset T^{-1}(I)\}.$$

We show first that $J' \subset J$ if a is large enough. Observe that this would be obvious if $z_j = T(z_j')$. However, in general a Möbius transformation will not map midpoints of circular arcs onto the midpoints of the image arcs. For $j \in J'$ we have $T(I_j') \subset I$ by definition. Now $|\arg(z_j') - \arg(-z_I)|$ is bounded away from 0 so that the same is true for $|1 + \overline{z_I} z_j'|$. It follows that

$$1 - |T(z_j')|^2 = \frac{(1-|z_j'|^2)(1-|z_I|^2)}{|1+\overline{z_I} z_j'|^2}$$
$$\geqslant c_0 |I_j'|\,|I| \geqslant c_{10}|I_j|.$$

The last inequality follows from (2.23). Next, $T(z_j') \in B_j$, hence

$$|f(z_j) - f(T(z_j'))| \leqslant c_{11}.$$

Thus $p(f(z_j) - f(z_I)) \in A$ for $a \geqslant a_0$ which proves that $J' \subset J$.

Since

$$|\mathbb{T} \setminus T^{-1}(I)| = 1 - \omega(z_I, I, \mathbb{D}) = \frac{\beta}{\pi},$$

an application of Theorem 2.2 to A_1 and b shows that

$$\sum_{j\in J} |I_j| \geqslant \sum_{j \in J'} |T(I_j')| \geqslant c_7 |I| \sum_{j\in J'} |I_j'|$$
$$\geqslant c_7 |I| \left(\sum_{p(b(z_j'))\in A_1} |I_j'| - \frac{2\beta}{\pi}\right)$$
$$\geqslant c_7 |I| \left(\left(1 - \frac{2\beta}{\pi}\right)\frac{|A|}{2} - \frac{2\beta}{\pi}\right),$$

finishing the proof of the corollary.

3. *The proof of Theorem* 1.1

We shall apply the following lower estimate for Hausdorff dimensions, due to Hungerford [4] and Makarov [8], see also [9, Chapter 10].

LEMMA HM. *Let $a > 0$ and $0 < c < 1$. Let $I_n^{(k)}$ $(n, k = 0, 1, 2, \ldots)$ be a family of arcs on $\mathbb{T}$ so that for every $I_n^{(k)}$ there is a $I_m^{(k-1)} \supset I_n^{(k)}$ such that*

$$|I_n^{(k)}| \leqslant e^{-a} |I_m^{(k-1)}| \tag{3.1}$$

and furthermore,

$$\sum_{I_m^{(k+1)} \subset I_n^{(k)}} |I_m^{(k+1)}| \geqslant c |I_n^{(k)}| \tag{3.2}$$

for all $n, k \geqslant 0$. Then

$$\dim \bigcap_{k\geqslant 0} \bigcup_n I_n^{(k)} \geqslant 1 - \frac{1}{a}\log\left(\frac{1}{c}\right). \tag{3.3}$$

Proof of Theorem 1.1. First we define inductively an increasing sequence $(t_n) \subset (0,1)$, $t_n \to 1$ as $n \to \infty$ and a sequence $\gamma_n \in \mathbb{C}$ so that

$$|\gamma_{n+1} - \gamma_n| = a, \quad \text{for } n = 0, 1, 2, \ldots \tag{3.4}$$

and

$$|\gamma(t) - \gamma_n| \leqslant 2a, \quad \text{with } t \in [t_n, t_{n+1}]. \tag{3.5}$$

To do so, let $t_0 = 0$, $\gamma_0 = 0$ and suppose that t_n and γ_n are already defined. If there is a $t \in (t_n, 1)$ with $|\gamma(t) - \gamma(t_n)| = a$ we set

$$t_{n+1} = \min\{t \in (t_n, 1) : |\gamma(t) - \gamma(t_n)| = a\}, \quad \gamma_{n+1} = \gamma(t_{n+1})$$

and continue the induction. Otherwise we have $|\gamma(t) - \gamma(t_n)| < a$ for all $t_n < t < 1$, and in this case we set inductively for $k > n$

$$t_{k+1} = \tfrac{1}{2}(1 + t_k), \quad \gamma_{k+1} = \gamma(t_k) + \tfrac{1}{2}((-1)^{k-n} + 1)\,a.$$

In either case (3.4) and (3.5) hold.

Our goal is to construct arcs $I_n^{(k)} \subset \mathbb{T}$ having the following properties (a), (b), (c). During the construction we shall finally fix the angle β on which the points $z_n^{(k)} = z(I_n^{(k)})$ depend.

(a) The arcs $I_n^{(k)}$ satisfy the assumptions (3.1) and (3.2) of Lemma HM (with a replaced by $\tfrac{1}{2}a$ and c being a universal constant to be determined later).

(b) The inequality $|f(z_n^{(k)}) - \gamma_k| < \tfrac{1}{2}a$ holds for all $n, k \geqslant 0$.

(c) For all $I_n^{(k)}$, $k \geqslant 1$, and for the corresponding $I_m^{(k-1)} \supset I_n^{(k)}$ we have

$$|f(z) - f(z_m^{(k-1)})| \leqslant 2a$$

for all $z \in [z_m^{(k-1)}, z_n^{(k)}]$.

For a moment, let us take the existence of such a family of arcs for granted and finish the proof of the theorem. Set

$$E = \bigcap_{k \geqslant 0} \bigcup_n I_n^{(k)}.$$

By (3.3) and Property (a) we have $\dim E \geqslant 1 - K/a$ and we shall be finished if we show that $E \subset E_{\gamma, 5a}$, (see (1.5) for the definition of $E_{\gamma,a}$). To this end let $\zeta \in E$ and denote by $I_k = I_{n_k}^{(k)}$ the arc with $\zeta \in I_k$. Let $z_k = z(I_k)$ again be the midpoint and consider the polygonal arc C consisting of the line segments $[z_k, z_{k+1}]$. The construction of the $I_n^{(k)}$ will start with $I_0^{(0)} = \mathbb{T}$ so that $z_0 = 0$. Hence C is a half open Jordan arc starting at 0 and ending at ζ. It is easy to see that C lies entirely in a Stolz cone of vertex ζ whose opening angle depends only on β. Let C be parametrized so that $|C(r)| = r$, for $0 \leqslant r < 1$, then it follows that

$$|f(C(r)) - f(r\zeta)| \leqslant c_1 \quad \text{for } 0 \leqslant r < 1. \tag{3.6}$$

Let $\phi = \phi_\zeta$ be the piecewise linear homeomorphism of $[0,1)$ satisfying $\phi(|z_k|) = t_k$, for $k = 0, 1, 2, \ldots$. Given $0 \leqslant r < 1$ we choose k so that $|z_k| \leqslant r < |z_{k+1}|$. Then, using (3.6), (c), (b) and (3.5), we have

$$\begin{aligned} |f(r\zeta) - \gamma(\phi(r))| &\leqslant |f(C(r)) - \gamma(\phi(r))| + c_1 \\ &\leqslant |f(z_k) - \gamma(\phi(r))| + 2a + c_1 \\ &\leqslant |\gamma_k - \gamma(\phi(r))| + \tfrac{5}{2}a + c_1 \leqslant \tfrac{9}{2}a + c_1 \leqslant 5a \end{aligned}$$

for $a > c_1$, hence $E \subset E_{\gamma, 5a}$.

It remains to construct the family $I_n^{(k)}$.

Let $I_0^{(0)} = \mathbb{T}$ and let us suppose that the arcs $I_j^{(k-1)}$ are already defined. Consider the disks

$$D_1 = \{w\colon |w-\gamma_{k-1}| < \tfrac{1}{2}a\}, \quad D_2 = \{w\colon |w-\gamma_k| < \tfrac{1}{2}a\}.$$

It follows from (b) that $w_j = f(z_j^{(k-1)}) \in D_1$ for any index j. Setting $d_j = |w_j - \gamma_k|$ we obtain

$$\tfrac{1}{2}a \leqslant d_j \leqslant \tfrac{3}{2}a. \tag{3.7}$$

Consider the domain $G_j = G_{d_j}(f, z_j^{(k-1)})$ defined in Chapter 2, together with the arcs $I_{n,j} \subset \mathbb{T}$, $B_{n,j} \subset \mathbb{D}$ and points $z_{n,j} \in B_{n,j}$. Let $I_m^{(k)}\, (m = 0,1,2,\ldots)$ denote an enumeration of those $I_{n,j}\, (n,j = 0,1,2,\ldots)$ for which

$$f(z_{n,j}) \in D_2, \tag{3.8}$$

$$I_{n,j} \subset I_j^{(k-1)}$$

hold. We shall show that these arcs satisfy (a) to (c) if a is large enough.

(a) Fix $I = I_j^{(k-1)}$ and set $z_I = z(I)$, $B = B(I)$. A simple geometric consideration shows that

$$M = \left\{ w\colon d_j - K_1 \leqslant |w-w_j| \leqslant d_j + K_1, \left| \arg \frac{w-w_j}{\gamma_k - w_j} \right| \leqslant \tfrac{1}{4} \right\} \subset D_2.$$

Apply Corollary 2.4 with a replaced by d_j to

$$A = \{e^{it}\colon |t - \arg(\gamma_k - w_j)| \leqslant \tfrac{1}{4}\}.$$

For $m \in J$, where J is defined by (2.20), it easily follows that $f(z_m) \in M$. Hence all the arcs I_m for $m \in J$ are arcs $I_n^{(k)} \subset I$ of our collection. Corollary 2.4 shows that the collection $I_n^{(k)}$ satisfies (3.1) with $(1/c(\beta))\, e^{-(a/2)}$ instead of e^{-a} and (3.2) holds with c replaced by the factor of $|I|$ in (2.21). Since $|A| = 1/(4\pi)$ we can choose β so small that this factor is positive, for example $\beta = \pi/165$ will do so.

(b) This is immediate from the definition of the $I_n^{(k)}$ by (3.8).

For proving (c) fix two arcs $I = I_m^{(k-1)}$ and $I_n = I_n^{(k)}$. Note that the hyperbolic geodesic from $z(I)$ to $z(I_n)$ lies in $G_d(f, z(I))$ with $d = |f(z(I)) - \gamma_k| \leqslant \frac{3}{2}a$ by (3.7). Hence $|f(z) - f(z(I))| \leqslant \frac{3}{2}a + K_1$ on the geodesic by (2.8). It remains to apply Theorem P to the hyperbolic geodesic and to $C = [z(I), z(I_n)]$ to obtain $|f(z) - f(z(I))| < 2a$ on C if a is large enough, thus finishing the proof.

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Department of Mathematics
University of Michigan
Ann Arbor
MI 48109
USA