ADDENDUM: AN ARCLENGTH PROBLEM FOR CLOSE-TO-CONVEX FUNCTIONS

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Professor Z. Lewandowski has pointed out that the definition of close-to-convex function given in the paper [**1**], and used in the proof of Theorem 1, is a rather restrictive one. It is more natural to say, essentially as in Kaplan's original paper [**3**], that a function $f(z) \in S$ is close-to-convex if there is a convex function $\phi(z)$ such that $\mathrm{Re}\{f'(z)/\phi'(z)\} > 0$. The function ϕ may be normalized so that $|\phi'(0)| = 1$, but the requirement $\phi'(0) = 1$ imposed in [**1**] leads to a smaller class of functions.

Nevertheless, the inequality $L_r(f) \leqslant L_r(k)$ remains true for all functions f which are close-to-convex in the more general sense. A proof is given below. It seems likely that the Koebe function and its rotations are still the only extremal functions, but this point is left unsettled.

If $f(z)$ is close-to-convex in the general sense, its derivative may be represented in the form

$$f'(z) = e^{i\alpha} \psi'(z)\, P(z),$$

where $\psi \in C$, $\mathrm{Re}\{P(z)\} > 0$, and $P(0) = e^{-i\alpha}$, $-\pi/2 < \alpha < \pi/2$. Such a function $P(z)$ has a representation

$$P(z) = \cos\alpha \int_0^{2\pi} \frac{1 + ze^{is}}{1 - ze^{is}}\, d\nu(s) - i \sin\alpha = e^{-i\alpha} \int_0^{2\pi} \frac{1 + ze^{i(s+2\alpha)}}{1 - ze^{is}}\, d\nu(s),$$

where $\nu(s)$ is a non-decreasing function of total variation 1 on $0 \leqslant s \leqslant 2\pi$. Proceeding as in [**1**], one finds

$$L_r(f) \leqslant r \int_0^{2\pi} d\nu(s) \int_0^{2\pi} d\mu(t) \int_0^{2\pi} \left| \frac{1 + re^{i(\theta+s+2\alpha)}}{1 - re^{i(\theta+s)}} \right| \frac{d\theta}{|1 - re^{i(\theta+t)}|^2},$$

$\mu(t)$ being the non-decreasing function of unit total variation in terms of which $\psi'(z)$ is represented. The inequality $L_r(f) \leqslant L_r(k)$ is therefore established if it can be shown that $I(\alpha, t) \leqslant I(0, 0)$, where

$$I(\alpha, t) = \int_{-\pi}^{\pi} \frac{|1 + re^{i(\theta+2\alpha)}|}{|1 - re^{i\theta}|} \frac{d\theta}{|1 - re^{i(\theta+t)}|^2}.$$

But this is an immediate consequence of a more general result on "rearrangements" of functions. Given a non-negative measurable function $F(x)$ on $[-a, a]$, let $F^*(x)$ denote its symmetrically decreasing rearrangement, as defined in [**2**; p. 278].

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LEMMA. *If $F(x)$, $G(x)$, and $H(x)$ are non-negative integrable functions on the interval $[-a, a]$, then*

$$\int_{-a}^{a} F(x)\, G(x)\, H(x)\, dx \leqslant \int_{-a}^{a} F^*(x)\, G^*(x)\, H^*(x)\, dx.$$

Proof. Following [**2**; p. 278], we first note that the statement is obviously true if F, G, and H are characteristic functions of measurable sets. Using this observation, we next prove the inequality for *simple* functions; that is, for functions which take only a finite number of values. Indeed, any such function F can be represented [**2**; p. 279] as a linear combination of characteristic functions:

$$F(x) = \alpha_1 F_1(x) + \alpha_2 F_2(x) + \ldots + \alpha_n F_n(x), \quad \alpha_k > 0,$$

in such a way that

$$F^*(x) = \alpha_1 F_1{}^*(x) + \alpha_2 F_2{}^*(x) + \ldots + \alpha_n F_n{}^*(x).$$

The inequality then reduces to a linear combination of inequalities involving characteristic functions. Finally, the general result is obtained by approximating F, G, and H by sequences of simple functions [**2**; p. 280].

References

1. P. L. Duren, " An arclength problem for close-to-convex functions ", *Journal London Math. Soc.*, 39 (1964), 757–761.
2. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Second edition (Cambridge University Press, 1952).
3. W. Kaplan, " Close-to-convex schlicht functions ", *Michigan Math. J.*, 1 (1952), 169–185.

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