DIOPHANTINE APPROXIMATION BY PRIME NUMBERS, III

By R. C. VAUGHAN

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1. Introduction

We continue our study of diophantine approximation by prime numbers. The problem that we study in this paper arises naturally from the subject matter of [**13**], but our approach will differ from that of either [**13**] or [**14**]. However, before passing to a description of the contents of this paper we make some remarks concerning [**14**]. Let $\lambda_1, \lambda_2, \ldots, \lambda_s$ be s non-zero real numbers, not all of the same sign and not all in rational ratio. Let η be any real number. As in [**14**] we define $\mathscr{D}(k)$ to be the least s for which there is a positive real number σ such that the inequality

$$\left|\eta + \sum_{j=1}^{s} \lambda_j p_j^{k}\right| < (\max p_j)^{-\sigma} \tag{1.1}$$

has infinitely many solutions in prime numbers p_j. The number $D(k)$ is the corresponding value for s if we insist only that the variables are natural numbers. The major interest in the results of [**14**] lies in the bounds for the number of variables rather than that on the right of (1.1) we have a power of the maximum value of the variables. For this reason bounds are given only when $k \geqslant 4$ and $k \geqslant 5$ respectively, that is, only when the bounds for the number of variables are better than those known previously even when there is only a fixed ε on the right of (1.1).

By combining the methods of [**13**] and [**14**] with Theorem 4 of Hua [**7**] we can easily show that $\mathscr{D}(2) \leqslant 5$ and $\mathscr{D}(3) \leqslant 9$. As far as D is concerned, when $k = 2$ a great deal is known and we have nothing new to add. When $k = 4$ the method of [**14**] will give $D(4) \leqslant 14$, and when $k = 3$ the method can be combined with that of Davenport and Roth [**4**] to give $D(3) \leqslant 8$. In each of these, of course, the bound for the number of variables is not new. What is new is that the right of (1.1) contains a power of the maximum of the variables.

In this paper we are concerned with the following question, posed by Halberstam in conversation. Suppose that λ_1/λ_2 is negative and irrational. Then can one use sieve methods to show that there exists a positive integer k such that the numbers of the form $\lambda_1 p + \lambda_2 P_k$, with p a prime number and P_k a natural number having at most k prime factors, are dense on the real line? We answer this question in the affirmative and show that 4 is a permissible value for k.

We use results of Richert [**10**] and a new idea of Chen, who has recently shown [**1**] (for a shorter proof see Ross [**11**]) that every large even integer is of the form $p+P_2$. We also require two analogues of the Bombieri–Vinogradov mean value theorem (see Chapter 24 of Davenport [**2**]) and these form the bulk of our work. Here the exponent $\frac{1}{2}$ which appears in the Bombieri–Vinogradov theorem has to be replaced by $\frac{1}{4}$ and is the cause of our being unable to do better than $k = 4$. It has been found that the most satisfactory approach is to adapt the arguments of [**15**].

Our main theorem is as follows.

THEOREM. *There is a positive number τ such that if λ_0, λ_1, and λ_2 are real numbers with $\lambda_1/\lambda_2 < 0$ and λ_1/λ_2 irrational, then there are infinitely many prime numbers p for which there exists a square free natural number P_4, having at most four prime factors, such that*

$$|\lambda_0+\lambda_1 p+\lambda_2 P_4| < p^{-\tau}. \tag{1.2}$$

It clearly suffices to prove the theorem with

$$\lambda_2 = -1, \quad \lambda_1 > 0, \tag{1.3}$$

which we assume henceforward.

2. Notation and assumed results

Throughout p is a prime number, a, g, k, m, n, q, r are natural numbers, j, b, h are integers, $t, \alpha, \lambda, \sigma$ are real numbers, $u, v, x, y, T, X, \eta, \xi$ are real numbers greater than or equal to 1, z is a real number with

$$0 < z < \min(\lambda_1, 1/\lambda_1),$$

δ is a real number satisfying $0 < \delta \leqslant \frac{1}{4}$, and ε is a sufficiently small positive real number in terms of δ. Implied constants in the O and $\ll$ notations depend at most on δ, ε, λ_0, and λ_1. We write $\|\alpha\| = \min_n|\alpha-n|$ and if $\alpha+\frac{1}{2}$ is not an integer we use $[\![\alpha]\!]$ to denote the integer nearest to α. We further define $\mathscr{L} = \log y$, and given an arbitrary function f, $B(\sigma, f)$ is used to denote the formal expression

$$\sum_{n\leqslant\eta} \sum_{\chi \bmod nr}^{*} \int_{-T}^{T} |f(\sigma+it, \chi)| \frac{dt}{1+|t|}.$$

Here $\sum_\chi^*$ denotes summation over all the primitive characters modulo nr. We reserve d, φ, ω, Λ, and Ω for respectively the divisor function, Euler's function, the number of different prime divisors, von Mangoldt's function, and the total number of prime divisors. As usual $\psi(x) = \sum_{n\leqslant x}\Lambda(n)$, $\pi(x) = \sum_{p\leqslant x} 1$, and $L(s,\chi)$, where $s = \sigma+it$, denotes the Dirichlet L-function formed from the character χ. The letters $\mathscr{A}$, $\mathscr{D}$, $\mathscr{M}$, and $\mathscr{N}$ denote sets of integers and $|\mathscr{A}|$ is the cardinality of $\mathscr{A}$. We further use $\mathscr{A}_r$ to denote $\{h: h \in \mathscr{A}, r\,|\,h\}$.

For our proof of (1.2) we require the following results from multiplicative number theory.

LEMMA 2.1. *Suppose that* $T \geqslant 2$. *Then*

$$\sum_{\chi \bmod n}{}^{*} \int_{-T}^{T} |L(\tfrac{1}{2}+it,\chi)|^4 \frac{dt}{1+|t|} \ll \varphi(n)\log^5 nT \tag{2.1}$$

and

$$\sum_{\chi \bmod n}{}^{*} \int_{-T}^{T} |L'(\tfrac{1}{2}+it,\chi)|^4 \frac{dt}{1+|t|} \ll \varphi(n)\log^9 nT. \tag{2.2}$$

Proof. Theorem 10.1 of Montgomery [8] states that

$$\sum_{\chi}{}^{*} \int_{-T}^{T} |L(\tfrac{1}{2}+it,\chi)|^4\,dt \ll \varphi(n)T\log^4 nT.$$

The inequality

$$\sum_{\chi}{}^{*} \int_{-T}^{T} |L'(\tfrac{1}{2}+it,\chi)|^4\,dt \ll \varphi(n)T\log^8 nT$$

can be shown in the same way as Corollary 10.2 of Montgomery [8]. The lemma now follows by partial summation.

LEMMA 2.2. *Let* $T \geqslant 2$ *and*

$$U(s,\chi) = \sum_m c_m\chi(m)m^{-it},$$

where the summation is over any set of positive integers m *for which* $\sum_m |c_m|^2 m$ *exists. Then*

$$B(0,U^2) \ll \sum_m |c_m|^2(m+\eta^2 r\log T).$$

Proof. By Theorem 1 of Davenport and Halberstam [3], with the x_j the set of points a/nr with $1 \leqslant a \leqslant nr$, $(a,nr) = 1$, and $n \leqslant \eta$, we have

$$\sum_{n\leqslant\eta} \sum_{\substack{a=1\\(a,nr)=1}}^{nr} \left|\sum_{m=M+1}^{M+N} c_m m^{-it} e(am/nr)\right|^2 \ll (N+\eta^2 r) \sum_{m=M+1}^{M+N} |c_m|^2.$$

We then complete the proof of the lemma by using the method of Theorem 3 of Gallagher [6] combined with (5) of Gallagher [5], and performing a partial integration.

LEMMA 2.3. *Suppose that*

$$0 < \alpha \leqslant 1, \tag{2.3}$$

$$1 < \alpha u \leqslant 4, \tag{2.4}$$

$$0 < \lambda < 1, \tag{2.5}$$

$$\sum_{n \leqslant x^\alpha} 3^{\omega(n)} \left| \sum_{b \in \mathscr{A}_n} 1 - xn^{-1} \right| < x/\log^2 x, \tag{2.6}$$

$$P(v) = \prod_{p<v} p, \tag{2.7}$$

and

$$W(\mathscr{A}, u, \lambda) = \sum_{\substack{b \in \mathscr{A} \\ (b, P(x^{\alpha/4}))=1}}' \left(1 - \sum_{\substack{x^{\alpha/4} \leqslant p < x^{1/u} \\ p \mid b}} \lambda\left(1 - \frac{u \log p}{\log x}\right)\right) + \sum_{p \geqslant x^{\alpha/4}} \sum_{\substack{b \in \mathscr{A} \\ p^2 \mid b}} 1, \tag{2.8}$$

where $\sum'$ *means that those* b *which have a repeated prime factor are not counted in the summation. Then*

$$W(\mathscr{A}, u, \lambda) \geqslant \frac{2x}{\alpha \log x}\left(\log 3 - \lambda\alpha \int_u^{4/\alpha} \frac{t-u}{t\alpha - 1}\frac{dt}{t} + O((\log x)^{-1/15})\right). \tag{2.9}$$

This,† apart from a few trivial modifications, is a special case of Theorem 1 of Richert [**10**].

LEMMA 2.4. *Suppose that* (2.3) *and* (2.6) *hold. Then*

$$S(\mathscr{A}, x^\alpha) = \sum_{\substack{b \in \mathscr{A} \\ (b, P(x^\alpha))=1}} 1 \leqslant \frac{2x}{\alpha \log x}(1 + O((\log x)^{-1/14})).$$

This follows from Theorem B of Richert [**10**].

LEMMA 2.5 (Pólya–Vinogradov). *Let* χ *be a non-principal character modulo* n. *Then*

$$\sum_{m \leqslant u} \chi(m) \ll n^{1/2} \log n.$$

For a proof of this see, for instance, p. 146 of Prachar [**9**].

LEMMA 2.6. *Let* $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ $(\sigma > 1)$ *where* $a_n \ll \log 2n$. *Then for every natural number* N *and every* $\theta > 1$,

$$\sum_{n \leqslant N} a_n = \frac{1}{2\pi i}\int_{\theta - iT}^{\theta + iT} f(s)\frac{(N+\frac{1}{2})^s}{s}\,ds + O\left(\frac{(N+\frac{1}{2})^\theta}{T}\left(\frac{1}{(\theta-1)^2} + \log^2 N\right)\right). \tag{2.10}$$

This is a special case of Lemma 3.12 of Titchmarsh [**12**].

† Note added in proof. Proofs of Lemmas 2.3 and 2.4 can be found in the recent book of Halberstam and Richert [**6a**].

3. An analogue of the Bombieri–Vinogradov theorem

Let

$$\mathscr{D} = \{[\![\lambda_0+\lambda_1 p]\!] : p \leqslant y,\ \|\lambda_0+\lambda_1 p\| \leqslant \tfrac{1}{2}z\}, \tag{3.1}$$

$$R_r = |\mathscr{D}_r| - xr^{-1}, \quad x = z\pi(y), \tag{3.2}$$

and

$$\psi(v,r,h,m) = \sum_{\substack{n\leqslant v\\ hn\equiv m (\mathrm{mod}\, r)}} \Lambda(n). \tag{3.3}$$

In (3.1), if $[\![\lambda_0+\lambda_1 p_1]\!] = [\![\lambda_0+\lambda_1 p_2]\!]$ with $p_1 \neq p_2$, then the elements $[\![\lambda_0+\lambda_1 p_j]\!]$ are considered as distinct. In order to sieve the set $\mathscr{D}$ we require information concerning R_r, which is provided by the following theorem.

THEOREM 3.1. *Suppose that* $0 < \delta \leqslant \frac{1}{4}$ *and* $|\lambda_1 - a/q| \leqslant q^{-2}$ *with* $(a,q) = 1$ *and* $q \geqslant q_0(\delta, \lambda_0, \lambda_1)$. *Let* $y = q^{2/(1+\delta)}$. *Then*

$$\sum_{r\leqslant y^{1/4-\delta}} |R_r| \ll y^{1-\delta/4}. \tag{3.4}$$

The proof of (3.4) rests on the following lemma.

LEMMA 3.1. *Let*

$$\Psi(u,v,r,h) = \sum_{\substack{m\leqslant u\\ (m,r)=1}} \left(\psi(v,r,h,m) - \frac{\psi(v)}{\varphi(r)}\right) \tag{3.5}$$

and

$$\Psi(r) = \max|\Psi(u,v,r,h)|, \tag{3.6}$$

where the maximum is taken over all h,u,v *with* $(h,r) = 1$, $u \leqslant q(2+|\lambda_0|)$, *and* $v \leqslant y$ *respectively. Then*

$$\sum_{m\leqslant y^{1/4-\delta}} \Psi(mq) \ll y^{1-(2\delta/7)}. \tag{3.7}$$

We defer the proof of this lemma until the next section and proceed with the deduction of (3.4) from (3.7).

Write $\lambda_0 = (b+\theta_0)/q$ and $\lambda_1 = a/q+\theta_1 q^{-2}$ where $|\theta_0|$, $|\theta_1| \leqslant 1$, and suppose that $p \leqslant y$. Then

$$\lambda_0+\lambda_1 p = (b+ap+\theta_0)/q+\theta_1 pq^{-2}.$$

Now write $b+ap = j+hq$ with $-\frac{1}{2}q < j \leqslant \frac{1}{2}q$. Then

$$\lambda_0+\lambda_1 p - h = (j+\theta_0)/q+\theta_1 pq^{-2}.$$

Thus $-\frac{1}{2}z \leqslant \lambda_0+\lambda_1 p - h \leqslant \frac{1}{2}z$ if and only if

$$-\tfrac{1}{2}zq \leqslant j+\theta_0+\theta_1 p/q \leqslant \tfrac{1}{2}zq. \tag{3.8}$$

Consider the inequality

$$|j| \leqslant \tfrac{1}{2}zq. \tag{3.9}$$

Clearly all but $O(y/q) = O(q^{1-\delta})$ possible values of j which satisfy (3.8) also satisfy (3.9) and vice versa. Moreover, if $-\frac{1}{2}z \leqslant \lambda_0+\lambda_1 p - h \leqslant \frac{1}{2}z$, we must have $h = [\![\lambda_0+\lambda_1 p]\!]$ and in this case $r\,|\,[\![\lambda_0+\lambda_1 p]\!]$ if and only if $r\,|\,h$, that is, if and only if $b+ap \equiv j \pmod{qr}$. If j is exceptional then h either is $[\![\lambda_0+\lambda_1 p]\!]$ or differs from $[\![\lambda_0+\lambda_1 p]\!]$ by 1. In the latter case, $r\,|\,[\![\lambda_0+\lambda_1 p]\!]$ implies that either $b+ap \equiv j+q \pmod{qr}$ or $b+ap \equiv j-q \pmod{qr}$. Thus, by (3.1),

$$|\mathscr{D}_r| = \sum_{|j|\leqslant \frac{1}{2}zq} \sum_{\substack{p\leqslant y \\ b+ap\equiv j \,(\mathrm{mod}\, qr)}} 1 + O\left(q^{1-\delta}\left(\frac{y(a,r)}{qr}+1\right)\right).$$

Therefore, by (3.2),

$$\sum_{r\leqslant y^{1/4-\delta}} |R_r| \ll yd(a)\mathscr{L}q^{-\delta} + \sum_{r\leqslant y^{1/4-\delta}} \left| zr^{-1}\pi(y) - \sum_{|j|\leqslant \frac{1}{2}zq} \sum_{\substack{p\leqslant y \\ b+ap\equiv j \,(\mathrm{mod}\, qr)}} 1 \right|. \tag{3.10}$$

Let

$$g = (a,r). \tag{3.11}$$

Then

$$\sum_{|j|\leqslant \frac{1}{2}zq} \sum_{\substack{p\leqslant y \\ b+ap\equiv j \,(\mathrm{mod}\, qr)}} 1 = \sum_{\substack{h \\ |b+h|\leqslant \frac{1}{2}zq}} \sum_{\substack{p\leqslant y \\ ap\equiv h \,(\mathrm{mod}\, qr)}} 1$$

$$= \sum_{\substack{j \\ |bg^{-1}+j|\leqslant zq/2g}} \sum_{\substack{p\leqslant y,\, p\nmid qr/g \\ ap/g\equiv j \,(\mathrm{mod}\, qr/g)}} 1 + O(\mathscr{L}).$$

Hence, by (3.10) and (3.11),

$$\sum_{r\leqslant y^{1/4-\delta}} |R_r|$$
$$\ll yd(a)\mathscr{L}q^{-\delta} + \sum_{n|a} \sum_{\substack{m\leqslant y^{1/4-\delta}/n \\ (m,a/n)=1}} \left(\left| \sum_{\substack{j,|bn^{-1}+j|\leqslant zq/2n \\ (j,mq)=1}} \left(\frac{\pi(y)}{\varphi(mq)} - \sum_{\substack{p\leqslant y \\ ap/n\equiv j \,(\mathrm{mod}\, mq)}} 1 \right) \right| \right.$$
$$\left. + \pi(y) \left| \frac{z}{mn} - \sum_{\substack{j,|bn^{-1}+j|\leqslant zq/2n \\ (j,mq)=1}} \frac{1}{\varphi(mq)} \right| \right).$$

It is easily verified that

$$\sum_{\substack{u<j\leqslant v \\ (j,r)=1}} 1 = \frac{\varphi(r)}{r}(v-u) + O(d(r)), \tag{3.12}$$

and we recall that $\lambda_0 = (b+\theta_0)/q$, so that $|b| \leqslant q(1+|\lambda_0|)$. Hence

$$\sum_{r\leqslant y^{1/4-\delta}} |R_r|$$
$$\ll yd(a)\mathscr{L}q^{-\delta} + d(a) \sum_{m\leqslant y^{1/4-\delta}} \max_{\substack{x\leqslant q(2+|\lambda_0|) \\ (h,mq)=1}} \left| \sum_{\substack{j\leqslant x \\ (j,mq)=1}} \left(\frac{\pi(y)}{\varphi(mq)} - \sum_{\substack{p\leqslant y \\ hp\equiv j \,(\mathrm{mod}\, mq)}} 1 \right) \right|$$
$$\ll yd(a)\mathscr{L}q^{-\delta} + d(a) \sum_{m\leqslant y^{1/4-\delta}} \Psi(mq), \tag{3.13}$$

where we have used the inequality

$$\sum_{2\leqslant n\leqslant y} c_n \ll \max_{v\leqslant y}\left|\sum_{n\leqslant v} c_n \log n\right|.$$

The theorem now follows from (3.13) and (3.7).

4. Proof of Lemma 3.1

Suppose that $(h, mq) = 1$, $mq \leqslant \xi \leqslant y$, $u \leqslant q(2+|\lambda_0|)$, and $v \leqslant y$. Consider the expression $\Psi(u, v, mq, h)$ given by (3.5). We use the inequality

$$\left|\psi(v) - \sum_{\substack{n\leqslant v\\(n,mq)=1}} \Lambda(n)\right| \ll \mathscr{L}^2$$

to replace $\psi(v)$ by $\sum_{n\leqslant v,(n,mq)=1} \Lambda(n)$. We then write the resulting expression in terms of characters modulo mq. Next we replace each character χ modulo mq by the primitive character χ^* that induces it, making use of the inequality

$$|\psi(v,\chi) - \psi(v,\chi^*)| \ll \mathscr{L}^2$$

and Lemma 2.5. Furthermore, we note that

$$\sum_{n\leqslant u} \chi(n) \ll d(mq) \max_{k\leqslant u}\left|\sum_{n\leqslant k} \chi^*(n)\right|. \tag{4.1}$$

Hence, by (3.6),

$$\Psi(mq) \ll \frac{d(mq)}{\varphi(mq)} \sum_{\chi\neq\chi_0 \bmod mq} \max_{\substack{u\leqslant q(2+|\lambda_0|)\\ v\leqslant y}} \left|\sum_{k\leqslant u} \chi^*(k)\psi(v,\chi^*)\right| + m^{1/2}q^{1/2}\mathscr{L}^3.$$

Thus

$$\sum_{m\leqslant\xi} \Psi(mq) \ll \xi^{3/2}q^{1/2}\mathscr{L}^3 + y^\varepsilon q^{-1} \sum_{r|q} \sum_{r^{-1}<n\leqslant\xi} n^{-1} \sum_{\chi \bmod nr}^{*} \max_{\substack{u\leqslant q(2+|\lambda_0|)\\ v\leqslant y}} \left|\sum_{k\leqslant u} \chi(k)\psi(v,\chi)\right|$$

$$\ll \xi^{3/2}q^{1/2}\mathscr{L}^3 + y^\varepsilon q^{-1} \sum_{r|q} \left(\xi^{-1}\Sigma_1(\xi) + \int_1^\xi \Sigma_1(\eta)\eta^{-2}\,d\eta\right), \tag{4.2}$$

where

$$\Sigma_1(\eta) = \sum_{r^{-1}<n\leqslant\eta} \sum_{\chi \bmod nr}^{*} \max_{\substack{u\leqslant q(2+|\lambda_0|)\\ v\leqslant y}} \left|\sum_{k\leqslant u} \chi(k)\psi(v,\chi)\right|.$$

By Hölder's inequality,

$$\Sigma_1(\eta) \leqslant \Sigma_2^{1/4}\Sigma_3^{3/4} \tag{4.3}$$

where

$$\Sigma_2 = \sum_{r^{-1}<n\leqslant\eta} \sum_{\chi \bmod nr}^{*} \max_{u\leqslant q(2+|\lambda_0|)} \left|\sum_{k\leqslant u} \chi(k)\right|^4 \tag{4.4}$$

and

$$\Sigma_3 = \sum_{r^{-1}<n\leqslant\eta} \sum_{\chi \bmod nr}^{*} \max_{v\leqslant y} |\psi(v,\chi)|^{4/3}. \tag{4.5}$$

Let

$$T = y^{100}, \tag{4.6}$$

$$\theta = 1+\mathscr{L}^{-1}, \tag{4.7}$$

and

$$u_0 = \tfrac{1}{2}+\max_{m\leqslant u} m. \tag{4.8}$$

Then, by Lemma 2.6,

$$\sum_{k\leqslant u}\chi(k) = \frac{1}{2\pi i}\int_{\theta-iT}^{\theta+iT} L(s,\chi)\frac{u_0^{\,s}}{s}ds+O(1).$$

We note that $L(s,\chi)$ is regular for $\sigma > 0$, and

$$L(s,\chi) \ll m^{1/2}q^{1/2}(1+|t|)^{1/2} \quad (\sigma \geqslant \tfrac{1}{2}).$$

Thus

$$\sum_{k\leqslant u}\chi(k) = \frac{1}{2\pi i}\int_{1/2-iT}^{1/2+iT} L(s,\chi)\frac{u_0^{\,s}}{s}ds+O(1). \tag{4.9}$$

Hence, by Hölder's inequality and (2.1),

$$\sum_{\chi \bmod nr}^{*}\ \max_{u\leqslant q(2+|\lambda_0|)}\left|\sum_{k\leqslant u}\chi(k)\right|^4 \ll q^2\mathscr{L}^3B(\tfrac{1}{2},L^4)+\varphi(nr)$$
$$\ll q^2\varphi(nr)\mathscr{L}^8.$$

Therefore, by (4.4),

$$\Sigma_2 \ll \eta^2q^2r\mathscr{L}^8. \tag{4.10}$$

We next treat Σ_3. Let

$$v_0 = \tfrac{1}{2}+\max_{m\leqslant v} m, \tag{4.11}$$

$$F(s,\chi) = \sum_{n\leqslant \eta r^{1/2}}\Lambda(n)\chi(n)n^{-s} \tag{4.12}$$

and

$$G(s,\chi) = \sum_{n\leqslant y^{1/2}}\mu(n)\chi(n)n^{-s}. \tag{4.13}$$

Then, by Lemma 2.6, for a non-principal character χ, we have

$$2\pi i\psi(v,\chi) = \int_{\theta-iT}^{\theta+iT}\left(\frac{L'}{L}+F\right)(LG-1)(s,\chi)\frac{v_0^{\,s}}{s}ds$$
$$+\int_{1/2-iT}^{1/2+iT}(F-L'G-LFG)(s,\chi)\frac{v_0^{\,s}}{s}ds+O(1). \tag{4.14}$$

Hence, by (4.5) and Hölder's inequality, applied several times,

$$\Sigma_3 \ll \mathscr{L}^{1/3}y^{4/3}B(\theta,|(L'/L+F)(LG-1)|^{4/3})$$
$$+\mathscr{L}^{1/3}y^{2/3}B(\tfrac{1}{2},|F-L'G-LFG|^{4/3})+\eta^2r$$
$$\ll \mathscr{L}^{1/3}y^{4/3}B(\theta,(L'/L+F)^4)^{1/3}B(\theta,(LG-1)^2)^{2/3}$$
$$+\mathscr{L}^{1/3}y^{2/3}B(\tfrac{1}{2},1+|L|^4+|L'|^4)^{1/3}B(\tfrac{1}{2},1+|F|^4)^{1/3}B(\tfrac{1}{2},1+|G|^4)^{1/3}.$$

Thus, by Lemmas 2.1 and 2.2,

$$\Sigma_3 \ll y^{4/3+\varepsilon}(1+\eta^2 r y^{-1/2})^{2/3} + y^{2/3+\varepsilon}(\eta^2 r)^{2/3}(y+\eta^2 r)^{1/3}$$
$$\ll y^{4/3+\varepsilon} + y^{1+\varepsilon}(\eta^2 r)^{2/3} + y^{2/3+\varepsilon}\eta^2 r.$$

Hence, by (4.3) and (4.10),

$$\Sigma_1(\eta) \ll y^{1+\varepsilon}\eta^{1/2}q^{1/2}r^{1/4} + y^{3/4+\varepsilon}\eta^{3/2}q^{1/2}r^{3/4} + y^{1/2+\varepsilon}\eta^2 q^{1/2} r.$$

Therefore, by (4.2),

$$\sum_{m\leqslant\xi} \Psi(mq) \ll y^{3\varepsilon}(yq^{-1/4} + y^{3/4}\xi^{1/2}q^{1/4} + y^{1/2}\xi q^{1/2} + \xi^{3/2}q^{1/2}).$$

This implies Lemma 3.1.

5. Another analogue of the Bombieri–Vinogradov theorem

Let

$$y_1 = \lambda_0 + \lambda_1 y, \tag{5.1}$$

$$\mathcal{N} = \mathcal{N}(X) = \Big\{p_1p_2p_3p_4p_5 \colon X \leqslant p_1 \leqslant y_1^{1/5},\ p_1 \leqslant p_2 \leqslant \left(\frac{y_1}{p_1}\right)^{1/4},$$
$$p_2 \leqslant p_3 \leqslant \left(\frac{y_1}{p_1p_2}\right)^{1/3},\ p_3 \leqslant p_4 \leqslant \left(\frac{y_1}{p_1p_2p_3}\right)^{1/2},\ X \leqslant p_5 \leqslant \frac{y_1}{p_1p_2p_3p_4}\Big\}, \tag{5.2}$$

$$\mathcal{D}^* = \mathcal{D}^*(X) = \{n \colon n \leqslant y, \|\lambda_0 + \lambda_1 n\| \leqslant \tfrac{1}{2}z, [[\lambda_0+\lambda_1 n]] \in \mathcal{N}\}, \tag{5.3}$$

$$\vartheta(k,j,h) = \sum_{\substack{n\in\mathcal{N}\\ hn\equiv j \,(\mathrm{mod}\, k)}} 1, \tag{5.4}$$

and

$$x_1 = \lambda_1^{-1} z |\mathcal{N}|. \tag{5.5}$$

Further write

$$R_r^* = |\mathcal{D}_r^*| - x_1 r^{-1}. \tag{5.6}$$

Theorem 5.1. *Suppose that* $0 < \delta \leqslant \frac{1}{50}$ *and* $y^{1/16-\delta/3} < X < y^{1/6}$. *Then on the hypothesis of Theorem* 3.1,

$$\sum_{r\leqslant y^{1/4-\delta}} |R_r^*| \ll y^{1-\delta/4}. \tag{5.7}$$

Our procedure is similar to that of §§ 3 and 4, but is sufficiently different because of the nature of $\mathcal{D}^*$ for it to be necessary to give the details. As in § 3 the proof depends on a lemma.

Lemma 5.1. *Let*

$$\Theta(u,r,h) = \sum_{\substack{m\leqslant u\\ (m,r)=1}} \left(\vartheta(r,m,h) - \frac{|\mathcal{N}|}{\varphi(r)}\right) \tag{5.8}$$

and

$$\Theta(r) = \max|\Theta(u,r,h)|, \tag{5.9}$$

where the maximum is taken over all u, h with $u \leqslant q(2+|\lambda_0|)$ and $(h,r) = 1$. Then

$$\sum_{m \leqslant y^{1/4-\delta}} \Theta(ma) \ll y^{1-2\delta/7}. \tag{5.10}$$

To deduce the theorem from the lemma we argue as follows. By (5.2) and (5.3),

$$|\mathscr{D}_r^*| = \sum_{\substack{m,n,n\in\mathscr{N}, r|m \\ |\lambda_0+\lambda_1 m - n| \leqslant \frac{1}{2}z}} 1 + O(1)$$

$$= \sum_{\substack{n\in\mathscr{N}, \|(n-\lambda_0)/\lambda_1\| \leqslant z/2\lambda_1 \\ r | [\![(n-\lambda_0)/\lambda_1]\!]}} 1 + O(1).$$

Write $-\lambda_0/\lambda_1 = (b+\theta_2)/a$ with $|\theta_2| \leqslant 1$ and note that

$$1/\lambda_1 = q/a + \theta_3/(\lambda_1 a q)$$

with $|\theta_3| \leqslant 1$. Thus by repeating the argument of § 3 we have

$$|\mathscr{D}_r^*| = \sum_{|j| \leqslant za/2\lambda_1} \vartheta(ar, j-b, q) + O\left(\frac{y(q,r)}{q^\delta r} + q^{1-\delta}\right)$$

$$= \sum_{h, |b+h| \leqslant za/2\lambda_1} \vartheta(ar, h, q) + O\left(\frac{y(q,r)}{q^\delta r} + q^{1-\delta}\right). \tag{5.11}$$

Let

$$g = (q, r). \tag{5.12}$$

Then, by (5.11),

$$|\mathscr{D}_r^*| = \sum_{j, |b/g+j| \leqslant za/2\lambda_1 g} \vartheta\left(a\frac{r}{g}, j, \frac{q}{g}\right) + O\left(\frac{yg}{q^\delta r} + q^{1-\delta}\right). \tag{5.13}$$

By (5.4),

$$\sum_{\substack{j, (j, ar/g)>1 \\ |b/g+j| \leqslant za/2\lambda_1 g}} \vartheta\left(a\frac{r}{g}, j, \frac{q}{g}\right) \ll \sum_{j, |b/g+j| \leqslant za/2\lambda_1 g} \sum_{\substack{pm\in\mathscr{N}, p|(j, ar/g) \\ (q/g)pm \equiv j \pmod{ar/g}}} 1$$

$$\ll \sum_{\substack{p | ar/g \\ p \geqslant X}} \sum_{\substack{h \\ |b/(gp)+h| \leqslant za/2\lambda_1 gp}} \sum_{\substack{m \leqslant y_1/p \\ (q/g)m \equiv h \pmod{ar/gp}}} 1.$$

This is easily seen to be

$$\ll \sum_{\substack{p|ar/g \\ p \geqslant X}} \left(\frac{za}{gp} + 1\right)\left(\frac{y_1 g}{ar} + 1\right)$$

$$\ll zy\mathscr{L}X^{-1}r^{-1} + y\mathscr{L}gq^{-1}r^{-1} + zq\mathscr{L}X^{-1} + \mathscr{L}.$$

Hence, by (5.13),

$$|\mathscr{D}_r^*| = \sum_{\substack{j, |b/g+j| \leqslant za/2\lambda_1 g \\ (j, ar/g)=1}} \vartheta\left(a\frac{r}{g}, j, \frac{q}{g}\right) + O\left(\frac{y\mathscr{L}g}{q^\delta r} + q^{1-\delta}\right).$$

Thus, by (5.6), (5.5), and (3.12),

$$R_r^* \ll \left| \sum_{\substack{j,\,(j,\,ar/g)=1\\ |b/g+j|\leqslant za/2\lambda_1 g}} \left(\vartheta\left(a\frac{r}{g}, j, \frac{q}{g}\right) - \frac{|\mathscr{N}|}{\varphi(ar/g)}\right)\right| + y\mathscr{L}gq^{-\delta}r^{-1}. \tag{5.14}$$

We note that $(za/2\lambda_1)+|b| \leqslant q+(a/\lambda_1)|\lambda_0| \leqslant q(2+|\lambda_0|)$. Theorem 5.1 now follows from (5.14) and Lemma 5.1.

6. The proof of Lemma 5.1

By (5.2),

$$\sum_{\substack{n\in\mathscr{N}\\ (n,r)>1}} 1 \ll \sum_{\substack{p|r\\ p\geqslant X}} \sum_{m\leqslant y_1/p} 1 \ll y\mathscr{L}X^{-1}.$$

Hence, by (5.8) and (5.4),

$$\Theta(u,r,h) = \frac{1}{\varphi(r)} \sum_{\chi\neq\chi_{0\bmod r}} \left(\sum_{m\leqslant u} \bar\chi(m)\right)\chi(h)\,\vartheta(\chi) + O(yu\mathscr{L}X^{-1}\varphi(r)^{-1}),$$

where $\vartheta(\chi) = \sum_{n\in\mathscr{N}}\chi(n)$. Thus, by the analogue of (4.1), and (5.9),

$$\Theta(ma) \ll yq\mathscr{L}X^{-1}\varphi(ma)^{-1} + \frac{d(ma)}{\varphi(ma)} \sum_{\chi\neq\chi_{0\bmod ma}} \max_{u\leqslant q(2+|\lambda_0|)} \left|\sum_{j\leqslant u}\chi^*(j)\vartheta_{ma}(\chi^*)\right|,$$

where

$$\vartheta_r(\chi) = \sum_{n\in\mathscr{N},(r,n)=1} \chi(n). \tag{6.1}$$

Hence, if $\xi \leqslant y^{1/4-\delta}$,

$$\sum_{m\leqslant\xi}\Theta(ma) \ll y^{1-\delta} + y^{\varepsilon}q^{-1}\sum_{r|a}\max_{k\leqslant y} \sum_{r^{-1}<n\leqslant\xi} n^{-1} \sum_{\chi_{\bmod nr}}^* \max_{u\leqslant q(2+|\lambda_0|)} \left|\sum_{j\leqslant u}\chi(j)\vartheta_k(\chi)\right|$$

$$\ll y^{1-\delta} + y^{\varepsilon}q^{-1}\sum_{r|a}\max_{k\leqslant y}\left(\xi^{-1}\Sigma_4(\xi,k) + \int_1^{\xi}\Sigma_4(\eta,k)\eta^{-2}\,d\eta\right), \tag{6.2}$$

where

$$\Sigma_4(\eta,k) = \sum_{r^{-1}<n\leqslant\eta} \sum_{\chi_{\bmod nr}}^* \max_{u\leqslant q(2+|\lambda_0|)} \left|\sum_{j\leqslant u}\chi(j)\,\vartheta_k(\chi)\right|. \tag{6.3}$$

Let

$$\Sigma_5 = \sum_{r^{-1}<n\leqslant\eta} \sum_{\chi_{\bmod nr}}^* |\vartheta_k(\chi)|^{4/3}. \tag{6.4}$$

Then, by (6.3), (4.4), and Hölder's inequality,

$$\Sigma_4(\eta,k) \leqslant \Sigma_2^{1/4}\Sigma_5^{3/4}. \tag{6.5}$$

Let

$$\mathscr{M} = \{m\colon m = p_1p_2p_3p_4, X\leqslant p_1\leqslant y_1^{1/5}, p_1\leqslant p_2\leqslant (y_1/p_1)^{1/4},$$
$$p_2\leqslant p_3\leqslant (y_1/(p_1p_2))^{1/3}, p_3\leqslant p_4\leqslant (y_1/(p_1p_2p_3))^{1/2}\}, \tag{6.6}$$

$$\mathscr{M}_1 = \{m\colon m\in\mathscr{M}, m\leqslant y^{1/2}\}, \tag{6.7}$$

and

$$\mathscr{M}_2 = \{m\colon m\in\mathscr{M}, m > y^{1/2}\}. \tag{6.8}$$

Then, by (6.1) and (5.2),

$$|\vartheta_k(\chi)|^{4/3} \ll |\vartheta_{1,k}(\chi)|^{4/3}+|\vartheta_{2,k}(\chi)|^{4/3}, \tag{6.9}$$

where

$$\vartheta_{j,k}(\chi) = \sum_{\substack{m,p,\, m\in\mathscr{M}_j\\ X\leqslant p\leqslant y_1/m\\ (mp,k)=1}} \chi(mp). \tag{6.10}$$

Let

$$D_j(s,\chi) = \sum_{\substack{m\in\mathscr{M}_j\\ (m,k)=1}} \chi(m)m^{-s}, \tag{6.11}$$

$$E_1(s,\chi) = \sum_{\substack{p\geqslant \max(\eta^2 r, X)\\ p\nmid k}} \chi(p)p^{-s} \quad (\sigma>1), \tag{6.12}$$

$$E_2(s,\chi) = \sum_{\substack{p\geqslant \max(\eta r^{1/2}, X)\\ p\nmid k}} \chi(p)p^{-s} \quad (\sigma>1), \tag{6.13}$$

$$H_1(s,\chi) = \sum_{\substack{X\leqslant p<\eta^2 r\\ p\nmid k}} \chi(p)p^{-s}, \tag{6.14}$$

and

$$H_2(s,\chi) = \sum_{\substack{X\leqslant p<\eta r^{1/2}\\ p\nmid k}} \chi(p)p^{-s}. \tag{6.15}$$

Further, let

$$y_0 = \tfrac{1}{2}+\max_{m\leqslant y_1} m. \tag{6.16}$$

Then for a non-principal character χ to a modulus not exceeding y, we have, by Lemma 2.6, (4.6), (4.7), and (6.10),

$$2\pi i\vartheta_{j,k}(\chi) = \int_{\theta-iT}^{\theta+iT} (D_jE_j)(s,\chi)\frac{y_0^{\,s}}{s}\,ds + \int_{1/2-iT}^{1/2+iT} (D_jH_j)(s,\chi)\frac{y_0^{\,s}}{s}\,ds + O(1).$$

Hence, by Hölder's inequality,

$$|\vartheta_{j,k}(\chi)|^{4/3} \ll y^{4/3}\mathscr{L}^{1/3}\int_{-T}^{T} |(D_jE_j)(\theta+it,\chi)|^{4/3}\frac{dt}{1+|t|} + y^{2/3}\mathscr{L}^{1/3}\int_{-T}^{T} |(D_jH_j)(\tfrac{1}{2}+it,\chi)|^{4/3}\frac{dt}{1+|t|}+1.$$

Thus, by Hölder's inequality,

$$\sum_{r^{-1}<n\leqslant\eta}\ \sum_{\chi \bmod nr}{}^{*} |\vartheta_{1,k}(\chi)|^{4/3} \ll y^{4/3}\mathscr{L}^{1/3}(B(\theta,D_1^{\,4}))^{1/3}(B(\theta,E_1^{\,2}))^{2/3} + y^{2/3}\mathscr{L}^{1/3}(B(\tfrac{1}{2},D_1^{\,4}))^{1/3}(B(\tfrac{1}{2},H_1^{\,2}))^{2/3}+\eta^2 r.$$

Therefore, by Lemma 2.2, (6.6), (6.7), (6.11), (6.12), and (6.14),

$$\sum_{r^{-1}<n\leqslant\eta}\ \sum_{\chi \bmod nr}{}^{*} |\vartheta_{1,k}(\chi)|^{4/3} \ll y^{4/3+\varepsilon}(1+\eta^2 rX^{-8})^{1/3} + y^{2/3+\varepsilon}(y+\eta^2 r)^{1/3}(\eta^2 r)^{2/3}+\eta^2 r$$
$$\ll g^{4/3+\varepsilon} + y^{7/6+\delta}(\eta^2 r)^{1/3} + y^{1+\varepsilon}(\eta^2 r)^{2/3} + y^{2/3+\varepsilon}\eta^2 r. \tag{6.17}$$

Similarly

$$\sum_{r^{-1}<n\leqslant\eta}\ \sum^*_{\chi \bmod nr} |\vartheta_{2,k}(\chi)|^{4/3}$$
$$\ll y^{4/3}\mathscr{L}^{1/3}(B(\theta,D_2^{\,2}))^{2/3}(B(\theta,E_2^{\,4}))^{1/3}$$
$$+y^{2/3}\mathscr{L}^{1/3}(B(\tfrac12,D_2^{\,2}))^{2/3}(B(\tfrac12,H_2^{\,4}))^{1/3}+\eta^2 r$$
$$\ll y^{4/3+\varepsilon}(1+\eta^2 r y^{-1/2})^{2/3}+y^{2/3+\varepsilon}(y^{4/5}+\eta^2 r)^{2/3}(\eta^2 r)^{1/3}+\eta^2 r$$
$$\ll y^{4/3+\varepsilon}+y^{6/5+\varepsilon}(\eta^2 r)^{1/3}+y^{1+\varepsilon}(\eta^2 r)^{2/3}+y^{2/3+\varepsilon}\eta^2 r.$$

Hence, by (6.4) and (6.9),

$$\Sigma_5 \ll y^{4/3+\varepsilon}+y^{6/5+\varepsilon}(\eta^2 r)^{1/3}+y^{1+\varepsilon}(\eta^2 r)^{2/3}+y^{2/3+\varepsilon}\eta^2 r.$$

Therefore, by (4.10) and (6.5),

$$\Sigma_4(\eta,k)$$
$$\ll y^{1+\varepsilon}\eta^{1/2}q^{1/2}r^{1/4}+y^{9/10+\varepsilon}\eta q^{1/2}r^{1/2}+y^{3/4+\varepsilon}\eta^{3/2}q^{1/2}r^{3/4}+y^{1/2+\varepsilon}\eta^2 q^{1/2}r.$$

Thus, by (6.2),

$$\sum_{m\leqslant\xi}\Theta(ma) \ll y^{3\varepsilon}(yq^{-1/4}+y^{9/10}+y^{3/4}(\xi^2 q)^{1/4}+y^{1/2}(\xi^2 q)^{1/2}).$$

Lemma 5.1 now follows easily.

7. The proof of the main theorem

Let

$$\delta = 10^{-5}, \tag{7.1}$$
$$\tau = \tfrac16\delta, \tag{7.2}$$
$$\alpha = \tfrac14-\delta, \tag{7.3}$$
$$|\lambda_1-a/q| \leqslant q^{-2},\quad q > q_0(\lambda_0,\lambda_1),\quad (a,q)=1, \tag{7.4}$$
$$y = q^{2/(1+\delta)}, \tag{7.5}$$
$$z = y^{-\delta/5}, \tag{7.6}$$
$$x = z\pi(y), \tag{7.7}$$
$$u = 4(1+\delta)/(1-4\delta), \tag{7.8}$$

and

$$\lambda = 1/(6-u-\delta). \tag{7.9}$$

Let

$$V = \sum''_{\substack{b\in\mathscr{D}\\(b,P(x^{\alpha/4}))=1}}\Big(1-\sum_{\substack{x^{\alpha/4}\leqslant p<x^{1/u}\\ p|b}}\lambda(1-(u(\log p)/\log x))\Big), \tag{7.10}$$

where the $''$ indicates that the summation is restricted to those elements of $\mathscr{D}$ which do not have repeated prime factors. By (3.1), $|\mathscr{D}_r| \ll 1+y/r$.

Hence, by Lemma 2.3 and Theorem 3.1,

$$V \geqslant \frac{2x}{\alpha \log x}$$
$$\times (\log 3 - \lambda(1+\delta)\log 4 + \lambda(1+\delta)\log(1+\delta) + \lambda\delta\log(3/\delta) + O((\log x)^{-1/15})). \qquad (7.11)$$

By (7.8) and (7.9), $0 < \lambda - \frac{1}{2} < 6\delta$. Hence

$$\lambda(1+\delta)\log 4 - \lambda(1+\delta)\log(1+\delta) - \lambda\delta\log(3/\delta)$$
$$< (\tfrac{1}{2} + 6\delta)(\log 4 + \delta\log 4 - 13\delta) < \log 2 + 4\delta.$$

Hence, by (7.11) and (7.3),

$$V > \frac{8\log\frac{3}{2} - 32\delta}{1-4\delta}\frac{x}{\log x}.$$

Thus, by (7.1),

$$V > 3{\cdot}243x/\log x. \qquad (7.12)$$

Consider the definition of V, (7.10). The weight in the sum satisfies

$$1 - \sum_{\substack{x^{\alpha/4} \leqslant p < x^{1/u} \\ p \mid b}} \lambda\left(1 - u\frac{\log p}{\log x}\right) \leqslant 1 - \lambda\left(\Omega(b) - u\frac{\log|b|}{\log x}\right).$$

Hence, by (3.1), (7.5), (7.7), (7.9), and (7.1),

$$1 - \sum_{\substack{x^{\alpha/4} \leqslant p < x^{1/u} \\ p \mid b}} \lambda\left(1 - u\frac{\log p}{\log x}\right) \leqslant \lambda\left(6 - \Omega(b) + u\frac{\log(\lambda_0 + \lambda_1 y + \frac{1}{2}z)}{\log x} - u - \delta\right)$$
$$< \lambda(6 - \Omega(b)).$$

Thus the weight is negative if $\Omega(b) > 5$ and is at most λ if $\Omega(b) = 5$. Moreover, every element of $\mathscr{D}$ for which there is a positive contribution to V has no prime factor less than $x^{\alpha/4}$, and is squarefree. It therefore suffices to show that the contribution to V from those elements of $\mathscr{D}$ having exactly five prime factors is at most $3{\cdot}041x/\log x$. By (3.1), (5.2), and (5.3) it is thus enough to show that

$$\lambda S(\mathscr{D}^*(x^{\alpha/4}), y^{1/4-\delta}) < 3{\cdot}041x/\log x. \qquad (7.13)$$

By Lemma 2.4, Theorem 5.1, and (7.3),

$$S(\mathscr{D}^*(x^{\alpha/4}), y^{1/4-\delta}) < \left(\frac{8}{1-4\delta} + \delta\right)\frac{x_1}{\log x_1}.$$

Hence, by (7.9), (7.8), (5.5), (5.2), (5.1) and (7.6),

$$\lambda S(\mathscr{D}^*(x^{\alpha/4}), y^{1/4-\delta}) < (4 + 100\delta)x_1/\log x. \qquad (7.14)$$

We now proceed to estimate x_1. By (5.2), (5.1), and the prime number theorem,

$$|\mathcal{N}(x^{\alpha/4})| < (1+\varepsilon)y_1 \int_{y_1^{1/16-\delta/3}}^{y_1^{1/5}} \frac{du_1}{u_1 \log u_1} \int_{u_1}^{(y_1/u_1)^{1/4}} \frac{du_2}{u_2 \log u_2} \int_{u_2}^{(y_1/u_1u_2)^{1/3}} \frac{du_3}{u_3 \log u_3} \times \int_{u_3}^{(y_1/u_1u_2u_3)^{1/2}} \frac{du_4}{u_4(\log u_4)\log(y_1/u_1u_2u_3u_4)}$$

$$< \lambda_1(1+\delta)\pi(y)(I+500\delta), \tag{7.15}$$

where

$$I = \int_{1/16}^{1/5} \frac{dv_1}{v_1} \int_{v_1}^{(1-v_1)/4} \frac{dv_2}{v_2} \int_{v_2}^{(1-v_1-v_2)/3} \frac{dv_3}{v_3} \int_{v_3}^{(1-v_1-v_2-v_3)/2} \frac{dv_4}{v_4(1-v_1-v_2-v_3-v_4)}. \tag{7.16}$$

The substitution $1-v_1-\ldots-v_{j-1} = v_j u_j$ gives

$$I = \int_5^{16} \frac{du_1}{u_1-1} \int_4^{u_1-1} \frac{du_2}{u_2-1} \int_3^{u_2-1} \frac{du_3}{u_3-1} \int_2^{u_3-1} \frac{du_4}{u_4-1}$$

$$= \int_4^{15} \frac{dv_1}{v_1} \int_4^{v_1} \frac{dv_2}{v_2-1} \int_4^{v_2} \frac{\log(v_3-3)}{v_3-2}\,dv_3$$

$$= \int_4^{15} \frac{\log(v_3-3)}{v_3-2}\,dv_3 \int_{v_3}^{15} \frac{dv_2}{v_2-1} \int_{v_2}^{15} \frac{dv_1}{v_1}$$

$$\leqslant \int_4^{15} \frac{u\log(u-3)}{(u-1)(u-2)}\,du \int_u^{15} \frac{1}{v}\log\frac{15}{v}\,dv$$

$$= \frac{1}{2}\int_4^{15} \frac{u\log(u-3)}{(u-1)(u-2)}\left(\log\frac{15}{u}\right)^2 du.$$

We compute an upper bound for I as follows. Let

$$J(j) = \frac{(j+1)(j-2)}{j(j-1)}\left(\log\frac{15}{j}\right)^2(\log^2(j-2)-\log^2(j-3)).$$

Then

$$I \leqslant \frac{1}{2}\sum_{j=4}^{14} \int_j^{j+1} \frac{u(u-3)}{(u-1)(u-2)}\left(\log\frac{15}{u}\right)^2 \frac{\log(u-3)}{u-3}\,du$$

$$\leqslant \frac{1}{4}\sum_{j=4}^{14} J(j). \tag{7.17}$$

In the following table, $\bar{J}(j)$ denotes a number such that $\bar{J}(j) \geqslant J(j)$.

j	$\bar{J}(j)$	j	$\bar{J}(j)$	j	$\bar{J}(j)$
4	0·70	8	0·24	12	0·03
5	0·79	9	0·15	13	0·01
6	0·57	10	0·09	14	0·01
7	0·38	11	0·05		

(This table was computed with the use of a table of five-figure natural logarithms. It was then checked on an HP65.) Hence, by (7.17), $4I \leqslant 3{\cdot}02$. Therefore, by (7.15), (5.5), (7.14), and (7.7), we have (7.13).

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Department of Mathematics
Imperial College
London S.W.7

and

The University of Michigan
Ann Arbor, Michigan 48104