ON THE FREIHEITSSATZ

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0. Summary

The classical proof of the Dehn–Magnus Freiheitssatz, due to Magnus, is recast in the language of combinatorial geometry. This provides an extension of the Freiheitssatz from free groups to groups that are free products of groups isomorphic to subgroups of the additive group of real numbers, and, at the same time, a sharpening of the author's Identity Theorem.

1. Introduction

The following theorem is our main result.

THEOREM 1. *Let G be a free product of groups each isomorphic to a subgroup of the additive group of real numbers. Let r be a cyclically reduced element of G, and let w be a non-trivial element of the normal closure of r in G. If the normal form for r contains a factor from a certain one of the components, then so does the normal form for w.*

This theorem clearly contains the classical Freiheitssatz, in which each component is an infinite cyclic group.

There is essentially only one proof known for the Freiheitssatz, that given by Magnus [5; see also 6]. Our starting point was an attempt to translate this proof into the language of combinatorial geometry, using a result of van Kampen [7] which we have discussed and exploited elsewhere [3]. (We have learned recently that Weinbaum [8] has already carried out a translation very similar to ours; however, his further applications of the method lie in a different direction from ours.) Our effort has been only partially successful. However, we were able to shift the emphasis of the proof, especially with regard to the induction argument, in such a way that it is easily adapted to prove the somewhat more general Theorem 1 above. The same argument without essential modification yields also a sharpened form of our Identity Theorem [1, 2], which we state as Theorem 4 below.

The result of van Kampen, as adapted to the group G, concerns what we have called *diagrams*. A diagram M consists of a connected and simply connected finite complex $|M|$ in the plane together with a function ϕ assigning to each directed edge (1-cell) as label a non-trivial element of one of the component groups G_α of the free product G, and assigning inverse elements to oppositely oriented edges. The result of van Kampen asserts that if R is any set of cyclically reduced elements of G, and w any non-trivial element of the normal closure of R in G, then there exists a diagram M such that the product of the labels on the boundary of M, starting at a suitable point, is w, while, for any face (2-cell) of M, the product of the labels on the boundary of the face, starting at a suitable point, is r or r^{-1} for some element r of R.

We may always suppose M chosen with the smallest possible number of faces for given R and w. Then M will be *reduced* in the following sense. If M_1 is any sub-diagram of M containing at least one face but not all faces of M, then the product of the labels on M_1, starting at any point, does not reduce to the trivial element 1 of G.

In view of van Kampen's theorem, we can restate Theorem 1 in the following form.

THEOREM 2. *Let R consist of a single element r, let w be a non-trivial element in the normal closure of R in G, and let M be a reduced diagram as above. If any edge of M bears a label from a certain component G_α, then some edge on the boundary of M bears a label from that component.*

Magnus' proof went by an induction, applied not to the Freiheitssatz itself, but to an extension of it. Similarly, we shall prove by induction an analogous extension of Theorem 2.

THEOREM 3. *Let certain among the components G_α be denoted as $H_a, H_{a+1}, ..., H_b$, and suppose that R consists of elements $r_1, r_2, ..., r_m$ where each r_i contains at least one factor from some H_j. For each r_i let a_i be the least index j such that r_i contains a factor from H_j, and let b_i be the greatest such index. Assume that $a_1 < a_2 < ... < a_m$ and $b_1 < b_2 < ... < b_m$. Let w be a non-trivial element in the normal closure of R in G, and let M be a reduced diagram as above. If j_0 is the largest index such that some edge of M bears a label from H_{j_0}, then some edge on the boundary of M bears a label from H_{j_0}.*

It is perhaps curious that, formulated thus, the extended Freiheitssatz takes the form of a combinatorial maximum principle (see [4]).

The main difference, although a minor one, between Magnus' argument and that to be given below seems to be that he argues by induction on the common length (in his case) of the r_i, while we argue by induction on the number of faces in the diagram M.

The planar diagrams M considered above could just as well be taken as diagrams on the sphere, but not covering the entire sphere. If, now, we start with a diagram M that exhausts the entire sphere, the arguments to be given below apply equally, and yield the following result.

THEOREM 4. *Let r be a cyclically reduced element of G. Then there does not exist a reduced diagram M on the sphere such that $|M|$ is a decomposition of the entire sphere, and such that the product of the labels on each face, starting at a suitable point, is r or r^{-1}.*

Theorem 4, in the case in which G is a free group, implies the Identity Theorem of the author [1, 2]. Theorem 4 admits an obvious extension to the situation of Theorem 3.

2. *The inductive step for $m > 1$*

We shall prove Theorem 3 by induction on the number of faces of M. The initial case, where M has only a single face, is immediate. In this Section we prove Theorem 3 in the case in which $m > 1$, assuming inductively that Theorem 3 holds for all diagrams

with fewer faces than M. The remainder of the paper is devoted to the case in which $m = 1$; the truth of Theorem 3 for given M with $m = 1$ will be reduced to that of Theorem 3 for a new diagram M'' with the same number of faces as M, but with $m > 1$.

We assume then the hypotheses of Theorem 3, with $m > 1$. To simplify notation, we may suppose that M contains faces with labels $r_1^{\pm 1}$ and $r_m^{\pm 1}$. We must show that some edge on the boundary of M bears a label from H_{b_m} (or, symmetrically, from H_{a_1}). In fact, we assume that no edge on the boundary bears a label from H_{b_m}, and from this derive a contradiction.

Let M_1 be a subcomplex of M, maximal with respect to the property that M_1 is connected and the union of a set of (closed) faces, each with boundary label $r_m^{\pm 1}$. Let e be an edge on the boundary of M_1. If e lies also on the boundary of M, then, by hypothesis, e does not bear a label from H_{b_m}. Otherwise e separates M_1 from a face F that does not belong to M_1. From the maximality of M_1 we conclude that F has boundary label $r_i^{\pm 1}$ for some $i < m$. Since r_i contains no factor from H_{b_m}, the label on e does not come from H_{b_m}. We have shown that no edge on the boundary of M_1 bears a label from H_{b_m}.

Suppose that M_1 is simply connected. Then M_1 is a subdiagram of M, containing at least one face but not all faces of M. Since M is reduced, M_1 is reduced, and has non-trivial boundary label. This contradicts the induction hypothesis, according to which Theorem 3 holds for the smaller diagram M_1.

Suppose that M_1 is not simply connected. Then one of the components of the complement of M_1 in M is a connected and simply connected subdiagram M_2 of M, with its boundary entirely contained in the boundary of M_1. From the maximality of M_1 it follows that M_2 contains a face with label $r_i^{\pm 1}$ for some $i < m$; let j be the least such i. Then M_2 contains an edge with label from H_{a_j}. But every edge on the boundary of M_2 lies on the boundary of M_1, and hence on the boundary of some face F of M_1; since the label $r_m^{\pm 1}$ on F contains no factor from H_{a_j}, the label on the edge cannot come from H_{a_j}. This contradicts Theorem 3 (in the symmetric form noted above) for the smaller diagram M_2.

3. The case $m = 1$

We assume that M is given, with more than one face, and satisfying the hypotheses of Theorem 3 with $m = 1$. We shall modify M to obtain a new diagram M'' with the same number of faces, and satisfying the hypotheses of Theorem 3 with $m > 1$. By the previous case, the conclusion of Theorem 3 follows for M''. We shall show that this implies the conclusion of Theorem 3 for M.

It is easy to see that the case $m = 1$ of Theorem 3 is equivalent to Theorem 2. We assume that G, r, w, and M are as in the hypothesis of Theorem 2.

Let μ be any homomorphism from G into $\mathbb{R}$, the additive group of reals, such that $\mu(r) = 0$. If π is any path in M, let $S(\pi)$ be the sum of values of μ on the labels on the edges of π. The condition $\mu(r) = 0$ implies that $S(\pi)$ depends only on the endpoints of π. Choose a vertex v_0 in M for base point, and, for each vertex v in M, define $h(v)$ to be the common value of $S(\pi)$ for all paths π from v_0 to v. Without loss of generality we can suppose that each component G_α of G is isomorphic to $\mathbb{R}$, and that a standard

isomorphism γ_α from $\mathbb{R}$ on to G_α is given. With this we can extend the function h by linearity to interior points of the edges of M.

Let K be the finite set of values assumed by the function h at the vertices of M; we can suppose the elements of K are $k_1 < k_2 < \ldots < k_n$. If, on a given edge e, the function h is not constant, it is strictly monotonic, hence assumes each of the values k_i at most once. We modify our diagram M by introducing as new vertices all those points on e where h assumes a value k_i from K. Each edge e is now subdivided into a succession of edges $e_1, e_2, \ldots, e_t$. Using the functions γ_α and linearity, we can factor the label g on e into labels $g_1, g_2, \ldots, g_t$ for $e_1, e_2, \ldots, e_t$. We have thus obtained a new diagram M', with the same number of faces as M.

The group G is the free product of components G_α, for α in some index set A. We define a new group G'' to be the free product of components $G_{\alpha, k}$, for α in A and k in K; we suppose given isomorphisms $\gamma_{\alpha, k}$ from G_α onto $G_{\alpha, k}$. If h is constant on an edge e, then $h(e) = k_i$ for some k_i in K; otherwise $h(e) = [k_i, k_{i+1}]$ for some k_i. In either case we define $k(e) = k_i$. Now let e be an edge, directed in such a way that its label g is 'positive', that is, if g is in G_α, then $\gamma_\alpha^{-1}(g)$ is a positive element in $\mathbb{R}$. We replace the label g on e by the new label $\gamma_{\alpha, k(e)}(g)$ in $G_{\alpha, k(e)}$. We have thus a new diagram M'', with labels from the group G''; it is evident that M'' is reduced and has non-trivial boundary label. We shall next examine the labels on the faces of M''.

Let F_1 and F_2 be two faces of M, with p_1 a vertex on the boundary of F_1 and p_2 on F_2. We shall say p_1 and p_2 are corresponding points if the boundary labels, on F_1 beginning at p_1 and on F_2 beginning at p_2, are the same or inverses. (Apart from the case in which r is a proper power, which can be excluded, p_1 uniquely determines p_2.) Let q_1 and q_2 be another pair of corresponding points. Then the path along the boundary of F_1 from p_1 to q_1, in the positive or negative sense according as the label on F_1 is r or r^{-1}, and the similar path on the boundary of F_2, will bear the same product-label, say u. From this we have that $h(q_1) = h(p_1) + \mu(u)$ and $h(q_2) = h(p_2) + \mu(u)$.

It follows that $h(q_2) - h(q_1) = h(p_2) - h(p_1)$. Thus there exists a real number d, depending only on F_1 and F_2, such that if p_1 and p_2 are any two corresponding points, p_1 on F_1 and p_2 on F_2, then $h(p_2) = h(p_1) + d$. Similarly, if e_1 and e_2 are corresponding edges, that is beginning at corresponding points, and have positive labels, then $k(e_2) = k(e_1) + d$. Now, in the new diagram M'', F_1 and F_2 will have boundary labels $r_1^{\pm 1}$ and $r_2^{\pm 1}$, where r_1 and r_2 are obtained from r by replacing each factor of r, from some G_α, by a corresponding factor from some $G_{\alpha, k}$. From the above we see that r_2 is obtainable from r_1 by replacing each factor of r_1, from some $G_{\alpha, k}$, by the corresponding factor from $G_{\alpha, k+d}$. We now arrange the finite set of r_i, arising as above, in order: $r_1, r_2, \ldots, r_m$, in such a way that in passing from an earlier r_i to a later r_j, the second subscript on the component $G_{\alpha, k}$ containing corresponding factors is always increased.

We now show that if Theorem 2 failed for M, then Theorem 3 would fail for M''. Suppose Theorem 2 fails for M. Then M contains an edge with a label from a certain G_β, but no such edge on its boundary. Now M'' will contain edges with labels from some of the $G_{\beta, k}$, but no such edge on its boundary. Let $H_1 = G_{\beta, k_1}$, $H_2 = G_{\beta, k_1}, \ldots, H_n = G_{\beta, k_n}$. The hypotheses of Theorem 3 are evidently satisfied, but the conclusion fails flagrantly. Now, *provided we know that $m > 1$*, this is a contradiction, and the proof is completed.

The next Section is devoted to showing that μ can always be chosen in such a way as to yield $m > 1$.

4. *Completion of the proof*

Let G, r, w, and M be, as before, as in the hypothesis of Theorem 2. We want to show that it is always possible to choose μ in such a way that the construction above yields M'' with $m > 1$.

There is in fact one degenerate case where μ cannot be so chosen; we dispose of this case by a direct argument. Let r be called *special* if it is of the form $r = (g_1 g_2 \cdots g_t)^n$ for some $t \geqslant 1$ and $n \geqslant 1$, and with each g_i from a different G_{α_i}. We assume that no boundary edge of M has a label from, say, $G_\beta = G_{\alpha_1}$, and will derive a contradiction. Note that this assumption implies that $t \geqslant 2$. Let B be that part of the 1-skeleton of M consisting of all edges with labels from G_β; then B is non-empty and contains no edge from the boundary of M. If any component of B were a tree, it would contain a vertex v with a single edge e of B emanating from it. Since e is not on the boundary of M, it separates two faces F_1 and F_2. Now it is easy to see that the subdiagram $M_1 = F_1 \cup F_2$ violates the assumption that M is reduced. We conclude that B contains a circuit, and hence that some component of the complement of B in M is a subdiagram M_2 with boundary contained entirely in B. It follows that no edge on the boundary of M_2 bears a label from G_{α_2}. Since M_2 is smaller than M, we can conclude by induction that this is impossible.

A *point* on r splits r into two parts: $r = uv$, product without cancellation; formally, we define a point a to be an ordered couple $a = (u, v)$ such that $r = uv$ without cancellation. Let p be a vertex of M, lying on the boundary of a face F. Then the boundary label of F, beginning at p, is $s^{\pm 1}$ for some cyclic conjugate s of r; now, for some u and v, we have $r = uv$ and $s = vu$. We say that p determines the point $a = (u, v)$ on r relative to F. It is clear how to extend this definition to the case in which p is an interior point of an edge. Let p lie on an edge separating faces F_1 and F_2. Then p determines two points a_1 and a_2 on r, relative to F_1 and F_2. We say that a_1 and a_2 are *immediately related*. We call two points a and b on r *related* if they are connected by a chain with consecutive members immediately related. We call two factors of r related (immediately related) if they contain interior points that are related (immediately related).

Suppose that, for some component G_α, not all the factors of r from G_α are related. Then we can divide these factors into two sets S_1 and S_2 such that no factor in S_1 is related to a factor in S_2. We modify G to G' by replacing the component G_α by two components $G_{\alpha, 1}$ and $G_{\alpha, 2}$. If an edge e of M separates faces F_1 and F_2, and has label from G_α, then the two factors of r that it determines relative to F_1 and F_2 are related and so belong to the same set S_i; we replace the label on e by the corresponding element of $G_{\alpha, i}$. We have now a new diagram M', all of whose faces have label $r'^{\pm 1}$, where r' is obtained from r by replacing each factor from S_i by the corresponding element of $G_{\alpha, i}$. If Theorem 2 holds for M', then it holds also for M. Finite iteration of this argument permits us to confine attention to the case in which all factors of r from the same component are related.

We return to a point p on an edge separating two faces F_1 and F_2, and the points a_1 and a_2 of r determined by p relative to faces F_1 and F_2. Let v be the segment of r (or a cyclic conjugate) running from a_1 to a_2; that is, suppose $r = uvz$ with $a_1 = (u, vz)$ and $a_2 = (uv, z)$. Suppose μ has been chosen so that $\mu(v) \neq 0$. The

point a_1 on r will be determined by a point q on the boundary of F_2, and the segment of the boundary of F_2 running from q to p will bear label v, whence

$$h(p) = h(q) + \mu(v) \neq h(q).$$

Thus the corresponding points p on F_1 and q on F_2 receive different values of h, and it follows that, in M'', F_1 and F_2 will bear labels $r_i^{\pm 1}$ for different i, whence $m > 1$. Suppose now that we have a chain $a_0, a_1, ..., a_n$ of points on r, with each pair a_i, a_{i+1} immediately related. We may write $a_i = (uv_1 v_2 ... v_i, v_{i+1} ... v_n z)$, hence $a_0 = (u, vz)$ and $a_n = (uv, z)$ for $v = v_1 v_2 ... v_n$. Now if $\mu(v) \neq 0$, then some $\mu(v_i) \neq 0$, and $m > 1$ as before. To obtain $m > 1$ then it suffices to choose μ so that $\mu(v) \neq 0$ where v is the segment of r determined by two related points of r.

Write $G = G_1 * G_2 * ... * G_n$. Using the given isomorphisms γ_α from $\mathbb{R}$ onto the G_α, we can write each element of G_α as $\gamma_\alpha(x)$, for x in $\mathbb{R}$. Clearly a homomorphism μ from G into $\mathbb{R}$ is uniquely determined by its values $\mu_\alpha = \mu(\gamma_\alpha(1))$. We can identify the commutator quotient group $\bar{G}$ of G with the direct product of the components, $\bar{G} = G_1 \times G_2 \times ... \times G_n$. Let the image of r in $\bar{G}$ be $r = \gamma_1(x_1) \gamma_2(x_2) ... \gamma_n(x_n)$. Then $\mu(r) = 0$ if and only if $\Sigma \mu_\alpha x_\alpha = 0$.

We call a component G_α *idle* if $x_\alpha = 0$, and otherwise *engaged*. Evidently, to choose μ with $\mu(r) = 0$, we may choose μ_α arbitrarily for all idle G_α, and also for all but one of the engaged G_α. Suppose first that r contains a factor from some idle component G_α. Then r must contain at least two factors g and h from G_α, and, choosing successive such factors, r has a part guh where u is a product of factors none of which lies in G_α. We may now suppose this configuration chosen with as few factors in u as possible; then u does not contain two factors from the same idle component. Suppose u contained a single factor f from an idle component, say $u = u_1 f u_2$. Then, by choosing $\mu(f)$ arbitrarily we make $\mu(u)$ arbitrary. Now if a in g and b in h are related points in g and h, we have in r a part v from a to b, where $v = g_2 u h_1$, $g = g_1 g_2$, $h = h_1 h_2$. If $\mu(u)$ is arbitrary, then so is $\mu(v)$; in particular, we can make $\mu(v) \neq 0$, and the proof is complete.

Suppose next that u contains no factor from an idle component. Again $\mu(u)$ is arbitrary, and the proof is complete, unless u contains factors from all the engaged components. Let k be the next factor in r from G_α after h; here we take r cyclically, and it may happen that k is g. Then r has a part hzk. If z contains a factor f_2 from an engaged component, then u also contains a factor f_1 from this component, and r has a part $f_1 q f_2$, where q contains the single factor h from G_α. Since we can choose $\mu(h)$, and so $\mu(q)$, arbitrarily, the proof is complete as before. The case remains in which all factors from z are idle. If we now choose a shortest segment hzk consisting of idle factors only, and with h and k factors from the same component, then z must contain a single factor f from some idle component. Now $\mu(f)$, and so $\mu(z)$, is arbitrary, and again the proof is complete. This completes the case in which r contains any factor from an idle component.

We may suppose now that all components are engaged. Suppose r contains two factors g and h from the same component, G_α and hence a part guh. Unless u contains a factor from each component other than G_α, we may choose $\mu(g_2 u h_1)$ arbitrarily, and the proof is complete. Thus we may assume that in r, between two factors from the same component there occurs a factor from each other component. This implies

that r has the form

$$r = \left(\gamma_1(x_{11})\,\gamma_2(x_{12})\ldots\gamma_n(x_{1n})\right)\left(\gamma_1(x_{21})\ldots\right)\ldots\left(\gamma_1(x_{t1})\,\gamma_2(x_{t2})\ldots\gamma_n(x_{tn})\right)$$

for some $t \geqslant 1$. We may put aside the case $n = 1$, where the theorem to be proved is trivial, and also the case $t = 1$, where r is special. Now, taking $\gamma_1(x_{11})$ and $\gamma_1(x_{21})$ in the role of g and h, we can still make $\mu(g_2 u h_1) \neq 0$ unless $x_{12}, x_{13}, \ldots x_{1n}$ are proportional to $x_2, x_3, \ldots, x_n$, say $x_{12} = c_1 x_2$, $x_{13} = c_1 x_3, \ldots, x_{1n} = c_1 x_n$. Next, taking $\gamma_2(x_{12})$ and $\gamma_2(x_{22})$ as g and h, we have, for some c_2, that

$$x_{13} = c_2 x_3, \quad x_{14} = c_2 x_4, \quad \ldots, \quad x_{21} = c_2 x_1.$$

If $n \geqslant 3$, we have $x_{13} = c_1 x_3 = c_2 x_3$, whence $c_1 = c_2 = c$. Continuation of this argument shows that each $x_{ij} = c x_j$, whence

$$r = \left(\gamma_1(c x_1)\,\gamma_2(c x_2)\ldots\gamma_n(c x_n)\right)^t,$$

and r is special.

The case remains in which $n = 2$, and this is a genuine exception to the argument above. To treat this case we remark first that, if there are two related points in the same factor, then the segment separating them lies in one of the components; then μ can be chosen not to vanish on this segment, and the proof is complete. We assume now that there are not two related points in the same factor. Pick a point a of r, interior to some factor, and a small segment u of that factor containing a in its interior. In each other factor from the same component G_α, the set of points related to points in u will be a finite union of intervals (possibly empty). We introduce all end points of such intervals, on all faces of M, as new vertices, and we relabel such intervals with elements from a new component $G_{\alpha,\,1}$. We have thus replaced M by a new diagram M' with labels $r'^{\pm 1}$ where r', in a new group G', has factors from three components. Evidently Theorem 2 for M' implies the same for M. Thus we have reduced the case $n = 2$ to the case $n \geqslant 3$, already treated. This completes the proof of the theorem.

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