HAUSDORFF DIMENSION AND QUASICONFORMAL MAPPINGS

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Dedicated to the memory of A. S. Besicovitch

1. *Introduction.* In this paper we study what happens to the Hausdorff dimension of a set A, denoted by H-dim A, under an n-dimensional quasiconformal mapping $f: D \to D'$ with $A \subset D$. It is clear that

$$\text{H-dim} f[A] = \text{H-dim } A \tag{1}$$

if f is a diffeomorphism or, more generally, if f and f^{-1} are locally Lipschitzian. We show first, however, that (1) need not hold if f is a general quasiconformal mapping. Next we give bounds for H-dim $f[A]$ in terms of H-dim A, n, and the maximal dilatation of f. In particular, we prove that H-dim $A = 0$ implies H-dim $f[A] = 0$, and we conjecture that H-dim $A = n$ implies H-dim $f[A] = n$, or equivalently that H-dim $A < n$ implies H-dim $f[A] < n$. We establish this conjecture for the case where $n = 2$ and then prove that, for general n, H-dim $f\ [A] < n$ whenever A is contained in an m-dimensional hyperplane with $m < n$. An example shows that H-dim $f[A]$ can be arbitrarily close to n, even when A is a 1-dimensional segment.

2. *Notation.* We shall use the terminology and notation for quasiconformal mappings given in **[16]**. Moreover, since we are concerned only with local properties which are invariant under Möbius transformations, we shall consider only quasiconformal mappings $f: D \to D'$ where D and D' are domains in the non-compact n-dimensional Euclidean space R^n.

For $a \in (0, \infty)$, the *Hausdorff a-dimensional outer measure* of a set $A \subset R^n$ is defined as

$$H_a(A) = \lim_{d \to 0} \left(\inf \sum_i \text{dia}\,(A_i)^a\right), \tag{2}$$

where the infimum is taken over all countable coverings of A by sets A_i with dia $(A_i) < d$. The *Hausdorff dimension* of A is then given by

$$\text{H-dim } A = \inf \{a : H_a(A) = 0\}. \tag{3}$$

Clearly $0 \leqslant$ H-dim $A \leqslant n$.

3. We shall need the following generalization of a result due to Mori **[13**; Lemma 4].

LEMMA. *Suppose that $f: D \to D'$ is an n-dimensional K-quasiconformal mapping, that U is a bounded domain with $\overline{U} \subset D$, and that $x \in U$. Let*

$$M = \max_{y \in \partial U} |y-x|,\ m = \min_{y \in \partial U} |y-x|,\ L = \max_{y \in \partial U} |f(y)-f(x)|,\ l = \min_{y \in \partial U} |f(y)-f(x)|\ .$$

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If the ball $B^n(f(x), L)$ is contained in D', then

$$L \leqslant Cl, \tag{4}$$

where C is a finite constant which depends only on n, K, and M/m.

Proof. Suppose that $l < L$ and let R denote the image under f^{-1} of the spherical ring

$$R' = \{y : l < |y - f(x)| < L\} \subset D'.$$

Then R is a ring which separates x and a point $y \in \partial U$ from ∞ and a point $z \in \partial U$. Hence if Γ is the family of arcs joining the components of $C(R)$ in R, it follows from the extremal property of the Teichmüller ring in R^n [**4** and **14**], or from [**16**; 11.9] that

$$M(\Gamma) \geqslant h_n\left(\frac{|z-x|}{|y-x|}\right) \geqslant h_n\left(\frac{M}{m}\right), \tag{5}$$

where $h_n : (0, \infty) \to (0, \infty)$ is positive and decreasing. Then since f is K-quasiconformal

$$M(\Gamma) \leqslant KM(f[\Gamma]) = K\omega_{n-1}\left(\log\frac{L}{l}\right)^{1-n}, \tag{6}$$

and (4) follows from (5) and (6).

4. *The Cantor sets C_s^n.* For each integer $n \geqslant 1$ and each $s \in (0, \frac{1}{2})$ we define a family of Cantor sets C_s^n as follows. Let Q denote the closed unit cube

$$Q = \{x = (x_1, \ldots, x_n) : 0 \leqslant x_i \leqslant 1\},$$

choose a collection of 2^n disjoint closed cubes Q_i of side s in int Q, $1 \leqslant i \leqslant 2^n$, oriented so that for each i there exists a similarity mapping

$$g_i(x) = sx + a_i,\ a_i \in Q,$$

which maps Q onto Q_i. Such collections of cubes Q_i obviously exist for each $s \in (0, \frac{1}{2})$. Next for each $j \geqslant 1$ let

$$F_j = \bigcup_{i_1, \ldots, i_j = 1}^{2^n} g_{i_1} \circ \ldots \circ g_{i_j}[Q].$$

Then $\{F_j\}$ is a decreasing sequence of compact sets, and each set F_j is the union of 2^{jn} disjoint closed cubes of side s^j. Hence

$$C_s^n = \bigcap_{j=1}^{\infty} F_j$$

is a compact set, and

$$\text{H-dim}\, C_s^n = n\,\frac{\log\frac{1}{2}}{\log s} \tag{7}$$

by, for example, [**1**; Theorem 3] or [**12**; Theorem III]. In particular,

$$0 < \text{H-dim}\, C_s^n < n$$

and

$$\lim_{s \to 0} \text{H-dim } C_s^n = 0, \quad \lim_{s \to \frac{1}{2}} \text{H-dim } C_s^n = n. \tag{8}$$

5. THEOREM. *For each integer $n \geq 2$ and each pair of such Cantor sets C_s^n and C_t^n there exists a quasiconformal mapping $f: R^n \to R^n$ which maps C_s^n onto C_t^n.*

Proof. Let g_i and F_j, g_i' and F_j' denote respectively the similarity mappings and sets corresponding to the constructions for C_s^n, C_t^n given in §4. Then it is not difficult to see that there exists a piecewise linear homeomorphism $f_1 : R^n \to R^n$ such that $f_1(x) = x$ if $x \in R^n \sim Q$ and such that for each i

$$f_1(x) = g_i' \circ g_i^{-1}(x)$$

if $x \in g_i[Q]$. Then f_1 is K-quasiconformal for some K and $f_1[F_1] = F_1'$. Next define $f_2 : R^n \to R^n$ by setting $f_2(x) = f_1(x)$ if $x \in R^n \sim F_1$ and

$$f_2(x) = g_i' \circ f_1 \circ g_i^{-1}(x)$$

if $x \in g_i[Q]$. Then f_2 is a piecewise linear K-quasiconformal mapping, $f_2[F_2] = F_2'$, and for each i and j

$$f_2(x) = g_i' \circ g_j' \circ g_j^{-1} \circ g_i^{-1}(x)$$

if $x \in g_i \circ g_j[Q]$. Continuing in this way, we obtain a sequence of piecewise linear K-quasiconformal mappings $f_j : R^n \to R^n$ such that $f_{j+1}(x) = f_j(x)$ in $R^n \sim F_j$ and $f_j[F_j] = F_j'$. This sequence converges to a K-quasiconformal mapping $f: R^n \to R^n$ which maps F_j onto F_j' for each j. Hence f maps C_s^n onto C_t^n.

6. COROLLARY. *For each integer $n \geq 2$ and each pair of numbers $\alpha, \beta \in (0, n)$, there exists a quasiconformal mapping $f: R^n \to R^n$ and a compact set $A \subset R^n$ such that*

$$\text{H-dim } A = \alpha, \ \text{H-dim} f[A] = \beta. \tag{9}$$

Proof. By (7) and (8) we can choose $s, t \in (0, \frac{1}{2})$ so that for any of the corresponding Cantor sets C_s^n, C_t^n,

$$\text{H-dim } C_s^n = \alpha, \ \text{H-dim } C_t^n = \beta.$$

Theorem 5 then yields a quasiconformal mapping f: $R^n \to R^n$ which maps C_s^n onto C_t^n, and (9) follows with $A = C_s^n$.

7. *Remark.* The above proof shows that for each $\alpha \in (0, n)$ there exists a set $A \subset R^n$ with H-dim $A = \alpha$ such that

$$\inf_f \text{H-dim} f[A] = 0, \ \sup_f \text{H-dim} f[A] = n, \tag{10}$$

where the infimum and supremum are taken over all quasiconformal mappings $f: D \to D'$ with $A \subset D$. We consider next what can be said if we take the infimum and supremum in (10) over the subclass of mappings $f: D \to D'$ which are K-quasiconformal for some fixed K.

8. THEOREM. *If $f: D \to D'$ is an n-dimensional K-quasiconformal mapping and if $A \subset D$ with* H-dim $A \geq \alpha > 0$, *then* H-dim $f[A] \geq \beta > 0$, *where*

$$\beta = \alpha K^{1/(1-n)} \geq \alpha/K. \tag{11}$$

Proof. Since A is the countable union of sets with compact closure in D, we may assume that A is contained in a compact subset of D. Then since f^{-1} is locally

Hölder continuous with exponent $K^{1/(1-n)}$ in D' ([**5**; Corollary 6] or [**10**; 3.2]), there exists a positive constant c such that

$$|f(x)-f(y)| \geqslant c|x-y|^{K^{1/(n-1)}} \tag{12}$$

for all $x, y \in A$. If $b > \text{H-dim} f[A]$, then (2), (3), and (12) imply that $H_a(A) = 0$, where $a = b\,K^{1/(n-1)}$. Hence $a \geqslant \alpha$ and (11) follows.

9. COROLLARY. *If $f: D \to D'$ is an n-dimensional quasiconformal mapping and if $A \subset D$ with* H-dim $A = 0$, *then* H-dim $f[A] = 0$.

10. CONJECTURE. *If $f: D \to D'$ is an n-dimensional K-quasiconformal mapping and if $A \subset D$ with* H-dim $A \leqslant \alpha < n$, *then* H-dim $f[A] \leqslant \beta < n$, *where β depends only on α, n, and K.*

11. We shall establish this conjecture for the case where $n = 2$. The proof is based on the following important result due to Bojarski [7; p. 226].

THEOREM. *If f is a 2-dimensional K-quasiconformal mapping, then its Jacobian J_f is locally L^q-integrable for $q \in [1, p(K)]$, where $p(K) > 1$ depends only on K.*

It is easy to see that $p(K) \leqslant K/(K-1)$, and it has been conjectured that Theorem 11 holds with $p(K) = K/(K-1)$.

12. THEOREM. *If $f: D \to D'$ is a 2-dimensional K-quasiconformal mapping and if $A \subset D$ with* H-dim $A \leqslant \alpha < 2$, *then* H-dim $f[A] \leqslant \beta < 2$, *where*

$$\beta = \frac{2p(K)\alpha}{2(p(K)-1)+\alpha} \tag{13}$$

and $p(K)$ is the constant given in Theorem 11.

Proof. As in the proof of Theorem 8, we may assume that A is contained in an open set with compact closure F in D. Then for each $a \in (\alpha, 2)$ and each $q \in (1, p(K))$ we must show that $H_b(f[A]) = 0$, where

$$b = \frac{2qa}{2(q-1)+a} .$$

Choose $\varepsilon > 0$ and $d > 0$. Then $H_a(A) = 0$ and by [**8**; Lemma 1] we can choose a covering of A by non-overlapping squares Q_i of side s_i such that $Q_i \subset F$,

$$\text{dia}\,(f[Q_i]) < d,$$

and

$$\sum_i s_i^{\,a} < \varepsilon. \tag{14}$$

Let x_i denote the centre of Q_i and set

$$L_i = \max_{y \in \partial Q_i} |f(y)-f(x_i)|,\ l_i = \min_{y \in \partial Q_i} |f(y)-f(x_i)| .$$

By choosing d sufficiently small, we may assume that the disks $B^2(f(x_i), L_i)$ all lie in D'. Then Lemma 3 implies that $L_i \leqslant Cl_i$, where C is a finite constant which depends only on K. Hence

$$\text{dia}\,(f[Q_i]) \leqslant 2L_i \leqslant 2Cl_i \leqslant C_1 m(f[Q_i])^{1/2},$$

where $C_1 = 2C\pi^{-1/2}$. On the other hand,

$$m(f[Q_i]) = \int\limits_{Q_i} J_f \, dm \leqslant s_i^{2(q-1)/q} \left(\int\limits_{Q_i} J_f^q \, dm \right)^{1/q}$$

by Hölder's inequality. Thus

$$\sum_i \operatorname{dia}(f[Q_i])^b \leqslant C_1^b \sum_i s_i^{b(q-1)/q} \left(\int\limits_{Q_i} J_f^q \, dm \right)^{b/(2q)},$$

and a second application of Hölder's inequality yields

$$\sum_i \operatorname{dia}(f[Q_i])^b \leqslant C_1^b \Big(\sum_i s_i^a \Big)^{b(q-1)/(aq)} \left(\int\limits_F J_f^q \, dm \right)^{b/(2q)}. \tag{15}$$

Finally, since d can be chosen arbitrarily small, (14) and (15) imply that

$$H_b(f[A]) \leqslant C_1^b \left(\int\limits_F J_f^q \, dm \right)^{b/(2q)} \varepsilon^{b(q-1)/(aq)},$$

and letting $\varepsilon \to 0$ yields $H_b(f[A]) = 0$.

13. COROLLARY. *If $f: D \to D'$ is a 2-dimensional quasiconformal mapping and if $A \subset D$ with* H-dim $A = 2$, *then* H-dim $f[A] = 2$.

14. *Remark.* If the conjecture that Theorem 11 holds with $p(K) = K/(K-1)$ is correct, then Theorem 12 would imply that

$$\frac{2\alpha}{2K-(K-1)\alpha} \leqslant \text{H-dim} f[A] \leqslant \frac{2K\alpha}{2+(K-1)\alpha}$$

for each 2-dimensional K-quasiconformal mapping $f: D \to D'$ and each set $A \subset D$ with H-dim $A = \alpha$. These bounds are asymptotic to those implied by Theorem 8 as $\alpha \to 0$.

15. Suppose that $f: D \to D'$ is an n-dimensional quasiconformal mapping and that J_f is locally L^q-integrable for $q \in [1, p)$ where $p > 1$. Then the proof for Theorem 12 shows that

$$\text{H-dim} f[A] \leqslant \beta = \frac{np\alpha}{n(p-1)+\alpha} < n$$

for each $A \subset D$ with H-dim $A \leqslant \alpha < n$. Unfortunately it is not known whether the analogue of Theorem 11 holds in higher dimensions, and hence we cannot use this argument to establish Conjecture 10 for general n.

We can, however, establish a weaker form of Conjecture 10 for general n by a different method. We require some additional notation. Suppose that $f: D \to D'$ is an n-dimensional homeomorphism. If $\bar{B}^n(x, r) \subset D$, we set

$$L(x,f,r) = \max_{|y-x|=r} |f(y)-f(x)|, \; l(x,f,r) = \min_{|y-x|=r} |f(y)-f(x)| .$$

Next we say that a closed cube $Q \subset D'$ is *f-admissible* if for each $x \in f^{-1}[Q]$,

$$\bar{B}^n(x,d) \subset D, \; \bar{B}^n(f(x), L(x,f,d)) \subset D'$$

where $d = \text{dia}(f^{-1}[Q])$. Since f is a homeomorphism, each point of D' is contained in the interior of some f-admissible cube Q.

16. LEMMA. *Suppose that $f: D \to D'$ is an n-dimensional K-quasiconformal mapping, that T is an $(n-1)$-plane in R^n, and that Q is an f-admissible closed cube of side s in D'. Then there exists an integer $p \geqslant 2$, which depends only on n and K, such that the subdivision of Q into p^n congruent closed cubes of side s/p contains a cube which does not meet $f[D \cap T]$.*

Proof. Let $C = C(n, K)$ denote the number given by Lemma 3 when $M/m = 1$. We shall show that the assertion is true for $p > \max(6, 3Cn^{1/2})$.

Fix such an integer p, let Q_0 denote a cube of the corresponding subdivision which contains the centre of Q, and let $S = f[D \cap T]$. If $S \cap Q_0 = \varnothing$, we are finished. Otherwise choose a point $z \in S \cap Q_0$, let $y = f^{-1}(z)$, and let e denote a unit normal to T. Then $B = B^n(z, s/3) \subset Q$ and we can choose $r > 0$ so that

$$x = y + re \in f^{-1}[\partial B] \subset f^{-1}[Q].$$

Since Q is f-admissible, Lemma 3 implies that

$$l(x,f,r) \geqslant (1/C)\, L(x,f,r) \geqslant (1/C)\, |f(y) - f(x)| = s/(3C).$$

Next since T is an $(n-1)$-plane, $B^n(x, r) \cap T = \varnothing$ while

$$B^n(f(x), s/(3C)) \subset f[B^n(x, r)].$$

Hence the ball $B^n(f(x), s/(3C))$ does not meet S, and since this ball contains a cube of the subdivision, the proof is complete.

17. *Definition.* A set $S \subset R^n$ is said to be a *K-quasiconformal m-ball* if there is a neighbourhood D of S and an n-dimensional K-quasiconformal mapping $f: D \to D'$ such that $f[S]$ is an ordinary m-dimensional (open or closed) ball. When $m = 1$, S is also said to be a K-quasiconformal arc. Finally S is said to be a quasiconformal m-ball if it is a K-quasiconformal m-ball for some K.

18. THEOREM. *If S is a K-quasiconformal m-ball in R^n and if $m < n$, then*

$$m \leqslant \text{H-dim}\, S \leqslant \beta < n, \tag{16}$$

where β depends only on n and K.

Proof. Since S is homeomorphic to an ordinary m-ball, S has topological dimension m, and the lower bound in (16) follows from [6; p. 107].

For the upper bound, there exists, by hypothesis, an n-dimensional K-quasiconformal mapping $f: D \to D'$ such that $S \subset f[D \cap T]$ for some $(n-1)$-plane $T \subset R^n$. Choose $p = p(n, K)$ as in Lemma 16 and set

$$a = (1 - p^{-n})^{1/2} < 1.$$

Then

$$ap^n > a^2 p^n = p^n - 1,$$

and we may choose $\beta \in (0, n)$ so that $ap^\beta = p^n - 1$. Then β depends only on n and K, and it suffices to show that $H_\beta(S) = 0$. Moreover since S can be covered by a countable collection of f-admissible cubes, it suffices to prove that $H_\beta(S \cap Q) = 0$ for each f-admissible closed cube $Q \subset D'$.

Let Q denote such a cube with side s, subdivide Q into p^n congruent closed cubes of side s/p, and let $Q_1, \ldots, Q_q$ denote the cubes of this subdivision which meet S. Since Q is f-admissible, Lemma 16 implies that $q \leqslant p^n - 1$ and hence that

$$\sum_{i=1}^{q} \operatorname{dia}(Q_i)^\beta = q((s/p)\, n^{1/2})^\beta \leqslant a(sn^{1/2})^\beta .$$

Next subdivide each cube Q_i into p^n congruent closed cubes of side s/p^2, and let $Q_{i1}, \ldots, Q_{iq_i}$ denote the cubes of this subdivision which meet S. Then since each Q_i is f-admissible, Lemma 16 implies that $q_i \leqslant p^n - 1$ for each i and hence that

$$\sum_{i=1}^{q} \left(\sum_{j=1}^{q_i} \operatorname{dia}(Q_{ij})^\beta \right) = \sum_{i=1}^{q} q_i((s/p^2)\, n^{1/2})^\beta \leqslant a^2\, (sn^{1/2})^\beta.$$

Continuing in this way, we see that for each integer $j \geqslant 1$, $S \cap Q$ can be covered by a finite number of closed cubes Q_i' of side s/p^j such that

$$\sum_i \operatorname{dia}(Q_i')^\beta \leqslant a^j (sn^{1/2})^\beta.$$

Then letting $j \to \infty$ we conclude that $H_\beta\ (S \cap Q) = 0$.

19. COROLLARY. *If $f: D \to D'$ is an n-dimensional K-quasiconformal mapping and if $A \subset D$ is contained in a countable union of K-quasiconformal $(n-1)$-balls, then* H-dim $f[A] \leqslant \beta < n$, *where β depends only on n and K.*

Proof. Since f is K-quasiconformal, $f[A]$ is contained in a countable union of K^2-quasiconformal $(n-1)$-balls, and the conclusion follows from Theorem 18.

20. THEOREM. *For each pair of integers $n \geqslant 2$ and $m \in [1, n-1]$ and each number $\beta \in [m, n)$, there exists a quasiconformal m-ball $S \subset R^n$ with* H-dim $S = \beta$.

Proof. Let $A = Q \cap T$, where Q is the closed unit cube and T is the m-plane

$$T = \{x = (x_1, \ldots, x_n) : x_{m+1} = \ldots = x_n = \tfrac{1}{2}\}, \tag{17}$$

and set $s = (4^n+1)^{-1}$. Then we can find 2^n disjoint oriented closed cubes Q_i of side s in int Q with centres in A. Following the construction in §4 for the corresponding Cantor set C_s^n, we see that for each j, the 2^{jn} disjoint closed cubes of side s^j in F_j all have their centres in A. Hence

$$C_s^n = \bigcap_{j=1}^{\infty} F_j \subset A. \tag{18}$$

Now choose $t \in (0, \frac{1}{2})$ and a Cantor set C_t^n so that H-dim $C_t^n = \beta$, let $f: R^n \to R^n$ be the quasiconformal mapping given in the proof of Theorem 5 which maps C_s^n onto C_t^n, and set $S = f[A]$. Since A is obviously a quasiconformal m-ball, so is S. Then (18) implies that

$$A = \left(\bigcup_{j=1}^{\infty} (A \sim F_j) \right) \cup C_s^n, \tag{19}$$

and hence that

$$S = \left(\bigcup_{j=1}^{\infty} f[A \sim F_j] \right) \cup C_t^n.$$

From the construction in the proof of Theorem 5, we see for each j that $f(x) = f_j(x)$ in $R^n \sim F_j$ and hence that $f|(R^n \sim F_j)$ is piecewise linear. Thus

$$\text{H-dim} f[A \sim F_j] = m \leqslant \beta$$

for each j and H-dim S = H-dim $C_t^n = \beta$.

21. *Remarks.* Theorem 20 shows that the upper bound β in Theorem 18 and Corollary 19 cannot be chosen so that it depends only on n.

We can also apply Theorem 20 to the theory of quasiregular mappings. Suppose that $f: D \to R^n$ is a quasiregular mapping, and let B_f denote the branch set of f. Then B_f and $f[B_f]$ are of n-dimensional measure zero [**9**; 2.27 and 8.3]. Suppose next that $B_f \neq \emptyset$. Then from [**11**; 3.4] it follows that $H_{n-2}(f[B_f]) > 0$. Hence in this case

$$n-2 \leqslant \text{H-dim} f[B_f] \leqslant n.$$

On the other hand, by a result of Černavskii [**2**] (see also [**15**]), the topological dimension of B_f is at most $n-2$, and the same is true for $f[B_f]$ by [**3**; 2.2].

22. Corollary. *For each integer $n \geqslant 3$ and each pair of numbers $\alpha, \beta \in [n-2, n)$, there exists a quasiregular mapping $f: R^n \to R^n$ such that*

$$\text{H-dim} B_f = \alpha, \ \text{H-dim} f[B_f] = \beta.$$

Proof. Set $m = n-2$ and let T be the $(n-2)$-plane in (17). Since (19) holds with A replaced by T, the proof of Theorem 20 shows that we can construct quasiconformal mappings $g_1 : R^n \to R^n$ and $g_2 : R^n \to R^n$ such that

$$\text{H-dim}\, g_1[T] = \alpha, \ \text{H-dim}\, g_2[T] = \beta.$$

Next define as in [**11**; 3.19] a quasiregular winding mapping $h: R^n \to R^n$ with $B_h = h[B_h] = T$, and set $f = g_2 \circ h \circ g_1^{-1}$. Then $f: R^n \to R^n$ is quasiregular and $B_f = g_1[T], f[B_f] = g_2[T]$.

23. *Final remarks.* The argument in the proof of Theorem 5 can be used to show that the lower bound in Theorem 8 is asymptotically sharp for sets of small Hausdorff dimension. More precisely, given $K \in (1, \infty)$, one can construct for each $\alpha \in (0, n)$ a K-quasiconformal mapping $f_\alpha : R^n \to R^n$ and a compact set $A_\alpha \subset R^n$ with

$$\text{H-dim}\, A_\alpha = \alpha$$

such that

$$\lim_{\alpha \to 0} \frac{\text{H-dim} f_\alpha[A_\alpha]}{\alpha} = K^{1/(1-n)}.$$

This argument also can be used to show that for each $K \in (1, \infty)$ there exists a K-quasiconformal mapping $f: R^n \to R^n$ and a compact set $A \subset R^n$ with H-dim $A = n$ such that f is differentiable with a vanishing Jacobian at each point of A.

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