Web-based Supplementary Materials for

Estimating the Average Treatment Effect on Survival Based on Observational Data and Using Partly Conditional Modeling

by

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1. Notation and XX Assumptions

We begin by reviewing the essential notation:

i: subject ($i = 1, \ldots, n$)

n: number of subjects

D_i: death time for the ith subject

C_i: independent censoring time for the ith subject

T_i: treatment time for the ith subject

X_i: $\min\{D_i, C_i\}$: observation time for the ith subject

$\boldsymbol{Z}_i^*(t)$: covariate for ith subject at follow-up time t

$\mathcal{E}_i(t)$: treatment eligibility indicator of ith subject at time t

$N_i^T(t) = I(T_i \leqslant t, T_i < X_i)$; note that $dN_i^T(t) = \mathcal{E}_i(t) dN_i^T(t)$

$\boldsymbol{\mathcal{H}}_i(t) = \{\boldsymbol{Z}_i^*(s), \mathcal{E}_i(s); s \in [0, t)\}$: history up to time t

τ: pre-specified constant satisfying $P(X_i \geqslant \tau) > 0$ for all i.

k: cross section ($k = 1, \ldots, K$); K: number of cross-sections

S_{ik}: follow-up time at calendar date of the kth cross section

$D_{ik} = D_i - S_{ik}$, death time measured from date of kth cross section

$T_{ik} = T_i - S_{ik}$, treatment time measured from date of kth cross section

$C_{ik} = C_i - S_{ik}$, independent censoring time measured from date of kth cross section

$N_{i0k}(t) = N_i(S_{ik} + t) I(T_i > S_{ik} + t)$

$\mathcal{E}_{ik} = \mathcal{E}_i(S_{ik})$

τ_{0k}: pre-specified constant satisfying $P(D_{ik} \wedge T_{ik} \wedge C_{ik} \geqslant \tau_{0k}) > 0$

$N_{i1}(t) = I(T_i < X_i) N_i(T_i + t)$

τ_1: pre-specified constant satisfying $P(D_i - T_i \geqslant \tau_1 | T_i, T_i < D_i) > 0$

$N_i^C(t) = I(C_i \leqslant t, C_i < D_i)$

The following hazard functions are modeled:

Pre-treatment death hazard: $\lambda_0(t; s|\boldsymbol{\mathcal{H}}_i(s), \mathcal{E}_i(s) = 1)$

Cross-section stratified version: $\lambda_{0k}(t; s|\boldsymbol{\mathcal{H}}_i(S_{ik}), \mathcal{E}_i(S_{ik}) = 1)$

Post-treatment death hazard: $\lambda_1(t; T_i|\boldsymbol{\mathcal{H}}_i(T_i), T_i)$

Treatment initiation hazard: $\lambda_i^T(t|\boldsymbol{\mathcal{H}}_i(t), \mathcal{E}_i(t))$

Independent censoring hazard: $\lambda_i^C(t)$.

The following models are assumed:

$$\begin{aligned}
\lambda_1(t; T_i|\boldsymbol{\mathcal{H}}_i(T_i), T_i) &= \lambda_{01}(t) \exp\{\boldsymbol{\beta}_1' \boldsymbol{Z}_{i1}(T_i)\} \\
\lambda_0(t; s|\boldsymbol{\mathcal{H}}_i(s), \mathcal{E}_i(s) = 1) &= \lambda_{00}(t) \exp\{\boldsymbol{\beta}_0' \boldsymbol{Z}_{i0}(s)\} \\
\lambda_{0k}(t; s|\boldsymbol{\mathcal{H}}_i(S_{ik}), \mathcal{E}_i(S_{ik}) = 1) &= \lambda_{00k}(t) \exp\{\boldsymbol{\beta}_0' \boldsymbol{Z}_{i0}(S_{ik})\} \\
\lambda_i^T(t|\boldsymbol{\mathcal{H}}_i(t), \mathcal{E}_i(t)) &= \mathcal{E}_i(t)\lambda_0^T(t) \exp\{\boldsymbol{\theta}_0' \boldsymbol{Z}_i(t)\} \\
\lambda_i^C(t) &= \lambda_0^C(t) \exp\{\boldsymbol{\alpha}_0' \boldsymbol{Z}_i(0)\}.
\end{aligned}$$

XX We assume strong ignorability, which essentially equates hazard functions corresponding to the counterfactuals (through which the target treatment effect is described) and hazard functions based on observed data. Specifically, we assume that

$$\begin{aligned}
&\lim_{dt\downarrow 0} P\{t \leqslant (D^1 - s) < t + dt|(D^1 - s) \geqslant t, \boldsymbol{\mathcal{H}}(s), T = s\} \\
= &\lim_{dt\downarrow 0} P\{t \leqslant (D - s) < t + dt|(D - s) \geqslant t, \boldsymbol{\mathcal{H}}(s), T \wedge D = s, T < D\}
\end{aligned}$$

and that

$$\begin{aligned}
&\lim_{dt\downarrow 0} P\{t \leqslant (D^0 - s) < t + dt|(D^0 - s) \geqslant t, \boldsymbol{\mathcal{H}}(s), T = s\} \\
= &\lim_{dt\downarrow 0} P\{t \leqslant (D \wedge T - s) < t + dt, D < T|(D \wedge T - s) \geqslant t, \boldsymbol{\mathcal{H}}(s), \mathcal{E}(s) = 1\}.
\end{aligned}$$

2. Regularity Conditions

In deriving the asymptotic properties of the proposed estimators the following conditions are assumed for $i = 1, ..., n$ and $k = 1, ..., K$

(a) The vectors $\{X_i, N_i(X_i), N_i^T(X_i), \boldsymbol{\mathcal{H}}_i(X_i \wedge T_i)\}$ are independent and identically distributed.

(b) $|Z_{il}^*(t)| < \kappa_l$, for $t \in [0, \tau]$ and $Z_{il}^*(t)$ is the lth element of $\boldsymbol{Z}_i^*(t)$.

(c) $\int_0^{\tau_{0k}} \lambda_{0k}(t)dt < \infty$, $\int_0^{\tau_1} \lambda_{01}(t)dt < \infty$, $\int_0^{\tau} \lambda_0^T(t)dt < \infty$ and $\int_0^{\tau} \lambda_0^C(t)dt < \infty$.

(d) Continuity of the following functions:

$$\begin{aligned} \boldsymbol{r}_T^{(1)}(t;\boldsymbol{\theta}) &= \frac{\partial}{\partial \boldsymbol{\theta}} r_T^{(0)}(t;\boldsymbol{\theta}), \\ \boldsymbol{r}_T^{(2)}(t;\boldsymbol{\theta}) &= \frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} r_T^{(0)}(t;\boldsymbol{\theta}), \end{aligned}$$

and $r_T^{(0)}(t;\boldsymbol{\theta})$, where

$$\boldsymbol{r}_T^{(p)}(t;\boldsymbol{\theta}) = E[\mathcal{E}_i(t)Y_i(t)\boldsymbol{Z}_i(t)^{\otimes p} \exp\{\boldsymbol{\theta}'\boldsymbol{Z}_i(t)\}],$$

is the limiting value of

$$\boldsymbol{R}_T^{(p)}(t;\boldsymbol{\theta}) = n^{-1} \sum_{i=1}^{n} \mathcal{E}_i(t)Y_i(t)\boldsymbol{Z}_i(t)^{\otimes p} \exp\{\boldsymbol{\theta}'\boldsymbol{Z}_i(t)\},$$

for $p = 0, 1, 2$, with $r_T^{(1)}(t;\boldsymbol{\theta})$ and $\boldsymbol{r}_T^{(2)}(t;\boldsymbol{\theta})$ bounded and $r_T^{(0)}(t;\boldsymbol{\theta})$ bounded away from 0 for $t \in [0, \tau]$ and $\boldsymbol{\theta}$ in an open set, with $z^{\otimes 0} = 1, \boldsymbol{z}^{\otimes 1} = \boldsymbol{z}$ and $\boldsymbol{z}^{\otimes 2} = \boldsymbol{z}\boldsymbol{z}'$ for a vector $\boldsymbol{z}$.

Continuity of the following functions:

$$\begin{aligned} \boldsymbol{r}_{0k}^{(1)}(t;\boldsymbol{\beta}, W) &= \frac{\partial}{\partial \boldsymbol{\beta}} r_{0k}^{(0)}(t;\boldsymbol{\beta}, W), \\ \boldsymbol{r}_{0k}^{(2)}(t;\boldsymbol{\beta}, W) &= \frac{\partial^2}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}'} r_{0k}^{(0)}(t;\boldsymbol{\beta}, W), \end{aligned}$$

and $r_{0k}^{(0)}(t;\boldsymbol{\beta}, W)$, where

$$\boldsymbol{r}_{0k}^{(p)}(t;\boldsymbol{\beta}, W) = E[\mathcal{E}_{ik} W_{ik}^A(t) \boldsymbol{Z}_{i0k}^{\otimes p} \exp(\boldsymbol{\beta}'\boldsymbol{Z}_{i0k})],$$

is the limiting value of

$$\boldsymbol{R}_{0k}^{(p)}(t;\boldsymbol{\beta},W) = n^{-1}\sum_{i=1}^{n}\mathcal{E}_{ik}W_{ik}^{A}(t)\boldsymbol{Z}_{i0k}^{\otimes p}\exp(\boldsymbol{\beta}'\boldsymbol{Z}_{i0k}),$$

for $p = 0, 1, 2$, with $\boldsymbol{r}_{0k}^{(1)}(t;\boldsymbol{\beta},W)$ and $\boldsymbol{r}_{0k}^{(2)}(t;\boldsymbol{\beta},W)$ bounded and $r_{0k}^{(0)}(t;\boldsymbol{\beta},W)$ bounded away from 0 for $t \in [0, \tau_{0k}]$ and $\boldsymbol{\beta}$ in an open set.

Continuity of the following functions:

$$\begin{aligned}\boldsymbol{r}_{1}^{(1)}(t;\boldsymbol{\beta}) &= \frac{\partial}{\partial\boldsymbol{\beta}}r_{1}^{(0)}(t;\boldsymbol{\beta}),\\ \boldsymbol{r}_{1}^{(2)}(t;\boldsymbol{\beta}) &= \frac{\partial^{2}}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'}r_{1}^{(0)}(t;\boldsymbol{\beta}),\end{aligned}$$

and $r_{1}^{(0)}(t;\boldsymbol{\beta})$ where

$$\boldsymbol{r}_{1}^{(p)}(t;\boldsymbol{\beta}_{1}) = E[Y_{i1}(t)\boldsymbol{Z}_{i1}^{\otimes p}\exp(\boldsymbol{\beta}_{1}'\boldsymbol{Z}_{i1})],$$

is the limiting value of

$$\boldsymbol{R}_{1}^{(p)}(t;\boldsymbol{\beta}_{1}) = n^{-1}\sum_{i=1}^{n}Y_{i1}(t)\boldsymbol{Z}_{i1}^{\otimes p}\exp(\boldsymbol{\beta}_{1}'\boldsymbol{Z}_{i1}),$$

for $p = 0, 1, 2$, with $\boldsymbol{r}_{1}^{(1)}(t;\boldsymbol{\beta}_{1})$ and $\boldsymbol{r}_{1}^{(2)}(t;\boldsymbol{\beta}_{1})$ bounded and $r_{1}^{(0)}(t;\boldsymbol{\beta}_{1})$ bounded away from 0 for $t \in [0, \tau_{1}]$ and $\boldsymbol{\beta}_{1}$ in an open set.

Continuity of the following functions:

$$\begin{aligned}\boldsymbol{r}_{C}^{(1)}(t;\boldsymbol{\alpha}) &= \frac{\partial}{\partial\boldsymbol{\alpha}}r_{C}^{(0)}(t;\boldsymbol{\alpha}),\\ \boldsymbol{r}_{C}^{(2)}(t;\boldsymbol{\alpha}) &= \frac{\partial^{2}}{\partial\boldsymbol{\alpha}\partial\boldsymbol{\alpha}'}r_{C}^{(0)}(t;\boldsymbol{\alpha}),\end{aligned}$$

and $r_{C}^{(0)}(t;\boldsymbol{\alpha})$where

$$\boldsymbol{r}_{C}^{(p)}(t;\boldsymbol{\alpha}) = E[Y_{i}(t)\boldsymbol{Z}_{i}(0)^{\otimes p}\exp\{\boldsymbol{\alpha}'\boldsymbol{Z}_{i}(0)\}],$$

is the limiting value of

$$\boldsymbol{R}_{C}^{(p)}(t;\boldsymbol{\alpha}) = n^{-1}\sum_{i=1}^{n}Y_{i}(t)\boldsymbol{Z}_{i}(0)^{\otimes p}\exp\{\boldsymbol{\alpha}'\boldsymbol{Z}_{i}(0)\},$$

for $p = 0, 1, 2$, with $\boldsymbol{r}_{C}^{(1)}(t;\boldsymbol{\alpha})$ and $\boldsymbol{r}_{C}^{(2)}(t;\boldsymbol{\alpha})$ bounded and $r_{C}^{(0)}(t;\boldsymbol{\alpha})$ bounded away from 0 for $t \in [0, \tau]$ and $\boldsymbol{\alpha}$ in an open set.

(e) Positive-definiteness of the matrices $\boldsymbol{\Omega}_T(\boldsymbol{\theta}_0)$, $\boldsymbol{\Omega}_0(\boldsymbol{\beta}_0)$, $\boldsymbol{\Omega}_1(\boldsymbol{\beta}_1)$ and $\boldsymbol{\Omega}_C(\boldsymbol{\alpha}_0)$, where

$$\begin{aligned}
\boldsymbol{\Omega}_T(\boldsymbol{\theta}) &= E\Big[\int_0^{\tau}\Big\{\frac{\boldsymbol{r}_T^{(2)}(t;\boldsymbol{\theta})}{r_T^{(0)}(t;\boldsymbol{\theta})} - \overline{\boldsymbol{z}}(t;\boldsymbol{\theta})^{\otimes 2}\Big\}dN_i^T(t)\Big],\\
\overline{\boldsymbol{z}}(t;\boldsymbol{\theta}) &= \boldsymbol{r}_T^{(1)}(t;\boldsymbol{\theta})/r_T^{(0)}(t;\boldsymbol{\theta}),\\
\boldsymbol{\Omega}_0(\boldsymbol{\beta}) &= E\Big[\sum_{k=1}^{K}\int_0^{\tau_{0k}}\Big\{\frac{\boldsymbol{r}_{0k}^{(2)}(t;\boldsymbol{\beta},W)}{r_{0k}^{(0)}(t;\boldsymbol{\beta},W)} - \overline{\boldsymbol{z}}_{0k}(t;\boldsymbol{\beta},W)^{\otimes 2}\Big\}dN_{i0k}(t)\Big].\\
\overline{\boldsymbol{z}}_{0k}(t;\boldsymbol{\beta},W) &= \boldsymbol{r}_{0k}^{(1)}(t;\boldsymbol{\beta},W)/r_{0k}^{(0)}(t;\boldsymbol{\beta},W),\\
\boldsymbol{\Omega}_1(\boldsymbol{\beta}) &= E\Big[\int_0^{\tau_1}\Big\{\frac{\boldsymbol{r}_1^{(2)}(t;\boldsymbol{\beta})}{r_1^{(0)}(t;\boldsymbol{\beta})} - \overline{\boldsymbol{z}}_1(t;\boldsymbol{\beta})^{\otimes 2}\Big\}dN_{i1}(t)\Big].\\
\overline{\boldsymbol{z}}_1(t;\boldsymbol{\beta}) &= \boldsymbol{r}_1^{(1)}(t;\boldsymbol{\beta})/r_1^{(0)}(t;\boldsymbol{\beta}),\\
\boldsymbol{\Omega}_C(\boldsymbol{\alpha}) &= E\Big[\int_0^{\tau}\Big\{\frac{\boldsymbol{r}_C^{(2)}(t;\boldsymbol{\alpha})}{r_C^{(0)}(t;\boldsymbol{\alpha})} - \overline{\boldsymbol{z}}_C(t;\boldsymbol{\alpha})^{\otimes 2}\Big\}dN_i^C(t)\Big],\\
\overline{\boldsymbol{z}}_C(t;\boldsymbol{\alpha}) &= \boldsymbol{r}_C^{(1)}(t;\boldsymbol{\alpha})/r_C^{(0)}(t;\boldsymbol{\alpha}),
\end{aligned}$$

(f) $P\{Y_i(t)=1\} > 0$ for $t \in (0,\tau]$

3. Outline of Asymptotic Derivation

We derive the influence functions of terms of interest as summations of independent and identical distributed (i.i.d.) terms plus a term which converges to zero in probability. The terms are as follows:

(1) $n^{\frac{1}{2}}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)$

(2) $n^{\frac{1}{2}}\{\widehat{\Lambda}_0^T(t) - \Lambda_0^T(t)\}$

(3) $n^{\frac{1}{2}}\{\widehat{\Lambda}_i^T(t) - \Lambda_i^T(t)\}$

(4) $n^{\frac{1}{2}}\{\widehat{W}_{ik}^A(t) - W_{ik}^A(t)\}$

(5) $n^{\frac{1}{2}}(\widehat{\boldsymbol{\beta}}_0 - \boldsymbol{\beta}_0)$

(6) $n^{\frac{1}{2}}\{\widehat{\Lambda}_{00}(t) - \Lambda_{00}(t)\}$

(7) $n^{\frac{1}{2}}\{\widehat{\Lambda}_{i0}(t;S_i) - \Lambda_{i0}(t;S_i)\}$

(8) $n^{\frac{1}{2}}\{\widehat{S}_{i0}(t;S_i) - S_{i0}(t;S_i)\}$

(9) $n^{\frac{1}{2}}\{\widehat{\mu}_{i0}(S_i) - \mu_{i0}(S_i)\}$

(10) $n^{\frac{1}{2}}(\widehat{\boldsymbol{\beta}}_1 - \boldsymbol{\beta}_1)$

(11) $n^{\frac{1}{2}}\{\widehat{\Lambda}_{01}(t) - \Lambda_{01}(t)\}$

(12) $n^{\frac{1}{2}}\{\widehat{\Lambda}_{i1}(t;T_i) - \Lambda_{i1}(t;T_i)\}$

(13) $n^{\frac{1}{2}}\{\widehat{S}_{i1}(t;T_i) - S_{i1}(t;T_i)\}$

(14) $n^{\frac{1}{2}}\{\widehat{\mu}_{i1}(T_i) - \mu_{i1}(T_i)\}$

(15) $n^{\frac{1}{2}}(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0)$

(16) $n^{\frac{1}{2}}\{\widehat{\Lambda}_0^C(t) - \Lambda_0^C(t)\}$

(17) $n^{\frac{1}{2}}\{\widehat{\Lambda}_i^C(t) - \Lambda_i^C(t)\}$

(18) $n^{\frac{1}{2}}\{\widehat{G}_i(t)^{-1} - G_i(t)^{-1}\}$

(19) $n^{\frac{1}{2}}\{\widehat{\delta}_i(t) - \delta_i(t)\}$

(20) $n^{\frac{1}{2}}\{\widehat{\Delta}_i(L) - \Delta_i(L)\}$

(21) $n^{\frac{1}{2}}\{\widehat{\delta}(t) - \delta(t)\}$

(22) $n^{\frac{1}{2}}(\widehat{\Delta}(L) - \Delta(L))$

The parameter $\boldsymbol{\theta}_0$ is consistently estimated by $\widehat{\boldsymbol{\theta}}$, the solution to $\boldsymbol{U}^T(\boldsymbol{\theta}) = \mathbf{0}$, where

$$\boldsymbol{U}^T(\boldsymbol{\theta}) = \sum_{i=1}^{n} \int_0^{\tau} \{\boldsymbol{Z}_i(t) - \overline{\boldsymbol{Z}}(t;\boldsymbol{\theta})\} dN_i^T(t),$$

where $\overline{\boldsymbol{Z}}(t;\boldsymbol{\theta}) = \boldsymbol{R}_T^{(1)}(t;\boldsymbol{\theta})/R_T^{(0)}(t;\boldsymbol{\theta})$, $\boldsymbol{R}_T^{(p)}(t;\boldsymbol{\theta}) = n^{-1}\sum_{i=1}^{n} Y_i(t)\mathcal{E}_i(t)\boldsymbol{Z}_i(t)^{\otimes p}\exp\{\boldsymbol{\theta}'\boldsymbol{Z}_i(t)\}$ for $p = 0, 1, 2,$. The Breslow (1972) estimator of $\Lambda_0^T(t)$ is given by $\widehat{\Lambda}_0^T(t) = n^{-1}\sum_{i=1}^{n}\int_0^t R_T^{(0)}(u;\widehat{\boldsymbol{\theta}})^{-1} dN_i^T(u)$.

The parameter $\boldsymbol{\beta}_1$ is consistently estimated by $\widehat{\boldsymbol{\beta}}_1$, the solution to $\boldsymbol{U}_1(\boldsymbol{\beta}) = \mathbf{0}$, where

$$\boldsymbol{U}_1(\boldsymbol{\beta}) = \sum_{i=1}^{n} N_i^T(\tau) \int_0^{\tau_1} \{\boldsymbol{Z}_{i1}(T_i) - \overline{\boldsymbol{Z}}_1(t;\boldsymbol{\beta})\} dN_{i1}(t), \quad (1)$$

with $N_{i1}(t) = N_i(T_i + t)I(C_i > T_i)$ denoting the counting process for post-treatment death;

τ is chosen to satisfy $P(C_i \geqslant \tau) > 0$ and will often be set to the maximum follow-up time; $\overline{\boldsymbol{Z}}_1(t;\boldsymbol{\beta}_1) = \boldsymbol{R}_1^{(1)}(t;\boldsymbol{\beta}_1)/R_1^{(0)}(t;\boldsymbol{\beta}_1)$, $Y_{i1}(t) = Y_i(T_i+t)I(C_i > T_i)$; $\boldsymbol{R}_1^{(p)}(t;\boldsymbol{\beta}_1) = n^{-1}\sum_{i=1}^n N_i^T(\tau)Y_{i1}(t)\boldsymbol{Z}_{i1}(T_i)^{\otimes p}\exp\{\boldsymbol{\beta}_1'\boldsymbol{Z}_{i1}(T_i)\}$, for $p = 0,1,2$, where, for a vector $\boldsymbol{z}$, $\boldsymbol{z}^{\otimes 0} = 1$, $\boldsymbol{z}^{\otimes 1} = \boldsymbol{z}$, $\boldsymbol{z}^{\otimes 2} = \boldsymbol{z}\boldsymbol{z}'$ and τ_1 satisfies $P\{Y_{i1}(\tau_1) = 1\} > 0$ and will typically be set to the maximum post-treatment follow-up time. The cumulative baseline hazard, $\Lambda_{01}(t)$ is consistently estimated by the Breslow (1972) estimator,

$$\widehat{\Lambda}_{01}(t) = n^{-1}\sum_{i=1}^n N_i^T(\tau)\int_0^t R_1^{(0)}(u;\widehat{\boldsymbol{\beta}}_1)^{-1}dN_{i1}(u).$$

The following quantities are pertinent to Theorem 1:

$$\begin{aligned}
\xi_j(t) &= V(\tau)^{-1}\left\{V_{1j}(t) + V_{2j}(t) + \int_0^\tau \{\delta(t;u) - \delta(t)\}G_j(u)^{-1}dN_j^T(u)\right\},\\
\eta_j(L) &= \int_0^L \xi_j(t)dt,
\end{aligned}$$

where

$$\begin{aligned}
V(\tau) &= E\left[\int_0^\tau G_i(u)^{-1}dN_i^T(u)\right],\\
V_{1j}(t) &= E\left[\int_0^\tau \varphi_{ij}^S(t)G_i(u)^{-1}dN_i^T(u)\right],\\
V_{2j}(t) &= E\left[\int_0^\tau \delta_i(t;u)\varphi_{ij}^C(u)dN_i^T(u)\right],\\
\varphi_{ij}^C(t) &= G_i(t)\{\boldsymbol{D}_i^{C'}(t)\boldsymbol{\Omega}_C(\boldsymbol{\alpha}_0)^{-1}\boldsymbol{U}_j^C(\boldsymbol{\alpha}_0) + J_{ij}^C(t)\},\\
\varphi_{ij}^S(t) &= S_{i0}(t)\{\Lambda_{i0}(t)\boldsymbol{Z}_i(S_i)'\boldsymbol{\Omega}_0^{-1}(\boldsymbol{\beta}_0)\boldsymbol{U}_{j0}(\boldsymbol{\beta}_0) - \exp\{\boldsymbol{\beta}_0'\boldsymbol{Z}_i(t)\}\Phi_{j0}(t)\},\\
&\quad -S_{i1}(t)\{\Lambda_{i1}(t)\boldsymbol{Z}_{i1}'\boldsymbol{\Omega}_1^{-1}(\boldsymbol{\beta}_1)\boldsymbol{U}_{j1}(\boldsymbol{\beta}_1) - \exp\{\boldsymbol{\beta}_1'\boldsymbol{Z}_{i1}\}\Phi_{j1}(t)\},
\end{aligned}$$

with $\boldsymbol{\Omega}_1(\boldsymbol{\beta}_1)$, $\boldsymbol{\Omega}_0(\boldsymbol{\beta}_0)$, $\boldsymbol{U}_{j1}(\boldsymbol{\beta}_1)$, $\boldsymbol{U}_{j0}(\boldsymbol{\beta}_0)$, $\mu_{i1}(t)$, $\mu_{i0}(t)$, $\Phi_{j1}(t)$, $\Phi_{j0}(t)$, $\boldsymbol{D}_i^C(t)$, $\boldsymbol{\Omega}_C(\boldsymbol{\alpha}_0)$, $\boldsymbol{U}_j^C(\boldsymbol{\alpha}_0)$ and $J_{ij}^C(t)$ are defined in derivation provided in the next subsection.

4. Derivation of Asymptotic Properties

Several parts of the proof regarding the proportional hazards model are well-established results. Therefore, they are simply listed without proof. For details, please refer to Andersen and Gill (1982), Fleming and Harrington (1991) and Andersen et al. (1993).

4.1 $n^{\frac{1}{2}}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)$

As $n \to \infty$, we have

$$n^{\frac{1}{2}}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) = \boldsymbol{\Omega}_T(\boldsymbol{\theta}_0)^{-1} n^{-\frac{1}{2}} \sum_{i=1}^{n} U_i^T(\boldsymbol{\theta}_0) + o_p(1),$$

where

$$\boldsymbol{U}_i^T(\boldsymbol{\theta}) = \int_0^\tau \{\boldsymbol{Z}_i(t) - \overline{\boldsymbol{z}}(t;\boldsymbol{\theta})\} dM_i^T(t;\boldsymbol{\theta}),$$
$$dM_i^T(t) = dN_i^T(t) - Y_i(t) d\Lambda_i^T(t),$$

This is now a well-established Cox model result, derived through Martingale theory.

4.2 $n^{\frac{1}{2}}\{\widehat{\Lambda}_0^T(t) - \Lambda_0^T(t)\}$

We induce the following decomposition:

$$\begin{aligned} n^{\frac{1}{2}}\{\widehat{\Lambda}_0^T(t) - \Lambda_0^T(t)\} & \\ &= n^{\frac{1}{2}}\{\widehat{\Lambda}_0^T(t;\widehat{\boldsymbol{\theta}}) - \widehat{\Lambda}_0^T(t;\boldsymbol{\theta}_0)\} \qquad (2) \\ &\quad + n^{\frac{1}{2}}\{\widehat{\Lambda}_0^T(t;\boldsymbol{\theta}_0) - \Lambda_0^T(t)\}. \qquad (3) \end{aligned}$$

We can express the first term as

$$\begin{aligned} (2) &= n^{-\frac{1}{2}} \sum_{i=1}^{n} \int_0^t \{R_T^{(0)}(u;\widehat{\boldsymbol{\theta}})^{-1} - R_T^{(0)}(u;\boldsymbol{\theta}_0)^{-1}\} dN_i^T(u) \\ &= \widehat{\boldsymbol{h}}'_T(t;\boldsymbol{\theta}_0)\boldsymbol{\Omega}_T(\boldsymbol{\theta}_0)^{-1} n^{-\frac{1}{2}} \sum_{i=1}^{n} \boldsymbol{U}_i^T(\boldsymbol{\theta}_0) \\ &= \boldsymbol{h}'_T(t;\boldsymbol{\theta}_0)\boldsymbol{\Omega}_T(\boldsymbol{\theta}_0)^{-1} n^{-\frac{1}{2}} \sum_{i=1}^{n} \boldsymbol{U}_i^T(\boldsymbol{\theta}_0) + o_p(1). \end{aligned}$$

where the third line follows from the convergence in probability of

$$\widehat{\boldsymbol{h}}'_T(t;\boldsymbol{\theta}) = -n^{-1} \sum_{i=1}^{n} \int_0^t R_T^{(0)}(u;\boldsymbol{\theta})^{-1} \overline{\boldsymbol{Z}}(u;\boldsymbol{\theta}) dN_i^T(u) = -\int_0^t \overline{\boldsymbol{Z}}(u;\boldsymbol{\theta}) d\widehat{\Lambda}_0^T(u;\boldsymbol{\theta}),$$
$$\widehat{\boldsymbol{\Omega}}_T(\boldsymbol{\theta}) = n^{-1} \sum_{i=1}^{n} \int_0^\tau \left\{ \frac{\boldsymbol{R}_T^{(2)}(t;\boldsymbol{\theta})}{R_T^{(0)}(t;\boldsymbol{\theta})} - \overline{\boldsymbol{Z}}(t,\boldsymbol{\theta})^{\otimes 2} \right\} dN_i^T(t),$$

where $\overline{\boldsymbol{Z}}(t;\boldsymbol{\theta}) = \boldsymbol{R}_T^{(1)}(t;\boldsymbol{\theta})/R_T^{(0)}(t;\boldsymbol{\theta})$, to the quantities

$$\boldsymbol{h}_T'(t;\boldsymbol{\theta}) = -\int_0^t \overline{\boldsymbol{z}}(u;\boldsymbol{\theta})d\Lambda_0^T(u),$$

and $\boldsymbol{\Omega}_T(\boldsymbol{\theta})$ respectively, with $\boldsymbol{\Omega}_T(\boldsymbol{\theta})$ defined in Regularity Condition (e).

With respect to the second term in the decomposition, we have,

$$\begin{aligned}(3) &= n^{-\frac{1}{2}}\sum_{i=1}^n \int_0^t R_T^{(0)}(u;\boldsymbol{\theta}_0)^{-1}dM_i^T(u) \\ &= n^{-\frac{1}{2}}\sum_{i=1}^n \int_0^t r_T^{(0)}(u;\boldsymbol{\theta}_0)^{-1}dM_i^T(u) + o_p(1),\end{aligned}$$

where the second line follows from continuity and Condition (d). Combining results, for the decomposition, we have

$$n^{\frac{1}{2}}\{\widehat{\Lambda}_0^T(t) - \Lambda_0^T(t)\} = n^{-\frac{1}{2}}\sum_{i=1}^n \Phi_i^T(t;\boldsymbol{\theta}_0) + o_p(1),$$

where

$$\Phi_i^T(t;\boldsymbol{\theta}) = \boldsymbol{h}_T'(t;\boldsymbol{\theta})\boldsymbol{\Omega}_T(\boldsymbol{\theta})^{-1}\boldsymbol{U}_i^T(\boldsymbol{\theta}) + \int_0^t r_T^{(0)}(u;\boldsymbol{\theta})^{-1}dM_i^T(u) = \int_0^t d\Phi_i^T(u;\boldsymbol{\theta}),$$

and

$$d\Phi_i^T(u;\boldsymbol{\theta}) = -\overline{\boldsymbol{z}}'(u;\boldsymbol{\theta})d\Lambda_0^T(u)\boldsymbol{\Omega}_T(\boldsymbol{\theta})^{-1}\boldsymbol{U}_i^T(\boldsymbol{\theta}) + r_T^{(0)}(u;\boldsymbol{\theta})^{-1}dM_i^T(u).$$

4.3 $n^{\frac{1}{2}}\{\widehat{\Lambda}_i^T(t) - \Lambda_i^T(t)\}$

We begin with another decomposition,

$$\begin{aligned}& n^{\frac{1}{2}}\{\widehat{\Lambda}_i^T(t) - \Lambda_i^T(t)\} \\ =\ & n^{\frac{1}{2}}\left\{\int_0^t \exp\{\widehat{\boldsymbol{\theta}}'\boldsymbol{Z}_i(u)\}d\widehat{\Lambda}_0^T(u) - \int_0^t \exp\{\boldsymbol{\theta}_0'\boldsymbol{Z}_i(u)\}d\widehat{\Lambda}_0^T(u)\right\} \qquad (4) \\ & + n^{\frac{1}{2}}\left\{\int_0^t \exp\{\boldsymbol{\theta}_0'\boldsymbol{Z}_i(u)\}d\widehat{\Lambda}_0^T(u) - \int_0^t \exp\{\boldsymbol{\theta}_0'\boldsymbol{Z}_i(u)\}d\Lambda_0^T(u)\right\}. \qquad (5)\end{aligned}$$

Considering the first term,

$$(4) = n^{\frac{1}{2}}\int_0^t \{\exp\{\widehat{\boldsymbol{\theta}}'\boldsymbol{Z}_i(u)\} - \exp\{\boldsymbol{\theta}_0'\boldsymbol{Z}_i(u)\}\}d\widehat{\Lambda}_0^T(u).$$

By a Taylor series expansion,

$$\begin{aligned} n^{\frac{1}{2}}\{\exp\{\widehat{\boldsymbol{\theta}}'\boldsymbol{Z}_i(u)\} - \exp\{\boldsymbol{\theta}'\boldsymbol{Z}_i(u)\}\} &= \boldsymbol{Z}_i'(u)\exp\{\boldsymbol{\theta}'\boldsymbol{Z}_i(u)\}n^{\frac{1}{2}}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) + o_p(1) \\ &= \boldsymbol{Z}_i'(u)\exp\{\boldsymbol{\theta}'\boldsymbol{Z}_i(u)\}\boldsymbol{\Omega}_T(\boldsymbol{\theta})^{-1}n^{-\frac{1}{2}}\sum_{l=1}^{n}\boldsymbol{U}_l^T(\boldsymbol{\theta}) + o_p(1). \end{aligned}$$

Since $\widehat{\Lambda}_0^T(t) \xrightarrow{p} \Lambda_0^T(t)$ for $t \in [0,\tau]$, we obtain

$$(4) \quad = \quad \int_0^t \boldsymbol{Z}_i'(u)d\Lambda_i^T(u)\boldsymbol{\Omega}_T(\boldsymbol{\theta})^{-1}n^{-\frac{1}{2}}\sum_{l=1}^{n}\boldsymbol{U}_l^T(\boldsymbol{\theta}) + o_p(1).$$

By using Result 4.2, the second term can be written as

$$\begin{aligned} (5) \quad &= \quad n^{\frac{1}{2}}\int_0^t \exp\{\boldsymbol{\theta}_0'\boldsymbol{Z}_i(u)\}d\{\widehat{\Lambda}_0^T(t) - \Lambda_0^T(t)\} \\ &= \int_0^t \exp\{\boldsymbol{\theta}_0'\boldsymbol{Z}_i(u)\}n^{-\frac{1}{2}}\sum_{l=1}^{n}d\Phi_l^T(u;\boldsymbol{\theta}_0) + o_p(1). \end{aligned}$$

Combining results from the decomposition leads to

$$\begin{aligned} n^{\frac{1}{2}}\{\widehat{\Lambda}_i^T(t) - \Lambda_i^T(t)\} &= \int_0^t \{\boldsymbol{Z}_i(u) - \overline{\boldsymbol{z}}(u;\boldsymbol{\theta}_0)\}'d\Lambda_i^T(u)\boldsymbol{\Omega}_T(\boldsymbol{\theta}_0)^{-1}n^{-\frac{1}{2}}\sum_{l=1}^{n}\boldsymbol{U}_l^T(\boldsymbol{\theta}_0) \\ &\quad + n^{-\frac{1}{2}}\sum_{l=1}^{n}\int_0^t \exp\{\boldsymbol{\theta}_0'\boldsymbol{Z}_i(u)\}r_T^{(0)}(u;\boldsymbol{\theta}_0)^{-1}dM_l^T(u) + o_p(1) \\ &= \boldsymbol{D}_i'(t;\boldsymbol{\theta}_0)\boldsymbol{\Omega}_T(\boldsymbol{\theta}_0)^{-1}n^{-\frac{1}{2}}\sum_{l=1}^{n}\boldsymbol{U}_l^T(\boldsymbol{\theta}_0) + n^{-\frac{1}{2}}\sum_{l=1}^{n}\boldsymbol{J}_{il}^T(t;\boldsymbol{\theta}_0) + o_p(1), \end{aligned}$$

where we define

$$\begin{aligned} \boldsymbol{D}_i(t;\boldsymbol{\theta}) &= \int_0^t \{\boldsymbol{Z}_i(u) - \overline{\boldsymbol{z}}(u;\boldsymbol{\theta})\}'d\Lambda_i^T(u) = \int_0^t d\boldsymbol{D}_i(u;\boldsymbol{\theta}), \\ \boldsymbol{J}_{il}^T(t;\boldsymbol{\theta}) &= \int_0^t \exp\{\boldsymbol{\theta}'\boldsymbol{Z}_i(u)\}r_T^{(0)}(u;\boldsymbol{\theta})^{-1}dM_l^T(u). \end{aligned}$$

4.4 $n^{\frac{1}{2}}\{\widehat{W}_{ik}^A(t) - W_{ik}^A(t)\}$

Consistent with the notation set in Section 1, when the subscript of quantities does not involve the cross section notation k, t refers the time from study entry. If k is present in the subscript, then t denotes the time from the kth cross section date.

Since $W_{ik}^A(t) = \exp\{\Lambda_i^T(t+S_{ik}) - \Lambda_i^T(S_{ik})\}$ and $\widehat{W}_{ik}^A(t) = \exp\{\widehat{\Lambda}_i^T(t+S_{ik}) - \widehat{\Lambda}_i^T(S_{ik})\}$, we then have

$$\begin{aligned}
& n^{\frac{1}{2}}\{\widehat{W}_{ik}^{A}(t) - W_{ik}^{A}(t)\} \\
= \;& n^{\frac{1}{2}}\{\exp\{\widehat{\Lambda}_i^T(t+S_{ik}) - \widehat{\Lambda}_i^T(S_{ik})\} - \exp\{\Lambda_i^T(t+S_{ik}) - \Lambda_i^T(S_{ik})\}\} \\
= \;& W_{ik}^{A}(t) n^{\frac{1}{2}}[\{\widehat{\Lambda}_i^T(t+S_{ik}) - \Lambda_i^T(t+S_{ik})\} - \{\widehat{\Lambda}_i^T(S_{ik}) - \Lambda_i^T(S_{ik})\}] + o_p(1) \\
= \;& W_{ik}^{A}(t) n^{-\frac{1}{2}} \sum_{l=1}^{n} \{\boldsymbol{D}'_{ik}(t;\boldsymbol{\theta}_0)\boldsymbol{\Omega}_T(\boldsymbol{\theta}_0)^{-1}\boldsymbol{U}_l^T(\boldsymbol{\theta}_0) + \boldsymbol{J}_{ikl}^T(t;\boldsymbol{\theta}_0)\} + o_p(1),
\end{aligned}$$

where we define

$$\begin{aligned}
\boldsymbol{D}_{ik}(t;\boldsymbol{\theta}) &= \int_{S_{ik}}^{S_{ik}+t} \{\boldsymbol{Z}_i(u) - \overline{\boldsymbol{z}}(u;\boldsymbol{\theta})\}' d\Lambda_i^T(u) = \int_0^t dD_i(u;\boldsymbol{\theta}), \\
\boldsymbol{J}_{ikl}^T(t;\boldsymbol{\theta}) &= \int_{S_{ik}}^{S_{ik}+t} \exp\{\boldsymbol{\theta}'\boldsymbol{Z}_i(u)\} r_T^{(0)}(u;\boldsymbol{\theta})^{-1} dM_l^T(u).
\end{aligned}$$

4.5 $n^{\frac{1}{2}}(\widehat{\boldsymbol{\beta}}_0 - \boldsymbol{\beta}_0)$

It is straightforward to show that

$$n^{\frac{1}{2}}(\widehat{\boldsymbol{\beta}}_0 - \boldsymbol{\beta}_0) = \boldsymbol{\Omega}_0^{-1}(\boldsymbol{\beta}_0) n^{-\frac{1}{2}} \sum_{i=1}^{n}\sum_{k=1}^{K} \mathcal{E}_{ik} \boldsymbol{U}_{i0k}(\boldsymbol{\beta}_0, \widehat{W}) + o_p(1),$$

where we define

$$\begin{aligned}
\boldsymbol{U}_{i0k}(\boldsymbol{\beta}, W) &= \int_0^{\tau_{0k}} \{\boldsymbol{Z}_{i0k} - \overline{\boldsymbol{z}}_{0k}(t;\boldsymbol{\beta}, W)\} W_{ik}^A(t) dM_{i0k}(t), \\
dM_{i0k}(t) &= dN_{i0k}(t) - Y_{i0k}(t) d\Lambda_{i0k}(t).
\end{aligned}$$

The term $n^{-\frac{1}{2}}\sum_{i=1}^{n}\sum_{k=1}^{K}\mathcal{E}_{ik}\boldsymbol{U}_{i0k}(\boldsymbol{\beta},\widehat{W})$ can be decomposed as follows,

$$\begin{aligned}
& n^{-\frac{1}{2}}\sum_{i=1}^{n}\sum_{k=1}^{K}\mathcal{E}_{ik}\boldsymbol{U}_{i0k}(\boldsymbol{\beta},\widehat{W}) \\
=\ & n^{-\frac{1}{2}}\sum_{i=1}^{n}\sum_{k=1}^{K}\mathcal{E}_{ik}\int_{0}^{\tau_{0k}}\{\boldsymbol{Z}_{i0k}-\overline{\boldsymbol{z}}_{0k}(t;\boldsymbol{\beta},\widehat{W})\}\widehat{W}_{ik}^{A}(t)dM_{i0k}(t) \\
=\ & n^{-\frac{1}{2}}\sum_{i=1}^{n}\sum_{k=1}^{K}\mathcal{E}_{ik}\int_{0}^{\tau_{0k}}\{\boldsymbol{Z}_{i0k}-\overline{\boldsymbol{z}}_{0k}(t;\boldsymbol{\beta},W)\}W_{ik}^{A}(t)dM_{i0k}(t) \qquad (6)\\
& -n^{-\frac{1}{2}}\sum_{i=1}^{n}\sum_{k=1}^{K}\mathcal{E}_{ik}\int_{0}^{\tau_{0k}}\{\overline{\boldsymbol{z}}_{0k}(t;\boldsymbol{\beta},\widehat{W})-\overline{\boldsymbol{z}}_{0k}(t;\boldsymbol{\beta},W)\}W_{ik}^{A}(t)dM_{i0k}(t) \qquad (7)\\
& +n^{-\frac{1}{2}}\sum_{i=1}^{n}\sum_{k=1}^{K}\mathcal{E}_{ik}\int_{0}^{\tau_{0k}}\{\boldsymbol{Z}_{i0k}-\overline{\boldsymbol{z}}_{0k}(t;\boldsymbol{\beta},\widehat{W})\}\{\widehat{W}_{ik}^{A}(t)-W_{ik}^{A}(t)\}dM_{i0k}(t) \qquad (8)\\
& +o_p(1).
\end{aligned}$$

Now, through the Functional Delta Method, combined with a lot of tedious algebra, (7) converges in probability to 0.

Using Result 4.4

$$\begin{aligned}
(8)\ =\ & n^{-\frac{1}{2}}\sum_{i=1}^{n}\sum_{k=1}^{K}\mathcal{E}_{ik}\int_{0}^{\tau_{0k}}\{\boldsymbol{Z}_{i0k}-\overline{\boldsymbol{z}}_{0k}(t;\boldsymbol{\beta},W)\}W_{ik}^{A}(t) \\
& \times n^{-1}\sum_{l=1}^{n}\boldsymbol{D}_{ik}'(t;\boldsymbol{\theta}_0)\boldsymbol{\Omega}_T(\boldsymbol{\theta}_0)^{-1}U_l^T(\boldsymbol{\theta}_0)dM_{i0k}(t) \qquad (9)\\
& +n^{-\frac{1}{2}}\sum_{i=1}^{n}\sum_{k=1}^{K}\mathcal{E}_{ik}\int_{0}^{\tau_{0k}}\{\boldsymbol{Z}_{i0k}-\overline{\boldsymbol{z}}_{0k}(t;\boldsymbol{\beta},W)\}W_{ik}^{A}(t) \\
& \times n^{-1}\sum_{l=1}^{n}\boldsymbol{J}_{ikl}^{T}(t;\boldsymbol{\theta}_0)dM_{i0k}(t). \qquad (10)
\end{aligned}$$

Switching the order of summation, we have

$$\begin{aligned}(9) &= n^{-1}\sum_{i=1}^{n}\sum_{k=1}^{K}\mathcal{E}_{ik}\int_0^{\tau_{0k}}\{\boldsymbol{Z}_{i0k}-\overline{\boldsymbol{z}}_{0k}(t;\boldsymbol{\beta},W)\}W_{ik}^{A}(t)D_{ik}'(t;\boldsymbol{\theta}_0)dM_{i0k}(t)\\ &\quad\times\boldsymbol{\Omega}_T(\boldsymbol{\theta}_0)^{-1}n^{-\frac{1}{2}}\sum_{l=1}^{n}\boldsymbol{U}_l^T(\boldsymbol{\theta}_0)\\ &= \widehat{\boldsymbol{H}}_0'(t;\boldsymbol{\beta},W)\boldsymbol{\Omega}_T(\boldsymbol{\theta}_0)^{-1}n^{-\frac{1}{2}}\sum_{l=1}^{n}\boldsymbol{U}_l^T(\boldsymbol{\theta}_0)\\ &= \boldsymbol{H}_0'(t;\boldsymbol{\beta},W)\boldsymbol{\Omega}_T(\boldsymbol{\theta})^{-1}n^{-\frac{1}{2}}\sum_{l=1}^{n}\boldsymbol{U}_l^T(\boldsymbol{\theta}_0)+o_p(1),\end{aligned}$$

where the last equality follows from the convergence in probability of

$$\widehat{\boldsymbol{H}}_0'(t;\boldsymbol{\beta},W)=n^{-1}\sum_{i=1}^{n}\sum_{k=1}^{K}\mathcal{E}_{ik}\int_0^{\tau_{0k}}\{\boldsymbol{Z}_{i0k}-\overline{\boldsymbol{z}}_{0k}(t;\boldsymbol{\beta},W)\}W_{ik}^{A}(t)\boldsymbol{D}_{ik}'(t;\boldsymbol{\theta}_0)dM_{i0k}(t),$$

to the quantity

$$\boldsymbol{H}_0'(t;\boldsymbol{\beta},W)=E\left[\sum_{k=1}^{K}\mathcal{E}_{ik}\int_0^{\tau_{0k}}\{\boldsymbol{Z}_{i0k}-\overline{\boldsymbol{z}}_{0k}(t;\boldsymbol{\beta},W)\}W_{ik}^{A}(t)\boldsymbol{D}_{ik}'(t;\boldsymbol{\theta}_0)dM_{i0k}(t)\right].$$

Switching the order of summation and integration

$$\begin{aligned}(10) &= n^{-\frac{1}{2}}\sum_{j=1}^{n}\int_{S_{ik}}^{\tau}\Bigg[n^{-1}\sum_{l=1}^{n}\sum_{k=1}^{K}\mathcal{E}_{ik}\exp\{\boldsymbol{\theta}_0'\boldsymbol{Z}_i(u)\}\int_{u-S_{ik}}^{\tau-S_{ik}}\{\boldsymbol{Z}_{i0k}-\overline{\boldsymbol{z}}_{0k}(t;\boldsymbol{\beta},W)\}\\ &\quad\times W_{ik}^{A}(t)dM_{i0k}(t)\Bigg]r_T^{(0)}(u;\boldsymbol{\theta}_0)^{-1}dM_l^T(u)\\ &= n^{-\frac{1}{2}}\sum_{l=1}^{n}\int_{S_{ik}}^{\tau}\widehat{G}_0(u,\tau;\boldsymbol{\beta})r_T^{(0)}(u;\boldsymbol{\theta}_0)^{-1}dM_l^T(u)\\ &= n^{-\frac{1}{2}}\sum_{l=1}^{n}\int_{S_{ik}}^{\tau}G_0(u,\tau;\boldsymbol{\beta})r_T^{(0)}(u;\boldsymbol{\theta}_0)^{-1}dM_l^T(u)+o_p(1),\end{aligned}$$

where the last equality follows from the convergence in probability of

$$\widehat{G}_0(t_1,t_2;\boldsymbol{\beta})=n^{-1}\sum_{i=1}^{n}\sum_{k=1}^{K}\mathcal{E}_{ik}\exp\{\boldsymbol{\theta}_0'\boldsymbol{Z}_i(t_1)\}\int_{t_1-S_{ik}}^{t_2-S_{ik}}\{\boldsymbol{Z}_{i0k}-\overline{\boldsymbol{z}}_{0k}(t;\boldsymbol{\beta},W)\}W_{ik}^{A}(t)dM_{i0k}(t),$$

to the quantity

$$G_0(t_1,t_2;\boldsymbol{\beta})=E\left[\sum_{k=1}^{K}\mathcal{E}_{ik}\exp\{\boldsymbol{\theta}_0'\boldsymbol{Z}_i(t_1)\}\int_{t_1-S_{ik}}^{t_2-S_{ik}}\{\boldsymbol{Z}_{i0k}-\overline{\boldsymbol{z}}_{0k}(t;\boldsymbol{\beta},W)\}W_{ik}^{A}(t)dM_{i0k}(t)\right].$$

Combining the equations (6) (9) and (10), we obtain

$$n^{\frac{1}{2}}(\widehat{\boldsymbol{\beta}}_0-\boldsymbol{\beta}_0)=\boldsymbol{\Omega}_0(\boldsymbol{\beta}_0)^{-1}n^{-\frac{1}{2}}\sum_{i=1}^{n}\boldsymbol{U}_{i0}(\boldsymbol{\beta}_0)+o_p(1),$$

where

$$\begin{aligned} \boldsymbol{U}_{i0}(\boldsymbol{\beta}_0) &= \sum_{k=1}^{K} \int_0^{\tau_{0k}} \mathcal{E}_{ik}\{\boldsymbol{Z}_{i0k} - \overline{\boldsymbol{z}}_{0k}(t;\boldsymbol{\beta}_0, W)\} W_{ik}^{A}(t) dM_{i0k}(t) \\ &+ \boldsymbol{H}_0'(t;\boldsymbol{\beta}_0, W)\boldsymbol{\Omega}_T(\boldsymbol{\theta}_0)^{-1}\boldsymbol{U}_i^T(\boldsymbol{\theta}_0) \\ &+ \int_{S_{ik}}^{\tau} G_0(t,\tau;\boldsymbol{\beta}_0) r_T^{(0)}(t;\boldsymbol{\theta}_0)^{-1} dM_i^T(t). \end{aligned}$$

4.6 $n^{\frac{1}{2}}\{\widehat{\Lambda}_{00}(t) - \Lambda_{00}(t)\}$

We define

$$\widehat{\Lambda}_{00}(t;\boldsymbol{\beta}_0) = n^{-1}\sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^t R_0^{(0)}(u;\boldsymbol{\beta}_0)^{-1}\mathcal{E}_{ik} W_{ik}^{A}(u) dN_{i0k}(u)$$

for $t \in (0, L]$, where $R_0^{(0)}(u;\boldsymbol{\beta}_0) = \sum_{k=1}^{K} R_{0k}^{(0)}(u;\boldsymbol{\beta}_0)$.

We begin another decomposition,

$$\begin{aligned} & n^{\frac{1}{2}}\{\widehat{\Lambda}_{00}(t) - \Lambda_{00}(t)\} \\ =\ & n^{\frac{1}{2}}[\widehat{\Lambda}_{00}\{t;\widehat{W}, R_0(\widehat{\boldsymbol{\beta}}_0,\widehat{W})\} - \widehat{\Lambda}_{00}\{t;\widehat{W}, R_0(\boldsymbol{\beta}_0,\widehat{W})\}] && (11) \\ & + n^{\frac{1}{2}}[\widehat{\Lambda}_{00}\{t;\widehat{W}, R_0(\boldsymbol{\beta}_0,\widehat{W})\} - \widehat{\Lambda}_{00}\{t;W, R_0(\boldsymbol{\beta}_0,\widehat{W})\}] && (12) \\ & + n^{\frac{1}{2}}[\widehat{\Lambda}_{00}\{t;W, R_0(\boldsymbol{\beta}_0,\widehat{W})\} - \widehat{\Lambda}_{00}\{t;W, R_0(\boldsymbol{\beta}_0,W)\}] && (13) \\ & + n^{\frac{1}{2}}[\widehat{\Lambda}_{00}\{t;W, R_0(\boldsymbol{\beta}_0,W)\} - \Lambda_{00}(t)] && (14) \end{aligned}$$

By using Result 4.5, we can express the first term as

$$\begin{aligned} (11) &= n^{-\frac{1}{2}}\sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^t \mathcal{E}_{ik}\{R_0^{(0)}(u;\widehat{\boldsymbol{\beta}}_0,\widehat{W})^{-1} - R_0^{(0)}(u;\boldsymbol{\beta}_0,\widehat{W})^{-1}\}\widehat{W}_{ik}^{A}(u) dN_{i0k}(u) \\ &= -\int_0^t \overline{\boldsymbol{Z}}_0'(u;\boldsymbol{\beta}_0, W) d\Lambda_{00}(u)\boldsymbol{\Omega}_0(\boldsymbol{\beta}_0)^{-1} n^{-\frac{1}{2}}\sum_{i=1}^{n}\boldsymbol{U}_{i0}(\boldsymbol{\beta}_0) + o_p(1) \\ &= \boldsymbol{h}_0'(t;\boldsymbol{\beta}_0, W)\boldsymbol{\Omega}_0(\boldsymbol{\beta}_0)^{-1} n^{-\frac{1}{2}}\sum_{i=1}^{n}\boldsymbol{U}_{i0}(\boldsymbol{\beta}_0) + o_p(1), \end{aligned}$$

where we define

$$\overline{\boldsymbol{Z}}_0(t;\boldsymbol{\beta},W) = R_0^{(1)}(t;\boldsymbol{\beta},W)/R_0^{(0)}(t;\boldsymbol{\beta},W).$$
$$R_0^{(p)}(t;\boldsymbol{\beta},W) = \sum_{k=1}^{K} R_{0k}^{(p)}(t;\boldsymbol{\beta},W),$$
$$\overline{\boldsymbol{z}}_0(t;\boldsymbol{\beta},W) = \boldsymbol{r}_0^{(1)}(t;\boldsymbol{\beta},W)/r_0^{(0)}(t;\boldsymbol{\beta},W).$$
$$\boldsymbol{r}_0^{(p)}(t;\boldsymbol{\beta},W) = \sum_{k=1}^{K} \boldsymbol{r}_{0k}^{(p)}(t;\boldsymbol{\beta},W),$$
$$\boldsymbol{h}_0(t;\boldsymbol{\beta},W) = -\int_0^t \overline{\boldsymbol{z}}_0'(u;\boldsymbol{\beta},W)d\Lambda_{00}(u).$$

By using Result 4.4, we have

$$\begin{aligned}
(12) \quad &= n^{-\frac{1}{2}}\sum_{i=1}^{n}\sum_{k=1}^{K}\mathcal{E}_{ik}\int_0^t \{\widehat{W}_{ik}^A(u) - W_{ik}^A(u)\}R_0^{(0)}(u;\boldsymbol{\beta}_0,\widehat{W})^{-1}dN_{i0k}(u) \\
&= n^{-1}\sum_{i=1}^{n}\sum_{k=1}^{K}\mathcal{E}_{ik}\int_0^t R_0^{(0)}(u;\boldsymbol{\beta}_0,W)^{-1}W_{ik}^A(u)\boldsymbol{D}_{ik}'(u;\boldsymbol{\theta}_0)\boldsymbol{\Omega}_T(\boldsymbol{\theta}_0)^{-1} \\
&\quad \times n^{-\frac{1}{2}}\sum_{l=1}^{n}\boldsymbol{U}_l^T(\boldsymbol{\theta}_0)dN_{i0k}(u) \qquad (15) \\
&\quad + n^{-1}\sum_{i=1}^{n}\sum_{k=1}^{K}\mathcal{E}_{ik}\int_0^t R_0^{(0)}(u;\boldsymbol{\beta}_0,W)^{-1}W_{ik}^A(u) \\
&\quad \times n^{-\frac{1}{2}}\sum_{l=1}^{n}\boldsymbol{J}_{ikl}^T(u;\boldsymbol{\theta}_0)dN_{i0k}(u) + o_p(1). \qquad (16)
\end{aligned}$$

Switching the order of summation, we have

$$\begin{aligned}
(15) \quad &= \widehat{\boldsymbol{B}}_0'(t;\boldsymbol{\beta}_0)\boldsymbol{\Omega}_T(\boldsymbol{\theta}_0)^{-1}n^{-\frac{1}{2}}\sum_{l=1}^{n}\boldsymbol{U}_l^T(\boldsymbol{\theta}_0) \\
&= \boldsymbol{B}_0'(t;\boldsymbol{\beta}_0)\boldsymbol{\Omega}_T(\boldsymbol{\theta}_0)^{-1}n^{-\frac{1}{2}}\sum_{l=1}^{n}\boldsymbol{U}_l^T(\boldsymbol{\theta}_0) + o_p(1),
\end{aligned}$$

where the last equality follows from the convergence in probability of

$$\widehat{\boldsymbol{B}}_0(t;\boldsymbol{\beta}) = n^{-1}\sum_{i=1}^{n}\sum_{k=1}^{K}\mathcal{E}_{ik}\int_0^t R_0^{(0)}(u;\boldsymbol{\beta},W)^{-1}W_{ik}^A(u)\boldsymbol{D}_{ik}'(u;\boldsymbol{\theta})dN_{i0k}(u)$$

to the quantity

$$\boldsymbol{B}_0(t;\boldsymbol{\beta}) = E\left[\mathcal{E}_{ik}\int_0^t r_0^{(0)}(u;\boldsymbol{\beta},W)^{-1}W_{ik}^A(u)\boldsymbol{D}_{ik}'(u;\boldsymbol{\theta})dN_{i0k}(u)\right].$$

Switching the order of summation and integration

$$
\begin{aligned}
(16) &= n^{-\frac{1}{2}} \sum_{l=1}^{n} \int_0^t \widehat{K}_0(u,t;\boldsymbol{\beta}_0) r_T^{(0)}(u;\boldsymbol{\theta}_0)^{-1} dM_l^T(u) \\
&= n^{-\frac{1}{2}} \sum_{l=1}^{n} \int_0^t K_0(u,t;\boldsymbol{\beta}_0) r_T^{(0)}(u;\boldsymbol{\theta}_0)^{-1} dM_l^T(u) + o_p(1),
\end{aligned}
$$

where the last equality follows from the convergence in probability of

$$
\widehat{K}_0(t_1,t_2;\boldsymbol{\beta}) = n^{-1} \sum_{i=1}^{n} \sum_{k=1}^{K} \exp\{\boldsymbol{\theta}' \boldsymbol{Z}_i(t_1)\} \int_{t_1-S_{ik}}^{t_2-S_{ik}} \mathcal{E}_{ik} W_{ik}^A(u) R_0^{(0)}(u;\boldsymbol{\beta},W)^{-1} dN_{i0k}(u),
$$

to the quantity

$$
K_0(t_1,t_2;\boldsymbol{\beta}) = E\left[\exp\{\boldsymbol{\theta}_0' \boldsymbol{Z}_i(t_1)\} \mathcal{E}_{ik} \int_{t_1-S_{ik}}^{t_2-S_{ik}} W_{ik}^A(u) r_0^{(0)}(u;\boldsymbol{\beta},W)^{-1} dN_{i0k}(u) \right].
$$

Combining equations (15) and (16), we obtain

$$
\begin{aligned}
(12) &= \boldsymbol{B}_0'(t;\boldsymbol{\beta}_0) \boldsymbol{\Omega}_T(\boldsymbol{\theta}_0)^{-1} n^{-\frac{1}{2}} \sum_{l=1}^{n} \boldsymbol{U}_l^T(\boldsymbol{\theta}_0) \\
&\quad + n^{-\frac{1}{2}} \sum_{l=1}^{n} \int_0^t K_0(u,t;\boldsymbol{\beta}_0) r_T^{(0)}(u;\boldsymbol{\theta}_0)^{-1} dM_l^T(u) + o_p(1).
\end{aligned}
$$

We can write

$$
(13) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \sum_{k=1}^{K} \mathcal{E}_{ik} \int_0^t W_{ik}^A(u) \{ R_0^{(0)}(u;\boldsymbol{\beta}_0,\widehat{W})^{-1} - R_0^{(0)}(u;\boldsymbol{\beta}_0,W)^{-1} \} dN_{i0k}(u).
$$

Now, through the Functional Delta Method,

$$
\begin{aligned}
& n^{\frac{1}{2}}\{R_0^{(0)}(u;\boldsymbol{\beta},\widehat{W})^{-1} - R_0^{(0)}(u;\boldsymbol{\beta},W)^{-1}\} \\
= & -R_0^{(0)}(u;\boldsymbol{\beta},W)^{-2} n^{-1}\sum_{i=1}^{n}\sum_{k=1}^{K}\mathcal{E}_{ik}\exp\{\boldsymbol{\beta}'\boldsymbol{Z}_{i0k}\} n^{\frac{1}{2}}\{\widehat{W}_{ik}^{A}(u) - W_{ik}^{A}(u)\} \\
= & -R_0^{(0)}(u;\boldsymbol{\beta},W)^{-2} n^{-1}\sum_{i=1}^{n}\sum_{k=1}^{K}\mathcal{E}_{ik}\exp\{\boldsymbol{\beta}'\boldsymbol{Z}_{i0k}\} W_{ik}^{A}(u) n^{-\frac{1}{2}} \\
& \times \sum_{l=1}^{n}\{\boldsymbol{D}_{ik}'(u)\boldsymbol{\Omega}_T(\boldsymbol{\theta}_0)^{-1}\boldsymbol{U}_l^T(\boldsymbol{\theta}_0) + \boldsymbol{J}_{ikl}^T(u)\} \\
= & \ R_0^{(0)}(u;\boldsymbol{\beta},W)^{-2}\widehat{\boldsymbol{F}}_0'(u;\boldsymbol{\beta})\boldsymbol{\Omega}_T(\boldsymbol{\theta}_0)^{-1} n^{-\frac{1}{2}}\sum_{l=1}^{n}\boldsymbol{U}_l^T(\boldsymbol{\theta}_0) \\
& + R_0^{(0)}(u;\boldsymbol{\beta},W)^{-2} n^{-\frac{1}{2}}\sum_{l=1}^{n}\int_0^{u+S_{ik}}\widehat{Q}_0(s,u;\boldsymbol{\theta}_0) r_T^{(0)}(s,\boldsymbol{\theta}_0)^{-1} dM_l^T(s) \\
= & \ R_0^{(0)}(u;\boldsymbol{\beta},W)^{-2}\boldsymbol{F}_0'(u;\boldsymbol{\beta})\boldsymbol{\Omega}_T(\boldsymbol{\theta}_0)^{-1} n^{-\frac{1}{2}}\sum_{l=1}^{n} U_l^T(\boldsymbol{\theta}_0) \\
& + R_0^{(0)}(u;\boldsymbol{\beta},W)^{-2} n^{-\frac{1}{2}}\sum_{l=1}^{n}\int_0^{u+S_{ik}} Q_0(s,u;\boldsymbol{\theta}_0) r_T^{(0)}(s,\boldsymbol{\theta}_0)^{-1} dM_l^T(s) + o_p(1),
\end{aligned}
$$

where the last line follows from the convergence in probability of

$$
\begin{aligned}
\widehat{\boldsymbol{F}}_0(u;\boldsymbol{\beta}) &= -n^{-1}\sum_{i=1}^{n}\sum_{k=1}^{K}\mathcal{E}_{ik}\exp\{\boldsymbol{\beta}'\boldsymbol{Z}_{i0k}\} W_{ik}^{A}(u)\boldsymbol{D}_{ik}'(u;\boldsymbol{\theta}), \\
\widehat{Q}_0(t_1,t_2;\boldsymbol{\theta}) &= -n^{-1}\sum_{i=1}^{n}\sum_{k=1}^{K}\mathcal{E}_{ik}\exp\{\boldsymbol{\theta}'\boldsymbol{Z}_i(t_1)\} e^{\boldsymbol{\beta}'\boldsymbol{Z}_{i0k}} W_{ik}^{A}(t_2),
\end{aligned}
$$

to the quantities

$$
\begin{aligned}
\boldsymbol{F}_0(u;\boldsymbol{\beta}) &= -E\Big[\mathcal{E}_{ik}\exp\{\boldsymbol{\beta}'\boldsymbol{Z}_{i0k}\} W_{ik}^{A}(u)\boldsymbol{D}_{ik}'(u;\boldsymbol{\theta})\Big], \\
Q_0(t_1,t_2;\boldsymbol{\theta}) &= -E\Big[\mathcal{E}_{ik}\exp\{\boldsymbol{\theta}'\boldsymbol{Z}_i(t_1)\} e^{\boldsymbol{\beta}'\boldsymbol{Z}_{i0k}} W_{ik}^{A}(t_2)\Big].
\end{aligned}
$$

Substituting this result into the expansion of (13), we obtain

$$
\begin{aligned}
(13) \quad = \quad & n^{-1}\sum_{i=1}^{n}\sum_{k=1}^{K}\mathcal{E}_{ik}\int_0^t W_{ik}^A(u)R_0^{(0)}(u;\boldsymbol{\beta}_0,W)^{-2}\boldsymbol{F}_0'(u;\boldsymbol{\beta}_0)\boldsymbol{\Omega}_T(\boldsymbol{\theta})^{-1}n^{-\frac{1}{2}} \\
& \times\sum_{l=1}^{n}\boldsymbol{U}_l^T(\boldsymbol{\theta}_0)dN_{i0k}(u) \\
& +n^{-1}\sum_{i=1}^{n}\sum_{k=1}^{K}\mathcal{E}_{ik}\int_0^t W_{ik}^A(u)R_0^{(0)}(u;\boldsymbol{\beta}_0,W)^{-2}n^{-\frac{1}{2}}\sum_{l=1}^{n}\int_0^{u+S_{ik}}Q_0(s,u;\boldsymbol{\theta}_0) \\
& \times r_T^{(0)}(s,\boldsymbol{\theta}_0)^{-1}dM_l^T(s)dN_{i0k}(u).
\end{aligned}
$$

Switching the order of summation for the first term, and the order of summation and integration in the second term, we have

$$
\begin{aligned}
(13) \quad = \quad & \widehat{\boldsymbol{E}}_0(t;\boldsymbol{\beta}_0)\boldsymbol{\Omega}_T(\boldsymbol{\theta}_0)^{-1}n^{-\frac{1}{2}}\sum_{l=1}^{n}\boldsymbol{U}_l^T(\boldsymbol{\theta}_0) \\
& +n^{-\frac{1}{2}}\sum_{l=1}^{n}\int_0^t \widehat{P}_0(u,t;\boldsymbol{\beta}_0)r_T^{(0)}(u,\boldsymbol{\theta}_0)^{-1}dM_l^T(u) \\
= \quad & \boldsymbol{E}_0(t;\boldsymbol{\beta}_0)\boldsymbol{\Omega}_T(\boldsymbol{\theta}_0)^{-1}n^{-\frac{1}{2}}\sum_{l=1}^{n}\boldsymbol{U}_l^T(\boldsymbol{\theta}_0) \\
& +n^{-\frac{1}{2}}\sum_{l=1}^{n}\int_0^t P_0(u,t;\boldsymbol{\beta}_0)r_T^{(0)}(u,\boldsymbol{\theta}_0)^{-1}dM_l^T(u)+o_p(1),
\end{aligned}
$$

where the last line follows from the convergence in probability of

$$
\begin{aligned}
\widehat{\boldsymbol{E}}_0(t;\boldsymbol{\beta}) &= n^{-1}\sum_{i=1}^{n}\sum_{k=1}^{K}\mathcal{E}_{ik}\int_0^t \frac{W_{ik}^A(u)\boldsymbol{F}_0(u;\boldsymbol{\beta})}{R_0^{(0)}(u;\boldsymbol{\beta},W)^2}dN_{i0k}(u), \\
\widehat{P}_0(t_1,t_2;\boldsymbol{\beta}) &= n^{-1}\sum_{i=1}^{n}\sum_{k=1}^{K}\mathcal{E}_{ik}\int_{t_1-S_{ik}}^{t_2-S_{ik}} \frac{W_{ik}^A(u)Q_0(t_1,u;\boldsymbol{\theta})}{R_0^{(0)}(u;\boldsymbol{\beta},W)^2}dN_{i0k}(u),
\end{aligned}
$$

to the quantities

$$
\begin{aligned}
\boldsymbol{E}_0(t;\boldsymbol{\beta}) &= E\left[\mathcal{E}_{ik}\int_0^t \frac{W_{ik}^A(u)\boldsymbol{F}_0(u;\boldsymbol{\beta})}{r_0^{(0)}(u;\boldsymbol{\beta},W)^2}dN_{i0k}(u)\right], \\
P_0(t_1,t_2;\boldsymbol{\beta}) &= E\left[\mathcal{E}_{ik}\int_{t_1-S_{ik}}^{t_2-S_{ik}} \frac{W_{ik}^A(u)Q_0(t_1,u;\boldsymbol{\theta})}{r_0^{(0)}(u;\boldsymbol{\beta},W)^2}dN_{i0k}(u)\right].
\end{aligned}
$$

We can also express

$$
\begin{aligned}
(14) &= n^{-\frac{1}{2}} \sum_{i=1}^{n} \sum_{k=1}^{K} \mathcal{E}_{ik} \int_0^t \frac{W_{ik}^A(u)}{R_0^{(0)}(u;\boldsymbol{\beta}_0, W)} dM_{i0k}(u), \\
&= n^{-\frac{1}{2}} \sum_{i=1}^{n} \sum_{k=1}^{K} \mathcal{E}_{ik} \int_0^t \frac{W_{ik}^A(u)}{r_0^{(0)}(u;\boldsymbol{\beta}_0, W)} dM_{i0k}(u) + o_p(1),
\end{aligned}
$$

Combining the results of equations (11) (12) (13) and (14), we obtain

$$
\begin{aligned}
& n^{\frac{1}{2}}\{\widehat{\Lambda}_{00}(t) - \Lambda_{00}(t)\} \\
= \;& \boldsymbol{h}_0'(t;\boldsymbol{\beta}_0, W)\boldsymbol{\Omega}_0^{-1}(\boldsymbol{\beta}_0) n^{-\frac{1}{2}} \sum_{i=1}^{n} \boldsymbol{U}_{i0}(\boldsymbol{\beta}_0) \\
& + [\boldsymbol{B}_0'(t;\boldsymbol{\beta}_0) + \boldsymbol{E}_0'(t;\boldsymbol{\beta}_0)]\boldsymbol{\Omega}_T(\boldsymbol{\theta})^{-1} n^{-\frac{1}{2}} \sum_{i=1}^{n} \boldsymbol{U}_i^T(\boldsymbol{\theta}_0) \\
& + n^{-\frac{1}{2}} \sum_{i=1}^{n} \int_0^t [K_0(u,t;\boldsymbol{\beta}_0) + P_0(u,t;\boldsymbol{\beta}_0)] r_T^{(0)}(u;\boldsymbol{\theta}_0)^{-1} dM_i^T(u) \\
& + n^{-\frac{1}{2}} \sum_{i=1}^{n} \sum_{k=1}^{K} \mathcal{E}_{ik} \int_0^t r_0^{(0)}(u;\boldsymbol{\beta}_0, W)^{-1} W_{ik}^A(u) dM_{i0k}(u) \\
= \;& n^{-\frac{1}{2}} \sum_{i=1}^{n} \int_0^t d\Phi_{i0}(u),
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi_{i0}(t) &= \boldsymbol{h}_0'(t;\boldsymbol{\beta}_0, W)\boldsymbol{\Omega}_0^{-1}(\boldsymbol{\beta}_0)\boldsymbol{U}_{i0}(\boldsymbol{\beta}_0) \\
& \quad + [\boldsymbol{B}_0'(t;\boldsymbol{\beta}_0) + \boldsymbol{E}_0'(t;\boldsymbol{\beta}_0)]\boldsymbol{\Omega}_T(\boldsymbol{\theta}_0)^{-1}\boldsymbol{U}_i^T(\boldsymbol{\theta}_0) \\
& \quad + \int_0^t [K_0(u,t;\boldsymbol{\beta}_0) + P_0(u,t;\boldsymbol{\beta}_0)] r_T^{(0)}(u;\boldsymbol{\theta}_0)^{-1} dM_i^T(u) \\
& \quad + \sum_{k=1}^{K} \mathcal{E}_{ik} \int_0^t r_0^{(0)}(u;\boldsymbol{\beta}_0, W)^{-1} W_{ik}^A(u) dM_{i0k}(u) \\
&= \int_0^t d\Phi_{i0}(u).
\end{aligned}
$$

4.7 $n^{\frac{1}{2}}\{\widehat{\Lambda}_{i0}(t;S_i)-\Lambda_{i0}(t;S_i)\}$

We begin with another decomposition

$$
\begin{aligned}
& n^{\frac{1}{2}}\{\widehat{\Lambda}_{i0}(t;S_i)-\Lambda_{i0}(t;S_i)\} \\
= \; & n^{\frac{1}{2}}\{\widehat{\Lambda}_{i0}(t,\widehat{\boldsymbol{\beta}};S_i)-\widehat{\Lambda}_{i0}(t,\boldsymbol{\beta}_0;S_i)\} \qquad (17)\\
& +n^{\frac{1}{2}}\{\widehat{\Lambda}_{i0}(t,\boldsymbol{\beta}_0;S_i)-\Lambda_{i0}(t)\}. \qquad (18)
\end{aligned}
$$

Consider the first term and using Result 4.5

$$
\begin{aligned}
(17) &= \widehat{\Lambda}_{00}(t)n^{\frac{1}{2}}\{\exp\{\widehat{\boldsymbol{\beta}}_0'\boldsymbol{Z}_i(S_i)\}-\exp\{\boldsymbol{\beta}_0'\boldsymbol{Z}_i(S_i)\}\} \\
&= \Lambda_{00}(t)\exp\{\boldsymbol{\beta}_0'\boldsymbol{Z}_i(S_i)\}\boldsymbol{Z}_i(S_i)'n^{\frac{1}{2}}(\widehat{\boldsymbol{\beta}}_0-\boldsymbol{\beta}_0)+o_p(1) \\
&= \Lambda_{00}(t)\exp\{\boldsymbol{\beta}_0'\boldsymbol{Z}_i(S_i)\}\boldsymbol{Z}_i(S_i)'\boldsymbol{\Omega}_0^{-1}(\boldsymbol{\beta}_0)n^{-\frac{1}{2}}\sum_{j=1}^{n}U_{j0}(\boldsymbol{\beta}_0)+o_p(1).
\end{aligned}
$$

By using Result 4.6, the second term can be written as

$$
\begin{aligned}
(18) &= \exp\{\boldsymbol{\beta}_0'\boldsymbol{Z}_i(S_i)\}n^{\frac{1}{2}}\{\widehat{\Lambda}_{00}(t)-\Lambda_{00}(t)\} \\
&= \exp\{\boldsymbol{\beta}_0'\boldsymbol{Z}_i(S_i)\}n^{-\frac{1}{2}}\sum_{j=1}^{n}\Phi_{j0}(t)+o_p(1).
\end{aligned}
$$

Combining equations (17) and (18), we obtain

$$
\begin{aligned}
& n^{\frac{1}{2}}\{\widehat{\Lambda}_{i0}(t;S_i)-\Lambda_{i0}(t;S_i)\} \\
= \; & \Lambda_{i0}(t;S_i)\boldsymbol{Z}_i(S_i)'\boldsymbol{\Omega}_0^{-1}(\boldsymbol{\beta}_0)n^{-\frac{1}{2}}\sum_{j=1}^{n}\boldsymbol{U}_{j0}(\boldsymbol{\beta}_0)+\exp\{\boldsymbol{\beta}_0'\boldsymbol{Z}_i(S_i)\}n^{-\frac{1}{2}}\sum_{j=1}^{n}\Phi_{j0}(t)+o_p(1).
\end{aligned}
$$

4.8 $n^{\frac{1}{2}}\{\widehat{S}_{i0}(t;S_i)-S_{i0}(t;S_i)\}$

Using the Functional Delta Method and Result 4.7, we have

$$
n^{\frac{1}{2}}\{\widehat{S}_{i0}(t;S_i)-S_{i0}(t;S_i)\}=-S_{i0}(t;S_i)n^{\frac{1}{2}}\{\widehat{\Lambda}_{i0}(t;S_i)-\Lambda_{i0}(t;S_i)\}+o_p(1).
$$

4.9 $n^{\frac{1}{2}}\{\widehat{\mu}_{i0}(S_i) - \mu_{i0}(S_i)\}$

Define $\mu_{i0}(S_i) = \int_0^L S_{i0}(u; S_i)du$. By continuity and Results 4.7 and 4.8, we have

$$
\begin{aligned}
& n^{\frac{1}{2}}\{\widehat{\mu}_{i0}(S_i) - \mu_{i0}(S_i)\} \\
= \;& n^{\frac{1}{2}} \int_0^L \{\widehat{S}_{i0}(t; S_i) - S_{i0}(t; S_i)\}dt \\
= \;& -\int_0^L S_{i0}(t; S_i) n^{\frac{1}{2}}\{\widehat{\Lambda}_{i0}(t; S_i) - \Lambda_{i0}(t; S_i)\}dt + o_p(1) \\
= \;& -\int_0^L S_{i0}(t; S_i)\Lambda_{i0}(t; S_i)\boldsymbol{Z}_i(S_i)'\boldsymbol{\Omega}_0^{-1}(\boldsymbol{\beta}_0) n^{-\frac{1}{2}} \sum_{j=1}^n \boldsymbol{U}_{j0}(\boldsymbol{\beta}_0)dt \qquad (19) \\
& -\int_0^L S_{i0}(t; S_i)\exp\{\boldsymbol{\beta}_0'\boldsymbol{Z}_i(S_i)\} n^{-\frac{1}{2}} \sum_{j=1}^n \int_0^t d\Phi_{j0}(u)dt + o_p(1). \qquad (20)
\end{aligned}
$$

For the second term, switching the order of integration and summation

$$
\begin{aligned}
(20) \;=\;& -n^{-\frac{1}{2}} \sum_{j=1}^n \int_0^L \int_u^L S_{i0}(t; S_i)\exp\{\boldsymbol{\beta}_0'\boldsymbol{Z}_i(S_i)\}dt d\Phi_{j0}(u) \\
=\;& -n^{-\frac{1}{2}} \sum_{j=1}^n \int_0^L \exp\{\boldsymbol{\beta}_0'\boldsymbol{Z}_i(S_i)\}\left\{\mu_{i0}(S_i) - \int_0^t S_{i0}(u; S_i)du\right\} d\Phi_{j0}(t).
\end{aligned}
$$

Combining equations (19) and (20), we have

$$
\begin{aligned}
n^{\frac{1}{2}}\{\widehat{\mu}_{i0}(S_i) - \mu_{i0}(S_i)\} \;=\;& \int_0^L S_{i0}(t; S_i)\Lambda_{i0}(t; S_i)\boldsymbol{Z}_i(S_i)'\boldsymbol{\Omega}_0^{-1}(\boldsymbol{\beta}_0) n^{-\frac{1}{2}} \sum_{j=1}^n \boldsymbol{U}_{j0}(\boldsymbol{\beta}_0)dt \\
& -n^{-\frac{1}{2}} \sum_{j=1}^n \int_0^L \exp\{\boldsymbol{\beta}_0'\boldsymbol{Z}_i(S_i)\}\left\{\mu_{i0}(S_i) - \int_0^t S_{i0}(u; S_i)du\right\} d\Phi_{j0}(t) \\
=\;& n^{-\frac{1}{2}} \sum_{j=1}^n \varphi_{ij0}(S_i).
\end{aligned}
$$

where

$$
\begin{aligned}
\varphi_{ij0}(S_i) \;=\;& \boldsymbol{Z}_i(S_i)'\boldsymbol{\Omega}_0^{-1}(\boldsymbol{\beta}_0)\boldsymbol{U}_{j0}(\boldsymbol{\beta}_0) \int_0^L S_{i0}(t; S_i)\Lambda_{i0}(t; S_i)dt \\
& -\exp\{\boldsymbol{\beta}_0'\boldsymbol{Z}_i(S_i)\} \int_0^L \left\{\mu_{i0}(S_i) - \int_0^t S_{i0}(u; S_i)du\right\} d\Phi_{j0}(t).
\end{aligned}
$$

4.10 $n^{\frac{1}{2}}(\widehat{\boldsymbol{\beta}}_1 - \boldsymbol{\beta}_1)$

It is straightforward to show that

$$
n^{\frac{1}{2}}(\widehat{\boldsymbol{\beta}}_1 - \boldsymbol{\beta}_1) = \boldsymbol{\Omega}_1^{-1}(\boldsymbol{\beta}_1) n^{-\frac{1}{2}} \sum_{i=1}^n \boldsymbol{U}_{i1}(\boldsymbol{\beta}_1) + o_p(1),
$$

where

$$\boldsymbol{U}_{i1}(\boldsymbol{\beta}_1) = \int_0^{\tau_1} \{\boldsymbol{Z}_{i1} - \overline{\boldsymbol{z}}_1(t;\boldsymbol{\beta}_1)\} dM_{i1}(t),$$
$$dM_{i1}(t) = dN_{i1}(t) - Y_{i1}(t) d\Lambda_{i1}(t).$$

This is now a well-established Cox model result, derived through Martingale theory.

4.11 $n^{\frac{1}{2}}\{\widehat{\Lambda}_{01}(t) - \Lambda_{01}(t)\}$

We begin with another decomposition,

$$\begin{aligned} & n^{\frac{1}{2}}\{\widehat{\Lambda}_{01}(t) - \Lambda_{01}(t)\} \\ = \; & n^{\frac{1}{2}}[\widehat{\Lambda}_{01}(t;\widehat{\boldsymbol{\beta}}_1) - \widehat{\Lambda}_{01}(t;\boldsymbol{\beta}_1)] \qquad (21) \\ & + n^{\frac{1}{2}}[\widehat{\Lambda}_{01}(t;\boldsymbol{\beta}_1) - \Lambda_{01}(t)]. \qquad (22) \end{aligned}$$

Consider the first term,

$$\begin{aligned} (21) &= n^{-\frac{1}{2}} \sum_{i=1}^n \int_0^t \{R_1^{(0)}(u;\widehat{\boldsymbol{\beta}}_1)^{-1} - R_1^{(0)}(u;\boldsymbol{\beta}_1)^{-1}\} dN_{i1}(u) \\ &= \widehat{\boldsymbol{h}}_1'(t;\boldsymbol{\beta}_1)\boldsymbol{\Omega}_1(\boldsymbol{\beta}_1)^{-1} n^{-\frac{1}{2}} \sum_{i=1}^n \boldsymbol{U}_{i1}(\boldsymbol{\beta}_1) \\ &= \boldsymbol{h}_1'(t;\boldsymbol{\beta}_1)\boldsymbol{\Omega}_1(\boldsymbol{\beta}_1)^{-1} n^{-\frac{1}{2}} \sum_{i=1}^n \boldsymbol{U}_{i1}(\boldsymbol{\beta}_1) + o_p(1), \end{aligned}$$

where the third line follows from the convergence in probability of

$$\widehat{\boldsymbol{h}}_1'(t;\boldsymbol{\beta}_1) = -\int_0^t \overline{\boldsymbol{z}}_1'(u;\boldsymbol{\beta}_1) d\widehat{\Lambda}_{01}(u),$$
$$\widehat{\boldsymbol{\Omega}}_1(\boldsymbol{\beta}_1) = n^{-1} \sum_{i=1}^n \int_0^{\tau_1} \left\{ \frac{\boldsymbol{R}_1^{(2)}(t;\boldsymbol{\beta}_1)}{R_1^{(0)}(t;\boldsymbol{\beta}_1)} - \overline{\boldsymbol{z}}_1(t;\boldsymbol{\beta}_1)^{\otimes 2} \right\} dN_{i1}(t),$$

to the quantities

$$\boldsymbol{h}_1'(t;\boldsymbol{\beta}_1) = -\int_0^t \overline{\boldsymbol{z}}_1'(u;\boldsymbol{\beta}_1) d\Lambda_{01}(u),$$

and $\boldsymbol{\Omega}_1(\boldsymbol{\beta}_1)$ respectively, with $\boldsymbol{\Omega}_1(\boldsymbol{\beta}_1)$ defined in Regularity Condition (e).

With respect to the second term in the decomposition, we have,

$$\begin{aligned}(22) &= n^{-\frac{1}{2}} \sum_{i=1}^{n} \int_0^t \frac{dM_{i1}(u)}{R_1^{(0)}(u;\boldsymbol{\beta}_1)}, \\ &= n^{-\frac{1}{2}} \sum_{i=1}^{n} \int_0^t \frac{dM_{i1}(u)}{r_1^{(0)}(u;\boldsymbol{\beta}_1)} + o_p(1),\end{aligned}$$

where the second line follows from continuity and Condition (d). Combining equations (21) and (22), for the decomposition, we have

$$\begin{aligned}& n^{\frac{1}{2}}\{\widehat{\Lambda}_{01}(t) - \Lambda_{01}(t)\} \\ =\ & \boldsymbol{h}_1'(t;\boldsymbol{\beta}_1)\boldsymbol{\Omega}_1(\boldsymbol{\beta}_1)^{-1} n^{-\frac{1}{2}} \sum_{i=1}^{n} \boldsymbol{U}_{i1}(\boldsymbol{\beta}_1) \\ & + n^{-\frac{1}{2}} \sum_{i=1}^{n} \int_0^t r_1^{(0)}(u;\boldsymbol{\beta}_1)^{-1} dM_{i1}(u) \\ =\ & n^{-\frac{1}{2}} \sum_{i=1}^{n} \int_0^t d\Phi_{i1}(u),\end{aligned}$$

where

$$\begin{aligned}\Phi_{i1}(t) &= \boldsymbol{h}_1'(t;\boldsymbol{\beta}_1)\boldsymbol{\Omega}_1(\boldsymbol{\beta}_1)^{-1}\boldsymbol{U}_{i1}(\boldsymbol{\beta}_1) \\ &\quad + \int_0^t r_1^{(0)}(u;\boldsymbol{\beta}_1)^{-1} dM_{i1}(u) \\ &= \int_0^t d\Phi_{i1}(u).\end{aligned}$$

4.12 $n^{\frac{1}{2}}\{\widehat{\Lambda}_{i1}(t;T_i) - \Lambda_{i1}(t;T_i)\}$

We begin with another decomposition

$$\begin{aligned}& n^{\frac{1}{2}}\{\widehat{\Lambda}_{i1}(t;T_i) - \Lambda_{i1}(t;T_i)\} \\ =\ & n^{\frac{1}{2}}\{\widehat{\Lambda}_{i1}(t,\widehat{\boldsymbol{\beta}}_1;T_i) - \widehat{\Lambda}_{i1}(t,\boldsymbol{\beta}_1;T_i)\} \qquad (23) \\ & + n^{\frac{1}{2}}\{\widehat{\Lambda}_{i1}(t,\boldsymbol{\beta}_1;T_i) - \Lambda_{i1}(t)\}. \qquad (24)\end{aligned}$$

Considering the first term, by a Taylor series expansion and Result 4.10, and $\widehat{\Lambda}_{01}(t) \xrightarrow{p} \Lambda_{01}(t)$ for $t \in [0, \tau]$, we obtain

$$
\begin{aligned}
(23) &= \widehat{\Lambda}_{01}(t) n^{\frac{1}{2}} \{\exp\{\widehat{\boldsymbol{\beta}}_1' \boldsymbol{Z}_{i1}\} - \exp\{\boldsymbol{\beta}_1' \boldsymbol{Z}_{i1}\}\} \\
&= \Lambda_{01}(t) \exp\{\boldsymbol{\beta}_1' \boldsymbol{Z}_{i1}\} \boldsymbol{Z}_{i1}' n^{\frac{1}{2}} (\widehat{\boldsymbol{\beta}}_1 - \boldsymbol{\beta}_1) + o_p(1) \\
&= \Lambda_{01}(t) \exp\{\boldsymbol{\beta}_1' \boldsymbol{Z}_{i1}\} \boldsymbol{Z}_{i1}' \boldsymbol{\Omega}_1(\boldsymbol{\beta}_1)^{-1} n^{-\frac{1}{2}} \sum_{j=1}^{n} \boldsymbol{U}_{j1}(\boldsymbol{\beta}_1) + o_p(1).
\end{aligned}
$$

By using Result 4.11, the second term can be written as

$$
\begin{aligned}
(24) &= \exp\{\boldsymbol{\beta}_1' \boldsymbol{Z}_{i1}\} n^{\frac{1}{2}} \{\widehat{\Lambda}_{01}(t) - \Lambda_{01}(t)\} \\
&= \exp\{\boldsymbol{\beta}_1' \boldsymbol{Z}_{i1}\} n^{-\frac{1}{2}} \sum_{j=1}^{n} \Phi_{j1}(t) + o_p(1).
\end{aligned}
$$

Combining results from the decomposition leads to,

$$
\begin{aligned}
& n^{\frac{1}{2}} \{\widehat{\Lambda}_{i1}(t; T_i) - \Lambda_{i1}(t; T_i)\} \\
= \; & \Lambda_{i1}(t; T_i) \boldsymbol{Z}_{i1}' \boldsymbol{\Omega}_1(\boldsymbol{\beta}_1)^{-1} n^{-\frac{1}{2}} \sum_{j=1}^{n} \boldsymbol{U}_{j1}(\boldsymbol{\beta}_1) + \exp\{\boldsymbol{\beta}_1' \boldsymbol{Z}_{i1}\} n^{-\frac{1}{2}} \sum_{j=1}^{n} \Phi_{j1}(t) + o_p(1).
\end{aligned}
$$

4.13 $n^{\frac{1}{2}} \{\widehat{S}_{i1}(t; T_i) - S_{i1}(t; T_i)\}$

Using the Functional Delta Method

$$
n^{\frac{1}{2}} \{\widehat{S}_{i1}(t; T_i) - S_{i1}(t; T_i)\} = -S_{i1}(t; T_i) n^{\frac{1}{2}} \{\widehat{\Lambda}_{i1}(t; T_i) - \Lambda_{i1}(t; T_i)\} + o_p(1).
$$

4.14 $n^{\frac{1}{2}} \{\widehat{\mu}_{i1}(T_i) - \mu_{i1}(T_i)\}$

Define $\mu_{i1}(T_i) = \int_0^L S_{i1}(u; T_i) du$. By continuity and Results 4.12 and 4.13,

$$
\begin{aligned}
& n^{\frac{1}{2}} \{\widehat{\mu}_{i1}(T_i) - \mu_{i1}(T_i)\} = n^{\frac{1}{2}} \int_0^L \{\widehat{S}_{i1}(t; T_i) - S_{i1}(t; T_i)\} dt \\
= & - \int_0^L S_{i1}(t; T_i) n^{\frac{1}{2}} \{\widehat{\Lambda}_{i1}(t; T_i) - \Lambda_{i1}(t; T_i)\} dt + o_p(1) \\
= & - \int_0^L S_{i1}(t; T_i) \Lambda_{i1}(t; T_i) \boldsymbol{Z}_{i1}' \boldsymbol{\Omega}_1(\boldsymbol{\beta}_1)^{-1} n^{-\frac{1}{2}} \sum_{j=1}^{n} \boldsymbol{U}_{j1}(\boldsymbol{\beta}_1) dt \qquad (25) \\
& - \int_0^L S_{i1}(t; T_i) \exp\{\boldsymbol{\beta}_1' \boldsymbol{Z}_{i1}\} n^{-\frac{1}{2}} \sum_{j=1}^{n} \int_0^t d\Phi_{j1}(u) dt. \qquad (26)
\end{aligned}
$$

For the second term, switching the order of integration and summation

$$
\begin{aligned}
(26) &= -n^{-\frac{1}{2}} \sum_{j=1}^{n} \int_0^L \int_u^L S_{i1}(t;T_i) \exp\{\boldsymbol{\beta}_1' \boldsymbol{Z}_{i1}\} dt d\Phi_{j1}(u) \\
&= -n^{-\frac{1}{2}} \sum_{j=1}^{n} \int_0^L \exp\{\boldsymbol{\beta}_1' \boldsymbol{Z}_{i1}\} \left\{ \mu_{i1}(T_i) - \int_0^t S_{i1}(u;T_i) du \right\} d\Phi_{j1}(t).
\end{aligned}
$$

Combining equations (25) and (26), we obtain

$$
\begin{aligned}
n^{\frac{1}{2}}\{\widehat{\mu}_{i1}(T_i) - \mu_{i1}(T_i)\} &= \int_0^L S_{i1}(t;T_i)\Lambda_{i1}(t;T_i)\boldsymbol{Z}_{i1}' \boldsymbol{\Omega}_1(\boldsymbol{\beta}_1)^{-1} n^{-\frac{1}{2}} \sum_{j=1}^{n} \boldsymbol{U}_{j1}(\boldsymbol{\beta}_1) dt \\
&\quad - n^{-\frac{1}{2}} \sum_{j=1}^{n} \int_0^L \exp\{\boldsymbol{\beta}_1' \boldsymbol{Z}_{i1}\} \left\{ \mu_{i1}(T_i) - \int_0^t S_{i1}(u;T_i) du \right\} d\Phi_{j1}(t) \\
&= n^{-\frac{1}{2}} \sum_{j=1}^{n} \varphi_{ij1}(T_i),
\end{aligned}
$$

where

$$
\begin{aligned}
\varphi_{ij1}(T_i) &= \boldsymbol{Z}_{i1}' \boldsymbol{\Omega}_1(\boldsymbol{\beta}_1)^{-1} \boldsymbol{U}_{j1}(\boldsymbol{\beta}_1) \int_0^L S_{i1}(t;T_i)\Lambda_{i1}(t;T_i) dt \\
&\quad - \exp\{\boldsymbol{\beta}_1' \boldsymbol{Z}_{i1}\} \int_0^L \left\{ \mu_{i1}(T_i) - \int_0^t S_{i1}(u;T_i) du \right\} d\Phi_{j1}(t).
\end{aligned}
$$

4.15 $n^{\frac{1}{2}}(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0)$

It is straightforward to show that

$$
n^{\frac{1}{2}}(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0) = \boldsymbol{\Omega}_C(\boldsymbol{\alpha}_0)^{-1} n^{-\frac{1}{2}} \sum_{i=1}^{n} \boldsymbol{U}_i^C(\boldsymbol{\alpha}_0) + o_p(1),
$$

where we define

$$
\begin{aligned}
&\boldsymbol{U}_i^C(\boldsymbol{\alpha}) = \int_0^\tau \{\boldsymbol{Z}_i(0) - \overline{\boldsymbol{z}}_C(t;\boldsymbol{\alpha})\} dM_i^C(t;\boldsymbol{\alpha}), \\
&dM_{i1}(t) = dN_{i1}(t) - Y_{i1}(t) d\Lambda_{ik}(t).
\end{aligned}
$$

This is now a well-established Cox model result, derived through Martingale theory.

4.16 $n^{\frac{1}{2}}\{\widehat{\Lambda}_0^C(t) - \Lambda_0^C(t)\}$

We start the following decomposition

$$\begin{aligned} & n^{\frac{1}{2}}\{\widehat{\Lambda}_0^C(t) - \Lambda_0^C(t)\} \\ = \; & n^{\frac{1}{2}}\{\widehat{\Lambda}_0^C(t;\widehat{\boldsymbol{\alpha}}) - \widehat{\Lambda}_0^C(t;\boldsymbol{\alpha}_0)\} \qquad (27) \\ & +n^{\frac{1}{2}}\{\widehat{\Lambda}_0^C(t;\boldsymbol{\alpha}_0) - \Lambda_0^C(t)\}. \qquad (28) \end{aligned}$$

We can express the first term as

$$\begin{aligned} (27) &= n^{-\frac{1}{2}}\sum_{i=1}^n \int_0^t \{R_C^{(0)}(u;\widehat{\boldsymbol{\alpha}})^{-1} - R_C^{(0)}(u;\boldsymbol{\alpha}_0)^{-1}\}dN_i^C(u) \\ &= \widehat{\boldsymbol{h}}'_C(t;\boldsymbol{\alpha}_0)\boldsymbol{\Omega}_C(\boldsymbol{\alpha}_0)^{-1}n^{-\frac{1}{2}}\sum_{i=1}^n \boldsymbol{U}_i^C(\boldsymbol{\alpha}_0) \\ &= \boldsymbol{h}'_C(t;\boldsymbol{\alpha}_0)\boldsymbol{\Omega}_C(\boldsymbol{\alpha}_0)^{-1}n^{-\frac{1}{2}}\sum_{i=1}^n \boldsymbol{U}_i^C(\boldsymbol{\alpha}_0) + o_p(1), \end{aligned}$$

where the third line follows from the convergence in probability of

$$\widehat{\boldsymbol{h}}'_C(t;\boldsymbol{\alpha}) = -n^{-1}\sum_{i=1}^n \int_0^t R_C^{(0)}(u;\boldsymbol{\alpha})^{-1}\overline{\boldsymbol{z}}_C(u;\boldsymbol{\alpha})dN_i^C(u) = -\int_0^t \overline{\boldsymbol{z}}_C(u;\boldsymbol{\alpha})d\widehat{\Lambda}_0^C(u;\boldsymbol{\alpha}),$$

$$\widehat{\boldsymbol{\Omega}}_C(\boldsymbol{\alpha}) = n^{-1}\sum_{i=1}^n \int_0^\tau \left\{\frac{\boldsymbol{R}_C^{(2)}(t;\boldsymbol{\alpha})}{R_C^{(0)}(t;\boldsymbol{\alpha})} - \overline{\boldsymbol{z}}_C(t;\boldsymbol{\beta}_1)^{\otimes 2}\right\}dN_i^C(t),$$

to the quantities

$$\boldsymbol{h}'_C(t;\boldsymbol{\alpha}) = -\int_0^t \overline{\boldsymbol{z}}_C(u;\boldsymbol{\alpha})d\Lambda_0^C(u),$$

and $\boldsymbol{\Omega}_C(\boldsymbol{\alpha})$ respectively, with $\boldsymbol{\Omega}_C(\boldsymbol{\alpha})$ defined in Regularity Condition (e).

With respect to the second term in the decomposition, we have

$$\begin{aligned} (28) &= n^{-\frac{1}{2}}\sum_{i=1}^n \int_0^t R_C^{(0)}(u;\boldsymbol{\alpha}_0)^{-1}dM_i^C(u) \\ &= n^{-\frac{1}{2}}\sum_{i=1}^n \int_0^t r_C^{(0)}(u;\boldsymbol{\alpha}_0)^{-1}dM_i^C(u) + o_p(1), \end{aligned}$$

where the second line follows from continuity and Condition (d). Combining equations (27) and (28), for the decomposition, we have

$$n^{\frac{1}{2}}\{\widehat{\Lambda}_0^C(t) - \Lambda_0^C(t)\} = n^{-\frac{1}{2}}\sum_{i=1}^n \Phi_i^C(t;\boldsymbol{\alpha}_0) + o_p(1),$$

where

$$\Phi_i^C(t;\boldsymbol{\alpha}) = \boldsymbol{h}_C'(t;\boldsymbol{\alpha})\boldsymbol{\Omega}_C(\boldsymbol{\alpha}_0)^{-1}\boldsymbol{U}_i^C(\boldsymbol{\alpha}_0) + \int_0^t r_C^{(0)}(u;\boldsymbol{\alpha}_0)^{-1}dM_i^C(u) = \int_0^t d\Phi_i^C(u;\boldsymbol{\alpha}_0),$$

and

$$d\Phi_i^C(u;\boldsymbol{\alpha}) = -\overline{\boldsymbol{z}}_C(u;\boldsymbol{\alpha})d\Lambda_0^C(u)\boldsymbol{\Omega}_C(\boldsymbol{\alpha})^{-1}\boldsymbol{U}_i^C(\boldsymbol{\alpha}) + r_C^{(0)}(u;\boldsymbol{\alpha})^{-1}dM_i^C(u).$$

4.17 $n^{\frac{1}{2}}\{\widehat{\Lambda}_i^C(t) - \Lambda_i^C(t)\}$

We start with another decomposition,

$$\begin{aligned} & n^{\frac{1}{2}}\{\widehat{\Lambda}_i^C(t) - \Lambda_i^C(t)\} \\ = \; & n^{\frac{1}{2}}\left\{\widehat{\Lambda}_i^C(t;\widehat{\boldsymbol{\alpha}}) - \widehat{\Lambda}_i^C(t;\boldsymbol{\alpha})\right\} \quad (29) \\ & + n^{\frac{1}{2}}\left\{\widehat{\Lambda}_i^C(t;\boldsymbol{\alpha}) - \Lambda_i^C(t)\right\}. \quad (30) \end{aligned}$$

Considering the first term,

$$(29) = \widehat{\Lambda}_0^C(t)n^{\frac{1}{2}}\{\exp\{\widehat{\boldsymbol{\alpha}}'\boldsymbol{Z}_i(0)\} - \exp\{\boldsymbol{\alpha}_0'\boldsymbol{Z}_i(0)\}\}.$$

By a Taylor series expansion,

$$\begin{aligned} n^{\frac{1}{2}}\{\exp\{\widehat{\boldsymbol{\alpha}}'\boldsymbol{Z}_i(0)\} - \exp\{\boldsymbol{\alpha}_0'\boldsymbol{Z}_i(0)\}\} &= \boldsymbol{Z}_i'(0)\exp\{\boldsymbol{\alpha}_0'\boldsymbol{Z}_i(0)\}n^{\frac{1}{2}}(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0) + o_p(1) \\ &= \boldsymbol{Z}_i'(0)\exp\{\boldsymbol{\alpha}_0'\boldsymbol{Z}_i(0)\}\boldsymbol{\Omega}_C(\boldsymbol{\alpha}_0)^{-1}n^{-\frac{1}{2}}\sum_{j=1}^n \boldsymbol{U}_j^C(\boldsymbol{\alpha}_0) + o_p(1). \end{aligned}$$

As $\widehat{\Lambda}_0^C(t) \xrightarrow{p} \Lambda_0^C(t)$ for $t \in [0,\tau]$, we obtain

$$(29) = \boldsymbol{Z}_i'(0)\Lambda_i^C(t)\exp\{\boldsymbol{\alpha}_0'\boldsymbol{Z}_i(0)\}\boldsymbol{\Omega}_C(\boldsymbol{\alpha}_0)^{-1}n^{-\frac{1}{2}}\sum_{j=1}^n \boldsymbol{U}_j^C(\boldsymbol{\alpha}_0) + o_p(1).$$

By using Result 4.16, the second term can be written as

$$\begin{aligned} (30) &= \exp\{\boldsymbol{\alpha}_0'\boldsymbol{Z}_i(0)\}n^{\frac{1}{2}}\{\widehat{\Lambda}_0^C(t) - \Lambda_0^C(t)\} \\ &= \exp\{\boldsymbol{\alpha}_0'\boldsymbol{Z}_i(0)\}n^{-\frac{1}{2}}\sum_{j=1}^n d\Phi_j^C(u;\boldsymbol{\alpha}_0) + o_p(1). \end{aligned}$$

Combining result leads to

$$\begin{aligned} n^{\frac{1}{2}}\{\widehat{\Lambda}_i^C(t)-\Lambda_i^C(t)\} &= \int_0^t \{\boldsymbol{Z}_i(0)-\overline{\boldsymbol{z}}_C(u;\boldsymbol{\alpha}_0)\}' d\Lambda_i^C(u)\boldsymbol{\Omega}_C(\boldsymbol{\alpha}_0)^{-1} n^{-\frac{1}{2}}\sum_{j=1}^n \boldsymbol{U}_j^C(\boldsymbol{\alpha}_0) \\ &\quad + n^{-\frac{1}{2}}\sum_{j=1}^n \int_0^t \exp\{\boldsymbol{\alpha}_0'\boldsymbol{Z}_i(0)\} r_C^{(0)}(u;\boldsymbol{\alpha}_0)^{-1} dM_j^C(u) + o_p(1) \\ &= \boldsymbol{D}_i^{C'}(t)\boldsymbol{\Omega}_C(\boldsymbol{\alpha}_0)^{-1} n^{-\frac{1}{2}}\sum_{j=1}^n \boldsymbol{U}_j^C(\boldsymbol{\alpha}_0) + n^{-\frac{1}{2}}\sum_{j=1}^n J_{ij}^C(t) + o_p(1), \end{aligned}$$

where we define

$$\begin{aligned} \boldsymbol{D}_i^{C'}(t) &= \int_0^t \{\boldsymbol{Z}_i(0)-\overline{\boldsymbol{z}}_C(u;\boldsymbol{\alpha}_0)\}' d\Lambda_i^C(u) = \int_0^t d\boldsymbol{D}_i^{C'}(t), \\ J_{ij}^C(t) &= \int_0^t \exp\{\boldsymbol{\alpha}_0'\boldsymbol{Z}_i(0)\} r_C^{(0)}(u;\boldsymbol{\alpha}_0)^{-1} dM_j^C(u). \end{aligned}$$

4.18 $n^{\frac{1}{2}}\{\widehat{G}_i(t)^{-1}-G_i(t)^{-1}\}$

Since $G_i(t)^{-1}=e^{\Lambda_i^C(t)}$ and $\widehat{G}_i(t)^{-1}=\exp\{\widehat{\Lambda}_i^C(t)\}$, we then have

$$\begin{aligned} n^{\frac{1}{2}}\{\widehat{G}_i(t)^{-1}-G_i(t)^{-1}\} &= n^{\frac{1}{2}}\{\exp\{\widehat{\Lambda}_i^C(t)\}-\exp\{\Lambda_i^C(t)\}\} \\ &= G_i(t)^{-1} n^{\frac{1}{2}}\{\widehat{\Lambda}_i^C(t)-\Lambda_i^C(t)\} + o_p(1) \\ &= G_i(t)^{-1} n^{-\frac{1}{2}}\sum_{j=1}^n \{\boldsymbol{D}_i^{C'}(t)\boldsymbol{\Omega}_C(\boldsymbol{\alpha}_0)^{-1}\boldsymbol{U}_j^C(\boldsymbol{\alpha}) + J_{ij}^C(t)\} + o_p(1) \\ &= n^{-\frac{1}{2}}\sum_{j=1}^n \varphi_{ij}^C(t) + o_p(1), \end{aligned}$$

where

$$\varphi_{ij}^C(t) = G_i(t)^{-1}\{\boldsymbol{D}_i^{C'}(t)\boldsymbol{\Omega}_C(\boldsymbol{\alpha}_0)^{-1}\boldsymbol{U}_j^C(\boldsymbol{\alpha}_0) + J_{ij}^C(t)\}.$$

4.19 $n^{\frac{1}{2}}\{\widehat{\delta}_i(t)-\delta_i(t)\}$

From Results 4.8 and 4.13, we have

$$n^{\frac{1}{2}}\{\widehat{S}_{i0}(t;T_i)-S_{i0}(t;T_i)\} = -S_{i0}(t;T_i) n^{\frac{1}{2}}\{\widehat{\Lambda}_{i0}(t;T_i)-\Lambda_{i0}(t;T_i)\} + o_p(1),$$

$$n^{\frac{1}{2}}\{\widehat{S}_{i1}(t;T_i)-S_{i1}(t;T_i)\} = -S_{i1}(t;T_i) n^{\frac{1}{2}}\{\widehat{\Lambda}_{i1}(t;T_i)-\Lambda_{i1}(t;T_i)\} + o_p(1),$$

then using Results 4.7 and 4.12, we obtain

$$\begin{aligned}
& n^{\frac{1}{2}}\{\widehat{\delta}_i(t)-\delta_i(t)\} \\
= \;& n^{\frac{1}{2}}\{\widehat{S}_{i1}(t;T_i)-S_{i1}(t;T_i)\}-n^{\frac{1}{2}}\{\widehat{S}_{i0}(t;T_i)-S_{i0}(t;T_i)\}+o_p(1) \\
= \;& S_{i0}(t;T_i)n^{\frac{1}{2}}\{\widehat{\Lambda}_{i0}(t;T_i)-\Lambda_{i0}(t;T_i)\}-S_{i1}(t;T_i)n^{\frac{1}{2}}\{\widehat{\Lambda}_{i1}(t;T_i)-\Lambda_{i1}(t;T_i)\}+o_p(1) \\
= \;& n^{-\frac{1}{2}}\sum_{j=1}^{n}\varphi_{ij}^{S}(t)+o_p(1),
\end{aligned}$$

where

$$\begin{aligned}
\varphi_{ij}^{S}(t) \;=\; & S_{i0}(t)\{\Lambda_{i0}(t)\boldsymbol{Z}_i(S_i)'\boldsymbol{\Omega}_0^{-1}(\boldsymbol{\beta}_0)\boldsymbol{U}_{j0}(\boldsymbol{\beta}_0)-\exp\{\boldsymbol{\beta}_0'\boldsymbol{Z}_i(t)\}\Phi_{j0}(t)\} \\
& -S_{i1}(t)\{\Lambda_{i1}(t)\boldsymbol{Z}_{i1}'\boldsymbol{\Omega}_1^{-1}(\boldsymbol{\beta}_1)\boldsymbol{U}_{j1}(\boldsymbol{\beta}_1)-\exp\{\boldsymbol{\beta}_1'\boldsymbol{Z}_{i1}\}\Phi_{j1}(t)\}.
\end{aligned}$$

4.20 $n^{\frac{1}{2}}\{\widehat{\Delta}_i(L)-\Delta_i(L)\}$

Since $\widehat{\Delta}_i(L)=\int_0^t\widehat{\delta}_i(u)du$ and $\Delta_i(L)=\int_0^t\delta_i(u)du$, using Result 4.19, we have

$$\begin{aligned}
n^{\frac{1}{2}}\{\widehat{\Delta}_i(L)-\Delta_i(L)\} &= n^{\frac{1}{2}}\int_0^L\{\widehat{\delta}_i(u)-\delta_i(u)\}du, \\
&= n^{-\frac{1}{2}}\int_0^L\sum_{j=1}^{n}\varphi_{ij}^{S}(u)du+o_p(1),
\end{aligned}$$

where switch the integration and summation sign, we obtain

$$n^{\frac{1}{2}}\{\widehat{\Delta}_i(L)-\Delta_i(L)\}=n^{-\frac{1}{2}}\sum_{j=1}^{n}\varphi_{ij}^{D}(L)+o_p(1),$$

where

$$\varphi_{ij}^{D}(L)=\int_0^L\varphi_{ij}^{S}(u)du.$$

4.21 $n^{\frac{1}{2}}\{\widehat{\delta}(t)-\delta(t)\}$

First define

$$\begin{aligned}
\widehat{V}(\tau) &= n^{-1}\sum_{i=1}^{n}\int_0^\tau\widehat{G}_i(u)^{-1}dN_i^T(u), \\
V(\tau) &= P(T_i\leqslant t,T_i<D_i),
\end{aligned}$$

and

$$\widehat{V}(t) \xrightarrow{p} \int_0^\tau E\left(\frac{dN_i^T(t)}{G_i(t)}\right) = P(T_i \leqslant t, T_i < D_i) = V(\tau).$$

Then by Slutsky's Theorem, we can write

$$\widehat{\delta}(t) = V(\tau)^{-1} n^{-1} \sum_{i=1}^{n} \int_0^\tau \widehat{\delta}_i(t;u) \widehat{G}_i(u)^{-1} dN_i^T(u) + o_p(1).$$

Since we define $\delta(t) = E[\delta(t; T, Z(T))]$, $\widehat{\delta}(t) - \delta(t)$ can then be decomposed as follows:

$$\begin{aligned}
& n^{\frac{1}{2}}\{\widehat{\delta}(t) - \delta(t)\} \\
= \; & n^{\frac{1}{2}}\left[\frac{\sum_{i=1}^n \int_0^\tau \widehat{\delta}_i(t;u)\widehat{G}_i(u)^{-1} dN_i^T(u)}{nV(\tau)} - \frac{\sum_{i=1}^n \int_0^\tau \delta_i(t;u)\widehat{G}_i(u)^{-1} dN_i^T(u)}{nV(\tau)}\right] \\
& + n^{\frac{1}{2}}\left[\frac{\sum_{i=1}^n \int_0^\tau \delta_i(t;u)\widehat{G}_i(u)^{-1} dN_i^T(u)}{nV(\tau)} - \frac{\sum_{i=1}^n \int_0^\tau \delta_i(t;u)G_i(u)^{-1} dN_i^T(u)}{nV(\tau)}\right] \\
& + n^{\frac{1}{2}}\left[\frac{\sum_{i=1}^n \int_0^\tau \delta_i(t;u)G_i(u)^{-1} dN_i^T(u)}{nV(\tau)} - \delta(t)\right] + o_p(1).
\end{aligned}$$

Then we can write

$$\begin{aligned}
& n^{\frac{1}{2}}\{\widehat{\delta}(t) - \delta(t)\} \\
= \; & V(\tau)^{-1} n^{-1} \sum_{i=1}^n \int_0^\tau n^{\frac{1}{2}}\{\widehat{\delta}_i(t;u) - \delta_i(t;u)\}\widehat{G}_i(u)^{-1} dN_i^T(u) && (31) \\
& + V(\tau)^{-1} n^{-1} \sum_{i=1}^n \int_0^\tau \delta_i(t;u) n^{\frac{1}{2}}\{\widehat{G}_i(u)^{-1} - G_i(u)^{-1}\} dN_i^T(u) && (32) \\
& + n^{-\frac{1}{2}} V(\tau)^{-1} \sum_{i=1}^n \int_0^\tau \{\delta_i(t;u) - \delta(t)\} G_i(u)^{-1} dN_i^T(u) + o_p(1). && (33)
\end{aligned}$$

By Result 4.19 and Slutsky Theorem, we have the following decomposition

$$\begin{aligned}
(31) \; & = n^{-\frac{1}{2}} V(\tau)^{-1} n^{-1} \sum_{i=1}^n \int_0^\tau \sum_{j=1}^n \varphi_{ij}^S(t) G_i(u)^{-1} dN_i^T(u) + o_p(1) \\
& = n^{-\frac{1}{2}} V(\tau)^{-1} \sum_{j=1}^n n^{-1} \sum_{i=1}^n \int_0^\tau \varphi_{ij}^S(t) G_i(u)^{-1} dN_i^T(u) + o_p(1) \\
& = n^{-\frac{1}{2}} V(\tau)^{-1} \sum_{j=1}^n \widehat{V}_{1j}(t) + o_p(1) \\
& = n^{-\frac{1}{2}} V(\tau)^{-1} \sum_{j=1}^n V_{1j}(t) + o_p(1)
\end{aligned}$$

where

$$\begin{aligned}\widehat{V}_{1j}(t) &= n^{-1}\sum_{i=1}^{n}\int_0^{\tau}\varphi_{ij}^{S}(t)G_i(u)^{-1}dN_i^T(u),\\ V_{1j}(t) &= E\Big[\int_0^{\tau}\varphi_{ij}^{S}(t)G_i(u)^{-1}dN_i^T(u)\Big].\end{aligned}$$

By Result 4.18 we have

$$\begin{aligned}(32) &= n^{-\frac{1}{2}}V(\tau)^{-1}n^{-1}\sum_{i=1}^{n}\int_0^{\tau}\delta_i(t;u)\sum_{j=1}^{n}\varphi_{ij}^{C}(u)dN_i^T(u)+o_p(1)\\ &= n^{-\frac{1}{2}}V(\tau)^{-1}\sum_{j=1}^{n}n^{-1}\sum_{i=1}^{n}\int_0^{\tau}\delta_i(t;u)\varphi_{ij}^{C}(u)dN_i^T(u)+o_p(1)\\ &= n^{-\frac{1}{2}}V(\tau)^{-1}\sum_{j=1}^{n}\widehat{V}_{2j}(t)+o_p(1)\\ &= n^{-\frac{1}{2}}V(\tau)^{-1}\sum_{j=1}^{n}V_{2j}(t)+o_p(1),\end{aligned}$$

where

$$\begin{aligned}\widehat{V}_{2j}(t) &= n^{-1}\sum_{i=1}^{n}\int_0^{\tau}\delta_i(t;u)\varphi_{ij}^{C}(u)dN_i^T(u),\\ V_{2j}(t) &= E\Big[\int_0^{\tau}\delta_i(t;u)\varphi_{ij}^{C}(u)dN_i^T(u)\Big].\end{aligned}$$

Combining all the results above, we can have

$$\begin{aligned}&n^{\frac{1}{2}}\{\widehat{\delta}(t)-\delta(t)\}\\ =\;& n^{-\frac{1}{2}}\sum_{j=1}^{n}V(\tau)^{-1}\Big\{V_{1j}(t)+V_{2j}(t)+\int_0^{\tau}\{\delta_j(t;u)-\delta(t)\}G_j(u)^{-1}dN_j^T(u)\Big\}+o_p(1)\\ =\;& n^{-\frac{1}{2}}\sum_{j=1}^{n}\xi_j(t)+o_p(1),\end{aligned}$$

where

$$\xi_j(t) = V(\tau)^{-1}\Big\{V_{1j}(t)+V_{2j}(t)+\int_0^{\tau}\{\delta_j(t;u)-\delta(t)\}G_j(u)^{-1}dN_j^T(u)\Big\}.$$

4.22 $n^{\frac{1}{2}}\{\widehat{\Delta(L)}-\Delta(L)\}$

Finally, since $\widehat{\Delta}(L)=\int_0^L\widehat{\delta}(t)dt$ and $\Delta(L)=\int_0^L\delta(t)dt$, we can have

$$n^{\frac{1}{2}}\{\widehat{\Delta}(L)-\Delta(L)\}=n^{-\frac{1}{2}}\sum_{j=1}^{n}\eta_j(L)+o_p(1),$$

where

$$\eta_j(L) = \int_0^L \xi_j(t)dt.$$

5. Simplified variance estimator

Simplified variance estimators for $\widehat{\delta}(t)$ and $\widehat{\Delta}(L)$ are given by $n^{-2}\sum_{i=1}^n \widehat{\xi}_i^*(t)^2$ and $n^{-2}\sum_{i=1}^n \widehat{\eta}_i^{*2}$ respectively, where

$$\begin{aligned}
\widehat{\xi}_j^*(t) &= \widehat{V}(\tau)^{-1}\sum_{i=1}^n \int_0^\tau \widehat{\varphi}_{ij}^{S*}(t)\widehat{G}_i(u)^{-1}dN_i^T(u) && (34)\\
&\quad +\widehat{V}(\tau)^{-1}\int_0^\tau \{\widehat{\delta}_j(t;u) - \widehat{\delta}(t)\}\widehat{G}_j(u)^{-1}dN_j^T(t),\\
\widehat{\eta}_j^* &= \int_0^L \widehat{\xi}_j^*(t)dt && (35)\\
\widehat{V}(\tau) &= n^{-1}\sum_{i=1}^n \int_0^\tau \widehat{G}_i(t)^{-1}dN_i^T(t),
\end{aligned}$$

with

$$\begin{aligned}
\varphi_{ij}^{S*}(t) &= S_{i0}(t)\{\Lambda_{i0}(t)\boldsymbol{Z}_i(S_i)'\boldsymbol{\Omega}_0^{-1}(\boldsymbol{\beta}_0)\boldsymbol{U}_{j0}(\boldsymbol{\beta}_0) - \exp\{\boldsymbol{\beta}_0'\boldsymbol{Z}_i(t)\}\Phi_{j0}^*(t)\}\\
&\quad -S_{i1}(t)\{\Lambda_{i1}(t)\boldsymbol{Z}_{i1}'\boldsymbol{\Omega}_1^{-1}(\boldsymbol{\beta}_1)\boldsymbol{U}_{j1}(\boldsymbol{\beta}_1) - \exp\{\boldsymbol{\beta}_1'\boldsymbol{Z}_{i1}\}\Phi_{j1}^*(t)\}\\
\Phi_{i0}^*(t) &= -\int_0^t \overline{\boldsymbol{z}}_0'(u;\boldsymbol{\beta},W)d\Lambda_{00}(u)\boldsymbol{\Omega}_0^{-1}(\boldsymbol{\beta}_0)\boldsymbol{U}_{i0}(\boldsymbol{\beta}_0) + \sum_{k=1}^K \int_0^t \frac{A_i(S_{ik})W_{ik}^A(u)dM_{i0k}(u)}{r_0^{(0)}(u;\boldsymbol{\beta}_0,W)},\\
\Phi_{i1}^*(t) &= -\int_0^t \overline{\boldsymbol{z}}_1'(u;\boldsymbol{\beta}_1)d\Lambda_{01}(u)\boldsymbol{\Omega}_1^{-1}(\boldsymbol{\beta}_1)\boldsymbol{U}_{i1}(\boldsymbol{\beta}_1) + \int_0^t \frac{N_i^T(\tau)dM_{i1}(u)}{r_1^{(0)}(u;\boldsymbol{\beta}_1)},\\
dM_{i0k}(t) &= dN_{i0k}(t) - Y_{i0k}(t)d\Lambda_{i0}(t),\\
dM_{i1}(t) &= dN_{i1}(t) - Y_{i1}(t)d\Lambda_{i1}(t),
\end{aligned}$$

where $\overline{\boldsymbol{z}}_0(t;\boldsymbol{\beta},W)$, $r_0^{(0)}(t;\boldsymbol{\beta}_0,W)$, $\overline{\boldsymbol{z}}_1(t;\boldsymbol{\beta}_1)$ and $\boldsymbol{r}_1^{(0)}(t;\boldsymbol{\beta}_1)$ are the limiting values of $\overline{Z}_0(t;\boldsymbol{\beta},W)$, $R_0^{(0)}(t;\boldsymbol{\beta}_0,W)$, $\overline{\boldsymbol{Z}}_1(t;\boldsymbol{\beta}_1)$ and $\boldsymbol{R}_1^{(0)}(t;\boldsymbol{\beta}_1)$, respectively.

6. Simulation results under model misspecification

In Tables 1 and 2 we evaluate the degree of bias introduced by mis-specifying the hazard model for either pre-transplant death (3rd last column), post-transplant death (2nd last column) or transplant (last column). In each case, a Bernoulli(0.5) is added to the data generator, but not the fitted models; the unobserved covariate has either a strong (Table) or moderate (Table 1) impact on the hazard process affected. Results are biased, although not considerably. The degree of bias seems to be impacted most through misspecification of the transplant hazard model.

[Table 1 about here.]

[Table 2 about here.]

7. Additional analysis of liver transplant data

[Table 3 about here.]

[Table 4 about here.]

Figure 1 present observed MELD trajectories by time since wait listing for 20 randomly selected patients with initial MELD (at listing) of 15.

[Figure 1 about here.]

Figure 2 presents MELD trajectories for 8 randomly selected patients from those from Figure 1 (initial MELD of 15).

[Figure 2 about here.]

[Figure 3 about here.]

The baseline hazard for death in the absence of transplantation is displayed in Figure 4. Note that the time axis is t time since transplantation (in days), such that the plot is interpreted as being averaged over the transplanted patients, and depicting the hazard to which their mortality experience would have been subject to had liver transplantation not existed as a treatment option.

[Figure 4 about here.]

The baseline transplant hazard is presented in Figure 5.

[Figure 5 about here.]

Average cumulative incidence of transplant is shown in Figure 6. The subdistribution seems to plateau at approximately 3 years post-wait-listing.

[Figure 6 about here.]

The post-transplant death hazard is presented in Figure 7.

[Figure 7 about here.]

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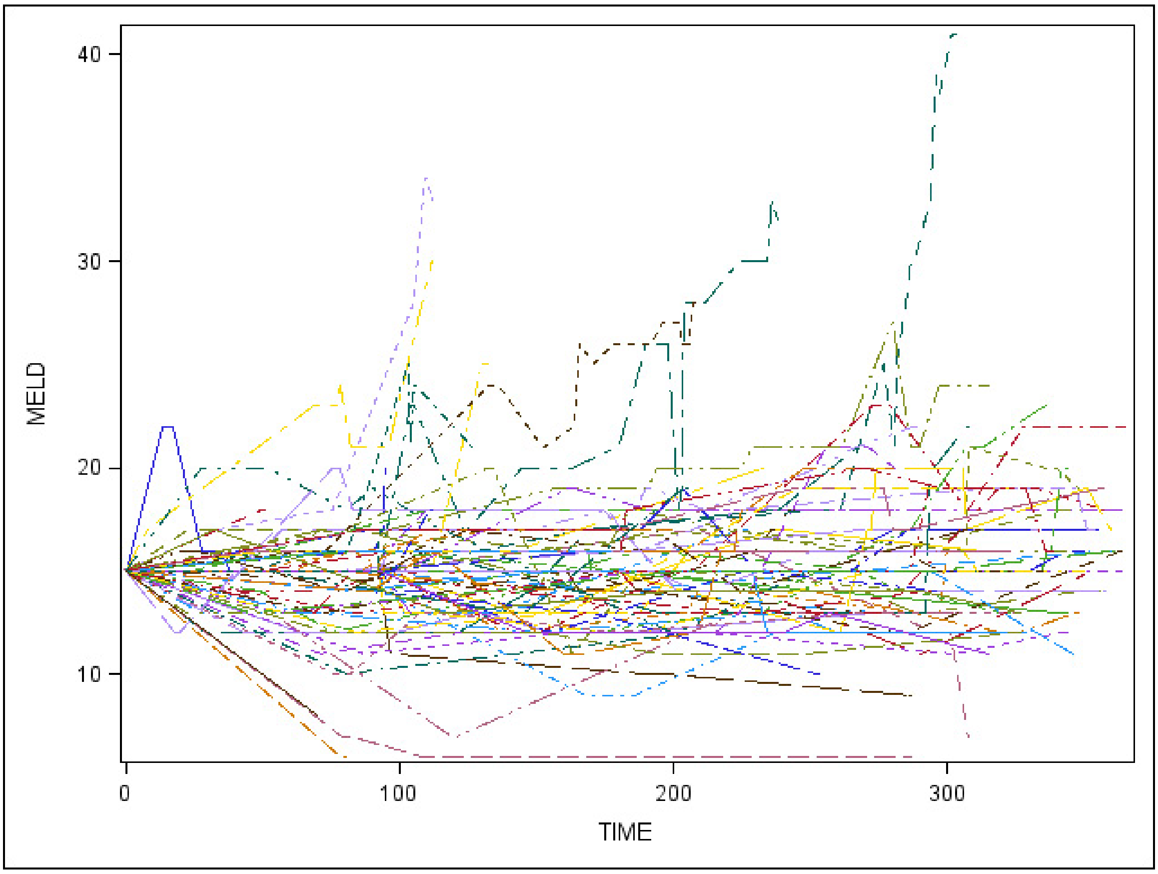

Figure 1. XX Spaghetti plot of MELD various randomly selected patients with MELD=15 at wait-listing ($s = 0$). The time axis is s, time since wait listing (in days).

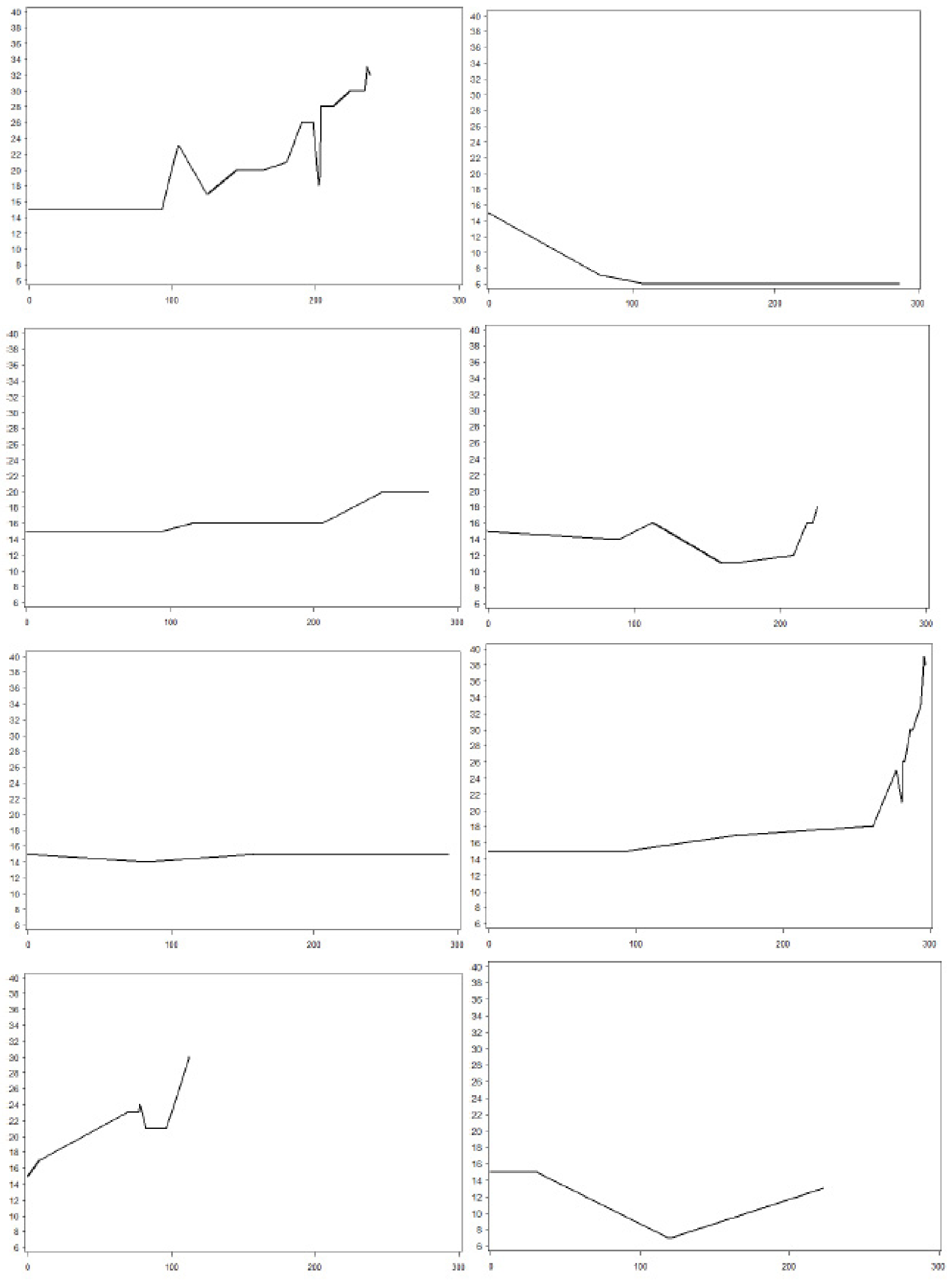

Figure 2. MELD trajectories for various randomly selected patients with MELD=15 at wait listing. The time axis is s, time since wait listing (in days).

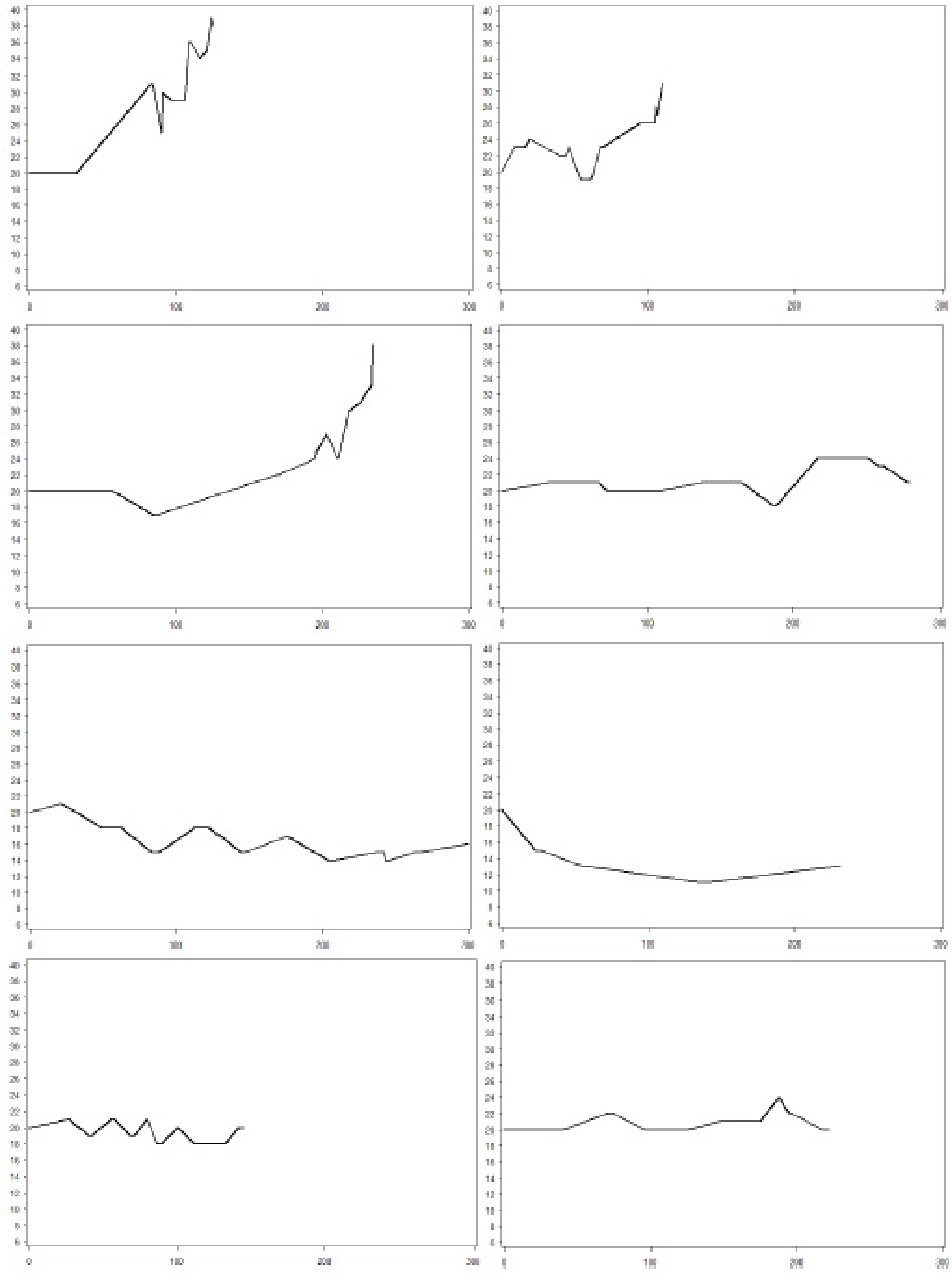

Figure 3. MELD trajectories for various randomly selected patients with MELD=20 at wait listing. The time axis is s, time since wait listing (in days).

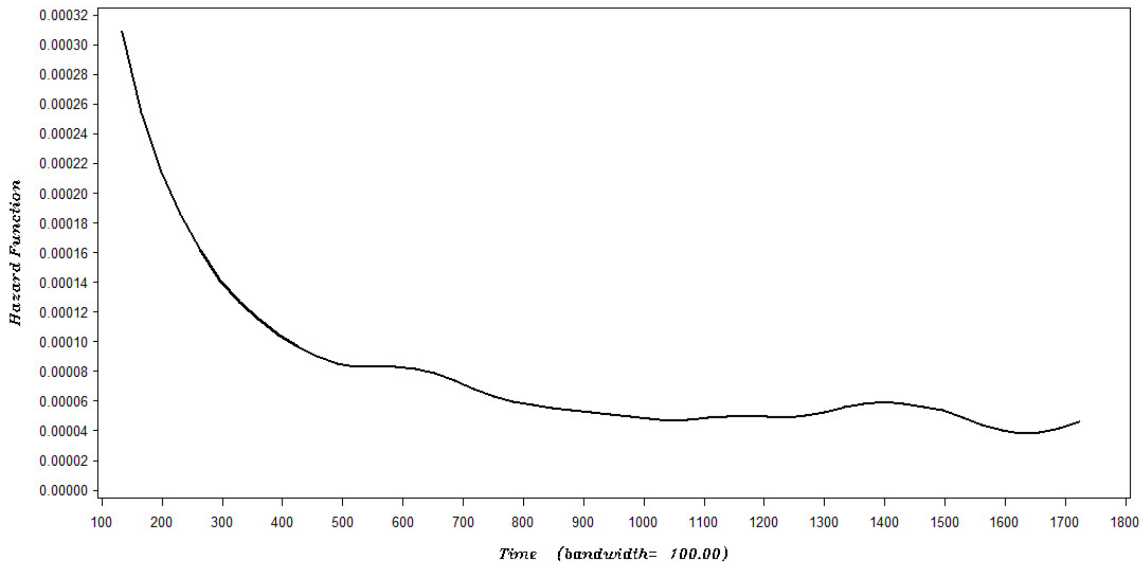

Figure 4. Smoothed estimator of $\lambda_{00}(t)$, the baseline hazard for death in the absence of transplantation, given by (12) in main manuscript; i.e., smooth increments of $\widehat{\Lambda}_{00}(t; \widehat{\boldsymbol{\beta}}_0)$, given by (19) in the main manuscript. The time axis is t, time since cross-section (in days).

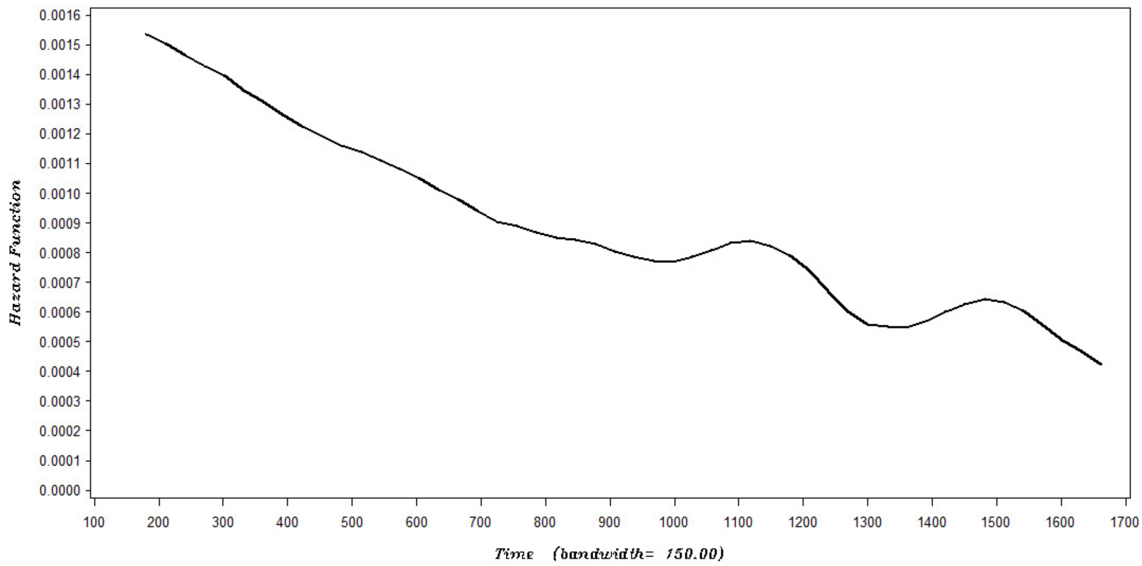

Figure 5. Smoothed estimator of $\lambda_0^T(t)$, given by (14) in the main manuscript: the baseline transplant hazard. The time axis is time since wait listing (in days).

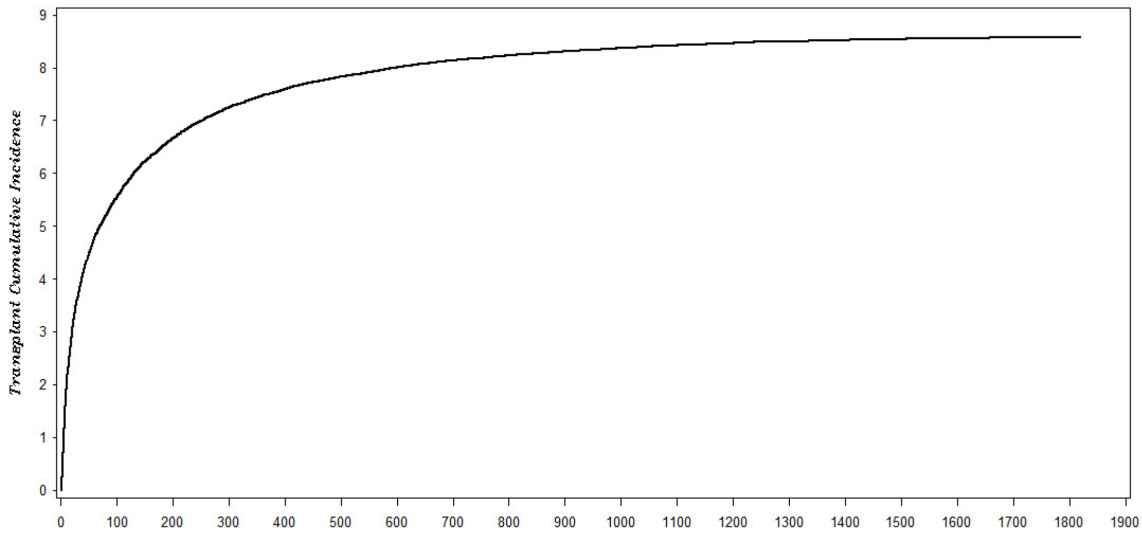

Figure 6. Average cumulative incidence of liver transplantation, given by (22) of the main manuscript. The time axis represents time since wait listing (in days).

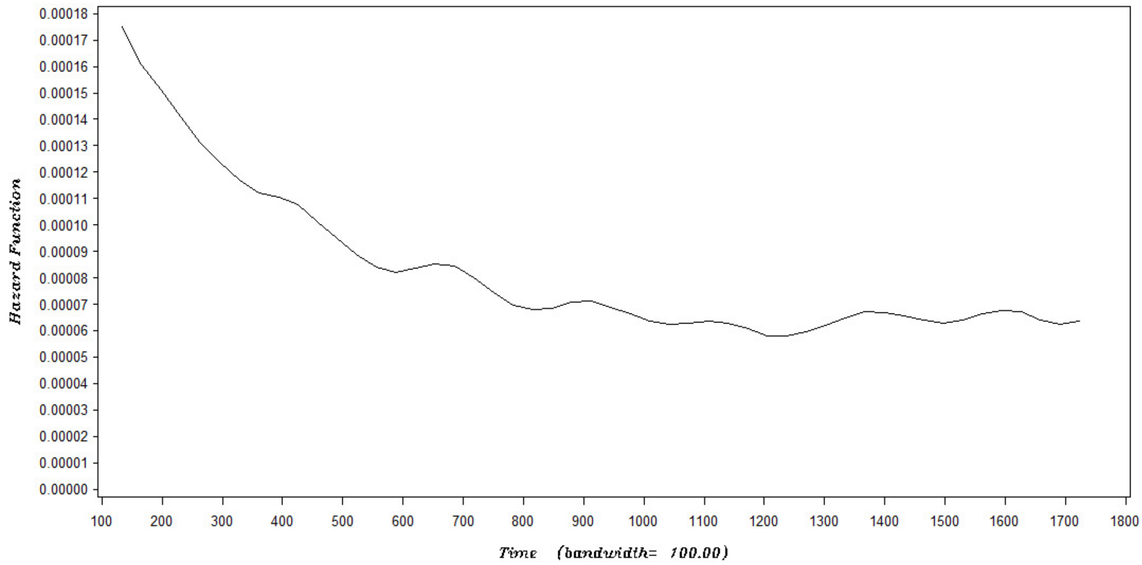

Figure 7. Smoothed estimator of $\lambda_{01}(t)$ from (11) in the main manuscript: the post-transplant death hazard. The time axis is time since liver transplant (in days)

Table 1

Raw Bias (year) Comparison for Moderate Mis-specified Models. β_M: The coefficient of the extra dichotomous variables in pre-/post-treatment/transplant model. Besides the two variables described in the manuscript, there is one extra dichotomous variable with coefficient β_M, which we mis-specified in the modeling process. $\beta_M = -0.70, -0.70, -0.70$ in pre, post and transplant model respectively.

$E[N_i^C(\tau)]$	$E[N_i^T(\tau)]$	Case	Term	Pre	Post	Tran
10%	10%	$\Delta > 0$	Δ	0.025	-0.004	0.050
			$\delta(1)$	0.011	-0.001	0.018
			$\delta(2)$	0.010	-0.001	0.021
			$\delta(3)$	0.010	0.001	0.019
15%	15%	$\Delta > 0$	Δ	0.016	0.029	0.009
			$\delta(1)$	0.007	0.007	0.009
			$\delta(2)$	0.008	0.012	0.001
			$\delta(3)$	0.007	0.016	-0.008
20%	20%	$\Delta > 0$	Δ	0.016	-0.008	0.013
			$\delta(1)$	0.003	-0.006	0.001
			$\delta(2)$	0.008	-0.002	0.007
			$\delta(3)$	0.011	-0.001	0.003
10%	10%	$\Delta = 0$	Δ	0.023	0.035	0.029
			$\delta(1)$	0.004	0.008	0.013
			$\delta(2)$	0.011	0.013	0.011
			$\delta(3)$	0.015	0.018	0.012
15%	15%	$\Delta = 0$	Δ	-0.003	0.022	0.033
			$\delta(1)$	0.006	0.006	0.012
			$\delta(2)$	-0.001	0.011	0.014
			$\delta(3)$	-0.006	0.019	0.011
20%	20%	$\Delta = 0$	Δ	0.016	-0.027	0.014
			$\delta(1)$	0.008	-0.006	0.007
			$\delta(2)$	0.006	-0.013	0.003
			$\delta(3)$	0.005	-0.007	0.005

Table 2

Raw Bias (year) Comparison for Mild Mis-specified Models. β_M: The coefficient of the extra dichotomous variables in pre-/post-treatment/transplant model. Besides the two variables described in the manuscript, there is one extra dichotomous variable with coefficient β_M, which we mis-specified in the modeling process. $\beta_M = -0.32, -0.20, -0.32$ in pre, post and transplant model respectively.

$E[N_i^C(\tau)]$	$E[N_i^T(\tau)]$	Case	Term	Pre	Post	Tran
10%	10%	$\Delta > 0$	Δ	0.017	-0.068	0.021
			$\delta(1)$	0.010	-0.006	-0.001
			$\delta(2)$	0.004	0.001	0.017
			$\delta(3)$	0.001	-0.004	0.017
15%	15%	$\Delta > 0$	Δ	0.008	0.029	-0.004
			$\delta(1)$	0.004	0.008	-0.001
			$\delta(2)$	0.005	0.012	0.001
			$\delta(3)$	-0.001	0.014	-0.001
20%	20%	$\Delta > 0$	Δ	0.001	0.012	0.030
			$\delta(1)$	0.004	0.008	0.012
			$\delta(2)$	0.001	0.004	0.007
			$\delta(3)$	-0.001	0.004	0.012
10%	10%	$\Delta = 0$	Δ	0.036	0.008	0.019
			$\delta(1)$	0.014	0.001	0.007
			$\delta(2)$	0.014	0.004	0.012
			$\delta(3)$	0.005	0.006	0.009
15%	15%	$\Delta = 0$	Δ	0.012	0.005	0.004
			$\delta(1)$	0.004	-0.001	0.003
			$\delta(2)$	0.003	-0.001	-0.004
			$\delta(3)$	0.008	0.009	0.006
20%	20%	$\Delta = 0$	Δ	0.011	0.020	0.066
			$\delta(1)$	0.003	0.001	0.007
			$\delta(2)$	0.010	0.011	0.010
			$\delta(3)$	0.001	0.011	0.019

Table 3
Mean and Total Patient Years (Out of 5 Years) Saved by Observed Transplants

MELD	$\widehat{\Delta}(5)$	Transplants	Total Years Saved
6-8	0.11	2,874	316
9-11	0.29	3,687	1,069
12-14	0.59	4,000	2,360
15-17	1.00	5,028	5,028
18-19	1.06	2,985	3,164
20-22	1.23	3,633	4,468
23-25	1.07	3,303	3,534
26-29	0.99	2,748	2,720
30-35	1.45	3,067	4,447
36-40	2.38	3,214	7,649
Total		34,539	34,757

Table 4

Results based on Sequential Stratification. The method is described in Schaubel et al. (2009), and the analysis followed closely the description in Sharma et al. (2015). Results are based on a stratified proportional hazards model with MELD-category-specific liver transplant effects, indexed by the parameter $\boldsymbol{\theta}_0$.

j	MELD	$\widehat{\theta}_j$	$\widehat{SE}$	p	$\exp\{\widehat{\theta}_j\}$
1	6-8	0.71	0.15	$< 10^{-4}$	2.04
2	9-11	-0.34	0.12	0.004	0.71
3	12-14	-0.82	0.08	$< 10^{-4}$	0.44
4	15-17	-1.22	0.06	$< 10^{-4}$	0.30
5	18-19	-1.40	0.07	$< 10^{-4}$	0.25
6	20-29	-1.50	0.04	$< 10^{-4}$	0.22
7	30-39	-1.88	0.06	$< 10^{-4}$	0.15
8	40	-2.05	0.07	$< 10^{-4}$	0.13