Retrospective Cost Methods for Combined State and Parameter Estimation

by

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A fanatic is one who can't change his mind and won't change the subject.

– Winston Churchill

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To Mom and Dad, for making my dreams a reality

To my wife, for bringing joy to every aspect of my life

To my advisor, whose guidance made this journey possible

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ABSTRACT

This dissertation is principally concerned with the combined state and parameter estimation problem, where the goal is to estimate the state of a discrete-time, linear time-invariant system with structured uncertainty in the system dynamics. First, we prove necessary and sufficient conditions for the identifiability of unknown parameters within a state-space realization. Next, we evaluate the performance of classical techniques for solving the combined state and parameter estimation problem. We then formulate and test the retrospective cost parameter estimation algorithm under the assumption that the initial states are known. Two variants of the retrospective cost parameter estimation and smoothing algorithm are formulated and tested in the case where the initial states are unknown. Finally, the retrospective cost Kalman filter algorithm is formulated and tested for state estimation despite uncertain noise covariances and potentially nonzero-mean sensor and process noise.

CHAPTER 1

Introduction

1.1 Combined State and Parameter Estimation

It is often the case in practice that state estimation is required for a linear time-invariant state-space system with unknown entries in the dynamics matrix A. This combined state and parameter estimation (CSPE) problem is nonlinear due to products of the unknown parameters and unmeasured states.

This dissertation is concerned solely with solving the CSPE problem. Our approach is to first identify the unknown parameters in the dynamics matrix using available measurements, and then perform state estimation using the identified dynamics matrix.

1.2 Identifiability of State-Space Realizations

Since the CSPE problem involves products of unmeasured states and unknown parameters, it is impossible in some cases to uniquely determine these variables. Consequently, we seek to determine conditions under which the CSPE problem is solvable. In particular, we consider the case where all components of C are known, the uncertainty in A is structured in the sense that each entry of A is specified as either known or unknown, and the unknown entries of A are independent in the sense that no assumption is made about the relationship among them. We say that the CSPE problem is *solvable* if the unmeasured states and unknown entries of A are uniquely determined by the measurements.

The fact that certain CSPE problems are not solvable may not be immediately evident. As a counter-argument, one may point to the use of subspace identification methods [1, 2], which can be used to estimate all of the entries of A and C along with the states. However, the resulting state space model is represented in an arbitrary basis, which obscures the meaning of the state components and does not distinguish between the known and unknown entries of A.

Another possibility is to use the least squares algorithm to identify a time-series representation of the system, express the time-series in a canonical realization, and then perform a similarity transform to the desired basis. However, it is not possible to obtain the change-of-basis matrix due to the uncertainty in A.

A convenient way to determine whether or not a specific CSPE problem is solvable is to assume that the input to the system is an impulse, in which case the output of the system is the impulse response of a state space model. The solvability of the CSPE problem is thus equivalent to determining the identifiability of a state space realization.

Identifiability of state-space realizations has been widely studied. Local identifiability and global identifiability are defined in [3, 4]. For continuous-time systems, necessary and sufficient conditions for testing local identifiability are provided in [5–7]. For discrete-time systems, necessary and sufficient conditions for local identifiability are provided in [8, 9]. Necessary and sufficient conditions for testing global identifiability of continuous-time systems are provided in [10]. This result is based on Ritt's algorithm [11], which is computationally intensive and thus is suitable only for simple cases, as shown in [12, 13]. Sufficient conditions for global identifiability of discrete-time systems are given in [14].

The first main contribution of this dissertation is the development of necessary and sufficient conditions for global identifiability of second-, third-, and fourth-order discrete-time systems. The tests derived in this dissertation are analytical and thus do not require any computation as in the case of Ritt's algorithm. The approach of the present paper is suggested in [15], but explicit results are not presented. Consequently, the results shown in

the present dissertation is the first to provide global, necessary and sufficient, and explicit conditions for the identifiability of discrete-time systems with structured uncertainty in the dynamics matrix. These conditions reveal the solvability of the corresponding CSPE problem.

1.3 Parameter Estimation without Full-State Measurements

For CSPE problems that are identifiable, the classical approach is to apply the extended Kalman filter [16]. Alternatively, the unscented Kalman filter (UKF) [17] can be used and applied to CSPE in [18]. Yet another approach to CSPE is based on the polynomial chaos series expansion [19, 20].

In this dissertation, we focus on the retrospective cost parameter estimation (RCPE) [21, 22] algorithm. The idea behind RCPE is to view the uncertain entries of the dynamics matrix as an uncertain subsystem connected in feedback. A performance metric based on the difference between the outputs of the true system and the model is then recursively optimized in order to update the estimates of the unknown parameters.

Note that if the state $x(k)$ is known for all $k \geq 0$, then it is straightforward to estimate the uncertain entries of A using variations of the least squares algorithm. Likewise, if all of the entries of A are known, then the Kalman filter can be used to estimate the state. Since RCPE can be used for parameter estimation without measurements of the state x, it can be viewed as an approach to solving CSPE.

Variants of RCPE have been applied to parameter estimation within the ionosphere-thermosphere [21–23], estimation of aeroelastic structural health [24], estimation of battery health [25–27], and online estimation of aircraft parameters [28, 29].

The key step in RCPE is the use of retrospective cost optimization, where the cost function is expressed as the sum of the measurement error and the filtered difference between prior estimates and retrospectively fitted estimates. While prior formulations of RCPE uti-

lize finite-impulse-response filters constructed from the Markov parameters of the system model, the RCPE algorithm presented in this dissertation utilizes a time-varying, infinite-impulse-response filter, which is continuously updated with prior estimates of the unknown parameters. This modification is motivated by the work in [30], where an interpretation of the filter as a target model is presented.

The second main contribution of this dissertation is thus the formulation of RCPE which uses the aforementioned time-varying filter. It is shown that this formulation provides accurate estimates of the unknown parameters under the assumption that the initial state of the system is known. Although this assumption is usually unrealistic in practice, the accuracy of the estimates is notable since knowledge of the initial state does not improve the performance of either EKF or UKF. This assumption leads to the development of the RCPE smoother (RCPES), which estimates both the unknown entries of the dynamics matrix and the unknown components of the initial state. Two variations of RCPES are given and assessed. Consequently, the third main contribution of the present paper is the development and assessment of RCPES.

1.4 State Estimation

The classical approach to state estimation is the Kalman Filter, which is the optimal state estimator for linear systems under zero-mean process and sensor noise with finite second moments. The optimality of the Kalman filter depends on knowledge of the system dynamics and noise covariances. When these estimates are unavailable or inaccurate, the performance of the Kalman filter degrades.

In this dissertation, we develop the retrospective cost Kalman filter (RCKF) algorithm, which is an alternative approach to state estimation which neither requires the process and sensor noise to be zero-mean and finite second moments nor requires knowledge of noise covariances. With RCKF, we view the innovations term $K(y_0 - E\hat{x})$ in the Kalman filter

as a static subsystem connected in feedback with a system model which is perturbed by a measured input. Using the same principles as RCPE, we estimate the parameters of the static subsystem in order to minimize the difference between the output of the true system and the model. The fourth main contribution of the present paper is thus the development and assessment of RCKF.

1.5 Dissertation Outline

In Chapter 2, we state and derive necessary and sufficient conditions for the global identifiability of second-, third-, and fourth-order discrete-time state-space systems. First, we state the CSPE problem and define identifiability. Next, we derive generic conditions for identifiability of second-, third-, and fourth-order systems where the measurement is an arbitrary scalar signal. Finally, we derive generic and non-generic conditions for identifiability of third- and fourth-order systems where the measurement is a vector of components of x.

In Chapter 3, we test classical methods for CSPE in order to establish a benchmark for the algorithms introduced in later chapters. First, we apply EKF to the second-order problem. Next, we constructed an augmented system and apply UKF to the same second-order problem. We then construct an augmented system using state-dependent coefficients and apply UKF to second- and third-order problems. Since one can construct infinitely many augmented systems using state-dependent coefficients, we test a range of these constructions and compare their performance.

In Chapter 4, we formulate and test the RCPE algorithm. First, we present the subsystem estimation framework. Next, we define a cost function based on the retrospective performance variable and then obtain minimizer. We then analyze the retrospective performance variable in order to arrive at an iterative update method for parameter estimation. To establish a comparison with classical methods for CSPE, we apply RCPE to the same

problems in Chapter 3. We first apply RCPE assuming the initial state is known. Then, we apply RCPE assuming there is a sufficiently large, measured excitation signal. Finally, we apply RCPE without excitation and assume the initial state is unknown. This motivates the need for an alternative formulation of RCPE.

In Chapter 5, we formulate and test the first variation of the RCPES algorithm. We start by presenting an augmented version of the subsystem estimation framework introduced in Chapter 4 in order to achieve concurrent estimation of the unknown parameters and the unmeasured components of the initial state. Then, to compare with classical approaches to CSPE, we apply RCPES to the same problems in Chapter 3 without excitation and assume that the initial state is unknown.

In Chapter 6, we formulate and test the second variation of RCPES. We start by presenting an alternate way to define the retrospective performance variable, which results in a polynomial cost function. We then present a convex approximation to this new cost function and obtain the solution. Once again, we apply RCPES to the same problems in Chapter 3 for comparison.

In Chapter 7, we formulate and test the RCKF algorithm. To start, we derive RCKF by showing that it is a form of static subsystem estimation. Then, assuming that the system is known and in the presence of process and sensor noise, we apply RCKF for state estimation without using knowledge of the noise covariance. The performance of RCKF is compared with the Kalman filter using inaccurate estimates of the noise covariance. We also compare the performance of RCKF with the Kalman filter for cases where the process and sensor noise are not zero-mean.

Finally, in Chapter 8, we summarize the contributions of this dissertation and discuss future work in identifiability, RCPE, RCPES, and RCKF.

CHAPTER 2

Global Identifiability of State-Space Realizations

2.1 Combined State and Parameter Estimation Problem

Consider the linear time-invariant system

$$x(k+1) = Ax(k), \tag{2.1}$$

$$x(0) = x_0, \tag{2.2}$$

$$y_0(k) = Ex(k), \tag{2.3}$$

where $n \geq 2$,

$$x(k) = \begin{bmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{bmatrix} \in \mathbb{R}^n, \quad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \in \mathbb{R}^{n \times n}, \tag{2.4}$$

$y_0(k) \in \mathbb{R}$ is the measurement, and

$$E = [\, e_1 \; e_2 \; \cdots \; e_n \,] \in \mathbb{R}^{1 \times n}. \tag{2.5}$$

We assume that E is known but A is structured in the sense that some entries of A are assumed to be known and others are assumed to be unknown. The objective is to use the measurements $y_0(k)$, where $k \geq 0$, to estimate the unknown entries of A and the

unmeasured components $x_1(k), x_2(k), \ldots, x_n(k)$ of the state $x(k)$.

2.2 Definition of Identifiability

To provide an equivalent formulation of (2.1) – (2.3), let $\delta(k)$ be the unit impulse signal, let

$$B = [\ b_1\ \cdots\ b_n\ ]^{\mathrm{T}} \in \mathbb{R}^n, \tag{2.6}$$

and consider the single-input, single-output system

$$\tilde{x}(k+1) = A\tilde{x}(k) + B\delta(k), \tag{2.7}$$

$$\tilde{x}(0) = 0, \tag{2.8}$$

$$\tilde{y}_0(k) = E\tilde{x}(k), \tag{2.9}$$

where E is given by (2.5). We assume that $B = x_0$, where x_0 is the initial condition in (7.2). It thus follows that (2.1) – (2.3) and (2.7) – (2.9) are equivalent in the sense that the outputs $y_0(k)$ and $\tilde{y}_0(k)$ are equal for all $k \geq 0$. This output is the free response of (3.1) – (3.3) as well as the impulse response of the transfer function corresponding to (A, B, E). Therefore, for (2.1) – (2.3), the feasibility of estimating the unmeasured states and unknown entries of A is equivalent to determining whether or not the unknown entries of A and B are identifiable, that is, uniquely specified given the transfer function corresponding to (A, B, E). The equivalence of (2.1) – (2.3) and (2.7) – (2.9) thus determines the feasibility of the combined state and parameter estimation problem in terms of the identifiability of the unknown entries of (A, B).

Now consider the transfer function

$$G(z) = \frac{\beta_{n-1}z^{n-1} + \cdots + \beta_1 z + \beta_0}{z^n + \cdots + \alpha_1 z + \alpha_0}, \tag{2.10}$$

whose numerator and denominator are coprime. Let $q \leq n^2 + n$ be the number of unknown entries of (A, B). We define $\mathcal{S} \subset \mathbb{R}^q$ to be the set of vectors of unknown entries of A and B such that (A, B, E) is a minimal realization of (2.10). If $\mathcal{S}$ contains exactly one element, then (A, B) is *identifiable*.

2.3 Generic Results on Global Identifiability

We now state necessary and sufficient conditions under which (A, B) is generically identifiable. For the remainder of this chapter, the notation $\bar{a}$ indicates that a is a known parameter.

Theorem 2.3.1: Let $n = 2$, let $q \leq 6$ denote the number of unknown entries of (A, B), and assume that (A, B, E) is controllable and observable, where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad E = \begin{bmatrix} \bar{e}_1 & \bar{e}_2 \end{bmatrix}. \tag{2.11}$$

Then the conditions under which (A, B) is generically identifiable are given in Figure 2.1.

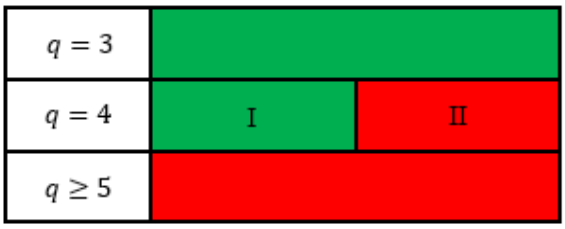

Figure 2.1: Cases in the green region are generically identifiable, whereas cases in the red region are generically not identifiable.

Region I contains the cases where both of the unknown entries of A are in either a single row or a single column. Region II contains the cases where both of the unknown entries of A are on either the diagonal or the anti-diagonal.

Proof: Set

$$[\,\bar{e}_1\ \ \bar{e}_2\,](zI_2 - A)^{-1}B = \frac{\beta_1 z + \beta_0}{z^2 + \alpha_1 z + \alpha_0}, \tag{2.12}$$

where I_2 is the 2×2 identity matrix. It follows that

$$\bar{e}_1 b_1 + \bar{e}_2 b_2 = \beta_1, \tag{2.13}$$

$$\bar{e}_1 b_2 a_{12} + \bar{e}_2 a_{21} b_1 - \bar{e}_1 b_1 a_{22} - \bar{e}_2 b_2 a_{11} = \beta_0, \tag{2.14}$$

$$a_{11} + a_{22} = -\alpha_1, \tag{2.15}$$

$$a_{11} a_{22} - a_{12} a_{21} = \alpha_0. \tag{2.16}$$

Consider the case where $q = 3$. Let a_{11} be unknown. We write (2.13)–(2.16) as

$$M \begin{bmatrix} a_{11} \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -\alpha_1 - \bar{a}_{22} \\ \frac{\alpha_0 + \bar{a}_{12}\bar{a}_{21}}{\bar{a}_{22}} \\ \beta_1 \\ \beta_0 \end{bmatrix}, \tag{2.17}$$

where

$$M \triangleq \begin{bmatrix} 1 & 0 & 0 \\ \bar{a}_{22} & 0 & 0 \\ 0 & \bar{e}_1 & \bar{e}_2 \\ 0 & \bar{e}_2\bar{a}_{21} - \bar{e}_1\bar{a}_{22} & \bar{e}_1\bar{a}_{12} - \bar{e}_2 a_{11} \end{bmatrix}. \tag{2.18}$$

Except for the case where

$$\begin{bmatrix} \bar{e}_1 & \bar{e}_2 \\ \bar{e}_2\bar{a}_{21} - \bar{e}_1\bar{a}_{22} & \bar{e}_1\bar{a}_{12} - \bar{e}_2 a_{11} \end{bmatrix} \tag{2.19}$$

is singular, M has full column rank, and thus (2.17) has a unique solution generically. Hence, $\mathcal{S}$ has exactly one element generically. In the case where a_{22} is unknown, a similar argument shows that $\mathcal{S}$ has exactly one element generically.

Next, let a_{12} be unknown. Then (2.15) constrains only known parameters and thus can be disregarded. Hence, (2.13) – (2.16) have the form

$$\bar{e}_1 b_1 + \bar{e}_2 b_2 = \beta_1, \tag{2.20}$$

$$\bar{e}_1 b_2 a_{12} + \bar{e}_2 \bar{a}_{21} b_1 - \bar{e}_1 b_1 \bar{a}_{22} - \bar{e}_2 b_2 \bar{a}_{11} = \beta_0, \tag{2.21}$$

$$\bar{a}_{11}\bar{a}_{22} - a_{12}\bar{a}_{21} = \alpha_0. \tag{2.22}$$

Except for the case where either $\bar{a}_{21} = 0$, $\bar{e}_1 = 0$, or $\bar{e}_1^2 a_{12} - \bar{e}_1\bar{e}_2\bar{a}_{11} + \bar{e}_1\bar{e}_2\bar{a}_{22} - \bar{e}_2^2\bar{a}_{21} = 0$, (2.20) – (2.22) imply

$$\begin{aligned} a_{12} &= \frac{\bar{a}_{11}\bar{a}_{22} - \alpha_0}{\bar{a}_{21}}, \quad b_1 = \frac{\beta_1 - \bar{e}_2 b_2}{\bar{e}_1} \\ b_2 &= \frac{\bar{e}_1\beta_0 + \bar{e}_1\bar{a}_{22}\beta_1 - \bar{e}_2\bar{a}_{21}\beta_1}{\bar{e}_1^2 a_{12} - \bar{e}_1\bar{e}_2\bar{a}_{11} + \bar{e}_1\bar{e}_2\bar{a}_{22} - \bar{e}_2^2\bar{a}_{21}}, \end{aligned} \tag{2.23}$$

and thus $\mathcal{S}$ has exactly one element generically. In the case where a_{21} is unknown, a similar argument shows that $\mathcal{S}$ has exactly one element generically.

Now consider the case where $q \geq 5$. Since either five or six unknown parameters satisfy (2.13) – (2.16), $\mathcal{S}$ has infinitely many elements.

Next, consider the case where $q = 4$. Let a_{11} and a_{12} be unknown. Then (2.13) – (2.16) have the form

$$\bar{e}_1 b_1 + \bar{e}_2 b_2 = \beta_1, \tag{2.24}$$

$$\bar{e}_1 b_2 a_{12} + \bar{e}_2 \bar{a}_{21} b_1 - \bar{e}_1 b_1 \bar{a}_{22} - \bar{e}_2 b_2 a_{11} = \beta_0, \tag{2.25}$$

$$a_{11} + \bar{a}_{22} = -\alpha_1, \tag{2.26}$$

$$a_{11}\bar{a}_{22} - a_{12}\bar{a}_{21} = \alpha_0. \tag{2.27}$$

Except for the cases where $\bar{a}_{21} = 0$, (2.26) – (2.27) imply

$$a_{11} = -\alpha_1 - \bar{a}_{22}, \quad a_{12} = \frac{a_{11}\bar{a}_{22} - \alpha_1}{\bar{a}_{21}}, \tag{2.28}$$

Furthermore, except for the cases where

$$\begin{bmatrix} \bar{e}_1 & \bar{e}_2 \\ \bar{e}_2\bar{a}_{21} - \bar{e}_1\bar{a}_{22} & \bar{e}_1 a_{12} - \bar{e}_2 a_{11} \end{bmatrix} \tag{2.29}$$

is singular, (2.24) – (2.25) imply

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \bar{e}_1 & \bar{e}_2 \\ \bar{e}_2\bar{a}_{21} - \bar{e}_1\bar{a}_{22} & \bar{e}_1 a_{12} - \bar{e}_2 a_{11} \end{bmatrix}^{-1} \begin{bmatrix} \beta_1 \\ \beta_0 \end{bmatrix}, \tag{2.30}$$

and thus $\mathcal{S}$ has exactly one element generically. In the cases where a_{12} and a_{22} are unknown, a_{11} and a_{21} are unknown, or a_{21} and a_{22} are unknown , a similar argument shows that $\mathcal{S}$ has exactly one element generically.

Let a_{11} and a_{22} be unknown. Then (2.13) – (2.16) have the form

$$\bar{e}_1 b_1 + \bar{e}_2 b_2 = \beta_1, \tag{2.31}$$

$$\bar{e}_1 b_2 \bar{a}_{12} + \bar{e}_2 \bar{a}_{21} b_1 - \bar{e}_1 b_1 a_{22} - \bar{e}_2 b_2 a_{11} = \beta_0, \tag{2.32}$$

$$a_{11} + a_{22} = -\alpha_1, \tag{2.33}$$

$$a_{11}a_{22} - \bar{a}_{12}\bar{a}_{21} = \alpha_0. \tag{2.34}$$

(2.33) – (2.34) imply

$$a_{11} = a_{22} = \frac{-\alpha_1 \pm \sqrt{\alpha_1^2 - 4(\alpha_0 + \bar{a}_{12}\bar{a}_{21})}}{2}, \tag{2.35}$$

and except for the cases where

$$\begin{bmatrix} \bar{e}_1 & \bar{e}_2 \\ \bar{e}_2\bar{a}_{21} - \bar{e}_1\bar{a}_{22} & \bar{e}_1 a_{12} - \bar{e}_2 a_{11} \end{bmatrix} \tag{2.36}$$

is singular, (2.31) – (2.32) imply

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \bar{e}_1 & \bar{e}_2 \\ \bar{e}_2\bar{a}_{21} - \bar{e}_1\bar{a}_{22} & \bar{e}_1 a_{12} - \bar{e}_2 a_{11} \end{bmatrix}^{-1} \begin{bmatrix} \beta_1 \\ \beta_0 \end{bmatrix}. \tag{2.37}$$

Thus, except for the case where $\alpha_1^2 - 4(\alpha_0 + \bar{a}_{12}\bar{a}_{21}) = 0$, $\mathcal{S}$ has exactly two elements generically. Note that both alternatives in (2.35) are real since A has real entries.

Let a_{12} and a_{21} be unknown. Note that (2.15) constrains only known parameters and thus can be disregarded. Hence, (2.13) – (2.16) have the form

$$\bar{e}_1 b_1 + \bar{e}_2 b_2 = \beta_1, \tag{2.38}$$

$$\bar{e}_1 b_2 a_{12} + \bar{e}_2 a_{21} b_1 - \bar{e}_1 b_1 \bar{a}_{22} - \bar{e}_2 b_2 \bar{a}_{11} = \beta_0, \tag{2.39}$$

$$\bar{a}_{11}\bar{a}_{22} - a_{12}a_{21} = \alpha_0. \tag{2.40}$$

Since four unknown parameters satisfy three equations, $\mathcal{S}$ has infinitely many elements. □

Example 1 Consider (2.7)-(2.9), where the diagonal entries of A are uncertain. Note that the uncertainty in A corresponds to the case $q = 3$ and falls in Region II in Figure 1. In order to explicitly demonstrate why the case of uncertain diagonal entries is not identifiable, let A_1 and A_2 denote special cases of A, and let G_1 and G_2 be the transfer functions defined by the realizations

$$G_1 \sim \left[\begin{array}{c|c} A_1 & B \\ \hline E & 0 \end{array}\right], \; G_2 \sim \left[\begin{array}{c|c} A_2 & B \\ \hline E & 0 \end{array}\right], \tag{2.41}$$

where

$$A_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},\ A_2 = \begin{bmatrix} a_{22} & a_{12} \\ a_{21} & a_{11} \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ E = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{2.42}$$

Note that A_1 and A_2 are identical except that their diagonal entries are swapped, which is consistent with the assumed uncertainty structure. The corresponding discrete-time transfer function of both systems is given by

$$G_1(z) = G_2(z) = \frac{a_{12}}{z^2 - (a_{11} + a_{22})z + (a_{12}a_{21} - a_{11}a_{22})}. \tag{2.43}$$

Since G_1 and G_2 are the same transfer function, it is impossible to distinguish between the two systems using only input and output data. Hence, the uncertain system is not identifiable. The lack of identifiability can be seen more directly by noting that $A_1 = SA_2S^{-1}$, $B = SB$, and $E = ES^{-1}$, where

$$S = -S^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{a_{22}-a_{11}}{a_{12}} & 1 \end{bmatrix}, \tag{2.44}$$

which also shows that $G_1 = G_2$.

Theorem 2.3.2: Let $n = 3$, let $q \leq 12$ be the number of unknown entries of (A, B), and assume that (A, B, E) is controllable and observable, where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad E = \begin{bmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \end{bmatrix}. \tag{2.45}$$

Then the conditions under which (A, B) is generically identifiable are given in Figure 2.2.

Region I contains the cases where at least one entry on the diagonal of A is unknown. Region II contains the cases where all of the entries on the diagonal of A are known and

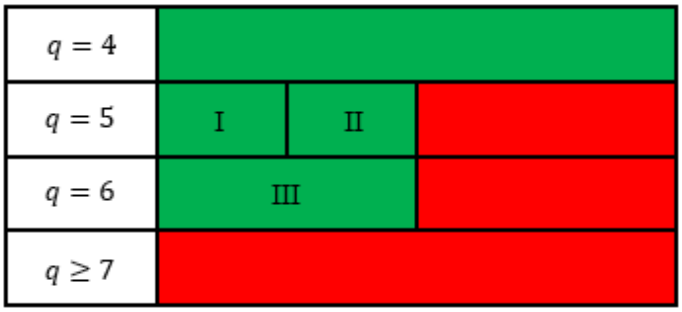

Figure 2.2: Cases in the green region are generically identifiable, whereas cases in the red region are generically not identifiable.

all of the unknown entries of A are in either a single row or a single column. Region III contains the cases where exactly one entry on the diagonal of A is unknown and all of the remaining unknown entries of A are in a single row or a single column.

The proof of Theorem 2.3.2 is similar to the proof of Theorem 2.3.1 and thus is omitted.

Theorem 2.3.3: Let $n = 4$, let $q \leq 20$ be the number of unknown entries of (A, B), and assume that (A, B, E) is controllable and observable, where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}, \quad E = \begin{bmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 & \bar{e}_4 \end{bmatrix}. \tag{2.46}$$

Then the conditions under which (A, B) is generically identifiable are given in Figure 2.3.

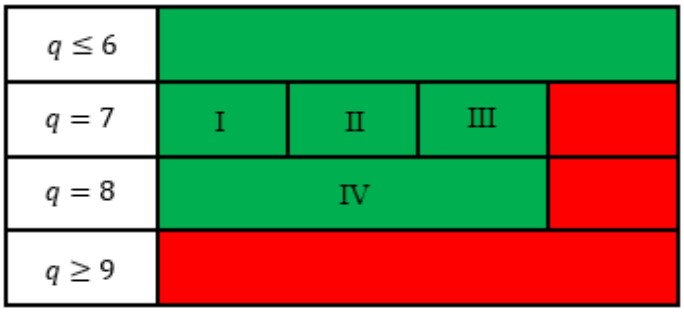

Figure 2.3: Cases in the green region are generically identifiable, whereas cases in the red region are generically not identifiable.

Region I contains the cases where all of the following conditions are satisfied: *i*) exactly two unknown entries on the diagonal of A are unknown; *ii*) exactly two unknown entries of A are in a single row; and *iii*) exactly two unknown entries of A are in a single column. Region II contains the cases where exactly one unknown entry on the diagonal of A is unknown. Region III contains the cases where all of the entries on the diagonal of A are known and all of the unknown entries are in either a single row or a single column. Region IV contains the cases where exactly one entry on the diagonal of A is unknown and all of the remaining unknown entries are in a single row or a single column.

The proof of Theorem 2.3.3 is similar to the proof of Theorem 2.3.1 and thus is omitted.

Extrapolating from Theorem 2.3.1–2.3.3 yields the following conjecture regarding the identifiability of (A, B) for arbitrary n.

Conjecture 2.3.4: Let $n > 2$, let $q \geq n^2 + n$ be the number of unknown entries of (A, B), and assume that (A, B, E) is observable and controllable, where

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}, \quad E = \begin{bmatrix} \bar{e}_1 & \cdots & \bar{e}_n \end{bmatrix}. \tag{2.47}$$

Then the following statements hold:

1. If $q \leq 2n - 2$, then (A, B) is generically identifiable.

2. If $q = 2n - 1$, all of the entries on the diagonal of A are known, and all of the unknown entries of A are in a single row or a single column, then (A, B) is generically identifiable.

3. If $q = 2n$, exactly one entry on the diagonal of A is unknown, and all of the remaining unknown entries are in a single row or a single column, then (A, B) is generically identifiable.

4. If $q \geq 2n + 1$, then (A, B) is generically not identifiable.

2.4 Specialized Results on Global Identifiability

In this section, we assume that E has a specific structure and derive further conditions on identifiability. Some of the identifiability conditions derived in this section include both generic and non-generic cases.

Theorem 2.4.1: Let $n = 3$,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \tag{2.48}$$

and assume that (A, B, E) is a minimal realization of

$$G(z) = \frac{\beta_2 z^2 + \beta_1 z + \beta_0}{z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0}, \tag{2.49}$$

whose numerator and denominator are coprime. Then (A, B) is not identifiable if and only if at least one of the following statements holds:

1. $\begin{bmatrix} a_{12} & a_{13} \\ a_{13}a_{32} - a_{12}a_{33} & a_{12}a_{33} - a_{13}a_{22} \end{bmatrix}$ is singular.
2. $q = 3$, a_{ij} is unknown, $a_{ji} = 0$, and $a_{kk}a_{ji} - a_{jk}a_{ki} = 0$, where $i \neq j \neq k \leq n$.
3. $q = 4$, a_{ii} and a_{jj} are unknown, and $a_{jk}a_{kj} = a_{ik}a_{ki}$, where $i \neq j \neq k \leq n$.
4. Generically, $q = 4$, a_{ij} and a_{kl} are unknown, where $i \neq k$, $j \neq l$, $i \neq j$, and $k \neq l$.
5. $q = 4$, a_{ii} and a_{jk} are unknown, $a_{kj} = 0$, and $a_{kj}a_{ll} - a_{kl}a_{lj} = 0$, where either $i = j \neq k \neq l \leq n$ or $i = k \neq l \neq l \leq n$.
6. $q = 4$, a_{ii} and a_{jk} are unknown, $a_{kj} = 0$, and $a_{kj}(\alpha_2 - a_{jj} - a_{kk}) + a_{ij}a_{ki} = 0$, where $i \neq j \neq k \leq n$.

7. $q = 4$, a_{ij} and a_{lm} are unknown, and $\begin{bmatrix} -a_{ji} & -a_{ml} \\ a_{ji}a_{kk} - a_{jk}a_{ki} & a_{ml}a_{oo} - a_{mo}a_{ol} \end{bmatrix}$ is singular, where $i = l \neq j \neq m \leq n$ or $j = m \neq i \neq l \leq n$, $i \neq j \neq k \leq n$, and $l \neq m \neq o \leq n$.

8. $q = 5$, a_{ii}, a_{jj}, and a_{kk} are known, where $i \neq j \neq k \leq n$.

9. Generically, $q = 5$ and a_{ii} and a_{jj} are unknown, where $i \neq j \leq n$.

10. Generically, $q = 5$, a_{ii}, a_{jk}, and a_{lm} are unknown, where $i \leq n$, $j \neq l \leq n$, and $k \neq m \leq n$.

11. $q = 5$, a_{ii}, a_{jk}, and a_{mo} are unknown, and $\begin{bmatrix} -a_{kj} & -a_{om} \\ a_{kj}a_{ll} - a_{kl}a_{lj} & a_{om}a_{pp} - a_{op}a_{pm} \end{bmatrix}$ is singular, where $i \leq n$, $j = m \neq k \neq o \leq n$ or $k = o \neq j \neq m \leq n$, $j \neq k \neq l \leq n$, and $m \neq o \neq p \leq n$.

12. $q \geq 6$.

Proof. Set

$$G(z) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} (zI_3 - A)^{-1}B. \tag{2.50}$$

It thus follows that

$$b_1 = \beta_2, \tag{2.51}$$

$$b_2 a_{12} + b_3 a_{13} - b_1 a_{22} - b_1 a_{33} = \beta_1, \tag{2.52}$$

$$\begin{aligned} &b_1 a_{22} a_{33} - b_1 a_{23} a_{32} + b_2 a_{13} a_{32} \\ &- b_2 a_{12} a_{33} + b_3 a_{12} a_{33} - b_3 a_{13} a_{22} = \beta_0, \end{aligned} \tag{2.53}$$

$$a_{11} + a_{22} + a_{33} = -\alpha_2, \tag{2.54}$$

$$\begin{aligned} &a_{11} a_{22} - a_{12} a_{21} + a_{11} a_{33} \\ &- a_{13} a_{31} + a_{22} a_{33} - a_{23} a_{32} = \alpha_1, \end{aligned} \tag{2.55}$$

$$\begin{aligned} &a_{13} a_{22} a_{31} - a_{13} a_{21} a_{32} - a_{12} a_{23} a_{31} \\ &+ a_{12} a_{21} a_{33} + a_{11} a_{23} a_{32} - a_{11} a_{22} a_{33} = \alpha_0. \end{aligned} \tag{2.56}$$

Since (2.51) constrains only known parameters, it can be disregarded.

Sufficiency:

1) We write (2.52) and (2.53) in matrix form and obtain

$$M \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \beta_1 + b_1(a_{22} + a_{33}) \\ \beta_2 + b_1(a_{23} a_{32} - a_{22} a_{33}) \end{bmatrix}, \tag{2.57}$$

where

$$M \triangleq \begin{bmatrix} a_{12} & a_{13} \\ a_{13} a_{32} - a_{12} a_{33} & a_{12} a_{33} - a_{13} a_{22} \end{bmatrix}. \tag{2.58}$$

Since b_2 and b_3 do not appear in any other equation and M is singular, $\mathcal{S}$ has infinitely many elements.

2) Consider the case where a_{12} is unknown. (2.54) constrains only known parameters

and thus can be disregarded. We write (2.55) and (2.56) in matrix form and obtain

$$Ma_{12} = \begin{bmatrix} \xi_1 \\ \xi_0 \end{bmatrix}, \tag{2.59}$$

where ξ_1 and ξ_0 are complex expressions of known parameters, and

$$M \triangleq \begin{bmatrix} -\bar{a}_{21} \\ \bar{a}_{21}\bar{a}_{33} - \bar{a}_{23}\bar{a}_{31} \end{bmatrix}. \tag{2.60}$$

Since $\bar{a}_{21} = 0$ and $\bar{a}_{21}\bar{a}_{33} - \bar{a}_{23}\bar{a}_{31} = 0$, M is not left invertible, thus $\mathcal{S}$ has infinitely many elements. In the case where exactly one of a_{21}, a_{13}, a_{31}, a_{23}, or a_{32} is unknown, a similar argument shows that $\mathcal{S}$ has infinitely many element.

3) Consider the case where a_{11} and a_{22} are unknown. (2.54) – (2.56) have the form

$$a_{11} + a_{22} = -\alpha_2 - \bar{a}_{33}, \tag{2.61}$$

$$\begin{aligned} a_{11}a_{22} = \alpha_1 - \bar{a}_{33}a_{11} - \bar{a}_{33}a_{22} \\ + \bar{a}_{12}\bar{a}_{21} + \bar{a}_{13}\bar{a}_{31} + \bar{a}_{23}\bar{a}_{32}, \end{aligned} \tag{2.62}$$

$$\begin{aligned} -\bar{a}_{33}a_{11}a_{22} = \alpha_0 - \bar{a}_{23}\bar{a}_{32}a_{11} - \bar{a}_{13}\bar{a}_{31}a_{22} \\ + \bar{a}_{13}\bar{a}_{21}\bar{a}_{32} + \bar{a}_{12}\bar{a}_{23}\bar{a}_{31} - \bar{a}_{12}\bar{a}_{21}\bar{a}_{33}. \end{aligned} \tag{2.63}$$

Substituting (2.62) into (2.63) and writing the remaining equations in matrix form, we obtain

$$M \begin{bmatrix} a_{11} \\ a_{22} \end{bmatrix} = \begin{bmatrix} \xi_2 \\ \xi_0 \end{bmatrix}, \tag{2.64}$$

where ξ_2 is a complex expression of known parameters, and

$$M \triangleq \begin{bmatrix} 1 & 1 \\ \bar{a}_{33}^2 + \bar{a}_{23}\bar{a}_{32} & \bar{a}_{33}^2 + \bar{a}_{13}\bar{a}_{31} \end{bmatrix}. \tag{2.65}$$

Since $\bar{a}_{23}\bar{a}_{32} = \bar{a}_{13}\bar{a}_{31}$, M is singular, thus $\mathcal{S}$ has infinitely many elements. In the case where either a_{11} and a_{33} are unknown or a_{22} and a_{33} are unknown, a similar argument shows that $\mathcal{S}$ has infinitely many elements.

4) Consider the case where a_{12} and a_{21} are unknown. (2.54) constrains only known parameters and thus can be disregarded. (2.55) and (2.56) have the form

$$\begin{aligned} -a_{12}a_{21} &= \alpha_1 - \bar{a}_{11}\bar{a}_{22} - \bar{a}_{11}\bar{a}_{33} \\ &\quad + \bar{a}_{13}\bar{a}_{31} - \bar{a}_{22}\bar{a}_{33} + \bar{a}_{23}\bar{a}_{32}, \end{aligned} \tag{2.66}$$

$$\begin{aligned} \bar{a}_{33}a_{12}a_{21} &= \alpha_0 + \bar{a}_{23}\bar{a}_{31}a_{12} + \bar{a}_{13}\bar{a}_{32}a_{21} \\ &\quad - \bar{a}_{23}\bar{a}_{32}\bar{a}_{11} - \bar{a}_{13}\bar{a}_{31}\bar{a}_{22} + \bar{a}_{33}\bar{a}_{11}\bar{a}_{22}. \end{aligned} \tag{2.67}$$

Substituting (2.66) into (2.68) reduces (2.51) – (2.56) to three equations with four unknown parameters, thus $\mathcal{S}$ has infinitely many elements. In the case where either a_{13} and a_{31} are unknown and a_{23} and a_{32} are unknown, a similar argument shows that $\mathcal{S}$ has infinitely many elements.

Consider the case where a_{12} and a_{23} are unknown. (2.54) constrains only known parameters and thus can be disregarded. (2.55) and (2.56) have the form

$$\begin{aligned} -\bar{a}_{21}a_{12} - \bar{a}_{32}a_{23} &= \alpha_1 - \bar{a}_{11}\bar{a}_{22} \\ &\quad - \bar{a}_{11}\bar{a}_{33} + \bar{a}_{13}\bar{a}_{31} - \bar{a}_{22}\bar{a}_{33}, \end{aligned} \tag{2.68}$$

$$\begin{aligned} -\bar{a}_{31}a_{12}a_{23} + \bar{a}_{33}\bar{a}_{21}a_{12} + \bar{a}_{32}\bar{a}_{11}a_{23} &= \alpha_0 \\ &\quad + \bar{a}_{13}\bar{a}_{32}a_{21} - \bar{a}_{13}\bar{a}_{31}\bar{a}_{22} + \bar{a}_{33}\bar{a}_{11}\bar{a}_{22}. \end{aligned} \tag{2.69}$$

Using (2.68), (2.69) is quadratic in either a_{12} or a_{23}, and thus $\mathcal{S}$ has exactly two elements generically. In the case where a_{12} and a_{31}, a_{13} and a_{21}, a_{13} and a_{32}, a_{21} and a_{32}, or a_{23} and a_{31} are unknown, a similar argument shows that $\mathcal{S}$ has exactly two elements generically.

5) Consider the case where a_{11} and a_{12} are unknown. (2.54) – (2.56) have the form

$$a_{11} = -\alpha_2 - \bar{a}_{22} - \bar{a}_{33}, \tag{2.70}$$

$$(\bar{a}_{22} + \bar{a}_{33})a_{11} - \bar{a}_{21}a_{12} = \alpha_1 + \bar{a}_{13}\bar{a}_{31} - a_{22}\bar{a}_{33} + \bar{a}_{23}\bar{a}_{32}, \tag{2.71}$$

$$(\bar{a}_{23}\bar{a}_{32} - \bar{a}_{33}\bar{a}_{22})a_{11} + (\bar{a}_{21}\bar{a}_{33} - \bar{a}_{23}\bar{a}_{31})a_{12} = \alpha_0 - \bar{a}_{13}\bar{a}_{31}a_{22} + \bar{a}_{13}\bar{a}_{21}\bar{a}_{32}. \tag{2.72}$$

Substituting (2.70) into (2.71) and (2.72) and writing the result in matrix form yield

$$Ma_{12} = \begin{bmatrix} \xi_1 \\ \xi_0 \end{bmatrix}, \tag{2.73}$$

where

$$M \triangleq \begin{bmatrix} -\bar{a}_{21} \\ \bar{a}_{21}\bar{a}_{33} - \bar{a}_{23}\bar{a}_{31} \end{bmatrix}. \tag{2.74}$$

Since $\bar{a}_{21} = 0$ and $\bar{a}_{21}\bar{a}_{33} - \bar{a}_{23}\bar{a}_{31} = 0$, M is not left invertible, thus $\mathcal{S}$ has infinitely many elements. In all other cases where a_{ii} and a_{jk} are unknown, $a_{kj} = 0$, and $a_{kj}a_{ll} - a_{kl}a_{lj} = 0$, where either $i = j \neq k \neq l \leq n$ or $i = k \neq l \neq l \leq n$, a similar argument shows that $\mathcal{S}$ has infinitely many elements.

6) A similar argument as the proof for 5) is used to prove this condition. The detailed formulation is omitted.

7) Consider the case where a_{12} and a_{13} are unknown. (2.54) constrains only known

parameters and thus can be disregarded. We write (2.55) and (2.56) in matrix form and obtain

$$M \begin{bmatrix} a_{12} \\ a_{13} \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_0 \end{bmatrix}, \tag{2.75}$$

where

$$M \triangleq \begin{bmatrix} -\bar{a}_{21} & -\bar{a}_{31} \\ \bar{a}_{21}\bar{a}_{33} - \bar{a}_{23}\bar{a}_{31} & \bar{a}_{22}\bar{a}_{31} - \bar{a}_{21}\bar{a}_{32} \end{bmatrix}. \tag{2.76}$$

Since M is singular, $\mathcal{S}$ has infinitely many elements. In every other case where a_{ij} and a_{lm} are unknown, and $\begin{bmatrix} -a_{ji} & -a_{ml} \\ a_{ji}a_{kk} - a_{jk}a_{ki} & a_{ml}a_{oo} - a_{mo}a_{ol} \end{bmatrix}$ is singular, where $i = l \neq j \neq m \leq n$ or $j = m \neq i \neq l \leq n$, $i \neq j \neq k \leq n$, and $l \neq m \neq o \leq n$, a similar argument shows that $\mathcal{S}$ has infinitely many elements.

8) Note that (2.54) constrains only known parameters and thus can be disregarded. Since five unknown parameters satisfy four equations, $\mathcal{S}$ has infinitely many elements.

9) Consider the case where a_{11}, a_{22}, and a_{33} are unknown. (2.54) – (2.56) have the form

$$a_{11} + a_{22} + a_{33} = -\alpha_2, \tag{2.77}$$

$$\begin{aligned} a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} &= \alpha_1 \\ &+ \bar{a}_{21}\bar{a}_{12} + \bar{a}_{13}\bar{a}_{31} + \bar{a}_{23}\bar{a}_{32}, \end{aligned} \tag{2.78}$$

$$\begin{aligned} -a_{11}a_{22}a_{33} + \bar{a}_{23}\bar{a}_{32}a_{11} + \bar{a}_{13}\bar{a}_{31}a_{22} \\ + \bar{a}_{21}\bar{a}_{12}a_{33} = \alpha_0 + \bar{a}_{13}\bar{a}_{21}\bar{a}_{32} + \bar{a}_{12}\bar{a}_{23}\bar{a}_{31}. \end{aligned} \tag{2.79}$$

Using (2.77) and (2.78), (2.79) may be two different cubic equations, thus $\mathcal{S}$ has exactly six elements generically.

Consider the case where a_{11}, a_{12}, and a_{22} are unknown. (2.54) – (2.56) have the form

$$a_{11} + a_{22} = -\alpha_2 - \bar{a}_{33}, \tag{2.80}$$

$$\begin{aligned} a_{11}a_{22} &= \alpha_1 + \bar{a}_{21}a_{12} - \bar{a}_{33}a_{11} \\ &\quad - \bar{a}_{33}a_{22} + \bar{a}_{13}\bar{a}_{31} + \bar{a}_{23}\bar{a}_{32}, \end{aligned} \tag{2.81}$$

$$\begin{aligned} -\bar{a}_{33}a_{11}a_{22} &= \alpha_0 - \bar{a}_{23}\bar{a}_{32}a_{11} - \bar{a}_{13}\bar{a}_{31}a_{22} \\ &\quad - \bar{a}_{21}\bar{a}_{33}a_{12} + \bar{a}_{13}\bar{a}_{21}\bar{a}_{32} + \bar{a}_{12}\bar{a}_{23}\bar{a}_{31}. \end{aligned} \tag{2.82}$$

Substituting (2.81) into (2.82) reduces (2.51) – (2.56) to four equations with five unknown parameters, thus $\mathcal{S}$ has infinitely many elements. In every other case where $q = 5$ and a_{ii} and a_{jj} are unknown, where $i \neq j \leq n$, a similar argument shows that $\mathcal{S}$ has infinitely many elements.

10) A similar argument as the proof for 9) is used to prove this condition. The detailed formulation is omitted.

11) A similar argument as the proof for 7) is used to prove this condition. The detailed formulation is omitted.

12) Since more than five unknown parameters satisfy five equations, $\mathcal{S}$ has infinitely many elements.

Necessity: Let

$$K \triangleq \begin{bmatrix} a_{12} & a_{13} \\ a_{13}a_{32} - a_{12}a_{33} & a_{12}a_{33} - a_{13}a_{22} \end{bmatrix}. \tag{2.83}$$

To establish necessity, we show that $\mathcal{S}$ has exactly one element in the case where one of the following mutually exclusive statements holds:

i) K is nonsingular, $q = 3$, a_{ii} is unknown, where $i \leq n$.

ii) K is nonsingular, $q = 3$, a_{ij} is unknown, and $a_{ji} \neq 0$ or $a_{kk}a_{ji} - a_{jk}a_{ki} \neq 0$, where

$i \neq j \neq k \leq n$.

iii) K is nonsingular, $q = 4$, a_{ii} and a_{jj} are unknown, and $a_{jk}a_{kj} \neq a_{ik}a_{ki}$, where $i \neq j \neq k \leq n$.

iv) K is nonsingular, $q = 4$, a_{ii} and a_{jk} are unknown, and $a_{kj} \neq 0$ or $a_{kj}a_{ll} - a_{kl}a_{lj} \neq 0$, where either $i = j \neq k \neq l \leq n$ or $i = k \neq l \neq l \leq n$.

v) K is nonsingular, $q = 4$, a_{ii} and a_{jk} are unknown, and $a_{kj} \neq 0$ or $a_{kj}(\alpha_2 - a_{jj} - a_{kk}) + a_{ij}a_{ki} \neq 0$, where $i \neq j \neq k \leq n$.

vi) K is nonsingular, $q = 4$, a_{ij} and a_{lm} are unknown, and $\begin{bmatrix} -a_{ji} & -a_{ml} \\ a_{ji}a_{kk} - a_{jk}a_{ki} & a_{ml}a_{oo} - a_{mo}a_{ol} \end{bmatrix}$ is nonsingular, where $i = l \neq j \neq m \leq n$ or $j = m \neq i \neq l \leq n$, $i \neq j \neq k \leq n$, and $l \neq m \neq o \leq n$.

vii) K is nonsingular, $q = 5$, a_{ii}, a_{jk}, and a_{mo} are unknown, and $\begin{bmatrix} -a_{kj} & -a_{om} \\ a_{kj}a_{ll} - a_{kl}a_{lj} & a_{om}a_{pp} - a_{op}a_{pm} \end{bmatrix}$ is nonsingular, where $i \leq n$, $j = m \neq k \neq o \leq n$ or $k = o \neq j \neq m \leq n$, $j \neq k \neq l \leq n$, and $m \neq o \neq p \leq n$.

In Figure 2.4, Eκ denotes condition κ) in Theorem 2.4.1. Note that $1) - 12)$ together correspond to the red region in Figure 2.4, and $i) - vi)$ together correspond to the green region in Figure 2.4. Therefore, $1) - 12)$ and $i) - vi)$ together cover all possible cases.

Figure 2.4: Cases in the green region are identifiable, whereas cases in the red region are not identifiable.

i) Consider the case where a_{11} is unknown. We write (2.54) – (2.56) in matrix form and obtain

$$Ma_{11} = \begin{bmatrix} \xi_2 \\ \xi_1 \\ \xi_0 \end{bmatrix}, \tag{2.84}$$

where

$$M \stackrel{\triangle}{=} \begin{bmatrix} 1 \\ \bar{a}_{22} + \bar{a}_{33} \\ \bar{a}_{23}\bar{a}_{32} - \bar{a}_{22}\bar{a}_{33} \end{bmatrix}. \tag{2.85}$$

Since M is left invertible, $\mathcal{S}$ has exactly one element. In the case where either a_{22} or a_{33} is unknown, a similar argument shows that $\mathcal{S}$ has exactly one element.

ii) Since $\bar{a}_{21} \neq 0$ or $\bar{a}_{21}\bar{a}_{33} - \bar{a}_{23}\bar{a}_{31} \neq 0$, the matrix M in (2.60) is left invertible, thus $\mathcal{S}$ has exactly one element. In the case where exactly one of a_{21}, a_{13}, a_{31}, a_{23}, or a_{32} is unknown, a similar argument shows that $\mathcal{S}$ has exactly one element.

iii) Since $\bar{a}_{23}\bar{a}_{32} \neq \bar{a}_{13}\bar{a}_{31}$, the matrix M in (2.65) is nonsingular, thus $\mathcal{S}$ has exactly one element. In the case where either a_{11} and a_{33} are unknown or a_{22} and a_{33} are unknown, a similar argument shows that $\mathcal{S}$ has exactly one element.

iv) Since $\bar{a}_{21} \neq 0$ or $\bar{a}_{21}\bar{a}_{33} - \bar{a}_{23}\bar{a}_{31} \neq 0$, the matrix M in (2.74) is left invertible, thus $\mathcal{S}$ has exactly one element. In all other cases where a_{ii} and a_{jk} are unknown, $a_{kj} \neq 0$ or $a_{kj}a_{ll} - a_{kl}a_{lj} \neq 0$, where either $i = j \neq k \neq l \leq n$ or $i = k \neq l \neq l \leq n$, a similar argument shows that $\mathcal{S}$ has exactly one element.

v) The detailed proof is omitted.

vi) Since the matrix M in (2.76) is nonsingular, $\mathcal{S}$ has exactly one element. In every other case where a_{ij} and a_{lm} are unknown, and

$\begin{bmatrix} -a_{ji} & -a_{ml} \\ a_{ji}a_{kk} - a_{jk}a_{ki} & a_{ml}a_{oo} - a_{mo}a_{ol} \end{bmatrix}$ is nonsingular, where $i = l \neq j \neq m \leq n$ or $j = m \neq i \neq l \leq n$, $i \neq j \neq k \leq n$, and $l \neq m \neq o \leq n$, a similar argument shows that $\mathcal{S}$ has exactly one element.

vii) The detailed proof is omitted.

□

Theorem 2.4.2: Let $n = 3$,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{2.86}$$

$q \leq n^2 + 1$ be the number of unknown entries in (A, B), and assume that (A, B, E) is a minimal realization of

$$G(z) = \begin{bmatrix} \frac{\beta_{21}z^2+\beta_{11}z+\beta_{01}}{z^3+\alpha_2 z^2+\alpha_1 z+\alpha_0} \\ \frac{\beta_{22}z^2+\beta_{12}z+\beta_{02}}{z^3+\alpha_2 z^2+\alpha_1 z+\alpha_0} \end{bmatrix}, \tag{2.87}$$

whose numerator and denominator are coprime. Then the conditions under which (A, B) is identifiable is summarized in Figure 2.5.

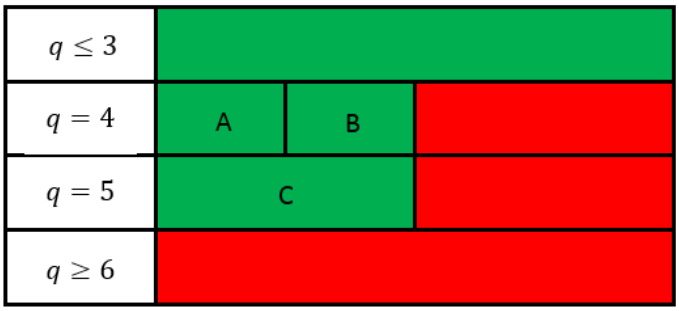

Figure 2.5: Cases in the green region are generically identifiable, whereas cases in the red region are generically not identifiable.

Region A contains the cases where the diagonal contains exactly one unknown param-

eter. Region B contains the following cases:

1. a_{11}, a_{22}, and a_{12} are unknown.
2. a_{11}, a_{22}, and a_{21} are unknown.
3. a_{11}, a_{33}, and a_{31} are unknown.
4. a_{22}, a_{33}, and a_{32} are unknown.
5. a_{12}, a_{13}, and a_{23} are unknown.
6. a_{12}, a_{13}, and a_{31} are unknown.
7. a_{12}, a_{13}, and a_{32} are unknown.
8. a_{12}, a_{23}, and a_{31} are unknown.
9. a_{12}, a_{31}, and a_{32} are unknown.
10. a_{13}, a_{23}, and a_{31} are unknown.
11. a_{13}, a_{31}, and a_{32} are unknown.
12. a_{21}, a_{23}, and a_{31} are unknown.
13. a_{21}, a_{23}, and a_{32} are unknown.
14. a_{21}, a_{31}, and a_{32} are unknown.

Region C contains the cases where the diagonal contains exactly one unknown parameter, either a_{21} is known or a_{12} and a_{13} are known, and none of the following statements are true:

1. a_{12} and a_{32} are unknown.
2. a_{12} and a_{23} are unknown.
3. a_{23} and a_{31} are unknown.

4. a_{13} and a_{21} are unknown.

The proof of Theorem 2.4.2 is similar to the proof of Theorem 2.4.1 and thus is omitted.

Theorem 2.4.3: Let $n = 4$,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \tag{2.88}$$

$q \leq n^2 + 2$ be the number of unknown entries in (A, B), and assume that (A, B, E) is a minimal realization of

$$G(z) = \begin{bmatrix} \frac{\beta_{31}z^3+\beta_{21}z^2+\beta_{11}z+\beta_{01}}{z^4+\alpha_3 z^3+\alpha_2 z^2+\alpha_1 z+\alpha_0} \\ \frac{\beta_{32}z^3+\beta_{22}z^2+\beta_{12}z+\beta_{02}}{z^4+\alpha_3 z^3+\alpha_2 z^2+\alpha_1 z+\alpha_0} \end{bmatrix}, \tag{2.89}$$

whose numerator and denominator are coprime. Then the conditions under which (A, B) is identifiable is summarized in Figure 2.6.

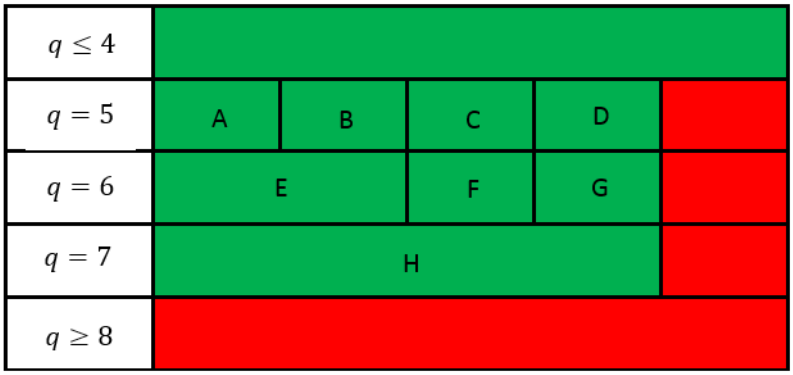

Figure 2.6: Cases in the green region are generically identifiable, whereas cases in the red region are generically not identifiable.

Region A contains the cases where the diagonal contains exactly two unknown parameters, and the remaining unknown parameter is in the same rows or columns as both unknown parameters on the diagonal. Region B contains the cases where the diagonal con-

tains exactly one unknown parameter. Region C contains the cases where every parameter on the diagonal are unknown, and all unknown parameters are in the same rows or columns. Region D contains the cases where all the following conditions are satisfied:

1. Two unknown parameters are in the same row or column as the third unknown parameter.

2. None of the unknown parameters are a_{13}, a_{14}, a_{23}, or a_{24}.

3. At least two of the unknown parameters are a_{31}, a_{32}, a_{41}, or a_{42}.

Region E contains the cases where the diagonal contains exactly one unknown parameter, and all other unknown parameters are in the same rows or columns. Region F contains the cases where all of the following conditions are satisfied:

1. The diagonal contains exactly one unknown parameter.

2. Apart from the unknown parameter on the diagonal, two unknown parameters are in the same row or column as the third unknown parameter.

3. None of the unknown parameters are a_{13}, a_{14}, a_{23}, or a_{24}.

4. At least two of the unknown parameters are a_{31}, a_{32}, a_{41}, or a_{42}.

Region G contains the case where the unknown parameters are a_{31}, a_{32}, a_{41}, and a_{42}. Region H contains the cases where the diagonal contains exactly one unknown parameter, and the remaining unknown parameters are a_{31}, a_{32}, a_{41}, and a_{42}.

The proof of Theorem 2.4.3 is similar to the proof of Theorem 2.4.1 and thus is omitted.

CHAPTER 3

Classical Approaches to CSPE

3.1 Combined State and Parameter Estimation

Consider a variation of the CSPE problem presented in Section 2.1 given by

$$x(k+1) = Ax(k) + Dw(k), \tag{3.1}$$

$$x(0) = x_0, \tag{3.2}$$

$$y_0(k) = Ex(k), \tag{3.3}$$

where

$$D = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \in \mathbb{R}^n, \tag{3.4}$$

and $w(k) \in \mathbb{R}$ is the measured excitation signal. We assume that E and D are known but A has structured uncertainty in the sense that some entries of A are known and others are unknown. Note that the assumption that E and D are known effectively fixes the basis in which A and the structure of its uncertainty are represented. Also, the fact that some entries in A are unknown makes it impossible to transform A into a canonical form. The objective is to use the measurement $y_0(k)$, where $k \geq 0$, to estimate the unknown entries of A and

the components $x_1(k), \ldots, x_n(k)$ of the state $x(k)$. This is the combined parameter and state estimation (CSPE) problem.

If the state $x(k)$ is known for all $k \geq 0$, then it is straightforward to estimate the uncertain entries of A. Likewise, if all of the entries of A are known, then standard techniques can be used to estimate the state. The difficulty of the CSPE problem stems from the fact that *both* states and parameters are unknown. Note that this problem formulation does not include either process noise or sensor noise, and thus the problem is deterministic.

3.2 Extended Kalman Filter

To provide a baseline for later developments, in this section we apply EKF to the CSPE problem.

3.2.1 Example 1: $n = 2$ and Two Unknown Entries in a Single Row

Consider (3.1)–(3.3) with

$$A = \begin{bmatrix} 0.27 & 1.17 \\ -0.8 & 0.2 \end{bmatrix}, \; x_0 = \begin{bmatrix} -23 \\ 17 \end{bmatrix}, \; E = \begin{bmatrix} 1 & 0 \end{bmatrix}, \tag{3.5}$$

assume that the entries $a_{11} = 0.27$ and $a_{12} = 1.17$ of A are unknown, and let $w(k) = 0$. To apply EKF, we first augment the dynamics (3.1) with additional equations that represent the fact that the unknown parameters are constant. The augmented system has the form

$$X(k+1) = \tilde{A}X(k), \tag{3.6}$$

$$X(0) = X_0, \tag{3.7}$$

$$y_0(k) = \tilde{E}X(k), \tag{3.8}$$

where

$$\tilde{A} \triangleq \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ -0.8 & 0.2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \tilde{E} \triangleq \begin{bmatrix} E & 0_{1\times 2} \end{bmatrix}, \quad X(k) \triangleq \begin{bmatrix} x_1(k) \\ x_2(k) \\ a_{11} \\ a_{12} \end{bmatrix}. \tag{3.9}$$

Forming the Jacobian matrix of (3.6) yields the augmented estimator system

$$\hat{X}(k+1) = \hat{\tilde{A}}(k)\hat{X}(k), \tag{3.10}$$

$$\hat{X}(0) = \hat{X}_0, \tag{3.11}$$

$$\hat{y}_0(k) = \tilde{E}\hat{X}(k), \tag{3.12}$$

where

$$\hat{\tilde{A}}(k) \triangleq \begin{bmatrix} \hat{a}_{11}(k) & \hat{a}_{12}(k) & \hat{x}_1(k) & \hat{x}_2(k) \\ -0.8 & 0.2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \hat{X}(k) \triangleq \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \\ \hat{a}_{11}(k) \\ \hat{a}_{12}(k) \end{bmatrix}, \tag{3.13}$$

$\hat{x}_1(k)$, $\hat{x}_2(k)$ denote estimates of $x_1(k)$, $x_2(k)$ and $\hat{a}_{11}(k)$, $\hat{a}_{12}(k)$ denote estimates of a_{11}, a_{12}. The Kalman filter is then applied to (3.10)–(3.12).

To evaluate the accuracy of EKF, we define the relative initial estimation errors

$$\xi_x \triangleq \frac{||\hat{x}_2(0) - x_2(0)||}{||x_2(0)||}, \quad \xi_a \triangleq \frac{||\hat{a}(0) - a||}{||a||}, \tag{3.14}$$

where the true parameter vector a and its estimate $\hat{a}$ are defined as

$$a \triangleq \begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix}, \quad \hat{a} \triangleq \begin{bmatrix} \hat{a}_{11} \\ \hat{a}_{12} \end{bmatrix}. \tag{3.15}$$

Note that $\xi_x = 0$ if and only if $\hat{x}_2(0) = x_2(0)$, and $\xi_a = 0$ if and only if $\hat{a}_{11}(0) = a_{11}$ and $\hat{a}_{12}(0) = a_{12}$.

In order to assess the performance of EKF, we consider 10000 randomly generated initial estimates of the unmeasured state and the uncertain entries of A. Since x_1 is measured, we set $\hat{x}_1(0) = x_1(0)$, and we choose initial estimates $(\hat{x}_2(0), \hat{a}_{11}(0), \hat{a}_{12}(0))$ such that $\xi_x, \xi_a \in (0, 2)$. Using the notation of [16], we set the initial covariance matrix to be $P(0) = 10000 I_4$ and choose the tuning parameters $Q = 10^{-2} I_{2l_x}$ and $R = 0$. Figure 3.1 shows that, for all 10000 initial estimates, none of the estimates $\hat{a}(1000)$ are within 10% of the true parameters a.

3.3 Unscented Kalman Filter

3.3.1 Example 2: Example 1 Revisited

In Example 1, the first row of the Jacobian matrix (3.10) gives an erroneous factor of 2 as compared to $\tilde{A}$, which is consistent with the resulting poor performance. Therefore, we revisit Example 1 by defining

$$\tilde{A}(k) \triangleq \begin{bmatrix} \hat{a}_{11}(k) & \hat{a}_{12}(k) & 0 & 0 \\ -0.8 & 0.2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.16}$$

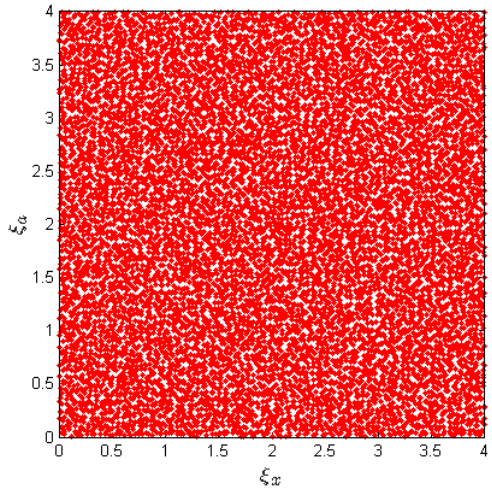

Figure 3.1: Application of EKF to Example 1. EKF is applied with 10000 randomly generated initial estimates $(\hat{x}_2(0), \hat{a}_{11}(0), \hat{a}_{12}(0))$ using the measurements $y_0(k) = x_1(k)$ over the interval $k \in [0, 1000]$. Trials where EKF estimates both components of a within 10% relative error at step $k = 1000$ are labeled with cyan; trials where EKF estimates exactly one component of a within 10% relative error at step $k = 1000$ are labeled with black; and trials where EKF estimates neither of the components of a within 10% relative error at step $k = 1000$ are labeled with red. 100% of the trials are red. Note: In all subsequent examples, cyan, black, and red indicate, respectively, trials where all, at least one, and none of the components of a satisfy the accuracy specification.

for (3.10)–(3.12) and applying the unscented Kalman filter to the augmented system. In order to assess the performance of UKF with (3.16), we consider 10000 randomly generated initial estimates of the unmeasured state and the uncertain entries of A. Since x_1 is measured, we set $\hat{x}_1(0) = x_1(0)$, and we choose initial estimates $(\hat{x}_2(0), \hat{a}_{11}(0), \hat{a}_{12}(0))$ such that $\xi_x, \xi_a \in (0, 2)$. Using the notation of [17], we set the initial covariance matrix to be $P(0) = 10000 I_4$ and choose the tuning parameters $\alpha = 1$, $\kappa = 0$, $\beta = 2$, $Q = 10^{-2} I_{2l_x}$, and $R = 0$. Figure 3.2 shows that, for all 10000 initial estimates, none of the estimates $\hat{a}(1000)$ are within 10% of the true parameters a.

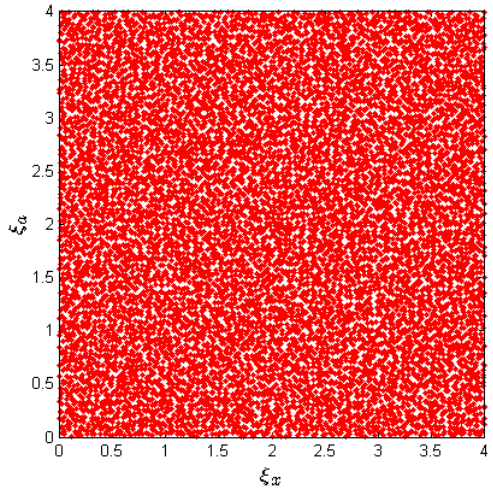

Figure 3.2: Application of UKF with (3.16) to Example 2. UKF with (3.16) is applied with 10000 randomly generated initial estimates $(\hat{x}_2(0), \hat{a}_{11}(0), \hat{a}_{12}(0))$ using the measurements $y_0(k) = x_1(k)$ over the interval $k \in [0, 1000]$. 100% of the trials are red.

3.4 Unscented Kalman Filter with State-Dependent Coefficients

3.4.1 Example 3: Example 1 Revisited

We revisit Example 1 by defining the state-dependent matrix

$$\tilde{A}(k) \triangleq \begin{bmatrix} \alpha_1\hat{a}_{11}(k) & \alpha_2\hat{a}_{12}(k) & (1-\alpha_1)\hat{x}_1(k) & (1-\alpha_2)\hat{x}_2(k) \\ -0.8 & 0.2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.17}$$

for (3.10)–(3.12), where $\alpha \in \mathbb{R}$, and applying UKF with (3.17). Note that (3.16) corresponds to setting $\alpha_1 = \alpha_2 = 1$. In order to assess the performance of UKF with (3.17), we reconsider the 10000 randomly generated initial estimates, initial covariance, and tuning parameters as in Example 2. Setting $\alpha_1 = \alpha_2 = 0.5$, Figure 3.3 shows that, for all 10000 initial estimates, all of the estimates $\hat{a}(1000)$ are within 10% of the true parameters

a. In most of the trials where estimation of the unknown entries is successful, the estimates converge within approximately 500 time steps.

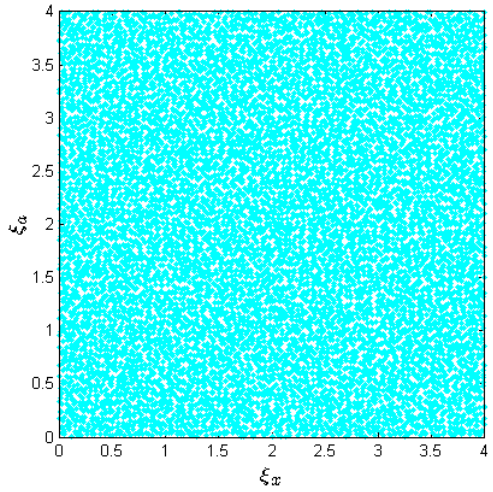

Figure 3.3: Application of UKF with (3.17) to Example 3. UKF with (3.17) is applied with 10000 randomly generated initial estimates $(\hat{x}_2(0), \hat{a}_{11}(0), \hat{a}_{12}(0))$ using the measurements $y_0(k) = x_1(k)$ over the interval $k \in [0, 1000]$. 100% of the trials are cyan.

To test the effect of α_1 and α_2, we consider 11 linearly spaced values of $\alpha_1 \in [-3, 3]$ and 11 linearly spaced values of $\alpha_2 \in [-3, 3]$. For each choice of α_1, α_2, we record the number of 10000 trials for which UKF with (3.17) estimates a within 10% relative error. Figure 3.4 shows that, generally, if $\alpha_1 < 1$ and $\alpha_2 < 1$, all of the estimates $\hat{a}(1000)$ are within 10% of the true parameters a. Otherwise, none of the estimates $\hat{a}(1000)$ are within 10% of the true parameters a. This example shows that, compared to Example 2, the state-dependent coefficient can significantly improve the performance of UKF depending on the choice of α_1 and α_2.

In all subsequent UKF examples, we set $\alpha_1 = \cdots = \alpha_p = 0.5$, where p is the number of unknown entries in A.

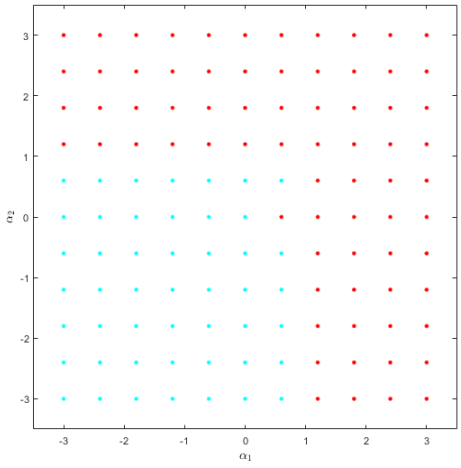

Figure 3.4: Application of UKF with (3.17) to Example 4. α_1 and α_2 is varied from -3 to 3 in increments of 0.6. For each pair of values (α_1, α_2) and 10000 randomly generated initial estimates $(\hat{x}_2(0), \hat{a}_{11}(0), \hat{a}_{12}(0))$, UKF with (3.17) is applied using the measurements $y_0(k) = x_1(k)$ over the interval $k \in [0, 1000]$. A cyan dot indicates that UKF estimates both components of a within 10% relative error at step $k = 1000$ in 100% of the trials, and a red dot indicates that UKF estimates neither of the components of a within 10% relative error in 100% of the trials. All of the trials are either cyan or red.

3.4.2 Example 4: $n = 3$ and One Unknown Entry

Consider (3.1)–(3.3) with

$$A = \begin{bmatrix} 0.51 & -0.285 & 0.05 \\ -0.012 & 0.34 & 1 \\ 0.03 & -0.88 & 0.34 \end{bmatrix}, \; x_0 = \begin{bmatrix} -23 \\ 67 \\ -31 \end{bmatrix}, \; E = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \tag{3.18}$$

assume that one entry in the first row of A is unknown, and let $w(k) = 0$. To apply UKF, we define the augmented system (3.10)–(3.12) with $\tilde{A}$ constructed as in (3.17) and $X, \tilde{E}$ constructed as in (3.13). Let $\hat{x}_1(k)$, $\hat{x}_2(k)$, $\hat{x}_3(k)$ be estimates of $x_1(k)$, $x_2(k)$, $x_3(k)$, and, for $i \in \{1, 2, 3\}$, let $\hat{a}_{1i}(k)$ be an estimate of a_{1i}. Define

$$\xi_x \triangleq \frac{||\hat{x}_{\mathrm{u}}(0) - x_{\mathrm{u}}(0)||}{||x_{\mathrm{u}}(0)||}, \quad \xi_a \triangleq \frac{||\hat{a}_{1i}(0) - a_{1i}||}{||a_{1i}||}, \tag{3.19}$$

where the unmeasured states and their estimates are defined by

$$x_{\mathrm{u}} \triangleq \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}, \quad \hat{x}_{\mathrm{u}} \triangleq \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix}. \tag{3.20}$$

Using the same UKF tuning parameters as in Example 3, we consider 10000 randomly generated initial estimates $(\hat{x}_2(0), \hat{x}_3(0), \hat{a}_{1i}(0))$ of initial estimates such that $\xi_x, \xi_a \in (0, 2)$. Figure 3.5(a) shows that 72.53% of the estimates $\hat{a}_{11}(1000)$ are within 10% of the true parameter a_{11}. In contrast, Figure 3.5(b) and Figure 3.5(c) show that 4.68% and 6.09% of the estimates $\hat{a}_{12}(1000)$ and $\hat{a}_{13}(1000)$ are within 10% of the true parameters a_{12} and a_{13}, respectively. In most of the trials where estimation of the unknown entries is successful, the estimates converge within approximately 500 time steps.

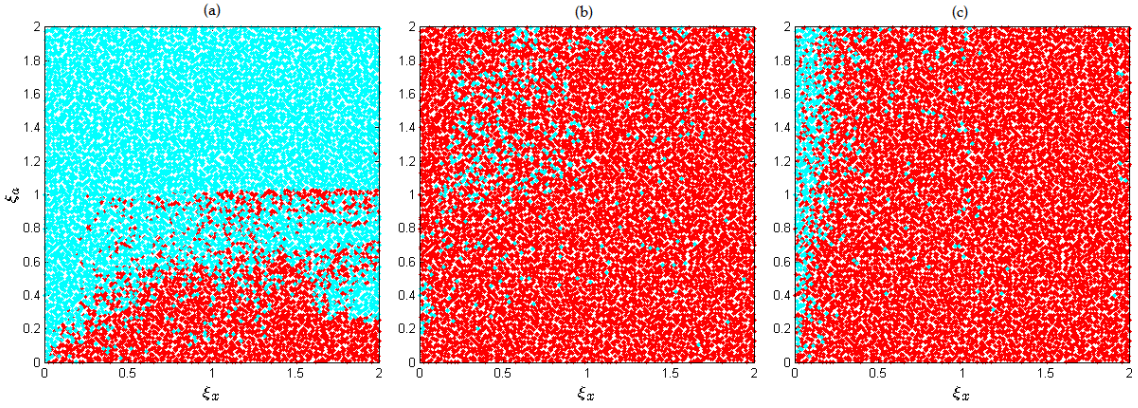

Figure 3.5: Application of UKF with (3.17) to Example 4. In (a), UKF with (3.17) is applied with 10000 randomly generated initial estimates $(\hat{x}_2(0), \hat{x}_3(0), \hat{a}_{11}(0))$. In (b), UKF with (3.17) is applied with 10000 randomly generated initial estimates $(\hat{x}_2(0), \hat{x}_3(0), \hat{a}_{12}(0))$. In (c), UKF with (3.17) is applied with 10000 random initial estimates $(\hat{x}_2(0), \hat{x}_3(0), \hat{a}_{13}(0))$. In all three cases, KF with (3.17) uses the measurements $y_0(k) = x_1(k)$ over the interval $k \in [0, 1000]$. 72.53%, 4.68%, and 6.09% of the trials in (a), (b), and (c), respectively, are cyan.

Note that both Example 3 and Example 4 involve a total of three unknown quantities in A and x_0. It is thus reasonable to expect that the performance of UKF would be similar for both examples. However, Example 3 involves two unknown constants and one unmeasured state, whereas Example 4 involves one unknown constant and two unmeasured states. This

distinction is consistent with the fact that UKF performs worse for Example 4 than for Example 3.

3.4.3 Example 5: $n = 3$ and Three Unknown Entries in a Single Row

We revisit Example 4 by assuming that all of the entries in the first row of A are jointly unknown. To apply UKF, we define the augmented system (3.10)–(3.12) with $\tilde{A}$ constructed as in (3.17) and $X, \tilde{E}$ constructed as in (3.13). Let $\hat{x}_1(k)$, $\hat{x}_2(k)$, $\hat{x}_3(k)$ denote estimates of $x_1(k)$, $x_2(k)$, $x_3(k)$, and let $\hat{a}_{11}(k)$, $\hat{a}_{12}(k)$, $\hat{a}_{13}(k)$ denote estimates of a_{11}, a_{12}, a_{13}. Define the true parameter vector a, its estimate $\hat{a}$, the unmeasured states x_{u}, and its estimates $\hat{x}_{\mathrm{u}}$, as

$$x_{\mathrm{u}} \triangleq \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}, \quad \hat{x}_{\mathrm{u}} \triangleq \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix}, \quad a \triangleq \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix}, \quad \hat{a} \triangleq \begin{bmatrix} \hat{a}_{11} \\ \hat{a}_{12} \\ \hat{a}_{13} \end{bmatrix}. \tag{3.21}$$

As in the case of Example 4 and using the same tuning parameters for UKF with (3.17), we consider 10000 randomly generated initial estimates $(\hat{x}_2(0), \hat{x}_3(0), \hat{a}_{11}(0), \hat{a}_{12}(0), \hat{a}_{13}(0))$ such that $\xi_x, \xi_a \in (0, 2)$.

Figure 3.6 shows that UKF with (3.17) estimates at least one component of a within 10% error in 0.20% of the trials and none of the components of a within 10% error in 99.80% of the trials. Note that, while Example 4 and Example 5 concern the same unknown entries, the three entries in Example 5 are estimated concurrently, whereas the three entries in Example 4 are estimated separately assuming the remaining entries are known. This distinction is consistent with the fact that UKF with (3.17) performs worse for Example 5 than for Example 4.

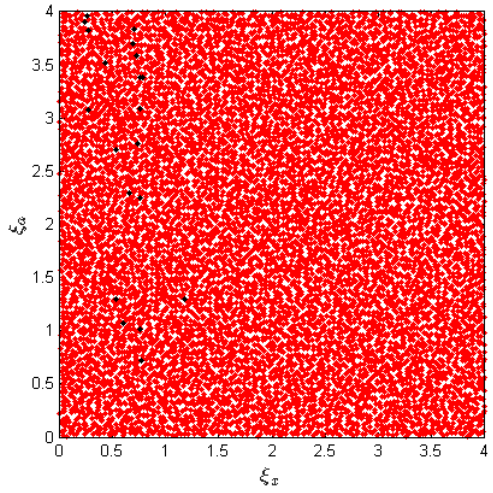

Figure 3.6: Application of UKF with (3.17) to Example 5. KF with (3.17) is applied with 10000 randomly generated initial estimates $(\hat{x}_2(0), \hat{x}_3(0), \hat{a}_{11}(0), \hat{a}_{12}(0), \hat{a}_{13}(0))$ using the measurements $y_0(k) = x_1(k)$ over the interval $k \in [0, 1000]$. 0.20% of the trials are black and 99.80% of the trials are red.

3.4.4 Example 6: Application to Linearized Longitudinal Aircraft Dynamics

We now consider the CSPE problem for linearized longitudinal aircraft dynamics. Consider the continuous-time linearized longitudinal aircraft dynamics matrix

$$A_{\mathrm{c}} = \begin{bmatrix} -0.0505 & -9.49 & -0.0127 & -32.2 \\ -0.00236 & -2.45 & 0.962 & 0 \\ 0.0179 & -42.0 & -3.44 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{3.22}$$

Discretizing the dynamics with the time step $T_s = 0.01$ s yields the discrete-time linearized longitudinal aircraft dynamics matrix

$$A = \begin{bmatrix} 0.999 & -0.0934 & -0.00216 & -0.322 \\ -1.49 \times 10^{-6} & 0.974 & 0.00933 & 2.86 \times 10^{-7} \\ 0.000176 & -0.408 & 0.964 & -2.85 \times 10^{-5} \\ 8.86 \times 10^{-7} & -0.00206 & 0.00982 & 1 \end{bmatrix}. \tag{3.23}$$

Consider (3.1)–(3.3) with

$$x_0 = \begin{bmatrix} -50 \\ 30 \\ -10 \\ 95 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \tag{3.24}$$

assume that the entries $a_{11} = 0.999$ and $a_{12} = -0.0934$ of A are unknown, and let $w(k) = 0$. To apply UKF, we define the augmented system (3.10)–(3.12) with $\tilde{A}$ constructed as in (3.17) and $X, \tilde{E}$ constructed as in (3.13). Furthermore, define the true parameter vector a, its estimate $\hat{a}$, the unmeasured states x_{u}, and its estimates $\hat{x}_{\mathrm{u}}$, as

$$x_{\mathrm{u}} \triangleq \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \hat{x}_{\mathrm{u}} \triangleq \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{bmatrix}, \quad a \triangleq \begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix}, \quad \hat{a} \triangleq \begin{bmatrix} \hat{a}_{11} \\ \hat{a}_{12} \end{bmatrix}. \tag{3.25}$$

We consider 10000 randomly generated initial estimates $(\hat{x}_2(0), \hat{x}_3(0), \hat{x}_4(0), \hat{a}_{11}(0), \hat{a}_{12})$ with UKF such that $\xi_x, \xi_a \in (0, 2)$. Using the notation of [17], we set the initial covariance matrix to be $P(0) = 10^{-4} I_4$ and choose the tuning parameters $\alpha = 1$, $\kappa = 0$, $\beta = 2$, $Q = 10^{-2} I_{2l_x}$, and $R = 0$. Figure 3.7 shows that 0.04% of the estimates $\hat{a}$ are within 10% of both components of the true parameters a, 2.76% of $\hat{a}$ are within 10% of at least one component of a, and 97.20% of $\hat{a}$ are within 10% of none of the components of a. In most

of the trials where estimation of the unknown entries is successful, the estimates converge within approximately 500 time steps, that is, 5 s.

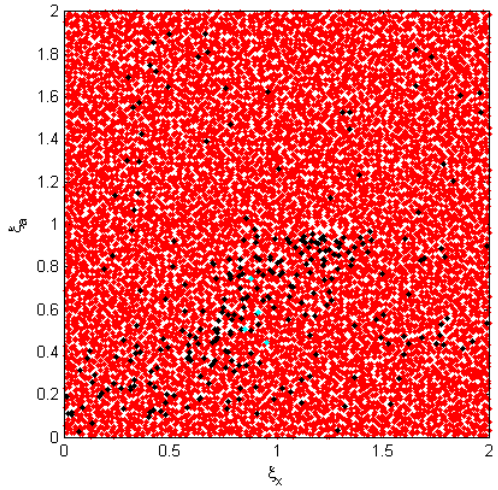

Figure 3.7: Application of KF with (3.17) to Example 11. KF with (3.17) is applied with 10000 randomly generated initial estimates $(\hat{x}_2(0), \hat{x}_3(0), \hat{x}_4(0), \hat{a}_{11}(0), \hat{a}_{12}(0))$ using the measurements $y_0(k) = x_1(k)$ over the interval $k \in [0, 1000]$. 0.04% of the trials are cyan, 2.76% of the trials are black, and 97.20% of the trials are red.

Examples 3–6 suggest that, while UKF with (3.17) can achieve reasonably accurate parameter estimation for CSPE with $n = 2$, the performance deteriorates drastically for CSPE with $n \geq 3$. This motivates the need to develop parameter estimation algorithms that are more effective for CSPE problems with $n \geq 3$.

CHAPTER 4

Retrospective Cost Parameter Estimation

4.1 Subsystem Estimation Framework

Consider the *main system* G shown in Figure 4.1 with the realization

$$x(k+1) = A_0 x(k) + Bu(k) + Dw(k), \tag{4.1}$$

$$y(k) = Cx(k), \tag{4.2}$$

$$y_0(k) = Ex(k), \tag{4.3}$$

where $x(k) \in \mathbb{R}^{l_x}$ is the main system state, $y(k) \in \mathbb{R}^{l_y}$ is the main system output, $u(k) \in \mathbb{R}^{l_u}$ is the main system input, $w(k) \in \mathbb{R}^{l_w}$ is the known excitation signal, and $y_0(k) \in \mathbb{R}^{l_z}$ is the main system measurement. The matrix A_0 is the *nominal dynamics matrix*. The main system (4.1)–(4.3) is interconnected with the *unknown subsystem* G_{s} modeled by

$$x_{\mathrm{s}}(k+1) = A_{\mathrm{s}} x_{\mathrm{s}}(k) + B_{\mathrm{s}} y(k), \tag{4.4}$$

$$u(k) = C_{\mathrm{s}} x_{\mathrm{s}}(k) + D_{\mathrm{s}} y(k), \tag{4.5}$$

where $x_{\mathrm{s}}(k) \in \mathbb{R}^{l_{x_{\mathrm{s}}}}$ is the unknown subsystem state. Together, (4.1)–(4.5) represent the *true system*.

Next, the *main system model* $\hat{G}$ has the realization

$$\hat{x}(k+1) = A_0\hat{x}(k) + B\hat{u}(k) + Dw(k), \tag{4.6}$$

$$\hat{y}(k) = C\hat{x}(k), \tag{4.7}$$

$$\hat{y}_0(k) = E\hat{x}(k), \tag{4.8}$$

where $\hat{x}(k) \in \mathbb{R}^{l_x}$ is the main system model state, $\hat{y}(k) \in \mathbb{R}^{l_y}$ is the main system model output, $\hat{u}(k) \in \mathbb{R}^{l_u}$ is the main system model input, and $\hat{y}_0(k) \in \mathbb{R}^{l_z}$ is the main system model measurement. The main system model is interconnected with the *subsystem model*

$$\hat{u}(k) = \hat{G}_{\mathrm{s}}(\mathbf{q})\hat{y}(k), \tag{4.9}$$

where $\mathbf{q}$ is the forward shift operator. Equations (4.6)–(4.9) represent the modeled system. The subsystem estimation problem is represented by the block diagram in Figure 4.1, where the goal is to estimate the subsystem model $\hat{G}_{\mathrm{s}}$ by minimizing a cost function based on the performance variable

$$z(k) \stackrel{\triangle}{=} \hat{y}_0(k) - y_0(k) \in \mathbb{R}^{l_z}. \tag{4.10}$$

For the subsystem estimation problem, we assume that the unknown subsystem input y and the unknown subsystem output u are not measured, and thus G_{s} is inaccessible. The input $\hat{y}$ of the subsystem model $\hat{G}_{\mathrm{s}}$ is computed, and the input $\hat{u}$ of the main system model $\hat{G}$ is estimated. Then, $\hat{u}$ and $\hat{y}$ are used to construct $\hat{G}_{\mathrm{s}}$, which is an estimate of G_{s}.

For parameter estimation, we assume that $G_{\mathrm{s}} = D_{\mathrm{s}}$ is static, and thus (4.4), (4.5) become

$$u(k) = D_{\mathrm{s}}y(k). \tag{4.11}$$

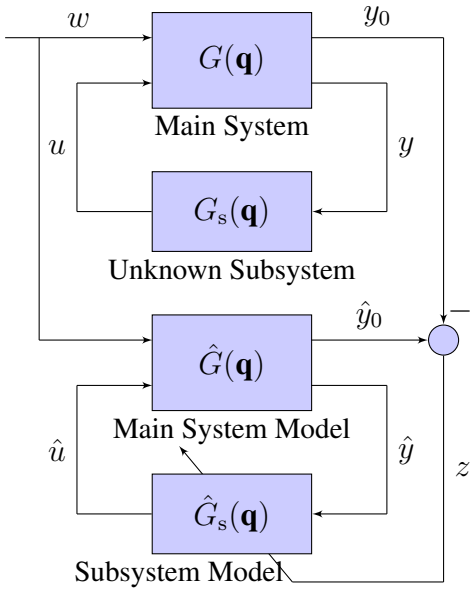

Figure 4.1: Subsystem estimation framework for RCPE.

In this case, x satisfies

$$x(k+1) = Ax(k) + Dw(k), \tag{4.12}$$

where the dynamics matrix of the true system is given by

$$A = A_0 + BD_{\mathrm{s}}C. \tag{4.13}$$

Note that the decomposition (4.13) represents the matrix A in Examples 1–5, where the uncertain entries of A are the entries of D_{s} and the corresponding entries of A_0 are set to zero. However, (4.13) can be used to model uncertain entries in A with nonzero nominal values, in which case each entry of D_{s} represents an offset from the nominal value. Consequently, the nominal values of the uncertain entries of A, which are given by the corresponding entries of A_0, can be viewed as estimates of the uncertain entries of A that correspond to $D_{\mathrm{s}} = 0$. Finally, if $w = 0$, then (4.1)–(4.5) is equivalent to the CSPE problem (3.1)–(3.3).

Let p be the number of uncertain entries in A, let q be the number of rows of A in which

they appear, and let r be the number of columns of A in which they appear. The expression (4.13) can be used to represent uncertain entries in A if and only if $p = qr$. This condition is equivalent to saying that, by reordering the rows and columns of A, the uncertain entries of A form a square or rectangular block of A. For example, uncertainty in a_{11} and a_{13} for a third-order system, which corresponds to $p = 2$, $q = 1$, and $r = 2$, can be represented using

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D_{\mathrm{s}} \in \mathbb{R}^{1\times 2}, \tag{4.14}$$

whereas uncertainty in a_{11} and a_{23}, which corresponds, to $p = 2$, $q = 2$, and $r = 2$, cannot be represented by (4.13). In the case where $p \neq qr$, (4.13) can be replaced by

$$A = A_0 + \begin{bmatrix} B_1 & \cdots & B_l \end{bmatrix} \begin{bmatrix} D_{\mathrm{s},1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_{\mathrm{s},l} \end{bmatrix} \begin{bmatrix} C_1 \\ \vdots \\ C_l \end{bmatrix} = A_0 + \tilde{B} D_{\mathrm{s}} \tilde{C}, \tag{4.15}$$

where $l \geq 2$. Note that, D_{s} in (4.15) has a block-diagonal structure, thus estimation of D_{s} entails estimation of both $D_{\mathrm{s},1}, \ldots, D_{\mathrm{s},l}$ and the zeros. In this case, we treat the block-diagonal matrix as fully populated, and we ignore the estimates of the off-block-diagonal entries, which are known to be zero.

4.2 Retrospective Cost Parameter Estimation

4.2.1 Subsystem Model

For static parameter estimation, the subsystem model is given by

$$\hat{u}(k) = \hat{D}_{\mathrm{s}}(k)\hat{y}(k). \tag{4.16}$$

We rewrite (4.16) as

$$\hat{u}(k) = \Phi(k)\hat{\theta}(k), \tag{4.17}$$

where the regressor matrix $\Phi(k)$ is defined by

$$\Phi(k) \triangleq \hat{y}(k)^{\mathrm{T}} \otimes I_{l_u} \in \mathbb{R}^{l_u \times l_\theta} \tag{4.18}$$

and the unknown entries of A are written as

$$\hat{\theta}(k) \triangleq \mathrm{vec}(\hat{D}_{\mathrm{s}}(k)) \in \mathbb{R}^{l_\theta}, \tag{4.19}$$

where $l_\theta \triangleq l_u l_y$, "$\otimes$" is the Kronecker product, and "vec" is the column-stacking operator.

4.2.2 Retrospective Performance Variable

We define the retrospective input

$$\tilde{u}(k-1) = \Phi(k-1)\hat{\theta} \tag{4.20}$$

and the corresponding retrospective performance variable

$$\hat{z}(k) \triangleq z(k) + \Phi_{\mathrm{f}}(k-1)\hat{\theta} - \hat{u}_{\mathrm{f}}(k-1), \tag{4.21}$$

where $\hat{\theta} \in \mathbb{R}^{l_\theta}$ is determined by optimization below, and $\Phi_{\mathrm{f}}(k-1) \in \mathbb{R}^{l_z \times l_\theta}$ and $\hat{u}_{\mathrm{f}}(k-1) \in \mathbb{R}^{l_z}$ are filtered versions of $\Phi(k-1)$ and $\hat{u}(k-1)$, respectively, defined by

$$\Phi_{\mathrm{f}}(k-1) \triangleq G_{\mathrm{f}}(\mathbf{q})\Phi(k-1), \quad \hat{u}_{\mathrm{f}}(k-1) \triangleq G_{\mathrm{f}}(\mathbf{q})\hat{u}(k-1). \tag{4.22}$$

The filter G_{f} has the form

$$G_{\mathrm{f}}(\mathbf{q}) \triangleq D_{\mathrm{f}}^{-1}(\mathbf{q})N_{\mathrm{f}}(\mathbf{q}), \tag{4.23}$$

where D_{f} and N_{f} are polynomial matrices and D_{f} is monic. The choice of these filters is discussed below.

4.2.3 Retrospective Cost Function

Using the retrospective performance variable $\hat{z}(k)$, we define the retrospective cost function

$$J(k,\hat{\theta}) \triangleq \sum_{i=1}^{k} \hat{z}^{\mathrm{T}}(i)R_z\hat{z}(i) + (\hat{\theta} - \theta(0))^{\mathrm{T}}R_\theta(\hat{\theta} - \theta(0)), \tag{4.24}$$

where R_z and R_θ are positive definite. The following result is a restatement of standard recursive least squares optimization.

Proposition: Let $P(0) = R_\theta^{-1}$. Then, for all $k \geq 1$, the retrospective cost function

(4.24) has a unique global minimizer $\theta(k)$, which is given by

$$\hat{\theta}(k) = \hat{\theta}(k-1) - P(k-1)\Phi_{\mathrm{f}}^{\mathrm{T}}(k-1)\Gamma^{-1}(k-1)[\Phi_{\mathrm{f}}(k-1)\hat{\theta}(k-1) + z_{\mathrm{f}}(k) - u_{\mathrm{f}}(k-1)], \tag{4.25}$$

$$P(k) = P(k-1) - P(k-1)\Phi_{\mathrm{f}}^{\mathrm{T}}(k-1)\Gamma^{-1}(k-1)\Phi_{\mathrm{f}}(k-1)P(k-1), \tag{4.26}$$

where

$$\Gamma(k-1) \stackrel{\triangle}{=} R_z^{-1} + \Phi_{\mathrm{f}}(k-1)P(k-1)\Phi_{\mathrm{f}-1}^{\mathrm{T}}(k-1). \tag{4.27}$$

4.2.4 Online Update of G_{f}

Note that the retrospective performance variable (4.21) can be rewritten as

$$\hat{z}(k) = z(k) - G_{\mathrm{f}}(\mathbf{q})\hat{\mu}(k-1), \tag{4.28}$$

where

$$\hat{\mu}(k-1) \stackrel{\triangle}{=} \hat{u}(k-1) - \tilde{u}(k-1). \tag{4.29}$$

The signal $\hat{\mu}$ can be viewed as a virtual exogenous input, as shown in Figure 4.2.

It can be seen from (4.28) that $\hat{z}$ is the residual of the fit between z and the output of G_{f} with input $\hat{\mu}$. However, the actual transfer function from $\hat{\mu}$ to z is given by

$$\tilde{G}_{\hat{y}_0\hat{\mu}}(\mathbf{q}) \sim \left[\begin{array}{c|c} A_0 + B\hat{D}_{\mathrm{s}}C & B \\ \hline E & 0 \end{array}\right]. \tag{4.30}$$

Consequently, minimizing $\hat{z}$ produces the value of $\hat{\theta}$ and thus the value of $\hat{D}_{\mathrm{s}}$ that optimally

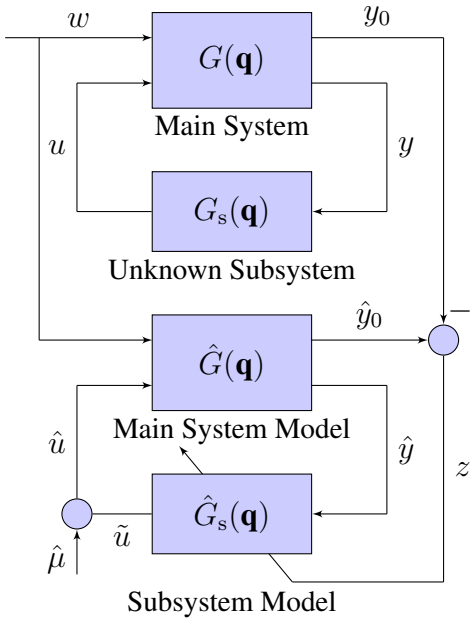

Figure 4.2: Subsystem estimation framework showing the virtual exogenous input $\hat{\mu}$.

fits $\tilde{G}_{\hat{y}_0\hat{\mu}}$ to G_{f}. Therefore, a desirable choice of G_{f} is

$$\tilde{G}^*(\mathbf{q}) \sim \left[\begin{array}{c|c} A_0 + BD_{\mathrm{s}}C & B \\ \hline E & 0 \end{array}\right]. \tag{4.31}$$

Since D_{s} is unknown, however, (4.31) cannot be implemented in practice. Thus, in all subsequent applications of RCPE, we use the time-varying filter

$$G_{\mathrm{f}}(\mathbf{q}, \hat{D}_{\mathrm{s}}(k-1)) \sim \left[\begin{array}{c|c} A_0 + B\hat{D}_{\mathrm{s}}(k-1)C & B \\ \hline E & 0 \end{array}\right]. \tag{4.32}$$

Note that, if $\hat{D}_{\mathrm{s}}(k-1) = D_{\mathrm{s}}$, then $G_{\mathrm{f}}(\mathbf{q}, D_{\mathrm{s}}) = \tilde{G}^*(\mathbf{q})$.

4.2.5 Data-window Reiteration

In order to enhance the accuracy of the estimates $\hat{D}_{\mathrm{s}}(k)$ of D_{s}, RCPE is applied multiple times to a given data set consisting of k_{f} data points. In the first iteration, we apply RCPE

with $G_{\mathrm{f}}(\mathbf{q}, \hat{D}_s(k-1))$ given by (4.32) initialized with $\hat{D}_s(0) = 0$. In addition, the entries of the nominal dynamics matrix A_0 in both (4.32) and the model (4.6) are set to the initial estimates of the unknown parameters. In subsequent iterations, we apply RCPE to the same data set with $\hat{D}_s(0)$ given by $\hat{D}_s(k_{\mathrm{f}})$ from the previous iteration and with A_0 replaced by $A_0 + B\hat{D}_s(k_{\mathrm{f}})C$.

4.3 RCPE with Known Initial State and No Excitation

In this section, we apply RCPE to the CSPE problem assuming the initial state is known and $w(k) = 0$.

4.3.1 Example 7: $n = 3$ and Three Unknown Entries in a Single Row

We revisit Example 5 with RCPE assuming that the initial state is known. We thus set $\hat{x}(0) = x_0$, which implies $\xi_x = 0$, and we choose 100 initial estimates $(\hat{a}_{11}(0), \hat{a}_{12}(0), \hat{a}_{13}(0))$ such that $\xi_a \in (0, 2)$. For all trials, we use the tuning parameters $R_\theta = I_{l_\theta}$, $k_{\mathrm{f}} = 100$, and $n_{\mathrm{u}} = 4$. Figure 4.3 shows that, in all trials, the RCPE estimates of both components of a are within 10% error. Note that, since $k_{\mathrm{f}} = 100$, in all trials where estimation of the unknown entries is successful, the estimates converge within 100 time steps.

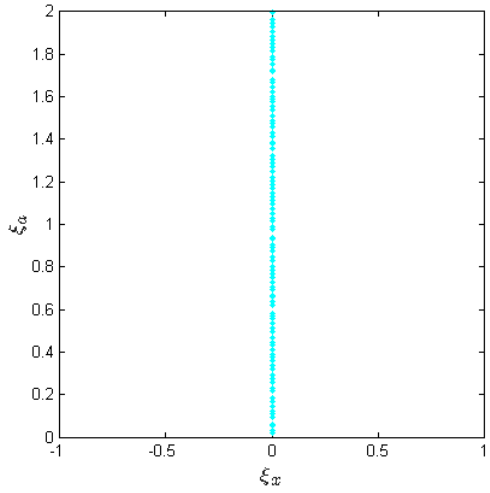

Figure 4.3: Application of RCPE to Example 7 assuming the initial state is unknown. RCPE is applied with 100 random initial estimates $(\hat{a}_{11}(0), \hat{a}_{12}(0)), \hat{a}_{13}(0))$ using the measurements $y_0(k) = x_1(k)$ over the interval $k \in [0, 100]$ and setting $\hat{x}(0) = x_0$. 100% of the trials are cyan.

4.3.2 Example 8: $n = 8$ and Eight Unknown Entries in a Single Row

Consider (3.1)–(3.3) with

$$A = \begin{bmatrix} 0.29 & 0.43 & 0.26 & 1.6 & 0.22 & -1.02 & -0.35 & -1.31 \\ 0.04 & 0.57 & 0.56 & 0.92 & -0.81 & -0.12 & 0.13 & -0.9 \\ 0.14 & 0.49 & 1.43 & 0.55 & -0.22 & -0.71 & -0.53 & -1.05 \\ -0.33 & -0.12 & -0.31 & -1.18 & 0.77 & 0.34 & 0.72 & 1.3 \\ -0.59 & 0.51 & 0.32 & 0.97 & 0.31 & -0.06 & -0.45 & -0.89 \\ 0.49 & -0.48 & -1.19 & -2.08 & 0.55 & 1.36 & 0.43 & 1.88 \\ 0.16 & -0.48 & -1.39 & -1.68 & 0.58 & 0.8 & 1.12 & 1.98 \\ 0 & 0.6 & 0.2 & 0.27 & -0.21 & -0.27 & -0.83 & 0.27 \end{bmatrix}, \; x_0 = \begin{bmatrix} -23 \\ 67 \\ -31 \\ 5 \\ 44 \\ -81 \\ 41 \\ -17 \end{bmatrix}, \tag{4.33}$$

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

assume that the entries $a_{11} = 0.29$, $a_{12} = 0.43$, $a_{13} = 0.26$, $a_{14} = 1.6$, $a_{15} = 0.22$, $a_{16} = -1.02$, $a_{17} = -0.35$, and $a_{18} = -1.31$ of A are unknown, and let $w(k) = 0$. Define

$$x_{\mathrm{u}} \triangleq \begin{bmatrix} x_2 \\ \vdots \\ x_8 \end{bmatrix}, \quad \hat{x}_{\mathrm{u}} \triangleq \begin{bmatrix} \hat{x}_2 \\ \vdots \\ \hat{x}_8 \end{bmatrix}, \quad a \triangleq \begin{bmatrix} a_{11} \\ \vdots \\ a_{18} \end{bmatrix}, \quad \hat{a} \triangleq \begin{bmatrix} \hat{a}_{11} \\ \vdots \\ \hat{a}_{18} \end{bmatrix}. \tag{4.34}$$

Assuming the initial state is known, we set $\hat{x}(0) = x_0$, which implies $\xi_x = 0$, and choose 100 initial estimates $(\hat{a}_{11}(0), \hat{a}_{12}(0), \hat{a}_{13}(0), \hat{a}_{14}(0), \hat{a}_{15}(0), \hat{a}_{16}(0), \hat{a}_{17}(0), \hat{a}_{18}(0))$ such that $\xi_a \in (0, 2)$. For all trials, we use the tuning parameters $R_\theta = 10000 I_{l_\theta}$, $k_{\mathrm{f}} = 25$, and $n_{\mathrm{u}} = 80$. Figure 4.4 shows that, in 97% of the trials, the RCPE estimates of all of the components of a are within 10% error. Since $k_{\mathrm{f}} = 25$, in all trials where estimation of the unknown entries is successful, the estimates converge within 25 time steps.

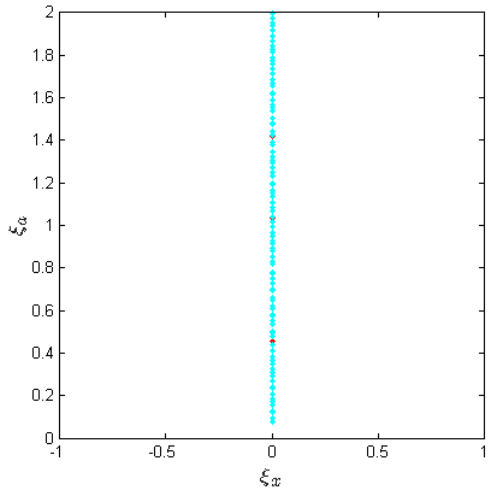

Figure 4.4: Application of RCPE to Example 8 assuming the initial state is known. RCPE is applied with 100 randomly generated initial estimates using the measurements $y_0(k) = x_1(k)$ over the interval $k \in [0, 25]$ and setting $\hat{x}(0) = x_0$. 97% of the trials are cyan.

4.4 RCPE with Unknown Initial State and Strong Excitation

In this section, we apply RCPE to the CSPE problem assuming the initial state is unknown and $w(k) \neq 0$. Let $y_0(k) = y_x(k) + y_w(k)$, where y_x and y_w are the main system outputs due to nonzero values of x_0 and w, respectively. We pick w such that, on average, $|y_w(k)|$ is significantly greater than $|y_x(k)|$.

4.4.1 Example 9: $n = 3$ and Three Unknown Entries in a Single Row

We revisit Example 7 with RCPE assuming that the initial state is unknown, w is measured Gaussian white noise with standard deviation of 10^4, and

$$D = \begin{bmatrix} 0.3 \\ -0.5 \\ 0.45 \end{bmatrix} \tag{4.35}$$

is known. Since x_1 is measured, we set $\hat{x}_1(0) = x_1(0)$ and choose 10000 randomly generated initial estimates $(\hat{x}_2(0), \hat{x}_3(0), \hat{a}_{11}(0), \hat{a}_{12}(0), \hat{a}_{13})$ such that $\xi_x, \xi_a \in (0, 2)$. For all trials, we use the tuning parameters $R_\theta = 10^8 I_{l_\theta}$, $k_{\mathrm{f}} = 100$, and $n_{\mathrm{u}} = 4$.

Figure 4.5 shows that 93.71% of the estimates $\hat{a}$ are within 10% of all three components of the true parameters a, 6.23% of $\hat{a}$ are within 10% of at least one component of a, and 0.06% of $\hat{a}$ are within 10% of none of the components of a. Since $k_{\mathrm{f}} = 100$, in all trials where estimation of the unknown entries is successful, the estimates converge within 100 time steps.

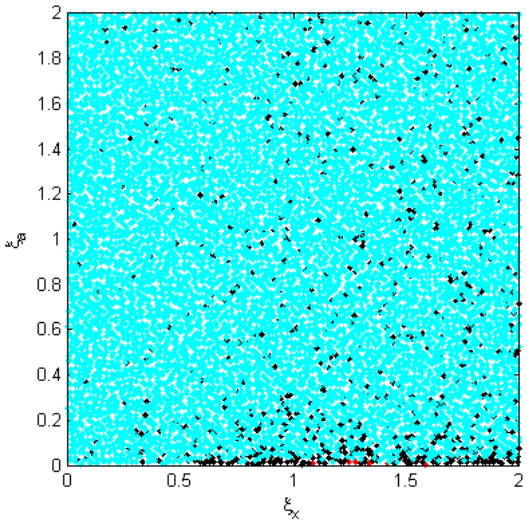

Figure 4.5: Application of RCPE to Example 9. RCPE is applied with 10000 randomly generated initial estimates $(\hat{x}_2(0), \hat{x}_3(0), \hat{a}_{11}(0), \hat{a}_{12}(0), \hat{a}_{13})$ using the measurements $y_0(k) = x_1(k)$ over the interval $k \in [0, 100]$. 93.71%, 6.23%, and 0.06% of the trials are cyan, black, and red, respectively.

4.4.2 Example 10: $n = 3$ and Eight Unknown Entries in a Single Row

We revisit Example 8 with RCPE assuming that the initial state is unknown, w is measured Gaussian white noise with standard deviation of 10^4, and

$$D = \begin{bmatrix} 0.3 \\ -0.5 \\ 0.45 \\ 0.1 \\ 0.15 \\ -0.3 \\ 0.6 \\ -0.1 \end{bmatrix} \tag{4.36}$$

is known. Since x_1 is measured, we set $\hat{x}_1(0) = x_1(0)$ and choose 10000 randomly generated initial estimates $(\hat{x}_2(0), \hat{x}_3(0), \hat{x}_4(0), \hat{x}_5(0), \hat{x}_6(0), \hat{x}_7(0), \hat{x}_8(0), \hat{a}_{11}(0), \hat{a}_{12}(0), \hat{a}_{13},$

$\hat{a}_{14}, \hat{a}_{15}, \hat{a}_{16}, \hat{a}_{17}, \hat{a}_{18})$ such that $\xi_x, \xi_a \in (0, 2)$. For all trials, we use the tuning parameters $R_\theta = 10^8 I_{l_\theta}$, $k_{\mathrm{f}} = 25$, and $n_{\mathrm{u}} = 40$.

Figure 4.6 shows that 73.29% of the estimates $\hat{a}$ are within 10% of all three components of the true parameters a, 12.92% of $\hat{a}$ are within 10% of at least one component of a, and 13.79% of $\hat{a}$ are within 10% of none of the components of a. Since $k_{\mathrm{f}} = 25$, in all trials where estimation of the unknown entries is successful, the estimates converge within 25 time steps.

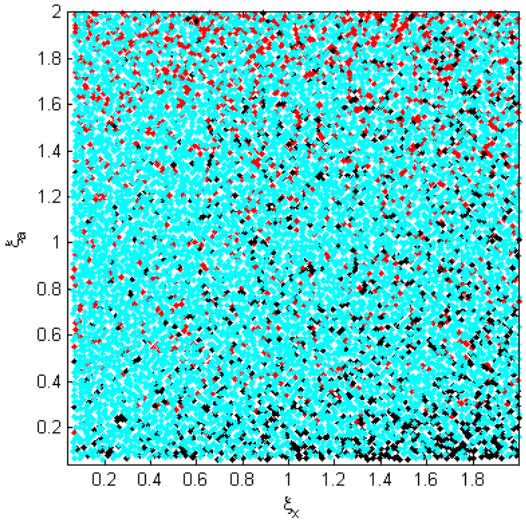

Figure 4.6: Application of RCPE to Example 10. RCPE is applied with 10000 randomly generated initial estimates using the measurements $y_0(k) = x_1(k)$ over the interval $k \in [0, 25]$. 73.29%, 12.92%, and 13.79% of the trials are cyan, black, and red, respectively.

4.5 RCPE with Unknown Initial State and Weak Excitation

In this section, we apply RCPE to the CSPE problem assuming the initial state is unknown. We pick w such that, on average, $|y_w(k)|$ is not significantly greater than $|y_x(k)|$.

4.5.1 Example 11: $n = 2$, Two Unknown Entries in a Single Row, Unknown Initial Condition

We revisit Example 1 with RCPE assuming that the initial state is unknown, w is measured Gaussian white noise with standard deviation of 10, and

$$D = \begin{bmatrix} 0.3 \\ -0.2 \end{bmatrix} \tag{4.37}$$

is known. Since x_1 is measured, we set $\hat{x}_1(0) = x_1(0)$ and choose 10000 randomly generated initial estimates $(\hat{x}_2(0), \hat{a}_{11}(0), \hat{a}_{12}(0))$ such that $\xi_x, \xi_a \in (0, 2)$. For all trials, we use the tuning parameters $R_\theta = 10^8 I_{l_\theta}$, $k_{\mathrm{f}} = 100$, and $n_{\mathrm{u}} = 4$.

Figure 4.7 shows that 40.29% of the estimates $\hat{a}$ are within 10% of both components of the true parameters a, 29.54% of $\hat{a}$ are within 10% of one component of a, and 30.17% of $\hat{a}$ are within 10% of neither components of a. Since $k_{\mathrm{f}} = 100$, in all trials where estimation of the unknown entries is successful, the estimates converge within 100 time steps.

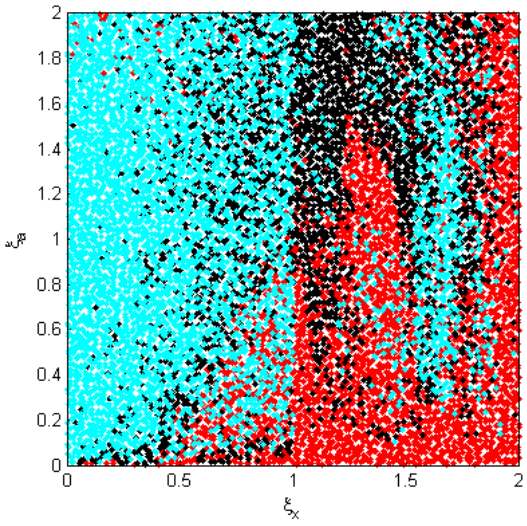

Figure 4.7: Application of RCPE to Example 9. RCPE is applied with 10000 randomly generated initial estimates $(\hat{x}_2(0), \hat{x}_3(0), \hat{a}_{11}(0), \hat{a}_{12}(0), \hat{a}_{13})$ using the measurements $y_0(k) = x_1(k)$ over the interval $k \in [0, 100]$. 40.29%, 29.54%, and 30.17% of the trials are cyan, black, and red, respectively.

4.5.2 Example 12: $n = 2$, Two Unknown Entries in a Single Row, Unknown Initial Condition

We revisit Example 1 with RCPE assuming the initial conditions are unknown and $w(k) = 0$. Since x_1 is measured, we set $\hat{x}_1(0) = x_1(0)$ and choose 10000 randomly generated initial estimates $(\hat{x}_2(0), \hat{a}_{11}(0), \hat{a}_{12}(0))$ such that $\xi_x, \xi_a \in (0, 2)$. For all trials, we use the tuning parameters $R_\theta = I_{l_\theta}$, $k_{\mathrm{f}} = 100$, and $n_{\mathrm{u}} = 4$. Figure 4.8 shows that, as ξ_x increases, the performance of RCPE degrades. In addition, 31.75% of the estimates $\hat{a}$ are within 10% of both components of the true parameters a, 36.42% are within 10% of exactly one component of a, and 31.83% are within 10% of none of the components of a. Since $k_{\mathrm{f}} = 100$, in all trials where estimation of the unknown entries is successful, the estimates converge within 100 time steps.

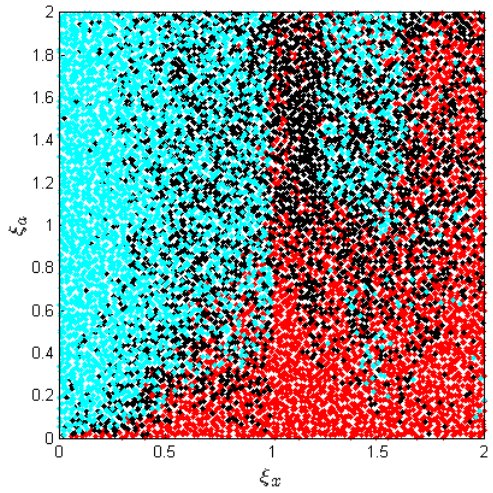

Figure 4.8: Application of RCPE to Example 12. RCPE is applied with 10000 randomly generated initial estimates $(\hat{x}_2(0), \hat{a}_{11}(0), \hat{a}_{12}(0))$ using the measurements $y_0(k) = x_1(k)$ over the interval $k \in [0, 100]$. 31.75%, 36.42%, and 31.83% of the trials are cyan, black, and red, respectively.

4.6 RCPE vs Classical Methods

4.6.1 Advantages of RCPE

Figure 3.6 suggests that UKF is not improved using knowledge of the initial conditions. In contrast, Examples 7–8 show that, if x_0 is known and we set $\hat{x}(0) = x_0$, then RCPE performs well. Furthermore, Examples 9–10 shows that, if x_0 is unknown and, on average, $|y_w|$ is significantly greater than $|y_x|$, then RCPE performs similarly well. Under these assumptions, not only is RCPE able to reliably estimate the unknown parameters of $n \geq 3$ systems, it is able to do so using significantly less data than UKF.

4.6.2 Deficiencies of RCPE

Example 11–12 shows that, in the case where x_0 is unknown and $|y_w|$ is not significantly greater than $|y_x|$ on average, the accuracy of RCPE degrades.

This can be explained by revisiting Section 2.2. Since a nonzero x_0 can be viewed as an impulse input, the fact that x_0 is unknown and $w(k) = 0$ implies the existence of an unmeasured impulse input. Extending this rationale, one can view y_x as the portion of the output due to an unmeasured impulse disturbance while y_w is the portion of the output due to a measured input. In the case where $|y_w|$ is not significantly greater than $|y_x|$, the effects of the unmeasured disturbance is non-negligible, and thus the performance of RCPE degrades.

Note that this is a consequence of estimating the unknown entries in A without estimating x_0. Thus, it motivates the development of a variation of RCPE that estimates both the unknown initial state and the unknown parameters.

CHAPTER 5

Retrospective Cost Parameter Estimation and Smoothing

5.1 RCPE Smoother

It was shown in the previous section that the RCPE estimates are reasonably accurate in the case where the initial state is known. In order to take advantage of this observation, we now formulate the RCPE smoother (RCPES) algorithm for concurrent parameter and initial state estimation.

5.1.1 Augmented Subsystem Estimation Framework

Let $\delta(k)$ be the unit impulse function and define $\delta_0(k) \triangleq \delta(k+1)$,

$$\mathcal{A} \triangleq \begin{bmatrix} A & x_0 \\ 0_{1\times l_x} & 0 \end{bmatrix}, \ \mathcal{D} \triangleq \begin{bmatrix} 0_{l_x\times 1} \\ 1 \end{bmatrix}, \ \mathcal{E} \triangleq \begin{bmatrix} E & 0 \end{bmatrix}, \ X \triangleq \begin{bmatrix} x \\ \delta_0 \end{bmatrix}. \tag{5.1}$$

Then, for all $k \geq -2$, (3.1)–(3.3) can be rewritten as the augmented system

$$X(k+1) = \mathcal{A}X(k) + \mathcal{D}\delta_0(k+2), \tag{5.2}$$

$$X(-2) = 0, \tag{5.3}$$

$$y_0(k) = \mathcal{E}X(k). \tag{5.4}$$

Now, assume that the mth component of E is 1 and all other components of E are zero, let $\hat{x}_0$ be the nominal initial state, and define

$$\mathcal{A}_0 \triangleq \begin{bmatrix} A_0 & \hat{x}_0 \\ 0_{1\times l_x} & 0 \end{bmatrix}, \ \mathcal{B} \triangleq \begin{bmatrix} B & B_0 \\ 0_{1\times l_u} & 0_{1\times(n-1)} \end{bmatrix}, \ \mathcal{C} \triangleq \begin{bmatrix} C & 0_{l_y\times 1} \\ 0_{1\times l_x} & 1 \end{bmatrix}, \tag{5.5}$$

where B_0 is I_n with the mth column deleted. Then, (5.2)–(5.4) can be written in the form of (4.1)–(4.3) as

$$X(k+1) = \mathcal{A}_0 X(k) + \mathcal{B}u(k) + \mathcal{D}\delta(k+2), \tag{5.6}$$

$$y(k) = \mathcal{C}X(k), \tag{5.7}$$

$$y_0(k) = \mathcal{E}X(k), \tag{5.8}$$

with known initial state $X(-2) = 0$. Using (4.11), it follows that the augmented dynamics matrix of the true system is given by

$$\mathcal{A} = \mathcal{A}_0 + \mathcal{B}D_{\mathrm{s}}\mathcal{C}. \tag{5.9}$$

Note that (5.6)–(5.8) has *known*, zero initial state and the augmented dynamics matrix $\mathcal{A}$ contains the unknown initial state x_0. This is a smoother problem since the goal is to estimate D_{s}, which contains the unknown entries of A along with the unknown components of x_0.

To construct an estimator based on (5.6)–(5.8), we define

$$\hat{X} \triangleq \begin{bmatrix} \hat{x} \\ \delta_0 \end{bmatrix} \tag{5.10}$$

and rewrite (4.6)–(4.8) as

$$\hat{X}(k+1) = \mathcal{A}_0\hat{X}(k) + \mathcal{B}\hat{u}(k) + \mathcal{D}\delta(k+2), \tag{5.11}$$

$$\hat{y}(k) = \mathcal{C}\hat{X}(k), \tag{5.12}$$

$$\hat{y}_0(k) = \mathcal{E}\hat{X}(k), \tag{5.13}$$

where the initial state $\hat{X}(-2) = 0$. For example, consider the case where $n = 2$, a_{11} is unknown, and $y(k) = x_1(k)$, and thus $x_2(0)$ is unknown. Let the $(1,1)$ entry of A_0 be zero and set $\hat{x}_2(0) = 0$. Then,

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix},\ B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \hat{D}_s = \begin{bmatrix} \hat{\theta}_1 & 0 \\ 0 & \hat{\theta}_2 \end{bmatrix}, \tag{5.14}$$

where $\hat{\theta}_1$ and $\hat{\theta}_2$ are estimates of a_{11} and $x_2(0)$, respectively. Note that, for this smoother problem, $\hat{D}_{\mathrm{s}}$ has the block-diagonal structure shown in (4.15).

5.1.2 Data Update

For concurrent parameter and initial state estimation, we apply RCPE to (5.11)–(5.13). At each step k, RCPE produces $\hat{D}_{\mathrm{s}}$, which contains estimates of the unknown components of A and x_0. Next, $\hat{y}(k)$ and $\hat{y}_0(k)$ are computed using

$$\hat{y}(k) = \mathcal{C}(\mathcal{A}_0 + \mathcal{B}\hat{D}_{\mathrm{s}}(k)\mathcal{C})^{k+1}\mathcal{D}, \quad \hat{y}_0(k) = \mathcal{E}(\mathcal{A}_0 + \mathcal{B}\hat{D}_{\mathrm{s}}(k)\mathcal{C})^{k+1}\mathcal{D}. \tag{5.15}$$

Since the values of $\hat{y}$ and $\hat{y}_0$ at previous steps are computed from prior estimates of $\hat{A}$ and $\hat{x}(0)$, there may be a mismatch between $P(k-1)$ and $\Phi_{\mathrm{f}}(k-1)$ in (4.26). To rectify this, at each step k, we use constant values of $\theta = \theta(k-1)$ to recompute $\hat{y}$, $\hat{y}_0$, $\hat{u}$, and Φ_{f} from steps -2 to $k-1$. Then, we rerun (4.25)–(4.26) from steps -2 to k with these updated values to obtain $\theta(k)$ and $P(k)$.

5.2 RCPES with Unknown Initial State and No Excitation

In this section, we apply RCPES to the CSPE problem for the case where the initial state is unknown and $w(k) = 0$.

5.2.1 Example 13: $n = 2$ and Two Unknown Entries in a Single Row

We revisit Example 1 with RCPES. Recall that in this case, the true parameter vector a is constructed only from the unknown entries of A while the unmeasured state vector x_{u} is constructed only from the unmeasured components of x. Since the uncertain entries in $\mathcal{A}$ cannot be represented by (4.13), we use (4.15). In this case, $l_\theta = 6$, where two components of θ are estimates of unknown parameters, one component is an estimate of the unknown component of the initial state, and three components are estimates of the known value zero and thus are ignored. For all trials, we use the tuning parameters $k_{\mathrm{f}} = 50$ and $n_{\mathrm{u}} = 10$, and we choose R_θ to be a diagonal matrix with diagonal entries 1, 1, 10^8, 10^8, 10^8, and 1, where the large entries correspond to the components of θ that are known to be zero. Figure 5.1 shows that, in all trials, RCPES estimates both components of a within 10% error. Since $k_{\mathrm{f}} = 50$, in all trials where estimation of the unknown entries is successful, the estimates converge within 50 time steps.

5.2.2 Example 14: $n = 3$ and Three Unknown Entries in a Single Row

We revisit Example 5 with RCPES. Once again, the uncertain entries in $\mathcal{A}$ must be represented with (4.15). In this case, $l_\theta = 12$, where three components of θ are estimates of unknown parameters, two components are estimates of the unknown components of the initial state, and seven components are estimates of the known value zero and thus are ignored. For all trials, we use the tuning parameters $k_{\mathrm{f}} = 50$, $n_{\mathrm{u}} = 10$, and set R_θ equal to a diagonal matrix with diagonal entries of 100, 100, 100, 10^8, 10^8, 10^8, 10^8, 10^{-4}, 10^8, 10^8, 10^8, and 10^{-4}. As in Example 13, the largest diagonal entries of R_θ correspond to the

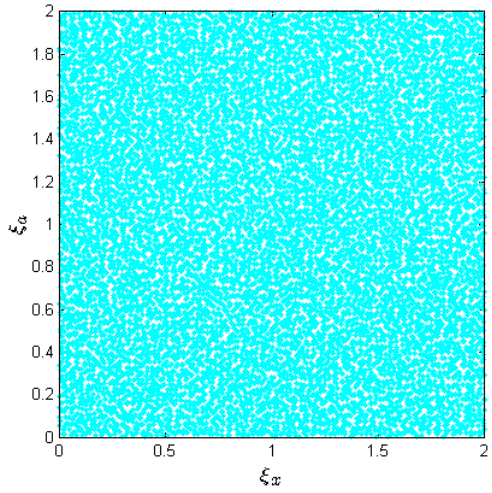

Figure 5.1: Application of RCPES to Example 13. RCPES is applied with 10000 randomly generated initial estimates $(\hat{x}_2(0), \hat{a}_{11}(0), \hat{a}_{12}(0))$ using the measurements $y_0(k) = x_1(k)$ over the interval $k \in [0, 50]$. 100% of the trials are cyan.

components of θ that are zero. Figure 5.2 shows that 70.92% of the estimates $\hat{a}$ are within 10% of all three components of the true parameters a, 11.62% of $\hat{a}$ are within 10% of at least one component of a, and 17.46% of $\hat{a}$ are within 10% of none of the components of a. Since $k_{\mathrm{f}} = 50$, in all trials where estimation of the unknown entries is successful, the estimates converge within 50 time steps.

5.2.3 Example 15: Application to Linearized Longitudinal Aircraft Dynamics

We revisit Example 6 with RCSES. Once again, the uncertain entries in $\mathcal{A}$ must be represented with (4.15). In this case, $l_\theta = 12$, two components of θ are estimates of the unknown parameter, three components are estimates of the unknown components of the initial state, and seven components are estimates of the known value zero and thus are ignored. For all trials, we use the tuning parameters $k_{\mathrm{f}} = 25$, $n_{\mathrm{u}} = 20$, and set R_θ equal to a diagonal matrix with diagonal entries of 0.1, 0.1, 10^8, 10^8, 10^8, 0.1, 10^8, 10^8, 0.1, 10^8, 10^8, and 0.1, where the largest entries correspond to components of θ that are zero. Figure

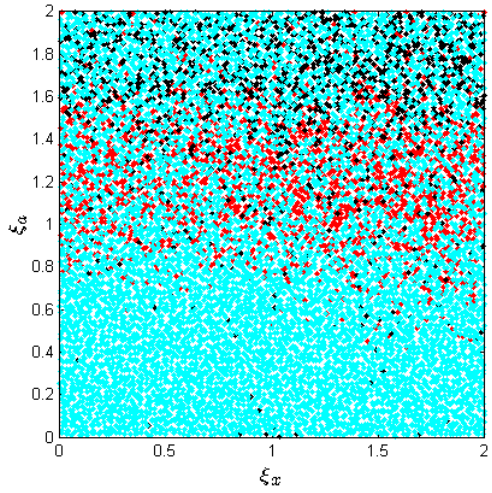

Figure 5.2: Application of RCPES to Example 14. RCPES is applied with 10000 randomly generated initial estimates $(\hat{x}_2(0), \hat{x}_3(0), \hat{a}_{11}(0), \hat{a}_{12}(0), \hat{a}_{13}(0))$ using the measurements $y_0(k) = x_1(k)$ over the interval $k \in [0, 50]$. 70.92% of the trials are cyan, 11.62% of the trials are black, and 17.46% of the trials are red.

5.3 shows that, 2.23% of the estimates $\hat{a}$ are within 10% of both components of the true parameters a, 58.79% of $\hat{a}$ are within 10% of at least one component of a, and 38.98% of $\hat{a}$ are within 10% of none of the components of a. Since $k_{\mathrm{f}} = 25$, in all trials where estimation of the unknown entries is successful, the estimates converge within 25 time steps.

5.3 RCPES vs Classical Methods

5.3.1 Advantages of RCPES

Example 13 shows that RCPES performs as well as UKF with (3.17) in the case where $n = 2$ and one row of A is unknown. Example 14 shows that RCPES performs better than UKF with (3.17) for the case where $n = 3$ and one row of A is unknown. Examples 15 shows that RCPES performs better than UKF for the case where two entries in the dynamics matrix of a linearized longitudinal aircraft model are unknown. In all trials that are successful, RCPES is able to estimate the unknown parameters using significantly less data than UKF.

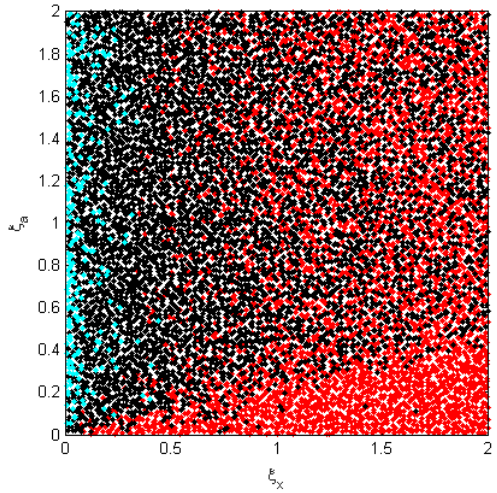

Figure 5.3: Application of RCSES to Example 12. RCSES is applied with 10000 randomly generated initial estimates $(\hat{x}_2(0), \hat{x}_3(0), \hat{x}_4(0), \hat{a}_{11}(0), \hat{a}_{12}(0))$ using the measurements $y_0(k) = x_1(k)$ over the interval $k \in [0, 25]$. 2.23% of the trials are cyan, 58.79% of the trials are black, and 38.98% of the trials are red.

5.3.2 Deficiencies of RCPES

While RCPES performs better than UKF in all cases, it often requires more computational power to execute. This is due to the data update method outlined in Section 5.1.2.

CHAPTER 6

An Alternative Formulation of RCPES

6.1 RCPE Smoother

In the previous chapter, we formulated a version of the RCPES algorithm based on an augmented main system model. In this chapter, we present an alternative formulation of RCPES based on a modified retrospective cost function.

6.1.1 Modified Retrospective Performance Variable

In the case where x_0 is unknown or inaccurate, we redefine the retrospective performance variable to account for the error between x_0 and $\hat{x}_0$. Let $\Delta\hat{x}_0$ be the estimate of $\Delta x_0 = \hat{x}_0 - x_0$. We then rewrite (4.21) as

$$\hat{z}(k) = z(k) + E(A_0 + B\hat{D}_{\mathrm{s}}C)^{k-1}\Delta\hat{x}_0 + G_{\mathrm{f}}(\mathbf{q})(\Phi(k)\hat{\theta} - \hat{u}(k)), \tag{6.1}$$

where

$$\Phi(k) = (y(k) + C(A_0 + B\hat{D}_{\mathrm{s}}C)^{k-1}\Delta\hat{x}_0)^{\mathrm{T}} \otimes I_{l_u}$$

and

$$G_{\mathrm{f}} \sim \left[\begin{array}{c|c} A_0 + B\hat{D}_{\mathrm{s}}C & B \\ \hline E & 0 \end{array}\right]. \tag{6.2}$$

Note that, with this new definition of (6.1), the cost function (4.24) is a polynomial function of $\hat{D}_{\rm s}$.

Optimization of a polynomial cost function is an open research problem and is outside the scope of this dissertation. Therefore, we simplify (6.1) by removing the term $B\hat{D}_{\rm s}C$ from both (6.1) and $\Phi(k)$. This yields the modified retrospective performance variable

$$\hat{z}'(k) \triangleq z(k) + \tilde{\Phi}_{\rm f}(k)\hat{\Theta} - \hat{u}_{\rm f}(k)), \tag{6.3}$$

where

$$\hat{\Theta} = \begin{bmatrix} \Delta\hat{x}_0 \\ \hat{\theta} \end{bmatrix}, \quad \tilde{\Phi}_{\rm f}(k) = \begin{bmatrix} EA_0^{k-1} & \bar{\Phi}_{\rm f}(k) \end{bmatrix}, \tag{6.4}$$

$\bar{\Phi}_{\rm f}(k) = G_{\rm f}(\mathbf{q})((y(k)+CA_0^{k-1}\Delta\hat{x}_0(k-1))^{\rm T} \otimes I_{l_u})$. $G_{\rm f}$ is implemented as the time-varying filter described in 4.2.4.

6.1.2 Revised Retrospective Cost Function

The new retrospective cost function is given by

$$J'(k,\hat{\Theta}) \triangleq \sum_{i=1}^{k} (\hat{z}'(i))^{\rm T} R_z (\hat{z}'(i)) + (\hat{\Theta} - \hat{\Theta}(0))^{\rm T} R_\Theta (\hat{\Theta} - \hat{\Theta}(0)), \tag{6.5}$$

which is quadratic in $\hat{\Theta}$. J' therefore has unique global minimizer given by

$$\bar{\Theta}(k) = \hat{\Theta}(k-1) - P(k-1)\tilde{\Phi}_{\rm f}^{\rm T}(k)\Gamma^{-1}(k) \cdot [\tilde{\Phi}_{\rm f}(k)\hat{\Theta}(k-1) + z_{\rm f}(k) - u_{\rm f}(k)], \tag{6.6}$$

$$P(k) = P(k-1) - P(k-1)\tilde{\Phi}_{\rm f}^{\rm T}(k)\Gamma^{-1}(k)\tilde{\Phi}_{\rm f}(k)P(k-1), \tag{6.7}$$

where

$$\Gamma(k) \triangleq R_z^{-1} + \tilde{\Phi}_{\mathrm{f}}(k)P(k-1)\tilde{\Phi}_{\mathrm{f}}^{\mathrm{T}}(k).$$

6.1.3 $\hat{\Theta}$ Update Logic

Define

$$\bar{J}(k, \Delta\hat{x}_0, \hat{\theta}) = \bar{J}(k, \hat{\Theta}) \triangleq \frac{1}{k}\sum_{i=1}^{k} E_1(A + B\hat{\theta}C)^{i-1}(\hat{x}_0 + \Delta\hat{x}_0) - y_0(i). \tag{6.8}$$

For all steps $k > 1$, we compute $\bar{J}(k, \bar{\Theta}(k))$ and then implement the logic

$$\hat{\Theta}(k) = \begin{cases} \bar{\Theta}(k), & \bar{J}(k, \bar{\Theta}(k)) < \bar{J}(k-1, \bar{\Theta}(k-1)) \\ \hat{\Theta}(k-1), & \bar{J}(k, \bar{\Theta}(k)) \geq \bar{J}(k-1, \bar{\Theta}(k-1)) \end{cases} \tag{6.9}$$

to obtain $\hat{\Theta}(k)$. We then compute (4.17), where $\hat{\theta}(k)$ is given by the last l_θ components of $\hat{\Theta}(k)$.

6.1.4 Data-window Reiteration

Like with RCPE, we enhance the accuracy of the estimates $\hat{\Theta}(k)$ by applying RCPES multiple times to a given data set consisting of k_{f} data points. In the first iteration, we apply RCPE with $G_{\mathrm{f}}(\mathbf{q}, \hat{D}_s(k-1))$ given by (4.32) initialized with $\hat{D}_s(0) = 0$. In addition, the entries of the nominal dynamics matrix A_0 in both (4.32) and the model (4.6) are set to the initial estimates of the unknown parameters. In subsequent iterations, we apply RCPE to the same data set with $\hat{D}_s(0)$ given by $\hat{D}_s(k_{\mathrm{f}})$ from the previous iteration, A_0 replaced by $A_0 + B\hat{D}_s(k_{\mathrm{f}})C$, and $\hat{x}_0$ replaced by $\hat{x}_0 + \Delta\hat{x}_0$, where $\Delta\hat{x}_0$ is given by the first l_x components of $\hat{\Theta}(k)$.

6.2 RCPES with Unknown Initial State and No Excitation

In this section, we apply the alternative formulation of RCPES to the CSPE problem for the case where the initial state is unknown and $w(k) = 0$.

6.2.1 Example 16: $n = 2$ and Two Unknown Entries in a Single Row

We revisit Example 1 with RCPES. Since x_1 is measured, we set $\hat{x}_1(0) = x_1(0)$ and choose 10000 randomly generated initial estimates $(\hat{x}_2(0), \hat{a}_{11}(0), \hat{a}_{12}(0))$ such that $\xi_x, \xi_a \in (0, 2)$. For all trials, we use the tuning parameters $R_\theta = 10^4 I_{l_\theta}$, $k_{\mathrm{f}} = 100$, and $n_{\mathrm{u}} = 4$.

Figure 6.1 shows that 44.86% of the estimates $\hat{a}$ are within 10% of both components of the true parameters a, 4.70% of $\hat{a}$ are within 10% of one component of a, and 50.44% of $\hat{a}$ are within 10% of neither components of a. Since $k_{\mathrm{f}} = 100$, in all trials where estimation of the unknown entries is successful, the estimates converge within 100 time steps.

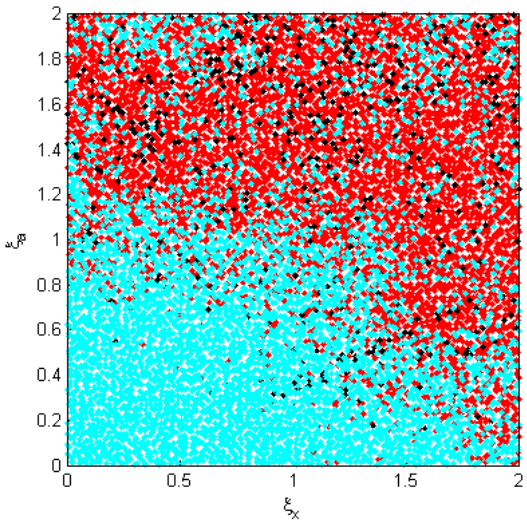

Figure 6.1: Application of RCPE to Example 16. RCPE is applied with 10000 randomly generated initial estimates $(\hat{x}_2(0), \hat{a}_{11}(0), \hat{a}_{12}(0))$ using the measurements $y_0(k) = x_1(k)$ over the interval $k \in [0, 100]$. 44.86%, 4.70%, and 50.44% of the trials are cyan, black, and red, respectively.

6.2.2 Example 17: $n = 2$ and Three Unknown Entries in a Single Row

We revisit Example 5 with RCPES. Since x_1 is measured, we set $\hat{x}_1(0) = x_1(0)$ and choose 10000 randomly generated initial estimates $(\hat{x}_2(0), \hat{x}_3(0), \hat{a}_{11}(0), \hat{a}_{12}(0), \hat{a}_{13}(0))$ such that $\xi_x, \xi_a \in (0, 2)$. For all trials, we use the tuning parameters $R_\theta = 10^4 I_{l_\theta}$, $k_{\mathrm{f}} = 100$, and $n_{\mathrm{u}} = 4$.

Figure 6.2 shows that 20.09% of the estimates $\hat{a}$ are within 10% of all three components of the true parameters a, 32.12% of $\hat{a}$ are within 10% of at least one component of a, and 47.79% of $\hat{a}$ are within 10% of none of the components of a. Since $k_{\mathrm{f}} = 100$, in all trials where estimation of the unknown entries is successful, the estimates converge within 100 time steps.

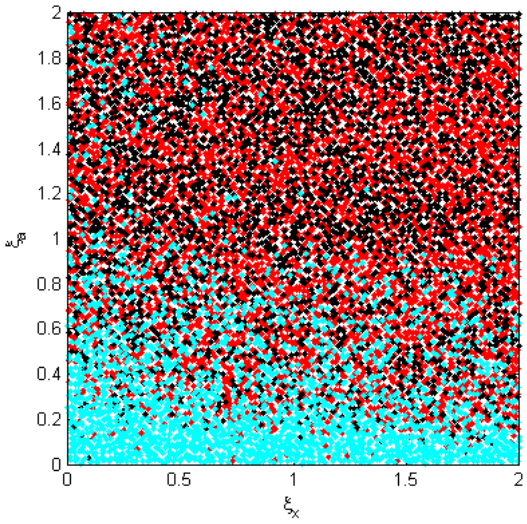

Figure 6.2: Application of RCPE to Example 17. RCPE is applied with 10000 randomly generated initial estimates $(\hat{x}_2(0), \hat{x}_3(0), \hat{a}_{11}(0), \hat{a}_{12}(0), \hat{a}_{13})$ using the measurements $y_0(k) = x_1(k)$ over the interval $k \in [0, 100]$. 20.09%, 32.12%, and 47.79% of the trials are cyan, black, and red, respectively.

6.3 RCPES vs Classical Methods

6.3.1 Advantages of RCPES

Example 17 shows that RCPES performs better than UKF with (3.17) for the case where $n = 3$ and one row of A is unknown. In all trials that are successful, RCPES is able to estimate the unknown parameters using significantly less data than UKF.

6.3.2 Deficiencies of RCPES

Example 16 shows that RCPES performs worse than UKF with (3.17) in the case where $n = 2$ and one row of A is unknown. Overall, this variant of RCPES performs worse than the first variant, although it requires less computational power to use.

CHAPTER 7

Retrospective Cost Kalman Filter

7.1 State Estimation Problem

Consider a variation of the equations presented in 2.1 given by

$$x(k+1) = Ax(k) + D_1 v_1(k), \tag{7.1}$$

$$x(0) = x_0, \tag{7.2}$$

$$y_0(k) = Ex(k) + D_2 v_2(k), \tag{7.3}$$

where

$$D_1 = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \in \mathbb{R}^n, \quad D_2 \in \mathbb{R}, \tag{7.4}$$

and $v_1(k), v_2(k) \in \mathbb{R}$ are the unmeasured process and sensor noise, respectively. We assume that A, E are known, D_1 and D_2 are unknown, and the goal is to estimate x.

7.2 RCKF

In this section, we formulate the RCKF algorithm for state estimation, assuming that the dynamics matrix A is known.

7.2.1 State Estimation as Static Subsystem Estimation

To start, consider the classical Kalman predictor, given by

$$K(k) = -AP(k-1)E^{\mathrm{T}}(EP(k-1)E^{\mathrm{T}} + R)^{-1}, \tag{7.5}$$

$$\hat{x}(k) = A\hat{x}(k-1) + K(k)(E\hat{x}(k-1) - y_0(k)), \tag{7.6}$$

$$P(k) = (I_{l_x} + K(k)E)(AP(k-1)A^{\mathrm{T}} + Q), \tag{7.7}$$

where $\hat{x}$ is the estimate of x, K is the Kalman gain, and Q, R contain the estimates of the noise covariances. Linear-quadratic estimation theory states that, assuming A is known and the sensor and process noises are zero-mean and have finite second moment, (7.5) will converge to $x(k)$ exponentially.

Let

$$\hat{u}(k) = K(k)\hat{y}_0(k), \tag{7.8}$$

$$\hat{y}_0(k) = E\hat{x}(k). \tag{7.9}$$

Then (7.6) can be represented as

$$\hat{x}(k) = A\hat{x}(k-1) + I_{l_x}\hat{u}(k) - K(k)y_0(k), \tag{7.10}$$

$$\hat{y}_0(k) = E\hat{x}(k), \tag{7.11}$$

$$\hat{u}(k) = K(k)\hat{y}(k). \tag{7.12}$$

In other words, the Kalman gain K can be viewed as a static parameter in feedback with a

model. Estimation of K can thus be viewed as a static parameter estimation problem with an unknown input matrix which is constrained to be equal to the unknown parameters.

Thus, we define

$$\hat{y}(k) = E\hat{x}(k), \tag{7.13}$$

apply RCPE with (4.6)–(4.8) replaced by (7.10)–(7.13), and use $\hat{x}$ as an estimate of x. This is the RCKF algorithm.

7.2.2 Choice of G_{f}

Similar to the derivation in Section 4.2.4, the optimal choice of G_{f} is given by

$$G_{\mathrm{f}}(\mathbf{q}) \sim \left[\begin{array}{c|c} A+KE & I_{l_x} \\ \hline E & 0 \end{array}\right], \tag{7.14}$$

where K is the Kalman gain for some sensor and process noise. However, we found that it suffices to use

$$G_{\mathrm{f}}(\mathbf{q}) \sim \left[\begin{array}{c|c} A+\bar{K}E & I_{l_x} \\ \hline E & 0 \end{array}\right], \tag{7.15}$$

where $\bar{K}$ is any nonzero matrix that stabilizes $A + \bar{K}E$. Note that our choice of G_{f} is constant. Furthermore, we do not use the data-window reiteration technique described in Section 4.2.5.

7.3 RCKF with Unknown Noise Covariances

In this section, we compare the performance of RCKF with the classical Kalman predictor, under the assumption that A is known and D_1, D_2 are uncertain.

7.3.1 Example 18: $n = 2$ and D_1, D_2 are Uncertain

Consider (7.1)–(7.3) with

$$A = \begin{bmatrix} 0.27 & 1.17 \\ -0.8 & 0.2 \end{bmatrix},\ x_0 = \begin{bmatrix} -23 \\ 17 \end{bmatrix},\ E = \begin{bmatrix} 1 & 0 \end{bmatrix},\ D_1 = \begin{bmatrix} 0.3 \\ -0.5 \end{bmatrix}, \tag{7.16}$$

$D_2 = 0.4$, and v_1, v_2 are Gaussian white noise with standard deviation of 1. The second component of x_0 is unmeasured. To apply RCKF, we use the tuning parameter $R_\theta = I_{l_\theta}$ and set

$$\bar{K} = \begin{bmatrix} 10 \\ 0.1 \end{bmatrix}. \tag{7.17}$$

To apply the Kalman predictor, we use the tuning parameter $P(0) = 10^4 I_{l_x}$ and set

$$Q = \hat{D}_1 \hat{D}_1^{\mathrm{T}}, \quad R = \hat{D}_2 \hat{D}_2^{\mathrm{T}}, \tag{7.18}$$

where $\hat{D}_1, \hat{D}_2$ are estimates of D_1, D_2, respectively.

In both RCKF and the Kalman predictor, we set the initial estimate of the second component of x_0 to be the arbitrary value of 5.

To evaluate the accuracy of $\hat{D}_1, \hat{D}_2$, we define the relative initial estimation errors

$$\xi_{D_1} \triangleq \frac{||\hat{D}_1 - D_1||}{||D_1||}, \quad \xi_{D_2} \triangleq \frac{||\hat{D}_2 - D_2||}{||D_2||}. \tag{7.19}$$

Note that $\xi_{D_1} = 0$ if and only if $\hat{D}_1 = D_1$, and $\xi_{D_2} = 0$ if and only if $\hat{D}_2 = D_2$.

To compare the performance of RCKF and the Kalman predictor, we consider 10000 randomly generated initial estimates $\hat{D}_1, \hat{D}_2$ such that $\xi_{D_1}, \xi_{D_2} \in (0, 1000)$. For all trials,

we evaluate the normalized relative state estimation error

$$\frac{||\hat{x}(k_{\mathrm{f}}) - x(k_{\mathrm{f}})||}{||x(k_{\mathrm{f}})||} \tag{7.20}$$

for both RCKF and the Kalman predictor. Figure 7.1 shows that, in 37.65% of all trials, RCKF produces more accurate estimates than the Kalman predictor while the reverse is true in 62.35% of all trials. In the case where RCKF performs better, it produces 18.5% lower relative error on average; in the case where the Kalman predictor performs better, it produces 6.88% lower relative error on average. Furthermore, we observe that, in the case where $\frac{\xi_{D_2}}{\xi_{D_1}}$ is large, RCKF produces more accurate estimates of x.

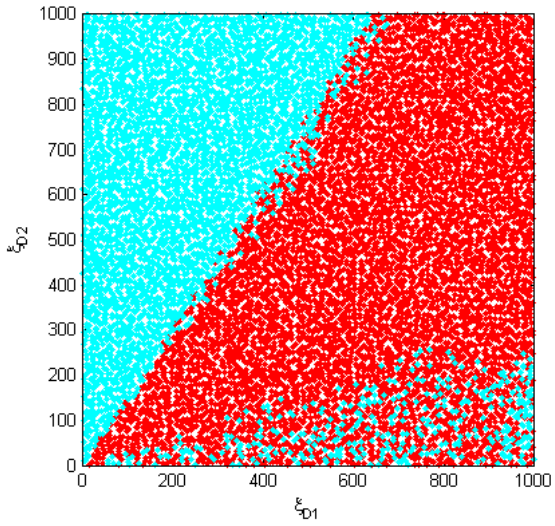

Figure 7.1: Comparison of RCKF and the Kalman predictor in Example 18. The Kalman predictor are applied with 10000 randomly generated initial estimates $\hat{D}_1, \hat{D}_2$ using the measurements $y_0(k) = x_1(k)$ over the interval $k \in [0, 500]$. Trials where RCKF produces more accurate estimates of x at step $k = 500$ are labeled with cyan and trials where the Kalman predictor produces more accurate estimates of x at step $k = 500$ are labeled with red. 37.65% of the trials are cyan and 62.35% of the trials are red. RCKF tends to produce more accurate estimates of x in trials where $\frac{\xi_{D_2}}{\xi_{D_1}}$ is large.

7.4 RCKF with Nonzero-Mean Noise Distribution

In this section, we compare the performance of RCKF with the classical Kalman predictor, under the assumption that A is known and v_1, v_2 are not zero-mean.

7.4.1 Example 19: $n = 2$ and v_1, v_2 are not Zero-Mean

We revisit Example 18 assuming that D_1, D_2 are known and v_1, v_2 are given by the function

$$v_1 = \xi_s + \xi_w \bar{v}_1, \tag{7.21}$$

$$v_2 = \xi_s + \xi_w \bar{v}_2, \tag{7.22}$$

where $\bar{v}_1, \bar{v}_2$ are gaussian white noise signals and $\xi_s, \xi_w \in \mathbb{R}$.

To compare the performance of RCKF and the Kalman predictor under nonzero-mean white noise, we consider 10000 linearly spaced values of $\xi_s, \xi_w \in (0, 2)$. Using the same tunings as in Example 18 for all trials, we evaluate the normalized relative state estimation error

$$\frac{||\hat{x}(k_{\mathrm{f}}) - x(k_{\mathrm{f}})||}{||x(k_{\mathrm{f}})||} \tag{7.23}$$

for both RCKF and the Kalman predictor. Figure 7.2 shows that, in 17.47% of all trials, RCKF produces more accurate estimates than the Kalman predictor while the reverse is true in 82.53% of all trials. In the case where RCKF performs better, it produces 4.68% lower relative error on average; in the case where the Kalman predictor performs better, it produces 10.2% lower relative error on average.

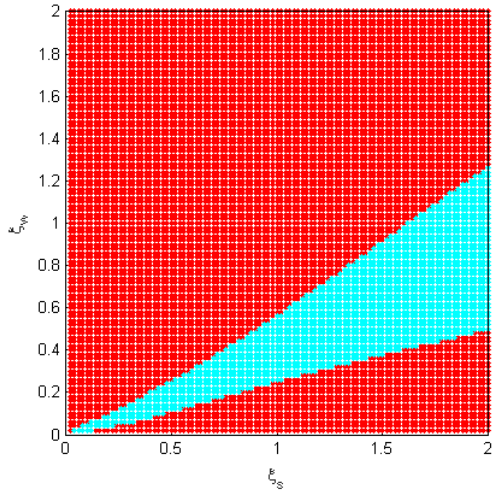

Figure 7.2: Comparison of RCKF and the Kalman predictor in Example 19. The Kalman predictor are applied with 10000 linearly-spaced values ξ_s, ξ_w using the measurements $y_0(k) = x_1(k)$ over the interval $k \in [0, 500]$. Trials where RCKF produces more accurate estimates of x at step $k = 500$ are labeled with cyan and trials where the Kalman predictor produces more accurate estimates of x at step $k = 500$ are labeled with red. 17.47% of the trials are cyan and 82.53% of the trials are red.

7.5 RCKF vs Kalman Predictor

7.5.1 Advantages of RCKF

Example 18 shows that, in the case where the noise covariances are unknown, RCKF is able to perform better than the Kalman predictor in 37.65% of the trials. In the trials where the sensor noise is greater than the process noise, RCKF typically performs better than the Kalman predictor. This suggests that RCKF is more suitable for applications where sensors are poor while actuators are accurate. We observe that, in trials where the Kalman predictor performs better, RCKF is able to deliver similar performance, whereas in trials where RCKF performs better, it is able to significantly out-perform the Kalman predictor.

7.5.2 Deficiencies of RCKF

Example 19 shows that, in the case where the sensor and process noises are not zero-mean, the Kalman predictor tends to perform better than RCKF. This suggests that RCKF

is less suitable for applications where the sensor and process noises are not zero-mean.

CHAPTER 8

Conclusion and Future Work

8.1 Conclusion

In this dissertation, we are principally concerned with the CSPE problem. In Chapter 2, we state and prove necessary and sufficient conditions for testing the feasibility of the CSPE problem. To do this, we relate the feasibility of the CSPE problem to the identifiability of state-space realizations. We derived generic conditions for arbitrary second-, third-, and fourth-order discrete-time, linear time-invariant state-space systems with scalar measurements. In addition, we derived generic conditions for third- and fourth-order discrete-time, linear time-invariant state-space systems with multiple measurements of the components of the state.

In Chapter 3, we test classical approaches to the CSPE problem, namely, EKF, UKF with augmented dynamics, and UKF with state-dependent coefficients. These algorithms are applied to second-, third-, and fourth-order systems, including application to a linearized longitudinal aircraft model. We found that both EKF and UKF with augmented dynamics are ineffective for second-order systems, while UKF with state-dependent coefficients are applicable to second- and third-order systems. We test multiple setups of UKF with state-dependent coefficients to determine the conditions under which they provide accurate estimates for the CSPE problem. The main drawbacks of these classical approaches is that their performance deteriorates drastically as the order of the system and the number

of unknown parameters increases. Furthermore, these algorithms typically require approximately 500 data points to converge onto the true values even without sensor and process noise. These results serve as a benchmark for future chapters.

In Chapter 4, we formulate and test the RCPE algorithm for parameter estimation without measurements of the state. This algorithm is applied to third- and eighth-order systems. We found that, under the assumption that the initial state is known, RCPE is highly effective for the CSPE problem. RCPE is similarly effective for the case where the initial state is unknown and the system is driven by a sufficiently large, measured excitation signal. Furthermore, RCPE typically requires approximately 100 data points to converge to the true values; on occasion, RCPE requires as little as 25 data points to converge to the true values. The main drawback of RCPE is that it is unreliable for cases where the initial state is unknown and the system is not driven by a sufficiently large excitation signal.

In Chapters 5 and 6, we formulate and test two variations of the RCPES algorithm, which performs concurrent parameter and initial state estimation without measurements of the state. These algorithms are applied to second-, third-, and fourth-order systems, including application to a linearized longitudinal aircraft model. The first variant of the RCPES algorithm performs better than the classical approaches in all cases. Furthermore, it typically requires approximately 100 data points to converge, requiring as little as 25 data points on occasion. The second variant of the RCPES algorithm performs worse than the classical approaches for second-order cases, but performs better for higher order cases. Once again, this algorithm requires approximately 100 data points to converge to the true values.

In Chapter 7, we formulate and test the RCKF algorithm, which performs state estimation without using knowledge of the noise covariances and does not require the sensor and process noise to be zero-mean and finite second moment. This algorithm is applied to second- and third- order systems, and the results are compared to the Kalman predictor. In the case where the error covariances are not known, if the sensor noise is greater

than the process noise, RCKF typically performs better than the Kalman predictor. In the case where the sensor and process noise is not zero-mean, RCKF performs better than the Kalman predictor on occasion.

8.2 Future Work

The main contributions of this dissertation is proving conditions on identifiability and the formulation of estimation algorithms based on the retrospective-cost, namely, RCPE, two variants of RCPES, and RCKF. We briefly outline the potential next steps for these research topics.

While the present dissertation derives necessary and sufficient conditions for identifiability for only second-, third-, and fourth-order cases with scalar measurements, the ultimate goal is to derive similar conditions for a system of arbitrary order. We believe that a viable approach for this is to apply proof by induction. First, establish the second-order identifiability conditions, which are given in this dissertation. Next, we need to show that a certain set of n^{th}-order conditions imply that the same conditions are true for $n+1^{\mathrm{th}}$-order systems. To obtain the n^{th}-order conditions, one may need to derive identifiability conditions for fifth- and sixth-order systems before a pattern can be observed. These derivations, while tedious, are not particularly difficult; one simply needs to utilize the same techniques outlined in this dissertation.

The RCPE results shown in this dissertation are predicated on the G_{f} update method outlined in Sections 4.2.4 and 4.2.5. While these results are already quite reliable, working well for arbitrary eighth-order systems, we believe that they can be further improved by applying a more sophisticated G_{f} update method.

The RCPES results shown in this dissertation, on the other hand, are less reliable. For the first RCPES variant, we believe the main difficulty stems from the fact that modifying $\hat{x}_0$ in $\mathcal{A}_0$ after the first step does not affect $\hat{y}_0$. This leads to development of the data

update method introduced in Section 5.1.2, which is arguably ad hoc. A potential area of improvement is to introduce a different matrix augmentation method that does not suffer from similar difficulties.

For the second RCPES variant, we believe the main difficulty comes from the fact that the proposed method approximates a polynomial cost function with a convex cost function. Currently, this approximation is based on experimental testing and not mathematical insight. Deeper analysis into the mathematical properties of the original polynomial cost function may yield an alternative approximation method that produces more reliable results.

In this dissertation, it was shown that RCKF can be viewed as static parameter estimation algorithm. Since RCPE is also a static parameter estimation algorithm, it might be fruitful to combine the two algorithms for concurrent parameter and state estimation. Perhaps this will yield the most reliable approach to solving the CSPE problem.

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