# A Mode Coupling Theory for Random Waveguides with Turning Points

by

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# LIST OF SYMBOLS

| | |
|---|---|
| $\partial_r$ | the partial derivative with respect to $r$ |
| $\delta(z)$ | the Dirac delta function |
| $\mathbb{R}$ | the real numbers |
| $\ll$ | much less than |
| $\widehat{f}$ | the Fourier transform of the function $f$ |
| $L^2(a,b)$ | the space of square-integrable functions on the interval $(a,b)$ |
| $\bar{z}$ | the complex conjugate of $z$ |
| $\lfloor\cdot\rfloor$ | integer part |
| $\setminus$ | set minus |
| $\lim_{z\to 0^+}$ | limit as $z$ approaches 0 from the right |
| $\lim_{z\to 0^-}$ | limit as $z$ approaches 0 from the left |
| $1_{(a,b)}(z)$ | the indicator function of the interval $(a,b)$ |
| $\mathbb{E}[\cdot]$ | expectation |
| $\Delta$ | Laplacian operator |
| $\gg$ | much greater than |
| $\sim$ | same order as |
| $\searrow$ | decreases to |
| $\nearrow$ | increases to |
| $\lesssim$ | less than up to multiplication by a positive constant |
| $\mathbf{arg}$ | argument of a complex number |
| $\delta_{jq}$ | Kronecker delta |
| $\mathbf{StD}[\cdot]$ | standard deviation |
| $\frac{d}{dz}$ | derivative with respect to z |
| $\mathbb{R}^p$ | $p$-dimensional Euclidean space |
| $\int \pi_q(dq)$ | integral with respect to the measure $\pi_q$ |
| $\mathbb{Z}$ | integers |
| $\mathbb{S}^p$ | $p$-dimensional torus |

| | |
|---|---|
| $\nabla_{\dot{\boldsymbol{\theta}}}$ | gradient with respect to $\dot{\boldsymbol{\theta}}$ |
| $\mathcal{C}_b^\infty(\mathbb{R}^d \times \mathbb{R}, \mathbb{R})$ | smooth bounded functions from $\mathbb{R}^d \times \mathbb{R}$ to $\mathbb{R}$ |
| $\sigma(\widehat{\boldsymbol{X}}^\varepsilon(s), s \leq z)$ | $\sigma$-algebra generated by $\widehat{\boldsymbol{X}}^\varepsilon(s)$ for $s \leq z$ |
| $*$ | convolution |
| $\ell^2(\mathbb{Z})$ | space of square-summable sequences |
| $\mathrm{Re}(\cdot)$ | real part of a complex number |
| $\mathrm{Im}(\cdot)$ | imaginary part of a complex number |

## ABSTRACT

We study acoustic waveguides with varying cross sections and slowly bending axes. In particular, we consider waveguides with rough walls and cross sectional width that varies slowly. Roughness means fast and small fluctuations that occur on the scale of the wavelength. The roughness in the walls is unknown in applications and so we model it as a random process to study the propagation of uncertainty in the walls to uncertainty in the wavefield. The slow variations occur on a scale much larger than the wavelength and cause jumps in the number of propagating modes supported by the guide. Here we present a mathematical analysis from first principles of waves in waveguides with an arbitrary but finite number of turning points.

We use our analysis to quantify randomization of the wavefield and transport of power in the guide. This is accomplished by obtaining a statistical description of coupled complex waveguide mode amplitudes in terms of the statistics of the fluctuations in the walls. Randomization is captured by decay of the means of the mode amplitudes with distance from the source. Transport of power is studied through differential equations for the second moments of the mode amplitudes. We show using these equations that the random fluctuations in the wall may increase or decrease net transmitted power depending upon the source excitation.

# CHAPTER I

# Introduction

A waveguide is a structure that directs the propagation of waves along a single direction. The waveguide effect may be due to reflecting boundaries as in [4] or variable wavespeed along an axis transverse to the direction of propagation as in [34]. Waveguides of both varieties appear across a diverse set of applications including underwater acoustics [51], electromagnetics [17, 46], optics [47, 37], and quantum waveguides [23]. The classic problem where the waveguide has reflecting straight walls and known wavespeed which varies only in the cross section of the waveguide is well-known. However, most applications do not fit this classical setting. Many waveguides have features such as varying wavespeed, varying cross section, and bends in the axis of the waveguide. We give in this dissertation a mathematical treatment of sound wave propagation in a waveguide with varying cross section, bends, and rough walls.

## 1.1 Review of Previous Work

We will refer to the classical setting where the waveguide has straight walls and is filled with a homogeneous medium as an ideal waveguide. One may solve the wave equation in an ideal waveguide by using separation of variables. This allows one to represent the wave field inside the waveguide as a superposition of uncoupled waveguide modes. These modes are either propagating or evanescent, have constant amplitude, and do not interact with one another.

If one allows for variation in the wavespeed due to filling the waveguide with a heterogeneous material, or variations in the geometry of the waveguide then the waveguide modes become coupled. There is a significant literature on solving such problems using numerical methods. These methods include multimodal techniques which rely on integrating Riccati equations for admittance or impedance matrices [42, 5, 24, 25, 41]. There are also finite difference methods as in [33] and numerous other approaches in the applied literature [15, 9, 18, 38, 16, 32, 8]. However, in the case where the mode coupling is induced by small fluctuations in wavespeed or roughness in the walls of the waveguide one can more precisely analyze the effects of the coupling using asymptotic methods.

Small-scale fluctuations in wavespeed or roughness in the walls of the waveguide are often unknowable in applications and introduce uncertainty. It then seems natural to model these features using randomness. One can describe the fluctuations in the wave speed or roughness in the walls using a random process. This allows for quantifying the cumulative scattering effects through determining how uncertainty in the model of the waveguide transfers to uncertainty in the wave field. This amounts to a statistical description of the pressure field in terms of the statistics of the driving random process in the model. Results of this sort were obtained for waveguides filled with random media both for sound [34, 20, 27, 28, 29] and electromagnetic waves [37, 3]. Further, waveguides with rough walls were studied in [4, 30, 10]. These types of results can then be used to inform imaging methods in the waveguide as in [12, 10, 14, 1].

If the walls of the waveguide slowly vary, then as is seen in [2, 50] one can use a modal decomposition and asymptotic methods to approximate the wavefield in the waveguide. Here propagating and evanescent mode amplitudes vary along the axis of the waveguide due to the influence of the variable geometry but are approximately independent of one another. In this setting, one must also account for turning points where the number of propagating modes supported by the waveguide jumps. This jump corresponds to a mode transitioning from propagating to evanescent and vice-versa as studied in [6]. Further, energy conservation implies that propagating modes will be reflected at turning points.

Combining the asymptotic methods used to study slowly varying waveguides with those used to study waveguides with rough walls is nontrivial due to the influence of the random fluctuations in the vicinity of the turning point. Recently, the case of weak random fluctuations in walls that affect only the turning modes was studied in [11]. Here we will consider the case of stronger random fluctuations which couple all of the waveguide modes.

## 1.2 Our Problem

We study time-harmonic sound waves in a 2-D waveguide which is slowly bending and has variations in its cross section. More precisely, we deal with a scalar Helmholtz equation at fixed frequency. The waveguide exhibits two types of variations in cross section. There are slow variations where slow means on a scale much larger than the wavelength. Additionally, there are fast fluctuations where fast means occurring on a scale comparable to the wavelength. The fast fluctuations correspond to roughness in the walls and are modeled as random as in [4, 30, 10]. Our goal is to quantify randomization of the wavefield and transport of power in the waveguide.

## 1.3 Outline

We address some preliminary material related to waveguides in chapter II. We begin with a review of wave propagation in an ideal waveguide and provide comments on the analysis of a waveguide with slowly varying cross section and bends. This chapter introduces the mode decomposition that we then generalize to the waveguide that we study when the modes become coupled.

We formulate our problem in chapter III and show how we can state it in an asymptotic framework. This involves the decomposition of the wavefield into forward/backward propagating modes and evanescent modes. The coefficients in the mode decomposition are random fields which satisfy a coupled system of equations driven by the random fluctuations of the boundary. We then use a technical construction to solve for the evanescent modes in terms of the propagating modes. This gives us a finite dimensional system for the propagating mode amplitudes and the tool for analyzing this system in an asymptotic setting is a diffusion limit theorem.

We then use our characterization of the limit mode amplitudes in chapter III to study the transport and reflection of power in the waveguide in chapter IV. We can use the infinitesimal generators of the limit mode amplitudes and the Kolmogorov backward equation to compute moments of the limit mode amplitudes. These moments allow us to quantify the effects of cumulative scattering at the random walls. We use these results to examine the power transmitted through the waveguide to the left of the source as well as the power going to the right of the source which is due both to the source excitation and reflection at turning points. We then demonstrate quantitatively how interactions with the random walls affect the net power transmitted through the waveguide with a few numerical illustrations at the end of the chapter.

In the remainder of this dissertation we address the mathematical tools that enabled our analysis in chapters III and IV. In chapter V we state and provide a proof of the diffusion limit theorem used in chapter III. The theorem is an extension of similar results found in [43, 26]. The proof of the theorem follows the format of the perturbed test function method described in [35, 26]. Chapter VI gives the analysis of the evanescent modes needed to close the system for the propagating modes in chapter III. Finally, we include as appendices material from stochastic analysis and the detailed computation of the infinitesimal generator used to calculate the moments of the mode amplitudes.

# CHAPTER II

# Preliminaries

Here we summarize the basic facts about wave propagation in waveguides. These form the foundation for our study of wave propagation in random waveguides with turning points in chapters III and IV.

## 2.1 The Ideal Waveguide

We begin with a review of sound wave propagation in a two-dimensional waveguide with straight walls and filled with a homogeneous medium. We refer to such a guide as an ideal waveguide. This is a classical problem whose solution can be found in many places but our presentation follows that of [14] as this setup is the most convenient later on.

### 2.1.1 Setup

We consider a two-dimensional acoustic waveguide with sound-soft boundary, filled with a homogeneous medium, and straight walls as illustrated in Figure 2.1. The acoustic pressure field in the guide $p(t, r, z)$ satisfies

$$\left(\partial_r^2 + \partial_z^2 - \frac{1}{c^2}\partial_t^2\right) p(t, r, z) = f(t)\delta(r - r_\star)\delta(z), \quad z \in \mathbb{R},\ r \in \left(-\frac{D}{2}, \frac{D}{2}\right), \tag{2.1}$$

where the right-hand side models a point source at $(r, z) = (r_\star, 0)$ emitting pulse $f(t)$. The parameter $c$ is the wavespeed which is constant since we assume the guide to

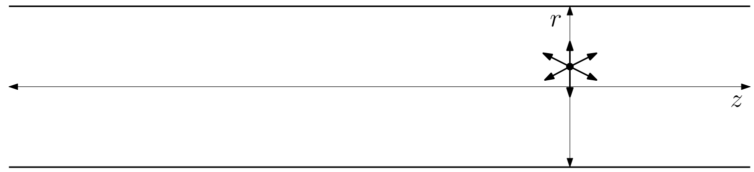


Figure 2.1: Illustration of an ideal waveguide

An ideal waveguide with horizontal axis $z$ and vertical axis $r$. The source of waves is at $z = 0$.

be filled with a homogeneous medium. The waveguide effect is due to the sound-soft boundaries, modeled mathematically using Dirichlet boundary conditions

$$p(t, -\frac{D}{2}, z) = p(t, \frac{D}{2}, z) = 0, \quad t, z \in \mathbb{R}. \tag{2.2}$$

Prior to the source excitation the medium is quiet

$$p(t, r, z) = 0, \quad t \ll 0. \tag{2.3}$$

We consider time-harmonic waves $p(t, r, z) = e^{-i\omega t}\widehat{p}(\omega, r, z)$ where $\widehat{p}(\omega, r, z)$ satisfies the Helmholtz equation

$$\left(\partial_r^2 + \partial_z^2 + k^2\right) \widehat{p}(\omega, r, z) = \widehat{f}(\omega)\delta(r - r_\star)\delta(z), \tag{2.4}$$

with Dirichlet conditions

$$\widehat{p}(\omega, \pm\frac{D}{2}, z) = 0, \tag{2.5}$$

and radiation conditions for $|z| \to \infty$. The parameter $k$ is given by $k = \omega/c$ and is called the wavenumber. This equation may be solved by separation of variables and decomposing $\widehat{p}$ into 1-D wavefields called modes.

### 2.1.2 Mode Decomposition

We decompose $\widehat{p}(\omega, r, z)$ into modes using Dirichlet eigenfunctions $y_j$ of the differential operator $\partial_r^2 + k^2$ with eigenvalues $\lambda_j(\omega)$. Each $y_j$ solves

$$(\partial_r^2 + k^2)y_j(r) = \lambda_j(\omega)y_j(r), \quad r \in \left(-\frac{D}{2}, \frac{D}{2}\right), \tag{2.6}$$

$$y_j\left(-\frac{D}{2}\right) = y_j\left(\frac{D}{2}\right) = 0. \tag{2.7}$$

Explicitly, the $y_j$ and $\lambda_j$ are given by

$$y_j(r) = \sqrt{\frac{2}{D}} \sin\left(\frac{\pi j}{D}\left(r + \frac{D}{2}\right)\right), \tag{2.8}$$

$$\lambda_j(\omega) = k^2 - \left(\frac{\pi j}{D}\right)^2. \tag{2.9}$$

Since (2.6)–(2.7) is a regular Sturm-Liouville problem, these eigenfunctions form a complete orthonormal set in the function space $L^2\left(-\frac{D}{2}, \frac{D}{2}\right)$. Thus, we have the decomposition

$$\widehat{p}(\omega, r, z) = \sum_{j=1}^{\infty} u_j(\omega, z) y_j(r). \tag{2.10}$$

where the modes are 1-D waves $u_j(\omega, z)e^{-i\omega t}$ with

$$u_j(\omega, z) := \langle \widehat{p}, y_j \rangle (\omega, z), \tag{2.11}$$

and $\langle \cdot, \cdot \rangle$ is the $L^2\left(-\frac{D}{2}, \frac{D}{2}\right)$ inner product given by

$$\langle f, g \rangle := \int_{-\frac{D}{2}}^{\frac{D}{2}} dr f(r)\overline{g(r)}. \tag{2.12}$$

The sign of the eigenvalues $\lambda_j$ determines whether the corresponding $j$-th mode is propagating or evanescent. The $\lambda_j(\omega)$ will be positive when $j < kD/\pi$. Thus, we can define the number of propagating modes $N(\omega)$ by

$$N(\omega) := \left\lfloor \frac{kD}{\pi} \right\rfloor. \tag{2.13}$$

We assume in what follows that none of the $\lambda_j(\omega) = 0$ so that there are no standing waves.

#### 2.1.3 Analysis of the Propagating and Evanescent Modes

From the decomposition (2.10), we obtain the 1-D Helmholtz equations

$$\left[\partial_z^2 + \beta_j^2(\omega)\right] u_j(\omega, z) = 0, \quad z \in \mathbb{R} \setminus \{0\} \tag{2.14}$$

where $\beta_j^2(\omega) := \lambda_j(\omega)$, and $j \leq N$. We rewrite these equations in first order system form as

$$\partial_z \begin{pmatrix} u_j(\omega, z) \\ v_j(\omega, z) \end{pmatrix} = i \begin{pmatrix} 0 & 1 \\ \beta_j^2(\omega) & 0 \end{pmatrix} \begin{pmatrix} u_j(\omega, z) \\ v_j(\omega, z) \end{pmatrix} \tag{2.15}$$

where $v_j(\omega, z) = -i\partial_z u_j$. We require this system to satisfy radiation conditions as $|z| \to \infty$ and jump conditions at $z = 0$

$$\lim_{z \to 0^+} \begin{pmatrix} u_j(\omega, z) \\ v_j(\omega, z) \end{pmatrix} - \lim_{z \to 0^-} \begin{pmatrix} u_j(\omega, z) \\ v_j(\omega, z) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{i}\widehat{f}(\omega) y_j(r_\star) \end{pmatrix} \tag{2.16}$$

The solution of (2.15) can be written explicitly using the propagator matrices

$$\mathbf{M}_j(\omega, z) := \begin{pmatrix} \frac{1}{\sqrt{\beta_j(\omega)}} e^{i\beta_j(\omega) z} & -\frac{1}{\sqrt{\beta_j(\omega)}} e^{-i\beta_j(\omega) z} \\ \sqrt{\beta_j(\omega)} e^{i\beta_j(\omega) z} & \sqrt{\beta_j(\omega)} e^{-i\beta_j(\omega) z} \end{pmatrix} \tag{2.17}$$

satisfying

$$\partial_z \mathbf{M}_j(\omega, z) = i \begin{pmatrix} 0 & 1 \\ \beta_j^2(\omega) & 0 \end{pmatrix} \mathbf{M}_j(\omega, z). \tag{2.18}$$

The solution is

$$\begin{pmatrix} u_j(\omega, z) \\ v_j(\omega, z) \end{pmatrix} = \mathbf{M}_j(\omega, z) \begin{pmatrix} a_j(\omega) \\ b_j(\omega) \end{pmatrix}. \tag{2.19}$$

where $a_j, b_j$ are mode amplitudes. Using the radiation conditions, which say that the wave is outgoing from the source we get

$$u_j(\omega, z) = 1_{(0,\infty)}(z) \frac{a_j(\omega)}{\sqrt{\beta_j(\omega)}} e^{i\beta_j(\omega) z} - 1_{(-\infty,0)}(z) \frac{b_j(\omega)}{\sqrt{\beta_j(\omega)}} e^{-i\beta_j(\omega) z}. \tag{2.20}$$

The mode amplitudes are obtained from (2.16) and (2.20) as

$$a_j(\omega) = -b_j(\omega) = \frac{1}{2i\sqrt{\beta_j(\omega)}} \widehat{f}(\omega) y_j(r_\star). \tag{2.21}$$

For $j > N$, the modes are evanescent and solve

$$\left[\partial_z^2 - \beta_j^2(\omega)\right] u_j(\omega, z) = 0, \quad z \in \mathbb{R} \setminus \{0\} \tag{2.22}$$

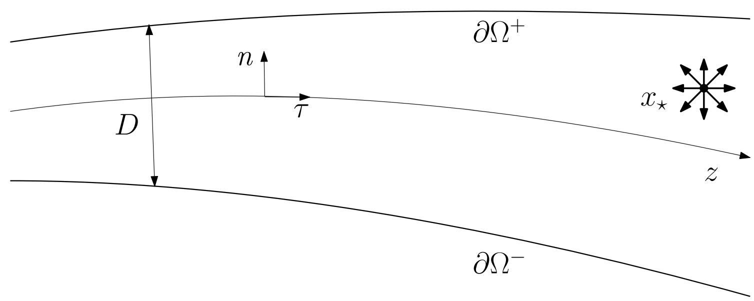


Figure 2.2: Illustration of a slowly varying waveguide

The guide has slowly varying width $D$ and bending axis parametrized by the arc length $z$. The boundary $\partial\Omega$ is the union of the curves $\partial\Omega^-$ (the bottom boundary) and $\partial\Omega^+$ (the top boundary). The unit tangent to the axis of the waveguide is denoted by $\boldsymbol{\tau}$ and the unit normal $\boldsymbol{n}$ points toward the upper boundary. The source of waves is at $\boldsymbol{x}_\star$.

where $\beta_j^2(\omega) := -\lambda_j(\omega)$. We require the $u_j$ to satisfy decay conditions $\lim_{|z|\to\infty} u_j(\omega, z) = 0$ as well as source conditions similar to (2.16). The evanescent waves both to the left and right of the source are given by

$$u_j(\omega, z) = \frac{e_j(\omega)}{\sqrt{\beta_j(\omega)}} e^{-\beta_j(\omega)|z|}, \quad z \in \mathbb{R} \tag{2.23}$$

where

$$e_j(\omega) = -\frac{1}{2\sqrt{\beta_j(\omega)}} \widehat{f}(\omega) y_j(r_\star). \tag{2.24}$$

Altogether, we have a full characterization of the time-harmonic wave field in the ideal waveguide. In particular, we may write

$$\begin{aligned} \widehat{p}(\omega, r, z) = \sum_{j=1}^{N} \Big[ 1_{(0,\infty)}(z) \frac{a_j(\omega)}{\sqrt{\beta_j(\omega)}} e^{i\beta_j(\omega)z} - 1_{(-\infty,0)}(z) \frac{b_j(\omega)}{\sqrt{\beta_j(\omega)}} e^{-i\beta_j(\omega)z} \Big] y_j(r) \\ + \sum_{j>N} \frac{e_j(\omega)}{\sqrt{\beta_j(\omega)}} e^{-\beta_j(\omega)|z|} y_j(r). \end{aligned} \tag{2.25}$$

Thus, the time-harmonic pressure field in the ideal waveguide is a superposition of constant amplitude 1-D waves that do not interact with one another.

## 2.2 Waveguides with Slowly Varying Geometry

One can extend the analysis in the previous sections to waveguides with slowly varying geometry (see Figure 2.2) as was done in [2, 50]. This case serves as a bridge between the problem in the ideal guide and the problem studied in this dissertation. We omit a detailed analysis as it can be recovered from the our analysis of a slowly varying guide with rough walls in the following chapter.

The wave equation will be the same as in (2.1) except that now the boundary changes along the axis of the guide. Also, since the waveguide may bend we work in curvilinear coordinates. The Sturm-Liouville operator in the radial direction remains the same and we have a local mode decomposition at each point along the axis of the guide. The slowly varying geometry means that changes in cross section or bending occur over distances that are very large with respect to the wavelength. This introduces a small parameter, the ratio of the wavelength to the length scale of the variations, that allows us to solve the problem using asymptotic methods. Here the mode amplitudes and eigenfunctions in the local mode decomposition will vary along the axis of the guide but the waveguide modes remain uncoupled [50]. When comparing to the ideal waveguide, the most notable difference in this setting is that the number of propagating modes in such a guide may change. These changes occur at turning points which were studied in [6].

# CHAPTER III

## Propagation in Random Waveguides with Turning Points

In this chapter we give the mathematical model for time-harmonic waves in a random waveguide with variable cross-section and bending axis. We begin in section 3.1 with the setup, and describe the scaling in section 3.2 in terms of a small, dimensionless parameter $\varepsilon$. We use this scaling in section 3.3 to write the wave problem in a form that can be analyzed in the asymptotic limit $\varepsilon \to 0$.

To analyze the solution of the perturbed wave equation obtained in section 3.3 we begin section 3.4 with the mode decomposition of the wavefield. These modes represent propagating and evanescent waves which are coupled by perturbation operators, as explained in section 3.5. We are interested in the propagating modes, which are left and right going waves with random amplitudes satisfying a stochastic system of equations derived in section 3.6. It is this system that we analyze in the asymptotic limit to quantify the cumulative scattering effects in the waveguide.

The limit is taken in each sector of the waveguide, bounded by two consecutive turning points, as explained in section 3.7. We introduce in section 3.8 a simplification, known as the forward scattering approximation, which applies to sufficiently smooth random fluctuations $\nu$. Finally, the limit of the mode amplitudes under this approximation is described in section 3.9.

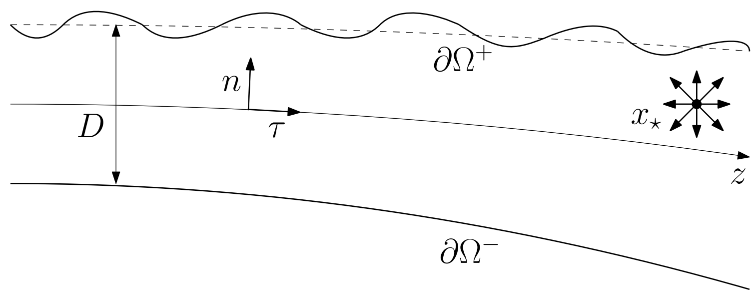


Figure 3.1: Illustration of a random waveguide with turning points

The guide has slowly varying width $D$ and bending axis parametrized by the arc length $z$. The boundary $\partial\Omega$ is the union of the curves $\partial\Omega^-$ (the bottom boundary) and $\partial\Omega^+$ (the top boundary). The top boundary is perturbed by small random fluctuations. The unit tangent to the axis of the waveguide is denoted by $\boldsymbol{\tau}$ and the unit normal $\boldsymbol{n}$ points toward the upper boundary. The source of waves is at $\boldsymbol{x}_\star$.

## 3.1 Setup

We consider a two-dimensional acoustic waveguide with sound-soft boundary. The waveguide occupies the semi-infinite domain $\Omega$, bounded above and below by two curves $\partial\Omega^+$ and $\partial\Omega^-$, as shown in Figure 3.1. The top boundary $\partial\Omega^+$ is perturbed by small random fluctuations about the curve $\partial\Omega_0^+$ shown in the figure with the dotted line. The axis of the waveguide is at half the distance $D$ between $\partial\Omega_0^+$ and $\partial\Omega^-$. It is a smooth curve parametrized by the arc length $z \in \mathbb{R}$, that bends slowly, meaning that its tangent $\boldsymbol{\tau}(z/L)$ and curvature $\kappa(z/L)$ vary on a scale $L$ which is large with respect to the waveguide width $D(z/L)$. The width function $D$ has bounded first two derivatives, and to avoid complications in the analysis of scattering of the waves at the random boundary, we also assume that it is monotonically increasing.

Because of the changing geometry, it is convenient to use orthogonal curvilinear coordinates with axes along $\boldsymbol{\tau}(z/L)$ and $\boldsymbol{n}(z/L)$, where $\boldsymbol{n}$ is the unit vector orthogonal to $\boldsymbol{\tau}$, pointing toward the upper boundary. We assume any $\boldsymbol{x} \in \Omega$, written henceforth as $\boldsymbol{x} = (r, z)$, can be written uniquely as

$$\boldsymbol{x} = \boldsymbol{x}_{\|}(z) + r\boldsymbol{n}\Big(\frac{z}{L}\Big), \tag{3.1}$$

where $\boldsymbol{x}_{\parallel}(z)$ is along the waveguide axis at arc length $z$, satisfying

$$\partial_z \boldsymbol{x}_{\parallel}(z) = \boldsymbol{\tau}\Big(\frac{z}{L}\Big), \tag{3.2}$$

and $r$ is the coordinate in the normal direction. This holds provided that the bends in the axis of the guide are mild. The domain $\Omega$ is the set

$$\Omega := \big\{(r,z): \ z \in \mathbb{R}, \ r \in (r^-(z), r^+(z))\big\}, \tag{3.3}$$

where

$$r^-(z) := -\frac{D(z/L)}{2}, \tag{3.4}$$

is at the bottom boundary $\partial\Omega^-$ and

$$r^+(z) := \frac{D(z/L)}{2}\left[1 + 1_{(-Z_M,Z_M)}(z)\,\sigma\nu\Big(\frac{z}{\ell}\Big)\right], \tag{3.5}$$

is at the randomly perturbed top boundary $\partial\Omega^+$. The perturbation is modeled by the random process $\nu$ and it extends over the interval $(-Z_M, Z_M)$, the support of the indicator function $1_{(-Z_M,Z_M)}(z)$, where $Z_M > L$ is a long scale needed to impose outgoing boundary conditions on the waves. This truncation is practically motivated by the relationship between distance over which the wave is influenced by the fluctuations and the duration of the observation time of the wave, using hyperbolicity of the wave equation in the time domain. The single-frequency wave that we analyze is the Fourier transform of the time-dependent wavefield. We let the boundaries of the waveguide be straight and parallel for $|z| > Z_M$.

The random process $\nu$ is stationary with zero mean

$$\mathbb{E}\big[\nu(\zeta)\big] = 0. \tag{3.6}$$

Its auto-correlation function is given by

$$\mathcal{R}(\zeta) := \mathbb{E}\big[\nu(0)\nu(\zeta)\big]. \tag{3.7}$$

We assume that $\nu$ is mixing, with rapidly decaying mixing rate, as defined for example in [43, section 2], and it is bounded, with bounded first two derivatives, almost surely. This implies in particular that $\mathcal{R}$ is integrable and has at least four bounded derivatives. We normalize $\nu$ by

$$\mathcal{R}(0) = 1, \tag{3.8}$$

so that $\sigma$ in (3.5) is the standard deviation of the fluctuations of $\partial\Omega^+$, and $\ell$ quantifies their correlation length.

The waves are generated by a point source at $\boldsymbol{x}_\star = (r_\star, z_\star = 0) \in \Omega$, which emits a complex signal $f(\omega)$ at frequency $\omega$. We take the origin of $z$ at the source, so that $z_\star = 0$. The waveguide is filled with a homogeneous medium with wave speed $c$, and the wavefield $p(\omega, \boldsymbol{x})$ satisfies the Helmholtz equation (we refer here to the principle of limit amplitude, see [21, 39, 40] and references therein)

$$\Delta p(\omega, \boldsymbol{x}) + k^2 p(\omega, \boldsymbol{x}) = f(\omega)\delta(\boldsymbol{x} - \boldsymbol{x}_\star), \quad \boldsymbol{x} = (r, z) \in \Omega, \tag{3.9}$$

where $k = \omega/c$ is the wavenumber.

We then make a change of variables to $(r, z)$-coordinates. The vectors in our coordinate frame are related through the Frenet-Serret formulas

$$\begin{aligned} \partial_z \boldsymbol{x}_{\|}(z) &= \boldsymbol{\tau}\Big(\frac{z}{L}\Big), \\ \partial_z \boldsymbol{\tau}\Big(\frac{z}{L}\Big) &= \frac{1}{L}\kappa\Big(\frac{z}{L}\Big)\boldsymbol{n}\Big(\frac{z}{L}\Big), \\ \partial_z \boldsymbol{n}\Big(\frac{z}{L}\Big) &= -\frac{1}{L}\kappa\Big(\frac{z}{L}\Big)\boldsymbol{\tau}\Big(\frac{z}{L}\Big), \end{aligned}$$

and from (3.1) we obtain that the vectors $\partial_r \boldsymbol{x} = \boldsymbol{n}\left(\frac{z}{L}\right)$ and $\partial_z \boldsymbol{x} = \left[1 - \frac{r}{L}\kappa\left(\frac{z}{L}\right)\right]\boldsymbol{\tau}\left(\frac{z}{L}\right)$ are orthogonal. Their norm defines the Lamé coefficients $h_r := |\partial_r \boldsymbol{x}| = 1$ and $h_z := |\partial_z \boldsymbol{x}| = \left|1 - \frac{r}{L}\kappa\left(\frac{z}{L}\right)\right|$, which in turn define the Laplacian operator in curvilinear coordinates [48] $\Delta = \frac{1}{h_r h_z}\left[\partial_r\left(\frac{h_z}{h_r}\partial_r\right) + \partial_x\left(\frac{h_r}{h_z}\partial_z\right)\right]$. We can also express the delta function on the right-hand side of (3.9) as $\delta(\boldsymbol{x} - \boldsymbol{x}_\star) = \frac{1}{h_r h_z}\delta(z)\delta(r - r_\star)$.

Thus, in curvilinear coordinates (3.9) takes the form

$$\left[\partial_r^2 - \frac{\frac{1}{L}\kappa\left(\frac{z}{L}\right)\partial_r}{1-\frac{r}{L}\kappa\left(\frac{z}{L}\right)} + \frac{\partial_z^2}{\left[1-\frac{r}{L}\kappa\left(\frac{z}{L}\right)\right]^2} + \frac{\frac{r}{L^2}\kappa'\left(\frac{z}{L}\right)\partial_z}{\left[1-\frac{r}{L}\kappa\left(\frac{z}{L}\right)\right]^3} + k^2\right] p(\omega,r,z)$$
$$= \left|1-\frac{r_\star}{L}\kappa(0)\right|^{-1} f(\omega)\delta(z)\delta(r-r_\star), \qquad (3.10)$$

where $\kappa'$ is the derivative of the curvature $\kappa$. The sound-soft boundary $\partial\Omega^+ \cup \partial\Omega^-$ is modeled by the homogeneous Dirichlet boundary conditions

$$p(\omega, r^+(z), z) = p(\omega, r^-(z), z) = 0, \qquad (3.11)$$

and at points $\boldsymbol{x} = (r,z)$ with $|z| > Z_M$ we have radiation conditions that state that $p(\omega, r, z)$ is outgoing and bounded.

### 3.2 Scaling

There are four length scales in the problem: The wavelength $\lambda = 2\pi/k$, the width of the waveguide $D$, the scale $L$ of the slow variations of the waveguide, and the correlation length $\ell$ of the random fluctuations of the boundary $\partial\Omega^+$. They satisfy

$$L \gg D \sim \lambda \sim \ell, \qquad (3.12)$$

where $\sim$ denotes "of the same order as"[1], and we model the separation of scales using the dimensionless parameter

$$\varepsilon := \frac{\ell}{L}, \quad 0 < \varepsilon \ll 1. \qquad (3.13)$$

Our analysis of the wavefield $p(\omega, r, z)$ is in the asymptotic limit $\varepsilon \to 0$.

As shown in section 3.5, the ratio of $D$ and $\lambda/2$ defines $N(z) := \lfloor 2D(z/L)/\lambda \rfloor$, the number of propagating components of the wave, called modes, where $\lfloor\ \rfloor$ denotes the integer part. The assumption $D \sim \lambda$ in (3.12) means that

$$N_{\min} \leq N(z) \leq N_{\max}, \qquad (3.14)$$

[1]To be precise, we write $a \sim b$ if there exists positive constants $c$ and $C$ such that $cb \leq a \leq Cb$.

for all $z$, where $N_{\min}$ and $N_{\max}$ are natural numbers, independent of $\varepsilon$.

The scales $\lambda$ and $\ell$ are of the same order in (3.12) so that the waves interact efficiently with the random fluctuations of the boundary. This interaction, called cumulative scattering, randomizes the wavefield as it propagates in the waveguide. The distance from the source at which the randomization occurs depends on the standard deviation $\sigma$ of the fluctuations. We scale $\sigma$ as

$$\sigma = \sqrt{\varepsilon}\tilde{\sigma}, \quad \tilde{\sigma} = O(1), \tag{3.15}$$

so that we observe the randomization at distances $z \sim L$.

The scaled variables are defined as follows: The arc length $z$ is scaled by $L$,

$$\tilde{z} := \frac{z}{L}, \tag{3.16}$$

and the similar lengths $D$, $r$ and $\lambda$ are scaled by $\ell$, to obtain

$$\tilde{D}(\tilde{z}) := \frac{D(z/L)}{\ell}, \; \tilde{r} := \frac{r}{\ell}, \; \tilde{k} := k\ell. \tag{3.17}$$

We also introduce the scaled bound $\tilde{Z}_M := Z_M/L$ of the support of the random fluctuations, which is a large number, independent of $\varepsilon$.

## 3.3 Asymptotic model

Let us multiply equation (3.10) by $L^2[1 - r\kappa/L]^2$ and use the scaling relations (3.15)-(3.17). Dropping the tilde to simplify notation, because all variables are scaled henceforth, we obtain

$$\left[\partial_z^2 + \frac{(1-\varepsilon r\kappa(z))^2}{\varepsilon^2}(\partial_r^2 + k^2) - \frac{\kappa(z)(1-\varepsilon r\kappa(z))}{\varepsilon}\partial_r + \frac{\varepsilon r\kappa'(z)}{(1-\varepsilon r\kappa(z))}\partial_z\right] p(\omega, r, z)$$
$$= \frac{f(\omega)[1-\varepsilon r_\star\kappa(0)]}{\varepsilon}\delta(r - r_\star)\delta(z), \tag{3.18}$$

with homogeneous Dirichlet boundary conditions (3.11) at

$$r^-(z) = -\frac{D(z)}{2}, \; r^+(z) = \frac{D(z)}{2}\left[1 + 1_{(-Z_M, Z_M)}(z)\sqrt{\varepsilon}\sigma\nu\left(\frac{z}{\varepsilon}\right)\right], \tag{3.19}$$

and appropriate radiation conditions for $|z| > Z_M$. These equations define the asymptotic model for the wavefield, and we wish to analyze it in the limit $\varepsilon \to 0$.

The boundary has $\varepsilon$-dependent fluctuations, so to ensure that the boundary conditions are satisfied at all orders of $\varepsilon$, we change variables to

$$r = \rho + \frac{[2\rho + D(z)]}{4}\sqrt{\varepsilon}\sigma\nu\Big(\frac{z}{\varepsilon}\Big), \tag{3.20}$$

for $|z| < Z_M$, and denote the transformed wavefield by

$$p^\varepsilon(\omega,\rho,z) := p\Big(\omega, \rho + \frac{(2\rho + D(z))}{4}\sqrt{\varepsilon}\sigma\nu\Big(\frac{z}{\varepsilon}\Big), z\Big). \tag{3.21}$$

Note that with this change of variables when $\rho = -D(z)/2$ we have $r = r^-(z)$, and when $\rho = D(z)/2$ we have $r = r^+(z)$. At $|z| > Z_M$ there are no fluctuations so the transformation is the identity $r = \rho$. We use the same notation $p^\varepsilon$ for the wave field at all $z \in \mathbb{R}$, and analyze it separately in the regions with the random fluctuations and without. The results are connected by continuity at $z = \pm Z_M$.

In the region $|z| < Z_M$, (3.20) and the chain rule give

$$\partial_r p = \frac{\partial_\rho p^\varepsilon(\omega,\rho,z)}{1 + \frac{\sqrt{\varepsilon}}{2}\sigma\nu\big(\frac{z}{\varepsilon}\big)}, \quad \partial_r^2 p = \frac{\partial_\rho^2 p^\varepsilon(\omega,\rho,z)}{\big[1 + \frac{\sqrt{\varepsilon}}{2}\sigma\nu\big(\frac{z}{\varepsilon}\big)\big]^2},$$

and

$$\partial_z p = \Big\{\partial_z - \frac{\big[[2\rho + D(z)]\frac{\sigma}{\sqrt{\varepsilon}}\nu'\big(\frac{z}{\varepsilon}\big) + D'(z)\sqrt{\varepsilon}\sigma\nu\big(\frac{z}{\varepsilon}\big)\big]}{2\big[2 + \sqrt{\varepsilon}\sigma\nu\big(\frac{z}{\varepsilon}\big)\big]}\partial_\rho\Big\}p^\varepsilon(\omega,\rho,z),$$

$$\partial_z^2 p = \Big\{\partial_z - \frac{\big[[2\rho + D(z)]\frac{\sigma}{\sqrt{\varepsilon}}\nu'\big(\frac{z}{\varepsilon}\big) + D'(z)\sqrt{\varepsilon}\sigma\nu\big(\frac{z}{\varepsilon}\big)\big]}{2\big[2 + \sqrt{\varepsilon}\sigma\nu\big(\frac{z}{\varepsilon}\big)\big]}\partial_\rho\Big\}^2 p^\varepsilon(\omega,\rho,z).$$

Substituting in (3.18) we get

$$
\begin{aligned}
&\partial_z^2 p^\varepsilon(\omega,\rho,z) + \frac{\Big[[2\rho+D(z)]\frac{\sigma}{\sqrt{\varepsilon}}\nu'\big(\frac{z}{\varepsilon}\big) + D'(z)\sqrt{\varepsilon}\sigma\nu\big(\frac{z}{\varepsilon}\big)\Big]^2}{4\Big[2+\sqrt{\varepsilon}\sigma\nu\big(\frac{z}{\varepsilon}\big)\Big]^2}\partial_\rho^2 p^\varepsilon(\omega,z)\\
&- \frac{\Big[[2\rho+D(z)]\frac{\sigma}{\sqrt{\varepsilon}}\nu'\big(\frac{z}{\varepsilon}\big) + D'(z)\sqrt{\varepsilon}\sigma\nu\big(\frac{z}{\varepsilon}\big)\Big]}{\Big[2+\sqrt{\varepsilon}\sigma\nu\big(\frac{z}{\varepsilon}\big)\Big]}\partial_{\rho z}^2 p^\varepsilon(\omega,z)\\
&+ \frac{\Big\{[2\rho+D(z)]\frac{\sigma^2}{\varepsilon}\nu'^{\,2}\big(\frac{z}{\varepsilon}\big) + D'(z)\sigma^2\nu'\big(\frac{z}{\varepsilon}\big)\nu\big(\frac{z}{\varepsilon}\big)\Big\}}{\Big[2+\sqrt{\varepsilon}\sigma\nu\big(\frac{z}{\varepsilon}\big)\Big]^2}\partial_\rho p^\varepsilon(\omega,\rho,z)\\
&- \frac{[2\rho+D(z)]\frac{\sigma}{\varepsilon^{3/2}}\nu''\big(\frac{z}{\varepsilon}\big) + 2D'(z)\frac{\sigma}{\sqrt{\varepsilon}}\nu'\big(\frac{z}{\varepsilon}\big) + D''(z)\sqrt{\varepsilon}\sigma\nu\big(\frac{z}{\varepsilon}\big)}{2\Big[2+\sqrt{\varepsilon}\sigma\nu\big(\frac{z}{\varepsilon}\big)\Big]}\partial_\rho p^\varepsilon(\omega,\rho,z)\\
&+ \frac{\Big\{1-\varepsilon\kappa(z)\big[\rho+\frac{[2\rho+D(z)]}{4}\sqrt{\varepsilon}\sigma\nu\big(\frac{z}{\varepsilon}\big)\big]\Big\}^2}{\varepsilon^2}\Bigg\{\frac{\partial_\rho^2 p^\varepsilon(\omega,\rho,z)}{\Big[1+\frac{\sqrt{\varepsilon}}{2}\sigma\nu\big(\frac{z}{\varepsilon}\big)\Big]^2} + k^2 p^\varepsilon(\omega,\rho,z)\Bigg\}\\
&- \frac{\kappa(z)\Big\{1-\varepsilon\kappa(z)\big[\rho+\frac{[2\rho+D(z)]}{4}\sqrt{\varepsilon}\sigma\nu\big(\frac{z}{\varepsilon}\big)\big]\Big\}}{\varepsilon[1+\frac{\sqrt{\varepsilon}}{2}\sigma\nu\big(\frac{z}{\varepsilon}\big)]}\partial_\rho p^\varepsilon(\omega,\rho,z)\\
&+ \frac{\varepsilon\kappa'(z)\big[\rho+\frac{[2\rho+D(z)]}{4}\sqrt{\varepsilon}\sigma\nu\big(\frac{z}{\varepsilon}\big)\big]}{\big\{1-\varepsilon\kappa(z)\big[\rho+\frac{[2\rho+D(z)]}{4}\sqrt{\varepsilon}\sigma\nu\big(\frac{z}{\varepsilon}\big)\big]\big\}}\Bigg\{\partial_z p^\varepsilon(\omega,\rho,z)\\
&\qquad\qquad - \frac{\Big[[2\rho+D(z)]\frac{\sigma}{\sqrt{\varepsilon}}\nu'\big(\frac{z}{\varepsilon}\big) + D'(z)\sqrt{\varepsilon}\sigma\nu\big(\frac{z}{\varepsilon}\big)\Big]}{2\Big[2+\sqrt{\varepsilon}\sigma\nu\big(\frac{z}{\varepsilon}\big)\Big]}\partial_\rho p^\varepsilon(\omega,\rho,z)\Bigg\}\\
&= \frac{f(\omega)\Big\{1-\varepsilon\Big[\rho_\star+\frac{[2\rho+D(0)]}{4}\sqrt{\varepsilon}\sigma\nu(0)\Big]\Big\}}{\varepsilon\Big[1+\frac{\sqrt{\varepsilon}}{2}\sigma\nu(0)\Big]}\delta(\rho-\rho_\star)\delta(z).
\end{aligned}
$$

By assumption $\nu$, $\nu'$ and $\nu''$ are bounded almost surely. Moreover, $\kappa'$ and $D'$, $D''$ are bounded uniformly in $\mathbb{R}$. Thus, we can expand the coefficients of the differential operator in powers of $\varepsilon$ and obtain after multiplying through by $\varepsilon$,

$$
\begin{aligned}
&\frac{1}{\varepsilon}\Big[(\varepsilon\partial_z)^2+\partial_\rho^2+k^2\Big]p^\varepsilon(\omega,\rho,z)-2\rho\kappa(z)\big[1+O(\sqrt{\varepsilon})\big]\big(\partial_\rho^2+k^2\big)\\
&-\kappa(z)\big[1+O(\sqrt{\varepsilon})\big]\partial_\rho p^\varepsilon(\omega,\rho,z)-\varepsilon\kappa'(z)\big[1+O(\sqrt{\varepsilon})\big](\varepsilon\partial_z)p^\varepsilon(\omega,\rho,z)\\
&-\frac{[2\rho+D(z)]}{2}\Big[\frac{\sigma}{\sqrt{\varepsilon}}\nu'\Big(\frac{z}{\varepsilon}\Big)-\frac{\sigma^2}{2}\nu'\Big(\frac{z}{\varepsilon}\Big)\nu\Big(\frac{z}{\varepsilon}\Big)+O(\sqrt{\varepsilon})\Big]\varepsilon\partial_{\rho z}^2p^\varepsilon(\omega,\rho,z)\\
&-\Big\{\frac{\sigma}{\sqrt{\varepsilon}}\nu\Big(\frac{z}{\varepsilon}\Big)-\frac{3\sigma^2}{4}\nu^2\Big(\frac{z}{\varepsilon}\Big)-\frac{[2\rho+D(z)]^2\sigma^2}{16}\nu'^{\,2}\Big(\frac{z}{\varepsilon}\Big)+O(\sqrt{\varepsilon})\Big\}\partial_\rho^2p^\varepsilon(\omega,\rho,z)\\
&-\frac{\big[2\rho+D(z)\big]}{4}\Big\{\frac{\sigma}{\sqrt{\varepsilon}}\nu''\Big(\frac{z}{\varepsilon}\Big)-\frac{\sigma^2}{2}\nu''\Big(\frac{z}{\varepsilon}\Big)\nu\Big(\frac{z}{\varepsilon}\Big)-\sigma^2\nu'^{\,2}\Big(\frac{z}{\varepsilon}\Big)+O(\sqrt{\varepsilon})\Big\}\partial_\rho p^\varepsilon(\omega,\rho,z)\\
&=f(\omega)\Big[1+O(\sqrt{\varepsilon})\Big]\delta(\rho-\rho_\star)\delta(z), \qquad (3.22)
\end{aligned}
$$

for $|z|<Z_M$, where for brevity we display only the terms in the expansions up to $O(\sqrt{\varepsilon})$.

We may express the wave equation (3.22) more compactly using an asymptotic series of differential operators which includes the full expansions of the coefficients

$$\sum_{j=0}^{\infty}\varepsilon^{j/2-1}\mathcal{L}_j^\varepsilon\, p^\varepsilon(\omega,\rho,z)=\widehat{f}(\omega)\big[1+O(\sqrt{\varepsilon})\big]\delta(\rho-\rho_\star)\delta(z), \qquad (3.23)$$

for $|\rho|<D(z)/2$ and $|z|<Z_M$, with the leading term in the expansion of the operator

$$\mathcal{L}_0^\varepsilon:=\big(\varepsilon\partial_z\big)^2+\partial_\rho^2+k^2. \qquad (3.24)$$

This is the Helmholtz operator in a perfect waveguide, with straight and parallel boundaries. The random fluctuations appear in the first perturbation operator,

$$\mathcal{L}_1^\varepsilon:=-\sigma\Big\{\nu\Big(\frac{z}{\varepsilon}\Big)\partial_\rho^2+\frac{[2\rho+D(z)]}{4}\Big[\nu''\Big(\frac{z}{\varepsilon}\Big)\partial_\rho+2\nu'\Big(\frac{z}{\varepsilon}\Big)\varepsilon\partial_{\rho z}^2\Big]\Big\}. \qquad (3.25)$$

The second perturbation operator has a deterministic part, due to the curvature of the axis of the waveguide, and a random part, quadratic in the random fluctuations,

$$
\begin{aligned}
\mathcal{L}_2^\varepsilon:=&-\kappa(z)\big[2\rho\big(\partial_\rho^2+k^2\big)+\partial_\rho\big]+\frac{\sigma^2}{4}\Big\{3\nu^2\Big(\frac{z}{\varepsilon}\Big)+\frac{\big[2\rho+D(z)\big]^2}{4}\nu'^{\,2}\Big(\frac{z}{\varepsilon}\Big)\Big\}\partial_\rho^2\\
&+\frac{[2\rho+D(z)]\sigma^2}{4}\Big\{\nu\Big(\frac{z}{\varepsilon}\Big)\nu'\Big(\frac{z}{\varepsilon}\Big)\varepsilon\partial_{\rho z}^2+\Big[\nu'^{\,2}\Big(\frac{z}{\varepsilon}\Big)+\frac{1}{2}\nu''\Big(\frac{z}{\varepsilon}\Big)\nu\Big(\frac{z}{\varepsilon}\Big)\Big]\partial_\rho\Big\}. \qquad (3.26)
\end{aligned}
$$

The remaining operators in the asymptotic series in (3.23) depend on higher powers of the fluctuations $\nu$, but play no role in the limit $\varepsilon \to 0$.

In the region $|z| > Z_M$, there are no variations of the waveguide and so the operator in the left hand side of (3.23) reduces to $\mathcal{L}_0^\varepsilon$ in this region.

In all, the change of variables (3.20) makes the boundary conditions independent of $\varepsilon$,

$$p^\varepsilon\left(\omega, \pm\frac{D(z)}{2}, z\right) = 0, \tag{3.27}$$

and maps the random fluctuations to the differential operator in the wave equation.

### 3.4 Mode decomposition

The second two terms in (3.24) are the Sturm-Liouville operator $\partial_\rho^2 + k^2$ acting on functions that vanish at $\rho = \pm D(z)/2$, for any given $z$. Its eigenvalues $\lambda_j$ are real and distinct

$$\lambda_j(z) = k^2 - \mu_j^2(z), \qquad \mu_j(z) := \frac{\pi j}{D(z)},\ j = 1, 2, \ldots \tag{3.28}$$

and the eigenfunctions

$$y_j(\rho, z) = \sqrt{\frac{2}{D(z)}} \sin\left[\frac{(2\rho + D(z))}{2}\mu_j(z)\right], \tag{3.29}$$

form an orthonormal $L^2$ basis in $[-D(z)/2, D(z)/2]$. We use this basis to decompose the solution of (3.24) in one dimensional waves $p_j^\varepsilon(\omega, z)$ called modes, for each $z$,

$$p^\varepsilon(\omega, \rho, z) = \sum_{j=1}^{\infty} p_j^\varepsilon(\omega, z) y_j(\rho, z). \tag{3.30}$$

Substituting (3.30) in (3.23), taking the inner product with $y_j(\rho, z)$ and using the identities (3.137)-(3.142) we obtain the following system of equations for the modes

$$
\begin{aligned}
&\frac{1}{\varepsilon}\Big[(\varepsilon\partial_z)^2 + k^2 - \mu_j^2(z)\Big]p_j^\varepsilon(\omega,z) + \frac{\sigma}{\sqrt{\varepsilon}}\Big[\mu_j^2(z)\nu\Big(\frac{z}{\varepsilon}\Big) + \frac{1}{4}\nu''\Big(\frac{z}{\varepsilon}\Big) + \frac{1}{2}\nu'\Big(\frac{z}{\varepsilon}\Big)\varepsilon\partial_z\Big]p_j^\varepsilon(\omega,z) \\
&\qquad - \frac{\sigma^2}{4}\Big\{3\mu_j^2(z)\nu^2\Big(\frac{z}{\varepsilon}\Big) + \Big[\frac{(\pi j)^2}{3} + \frac{1}{2}\Big]\nu'^2\Big(\frac{z}{\varepsilon}\Big) + \frac{1}{2}\nu\Big(\frac{z}{\varepsilon}\Big)\nu''\Big(\frac{z}{\varepsilon}\Big)\Big]p_j^\varepsilon(\omega,z) \\
&\qquad - \frac{\sigma^2}{4}\nu\Big(\frac{z}{\varepsilon}\Big)\nu'\Big(\frac{z}{\varepsilon}\Big)\varepsilon\partial_z p_j^\varepsilon(\omega,z) \approx C_j^\varepsilon(\omega,z) + f(\omega)y_j(\rho_\star,0)\delta(z), \qquad (3.31)
\end{aligned}
$$

at $|z| < Z_M$, where the approximation is because we neglect the $O(\sqrt{\varepsilon})$ terms that vanish in the limit $\varepsilon \to 0$. The coupling term is

$$
\begin{aligned}
C_j^\varepsilon(\omega,z) := \sum_{q=1,q\neq j}^{\infty}\Big\{ &\frac{2jq(-1)^{j+q}}{(q^2-j^2)}\Big[\frac{\sigma}{\sqrt{\varepsilon}}\nu'\Big(\frac{z}{\varepsilon}\Big) - \frac{\sigma^2}{2}\nu\Big(\frac{z}{\varepsilon}\Big)\nu'\Big(\frac{z}{\varepsilon}\Big)\Big]\varepsilon\partial_z p_q^\varepsilon(\omega,z) \\
&+ \frac{D'(z)}{D(z)}\frac{2jq[1+(-1)^{j+q}]}{(q^2-j^2)}\varepsilon\partial_z p_q^\varepsilon(\omega,z) \\
&+ \frac{jq(-1)^{j+q}}{(q^2-j^2)}\Big[\frac{\sigma}{\sqrt{\varepsilon}}\nu''\Big(\frac{z}{\varepsilon}\Big) - \frac{\sigma^2}{2}\nu\Big(\frac{z}{\varepsilon}\Big)\nu''\Big(\frac{z}{\varepsilon}\Big)\Big]p_q^\varepsilon(\omega,z) \\
&+ \frac{jq(j^2+q^2)(-1)^{j+q}}{(q^2-j^2)^2}\sigma^2\nu'^2\Big(\frac{z}{\varepsilon}\Big)p_q^\varepsilon(\omega,z) \\
&+ \frac{\kappa(z)}{D(z)}\frac{2jq[1-(-1)^{j+q}]\big[j^2+3q^2-4\big(\frac{kD(z)}{\pi}\big)^2\big]}{(q^2-j^2)^2}p_q^\varepsilon(\omega,z)\Big\}, \qquad (3.32)
\end{aligned}
$$

where we obtained from (3.143)-(3.147) that

$$
\begin{aligned}
\big\langle(2\rho+D)y_j, \partial_\rho y_q\big\rangle &= \frac{4jq(-1)^{j+q}}{q^2-j^2}, \\
\langle y_j, \partial_z y_q\rangle &= \frac{D'(z)}{D(z)}\frac{jq[1+(-1)^{j+q}]}{j^2-q^2}, \\
\frac{\mu_q^2(z)\,\langle(2\rho+D)^2 y_j, y_q\rangle}{16} - \frac{\big\langle(2\rho+D)y_j, \partial_\rho y_q\big\rangle}{4} &= \frac{jq(j^2+q^2)(-1)^{j+q}}{(q^2-j^2)^2}, \\
\big(k^2-\mu_q^2(z)\big)\,\langle(2\rho+D)y_j, y_q\rangle + \langle y_j, \partial_\rho y_q\rangle &= \frac{2jq[1-(-1)^{j+q}]}{D(z)(q^2-j^2)^2} \\
&\quad\times\Big[j^2+3q^2-4\Big(\frac{kD(z)}{\pi}\Big)^2\Big].
\end{aligned}
$$

We now use integrating factors to simplify equations (3.31). Specifically, we define

$$
u_j^\varepsilon(\omega,z) := p_j^\varepsilon(\omega,z)\exp\Big[\frac{\sigma\sqrt{\varepsilon}}{4}\nu\Big(\frac{z}{\varepsilon}\Big) - \frac{\sigma^2\varepsilon}{16}\nu^2\Big(\frac{z}{\varepsilon}\Big)\Big] = p_j^\varepsilon(\omega,z)[1+O(\sqrt{\varepsilon})], \qquad (3.33)
$$

and obtain after substituting in (3.31) that these modes satisfy a coupled system of one dimensional wave equations

$$\frac{1}{\varepsilon}\Big[(\varepsilon\partial_z)^2 + k^2 - \mu_j^2(z)\Big]u_j^\varepsilon(\omega,z) + \frac{\sigma}{\sqrt{\varepsilon}}\mu_j^2(z)\nu\Big(\frac{z}{\varepsilon}\Big)u_j^\varepsilon(\omega,z) + \sigma^2 g_j^\varepsilon(\omega,z)u_j^\varepsilon(\omega,z)$$
$$\approx \mathcal{C}_j^\varepsilon(\omega,z) + f(\omega)y_j(\rho_\star,0)\delta(z). \tag{3.34}$$

The coefficient $g_j^\varepsilon$ in the left-hand side is

$$g_j^\varepsilon(\omega,z) := -\frac{3}{4}\mu_j^2(z)\nu^2\Big(\frac{z}{\varepsilon}\Big) - \Big[\frac{(\pi j)^2}{12} + \frac{1}{16}\Big]\nu'^2\Big(\frac{z}{\varepsilon}\Big), \tag{3.35}$$

and

$$\mathcal{C}_j^\varepsilon(\omega,z) = \sum_{q=1,q\neq j}^\infty \Big[\frac{\sigma\Gamma_{jq}}{\sqrt{\varepsilon}}\nu''\Big(\frac{z}{\varepsilon}\Big) + \sigma^2\gamma_{jq}\Big(\frac{z}{\varepsilon}\Big) + \gamma_{jq}^o(z)\Big]u_q^\varepsilon(\omega,z)$$
$$+ \sum_{q=1,q\neq j}^\infty \Big[\frac{\sigma\Theta_{jq}}{\sqrt{\varepsilon}}\nu'\Big(\frac{z}{\varepsilon}\Big) + \sigma^2\theta_{jq}\Big(\frac{z}{\varepsilon}\Big) + \theta_{jq}^o(z)\Big]\varepsilon\partial_z u_q^\varepsilon(\omega,z), \tag{3.36}$$

models the coupling between the modes. The leading coupling coefficients are the constants

$$\Gamma_{jq} := \frac{jq(-1)^{j+q}}{(q^2-j^2)}, \quad \Theta_{jq} := \frac{2jq(-1)^{j+q}}{(q^2-j^2)}. \tag{3.37}$$

The second order coefficients, due to the random fluctuations, are

$$\gamma_{jq}\Big(\frac{z}{\varepsilon}\Big) := \frac{jq(-1)^{j+q}}{2(q^2-j^2)}\Big[\frac{(3j^2+q^2)}{(q^2-j^2)}\nu'^2\Big(\frac{z}{\varepsilon}\Big) - \nu\Big(\frac{z}{\varepsilon}\Big)\nu''\Big(\frac{z}{\varepsilon}\Big)\Big], \tag{3.38}$$

$$\theta_{jq}\Big(\frac{z}{\varepsilon}\Big) := -\frac{jq(-1)^{j+q}}{(q^2-j^2)}\nu\Big(\frac{z}{\varepsilon}\Big)\nu'\Big(\frac{z}{\varepsilon}\Big), \tag{3.39}$$

and those due to the slow changes in the waveguide are

$$\gamma_{jq}^o(z) := \frac{\kappa(z)}{D(z)}\frac{2jq[1-(-1)^{j+q}]\big[j^2+3q^2-4\big(\frac{kD(z)}{\pi}\big)^2\big]}{(q^2-j^2)^2}, \tag{3.40}$$

$$\theta_{jq}^o(z) := \frac{D'(z)}{D(z)}\frac{2jq[1+(-1)^{j+q}]}{(q^2-j^2)}. \tag{3.41}$$

In the region $|z| > Z_M$, where the waveguide has straight and parallel boundaries, the wave equation simplifies to

$$\frac{1}{\varepsilon}\Big[(\varepsilon\partial_z)^2 + k^2 - \mu_j^2(z)\Big]u_j^\varepsilon(\omega, z) = 0. \tag{3.42}$$

Depending on the index $j$, its solution is either an outgoing propagating wave or a decaying evanescent wave. This wave is connected to the solution of (3.34) by the continuity of $u_j^\varepsilon$ and $\partial_z u_j^\varepsilon$ at $z = \pm Z_M$.

## 3.5 Random mode amplitudes

Equations (3.34) are perturbations of the wave equation with operator $(\varepsilon\partial_z)^2+k^2-\mu_j^2(z)$, where the perturbation term models the coupling of the modes. This coupling is similar to that in waveguides with randomly perturbed parallel boundaries, studied in [4, 14], but the slow variation of the waveguide introduces two differences: The first is the presence of the extra terms $\gamma_{jq}^o(z)$ and $\theta_{jq}^o(z)$ in (3.36), given by (3.40)-(3.41), which turn out to play no role in the limit $\varepsilon \to 0$. The second difference is important, as it gives a $z$ dependent number

$$N(z) = \left\lfloor \frac{kD(z)}{\pi} \right\rfloor \tag{3.43}$$

of mode indices $j = 1, \ldots, N(z)$ for which $k^2 - \mu_j^2(z) > 0$. These modes are oscillatory functions in $z$, and represent left and right going waves. For indices $j > N(z)$ the modes are decaying evanescent waves.

### 3.5.1 Turning points

The function (3.43) that defines the number of propagating modes is piecewise constant. Starting from the origin, where we denote the number of propagating modes by $N^{(0)} := N(0)$, the function (3.43) increases by 1 at arc lengths $z_+^{(t)} > 0$, for

$t = 1, \ldots, t_M^+$, and decreases by 1 at $z_-^{(t)} < 0$, for $t = 1, \ldots, t_M^-$. The jump locations $z_\pm^{(t)}$, ordered as

$$-Z_M < \ldots < z_-^{(2)} < z_-^{(1)} < 0 < z_+^{(1)} < z_+^{(2)} < \ldots < Z_M,$$

are the zeroes of the eigenvalues (3.28), and are called turning points [36, 6]. We assume henceforth that the monotonically increasing $D(z)$ satisfies

$$D'\big(z_\pm^{(t)}\big) > 0, \quad \forall\, t \geq 1, \tag{3.44}$$

so that the turning points are simple and isolated. Consistent with our scaling, they are spaced at order one scaled distances.

Between any two consecutive turning points $z_\pm^{(t-1)}$ and $z_\pm^{(t)}$, where we set by convention $z_\pm^{(0)} = 0$, the number of propagating modes equals the constant

$$N_\pm^{(t-1)} := N^{(0)} \pm (t-1). \tag{3.45}$$

This number is bounded above and below as in (3.14), with $N_{\min} = N(-Z_M)$ and $N_{\max} = N(Z_M)$, so the bounds $t_M^+$ and $t_M^-$ on the indices $t$ are

$$t_M^- = N^{(0)} - N_{\min} + 1 \text{ and } t_M^+ = N_{\max} - N^{(0)} + 1. \tag{3.46}$$

Beginning from the source location $z = 0$, which we assume is not a turning point, $z_-^{(t)}$ is defined as the unique, negative arc-length satisfying

$$k = \frac{\pi N_-^{(t-1)}}{D\big(z_-^{(t)}\big)}, \quad t = 1, \ldots, t_M^-, \tag{3.47}$$

where the uniqueness is due to the monotonicity of $D(z)$. Similarly, the jump location $z_+^{(t)}$ is defined as the unique, positive arc length satisfying

$$k = \frac{\pi\big(N_+^{(t-1)} + 1\big)}{D\big(z_+^{(t)}\big)}, \quad t = 1, \ldots, t_M^+. \tag{3.48}$$

The analysis of the modes is similar on the left and right of the source, so we focus attention in this section on a sector $z \in \left(z_-^{(t)}, z_-^{(t-1)}\right)$ of the waveguide, for some $1 \le t \le t_M^-$, and simplify the notation for the number (3.45) of propagating modes

$$\mathcal{N} := N_-^{(t-1)}. \tag{3.49}$$

These modes are a superposition of right and left going waves, with random amplitudes that model cumulative scattering in the waveguide, as we explain in the next section.

### 3.5.2 The left and right going waves

We decompose the propagating modes in left and right going waves, using a flow of smooth and invertible matrices $\mathbf{M}_j^\varepsilon(\omega, z)$,

$$\begin{pmatrix} a_j^\varepsilon(\omega,z) \\ b_j^\varepsilon(\omega,z) \end{pmatrix} := \mathbf{M}_j^{\varepsilon,-1}(\omega,z) \begin{pmatrix} u_j^\varepsilon(\omega,z) \\ v_j^\varepsilon(\omega,z) \end{pmatrix}, \tag{3.50}$$

where $\mathbf{M}_j^{\varepsilon,-1}$ denotes the inverse of $\mathbf{M}_j^\varepsilon$ and

$$v_j^\varepsilon(\omega,z) := -i\varepsilon\partial_z u_j^\varepsilon(\omega,z), \qquad j = 1,\ldots,\mathcal{N}. \tag{3.51}$$

We obtain from (3.34) that

$$\begin{aligned}\partial_z \begin{pmatrix} a_j^\varepsilon(\omega,z) \\ b_j^\varepsilon(\omega,z) \end{pmatrix} &= \mathbf{M}_j^{\varepsilon,-1}(\omega,z) \left\{ \frac{i}{\varepsilon} \begin{pmatrix} 0 & 1 \\ k^2(\omega) - \mu_j^2(z) & 0 \end{pmatrix} \mathbf{M}_j^\varepsilon(\omega,z) - \partial_z \mathbf{M}_j^\varepsilon(\omega,z) \right.\\ &\quad \left. + \left[ \frac{i\sigma}{\sqrt{\varepsilon}} \mu_j^2(z) \nu\left(\frac{z}{\varepsilon}\right) + i\sigma^2 g_j^\varepsilon(\omega,z) \right] \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{M}_j^\varepsilon(\omega,z) \right\} \begin{pmatrix} a_j^\varepsilon(\omega,z) \\ b_j^\varepsilon(\omega,z) \end{pmatrix} \\ &\quad - i\,\mathcal{C}_j^\varepsilon(\omega,z) \mathbf{M}_j^{\varepsilon,-1}(\omega,z) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{aligned} \tag{3.52}$$

and the decomposition is achieved by a flow $\mathbf{M}_j^\varepsilon(\omega,z)$ that removes to leading order the large deterministic term in (3.52), the first line in the right hand side.

The matrix $\mathbf{M}_j^\varepsilon(\omega, z)$ has the structure

$$\mathbf{M}_j^\varepsilon(\omega, z) = \begin{pmatrix} M_{j,11}^\varepsilon(\omega, z) & -\overline{M_{j,11}^\varepsilon}(\omega, z) \\ M_{j,21}^\varepsilon(\omega, z) & \overline{M_{j,21}^\varepsilon}(\omega, z) \end{pmatrix}, \tag{3.53}$$

where the bar denotes complex conjugate, so that the decomposition (3.50) conserves energy. The expression of the components in (3.53) depends on the mode index, more precisely on the mode wavenumber denoted by

$$\beta_j(\omega, z) := \sqrt{k^2 - \mu_j^2(z)}. \tag{3.54}$$

Note that $\beta_j$ is bounded away from zero for all $j = 1, \ldots, \mathcal{N} - 1$, and it approaches zero as $z \searrow z_-^{(t)}$, for $j = \mathcal{N}$. This last mode is a turning wave which transitions from a propagating wave at $z \in (z_-^{(t)}, z_-^{(t-1)})$ to an evanescent wave at $z < z_-^{(t)}$, as described in section 3.5.3. Here we give the decomposition of the modes indexed by $j \leq \mathcal{N} - 1$.

The entries of (3.53) are defined by

$$\begin{aligned} M_{j,11}^\varepsilon(\omega, z) &:= \frac{1}{\sqrt{\beta_j(\omega, z)}} \exp\Big[\frac{i}{\varepsilon} \int_0^z dz' \beta_j(\omega, z')\Big], \\ M_{j,21}^\varepsilon(\omega, z) &:= \beta_j(\omega, z) M_{j,11}^\varepsilon(\omega, z), \end{aligned} \tag{3.55}$$

for $j = 1, \ldots, \mathcal{N} - 1$. This definition looks the same as in perfect waveguides with straight and parallel boundary, except that the mode wavenumber $\beta_j$ varies with $z$. We obtain from (3.53)-(3.55) that the determinant of $\mathbf{M}_j^\varepsilon(\omega, z)$ is constant

$$\det \mathbf{M}_j^\varepsilon(\omega, z) = 2, \qquad \forall\, z \in \big(z_-^{(t)}, z_-^{(t-1)}\big), \tag{3.56}$$

so the matrix is invertible, and the decomposition (3.50) can be rewritten as

$$u_j^\varepsilon(\omega, z) = \frac{1}{\sqrt{\beta_j(\omega, z)}} \Big[ a_j^\varepsilon(\omega, z) e^{\frac{i}{\varepsilon} \int_0^z dz' \beta_j(\omega, z')} - b_j^\varepsilon(\omega, z) e^{-\frac{i}{\varepsilon} \int_0^z dz' \beta_j(\omega, z')} \Big], \tag{3.57}$$

and

$$\varepsilon\partial_z u_j^\varepsilon(\omega,z) = i\sqrt{\beta_j(\omega,z)}\Big[a_j^\varepsilon(\omega,z)e^{\frac{i}{\varepsilon}\int_0^z dz'\beta_j(\omega,z')} + b_j^\varepsilon(\omega,z)e^{-\frac{i}{\varepsilon}\int_0^z dz'\beta_j(\omega,z')}\Big]. \tag{3.58}$$

The equations (3.57)-(3.58) are precisely what one would obtain using the method of variation of parameters for the perturbed wave equation satisfied by the $j$-th mode subject to the appropriate source and radiation conditions, which are given in detail in a later section. They decompose the mode in a right going wave with amplitude $a_j^\varepsilon$ and a left going wave with amplitude $b_j^\varepsilon$. In perfect waveguides these amplitudes would be constant, meaning physically that the waves are independent. In our case the amplitudes are random fields, satisfying the system of stochastic differential equations

$$\partial_z \begin{pmatrix} a_j^\varepsilon(\omega,z) \\ b_j^\varepsilon(\omega,z)\end{pmatrix} = \mathbf{H}_j^\varepsilon(\omega,z)\begin{pmatrix} a_j^\varepsilon(\omega,z) \\ b_j^\varepsilon(\omega,z)\end{pmatrix} - \frac{i}{2}\mathcal{C}_j^\varepsilon(\omega,z)\begin{pmatrix} \overline{M_{j,11}^\varepsilon}(\omega,z) \\ M_{j,11}^\varepsilon(\omega,z)\end{pmatrix}, \tag{3.59}$$

obtained by substituting (3.53) and (3.55) in (3.52). Here $\mathbf{H}_j^\varepsilon(\omega,z)$ is the matrix-valued random process

$$\mathbf{H}_j^\varepsilon(\omega,z) := \begin{pmatrix} H_j^{\varepsilon(aa)}(\omega,z) & H_j^{\varepsilon(ab)}(\omega,z) \\ H_j^{\varepsilon(ba)}(\omega,z) & H_j^{\varepsilon(bb)}(\omega,z)\end{pmatrix}, \tag{3.60}$$

with entries satisfying the relations

$$H_j^{\varepsilon(ba)}(\omega,z) = \overline{H_j^{\varepsilon(ab)}}(\omega,z), \quad H_j^{\varepsilon(bb)}(\omega,z) = \overline{H_j^{\varepsilon(aa)}}(\omega,z), \tag{3.61}$$

and taking the values

$$H_j^{\varepsilon(aa)}(\omega,z) \approx \frac{i}{2\beta_j(\omega,z)}\Big[\frac{\sigma}{\sqrt{\varepsilon}}\mu_j^2(z)\nu\Big(\frac{z}{\varepsilon}\Big) + \sigma^2 g_j^\varepsilon(\omega,z)\Big], \tag{3.62}$$

and

$$H_j^{\varepsilon(ab)}(\omega,z) \approx \Big[\overline{H_j^{\varepsilon(aa)}}(\omega,z) - \frac{\partial_z\beta_j(\omega,z)}{2\beta_j(\omega,z)}\Big]\exp\Big[-\frac{2i}{\varepsilon}\int_0^z dz'\beta_j(\omega,z')\Big]. \tag{3.63}$$

As before, the approximation means up to negligible terms in the limit $\varepsilon \to 0$.

Equations (3.59) show that the amplitudes of the $j$-th mode are coupled to each other by the process $\mathbf{H}_j^\varepsilon$, and to the other modes by $\mathcal{C}_j^\varepsilon$, defined by the series (3.36). The first terms in this series involve the propagating waves $u_q^\varepsilon(\omega, z)$, for $q \neq j$, decomposed as in (3.57)-(3.58). We describe in the next two sections the turning and the evanescent waves.

### 3.5.3 The turning waves

The mode indexed by $j = \mathcal{N}$ transitions at $z = z_-^{(t)}$ from propagating to evanescent. This transition is captured by the matrix $\mathbf{M}_\mathcal{N}^\varepsilon(\omega, z)$, which has the same structure as in (3.53), but its entries are defined in terms of Airy functions [19, chapter 9]. This is because near the simple turning point $z_-^{(t)}$, equation (3.34) for $j = \mathcal{N}$ is a perturbation of Airy's equation. We refer to [6, 47] for classic studies of turning waves in waveguides, and to [11] for an analysis of their interaction with the random boundary. The setup in [11] is the same as here, with the exception that we consider a larger standard deviation of the random fluctuations, to observe mode coupling in the waveguide.

We use the same $\mathbf{M}_\mathcal{N}^\varepsilon(\omega, z)$ as in [11], with entries

$$
\begin{aligned}
M_{\mathcal{N},11}^\varepsilon(\omega, z) := \varepsilon^{-1/6}\sqrt{\pi} Q_\mathcal{N}(\omega, z) \exp\Big[-i\frac{\phi_\mathcal{N}\big(\omega, 0\big)}{\varepsilon} + \frac{i\pi}{4}\Big] \\
\times \Big[\mathrm{Ai}\big(-\eta_\mathcal{N}^\varepsilon(\omega, z)\big) - i\mathrm{Bi}\big(-\eta_\mathcal{N}^\varepsilon(\omega, z)\big)\Big],
\end{aligned}
\tag{3.64}
$$

and

$$
M_{\mathcal{N},21}^\varepsilon(\omega, z) := -i\varepsilon\partial_z M_{\mathcal{N},11}^\varepsilon(\omega, z), \tag{3.65}
$$

for $z \in \big(z_-^{(t)} - \delta, z_-^{(t-1)}\big)$, where $\delta$ is a small, positive number, independent of $\varepsilon$. We go slightly beyond the turning point to capture the transition of the wave to an evanescent one. The phase in definition (3.64) is given by the function

$$\phi_{\mathcal{N}}(\omega, z) := \int_{z_-^{(t)}}^{z} dz' \sqrt{|k^2 - \mu_{\mathcal{N}}^2(z')|}, \tag{3.66}$$

evaluated at the source location $z = 0$, and the amplitude factor

$$Q_{\mathcal{N}}(\omega, z) := \frac{\left|3\phi_{\mathcal{N}}(\omega, z)/2\right|^{1/6}}{\left|k^2 - \mu_{\mathcal{N}}^2(z)\right|^{1/4}}, \tag{3.67}$$

is shown in [11, Section 3.1.1] to be bounded, and at least twice continuously differentiable. The Airy functions Ai and Bi are evaluated at

$$\eta_{\mathcal{N}}^{\varepsilon}(\omega, z) := \text{sign}\left(z - z_-^{(t)}\right) \left[\frac{3|\phi_{\mathcal{N}}(\omega, z)|}{2\varepsilon}\right]^{2/3}, \tag{3.68}$$

where $|\eta^{\varepsilon}(\omega, z)|$ is of order one in the vicinity $\left|z - z_-^{(t)}\right| \leq O\left(\varepsilon^{2/3}\right)$ of the turning point, and it is much larger than one in the rest of the domain $\left(z_-^{(t)} - \delta, z_-^{(t-1)}\right)$.

We summarize here a few facts about $\mathbf{M}_{\mathcal{N}}^{\varepsilon}(\omega, z)$ from [11] and refer to section 3.10.2 in an appendix included at the end of this chapter for details. We have from [11, Lemma 3.1] that the matrix $\mathbf{M}_{\mathcal{N}}^{\varepsilon}(\omega, z)$ is invertible, with constant determinant

$$\det \mathbf{M}_{\mathcal{N}}^{\varepsilon}(\omega, z) = 2, \qquad \forall\, z \in \left(z_-^{(t)} - \delta, z_-^{(t-1)}\right), \tag{3.69}$$

so the decomposition (3.50) is well defined. Moreover, [11, Lemma 3.2] shows that at $z - z_-^{(t)} \gg \varepsilon^{2/3}$ the expressions (3.64)-(3.65) become like (3.55),

$$\begin{aligned} M_{\mathcal{N},11}^{\varepsilon}(\omega, z) &= \frac{1}{\sqrt{\beta_{\mathcal{N}}(\omega, z)}} \exp\left[\frac{i}{\varepsilon}\int_0^z dz' \beta_{\mathcal{N}}(\omega, z')\right] + O(\varepsilon), \\ M_{\mathcal{N},21}^{\varepsilon}(\omega, z) &= \beta_{\mathcal{N}}(\omega, z) M_{\mathcal{N},11}^{\varepsilon}(\omega, z) + O(\varepsilon), \end{aligned} \tag{3.70}$$

so the turning wave behaves just like any other propagating wave until it reaches the vicinity of the turning point from the right. On the left side of the turning point, at $z_-^{(t)} - z \gg \varepsilon^{2/3}$, the entries of $\mathbf{M}_{\mathcal{N}}^{\varepsilon}(\omega, z)$ grow exponentially, as given in [11, Lemma

3.3]. The wave is evanescent in this region, and must be decaying in order to have energy conservation. This is ensured by the radiation condition

$$a^{\varepsilon}_{\mathcal{N}}\big(\omega, z^{(t)}_{-} - \delta\big) = i \exp\Big[\frac{2i}{\varepsilon}\phi_{\mathcal{N}}\big(\omega, 0\big)\Big] b^{\varepsilon}_{\mathcal{N}}\big(\omega, z^{(t)}_{-} - \delta\big), \tag{3.71}$$

which sets to zero the coefficients of the growing Airy function Bi and its derivative Bi$'$ in the expression of $u^{\varepsilon}_{\mathcal{N}}$ and $\partial_z u^{\varepsilon}_{\mathcal{N}}$ at the end $z^{(t)}_{-} - \delta$ of the domain. We refer to [11, Section 3.1] for more details, and for the proof that the result does not depend on the particular choice of $\delta$ which is small, but bounded away from 0 in the limit $\varepsilon \to 0$.

The evolution equation of the turning mode amplitudes is of the same form as in (3.59), with the following entries of the matrix (3.60)-(3.61) indexed by $j = \mathcal{N}$,

$$H^{\varepsilon(aa)}_{\mathcal{N}}(\omega, z) \approx \frac{i\big|M^{\varepsilon}_{\mathcal{N},11}(\omega, z)\big|^2}{2}\Big[\frac{\sigma}{\sqrt{\varepsilon}}\mu^2_j(z)\nu\Big(\frac{z}{\varepsilon}\Big) + \sigma^2 g^{\varepsilon}_j(\omega, z)\Big], \tag{3.72}$$

and

$$H^{\varepsilon(ab)}_{\mathcal{N}}(\omega, z) \approx -\frac{i\big[M^{\varepsilon}_{\mathcal{N},11}(\omega, z)\big]^2}{2}\Big[\frac{\sigma}{\sqrt{\varepsilon}}\mu^2_j(z)\nu\Big(\frac{z}{\varepsilon}\Big) + \sigma^2 g^{\varepsilon}_j(\omega, z)\Big]. \tag{3.73}$$

These expressions reduce to those in (3.62)-(3.63) at $z - z^{(t)}_{-} \gg \varepsilon^{2/3}$, with the extra term involving $\partial_z \beta_{\mathcal{N}}$ in (3.63) coming from an $O(\varepsilon)$ correction of the amplitudes, induced by the residual in (3.70).

#### 3.5.4 Coupling with the evanescent waves

The modes indexed by $j > \mathcal{N}$ in equations (3.34) are evanescent waves, with wavenumber

$$\beta_j(\omega, z) := \sqrt{\mu^2_j(z) - k^2}. \tag{3.74}$$

In order to close our system for the propagating modes we will show that these waves can be expressed in terms of the propagating ones.

Let us begin by rewriting equation (3.34) in first order system form, for the unknown vector with components $u_j^\varepsilon(\omega,z)$ and

$$v_j^\varepsilon(\omega,z) := \frac{\varepsilon}{\beta_j(\omega,z)}\partial_z u_j^\varepsilon(\omega,z), \tag{3.75}$$

where $j > \mathcal{N}$ and $z \in \left(z_-^{(t)}, z_-^{(t-1)}\right)$. The mode wavenumber $\beta_j$ is defined in (3.74), and the system is

$$\left\{\partial_z - \frac{\beta_j(\omega,z)}{\varepsilon}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \Big[\frac{\sigma\mu_j^2(z)}{\sqrt{\varepsilon}\beta_j(\omega,z)}\nu\big(\frac{z}{\varepsilon}\big) + \frac{\sigma^2 g_j^\varepsilon(\omega,z)}{\beta_j(\omega,z)}\Big]\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right\}\begin{pmatrix}u_j^\varepsilon(\omega,z)\\ v_j^\varepsilon(\omega,z)\end{pmatrix}$$
$$= \frac{\mathcal{C}_j^\varepsilon(\omega,z)}{\beta_j(\omega,z)}\begin{pmatrix}0\\ 1\end{pmatrix}. \tag{3.76}$$

The matrix $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ in the leading term has the eigenvalues $\pm 1$, and the orthonormal eigenfunctions $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm 1\end{pmatrix}$. Expanding the solution in the basis of these eigenfunctions

$$\begin{pmatrix}u_j^\varepsilon(\omega,z)\\ v_j^\varepsilon(\omega,z)\end{pmatrix} = \frac{\alpha_j^+(\omega,z)}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} + \frac{\alpha_j^-(\omega,z)}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}, \tag{3.77}$$

and substituting in (3.76) gives the following equations for the coefficients

$$\left[\partial_z \mp \frac{\beta_j(\omega,z)}{\varepsilon}\right]\alpha_j^\pm(\omega,z) = \pm\frac{\mathcal{C}_j^\varepsilon(\omega,z)}{\sqrt{2}\beta_j(\omega,z)} \mp \frac{\left[\alpha_j^+(\omega,z)+\alpha_j^-(\omega,z)\right]}{2\beta_j(\omega,z)}$$
$$\times\left[\frac{\sigma}{\sqrt{\varepsilon}}\mu_j^2(z)\nu\left(\frac{z}{\varepsilon}\right) + \sigma^2 g_j^\varepsilon(\omega,z)\right]. \tag{3.78}$$

These are complemented with the boundary conditions

$$\alpha_j^+\left(\omega, z_-^{(t-1)}\right) = \sqrt{2}c_j^{(t)+}, \quad \alpha_j^-\left(\omega, z_-^{(t)}\right) = 0, \tag{3.79}$$

with constant $c_j^{(t)}$ to be determined later, indexed by $t$ to remind us that we work in the sector $z \in \left(z_-^{(t)}, z_-^{(t-1)}\right)$. In (3.79) we set to zero the component $\alpha_j^-$ at the farther end $z_-^{(t)}$ from the source, to suppress the growing part of the solution.

We obtain after integration of (3.78) that

$$\alpha_j^+(\omega,z) = \sqrt{2}c_j^{(t)} \exp\Big[-\frac{1}{\varepsilon}\int_z^{z_-^{(t-1)}} d\zeta\, \beta_j(\omega,\zeta)\Big] - \int_z^{z_-^{(t-1)}} d\zeta \frac{\exp\Big[-\frac{1}{\varepsilon}\int_z^\zeta ds\, \beta_j(\omega,s)\Big]}{\sqrt{2}\beta_j(\omega,\zeta)}$$
$$\times \Bigg\{\mathcal{C}_j^\varepsilon(\omega,\zeta) - \frac{\big[\alpha_j^+(\omega,\zeta)+\alpha_j^-(\omega,\zeta)\big]}{\sqrt{2}}\Big[\frac{\sigma}{\sqrt{\varepsilon}}\mu_j^2(\zeta)\nu\Big(\frac{\zeta}{\varepsilon}\Big) + \sigma^2 g_j^\varepsilon(\omega,\zeta)\Big]\Bigg\}, \qquad (3.80)$$

and

$$\alpha_j^-(\omega,z) = -\int_{z_-^{(t)}}^z d\zeta \frac{\exp\Big[-\frac{1}{\varepsilon}\int_\zeta^z ds\, \beta_j(\omega,s)\Big]}{\sqrt{2}\beta_j(\omega,\zeta)} \Bigg\{\mathcal{C}_j^\varepsilon(\omega,\zeta) - \frac{\big[\alpha_j^+(\omega,\zeta)+\alpha_j^-(\omega,\zeta)\big]}{\sqrt{2}}$$
$$\times\Big[\frac{\sigma}{\sqrt{\varepsilon}}\mu_j^2(\zeta)\nu\Big(\frac{\zeta}{\varepsilon}\Big) + \sigma^2 g_j^\varepsilon(\omega,\zeta)\Big]\Bigg\}. \qquad (3.81)$$

All the exponential terms in these equations are decaying in $z$, so we can change the variable of integration as $\zeta = z + \varepsilon\xi$, and note that only $\xi = O(1)$ contributes to the result. Equation (3.80) becomes

$$\alpha_j^+(\omega,z) \approx \sqrt{2}c_j^{(t)} \exp\Big[-\frac{1}{\varepsilon}\int_z^{z_-^{(t-1)}} d\zeta\, \beta_j(\omega,\zeta)\Big] - \frac{\varepsilon}{\sqrt{2}\beta_j(\omega,z)}\int_0^\infty d\xi\, e^{-\xi\beta_j(\omega,z)}$$
$$\times \Bigg\{\mathcal{C}_j^\varepsilon(\omega,z+\varepsilon\xi) - u_j^\varepsilon(\omega,z+\varepsilon\xi)\Big[\frac{\sigma}{\sqrt{\varepsilon}}\mu_j^2(z)\nu\Big(\frac{z}{\varepsilon}+\xi\Big) + \sigma^2 g_j^\varepsilon(\omega,z+\varepsilon\xi)\Big]\Bigg\}, \quad (3.82)$$

where we used (3.77) in the integrand, and the approximation means that we neglect terms that vanish in the limit $\varepsilon \to 0$. Similarly, equation (3.81) becomes

$$\alpha_j^-(\omega,z) \approx -\frac{\varepsilon}{\sqrt{2}\beta_j(\omega,z)}\int_{-\infty}^0 d\xi\, e^{\xi\beta_j(\omega,z)} \big\{\mathcal{C}_j^\varepsilon(\omega,z+\varepsilon\xi) - u_j^\varepsilon(\omega,z+\varepsilon\xi)$$
$$\times \Big[\frac{\sigma}{\sqrt{\varepsilon}}\mu_j^2(z)\nu\Big(\frac{z}{\varepsilon}+\xi\Big) + \sigma^2 g_j^\varepsilon(\omega,z+\varepsilon\xi)\Big]\Bigg\}. \quad (3.83)$$

The expression of $u_j^\varepsilon$ follows from these equations and (3.77),

$$u_j^\varepsilon(\omega,z) \approx c_j^{(t)}(\omega) \exp\Big[-\frac{1}{\varepsilon}\int_z^{z_-^{(t-1)}} d\zeta\, \beta_j(\omega,\zeta)\Big] - \frac{\varepsilon}{2\beta_j(\omega,z)}\int_{-\infty}^\infty d\xi\, e^{-|\xi|\beta_j(\omega,z)}$$
$$\times \Bigg\{\mathcal{C}_j^\varepsilon(\omega,z+\varepsilon\xi) - u_j^\varepsilon(\omega,z+\varepsilon\xi)\Big[\frac{\sigma}{\sqrt{\varepsilon}}\mu_j^2(z)\nu\Big(\frac{z}{\varepsilon}+\xi\Big) + \sigma^2 g_j^\varepsilon(\omega,z+\varepsilon\xi)\Big]\Bigg\}. \quad (3.84)$$

The derivative $\varepsilon\partial_z u_j^\varepsilon$ is obtained from (3.75), (3.77), (3.82)-(3.83) and integration by parts

$$\varepsilon\partial_z u_j^\varepsilon(\omega,z) \approx \beta_j(\omega,z)c_j^{(t)}(\omega)\exp\Big[\frac{1}{\varepsilon}\int_{z_-^{(t-1)}}^{z} d\zeta\,\beta_j(\omega,\zeta)\Big] - \frac{\varepsilon}{2\beta_j(\omega,z)}\int_{-\infty}^{\infty} d\xi\, e^{-|\xi|\beta_j(\omega,z)}$$
$$\times\,\varepsilon\partial_z\Big\{\mathcal{C}_j^\varepsilon(\omega,z+\varepsilon\xi) - u_j^\varepsilon(\omega,z+\varepsilon\xi)\Big[\frac{\sigma}{\sqrt{\varepsilon}}\mu_j^2(z)\nu\Big(\frac{z}{\varepsilon}+\xi\Big) + \sigma^2 g_j^\varepsilon(\omega,z+\varepsilon\xi)\Big]\Big\}. \quad (3.85)$$

Now let us recall the expression (3.36) of $\mathcal{C}_j^\varepsilon(\omega,z)$, which models the coupling with the other modes, and write it as the sum of two terms:

$$\mathcal{C}_j^\varepsilon(\omega,z) =: \mathcal{C}_j^{\varepsilon(p)}(\omega,z) + \mathcal{C}_j^{\varepsilon(e)}(\omega,z). \quad (3.86)$$

The first term is the coupling with the propagating modes, and is given by restricting the sum in (3.36) to $q \le \mathcal{N}$. The second term is the remaining series, with terms indexed by $q > \mathcal{N}$, and $q \ne j$. Each term in this series involves $u_q^\varepsilon(\omega,z)$ and $\varepsilon\partial_z u_q^\varepsilon(\omega,z)$ that have expressions like (3.84)-(3.85). Stringing all the unknowns in the infinite-dimensional vector $\boldsymbol{U} := \big(\boldsymbol{U}_{\mathcal{N}+1}, \boldsymbol{U}_{\mathcal{N}+2},\ldots,\big)$ where $\boldsymbol{U}_j := (u_j^\varepsilon, \varepsilon\partial_z u_j^\varepsilon)$, for $j > \mathcal{N}$, we can write equations (3.84)-(3.85) in compact form as

$$\Big(\mathbf{I} - \sqrt{\varepsilon}\mathbf{K}\Big)\boldsymbol{U}(\omega,z) = \boldsymbol{F}(\omega,z), \quad (3.87)$$

with right hand side given by the concatenation of

$$\boldsymbol{F}_j(\omega,z) := \begin{pmatrix} 1 \\ \beta_j(\omega,z)\end{pmatrix} c_j^{(t)}(\omega)\exp\Big[-\frac{1}{\varepsilon}\int_z^{z_-^{(t-1)}} d\zeta\,\beta_j(\omega,\zeta)\Big]$$
$$- \frac{\varepsilon}{2\beta_j(\omega,z)}\int_{-\infty}^{\infty} d\xi e^{-|\xi|\beta_j(\omega,z)}\begin{pmatrix} 1 \\ \varepsilon\partial_z\end{pmatrix}\mathcal{C}_j^{\varepsilon(p)}(\omega,z+\varepsilon\xi), \quad (3.88)$$

for $j \ge \mathcal{N}$. In the left hand side of (3.87) we have the perturbation of the identity $\mathbf{I}$ by the integral operator $\mathbf{K}$, whose kernel follows easily from the $(u_q^\varepsilon)_{q>\mathcal{N}}$ dependent terms in the integrand in (3.84)-(3.85), including those in $\mathcal{C}_j^{\varepsilon(e)}$. This integral operator is similar to that analyzed in [4, Lemma 3.1], and we show in chapter VI it is bounded independent of $\varepsilon$ sufficiently small with respect to an appropriate norm. This means that we can solve (3.87) using Neumann series and obtain

$$\boldsymbol{U}(\omega,z) = \boldsymbol{F}(\omega,z) + O(\sqrt{\varepsilon}). \tag{3.89}$$

The first term in (3.88) matters only in the $O(\varepsilon)$ vicinity of $z_-^{(t-1)}$, over which the mode coupling is negligible. The constant $c_j^{(t)}$ is determined by continuity conditions at $z_-^{(t-1)}$ as follows: If $t=1$, so that $z_-^{(t-1)}=0$, $c_j^{(1)}$ is determined by the source excitation, and it equals the coefficient of the $j$-th evanescent mode in the perfect waveguide with width $D(0)$. If $t>1$, then $c_j^{(t)}$ is determined by continuity of the wavefield at the turning point $z_-^{(t-1)}$.

Assuming that $z_-^{(t-1)} - z \gg \varepsilon$, so we can neglect the first term in (3.88), we have

$$\begin{pmatrix} u_j^\varepsilon(\omega,z) \\ \varepsilon\partial_z u_j^\varepsilon(\omega,z)\end{pmatrix} \approx -\frac{\varepsilon}{2\beta_j(\omega,z)}\int_{-\infty}^{\infty} d\xi e^{-|\xi|\beta_j(\omega,z)} \begin{pmatrix} 1 \\ \varepsilon\partial_z\end{pmatrix} \mathcal{C}_j^{\varepsilon(p)}(\omega,z+\varepsilon\xi), \tag{3.90}$$

with

$$\varepsilon\mathcal{C}_j^{\varepsilon(p)}(\omega,z+\varepsilon\xi) \approx \sigma\sqrt{\varepsilon}\sum_{q=1}^{\mathcal{N}}\left[\Gamma_{jq}\nu''\Big(\frac{z}{\varepsilon}+\xi\Big)+\Theta_{jq}\nu'\Big(\frac{z}{\varepsilon}+\xi\Big)\varepsilon\partial_z\right]u_q^\varepsilon(\omega,z+\varepsilon\xi), \tag{3.91}$$

obtained from (3.36). Here the modes $u_q^\varepsilon$ and their derivative $\varepsilon\partial_z u_q^\varepsilon(\omega,z)$ are given in (3.57)-(3.58), and the constant coefficients $\Gamma_{jq}$ and $\Theta_{jq}$ are defined in (3.37). Substituting in (3.91) and then (3.90), and using that the derivatives of the mode amplitudes given in (3.59) are at most $O(\varepsilon^{-1/2})$, we obtain

$$\begin{aligned} u_j^\varepsilon(\omega,z) \approx &-\frac{\sigma\sqrt{\varepsilon}}{2\beta_j(\omega,z)}\sum_{q=1}^{\mathcal{N}}\int_{-\infty}^{\infty}d\xi\left[\gamma_{jq}^{(e)}\Big(\omega,\frac{z}{\varepsilon}+\xi\Big)\frac{a_q^\varepsilon(\omega,z)}{\sqrt{\beta_q(\omega,z)}}e^{\frac{1}{\varepsilon}\int_0^z dz'\beta_q(\omega,z')+i\xi\beta_q(\omega,z)}\right. \\ &\left.-\overline{\gamma_{jq}^{(e)}}\Big(\omega,\frac{z}{\varepsilon}+\xi\Big)\frac{b_q^\varepsilon(\omega,z)}{\sqrt{\beta_q(\omega,z)}}e^{-\frac{1}{\varepsilon}\int_0^z dz'\beta_q(\omega,z')-i\xi\beta_q(\omega,z)}\right]e^{-|\xi|\beta_j(\omega,z)}. \end{aligned} \tag{3.92}$$

Here we introduced the notation

$$\gamma_{jq}^{(e)}\Big(\omega, \frac{z}{\varepsilon}+\xi\Big) := \Gamma_{jq}\nu''\Big(\frac{z}{\varepsilon}+\xi\Big) + i\beta_q(\omega,z)\Theta_{jq}\nu'\Big(\frac{z}{\varepsilon}+\xi)\Big), \tag{3.93}$$

with coefficients $\Gamma_{jq}$ and $\Theta_{jq}$ defined in (3.37), and recall that the bar denotes complex conjugate.

The derivative in the integrand in (3.90) is

$$\varepsilon\partial_z\Big[\varepsilon\mathcal{C}_j^{\varepsilon(p)}(\omega, z+\varepsilon\xi)\Big] = \sigma\sqrt{\varepsilon}\sum_{q=1}^{\mathcal{N}}\Big[\Gamma_{jq}\nu'''\Big(\frac{z}{\varepsilon}+\xi\Big) + \big(\Gamma_{jq}+\Theta_{jq}\big)\nu''\Big(\frac{z}{\varepsilon}+\xi\Big)\varepsilon\partial_z \\ - \beta_q^2(\omega,z)\Theta_{jq}\nu'\Big(\frac{z}{\varepsilon}+\xi\Big)\Big]u_q^\varepsilon(\omega, z+\varepsilon\xi), \tag{3.94}$$

where we used equation (3.34) for $(\varepsilon\partial_z)^2 u_q^\varepsilon$. Substituting (3.57)-(3.58) in (3.94) and then in (3.90), we obtain

$$\varepsilon\partial_z u_j^\varepsilon(\omega,z) \approx -\frac{\sigma\sqrt{\varepsilon}}{2\beta_j(\omega,z)}\sum_{q=1}^{\mathcal{N}}\int_{-\infty}^{\infty}d\xi\Bigg[\theta_{jq}^{(e)}\Big(\omega,\frac{z}{\varepsilon}+\xi\Big)\frac{a_q^\varepsilon(\omega,z)}{\sqrt{\beta_q(\omega,z)}}e^{\frac{1}{\varepsilon}\int_0^z dz'\beta_q(\omega,z')+i\xi\beta_q(\omega,z)} \\ -\overline{\theta_{jq}^{(e)}}\Big(\omega,\frac{z}{\varepsilon}+\xi\Big)\frac{b_q^\varepsilon(\omega,z)}{\sqrt{\beta_q(\omega,z)}}e^{-\frac{1}{\varepsilon}\int_0^z dz'\beta_q(\omega,z')-i\xi\beta_q(\omega,z)}\Bigg]e^{-|\xi|\beta_j(\omega,z)}, \tag{3.95}$$

with notation

$$\theta_{jq}^{(e)}\Big(\omega,\frac{z}{\varepsilon}+\xi\Big) := \Gamma_{jq}\nu'''\Big(\frac{z}{\varepsilon}+\xi\Big) - \beta_q^2(\omega,z)\Theta_{jq}\nu'\Big(\frac{z}{\varepsilon}+\xi\Big) \\ +i\beta_q(\omega,z)(\Gamma_{jq}+\Theta_{jq})\nu''\Big(\frac{z}{\varepsilon}+\xi)\Big). \tag{3.96}$$

### 3.6 Closed system for the propagating modes

The propagating mode amplitudes satisfy the system of equations (3.59), with coupling modeled by the series (3.36). Substituting the expressions (3.92) and (3.95) of the evanescent waves in (3.36), we obtain a closed system of equations for the propagating modes, as we now explain.

### 3.6.1 Propagation between turning points

We begin with $z \in \left(z_-^{(t)}, z_-^{(t-1)}\right)$ satisfying $z - z_-^{(t)} \gg \varepsilon^{2/3}$ and $z_-^{(t-1)} - z \gg \varepsilon$. In this region the turning wave indexed by $j = \mathcal{N}$ behaves like all the other propagating modes, and the evanescent modes have the expression (3.92) and (3.95). The system of equations for the right- and left-going amplitudes is

$$\partial_z \begin{pmatrix} \boldsymbol{a}^\varepsilon(\omega, z) \\ \boldsymbol{b}^\varepsilon(\omega, z) \end{pmatrix} = \boldsymbol{\Upsilon}^\varepsilon(\omega, z) \begin{pmatrix} \boldsymbol{a}^\varepsilon(\omega, z) \\ \boldsymbol{b}^\varepsilon(\omega, z) \end{pmatrix}, \tag{3.97}$$

where $\boldsymbol{a}^\varepsilon$ and $\boldsymbol{b}^\varepsilon$ are the complex column vectors in $\mathbb{C}^{\mathcal{N}}$ with entries $a_j^\varepsilon$ and $b_j^\varepsilon$, for $1 \le j \le \mathcal{N}$. The complex matrix $\boldsymbol{\Upsilon}^\varepsilon(\omega, z)$ depends on the random fluctuations $\nu$ and the slow changes of the waveguide, and has the block structure

$$\boldsymbol{\Upsilon}^\varepsilon(\omega, z) := \begin{pmatrix} \boldsymbol{\Upsilon}^{\varepsilon(aa)}(\omega, z) & \boldsymbol{\Upsilon}^{\varepsilon(ab)}(\omega, z) \\ \boldsymbol{\Upsilon}^{\varepsilon(ba)}(\omega, z) & \boldsymbol{\Upsilon}^{\varepsilon(bb)}(\omega, z) \end{pmatrix}, \tag{3.98}$$

with $\mathcal{N} \times \mathcal{N}$ blocks satisfying the relations

$$\boldsymbol{\Upsilon}^{\varepsilon(ba)}(\omega, z) = \overline{\boldsymbol{\Upsilon}^{\varepsilon(ab)}}(\omega, z), \quad \boldsymbol{\Upsilon}^{\varepsilon(bb)}(\omega, z) = \overline{\boldsymbol{\Upsilon}^{\varepsilon(aa)}}(\omega, z). \tag{3.99}$$

Their entries are defined as follows: Off the diagonal, we have

$$\begin{aligned}\Upsilon_{jq}^{\varepsilon(aa)}(\omega, z) := & -\frac{i e^{\frac{i}{\varepsilon}\int_0^z dz' \left(\beta_q(\omega, z') - \beta_j(\omega, z')\right)}}{2\sqrt{\beta_j(\omega, z)\beta_q(\omega, z)}} \left\{ \frac{\sigma}{\sqrt{\varepsilon}} \left[ \Gamma_{jq} \nu''\left(\frac{z}{\varepsilon}\right) + i\beta_q(\omega, z)\Theta_{jq}\nu'\left(\frac{z}{\varepsilon}\right) \right] \right. \\ & \left. + \sigma^2 \left[ \widetilde{\gamma}_{jq}\left(\omega, \frac{z}{\varepsilon}\right) + i\beta_q \widetilde{\theta}_{jq}\left(\omega, \frac{z}{\varepsilon}\right) \right] + \gamma_{jq}^o(z) + i\beta_q(\omega, z)\theta_{jq}^o(z) \right\}, \qquad j \neq q, \end{aligned} \tag{3.100}$$

and

$$\Upsilon_{jq}^{\varepsilon(ab)}(\omega, z) := \overline{\Upsilon_{jq}^{\varepsilon(aa)}}(\omega, z) e^{-\frac{2i}{\varepsilon}\int_0^z dz' \, \beta_j(\omega, z')}, \qquad j \neq q, \tag{3.101}$$

and on the diagonal we have

$$\Upsilon_{jj}^{\varepsilon(aa)}(\omega, z) := H_j^{\varepsilon(aa)}(\omega, z) + \frac{i\sigma^2}{2\beta_j(\omega, z)} \eta_j\left(\omega, \frac{z}{\varepsilon}\right), \tag{3.102}$$

and

$$\Upsilon_{jj}^{\varepsilon(ab)}(\omega,z) := \Big[\overline{\Upsilon_{jj}^{\varepsilon(aa)}}(\omega,z) - \frac{\partial_z \beta_j(\omega,z)}{2\beta_j(\omega,z)}\Big] e^{-\frac{2i}{\varepsilon}\int_0^z dz'\,\beta_j(\omega,z')}. \tag{3.103}$$

The coefficients in these definitions are given in (3.62), and (3.37)-(3.41), except for $\eta_j$, $\widetilde{\gamma}_{jq}$ and $\widetilde{\theta}_{jq}$, which include the interaction with the evanescent modes. These are defined by

$$\widetilde{\gamma}_{jq}\Big(\omega,\frac{z}{\varepsilon}\Big) := \gamma_{jq}\Big(\frac{z}{\varepsilon}\Big) - \sum_{l>\mathcal{N}} \frac{\Gamma_{jl}}{2\beta_l(\omega,z)}\nu''\Big(\frac{z}{\varepsilon}\Big)\int_{-\infty}^{\infty} d\xi\, \gamma_{lq}^{(e)}\Big(\omega,\frac{z}{\varepsilon}+\xi\Big)e^{-|\xi|\beta_l(\omega,z)+i\xi\beta_q(\omega,z)},$$

and

$$\widetilde{\theta}_{jq}\Big(\omega,\frac{z}{\varepsilon}\Big) := \theta_{jq}\Big(\frac{z}{\varepsilon}\Big) - \sum_{l>\mathcal{N}} \frac{\Theta_{jl}}{2\beta_l(\omega,z)}\nu'\Big(\frac{z}{\varepsilon}\Big)\int_{-\infty}^{\infty} d\xi\, \theta_{lq}^{(e)}\Big(\omega,\frac{z}{\varepsilon}+\xi\Big)e^{-|\xi|\beta_l(\omega,z)+i\xi\beta_q(\omega,z)},$$

and

$$\begin{aligned}\eta_j\Big(\omega,\frac{z}{\varepsilon}\Big) :=& \sum_{l>\mathcal{N}} \frac{1}{2\beta_l(\omega,z)}\int_{-\infty}^{\infty} d\xi\, e^{-|\xi|\beta_l(\omega,z)+i\xi\beta_j(\omega,z)} \\ &\times \Big[\Gamma_{jl}\nu''\Big(\frac{z}{\varepsilon}\Big)\gamma_{lj}^{(e)}\Big(\omega,\frac{z}{\varepsilon}+\xi\Big) + \Theta_{jl}\nu'\Big(\frac{z}{\varepsilon}\Big)\theta_{lj}^{(e)}\Big(\omega,\frac{z}{\varepsilon}+\xi\Big)\Big],\end{aligned}$$

with $\gamma_{jq}$ and $\theta_{jq}$ given in (3.38)-(3.39) and $\gamma_{lq}^{(e)}$, $\theta_{lq}^{(e)}$ given in (3.93) and (3.96). Note that the coefficients $\Gamma_{jl}/\beta_l$ and $\Theta_{jl}/\beta_l$ decay as $1/l^2$ for $l \gg 1$, and the integrals in $\xi$ are bounded, so the series defining $\widetilde{\gamma}_{jq}$, $\widetilde{\theta}_{jq}$ and $\eta_j$ are absolutely convergent.

### 3.6.2 Vicinity of turning points

Let us consider a vicinity $|z - z_-^{(t)}| = O(\varepsilon^s)$ of the turning point $z_-^{(t)}$, for some $s > 0$, and change for a moment variables to $z = z_-^{(t)} + \varepsilon^s\zeta$, so that $\zeta = O(1)$. In the new variable, we observe that the coupling terms in the evolution equations (3.59) for the turning wave indexed by $j = \mathcal{N}$, modeled by the series (3.36), are proportional to

$$\frac{\varepsilon^{s/2}}{\sqrt{\varepsilon^{1-s}}}\widetilde{\nu}\Big(\frac{\zeta}{\varepsilon^{1-s}}\Big) + O(\varepsilon^s), \quad \widetilde{\nu} := \nu'' \text{ or } \nu'. \tag{3.104}$$

In the limit $\varepsilon \to 0$, described in detail in sections 3.7-3.9, all these terms tend to zero. Thus, the turning wave does not interact with the other modes near the turning point.

We also obtain that the right-hand side of equation (3.59) for $1 \le j \le \mathcal{N} - 1$ tends to zero as $\varepsilon \to 0$, so the propagating mode amplitudes remain constant as they traverse the vicinity of the turning point $z_-^{(t)}$. A similar argument shows that the propagating mode amplitudes remain constant as they traverse the vicinity of the turning point $z_-^{(t-1)}$ at the other end of the interval.

It remains to describe the turning mode that undergoes a transition near $z_-^{(t)}$. To do so, we return to the original coordinate $z$, but stay in the vicinity of $z_-^{(t)}$. We obtain from (3.59) with $j = \mathcal{N}$, after neglecting the coupling terms, that

$$\partial_z \begin{pmatrix} a_{\mathcal{N}}^{\varepsilon}(\omega, z) \\ b_{\mathcal{N}}^{\varepsilon}(\omega, z) \end{pmatrix} \approx \mathbf{H}_{\mathcal{N}}^{\varepsilon}(\omega, z) \begin{pmatrix} a_{\mathcal{N}}^{\varepsilon}(\omega, z) \\ b_{\mathcal{N}}^{\varepsilon}(\omega, z) \end{pmatrix}, \tag{3.105}$$

where the matrix $\mathbf{H}_{\mathcal{N}}^{\varepsilon}$ is defined by (3.60) and (3.72)-(3.73). These equations give

$$\partial_z \Big[ \big| a_{\mathcal{N}}^{\varepsilon}(\omega, z) \big|^2 - \big| b_{\mathcal{N}}^{\varepsilon}(\omega, z) \big|^2 \Big] \approx 0, \tag{3.106}$$

and using the radiation condition (3.71), we conclude that near the turning point we have energy conservation

$$\big| a_{\mathcal{N}}^{\varepsilon}(\omega, z) \big|^2 \approx \big| b_{\mathcal{N}}^{\varepsilon}(\omega, z) \big|^2. \tag{3.107}$$

We note that all the energy conservation relations are approximate at a finite $\varepsilon$, due to the interaction with the evanescent modes. However, we will see in section 3.9 that there is no energy loss in the limit $\varepsilon \to 0$. Due to the energy conservation, the impinging left going wave with amplitude $b^{\varepsilon}$ is reflected back at the turning point to give the right going wave with amplitude $a^{\varepsilon}$, determined by the reflection coefficient

$$\mathrm{R}_{\mathcal{N}}^{\varepsilon}(\omega, z) := \frac{a_{\mathcal{N}}^{\varepsilon}(\omega, z)}{b_{\mathcal{N}}^{\varepsilon}(\omega, z)} \approx i \exp \big[ \frac{2i}{\varepsilon} \phi_{\mathcal{N}}(\omega, 0) + i \vartheta_{\mathcal{N}}^{\varepsilon}(\omega, z) \big]. \tag{3.108}$$

This is a complex number with modulus $\left|\mathrm{R}^{\varepsilon}_{\mathcal{N}}(\omega,z)\right| \approx 1$, because there is no loss of power in the limit $\varepsilon \to 0$, and with random phase $\vartheta^{\varepsilon}_{\mathcal{N}}(\omega,z)$.

The phase $\vartheta^{\varepsilon}_{\mathcal{N}}$ is described in detail in [11, Lemmas 4.1 and 4.2], for the purpose of characterizing the reflection of a broad-band pulse at the turning point. The standard deviation of the random boundary fluctuations considered in [11] is smaller than what we have in (3.15), by a factor of $|\ln\varepsilon|^{1/2}$, so that as $\varepsilon \to 0$ there is no mode coupling at any $z$, small or order one. Here we have mode coupling away from the turning points, due to the stronger random boundary fluctuations, and we are interested in the transport of energy by single frequency modes in the waveguide. The mode powers are not affected by the phase, so the details of $\vartheta^{\varepsilon}_{\mathcal{N}}(\omega,z)$ are not important in the context of this dissertation.

#### 3.6.3 Source excitation and matching conditions

The evolution equations of the left and right going mode amplitudes, described above, are complemented by matching conditions at the turning points, by radiation conditions at $|z| > Z_M$, and by initial conditions at $z = 0$, where the source lies.

Starting from the source location $z = 0$, which is not a turning point, we have the jump conditions,

$$\begin{aligned} a^{\varepsilon}_j(\omega,0+) - a^{\varepsilon}_j(\omega,0-) &= \frac{\widehat{f}(\omega) y_j(\rho_\star,0)}{2i\sqrt{\beta_j(\omega,0)}}, \\ b^{\varepsilon}_j(\omega,0+) - b^{\varepsilon}_j(\omega,0-) &= \frac{\widehat{f}(\omega) y_j(\rho_\star,0)}{2i\sqrt{\beta_j(\omega,0)}}, \quad 1 \le j \le N^{(0)}, \end{aligned} \tag{3.109}$$

where we recall that $N^{(0)}$ is the number of propagating modes at $z = 0$ and we denote $a(0+) = \lim_{z\searrow 0} a(z)$ and $a(0-) = \lim_{z\nearrow 0} a(z)$.

On the left of the source, at turning points $z^{(t)}_{-}$, for $1 \le t \le t^{-}_{M}$, we have the continuity relations

$$a_j^\varepsilon(\omega, z_-^{(t)}+) = a_j^\varepsilon(\omega, z_-^{(t)}-), \quad b_j^\varepsilon(\omega, z_-^{(t)}+) = b_j^\varepsilon(\omega, z_-^{(t)}-), \tag{3.110}$$

for $1 \leq j \leq N_-^{(t-1)} - 1$, where we recall definition (3.45) of $N_-^{(t-1)}$. We also have the reflection of the turning mode, modeled by the equation

$$a_{N_-^{(t-1)}}^\varepsilon(\omega, z_-^{(t)}+) = \mathrm{R}_{N_-^{(t-1)}}^\varepsilon(\omega, z_-^{(t)}) b_{N_-^{(t-1)}}^\varepsilon(\omega, z_-^{(t)}+), \tag{3.111}$$

where $\mathrm{R}_{N_-^{(t-1)}}^\varepsilon$ is the complex reflection coefficient defined as in (3.108).

At $z < -Z_M$, where the waveguide has straight and parallel boundaries and supports $N_{\min}$ propagating modes, the wave is outgoing (propagating to the left), so we have the conditions

$$a_j(z) = a_j(-Z_M+) = 0, \quad b_j(z) = b_j(-Z_M+), \qquad z < -Z_M, \tag{3.112}$$

for $j = 1, \ldots, N_{\min}$.

Similarly, on the right of the source, at turning points $z_+^{(t)}$, for $1 \leq t \leq t_M^+$, we have the continuity relations

$$a_j^\varepsilon(\omega, z_+^{(t)}+) = a_j^\varepsilon(\omega, z_+^{(t)}-), \quad b_j^\varepsilon(\omega, z_+^{(t)}+) = b_j^\varepsilon(\omega, z_+^{(t)}-), \tag{3.113}$$

for $1 \leq j \leq N_+^{(t-1)}$, where we recall definition (3.45) of $N_+^{(t-1)}$. The number of propagating modes increases by one at $z_+^{(t)}$, to equal $N_+^{(t)}$, and the amplitude of the turning wave, indexed by $j = N_+^{(t)}$, starts from zero there

$$a_{N_+^{(t)}}^\varepsilon(\omega, z_+^{(t)}) = b_{N_+^{(t)}}^\varepsilon(\omega, z_+^{(t)}+) = 0. \tag{3.114}$$

At $z > Z_M$, where the waveguide has straight and parallel boundaries and supports $N_{\max}$ propagating modes, the wave is outgoing (propagating to the right), so we have the conditions

$$a_j(z) = a_j(Z_M-), \quad b_j(z) = b_j(Z_M-) = 0, \qquad z > Z_M, \tag{3.115}$$

for $j = 1, \ldots, N_{\max}$.

### 3.7 The propagator matrix

The discussion below applies to any sector of the waveguide, so let us consider as in section 3.5.2 the sector $z \in \left(z_-^{(t)}, z_-^{(t-1)}\right)$, supporting $\mathcal{N} = N_-^{(t-1)}$ propagating modes.

The mode amplitudes satisfy the system of equations (3.97), with $2\mathcal{N} \times 2\mathcal{N}$ random propagator matrix $\mathbf{P}^\varepsilon(\omega, z; z_-^{(t-1)})$. This solves the equation

$$\partial_z \mathbf{P}^\varepsilon(\omega, z; z_-^{(t-1)}) = \mathbf{\Upsilon}^\varepsilon(\omega, z)\mathbf{P}^\varepsilon(\omega, z; z_-^{(t-1)}), \tag{3.116}$$

backward in $z$, starting from

$$\mathbf{P}^\varepsilon(\omega, z_-^{(t-1)}; z_-^{(t-1)}) = \mathbf{I}, \tag{3.117}$$

where $\mathbf{I}$ is the $2\mathcal{N} \times 2\mathcal{N}$ identity matrix and $\mathbf{\Upsilon}^\varepsilon(\omega, z)$ is defined in (3.98)-(3.103).

The propagator defines the solution of (3.97),

$$\begin{pmatrix} \boldsymbol{a}^\varepsilon(\omega, z) \\ \boldsymbol{b}^\varepsilon(\omega, z) \end{pmatrix} = \mathbf{P}^\varepsilon(\omega, z; z_-^{(t-1)}) \begin{pmatrix} \boldsymbol{a}^\varepsilon(\omega, z_-^{(t-1)}) \\ \boldsymbol{b}^\varepsilon(\omega, z_-^{(t-1)}) \end{pmatrix}, \tag{3.118}$$

and due to the symmetry relations (3.99) of the blocks of $\mathbf{\Upsilon}^\varepsilon$, we note that

$$\begin{pmatrix} \overline{\boldsymbol{b}^\varepsilon}(\omega, z) \\ \overline{\boldsymbol{a}^\varepsilon}(\omega, z) \end{pmatrix} = \mathbf{P}^\varepsilon(\omega, z; z_-^{(t-1)}) \begin{pmatrix} \overline{\boldsymbol{b}^\varepsilon}(\omega, z_-^{(t-1)}) \\ \overline{\boldsymbol{a}^\varepsilon}(\omega, z_-^{(t-1)}) \end{pmatrix} \tag{3.119}$$

is also a solution. Writing explicitly these equations, and using the uniqueness of solution of (3.97), we conclude that the propagator has the block form

$$\mathbf{P}^\varepsilon(\omega, z; z_-^{(t-1)}) = \begin{pmatrix} \overline{\mathbf{P}^{\varepsilon(bb)}}(\omega, z; z_-^{(t-1)}) & \overline{\mathbf{P}^{\varepsilon(ba)}}(\omega, z; z_-^{(t-1)}) \\ \mathbf{P}^{\varepsilon(ba)}(\omega, z; z_-^{(t-1)}) & \mathbf{P}^{\varepsilon(bb)}(\omega, z; z_-^{(t-1)}) \end{pmatrix}. \tag{3.120}$$

The blocks are $\mathcal{N} \times \mathcal{N}$ complex matrices, where $\mathbf{P}^{\varepsilon(bb)}$ describes the coupling between the components of $\boldsymbol{b}^\varepsilon$, the vector of left-going mode amplitudes, and $\mathbf{P}^{\varepsilon(ba)}$ describes the coupling between the components of $\boldsymbol{b}^\varepsilon$ and of $\boldsymbol{a}^\varepsilon$, the vector of right-going mode amplitudes.

The limit of $\mathbf{P}^\varepsilon(\omega, z; z_-^{(t-1)})$ as $\varepsilon \to 0$ can be obtained and identified as a multi-dimensional diffusion process, meaning that the entries of the limit matrix satisfy a system of linear stochastic equations. This follows from the application of an extension of the diffusion approximation theorem proved in [43] and presented in [26, Chapter 6]. This extension is stated in Theorem V.1 and is proved in section V for a general system of equations with real-valued unknown vector $\boldsymbol{X}^\varepsilon$. In our case $\boldsymbol{X}^\varepsilon$ is obtained by concatenating the moduli and arguments of the entries in $\mathbf{P}^{\varepsilon(bb)}$ and $\mathbf{P}^{\varepsilon(ba)}$.

### 3.8 The forward scattering approximation

When we use Theorem V.1, we obtain that the limit entries of $\mathbf{P}^{\varepsilon(bb)}$ are coupled to the limit entries of $\mathbf{P}^{\varepsilon(ba)}$ through coefficients that are proportional to the power spectral density[2] $\widehat{\mathcal{R}}$ of the random fluctuations $\nu$, evaluated at the sum of the mode wavenumbers,

$$\widehat{\mathcal{R}}\big(\beta_j(\omega, z) + \beta_l(\omega, z)\big) = 2\int_0^\infty d\zeta\, \mathcal{R}(\zeta) \cos[(\beta_j(\omega, z) + \beta_l(\omega, z))\zeta]\,, \tag{3.121}$$

for $j, l = 1, \ldots, \mathcal{N}$. This can be traced back to the phase factors

$$\frac{1}{\varepsilon}\int_0^z dz' \big[\beta_j(\omega, z') + \beta_l(\omega, z')\big]$$

in the matrix $\boldsymbol{\Upsilon}^{\varepsilon(ba)}(\omega, z)$ defined in (3.101). The limit entries of $\mathbf{P}^{\varepsilon(bb)}(z)$ are coupled to each other through $\widehat{\mathcal{R}}\big(\beta_j(\omega, z) - \beta_l(\omega, z)\big)$, because the phase factors in $\boldsymbol{\Upsilon}^{\varepsilon(bb)}(\omega, z)$ defined in (3.99)-(3.100) are

$$\frac{1}{\varepsilon}\int_0^z dz' \big[\beta_j(\omega, z') - \beta_l(\omega, z')\big],$$

[2] The power spectral density is the Fourier transform of the auto-correlation function $\mathcal{R}$ defined in (3.7). It is a non-negative and even function that decays rapidly when $\mathcal{R}$ and therefore $\nu$ are smooth in $z$.

for $j, l = 1, \ldots, \mathcal{N}$.

To simplify the analysis of the cumulative scattering effects in the limit $\varepsilon \to 0$, we assume that the power spectral density $\widehat{\mathcal{R}}$ peaks at zero and decays rapidly away from it[3], so that

$$\widehat{\mathcal{R}}\big(\beta_j(\omega, z) + \beta_l(\omega, z)\big) \approx 0\,, \quad \forall\, j, l = 1, \ldots, \mathcal{N}. \tag{3.122}$$

With this assumption we can neglect the coupling between the blocks $\mathbf{P}^{\varepsilon(bb)}(\omega, z)$ and $\mathbf{P}^{\varepsilon(ba)}(\omega, z)$ of the propagator. Since $\mathbf{P}^{\varepsilon(ba)}$ starts from zero at $z = z_-^{(t-1)}$, we obtain

$$\mathbf{P}^{\varepsilon}(\omega, z; z_-^{(t-1)}) \approx \begin{pmatrix} \overline{\mathbf{P}^{\varepsilon(bb)}}(\omega, z; z_-^{(t-1)}) & \mathbf{0} \\ \mathbf{0} & \mathbf{P}^{\varepsilon(bb)}(\omega, z; z_-^{(t-1)}) \end{pmatrix}, \tag{3.123}$$

and equation (3.118) gives

$$\boldsymbol{b}^{\varepsilon}(\omega, z) \approx \mathbf{P}^{\varepsilon(bb)}(\omega, z; z_-^{(t-1)})\boldsymbol{b}^{\varepsilon}(\omega, z_-^{(t-1)})\,, \quad z < z_-^{(t-1)}. \tag{3.124}$$

This is the forward scattering approximation. It describes the left going amplitudes $\boldsymbol{b}^{\varepsilon}(\omega, z)$ of the waves, propagating forward from the source, independent of the right-going amplitudes $\boldsymbol{a}^{\varepsilon}(\omega, z)$ of the waves, propagating backward, toward the source.

Note that since $\beta_j$ decrease monotonically with $j$, the smallest argument of the power spectral density in (3.122) is at $j = l = \mathcal{N}$. The wavenumber $\beta_{\mathcal{N}}(z)$ is of order $k/\sqrt{\mathcal{N}}$ away from the turning point $z_-^{(t)}$, but tends to zero as $z \searrow z_-^{(t)}$. The left- and right-going amplitudes of the turning mode are coupled near $z_-^{(t)}$, as described by the reflection coefficient (3.108). We assume that this coupling holds only at $z - z_-^{(t)} < \delta$, where $\delta$ is a small and positive number, independent of $\varepsilon$. Over the small distance $\delta$ there is negligible interaction between the turning mode and the others, as explained

[3]An example is the Fourier transform of the Gaussian auto-correlation function used in the numerical simulations in section IV.

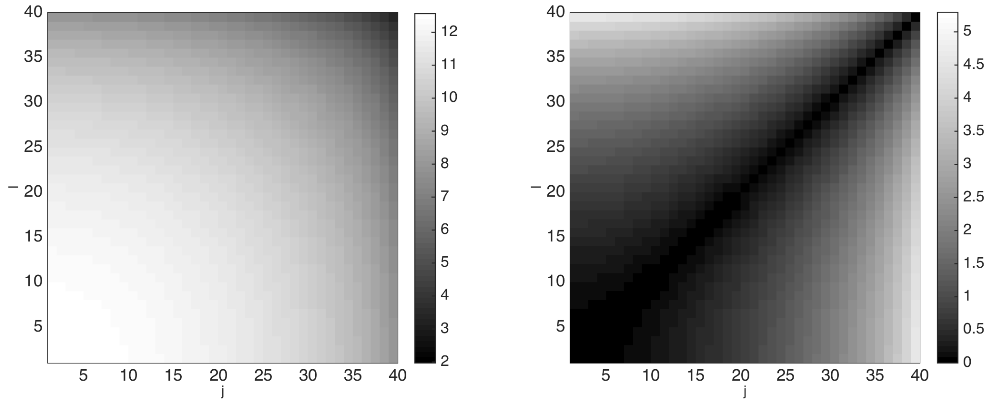


Figure 3.2: Plots of the matrices with entries $\beta_j + \beta_l$ and $|\beta_j - \beta_l|$

The matrix with entries $\beta_j+\beta_l$ is on the left and $|\beta_j-\beta_l|$ is on the right, both are plotted v.s. $j, l = 1, \ldots, \mathcal{N}$, for the case of $\mathcal{N} = 40$ propagating modes. The scaled wavenumber is $k = 2\pi$ and the waveguide width is $D = 20.25$. Note that the entries in the left plot are larger than $2\beta_{\mathcal{N}} = 1.97$, whereas the entries near the diagonal in the right plot are small.

in section 3.6.2. In the remaining interval $z \in (z_-^{(t)} + \delta, z_-^{(t-1)})$ we have

$$\mathcal{R}\big(2\beta_{\mathcal{N}}(\omega, z)\big) \lesssim \widehat{\mathcal{R}}\big(2\beta_{\mathcal{N}}(\omega, z_-^{(t)} + \delta)\big) \approx 0, \tag{3.125}$$

so we can use the forward scattering approximation.

Note that there is mode coupling in this approximation, but only between the forward-going mode amplitudes. This is due to the fact that $|\beta_j(\omega, z) - \beta_l(\omega, z)|$ is small at least for nearby indices $j, l$, as illustrated in Figure 3.2. The power spectral density evaluated at such differences is not negligible, and the net coupling effect is described in the next section.

## 3.9 The coupled mode diffusion process

The $\varepsilon \to 0$ limit of the forward-going mode amplitudes is stated in the next theorem. We derive it using Theorem V.1 for the vector $\boldsymbol{X}^\varepsilon \in \mathbb{R}^{2\mathcal{N}}$ obtained by concatenating the moduli and arguments of $b_j^\varepsilon$, with $j = 1, \ldots, \mathcal{N}$. The differential equations for $\boldsymbol{X}^\varepsilon$ follow from the system

$$\partial_z \boldsymbol{b}^\varepsilon(\omega, z) \approx \boldsymbol{\Upsilon}^{\varepsilon(bb)}(\omega, z)\boldsymbol{b}^\varepsilon(\omega, z)\,, \quad z < z_-^{(t-1)}, \tag{3.126}$$

with given $\boldsymbol{b}^\varepsilon(\omega, z_-^{(t-1)})$. As explained in the previous section, the approximation in (3.126) means that there is an error that vanishes in the limit $\varepsilon \to 0$.

**Theorem III.1.** *The complex mode amplitudes* $\{b_j^\varepsilon(\omega, z)\}_{j=1}^{\mathcal{N}}$ *converge in distribution as* $\varepsilon \to 0$ *to an inhomogeneous diffusion Markov process* $\{b_j(\omega, z)\}_{j=1}^{\mathcal{N}}$ *with generator* $-\mathcal{L}_z^{\mathcal{N}}$ *given below.*[4]

Let us write the limit process as

$$b_j(\omega, z) = P_j^{1/2}(\omega, z) e^{i\psi_j(\omega,z)}, \quad j = 1, \dots, \mathcal{N},$$

in terms of the power $P_j := |b_j|^2$ and the phase $\psi_j := \arg b_j$. Then, we can express the infinitesimal generator of the limit diffusion as the sum of two operators

$$\mathcal{L}_z^{\mathcal{N}} = \mathcal{L}_{P,z}^{\mathcal{N}} + \mathcal{L}_{\psi,z}^{\mathcal{N}}. \tag{3.127}$$

The first is a partial differential operator in the powers

$$\mathcal{L}_{P,z}^{\mathcal{N}} := \sum_{\substack{j,l=1 \\ j \neq l}}^{\mathcal{N}} G_{jl}^{(c)}(\omega, z) \left[ P_l P_j \left( \frac{\partial}{\partial P_j} - \frac{\partial}{\partial P_l} \right) \frac{\partial}{\partial P_j} + (P_l - P_j) \frac{\partial}{\partial P_j} \right], \tag{3.128}$$

with symmetric matrix $\mathbf{G}^{(c)}(\omega, z) := \left( G_{jl}^{(c)}(\omega, z) \right)_{j,l=1}^{\mathcal{N}}$ of coefficients that are non-negative off the diagonal

$$G_{jl}^{(c)}(\omega, z) := \frac{\sigma^2 \mu_j^2(z) \mu_l^2(z)}{4\beta_j(\omega, z)\beta_l(\omega, z)} \widehat{\mathcal{R}}[\beta_j(\omega, z) - \beta_l(\omega, z)], \quad j \neq l, \tag{3.129}$$

and sum to zero in the rows

$$G_{jj}^{(c)}(\omega, z) := -\sum_{\substack{l=1 \\ l \neq j}}^{\mathcal{N}} G_{jl}^{(c)}(\omega, z). \tag{3.130}$$

[4]The minus sign in front of the generator is because we solve the Kolmogorov equation for the moments of the limit process backward in $z$, starting from $z_-^{(t-1)}$.

The second partial differential operator in (3.127) is with respect to the phases

$$\mathcal{L}^{\mathcal{N}}_{\psi,z} := \frac{1}{8}\sum_{\substack{j,l=1\\ j\neq l}}^{\mathcal{N}} G^{(c)}_{jl}(\omega,z)\left[\frac{P_j}{P_l}\frac{\partial^2}{\partial\psi_l^2} + \frac{P_l}{P_j}\frac{\partial^2}{\partial\psi_j^2} + 2\frac{\partial^2}{\partial\psi_j\partial\psi_l}\right] + \frac{1}{2}\sum_{j,l=1}^{\mathcal{N}} G^{(0)}_{jl}(\omega,z)\frac{\partial^2}{\partial\psi_j\partial\psi_l}$$
$$+ \frac{1}{2}\sum_{\substack{j,l=1\\ j\neq l}}^{\mathcal{N}} G^{(s)}_{jl}(\omega,z)\frac{\partial}{\partial\psi_j} + \sum_{j=1}^{\mathcal{N}}\kappa^{\mathcal{N}}_j(\omega,z)\frac{\partial}{\partial\psi_j}\,, \tag{3.131}$$

with coefficients

$$G^{(0)}_{jl}(\omega,z) := \frac{\sigma^2\mu_j^2(z)\mu_l^2(z)}{4\beta_j(\omega,z)\beta_l(\omega,z)}\widehat{\mathcal{R}}(0)\,, \quad j,l = 1,\dots,\mathcal{N}, \tag{3.132}$$

and

$$G^{(s)}_{jl}(\omega,z) := \frac{\sigma^2\mu_j^2(z)\mu_l^2(z)}{2\beta_j(\omega,z)\beta_l(\omega,z)}\int_0^\infty d\zeta\,\mathcal{R}(\zeta)\sin\left[(\beta_j(\omega,z)-\beta_l(\omega,z))\zeta\right]\,, \tag{3.133}$$

for $j,l = 1,\dots,\mathcal{N}$ and $j\neq l$. The coefficient $\kappa^{\mathcal{N}}_j$ in the last term of (3.131) is

$$\kappa^{\mathcal{N}}_j(\omega,z) := \frac{\sigma^2}{2\beta_j(\omega,z)}\Big\{\Big(\frac{\pi^2 j^2}{12}+\frac{1}{16}\Big)\mathcal{R}''(0) - \frac{3\mu_j^2(z)}{4}\mathcal{R}(0)\Big\}$$
$$-\sum_{\substack{l=1\\ j\neq l}}^{\mathcal{N}}\frac{\mu_j^2(z)\mu_l^2(z)}{4\beta_j(\omega,z)\beta_l(\omega,z)[\beta_j(\omega,z)-\beta_l(\omega,z)]}\Big[\mathcal{R}(0)+\frac{\mathcal{R}''(0)}{[\beta_j(\omega,z)+\beta_l(\omega,z)]^2}\Big]$$
$$+\sum_{l>\mathcal{N}}\frac{\sigma^2\mu_j^2(z)\mu_l^2(z)}{2\beta_j\beta_l[\beta_j^2(\omega,z)+\beta_l^2(\omega,z)]^2}\Big\{-\beta_l(\omega,z)\mathcal{R}''(0)+\int_0^\infty d\zeta\,\mathcal{R}''(\zeta)e^{-\beta_l(\omega,z)\zeta}$$
$$\times\Big[[\beta_l^2(\omega,z)-\beta_j^2(\omega,z)]\cos(\beta_j(\omega,z)\zeta) - 2\beta_j(\omega,z)\beta_l(\omega,z)\sin(\beta_j(\omega,z)\zeta)\Big]\Big\}. \tag{3.134}$$

Note that the coefficients of the partial derivatives with respect to the mode powers $P_j$ are independent of the phases $\psi_j$. This implies that $\{|b^\varepsilon_j(\omega,z)|^2\}_{j=1}^{\mathcal{N}}$ converge in distribution in the limit $\varepsilon\to 0$ to the inhomogeneous diffusion Markov process $\{P_j(\omega,z)\}_{j=1}^{\mathcal{N}}$ with infinitesimal generator $-\mathcal{L}^{\mathcal{N}}_{P,z}$ defined in (3.128). The total power of the propagating modes satisfies

$$\mathcal{L}^{\mathcal{N}}_{P,z}\Big[\sum_{j=1}^{\mathcal{N}} P_j(\omega,z)\Big] = \sum_{\substack{j,l=1\\ j\neq l}}^{\mathcal{N}} G^{(c)}_{jl}(\omega,z)\Big[P_l(\omega,z)-P_j(\omega,z)\Big] = 0, \tag{3.135}$$

where we used (3.130) and the symmetry of matrix $\mathbf{G}^{(c)}(\omega, z)$. This implies that the total power is conserved

$$\sum_{j=1}^{\mathcal{N}} P_j(\omega, z) = \text{constant}, \qquad z \in \left(z_-^{(t)}, z_-^{(t-1)}\right). \tag{3.136}$$

The evanescent waves do not contribute to the expression of the infinitesimal generator $\mathcal{L}_{P,z}^{\mathcal{N}}$, so they do not exchange energy with the propagating modes in the limit $\varepsilon \to 0$. However, they appear in the last coefficient (3.134) of the operator $\mathcal{L}_{\psi,z}^{\mathcal{N}}$, so they affect the phases of the mode amplitudes.

The limit Markov process $\{b_j(\omega, z)\}_{j=1}^{\mathcal{N}}$ is inhomogeneous due to the slow variations of the waveguide which make the coefficients of the operators (3.128) and (3.131) $z$ dependent. The slow variations also change the number of propagating modes at the turning points, and this leads to partial reflection of power, as described in the next chapter.

## 3.10 Appendix

Here we collect some integral identities and facts about the matrix $\mathbf{M}_{\mathcal{N}}^{\varepsilon}(\omega, z)$ that were used earlier in this chapter. The integral identities are for the eigenfunctions (3.29) and were used in the derivation of (3.31). The facts about $\mathbf{M}_{\mathcal{N}}^{\varepsilon}(\omega, z)$ were used in our discussion of the turning waves in 3.5.3.

### 3.10.1 A Few Useful Identities

The first identity is just the statement that the eigenfunctions are orthonormal

$$\int_{-D(z)/2}^{D(z)/2} d\rho \, y_j(\rho, z) y_q(\rho, z) = \delta_{jq}, \tag{3.137}$$

where $\delta_{jq}$ is the Kronecker delta symbol. The second identity

$$\int_{-D(z)/2}^{D(z)/2} d\rho \, \rho y_j^2(\rho, z) = 0, \tag{3.138}$$

is because the integrand is odd. The third identity follows from the fundamental theorem of calculus,

$$\int_{-D(z)/2}^{D(z)/2} d\rho\, y_j(\rho,z)\partial_\rho y_j(\rho,z) = \frac{1}{2}\int_{-D(z)/2}^{D(z)/2} d\rho\, \partial_\rho y_j^2(\rho,z) = 0, \tag{3.139}$$

because the eigenfunctions vanish at $\rho = \pm D(z)/2$. The fourth identity is

$$\begin{aligned}\int_{-D(z)/2}^{D(z)/2} d\rho\,[2\rho + D(z)]y_j(\rho,z)\partial_\rho y_j(\rho,z) &= \int_{-D(z)/2}^{D(z)/2} d\rho\, \rho\partial_\rho y_j^2(\rho,z) \\ &= \int_{-D(z)/2}^{D(z)/2} d\rho\left\{\partial_\rho\left[\rho y_j^2(\rho,z)\right] - y_j^2(\rho,z)\right\} = -1,\end{aligned} \tag{3.140}$$

where we used integration by parts. The fifth identity is

$$\int_{-D(z)/2}^{D(z)/2} d\rho\, y_j(\rho,z)\partial_z y_j(\rho,z) = 0. \tag{3.141}$$

To derive it, we take the $z$ derivative in (3.137), for $q = j$, and obtain that

$$\begin{aligned}0 =& \partial_z \int_{-D(z)/2}^{D(z)/2} d\rho\, y_j^2(\rho,z) = 2\int_{-D(z)/2}^{D(z)/2} d\rho\, y_j(\rho,z)\partial_z y_j(\rho,z) \\ &+ \frac{D'(z)}{2}\left[y_j^2(D(z)/2,z) - y_j^2(-D(z)/2,z)\right] = 2\int_{-D(z)/2}^{D(z)/2} d\rho\, y_j(\rho,z)\partial_z y_j(\rho,z).\end{aligned}$$

We also have from (3.137), (3.138), and definition (3.29) that

$$\begin{aligned}\int_{-D(z)/2}^{D(z)/2} d\rho[2\rho + D(z)]^2 y_j^2(\rho,z) &= D^2(z) + \frac{8}{D(z)}\int_{-D(z)/2}^{D(z)/2} d\rho\, \rho^2 \sin^2\left[\left(\frac{\rho}{D(z)} + \frac{1}{2}\right)\pi j\right] \\ &= D^2(z)\left[\frac{4}{3} - \frac{2}{(\pi j)^2}\right].\end{aligned} \tag{3.142}$$

For $j \neq q$ we have from definition (3.29) of the eigenfunctions that

$$\begin{aligned}\int_{-D(z)/2}^{D(z)/2} d\rho\,[2\rho + D(z)]y_j(\rho,z)\partial_\rho y_q(\rho,z) = 2\pi q\int_{-D(z)/2}^{D(z)/2} \frac{d\rho}{D(z)}\left[\frac{2\rho}{D(z)} + 1\right] \\ \times\sin\left[\left(\frac{\rho}{D(z)} + \frac{1}{2}\right)\pi j\right]\cos\left[\left(\frac{\rho}{D(z)} + \frac{1}{2}\right)\pi q\right] = -\frac{4jq(-1)^{j+q}}{j^2 - q^2}.\end{aligned} \tag{3.143}$$

Similarly, we obtain after taking the derivative with respect to $z$ of $y_q(\rho,z)$ and substituting in the integral below that

$$\int_{-D(z)/2}^{D(z)/2} d\rho\, y_j(\rho,z)\partial_z y_q(\rho,z) = \frac{D'(z)}{D(z)}\frac{jq\left[(-1)^{j+q} + 1\right]}{j^2 - q^2}. \tag{3.144}$$

We also calculate using the expression (3.29) that

$$\int_{-D(z)/2}^{D(z)/2} d\rho\, [2\rho + D(z)]^2 y_j(\rho, z) y_q(\rho, z) = \frac{32D^2(z)}{\pi^2} \frac{jq(-1)^{j+q}}{(j^2 - q^2)}, \tag{3.145}$$

and

$$\int_{-D(z)/2}^{D(z)/2} d\rho\, y_j(\rho, z) \partial_\rho y_q(\rho, z) = \frac{2jq[1 - (-1)^{j+q}]}{D(z)(j^2 - q^2)}, \tag{3.146}$$

and

$$\int_{-D(z)/2}^{D(z)/2} d\rho\, (2\rho + D(z)) y_j(\rho, z) y_q(\rho, z) = -\frac{8D(z)jq[1 - (-1)^{j+q}]}{\pi^2 (j^2 - q^2)^2}. \tag{3.147}$$

### 3.10.2 Properties of $\mathbf{M}^{\varepsilon}_{\mathcal{N}}(\omega, z)$

We restate some lemmas from [11] for ease of reference. They are formulated here to fit directly their application in section 3.5.3.

**Lemma III.2.** *Let* $\mathbf{M}^{\varepsilon}_{\mathcal{N}}(\omega, z)$ *be as defined in* (3.64) *and* (3.65). *Then*

$$det\, \mathbf{M}^{\varepsilon}_{\mathcal{N}}(\omega, z) = 2 \tag{3.148}$$

*for all* $z \in \left(z_-^{(t)} - \delta, z_-^{(t-1)}\right)$, *where* $\delta$ *is a small, positive number, independent of* $\varepsilon$.

*Proof.* We will compute the determinant of $\mathbf{M}^{\varepsilon}_{\mathcal{N}}(\omega, z)$ directly, though one could also obtain the result through an application of Abel's theorem. We have that

$$\det \mathbf{M}^{\varepsilon}_{\mathcal{N}}(\omega, z) = 2\, \mathrm{Re} \left[ M^{\varepsilon}_{\mathcal{N},11}(\omega, z) \overline{M^{\varepsilon}_{\mathcal{N},21}(\omega, z)} \right].$$

Using the definitions (3.64) and (3.65) of the entries of $\mathbf{M}^{\varepsilon}_{\mathcal{N}}(\omega, z)$ we have

$$\begin{aligned} M^{\varepsilon}_{\mathcal{N},11}(\omega, z) \overline{M^{\varepsilon}_{\mathcal{N},21}(\omega, z)} = & - i\pi \left[ \mathrm{Ai}\big( - \eta^{\varepsilon}_{\mathcal{N}}(\omega, z)\big) - i\mathrm{Bi}\big( - \eta^{\varepsilon}_{\mathcal{N}}(\omega, z)\big) \right] \\ & \times \left[ \mathrm{Ai}'\big( - \eta^{\varepsilon}_{\mathcal{N}}(\omega, z)\big) + i\mathrm{Bi}'\big( - \eta^{\varepsilon}_{\mathcal{N}}(\omega, z)\big) \right] \\ & + i\pi\varepsilon^{2/3} Q_{\mathcal{N}}(\omega, z) Q'_{\mathcal{N}}(\omega, z) \\ & \times \left[ \mathrm{Ai}^2\big( - \eta^{\varepsilon}_{\mathcal{N}}(\omega, z)\big) + \mathrm{Bi}^2\big( - \eta^{\varepsilon}_{\mathcal{N}}(\omega, z)\big) \right]. \end{aligned}$$

Taking the real part we obtain

$$\operatorname{Re}\left[M^{\varepsilon}_{\mathcal{N},11}(\omega,z)\overline{M^{\varepsilon}_{\mathcal{N},21}(\omega,z)}\right] = \pi\big[\operatorname{Ai}\big(-\eta^{\varepsilon}_{\mathcal{N}}(\omega,z)\big)\operatorname{Bi}'\big(-\eta^{\varepsilon}_{\mathcal{N}}(\omega,z)\big) - \operatorname{Ai}'\big(-\eta^{\varepsilon}_{\mathcal{N}}(\omega,z)\big)\operatorname{Bi}\big(-\eta^{\varepsilon}_{\mathcal{N}}(\omega,z)\big)\big]$$

where the bracketed terms are the Wronskian of the Airy functions. This is constant and equal to $1/\pi$ and thus the desired result is achieved.

□

**Lemma III.3.** *For $z - z_-^{(t)} \gg \varepsilon^{2/3}$, we have that the entries of $\mathbf{M}^{\varepsilon}_{\mathcal{N}}(\omega,z)$ are given by*

$$M^{\varepsilon}_{\mathcal{N},11}(\omega,z) = \frac{1}{\sqrt{\beta_{\mathcal{N}}(\omega,z)}}\exp\left[\frac{i}{\varepsilon}\int_0^z dz'\beta_{\mathcal{N}}(\omega,z')\right] + O(\varepsilon),$$
$$M^{\varepsilon}_{\mathcal{N},21}(\omega,z) = \beta_{\mathcal{N}}(\omega,z)M^{\varepsilon}_{\mathcal{N},11}(\omega,z) + O(\varepsilon). \tag{3.149}$$

*Proof.* For $z - z_-^{(t)} \gg \varepsilon^{2/3}$, the functions $\phi_{\mathcal{N}}(\omega,z)$ and $\eta^{\varepsilon}_{\mathcal{N}}(\omega,z)$ satisfy $\phi_{\mathcal{N}}(\omega,z) = O(1)$ and $\eta^{\varepsilon}_{\mathcal{N}}(\omega,z) = O(\varepsilon^{-2/3}) \gg 1$. Asymptotic expansions for the Airy functions at large, negative arguments as given in [19, chapter 9] yield

$$\operatorname{Ai}\big(-\eta^{\varepsilon}_{\mathcal{N}}(\omega,z)\big) = \frac{1}{\sqrt{\pi}\big(\eta^{\varepsilon}_{\mathcal{N}}(\omega,z)\big)^{1/4}}\left\{\sin\left[\frac{2}{3}\big(\eta^{\varepsilon}_{\mathcal{N}}(\omega,z)\big)^{3/2} + \frac{\pi}{4}\right] + O\left(\big(\eta^{\varepsilon}_{\mathcal{N}}(\omega,z)\big)^{-3/2}\right)\right\}, \tag{3.150}$$

$$\operatorname{Ai}'\big(-\eta^{\varepsilon}_{\mathcal{N}}(\omega,z)\big) = -\frac{\big(\eta^{\varepsilon}_{\mathcal{N}}(\omega,z)\big)^{1/4}}{\sqrt{\pi}}\left\{\cos\left[\frac{2}{3}\big(\eta^{\varepsilon}_{\mathcal{N}}(\omega,z)\big)^{3/2} + \frac{\pi}{4}\right] + O\left(\big(\eta^{\varepsilon}_{\mathcal{N}}(\omega,z)\big)^{-3/2}\right)\right\}, \tag{3.151}$$

and

$$\mathrm{Bi}\big(-\eta_{\mathcal{N}}^{\varepsilon}(\omega,z)\big) = \frac{1}{\sqrt{\pi}\big(\eta_{\mathcal{N}}^{\varepsilon}(\omega,z)\big)^{1/4}}\left\{\cos\left[\frac{2}{3}\big(\eta_{\mathcal{N}}^{\varepsilon}(\omega,z)\big)^{3/2}+\frac{\pi}{4}\right]\right.$$
$$\left.+O\left(\big(\eta_{\mathcal{N}}^{\varepsilon}(\omega,z)\big)^{-3/2}\right)\right\}, \tag{3.152}$$
$$\mathrm{Bi}'\big(-\eta_{\mathcal{N}}^{\varepsilon}(\omega,z)\big) = \frac{\big(\eta_{\mathcal{N}}^{\varepsilon}(\omega,z)\big)^{1/4}}{\sqrt{\pi}}\left\{\sin\left[\frac{2}{3}\big(\eta_{\mathcal{N}}^{\varepsilon}(\omega,z)\big)^{3/2}+\frac{\pi}{4}\right]\right.$$
$$\left.+O\left(\big(\eta_{\mathcal{N}}^{\varepsilon}(\omega,z)\big)^{-3/2}\right)\right\}. \tag{3.153}$$

We also have

$$\big(\eta_{\mathcal{N}}^{\varepsilon}(\omega,z)\big)^{1/4} = \varepsilon^{-1/6}\big[k^2-\mu_{\mathcal{N}}^2(z)\big]^{1/4}Q_{\mathcal{N}}(\omega,z) \tag{3.154}$$

and

$$\frac{2}{3}\big(\eta_{\mathcal{N}}^{\varepsilon}(\omega,z)\big)^{3/2} = \varepsilon^{-1}\phi_{\mathcal{N}}(\omega,z). \tag{3.155}$$

The result follows from the definition of the entries of $\mathbf{M}_{\mathcal{N}}^{\varepsilon}(\omega,z)$ in (3.64) and (3.65) and the expressions above.

□

**Lemma III.4.** *For $z_-^{(t)} - z \gg \varepsilon^{2/3}$, the entries of $\mathbf{M}_{\mathcal{N}}^{\varepsilon}(\omega,z)$ are given by*

$$M_{\mathcal{N},11}^{\varepsilon}(\omega,z) \approx \big[\mu_{\mathcal{N}}^2(z)-k^2\big]^{-1/4}\exp\left[\frac{1}{\varepsilon}\int_z^{z_-^{(t)}} dz'\sqrt{\mu_{\mathcal{N}}^2(z')-k^2} - i\frac{\phi_{\mathcal{N}}(\omega,0)}{\varepsilon} - \frac{i\pi}{4}\right],$$
$$M_{\mathcal{N},21}^{\varepsilon}(\omega,z) \approx \big[\mu_{\mathcal{N}}^2(z)-k^2\big]^{-1/4}\exp\left[\frac{1}{\varepsilon}\int_z^{z_-^{(t)}} dz'\sqrt{\mu_{\mathcal{N}}^2(z')-k^2} - i\frac{\phi_{\mathcal{N}}(\omega,0)}{\varepsilon} + \frac{i\pi}{4}\right],$$

*with relative error of order $\varepsilon$.*

*Proof.* For $z_-^{(t)} - z \gg \varepsilon^{2/3}$, the function $\eta_{\mathcal{N}}^{\varepsilon}(\omega,z)$ is negative and satisfies $|\eta_{\mathcal{N}}^{\varepsilon}(\omega,z)| = O(\varepsilon^{-2/3}) \gg 1$. Asymptotic expansions for the Airy functions at large, positive arguments as in [19, chapter 9] yield

$$\mathrm{Ai}\big(|\eta_{\mathcal{N}}^{\varepsilon}(\omega,z)|\big) = \frac{e^{-\frac{2}{3}\left|\eta_{\mathcal{N}}^{\varepsilon}(\omega,z)\right|^{3/2}}}{2\sqrt{\pi}\left|\eta_{\mathcal{N}}^{\varepsilon}(\omega,z)\right|^{1/4}}\left[1+O\left(\left|\eta_{\mathcal{N}}^{\varepsilon}(\omega,z)\right|^{-3/2}\right)\right], \tag{3.156}$$
$$\mathrm{Ai}'\big(|\eta_{\mathcal{N}}^{\varepsilon}(\omega,z)|\big) = -\frac{\left|\eta_{\mathcal{N}}^{\varepsilon}(\omega,z)\right|^{1/4}}{2\sqrt{\pi}}e^{-\frac{2}{3}\left|\eta_{\mathcal{N}}^{\varepsilon}(\omega,z)\right|^{3/2}}\left[1+O\left(\left|\eta_{\mathcal{N}}^{\varepsilon}(\omega,z)\right|^{-3/2}\right)\right], \tag{3.157}$$

and

$$\mathrm{Bi}\big(\,|\eta^{\varepsilon}_{\mathcal{N}}(\omega,z)|\,\big)=\frac{e^{\frac{2}{3}\left|\eta^{\varepsilon}_{\mathcal{N}}(\omega,z)\right|^{3/2}}}{\sqrt{\pi}\left|\eta^{\varepsilon}_{\mathcal{N}}(\omega,z)\right|^{1/4}}\left[1+O\left(\left|\eta^{\varepsilon}_{\mathcal{N}}(\omega,z)\right|^{-3/2}\right)\right], \tag{3.158}$$

$$\mathrm{Bi}'\big(\,|\eta^{\varepsilon}_{\mathcal{N}}(\omega,z)|\,\big)=\frac{\left|\eta^{\varepsilon}_{\mathcal{N}}(\omega,z)\right|^{1/4}}{\sqrt{\pi}}e^{\frac{2}{3}\left|\eta^{\varepsilon}_{\mathcal{N}}(\omega,z)\right|^{3/2}}\left[1+O\left(\left|\eta^{\varepsilon}_{\mathcal{N}}(\omega,z)\right|^{-3/2}\right)\right]. \tag{3.159}$$

We also have

$$\left|\eta^{\varepsilon}_{\mathcal{N}}(\omega,z)\right|^{1/4}=\varepsilon^{-1/6}\left[\mu^{2}_{\mathcal{N}}(z)-k^{2}\right]^{1/4}Q_{\mathcal{N}}(\omega,z) \tag{3.160}$$

and

$$\frac{2}{3}\left|\eta^{\varepsilon}_{\mathcal{N}}(\omega,z)\right|^{3/2}=\varepsilon^{-1}\left|\phi_{\mathcal{N}}(\omega,z)\right|. \tag{3.161}$$

The result follows from the definition of the entries of $\mathbf{M}^{\varepsilon}_{\mathcal{N}}(\omega,z)$ in (3.64) and (3.65) and the expressions above.

□

# CHAPTER IV

# Transport and Reflection of Power

We now use the infinitesimal generator (3.127) to quantify the cumulative scattering effects in the waveguide. We begin in section 4.1 with the modes transmitted through the left part of the waveguide. The right-going modes are discussed in 4.4. They are defined by the direct excitation from the source and the reflection at the turning points. We end with some numerical illustrations in section 4.7.

## 4.1 The left-going waves

The wave propagation from the source at $z = 0$ to the end $z = -Z_M$ of the support of variations of the waveguide can be described in the limit $\varepsilon \to 0$ as follows:

The left-going mode amplitudes start with the values

$$b_j(\omega, 0-) = b_{j,0}(\omega) = -\frac{f(\omega) y_j(\rho_\star, 0)}{2i\sqrt{\beta_j(\omega, 0)}}, \qquad j = 1, \ldots, N^{(0)}, \tag{4.1}$$

obtained from equation (3.109) and the observation that at $z > 0$, where the opening $D(z)$ increases, the waves are right going.

In the sector $\left(z_-^{(1)}, 0\right)$ the amplitudes $\{b_j(\omega, z)\}_{j=1}^{N^{(0)}}$ evolve according to the diffusion Markovian dynamics with generator $-\mathcal{L}_z^{N^{(0)}}$, starting from $\{b_{j,o}(\omega)\}_{j=1}^{N^{(0)}}$. The first $N^{(0)} - 1$ left-going modes pass through the turning point

$$b_j(\omega, z_-^{(1)}-) = b_j(\omega, z_-^{(1)}+), \qquad j = 1, \ldots, N^{(0)} - 1, \tag{4.2}$$

but the last mode is reflected back.

In the sector $\left(z_-^{(2)}, z_-^{(1)}\right)$ there are $N_-^{(1)} = N^{(0)} - 1$ left going modes, with amplitudes evolving according to the diffusion Markovian dynamics with generator $-\mathcal{L}_z^{N_-^{(1)}}$, starting from the values (4.2) at $z = z_-^{(1)}-$. At the next turning point $z_-^{(2)}$, only the first $N_-^{(1)} - 1$ modes pass through

$$b_j(\omega, z_-^{(2)}-) = b_j(\omega, z_-^{(2)}+), \qquad j = 1, \dots, N_-^{(1)} - 1, \tag{4.3}$$

and the last mode is reflected back.

We continue this way until we reach $z = -Z_M$, with amplitudes $\{b_j(\omega, -Z_M)\}_{j=1}^{N_{\min}}$ obtained from the diffusion Markovian dynamics with generator $-\mathcal{L}_z^{N_{\min}}$ over the interval $(-Z_M, z_-^{(t_M^-)})$, starting with the values $\{b_j(\omega, z_-^{(t_M^-)}-)\}_{j=1}^{N_{\min}}$ determined as explained above, from the previous waveguide sectors.

The waveguide has no variations at $z < -Z_M$, so the left-going mode amplitudes remain equal to their values at $-Z_M$, as stated in equation (3.112). The emerging wave is obtained from (3.30) and (3.57),

$$p^\varepsilon(\omega, \rho, z) \approx -\sum_{j=1}^{N_{\min}} \frac{y_j(\rho, -Z_M) b_j(\omega, -Z_M)}{\sqrt{\beta_j(\omega, -Z_M)}} \exp\Big[ -\frac{i}{\varepsilon}\int_0^{-Z_M} dz' \beta_j(\omega, z') \\ - \frac{i}{\varepsilon}\beta_j(\omega, -Z_M)(z + Z_M)\Big], \quad \text{for } z < -Z_M. \tag{4.4}$$

## 4.2 The mean transmitted wave field

With the infinitesimal generator (3.127) and Kolmogorov's equation we can calculate the mean mode amplitudes

$$\big\langle b_j(\omega, z)\big\rangle := \mathbb{E}\big[b_j(\omega, z)\big]. \tag{4.5}$$

We first recall the Kolmogorov backward equation (A.4). If we consider the diffusion process $(Z(-z), \mathbf{b}(-z))$ where $Z(z) := z$ this will be a homogeneous process with generator $\mathcal{L} := -\partial_Z + \mathcal{L}_z^{N^{(0)}}$ which must have a corresponding solution to (A.4). We can use this equation to obtain equations for the mean mode amplitudes by choosing test functions $f(Z, \boldsymbol{b}) := b_j$ for each $1 \le j \le N^{(0)}$. We note equations satisfied by higher order moments of the mode amplitudes $b_j$ can be obtained similarly as one only needs to make a different choice of test function. The backward equation with our current choice of test function is given by

$$\partial_z \big\langle b_j(\omega, z) \big\rangle = -\mathcal{L} \big\langle b_j(\omega, z) \big\rangle. \tag{4.6}$$

Then by exchanging the order of expectation and the generator we can simplify the right-hand side

$$\begin{aligned} -\mathcal{L}\big\langle b_j(\omega, z)\big\rangle &= -\mathbb{E}[\mathcal{L} b_j(\omega, z)] \\ &= -\mathbb{E}[(-\partial_Z + \mathcal{L}_z^{N^{(0)}}) b_j(\omega, z)] \\ &= \mathbb{E}[-\mathcal{L}_z^{N^{(0)}} b_j(\omega, z)] \\ &= -\Big[ G_{jj}^{(c)}(\omega, z) - G_{jj}^{(0)}(\omega, z) \\ &\qquad + iG_{jj}^{(s)}(\omega, z) + 2i\kappa_j^{N^{(0)}}(\omega, z) \Big] \frac{\mathbb{E}[b_j(\omega, z)]}{2}. \end{aligned} \tag{4.7}$$

In the first sector $(z_-^{(1)}, 0)$, the mean mode amplitudes then satisfy the evolution equations

$$\partial_z \big\langle b_j(\omega, z) \big\rangle = -\Big[ G_{jj}^{(c)}(\omega, z) - G_{jj}^{(0)}(\omega, z) + iG_{jj}^{(s)}(\omega, z) + 2i\kappa_j^{N^{(0)}}(\omega, z) \Big] \frac{\big\langle b_j(\omega, z) \big\rangle}{2}, \tag{4.8}$$

solved backward in $z$, for $z \in \big(z_-^{(1)}, 0\big)$, starting from the values

$$\big\langle b_j(\omega, 0-) \big\rangle = b_{j,0}(\omega), \quad j = 1, \ldots, N^{(0)}. \tag{4.9}$$

The coefficients in (4.8) are defined by (3.130), (3.132), (3.134) and

$$G_{jj}^{(s)}(\omega, z) := -\sum_{\substack{l=1 \\ l \neq j}}^{N^{(0)}} G_{lj}^{(s)}(\omega, z), \tag{4.10}$$

with $G^{(s)}_{lj}(\omega, z)$ given in (3.133). Because $-G^{(c)}_{jj}(\omega, z)+G^{(0)}_{jj}(\omega, z) > 0$ (by the Wiener-Khintchine theorem), we conclude from (4.18) that the mean mode amplitudes decay with $|z|$, and therefore

$$\left|\left\langle b_j(\omega, z_-^{(1)})\right\rangle\right| < \left|b_{j,0}(\omega)\right|, \qquad 1 \le j \le N^{(0)}. \tag{4.11}$$

This decay models the randomization of the left-going modes, and occurs on a $j$ dependent length scale, as illustrated in section 4.7. Similar to the case of waveguides with random perturbations of straight boundaries [4, Section 5], the modes with larger index $j$ randomize faster. Intuitively, this is because these modes propagate slowly along $z$, at group velocity $1/\partial_\omega \beta_j(\omega, z)$ that is small with respect to the wave speed, and bounce more often at the random boundary.

A similar calculation applies to the other sectors $\left(z_-^{(t)}, z_-^{(t-1)}\right)$ of the waveguide, indexed by $t = 1, \dots t_M^-$. The only difference is that the starting values of the mode amplitudes are random, so we use conditional expectations

$$\left\langle b_j(\omega, z)\right\rangle := \mathbb{E}\left[\mathbb{E}\left[b_j(\omega, z)\middle|\mathcal{F}_{z_-^{(t-1)}}\right]\right], \qquad z < z_-^{(t-1)}, \tag{4.12}$$

where $\mathcal{F}_{z_-^{(t-1)}}$ denotes the $\sigma$-algebra (information) generated by the Markov limit process $\{b_q(\omega, z)\}_{q=1}^{N_-^{(t-1)}}$ at $z = z_-^{(t-1)}$. We obtain that $\left\langle b_j(\omega, z)\right\rangle$ satisfies an equation like (4.8), with redefined coefficients for the $N_-^{(t-1)}$ number of propagating modes, and starting value $\left\langle b_j(\omega, z_-^{(t-1)})\right\rangle$ calculated in the previous waveguide sector.

Proceeding this way we reach $z = -Z_M$. The mean transmitted wave is the expectation of (4.4), with $\left\langle b_j(\omega, -Z_M)\right\rangle$ obtained by solving equations (4.8) for all the sectors of the waveguide. The scattering effects at the random boundary add up in each sector, and the mean mode amplitudes decay, as explained above,

$$\left|\left\langle b_j(\omega, -Z_M)\right\rangle\right| < \left|\left\langle b_j(\omega, -z_-^{(t_M^-)}\right\rangle\right| < \ldots < \left|b_{j,0}(\omega)\right|, \qquad 1 \le j \le N_{\min}. \tag{4.13}$$

## 4.3 The transmitted power

Using the infinitesimal generator (3.128) of the Markov process $\{P_j(\omega, z)\}$, the $\varepsilon \to 0$ limit of the left-going mode powers, we now calculate the mean and standard deviation of the transmitted power at $z < 0$.

We proceed as in the previous section, one sector of the waveguide at a time, starting from the source. In the first sector $z \in \left(z_-^{(1)}, 0\right)$, the mean powers

$$\left\langle P_j(\omega, z)\right\rangle := \mathbb{E}\left[P_j(\omega, z)\right], \qquad j = 1, \ldots, N^{(0)}, \tag{4.14}$$

evolve from the initial values $\left\langle P_j(\omega, 0-)\right\rangle = |b_{j,0}(\omega)|^2$ according to equation

$$\partial_z \begin{pmatrix} \left\langle P_1(\omega, z)\right\rangle \\ \vdots \\ \left\langle P_{N^{(0)}}(\omega, z)\right\rangle \end{pmatrix} = -\mathbf{G}^{(c)}(\omega, z) \begin{pmatrix} \left\langle P_1(\omega, z)\right\rangle \\ \vdots \\ \left\langle P_{N^{(0)}}(\omega, z)\right\rangle \end{pmatrix}, \tag{4.15}$$

with matrix $\mathbf{G}^{(c)}(\omega, z)$ defined in (3.129)-(3.130), for $\mathcal{N} = N^{(0)}$.

In the next sectors $(z_-^{(t)}, z_-^{(t-1)})$ we use conditional expectations

$$\left\langle P_j(\omega, z)\right\rangle := \mathbb{E}\left[\mathbb{E}\left[P_j(\omega, z)\middle|\mathcal{F}_{z_-^{(t-1)}}\right]\right], \qquad z < z_-^{(t-1)}, \tag{4.16}$$

and obtain that the mean powers satisfy an equation like (4.15), with $N_-^{(t-1)}$ unknowns and $N_-^{(t-1)} \times N_-^{(t-1)}$ matrix $\mathbf{G}^{(c)}(\omega, z)$. These equations are solved backward in $z$, starting from the values $\left\langle P_j(\omega, z_-^{(t-1)})\right\rangle$ computed in the previous sectors. Proceeding this way, we reach $z = -Z_M$, and obtain $\left\langle P_j(\omega, -Z_M)\right\rangle$, for $j = 1, \ldots, N_{\min}$.

Note that unlike the expectations (4.5), the mean powers are coupled by the matrix $\mathbf{G}^{(c)}(\omega, z)$. This coupling models the exchange of power between the left-going modes, induced by cumulative scattering at the random boundary of the waveguide. The exchange depends on the mode index, as illustrated in section 4.7. Specifically, the higher indexed modes transfer power more quickly than the others.

How much power is exchanged depends on the length of the sectors $\left(z_-^{(t)}, z_-^{(t-1)}\right)$ of the waveguide. In short sectors, the exchange is mostly among the higher indexed modes. The longer the sectors, the more modes participate in the exchange and the power may become evenly distributed among the modes, independent of the starting value at $z_-^{(t-1)}$. This equipartition of energy has been explained in waveguides with straight walls in [26, Section 20.3], for a matrix $\mathbf{G}^{(c)}$ with non-zero off diagonal entries. By the Perron-Frobenius theorem, and due to energy conservation, such a matrix has a simple eigenvalue equal to zero, and the other eigenvalues are negative. It is straightforward to see from equation (4.15) that the solution converges at large $|z|$ to a vector in the nullspace of $\mathbf{G}^{(c)}$. Equation (3.130) gives that this space is spanned by the vector of all ones, so the power becomes evenly distributed at distances that exceed the equipartition distance. This length scale is defined by the inverse of the absolute value of the largest, non-zero eigenvalue of $\mathbf{G}^{(c)}$.

By the energy conservation (3.136), the transmitted power in the first sector of the waveguide is

$$\mathcal{P}_{\text{trans}}(\omega, z) := \sum_{j=1}^{N^{(0)}} P_j(\omega, z) = \sum_{j=1}^{N^{(0)}} |b_{j,0}(\omega)|^2, \qquad z \in (z_-^{(1)}, 0), \tag{4.17}$$

where the right hand side is the deterministic, total left going power emitted by the source. At the turning point $z_-^{(1)}$ the $N^{(0)}$-th mode is reflected back. The transmitted power to the next sector of the waveguide, carried by the remaining $N_-^{(1)} = N^{(0)} - 1$ modes, is random and given by

$$\mathcal{P}_{\text{trans}}(\omega, z) = \sum_{j=1}^{N_-^{(1)}} P_j(\omega, z) = \sum_{j=1}^{N_-^{(1)}} P_j(\omega, z_-^{(1)}), \qquad z \in (z_-^{(2)}, z_-^{(1)}). \tag{4.18}$$

This repeats for the other sectors, and beyond $z = -Z_M$ we have

$$\mathcal{P}_{\text{trans}}(\omega, z) = \sum_{j=1}^{N_{\min}} P_j(\omega, -Z_M), \qquad z \leq -Z_M. \tag{4.19}$$

In summary, the transmitted power is a piecewise constant function with jumps at the turning points, and random values determined by the sum of the mode powers entering each sector of the waveguide. Its mean is obtained by taking expectations in (4.17)-(4.19), and using the mean mode powers calculated as explained above.

The random fluctuations of $\mathcal{P}_{\rm trans}(\omega, z)$ about the mean are quantified by its standard deviation

$$\mathrm{StD}\left[\mathcal{P}_{\rm trans}(\omega,z)\right] = \left\{ \sum_{j,l=1}^{N_-^{(t-1)}} \left[ \left\langle \mathcal{P}_{jl}(\omega,z)\right\rangle - \left\langle P_j(\omega,z)\right\rangle \left\langle P_l(\omega,z)\right\rangle \right] \right\}^{1/2} \qquad (4.20)$$

for $z \in (z_-^{(t)}, z_-^{(t-1)})$ and $1 \le t \le t_M^-$. To calculate it we need the second moments

$$\left\langle \mathcal{P}_{jl}(\omega,z)\right\rangle := \mathbb{E}\left[P_j(\omega,z)P_l(\omega,z)\right]. \qquad (4.21)$$

Again, these are obtained in one sector of the waveguide at a time, starting from the source, where

$$\left\langle \mathcal{P}_{jl}(\omega,0)\right\rangle = |b_{j,0}(\omega)|^2 |b_{l,0}(\omega)|^2, \qquad j,l = 1,\dots,N^{(0)}. \qquad (4.22)$$

The evolution equations of the moments (4.21) at $z \in \left(z_-^{(t)}, z_-^{(t-1)}\right)$ are

$$\partial_z \left\langle \mathcal{P}_{jj}(\omega,z)\right\rangle = 2G_{jj}^{(c)}(\omega,z)\left\langle \mathcal{P}_{jj}(\omega,z)\right\rangle - 4\sum_{l=1}^{N_-^{(t-1)}} G_{jl}^{(c)}(\omega,z)\left\langle \mathcal{P}_{lj}(\omega,z)\right\rangle, \qquad (4.23)$$

and

$$\partial_z \left\langle \mathcal{P}_{jq}(\omega,z)\right\rangle = 2G_{jq}^{(c)}(\omega,z)\left\langle \mathcal{P}_{jq}(\omega,z)\right\rangle - \sum_{l=1}^{N_-^{(t-1)}} \Big[ G_{jl}^{(c)}(\omega,z)\left\langle \mathcal{P}_{lq}(\omega,z)\right\rangle + G_{lq}^{(c)}(\omega,z)\left\langle \mathcal{P}_{jl}(\omega,z)\right\rangle \Big], \qquad (4.24)$$

for $j,q = 1,\dots,N_-^{(t-1)}$ and $j \neq q$. These equations are solved backward in $z$, with the starting values $\left\langle \mathcal{P}_{jq}(\omega, z_-^{(t-1)})\right\rangle$ calculated from the previous sector.

## 4.4 The right-going waves

Even though we consider the forward scattering approximation in each sector of the waveguide, there are both left- and right-going modes at $z < 0$, due to reflection at the turning points. At $z > 0$ we also have the right-going waves emitted from the source. The analysis of the reflected mode amplitudes is more complicated, because they quantify cumulative scattering in the waveguide sectors traversed both ways: to the left by the incoming wave and to the right by the reflected wave.

In each sector $\left(z_-^{(t)}, z_-^{(t-1)}\right)$ we obtain from (3.123) that the right-going mode amplitudes satisfy

$$\boldsymbol{a}^\varepsilon(\omega, z) \approx \overline{\mathbf{P}^{\varepsilon(bb)}}(\omega, z; z_-^{(t-1)})\boldsymbol{a}^\varepsilon(\omega, z_-^{(t-1)}), \qquad t = 1, \ldots, t_M^-. \tag{4.25}$$

This looks similar to equation (3.124) that describes the evolution of the left-going waves, but we have different boundary conditions, as we now explain.

Starting from the leftmost turning point $z_-^{(t_M^-)}$, and denoting $\mathcal{N} = N_-^{(t_M^- - 1)}$, we obtain from (3.111) the initial condition

$$a_j^\varepsilon\left(\omega, z_-^{(t_M^-)}\right) = \mathrm{R}_\mathcal{N}^\varepsilon\left(\omega, z_-^{(t_M^-)}\right) b_\mathcal{N}^\varepsilon\left(\omega, z_-^{(t_M^-)}\right)\delta_{j\mathcal{N}}, \qquad j = 1, \ldots, \mathcal{N}, \tag{4.26}$$

for the vector $\boldsymbol{a}^\varepsilon(\omega, z) \in \mathbb{C}^\mathcal{N}$, where $\delta_{j\mathcal{N}}$ is the Kronecker delta symbol and $\mathrm{R}_\mathcal{N}^\varepsilon$ is reflection coefficient defined in (3.108). The amplitudes of the right-going modes impinging on the next turning point are obtained from (4.25)

$$\boldsymbol{a}^\varepsilon(\omega, z_-^{(t_M^- - 1)} -) \approx \left[\mathbf{P}^{\varepsilon(bb)}(\omega, z_-^{(t_M^-)}; z_-^{(t_M^- - 1)})\right]^T \boldsymbol{a}^\varepsilon(\omega, z_-^{(t_M^-)}), \tag{4.27}$$

using that the propagator $\mathbf{P}^{\varepsilon(bb)}$ is approximately unitary. This follows from the energy conservation relation (3.136), which holds in the limit $\varepsilon \to 0$, independent of the initial conditions.

On the right of the turning point $z_-^{(t_M^- -1)}$ there is an extra right-going mode. Renaming $\mathcal{N} = N_-^{(t_M^- -2)}$, we obtain the following initial condition for the vector $\boldsymbol{a}^\varepsilon(\omega, z)$: Its first $\mathcal{N}-1$ components are given in (4.27), and the last component is

$$a_{\mathcal{N}}^\varepsilon\big(\omega, z_-^{(t_M^- -1)}\big) = \mathrm{R}_{\mathcal{N}}^\varepsilon\big(\omega, z_-^{(t_M^- -1)}\big) b_{\mathcal{N}}^\varepsilon\big(\omega, z_-^{(t_M^- -1)}\big). \tag{4.28}$$

These amplitudes and the $\mathcal{N} \times \mathcal{N}$ propagator $\mathbf{P}^{\varepsilon(bb)}(\omega, z; z_-^{(t_M^- -2)})$ determine the amplitudes of the right-going modes impinging on the turning point $z_-^{(t_M^- -2)}$ and so on.

Proceeding this way we obtain the amplitudes $\{a_j^\varepsilon(\omega, 0-)\}_{j=1}^{N^{(0)}}$ on the left of the source. The amplitudes at $z = 0+$ are given by these and the source conditions (3.109). The analysis of forward propagation at $z > 0$ is similar to that in section 4.1, with the exception that at the turning points $z_+^{(t)}$, for $1 \leq t \leq t_M^+$, there is no reflection. We add instead a new mode with zero initial condition, as stated in (3.114).

### 4.5 The net reflected power

The calculation of the statistical moments of the right-going mode amplitudes in the limit $\varepsilon \to 0$ requires the infinitesimal generator of the limit propagator $\mathbf{P}^{\varepsilon(bb)}$, in each sector of the waveguide. This operator can be obtained using Theorem V.1, but the calculation is complex. Here we quantify only the net reflected power at each turning point, without asking how this power gets distributed among the modes as they propagate toward the right. This is an easier task as the net reflected power can be completely characterized in terms of the initial power to the left of the source and the power transmitted through the left part of the waveguide, whose statistics we have previously computed.

The net reflected power is determined by the transmitted power in the left part of the waveguide, using energy conservation. Specifically, starting from the leftmost turning point, the net reflected power is

$$\mathcal{P}_{\text{refl}}(\omega, z) = P_{N_-^{(t_M^- - 1)}}(\omega, z_-^{(t_M^-)}+), \qquad z \in \left(z_-^{(t_M^-)}, z_-^{(t_M^- - 1)}\right), \tag{4.29}$$

where the right hand side is the power of the left-going turning mode, analyzed in section 4.1. Here we used the conservation relation

$$\lim_{\varepsilon \to 0} \sum_{j=1}^{N_-^{(t_M^- - 1)}} |a_j^\varepsilon(\omega, z)|^2 = \text{constant}, \quad \text{for } z \in \left(z_-^{(t_M^-)}, z_-^{(t_M^- - 1)}\right),$$

derived the same way as (3.136), equation (4.26) and $\lim_{\varepsilon \to 0} |\mathrm{R}_{\mathcal{N}}^\varepsilon| = 1$.

At the next turning point $z_-^{(t_M^- - 1)}$ we add a new mode amplitude, and the net reflected power increases to

$$\mathcal{P}_{\text{refl}}(\omega, z) = P_{N_-^{(t_M^- - 1)}}(\omega, z_-^{(t_M^-)}+) + P_{N_-^{(t_M^- - 2)}}(\omega, z_-^{(t_M^- - 1)}+), \tag{4.30}$$

for $z \in \left(z_-^{(t_M^- - 1)}, z_-^{(t_M^- - 2)}\right)$, and so on. Proceeding this way we obtain that the net reflected power is a piecewise constant function at $z < 0$, with jumps at the turning points $z_-^{(t)}$ indexed by $1 \le t \le t_M^-$. At the source location this equals

$$\mathcal{P}_{\text{refl}}(\omega, 0) = \sum_{t=1}^{t_M^-} P_{N_-^{(t-1)}}(\omega, z_-^{(t)}+), \tag{4.31}$$

and its mean and standard deviation are determined by those of the turning wave powers, calculated in section 4.1. By comparing with (4.17-4.19) we obtain the global conservation of energy relation

$$\mathcal{P}_{\text{refl}}(\omega, 0) + \mathcal{P}_{\text{trans}}(\omega, -Z_M) = \sum_{j=1}^{N^{(0)}} |b_{j,0}(\omega)|^2. \tag{4.32}$$

Therefore the first two moments of the net transmitted and reflected powers are related through:

$$\begin{aligned}\langle \mathcal{P}_{\text{refl}}(\omega,0)\rangle &= \sum_{j=1}^{N^{(0)}} |b_{j,0}(\omega)|^2 - \langle \mathcal{P}_{\text{trans}}(\omega,-Z_M)\rangle\,, &(4.33)\\ \text{StD}\left[\mathcal{P}_{\text{refl}}(\omega,0)\right] &= \text{StD}\left[\mathcal{P}_{\text{trans}}(\omega,-Z_M)\right]. &(4.34)\end{aligned}$$

## 4.6 The net power transmitted to the right

There is no mode reflection at $z > 0$, and the net transmitted power to the right is

$$\mathcal{P}_{\text{trans,right}}(\omega,z) = \lim_{\varepsilon\to 0}\sum_{j=1}^{N^{(0)}} |a_j^\varepsilon(\omega,0+)|^2, \qquad z>0, \tag{4.35}$$

where the equality means having the same statistical distribution, and

$$a_j^\varepsilon(\omega,0+) = a_j^\varepsilon(\omega,0-) + a_{j,o}(\omega), \qquad a_{j,o}(\omega) = \frac{\widehat{f}(\omega)y_j(\rho_\star,0)}{2i\sqrt{\beta_j(\omega,0)}}. \tag{4.36}$$

The calculation of the statistical moments of (4.35) is as complicated as the calculation of the moments of the limit right-going mode amplitudes. Specifically, it requires the infinitesimal generator of the $\varepsilon\to 0$ limit of the propagator $\mathbf{P}^{\varepsilon(bb)}$, in particular, we need to characterize the phases of the reflection coefficients $\mathrm{R}^\varepsilon_{N_-^{(t-1)}}\big(\omega, z_-^{(t)}\big)$, $t = 1,\dots,t_M^-$. By extrapolating the results given in [11] (in which the standard deviation of the fluctuations of the boundary was smaller), we expect that these phases are independent and uniformly distributed over $[0,2\pi]$. We could then expect that the mean power transmitted to the right is

$$\begin{aligned}\langle \mathcal{P}_{\text{trans,right}}(\omega,z)\rangle &= \langle \mathcal{P}_{\text{refl}}(\omega,0)\rangle + \sum_{j=1}^{N^{(0)}} |a_{j,o}(\omega)|^2\\ &= \sum_{j=1}^{N^{(0)}} |a_{j,o}(\omega)|^2 + \sum_{j=1}^{N^{(0)}} |b_{j,0}(\omega)|^2 - \langle \mathcal{P}_{\text{trans}}(\omega,-Z_M)\rangle\,, &(4.37)\end{aligned}$$

for any $z > 0$.

## 4.7 Numerical illustration

In this section we illustrate with some plots the exchange of power among the propagating modes in the left part $z < 0$ of the waveguide, due to a point source at $\boldsymbol{x}_\star = (D(0)/7, 0)$. For comparison, we also consider other initial conditions, where the excitation at $z = 0$ is for a single mode at a time.

We take a waveguide with a straight axis that has a single turning point, at arc length $z_-^{(1)} = -L = -1000\lambda$, where $\lambda$ is the wavelength. The waveguide opening $D(z/L)$ increases linearly in $z$ in the interval $[-L, 0]$, from the value $20\lambda$ to $20.49\lambda$, and transitions as a cubic polynomial to the constant $19.999\lambda$ at $z < -L - 0.2\lambda$ and $20.491\lambda$ at $z > 0.2\lambda$. Thus, there are $N^{(0)} = 40$ propagating modes at $z > -L$ and $N_-^{(1)} = 39$ modes at $z < -L$. The top and bottom boundaries of the waveguide are straight and parallel at $z \in (-\infty, -L - 0.2\lambda) \cup (0.2\lambda, \infty)$.

The auto-correlation function $\mathcal{R}$ of the process $\nu(\zeta)$ is a Gaussian with standard deviation 1. The correlation length of the fluctuations is $\ell = 3\lambda$, so $\varepsilon = \ell/L = 0.003$, and the standard deviation $\sigma$ of the fluctuations equals $\sqrt{\varepsilon}$.

We can describe approximately what to expect in terms of the randomization of the mode amplitudes and the exchange of power among the modes by looking at the following length scales calculated in a waveguide with constant opening equal to $D(0)$:

1. The mode-dependent scattering mean free path

$$L_{j,\text{smf}} := \frac{2}{G_{jj}^{(0)}(\omega, 0) - G_{jj}^{(c)}(\omega, 0)}, \qquad j = 1, \ldots, 40, \tag{4.38}$$

which is the scale of decay of the mean mode amplitudes, as seen from (4.8).

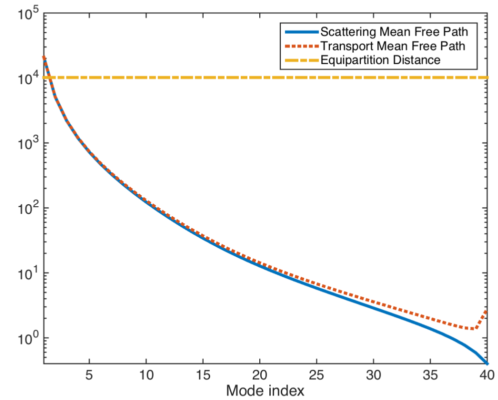


Figure 4.1: Plot of the characteristic length scales

These three length scales quantify net scattering in a waveguide with constant opening $D(0)$. The solid blue line is for the scattering mean free path (4.38). The dashed red line is for the transport mean free path (4.39). The yellow dashed line is for the equipartition distance. The abscissa is the mode index $j = 1, \ldots, 40$ and the ordinate is in units of $\lambda$.

2. The mode-dependent transport mean free paths,

$$L_{j,\mathrm{tmf}} := -\frac{2}{G_{jj}^{(c)}(\omega, 0)}, \qquad j = 1, \ldots, 40, \tag{4.39}$$

defined in terms of the diffusion coefficient $-G_{jj}^{(c)}$ of the mode power infinitesimal generator (3.128). The modes exchange power with their neighbors as they propagate at distances of order (4.39).

3. The equipartition distance $L_{\mathrm{eq}}$, which is defined as the inverse of the absolute value of the largest, non-zero eigenvalue of matrix $\mathbf{G}^{(c)}(\omega, 0)$. At distances of order $L_{\mathrm{eq}}$, we expect that the power gets evenly distributed among the modes, independent of the excitation at $z = 0$.

We display these scales in Figure 4.1 and observe that at the distance $L = 1000\lambda$ between the source and the turning point, we have

$$L \geq L_{j,\mathrm{smf}}, L_{j,\mathrm{tmf}}, \qquad j = 5, \ldots, 40.$$

Thus, these modes should be randomized and moreover, they should share their power with the other modes. Because $L < L_{\mathrm{eq}}$, we expect that at least the first five

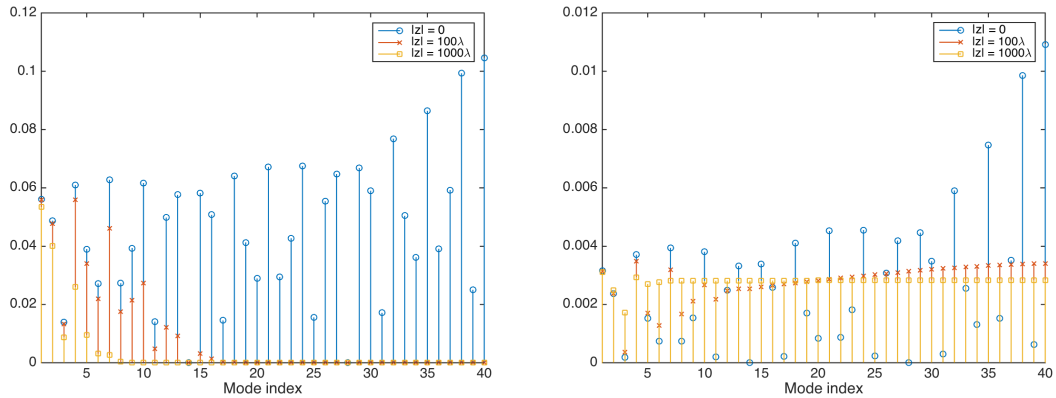


Figure 4.2: Display of $|\langle b_j(\omega, z)\rangle|$ and $\langle P_j(\omega, z)\rangle$ v.s. the mode index $j$

The absolute value of the mean mode amplitudes $|\langle b_j(\omega, z)\rangle|$ (left) and the mean mode powers $\langle P_j(\omega, z)\rangle$ v.s. the mode index $j$ at three different distances from the source: The blue circles correspond to the initial values at $z = 0$, due to a point source at location $(D(0)/7, 0)$. The red crosses are for $|z| = 100\lambda$ and the yellow squares are for $|z| = L = 1000\lambda$. The abscissa is the mode index $j = 1, \dots, 40$.

modes have not shared all their power with the other modes.

These expectations are confirmed by the results displayed in Figure 4.2, where we show the absolute values $|\langle b_j(\omega, z)\rangle|$ of the mean mode amplitudes (left plot) and the mean mode powers $\langle P_j(\omega, z)\rangle$ (right plot) at three distances from the point source. The dashed blue line is for $z = 0$, so it corresponds to the initial values (4.1) of the mode amplitudes, which oscillate in $j$ due to the factor

$$y_j(\rho_\star, 0) = \sqrt{\frac{2}{D(0)}} \sin\left[\left(\frac{\rho_\star}{D(0)} + \frac{1}{2}\right)\pi j\right], \qquad j = 1, \dots, N^{(0)}, \ \rho_\star = \frac{D(0)}{7}.$$

As we increase the distance $|z|$ from the source, the left plot in Figure 4.2 illustrates the decay of the mean mode amplitudes. We note that at $|z| = 100\lambda$, the modes indexed by $j > 15$ have negligible mean, and at the turning point $|z| = L = 1000\lambda$, the modes indexed by $j > 5$ have negligible mean. This is as expected from Figure 4.1, because because $L_{j,\text{smf}} < 100\lambda$ for $j > 15$ and $L_{j,\text{smf}} < 1000\lambda$ for $j > 5$. The right plot in Figure 4.2 illustrates the effect of exchange of power among the modes. The scattering mean free path and the transport mean free path are almost the same in this simulation, as shown in Figure 4.1, and we note that at the turning point

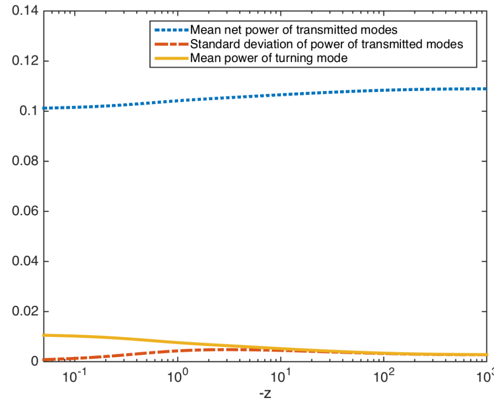


Figure 4.3: Display of mode power statistics for a point source

The mean net power of the transmitted modes is in dashed blue line, the standard deviation of this power in dashed red line, and the mean power of the turning mode, indexed by $j = 40$. The abscissa is the arc length in units of $\lambda$ (in logarithmic scale).

$|z| = L = 1000\lambda$ the modes indexed by $j > 5$ have almost the same power.

In Figure 4.3 we display the mean and standard deviation of the net power $\sum_{j=1}^{39} P_j(\omega, z)$ of the modes that are transmitted through the turning point, and the mean power of the turning mode, as functions of $z$. At $|z| = L = 1000\lambda$, these determine the transmitter power (4.18) beyond the turning point, and the reflected power (4.31). Note that in this case cumulative scattering at the random boundary is beneficial for power transmission through the waveguide. In the absence of the random fluctuations there would be no power exchange between the modes, and the transmitted power would equal $\sum_{j=1}^{39} P_j(\omega, 0)$. As seen in Figure 4.2, the turning mode has the largest mode amplitude initially, and all its power would be reflected back. The cumulative scattering at the random boundary leads to rapid exchange of the power of the turning mode, as shown in the right plot of Figure 4.2, and much less power is reflected. The standard deviation of the net power of the first 39 modes, shown with the red dashed line in Figure 4.3, is smaller than its mean. Thus, $\sum_{j=1}^{39} P_j(\omega, z) \approx \sum_{j=1}^{39} \left\langle P_j(\omega, z)\right\rangle$, with less than 10% relative error (i.e., random

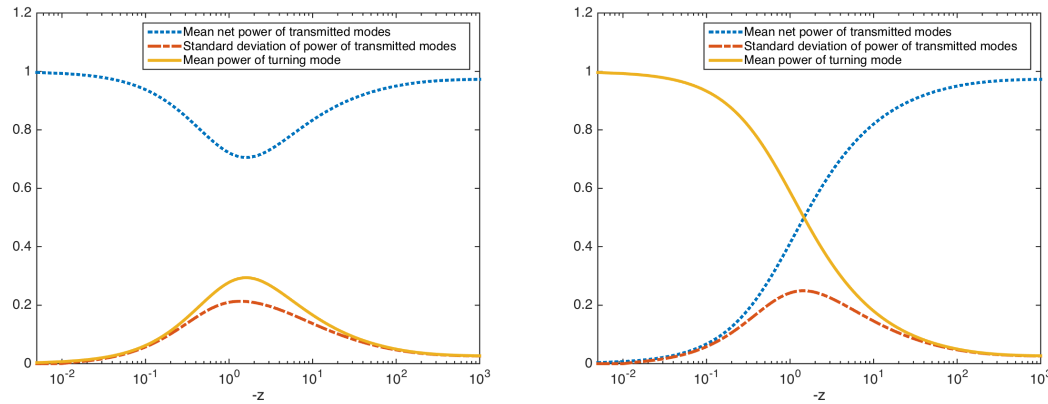


Figure 4.4: Display of mode power statistics for single mode excitations

The mean net power of the transmitted modes is in dashed blue line, of the standard deviation of this power in dashed red line, and the mean power of the turning mode, indexed by $j = 40$. The abscissa is the arc length in units of $\lambda$ (in logarithmic scale). Only one mode was excited initially, the 39-th one in the left plot and the 40-th one in the right plot.

fluctuations).

The last illustration, in Figure 4.4, shows the mean and standard deviation of $\sum_{j=1}^{39} P_j(\omega, z)$, and the mean power $\left\langle P_{40}(\omega, z)\right\rangle$ of the turning mode, as functions of $z$, for initial excitations of a single mode. In the left plot the 39-th mode is excited, and in the right plot the 40-th mode is excited. In the absence of the random fluctuations, these initial conditions would determine the transmitted power at the turning point. Specifically, in the first case the power would stay in the 39-th mode and would propagate through, whereas in the second case the power of the 40-th mode would be totally reflected. The cumulative scattering in the random waveguide distributes the power among the modes, and we note in the left plot of Figure 4.4 that slightly less power is transmitted, due to the power transfer to the turning mode, whereas in the right plot, most of the power is transmitted, due to the transfer of power from the turning mode to the other modes.

## 4.8 Universal transmission properties for strong scattering

In case of strong scattering, the mean transmitted power through the left part of the waveguide becomes universal and equal to $\mathcal{P}_0 N_{\min}/N^{(0)}$, where $\mathcal{P}_0 := \sum_{j=1}^{N^{(0)}} |b_{j,0}(\omega)|^2$ is the power transmitted to the left by the source. More exactly, if scattering is so strong that equipartition is reached in each section between two turning points, in the sense that $z_-^{(t-1)} - z_-^{(t)} > L_{\rm eq}^t$ for all $t = 1, \ldots, t_M^-$ (where $L_{\rm eq}^t$ is the equipartition distance in the section $(z_-^{(t)}, z_-^{(t-1)})$), then the fraction of mean power transmitted through the $t$-th turning point $z_-^{(t)}$ is $1 - 1/N_-^{(t-1)}$, because the $N_-^{(t-1)}$-th mode carrying a fraction $1/N_-^{(t-1)}$ of the mean power is reflected. By denoting $\left\langle \mathcal{P}_{\rm trans}^{(t-1)} \right\rangle$ the net transmitted power in the $t$-th section $(z_-^{(t)}, z_-^{(t-1)})$, we get the recursive relation

$$\left\langle \mathcal{P}_{\rm trans}^{(t)} \right\rangle = \left\langle \mathcal{P}_{\rm trans}^{(t-1)} \right\rangle (N_-^{(t-1)} - 1)/N_-^{(t-1)}, \quad t = 1, \ldots, t_M^-, \tag{4.40}$$

which gives that the mean transmitted power at $-Z_M$ is $\mathcal{P}_0 N_{\min}/N^{(0)}$.

# CHAPTER V

# Diffusion Approximation Theorem

In this chapter we state and prove the diffusion approximation theorem used to obtain the asymptotic limit of the mode amplitudes in sections 3.7 - 3.9. Similar results were proven in [43, 7] and summarized in [26, chapter 6]. The proof relies upon the perturbed test function method of [44].

## 5.1 Statement of the Theorem

We state the theorem for a general system of random differential equations

$$\frac{d\boldsymbol{X}^\varepsilon(z)}{dz} = \frac{1}{\sqrt{\varepsilon}}\boldsymbol{F}\big(\boldsymbol{X}^\varepsilon(z), q^\varepsilon(z), \boldsymbol{\theta}^\varepsilon(z), z\big), \quad z > 0, \quad \boldsymbol{X}^\varepsilon(0) = \boldsymbol{x}_0, \tag{5.1}$$

with unknown vector $\boldsymbol{X}^\varepsilon \in \mathbb{R}^d$, and right hand side defined by a function of the form

$$\boldsymbol{F}\big(\boldsymbol{X}, q, \boldsymbol{\theta}, z\big) := \sum_{j=1}^{p} \boldsymbol{F}^{(j)}\big(\boldsymbol{X}, q, \theta_j, z\big), \quad \text{for } \boldsymbol{\theta} := \big(\theta_j\big)_{j=1}^p \in \mathbb{R}^p. \tag{5.2}$$

The second argument of $\boldsymbol{F}$ is defined by $q^\varepsilon(z) := q(z/\varepsilon)$, where $q(z)$ is a stationary and ergodic Markov process taking values in a space $E$, with generator $Q$ and stationary distribution $\pi_q$. We assume that $Q$ satisfies the Fredholm alternative, which holds true for many different classes of Markov processes [26, section 6.3.3]. Note that the Markovian assumption on the driving process $q$ is convenient for the proof, but the statement of the diffusion approximation theorem V.1 generalizes to a process $q$ that is not Markovian, but $\phi$-mixing with $\phi \in L^{1/2}$ [35, Sec. 4.6.2].

The third argument of $\boldsymbol{F}$ is the vector valued function $\boldsymbol{\theta}^\varepsilon(z)$ taking values in $\mathbb{R}^p$, with components satisfying the equation

$$\frac{d\theta_j^\varepsilon}{dz} = \frac{1}{\varepsilon}\beta_j(z), \qquad j = 1, \ldots, p,$$

where $\beta_j(z)$ is a $\mathbb{R}$-valued smooth function, bounded as $C \leq \beta_j(z) \leq 1/C$ for some constant $C > 0$.

We assume that the components $\boldsymbol{F}^{(j)}$ in (5.2) satisfy the following conditions, for all $j = 1, \ldots, p$:

1. The mappings $(\boldsymbol{x}, z) \in \mathbb{R}^d \times \mathbb{R} \mapsto \boldsymbol{F}^{(j)}(\boldsymbol{x}, q, \theta_j, z) \in \mathbb{R}^d$ are smooth for all $q \in E$ and $\theta_j \in \mathbb{R}$.
2. The mappings $q \in E \mapsto \boldsymbol{F}^{(j)}(\boldsymbol{x}, q, \theta_j, z)$ are centered with respect to the stationary distribution $\pi_q$,

$$\mathbb{E}[\boldsymbol{F}^{(j)}(\boldsymbol{x}, q(0), \theta, z)] = \int_E \boldsymbol{F}^{(j)}(\boldsymbol{x}, q, \theta_j, z)\pi_q(dq) = 0,$$

   for any $\boldsymbol{x} \in \mathbb{R}^d$, $\theta_j \in \mathbb{R}$ and $z \in \mathbb{R}$.
3. The mappings $\theta_j \in \mathbb{R} \mapsto \boldsymbol{F}^{(j)}(\boldsymbol{x}, q, \theta_j, z)$ are periodic with period 1 for all $\boldsymbol{x} \in \mathbb{R}^d$ and $q \in E$.

**Theorem V.1.** *Let $\boldsymbol{X}^\varepsilon(z)$ be the solution of (5.1), with right-hand side $\boldsymbol{F}$ defined in terms of the functions $\boldsymbol{F}^{(j)}$ as in (5.2), and $\boldsymbol{F}^{(j)}$ satisfying the three properties above. In the limit $\varepsilon \to 0$, the continuous processes $(\boldsymbol{X}^\varepsilon(z))_{z\geq 0}$ converge in distribution to* *the Markov diffusion process* $(\boldsymbol{X}(z))_{z\geq 0}$ *with the inhomogeneous generator*

$$\mathcal{L}_z f(\boldsymbol{x}) = \sum_{m=1}^{d} \mathrm{h}_m(\boldsymbol{x}, z)\partial_{x_m} f(\boldsymbol{x}) + \sum_{m,n=1}^{d} \mathrm{a}_{m,n}(\boldsymbol{x}, z)\partial^2_{x_m x_n} f(\boldsymbol{x}), \tag{5.3}$$

$$\mathrm{h}_m(\boldsymbol{x}, z) := \sum_{n=1}^{d} \Big\langle \int_0^\infty \mathbb{E}\left[F_n(\boldsymbol{x}, q(0), \cdot, z)\partial_{x_n} F_m(\boldsymbol{x}, q(\zeta), \cdot + \boldsymbol{\beta}(z)\zeta, z)\right] d\zeta \Big\rangle_{\boldsymbol{\beta}(z)}, \tag{5.4}$$

$$\mathrm{a}_{m,n}(\boldsymbol{x}, z) := \Big\langle \int_0^\infty \mathbb{E}\left[F_n(\boldsymbol{x}, q(0), \cdot, z)F_m(\boldsymbol{x}, q(\zeta), \cdot + \boldsymbol{\beta}(z)\zeta, z)\right] d\zeta \Big\rangle_{\boldsymbol{\beta}(z)}, \tag{5.5}$$

*where* $\langle . \rangle_{\boldsymbol{\beta}}$ *is the mean value for almost periodic functions,*

$$\big\langle \boldsymbol{H}(\cdot) \big\rangle_{\boldsymbol{\beta}} := \lim_{S\to\infty} \frac{1}{S} \int_0^S \boldsymbol{H}(\boldsymbol{\theta} + \boldsymbol{\beta}s) ds.$$

Note that the mean values for the terms involved in (5.4-5.5) exist and are independent of $\boldsymbol{\theta}$, since the functions

$$G_{n,m}(s) := \int_0^\infty \mathbb{E}\left[F_n(\boldsymbol{x}, q(0), \boldsymbol{\theta} + \boldsymbol{\beta}s, z)F_m(\boldsymbol{x}, q(\zeta), \boldsymbol{\theta} + \boldsymbol{\beta}s + \boldsymbol{\beta}\zeta, z)\right] d\zeta,$$
$$\tilde{G}_{n,m}(s) := \int_0^\infty \mathbb{E}\left[F_n(\boldsymbol{x}, q(0), \boldsymbol{\theta} + \boldsymbol{\beta}s, z)\partial_{x_n} F_m(\boldsymbol{x}, q(\zeta), \boldsymbol{\theta} + \boldsymbol{\beta}s + \boldsymbol{\beta}\zeta, z)\right] d\zeta,$$

are periodic or almost periodic in $s$, for any fixed $\boldsymbol{x}$ and $q$.

## 5.2 The Proof

*Proof.* Let us define the projection on the torus $\mathbb{S} \simeq \mathbb{R}/\mathbb{Z}$:

$$\theta \in \mathbb{R} \mapsto \dot{\theta} := \theta \bmod 1 \in \mathbb{S},$$

and observe that if a function $f : \mathbb{R} \to \mathbb{R}$ is periodic with period 1, then $f(\theta) = f(\dot{\theta})$. We also define $\dot{\boldsymbol{\theta}}^\varepsilon(z) := \boldsymbol{\theta}^\varepsilon(z) \bmod 1$, and $Z(z) := z$. The joint process $(\boldsymbol{X}^\varepsilon(z), q^\varepsilon(z), \dot{\boldsymbol{\theta}}^\varepsilon(z), Z(z))_{z\geq 0}$ is a Markov process with values in $\mathbb{R}^d \times E \times \mathbb{S}^p \times \mathbb{R}$ and infinitesimal generator

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon}\big(Q + \boldsymbol{\beta}(Z) \cdot \nabla_{\dot{\boldsymbol{\theta}}}\big) + \frac{1}{\sqrt{\varepsilon}} \boldsymbol{F}(\boldsymbol{X}, q, \dot{\boldsymbol{\theta}}, Z) \cdot \nabla_{\boldsymbol{X}} + \partial_Z \,. \tag{5.6}$$

One can show by the perturbed test function method [26, Section 6.3.2] (see also the appendix at the end of this chapter) and Lemma V.3 that the continuous processes $(\boldsymbol{X}^\varepsilon(z), Z(z))_{z\geq 0}$ converge in distribution to the Markov diffusion process $(\boldsymbol{X}(z), Z(z))_{z\geq 0}$ with the homogeneous generator:

$$\begin{aligned}\mathcal{L}f(\boldsymbol{x}, Z) &= \partial_Z f(\boldsymbol{x}, Z) \\ &+ \Big\langle \int_0^\infty \mathbb{E}\left[\boldsymbol{F}(\boldsymbol{x}, q(0), \cdot, Z) \cdot \nabla_{\boldsymbol{x}} \left(\boldsymbol{F}(\boldsymbol{x}, q(\zeta), \cdot + \boldsymbol{\beta}(Z)\zeta, Z) \cdot \nabla_{\boldsymbol{x}} f(\boldsymbol{x}, Z)\right)\right] d\zeta \Big\rangle_{\boldsymbol{\beta}(z)}.\end{aligned} \tag{5.7}$$

Since $(Z(z))_{z\geq 0}$ is deterministic, we conclude that $(\boldsymbol{X}(z))_{z\geq 0}$ is a Markov process and its inhomogeneous generator is

$$\mathcal{L}_z f(\boldsymbol{x}) = \Big\langle \int_0^\infty \mathbb{E}\left[\boldsymbol{F}(\boldsymbol{x}, q(0), \cdot, z) \cdot \nabla_{\boldsymbol{x}} \left(\boldsymbol{F}(\boldsymbol{x}, q(\zeta), \cdot + \boldsymbol{\beta}(z)\zeta, z) \cdot \nabla_{\boldsymbol{x}} f(\boldsymbol{x})\right)\right] d\zeta \Big\rangle_{\boldsymbol{\beta}(z)} \tag{5.8}$$

or equivalently (5.3).

□

**Lemma V.2.** *We have the following two statements:*

1. *Let $\beta \in \mathbb{R}\backslash\{0\}$. Let $g(q, \theta)$ be a bounded function, periodic in $\theta \in \mathbb{R}$ with period* 1, *such that*

$$\mathbb{E}[g(q(0), \theta)] = 0 \text{ for all } \theta \in \mathbb{R}\,.$$

*The Poisson equation*

$$\big(Q + \beta\partial_{\dot\theta}\big) f = g$$

*has a unique solution $f$, periodic in $\theta$, up to an additive constant. The solution with mean zero is*

$$f(q, \dot\theta) = -\int_0^\infty \mathbb{E}[g(q(\zeta), \dot\theta + \beta\zeta) | q(0) = q] d\zeta\,. \tag{5.9}$$

*2. Let* $\boldsymbol{\beta} \in \mathbb{R}^2$ *with non-zero entries. Let* $g(q, \boldsymbol{\theta})$ *be a bounded function, periodic in* $\boldsymbol{\theta} \in \mathbb{R}^2$ *with period* 1*, such that*

$$\mathbb{E}[g(q(0), \boldsymbol{\theta})] = 0 \textit{ for all } \boldsymbol{\theta} \in \mathbb{R}^2 .$$

*The Poisson equation*

$$\big(Q + \boldsymbol{\beta} \cdot \nabla_{\dot{\boldsymbol{\theta}}}\big) f = g$$

*has a unique solution* $f$*, periodic in* $\boldsymbol{\theta}$*, up to an additive constant. The solution with mean zero is*

$$f(q, \dot{\boldsymbol{\theta}}) = -\int_0^\infty \mathbb{E}[g(q(\zeta), \dot{\boldsymbol{\theta}} + \boldsymbol{\beta}\zeta)|q(0) = q] d\zeta . \tag{5.10}$$

Note that in the second item of Lemma V.2 it is important to assume that $\mathbb{E}[g(q(0), \boldsymbol{\theta})] = 0$ for all $\boldsymbol{\theta} \in \mathbb{R}^2$, and not only that $\int_{\mathbb{S}^2} \mathbb{E}[g(q(0), \dot{\boldsymbol{\theta}})] d\dot{\boldsymbol{\theta}} = 0$. The latter weaker hypothesis ensures the desired result only when $\beta_1/\beta_2$ is irrational.

*Proof.* To prove statement 1. let $\beta \in \mathbb{R}$ be fixed. We denote by $\theta_\beta(\zeta)$ the solution to $\frac{d\theta_\beta}{d\zeta} = \beta$ and by $\dot{\theta}_\beta(\zeta) := \theta_\beta(\zeta)$ mod 1. The process $(q(\zeta), \dot{\theta}_\beta(\zeta))_{\zeta \geq 0}$ is a Markov process with values in $E \times \mathbb{S}$ and with generator $Q + \beta\partial_{\dot{\theta}}$. It is a stationary process with the stationary distribution $\pi_q \otimes \nu_{\mathbb{S}}$ where $\nu_{\mathbb{S}}$ is the uniform distribution over the torus $\mathbb{S}$. It is also an ergodic process with respect to the stationary distribution $\pi_q \otimes \nu_{\mathbb{S}}$. Since $g$ satisfies $\int g(q, \dot{\theta}) \pi_q(dq) = 0$ for all $\dot{\theta}$, it a fortiori satisfies $\iint g(q, \dot{\theta}) \pi_q(dq) \nu_{\mathbb{S}}(d\dot{\theta}) = 0$, and the result then follows from standard arguments [26, section 6.5.2]:

$$f(q_0, \dot{\theta}_0) = -\int_0^\infty \mathbb{E}\big[g(q(\zeta), \dot{\theta}_\beta(\zeta))|q(0) = q_0, \dot{\theta}_\beta(0) = \dot{\theta}_0\big] d\zeta ,$$

which gives (5.9).

To prove statement 2. let $\boldsymbol{\beta} \in \mathbb{R}^2$ be fixed. We denote by $\boldsymbol{\theta}_{\boldsymbol{\beta}}(\zeta)$ the solution to $\frac{d\boldsymbol{\theta}_{\boldsymbol{\beta}}}{d\zeta} = \boldsymbol{\beta}$ and by $\dot{\boldsymbol{\theta}}_{\boldsymbol{\beta}}(\zeta) := \boldsymbol{\theta}_{\boldsymbol{\beta}}(\zeta) \bmod 1$. The process $(q(\zeta), \dot{\boldsymbol{\theta}}_{\boldsymbol{\beta}}(\zeta))_{\zeta \geq 0}$ is a Markov process with values in $E \times \mathbb{S}^2$ and with generator $Q + \boldsymbol{\beta} \cdot \nabla_{\dot{\boldsymbol{\theta}}}$.

If the ratio $\beta_1/\beta_2$ of the entries of $\beta_2$ of $\boldsymbol{\beta}$ is irrational, the process $(q(\zeta), \dot{\boldsymbol{\theta}}_{\boldsymbol{\beta}}(\zeta))_{\zeta \geq 0}$ is stationary and ergodic, with the stationary distribution $\pi_q \otimes \nu_{\mathbb{S}^2}$, where $\nu_{\mathbb{S}^2}$ is the uniform distribution over the torus $\mathbb{S}^2$. Since $g$ satisfies $\int g(q, \dot{\boldsymbol{\theta}}) \pi_q(dq) = 0$ for all $\dot{\boldsymbol{\theta}}$, it a fortiori satisfies $\iint g(q, \dot{\boldsymbol{\theta}}) \pi_q(dq) \nu_{\mathbb{S}^2}(d\dot{\boldsymbol{\theta}}) = 0$, and the result then follows from standard arguments [26, section 6.5.2]:

$$f(q_0, \dot{\boldsymbol{\theta}}_0) = -\int_0^\infty \mathbb{E}[g(q(\zeta), \dot{\boldsymbol{\theta}}_{\boldsymbol{\beta}}(\zeta)) | q(0) = q_0, \dot{\boldsymbol{\theta}}_{\boldsymbol{\beta}}(0) = \dot{\boldsymbol{\theta}}_0] d\zeta \,,$$

which gives (5.10).

If the ratio $\beta_1/\beta_2$ of the entries of $\boldsymbol{\beta}$ is rational, that is to say, if there exist nonzero integers $n_1, n_2$ such that $n_1\beta_1 = n_2\beta_2$, then $(\dot{\boldsymbol{\theta}}_{\boldsymbol{\beta}}(\zeta))_{\zeta \geq 0}$ is not ergodic over the torus $\mathbb{S}^2$. However, for a given starting point $\dot{\boldsymbol{\theta}}_0$, it satisfies the ergodic theorem over the compact manifold $\mathbb{S}^1_{\dot{\boldsymbol{\theta}}_0} := \{\dot{\boldsymbol{\theta}}_0 + \boldsymbol{\beta} s \bmod 1, \ s \in \mathbb{R}\}$, with the uniform distribution over the manifold $\mathbb{S}^1_{\dot{\boldsymbol{\theta}}_0}$. Since $g$ satisfies $\int g(q, \dot{\boldsymbol{\theta}}) \pi_q(dq) = 0$ for all $\dot{\boldsymbol{\theta}}$, it a fortiori satisfies $\iint g(q, \dot{\boldsymbol{\theta}}) \pi_q(dq) \nu_{\mathbb{S}^1_{\dot{\boldsymbol{\theta}}_0}}(d\dot{\boldsymbol{\theta}}) = 0$. We can then define

$$f(q_0, \dot{\boldsymbol{\theta}}_0) = -\int_0^\infty \mathbb{E}[g(q(\zeta), \dot{\boldsymbol{\theta}}_{\boldsymbol{\beta}}(\zeta)) | q(0) = q_0, \dot{\boldsymbol{\theta}}_{\boldsymbol{\beta}}(0) = \dot{\boldsymbol{\theta}}_0] d\zeta \,,$$

which gives (5.10).

□

We can now state the lemma used in the proof of Theorem III.1:

**Lemma V.3.** *For all $f \in \mathcal{C}_b^\infty(\mathbb{R}^d \times \mathbb{R}, \mathbb{R})$, and all compact sets $K$ of $\mathbb{R}^d \times \mathbb{R}$, there* *exists a family $f^\varepsilon$ such that:*

$$\sup_{(\boldsymbol{x},Z)\in K, q\in E, \dot{\boldsymbol{\theta}}\in\mathbb{S}^p} |f^\varepsilon(\boldsymbol{x},q,\dot{\boldsymbol{\theta}},Z) - f(\boldsymbol{x},Z)| \overset{\varepsilon\to 0}{\longrightarrow} 0, \tag{5.11}$$

$$\sup_{(\boldsymbol{x},Z)\in K, q\in E, \dot{\boldsymbol{\theta}}\in\mathbb{S}^p} |\mathcal{L}^\varepsilon f^\varepsilon(\boldsymbol{x},q,\dot{\boldsymbol{\theta}},Z) - \mathcal{L}f(\boldsymbol{x},Z)| \overset{\varepsilon\to 0}{\longrightarrow} 0, \tag{5.12}$$

*where $\mathcal{L}^\varepsilon$ is the generator (5.6) and $\mathcal{L}$ is the generator (5.7).*

*Proof.* Let $f \in \mathcal{C}_b^\infty(\mathbb{R}^d \times \mathbb{R}, \mathbb{R})$, and define

$$f^\varepsilon(\boldsymbol{x},q,\dot{\boldsymbol{\theta}},Z) := f(\boldsymbol{x},Z) + \sqrt{\varepsilon} f_1(\boldsymbol{x},q,\dot{\boldsymbol{\theta}},Z) + \varepsilon f_2(\boldsymbol{x},q,\dot{\boldsymbol{\theta}},Z) + \varepsilon f_3^\varepsilon(\boldsymbol{x},\dot{\boldsymbol{\theta}},Z), \tag{5.13}$$

where $f_1$, $f_2$, and $f_3^\varepsilon$ will be specified later on. Applying $\mathcal{L}^\varepsilon$ to $f^\varepsilon$, we get

$$\begin{aligned}\mathcal{L}^\varepsilon f^\varepsilon(\boldsymbol{x},q,\dot{\boldsymbol{\theta}},Z) = \frac{1}{\sqrt{\varepsilon}} &\Big( \big(Q + \boldsymbol{\beta}(Z)\cdot\nabla_{\dot{\boldsymbol{\theta}}}\big) f_1 + \boldsymbol{F}(\boldsymbol{x},q,\dot{\boldsymbol{\theta}},Z)\cdot\nabla_{\boldsymbol{x}} f(\boldsymbol{x},Z)\Big) \\ &+ \Big( \big(Q + \boldsymbol{\beta}(Z)\cdot\nabla_{\dot{\boldsymbol{\theta}}}\big) f_2 + \boldsymbol{F}(\boldsymbol{x},q,\dot{\boldsymbol{\theta}},Z)\cdot\nabla_{\boldsymbol{x}} f_1(\boldsymbol{x},q,\dot{\boldsymbol{\theta}},Z)\Big) \\ &+ \boldsymbol{\beta}(Z)\cdot\nabla_{\dot{\boldsymbol{\theta}}} f_3^\varepsilon(\boldsymbol{x},\dot{\boldsymbol{\theta}},Z) + \partial_Z f(\boldsymbol{x},Z) + O(\sqrt{\varepsilon}).\end{aligned} \tag{5.14}$$

Now let us define the correction $f_1$ as

$$f_1(\boldsymbol{x},q,\dot{\boldsymbol{\theta}},Z) := \sum_{j=1}^p f_1^{(j)}(\boldsymbol{x},q,\dot{\theta}_j,Z), \tag{5.15}$$

where

$$f_1^{(j)}(\boldsymbol{x},q,\dot{\theta}_j,Z) = -\big(Q + \beta_j(Z)\partial_{\dot{\theta}_j}\big)^{-1}\Big(\boldsymbol{F}^{(j)}(\boldsymbol{x},q,\dot{\theta}_j,Z)\cdot\nabla_{\boldsymbol{x}} f(\boldsymbol{x},Z)\Big).$$

These functions are well-defined and admit the representation

$$f_1^{(j)}(\boldsymbol{x},q,\dot{\theta}_j,Z) = \int_0^\infty \mathbb{E}\big[\boldsymbol{F}^{(j)}(\boldsymbol{x},q(\zeta),\dot{\theta}_j + \beta_j(Z)\zeta,Z)\cdot\nabla_{\boldsymbol{x}} f(\boldsymbol{x},Z)|q(0)=q\big]d\zeta,$$

by Lemma V.2.

The second correction $f_2$ is defined by

$$f_2(\boldsymbol{x},q,\dot{\boldsymbol{\theta}},Z) := \sum_{j,l=1}^p f_2^{(jl)}(\boldsymbol{x},q,\dot{\theta}_j,\dot{\theta}_l,Z), \tag{5.16}$$

where

$$f_2^{(jl)}(\boldsymbol{x},q,\dot\theta_j,\dot\theta_l,Z) = -\big(Q+\beta_j(Z)\partial_{\dot\theta_j}+\beta_l(Z)\partial_{\dot\theta_l}\big)^{-1}$$
$$\times\Big(\boldsymbol{F}^{(j)}(\boldsymbol{x},q,\dot\theta_j,Z)\cdot\nabla_{\boldsymbol{x}}f_1^{(l)}(\boldsymbol{x},q,\dot\theta_l,Z)-\mathbb{E}\big[\boldsymbol{F}^{(j)}(\boldsymbol{x},q(0),\dot\theta_j,Z)\cdot\nabla_{\boldsymbol{x}}f_1^{(l)}(\boldsymbol{x},q(0),\dot\theta_l,Z)\big]\Big).$$

These functions are well defined by Lemma V.2 since the argument of the operator $\big(Q+\beta_j(Z)\partial_{\dot\theta_j}+\beta_l(Z)\partial_{\dot\theta_l}\big)^{-1}$ has mean zero for all $\boldsymbol{\theta}$.

Substituting (5.15) and (5.16) in (5.14) we obtain

$$\mathcal{L}^\varepsilon f^\varepsilon(\boldsymbol{x},q,\dot{\boldsymbol{\theta}},Z) = \sum_{j,l=1}^{p} g_3^{(jl)}(\boldsymbol{x},\dot\theta_j,\dot\theta_l,Z)+\boldsymbol{\beta}(Z)\cdot\nabla_{\dot{\boldsymbol{\theta}}}f_3^\varepsilon(\boldsymbol{x},\dot{\boldsymbol{\theta}},Z)$$
$$+\partial_Z f(\boldsymbol{x},Z)+O(\sqrt{\varepsilon}), \qquad (5.17)$$

with

$$g_3^{(jl)}(\boldsymbol{x},\dot\theta_j,\dot\theta_l,Z) := \mathbb{E}\big[\boldsymbol{F}^{(j)}(\boldsymbol{x},q(0),\dot\theta_j,Z)\cdot\nabla_{\boldsymbol{x}}f_1^{(l)}(\boldsymbol{x},q(0),\dot\theta_l,Z)\big]. \qquad (5.18)$$

We now define the third correction function

$$f_3^\varepsilon(\boldsymbol{x},\dot{\boldsymbol{\theta}},Z) := \sum_{j,l=1}^{p} f_3^{(jl),\varepsilon}(\boldsymbol{x},\dot\theta_j,\dot\theta_l,Z), \qquad (5.19)$$

with terms

$$f_3^{(jl),\varepsilon}(\boldsymbol{x},\dot\theta_j,\dot\theta_l,Z) := \int_0^\infty e^{-\sqrt{\varepsilon}s}\,\widetilde{g}_3^{(jl)}(\boldsymbol{x},\dot\theta_j+\beta_j(Z)s,\dot\theta_l+\beta_l(Z)s,Z)ds,$$

defined by

$$\widetilde{g}_3^{(jl)}(\boldsymbol{x},\dot\theta_j,\dot\theta_l,Z) := g_3^{(jl)}(\boldsymbol{x},\dot\theta_j,\dot\theta_l,Z)-\mathcal{G}_3^{(jl)}(\boldsymbol{x},Z),$$

where

$$\mathcal{G}_3^{(jl)}(\boldsymbol{x},Z) := \lim_{S\to\infty}\frac{1}{S}\int_0^S g_3^{(jl)}(\boldsymbol{x},\dot\theta_j+\beta_j(Z)s,\dot\theta_l+\beta_l(Z)s,Z)ds. \qquad (5.20)$$

These are are well defined because $s \mapsto g_3^{(jl)}(\boldsymbol{x}, \dot{\theta}_j + \beta_j(Z)s, \dot{\theta}_l + \beta_l(Z)s, Z)$ are almost periodic mappings.

Note that $\sqrt{\varepsilon} f_3^{(jl),\varepsilon}$ is uniformly bounded because $\widetilde{g}_3^{(jl)}$ is bounded. This and definitions (5.15), (5.16) of the corrections $f_1$ and $f_2$ used in equation (5.13) imply that $f^\varepsilon$ satisfies (5.11). Note also that $\sqrt{\varepsilon} f_3^{(jl),\varepsilon}$ goes to zero as $\varepsilon \to 0$, because the mapping $s \mapsto \widetilde{g}_3^{(jl)}(\boldsymbol{x}, \dot{\theta}_j + \beta_j(Z)s, \dot{\theta}_l + \beta_l(Z)s, Z)$ is almost periodic and with mean zero. Moreover, using the chain rule and integration by parts, we obtain

$$
\begin{aligned}
\big(\beta_j(Z)\partial_{\dot{\theta}_j} + \beta_l(Z)\partial_{\dot{\theta}_l}\big) & f_3^{(jl),\varepsilon}(\boldsymbol{x}, \dot{\theta}_j, \dot{\theta}_l, Z) \\
&= \int_0^\infty e^{-\sqrt{\varepsilon}s} \partial_s \big[\widetilde{g}_3^{(jl)}(\boldsymbol{x}, \dot{\theta}_j + \beta_j(Z)s, \dot{\theta}_l + \beta_l(Z)s, Z)\big] ds \\
&= -\widetilde{g}_3^{(jl)}(\boldsymbol{x}, \dot{\theta}_j, \dot{\theta}_l, Z) + \sqrt{\varepsilon} f_3^{(jl),\varepsilon}(\boldsymbol{x}, \dot{\theta}_j, \dot{\theta}_l, Z).
\end{aligned}
$$

Gathering the results, equation (5.17) becomes

$$
\mathcal{L}^\varepsilon f^\varepsilon = \sum_{j,l=1}^{p} \mathcal{G}_3^{(jl)}(\boldsymbol{x}, Z) + \partial_Z f(\boldsymbol{x}, Z) + \sqrt{\varepsilon} f_3^\varepsilon(\boldsymbol{x}, \dot{\boldsymbol{\theta}}, Z) + O(\sqrt{\varepsilon}).
$$

The result (5.12) follows from this equation and definitions (5.18), (5.20) and (5.15), because $\sqrt{\varepsilon} f_3^\varepsilon$ goes to zero as $\varepsilon \to 0$.

□

## 5.3 Appendix: The Perturbed Test Function Method

The perturbed test function method consists of two main steps as was stated in [44, 7, 35, 26]. The first is showing tightness of the laws of the processes $\widehat{\boldsymbol{X}}^\varepsilon(z) := (\boldsymbol{X}^\varepsilon(z), q^\varepsilon(z), \dot{\boldsymbol{\theta}}^\varepsilon(z), Z(z))$. The second is to show that the martingale problem associated with $\mathcal{L}^\varepsilon$ yields the martingale problem associated with $\mathcal{L}$ in the limit $\varepsilon \to 0$. These two steps combined yield convergence in distribution. We outline these two steps here and give references for where one can find more detailed accounts.

With regard to proving tightness, a sufficient condition is that the family of processes $\widehat{\boldsymbol{X}}^{\varepsilon}(z)$ satisfy the Kolmogorov moment estimate

$$\mathbb{E}[|\widehat{\boldsymbol{X}}^{\varepsilon}(z) - \widehat{\boldsymbol{X}}^{\varepsilon}(z')|^{\alpha}] \leq C|z - z'|^{1+\gamma} \tag{5.21}$$

for $0 \leq z' \leq z \leq Z$, constant independent of $\varepsilon$, and $\alpha, \gamma > 0$ [45]. Showing (5.21) requires $\varepsilon$-independent estimates for moments of $\widehat{\boldsymbol{X}}^{\varepsilon}(z)$. One can prove such estimates by using that

$$M_{f^{\varepsilon}}(z) := f^{\varepsilon}(\widehat{\boldsymbol{X}}^{\varepsilon}(z)) - f^{\varepsilon}(\widehat{\boldsymbol{x}}_0) - \int_0^z ds \mathcal{L}^{\varepsilon} f^{\varepsilon}(\widehat{\boldsymbol{X}}^{\varepsilon}(s)) \tag{5.22}$$

is a martingale for all test functions $f^{\varepsilon}$ where $\widehat{\boldsymbol{X}}^{\varepsilon}(0) = \widehat{\boldsymbol{x}}_0$.

In particular, if we take $f^{\varepsilon} = \widehat{\boldsymbol{x}} + \sqrt{\varepsilon} f_1(\widehat{\boldsymbol{x}})$ where $\widehat{\boldsymbol{x}} = (\boldsymbol{x}, q, \dot{\boldsymbol{\theta}}, Z)$ and $f_1$ is as in (5.15) we can write

$$\widehat{\boldsymbol{X}}^{\varepsilon}(z) = \widehat{\boldsymbol{x}}_0 - \sqrt{\varepsilon}(f_1(\widehat{\boldsymbol{X}}^{\varepsilon}(z)) - f_1(\widehat{\boldsymbol{x}}_0)) + \int_0^z ds \mathcal{L}^{\varepsilon} f^{\varepsilon}(\widehat{\boldsymbol{X}}^{\varepsilon}(s)) + M_{f^{\varepsilon}}(z). \tag{5.23}$$

Moments of $\widehat{\boldsymbol{X}}^{\varepsilon}(z)$ can then be estimated by estimating the right hand side of (5.23) using growth properties of the test functions which are inherited from $\boldsymbol{F}$ in (5.1), Gronwall's inequality, and Doob's martingale inequality as was done in [26, Section 6.3.5]. It then remains to show (5.21). However, the test function $f_1$ in the representation (5.23) for $\widehat{\boldsymbol{X}}^{\varepsilon}(z)$ prevents us from obtaining such an estimate. Instead, one can prove (5.21) for a process which is uniformly close to $\widehat{\boldsymbol{X}}^{\varepsilon}(z)$ in probability and this will suffice for tightness. Such a process can be obtained by omitting the terms involving $f_1$ in (5.23) and this process will be close to $\widehat{\boldsymbol{X}}^{\varepsilon}(z)$ since their difference $-\sqrt{\varepsilon}(f_1(\widehat{\boldsymbol{X}}^{\varepsilon}(z)) - f_1(\widehat{\boldsymbol{x}}_0))$ will be small [26, Section 6.3.5].

The argument for showing that the martingale problem associated with $\mathcal{L}^{\varepsilon}$ converges to that of $\mathcal{L}$ is as in [26, Section 6.3.4]. We use that (5.22) to say that in particular for $f^{\varepsilon}$ as in Lemma V.3 we have from the martingale property that

$$\mathbb{E}[M_{f^{\varepsilon}}(z') - M_{f^{\varepsilon}}(z)|\widehat{\mathcal{F}}_z] = 0 \tag{5.24}$$

where $\widehat{\mathcal{F}}_z := \sigma(\widehat{\boldsymbol{X}}^\varepsilon(s), s \le z)$. Further, by restricting to the $\sigma$-algebra generated by $((\boldsymbol{X}^\varepsilon(s), Z(s)))$ for $s \le z$ and properties of conditional expectation we have

$$\mathbb{E}[(M_{f^\varepsilon}(z') - M_{f^\varepsilon}(z)) \prod_{j=1}^n h_j(\boldsymbol{X}^\varepsilon(z_j), Z(z_j))] = 0 \tag{5.25}$$

for all bounded, continuous $h_j$, $0 \le z_1 \le \ldots \le z_n \le z \le z'$. Then using Lemma V.3 and $\varepsilon$-independent moment estimates obtained in the proof of tightness one can write (5.25) as

$$\mathbb{E}[(M_f(z') - M_f(z)) \prod_{j=1}^n h_j(\boldsymbol{X}^\varepsilon(z_j), Z(z_j))] = O(\varepsilon) \tag{5.26}$$

where $f$ is as in Lemma V.3.

Expanding $M_f$ we have

$$\begin{aligned}\mathbb{E}[(f(\boldsymbol{X}^\varepsilon(z'), Z(z')) - f(\boldsymbol{X}^\varepsilon(z), Z(z)) - \int_z^{z'} ds \mathcal{L} f(\boldsymbol{X}^\varepsilon(s), Z(s)))\\ \times \prod_{j=1}^n h_j(\boldsymbol{X}^\varepsilon(z_j), Z(z_j))] = 0.\end{aligned} \tag{5.27}$$

Then the limit $(\boldsymbol{X}(z), Z(z))$ of any weakly convergent subsequence of $(\boldsymbol{X}^\varepsilon(z), Z(z))$ satisfies

$$\begin{aligned}\mathbb{E}[(f(\boldsymbol{X}(z'), Z(z')) - f(\boldsymbol{X}(z), Z(z)) - \int_z^{z'} ds \mathcal{L} f(\boldsymbol{X}(s), Z(s)))\\ \times \prod_{j=1}^n h_j(\boldsymbol{X}(z_j), Z(z_j))] = 0.\end{aligned} \tag{5.28}$$

for all bounded, continuous $h_j$, $0 \le z_1 \le \ldots \le z_n \le z \le z'$. This in turn implies that

$$\mathbb{E}[f(\boldsymbol{X}(z'), Z(z')) - f(\boldsymbol{X}(z), Z(z)) - \int_z^{z'} ds \mathcal{L} f(\boldsymbol{X}(s), Z(s)) | \mathcal{F}_z] = 0. \tag{5.29}$$

where $\mathcal{F}_z := \sigma((\boldsymbol{X}(s), Z(s)), s \le z)$. Thus,

$$f(\boldsymbol{X}(z), Z(z)) - f(\boldsymbol{X}(0), Z(0)) - \int_0^z ds \mathcal{L} f(\boldsymbol{X}(s), Z(s)), \quad z \ge 0 \tag{5.30}$$

is a martingale for any limit process $(\boldsymbol{X}(z), Z(z))$ and test function $f \in \mathcal{C}_b^\infty(\mathbb{R}^d \times \mathbb{R}, \mathbb{R})$.

If the generator $\mathcal{L}$ given in (5.7) has diffusion coefficients which are at most quadratically growing in $\boldsymbol{x}$ and drift coefficients are at most linearly growing in $\boldsymbol{x}$ then the limit process $(\boldsymbol{X}(z), Z(z))$ will be the unique diffusion process with generator $\mathcal{L}$. Combining this with tightness yields the desired result (see [22, Section 4.8]).

# CHAPTER VI

# Boundedness of the Operator K

Here we prove that the integral operator **K** which appears in the analysis of the evanescent modes in section 3.5.4 is bounded in an appropriate function space. Boundedness of the operator **K** allows us to invert the the operator $\left(\mathbf{I} - \sqrt{\varepsilon}\mathbf{K}\right)$ using a Neumann series which in turn gives us a way of expressing the evanescent modes in terms of the propagating modes and a term which depends on the source. We include this as a separate chapter so as not to interrupt the flow of chapter III with a long technical aside.

## 6.1 Setup

In (3.87) the integral operator **K** is given component-wise by

$$[\mathbf{K}\mathbf{U}]_j(\omega, z) := -\frac{1_I(z)}{2\beta_j(\omega, z)} \int_{-\infty}^{\infty} d\xi e^{-\beta_j(\omega,z)|\xi|} \times \left[ \begin{pmatrix} 1 \\ \varepsilon\partial_z \end{pmatrix} \sqrt{\varepsilon}\mathcal{C}_j^{\varepsilon(e)}(\omega, z+\varepsilon\xi) - \sigma\mu_j^2(z)\nu\left(\frac{z}{\varepsilon} + \xi\right)\boldsymbol{U}_j \right], \qquad (6.1)$$

where $I := (z_-^{(t)}, z_-^{(t-1)})$ and

$$\sqrt{\varepsilon}\mathcal{C}_j^{\varepsilon(e)}(\omega, z) = \mathcal{C}_j^{(e)}(\omega, z) + O(\sqrt{\varepsilon}), \tag{6.2}$$

$$\varepsilon^{3/2}\partial_z\mathcal{C}_j^{\varepsilon(e)}(\omega, z) = \varepsilon\partial_z\mathcal{C}_j^{(e)}(\omega, z) + O(\sqrt{\varepsilon}), \tag{6.3}$$

$$\mathcal{C}_j^{(e)}(\omega, z) := \sum_{\substack{q>\mathcal{N} \\ q\neq j}}^{\infty} \sigma\Gamma_{jq}\nu''\Big(\frac{z}{\varepsilon}\Big)u_q^{\varepsilon}(\omega, z) + \sigma\Theta_{jq}\nu'\Big(\frac{z}{\varepsilon}\Big)\varepsilon\partial_z u_q^{\varepsilon}(\omega, z) \tag{6.4}$$

$$\begin{aligned} \varepsilon\partial_z\mathcal{C}_j^{(e)}(\omega, z) := &\sum_{\substack{q>\mathcal{N} \\ q\neq j}} \sigma\Gamma_{jq}[\nu'''\Big(\frac{z}{\varepsilon}\Big)u_q^{\varepsilon}(\omega, z) + \nu''\Big(\frac{z}{\varepsilon}\Big)\varepsilon\partial_z u_q^{\varepsilon}(\omega, z)] \\ &+ \sum_{\substack{q>\mathcal{N} \\ q\neq j}} \sigma\Theta_{jq}[\nu''\Big(\frac{z}{\varepsilon}\Big)\varepsilon\partial_z u_q^{\varepsilon}(\omega, z) + \nu'\Big(\frac{z}{\varepsilon}\Big)\varepsilon^2\partial_z^2 u_q^{\varepsilon}(\omega, z)]. \end{aligned} \tag{6.5}$$

In what follows, we will neglect the $O(\sqrt{\varepsilon})$ parts of (6.2) and (6.3) as they can be shown to decay to 0 as $\varepsilon \to 0$ using similar arguments to those we give in Section 6.3. We will also suppress the dependence on $\omega$ since it is assumed fixed.

It will be helpful to break up the first component of $\mathbf{KU}_j(z)$, which we denote by $[\mathbf{KU}]_j^{(1)}(z)$, into three terms. We have that

$$\begin{aligned} [\mathbf{KU}]_j^{(1)}(z) &= -\frac{1_I(z)}{2\beta_j(z)}\int_{-\infty}^{\infty} d\xi e^{-\beta_j(z)|\xi|}\left[\mathcal{C}_j^{(e)}(z+\varepsilon\xi) - \sigma\mu_j^2(z)\nu\Big(\frac{z}{\varepsilon}+\xi\Big)u_j^{\varepsilon}(z+\varepsilon\xi)\right] \\ &=: [\mathbf{KU}]_j^{(1,1)}(z) + [\mathbf{KU}]_j^{(1,2)}(z) + [\mathbf{KU}]_j^{(1,3)}(z) \end{aligned} \tag{6.6}$$

where $[\mathbf{KU}]_j^{(1,1)}(z)$, $[\mathbf{KU}]_j^{(1,2)}(z)$ , and $[\mathbf{KU}]_j^{(1,3)}(z)$ are as given below. We have

$$\begin{aligned} [\mathbf{KU}]_j^{(1,1)}(z) &:= -\frac{\sigma 1_I(z)}{2\beta_j(z)}\int_{-\infty}^{\infty} d\xi e^{-\beta_j(z)|\xi|}\sum_{\substack{q>\mathcal{N} \\ q\neq j}}\Gamma_{jq}\nu''\Big(\frac{z+\varepsilon\xi}{\varepsilon}\Big)u_q^{\varepsilon}(z+\varepsilon\xi) \\ &= \frac{\sigma 1_I(z)}{2\beta_j(z)\varepsilon}\int_{-\infty}^{\infty} ds e^{-\beta_j(z)|s|/\varepsilon}\sum_{\substack{q>\mathcal{N} \\ q\neq j}}\frac{jq(-1)^{j+q}}{j^2-q^2}\nu''\Big(\frac{z+s}{\varepsilon}\Big)u_q^{\varepsilon}(z+s) \end{aligned} \tag{6.7}$$

$$[\mathbf{KU}]_j^{(1,2)}(z) := -\frac{\sigma 1_I(z)}{2\beta_j(z)} \int_{-\infty}^{\infty} d\xi e^{-\beta_j(z)|\xi|} \sum_{\substack{q>\mathcal{N}\\ q\neq j}} \Theta_{jq} \nu'\Big(\frac{z+\varepsilon\xi}{\varepsilon}\Big) \varepsilon \partial_z u_q^\varepsilon(z+\varepsilon\xi)$$

$$= \frac{\sigma 1_I(z)}{\beta_j(z)\varepsilon} \int_{-\infty}^{\infty} dse^{-\beta_j(z)|s|/\varepsilon} \sum_{\substack{q>\mathcal{N}\\ q\neq j}} \frac{jq(-1)^{j+q}}{j^2-q^2} \nu'\Big(\frac{z+s}{\varepsilon}\Big) \varepsilon \partial_z u_q^\varepsilon(z+s) \quad (6.8)$$

$$[\mathbf{KU}]_j^{(1,3)}(z) := \frac{\sigma 1_I(z)}{2\beta_j(z)} \int_{-\infty}^{\infty} d\xi e^{-|\xi|\beta_j(z)} \mu_j^2(z) \nu\Big(\frac{z}{\varepsilon}+\xi\Big) u_j^\varepsilon(z+\varepsilon\xi)$$

$$= \frac{\sigma 1_I(z)}{2\beta_j(z)\varepsilon} \int_{-\infty}^{\infty} dse^{-|s|\beta_j(z)/\varepsilon} \Big(\frac{\pi j}{D(z)}\Big)^2 \nu\Big(\frac{z+s}{\varepsilon}\Big) u_j^\varepsilon(z+s). \quad (6.9)$$

We decompose similarly the second component of $\mathbf{KU}_j(z)$, which we denote by $[\mathbf{KU}]_j^{(2)}(z)$. We have that

$$[\mathbf{KU}]_j^{(2)}(z) = -\frac{1_I(z)}{2\beta_j(z)} \int_{-\infty}^{\infty} d\xi e^{-|\xi|\beta_j(z)} \Big[\varepsilon \partial_z \mathcal{C}_j^{(e)}(z+\varepsilon\xi)$$

$$- \sigma \mu_j^2(z) \nu\Big(\frac{z}{\varepsilon}+\xi\Big) \varepsilon \partial_z u_j^\varepsilon(z+\varepsilon\xi)\Big]$$

$$= -\frac{\sigma 1_I(z)}{2\beta_j(z)\varepsilon} \int_{-\infty}^{\infty} dse^{-\beta_j(z)|s|/\varepsilon} \Bigg[\sum_{\substack{q>\mathcal{N}\\ q\neq j}} \Big[\beta_j(z)\,\mathrm{sgn}(s)\big(\Gamma_{jq}\nu''\big(\frac{z+s}{\varepsilon}\big) u_q^\varepsilon(z+s)$$

$$+ \Theta_{jq}\nu'\big(\frac{z+s}{\varepsilon}\big)\varepsilon\partial_z u_q^\varepsilon(z+s)\big)\Big] - \mu_j^2(z)\nu\big(\frac{z+s}{\varepsilon}\big)\varepsilon\partial_z u_j^\varepsilon(z+s)\Bigg] \quad (6.10)$$

$$=: [\mathbf{KU}]_j^{(2,1)}(z) + [\mathbf{KU}]_j^{(2,2)}(z) + [\mathbf{KU}]_j^{(2,3)}(z) \quad (6.11)$$

where we used integration by parts to obtain (6.10). The three terms in (6.11) are given by

$$[\mathbf{KU}]_j^{(2,1)}(z) := \frac{\sigma 1_I(z)}{2\varepsilon}\int_{-\infty}^{\infty} ds\,\mathrm{sgn}(s)e^{-\beta_j(z)|s|/\varepsilon} \times \sum_{\substack{q>\mathcal{N}\\ q\neq j}} \frac{jq(-1)^{j+q}}{j^2-q^2}\nu''\Big(\frac{z+s}{\varepsilon}\Big)u_q^\varepsilon(z+s) \tag{6.12}$$

$$[\mathbf{KU}]_j^{(2,2)}(z) := \frac{\sigma 1_I(z)}{\varepsilon}\int_{-\infty}^{\infty} ds e^{-\beta_j(z)|s|/\varepsilon} \times \sum_{\substack{q>\mathcal{N}\\ q\neq j}} \frac{jq(-1)^{j+q}}{j^2-q^2}\nu'\Big(\frac{z+s}{\varepsilon}\Big)\varepsilon\partial_z u_q^\varepsilon(z+s) \tag{6.13}$$

$$\begin{aligned}[\mathbf{KU}]_j^{(2,3)}(z) &:= \frac{\sigma 1_I(z)}{2\beta_j(z)}\int_{-\infty}^{\infty} d\xi e^{-|\xi|\beta_j(z)}\mu_j^2(z)\nu\Big(\frac{z}{\varepsilon}+\xi\Big)\varepsilon\partial_z u_j^\varepsilon(z+\varepsilon\xi)\\ &= \frac{\sigma 1_I(z)}{2\beta_j(z)\varepsilon}\int_{-\infty}^{\infty} ds e^{-|s|\beta_j(z)/\varepsilon}\Big(\frac{\pi j}{D(z)}\Big)^2\nu\Big(\frac{z+s}{\varepsilon}\Big)\varepsilon\partial_z u_j^\varepsilon(z+s).\end{aligned} \tag{6.14}$$

### 6.2 A Few Useful Estimates

As was the case in [4], we can estimate the operator $\mathbf{K}$ in terms of operators for which there are well known estimates, such as convolutions and discrete Hilbert transforms. Here we will define a few of these simpler operators and prove some intermediate estimates on them. Many of these operators and estimates are either analogous or identical to those that appeared in [4]. These will all be put together in section 6.3 to show boundedness of $\mathbf{K}$.

The discrete Hilbert transform of a sequence $\{a_n\}_{n\in\mathbb{Z}}$ is given by

$$\boldsymbol{a} * \frac{1}{n} := \sum_{n\neq j}\frac{a_n}{j-n}. \tag{6.15}$$

The discrete Hilbert transform is a bounded operator from $\ell^2(\mathbb{Z})$ to $\ell^2(\mathbb{Z})$ satisfying the estimate

$$\big\|\boldsymbol{a} * \frac{1}{n}\big\|_{\ell^2} \leq \pi\big\|\boldsymbol{a}\big\|_{\ell^2} \tag{6.16}$$

an elementary proof of which can be found in [31].

The operator $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is defined by

$$[T\boldsymbol{a}]_j := \sum_{q \neq \pm j} \frac{jq}{j^2 - q^2} a_q. \tag{6.17}$$

As was noted in [4], this operator is essentially a sum of discrete Hilbert transforms and we may write

$$[T\boldsymbol{a}]_j = \frac{1}{2}\Big[(-qa_{-q}) * \frac{1}{q} + (qa_q) * \frac{1}{q}\Big]_j + \frac{1}{4}(a_{-j} - a_j). \tag{6.18}$$

Then we have using the estimate on the discrete Hilbert transform that

$$\|T\boldsymbol{a}\|_{\ell^2} \leq \Big(\frac{1}{2} + \pi\Big)\|j\boldsymbol{a}\|_{\ell^2}. \tag{6.19}$$

The the convolution operator $\widetilde{T}$ will be given component-wise by

$$[\widetilde{T}\boldsymbol{a}]_j(z) := \big([|T\boldsymbol{a}|]_j * e^{-\widetilde{\beta}_j|\cdot|/\varepsilon}\big)(z) 1_{\{j > \mathcal{N}\}} \tag{6.20}$$

where $\widetilde{\beta}_j := \beta_j(z_-^{(t-1)})$. Then Young's inequality for convolutions and $\big\|e^{-\widetilde{\beta}_j|\cdot|/\varepsilon}\big\|_{L^1(\mathbb{R})} = 2\varepsilon/\widetilde{\beta}_j$ implies

$$\|[\widetilde{T}\boldsymbol{a}]_j\|_{L^2(\mathbb{R})} \leq \frac{2\varepsilon}{j\widetilde{C}} \|[T\boldsymbol{a}]_j\|_{L^2(\mathbb{R})} \tag{6.21}$$

where we used that for $j > \mathcal{N}$

$$\widetilde{\beta}_j \geq \frac{j\pi}{\widetilde{D}} \sqrt{1 - \Big(\frac{k\widetilde{D}}{(\mathcal{N}+1)\pi}\Big)^2} =: j\widetilde{C} \tag{6.22}$$

and $\widetilde{D} := D(z_-^{(t-1)})$.

If we consider a slight variant on $\widetilde{T}$ given by

$$[\mathcal{T}\boldsymbol{a}]_j(z) := \int_{-\infty}^{\infty} ds [T\boldsymbol{a}]_j(z+s) 1_I(z) e^{-\beta_j(z)|s|/\varepsilon} 1_{\{j > \mathcal{N}\}} \tag{6.23}$$

we can obtain a similar estimate on its $L^2(\mathbb{R})$ norm. In particular, we can use

$$1_I(z) e^{-\beta_j(z)|s|/\varepsilon} \leq e^{-\widetilde{\beta}_j|s|/\varepsilon} \tag{6.24}$$

to obtain

$$\|[\mathcal{T}\boldsymbol{a}]_j\|_{L^2(\mathbb{R})} \le \|[\widetilde{T}\boldsymbol{a}]_j\|_{L^2(\mathbb{R})}. \tag{6.25}$$

Now let

$$[\widetilde{K}\boldsymbol{a}]_j(z) := \frac{1_I(z)}{\varepsilon}\int_{-\infty}^{\infty} ds e^{-\beta_j(z)|s|/\varepsilon} \times (-1)^j \sum_{\substack{q>\mathcal{N}\\ q\neq j}} \frac{jq}{j^2-q^2}(-1)^q\widetilde{\nu}\Big(\frac{z+s}{\varepsilon}\Big)a_q(z+s). \tag{6.26}$$

We can write this in terms of the intermediate operator $\mathcal{T}$

$$[\widetilde{K}\boldsymbol{a}]_j(z) = \frac{1_I(z)}{\varepsilon}(-1)^j[\mathcal{T}(-1)^q\widetilde{\nu}(\cdot/\varepsilon)a_q 1_{\{q>\mathcal{N}\}}]_j(z). \tag{6.27}$$

Then

$$\|[\widetilde{K}\boldsymbol{a}]_j\|_{L^2(\mathbb{R})} \le \frac{2}{j\widetilde{C}}\|[T(-1)^q\widetilde{\nu}(\cdot/\varepsilon)a_q 1_{\{q>\mathcal{N}\}}]_j\|_{L^2(\mathbb{R})} \tag{6.28}$$

and so

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} j^2\|[\widetilde{K}\boldsymbol{a}]_j\|^2_{L^2(\mathbb{R})} &\le \frac{4}{\widetilde{C}^2}\sum_{j\in\mathbb{Z}}\|[T(-1)^q\widetilde{\nu}(\cdot/\varepsilon)a_q 1_{\{q>\mathcal{N}\}}]_j\|^2_{L^2(\mathbb{R})}\\
&= \frac{4}{\widetilde{C}^2}\int_{-\infty}^{\infty} dz \sum_{j\in\mathbb{Z}}\left|[T(-1)^q\widetilde{\nu}(z/\varepsilon)a_q(z)1_{\{q>\mathcal{N}\}}]_j\right|^2\\
&\le \frac{4}{\widetilde{C}^2}\Big(\frac{1}{2}+\pi\Big)^2\int_{-\infty}^{\infty} dz \sum_{j\in\mathbb{Z}}\left|\widetilde{\nu}(z/\varepsilon)ja_j(z)1_{\{j>\mathcal{N}\}}\right|^2\\
&= \frac{4}{\widetilde{C}^2}\Big(\frac{1}{2}+\pi\Big)^2\sum_{j\in\mathbb{Z}} j^2\|\widetilde{\nu}(\cdot/\varepsilon)a_j 1_{\{j>\mathcal{N}\}}\|^2_{L^2(\mathbb{R})}\\
&\le \frac{4}{\widetilde{C}^2}\Big(\frac{1}{2}+\pi\Big)^2\|\widetilde{\nu}\|^2_{L^\infty(\mathbb{R})}\sum_{j\in\mathbb{Z}} j^2\|a_j\|^2_{L^2(\mathbb{R})}
\end{aligned} \tag{6.29}$$

### 6.3 Proof of Boundedness

We will show that $\mathbf{K}$ is bounded in the Banach space $X := \ell^2(L^2(\mathbb{R};\mathbb{R}^2), w)$, that is the space of square summable sequences of the $L^2(\mathbb{R})$ functions taking values in

$\mathbb{R}^2$ with linear weights $w := \{j\}_{j\in\mathbb{Z}}$, equipped with the norm

$$\|\boldsymbol{V}\|_X := \Big(\sum_{j\in\mathbb{Z}} j^2(\|V_j^{(1)}\|^2_{L^2(\mathbb{R})} + \|V_j^{(2)}\|^2_{L^2(\mathbb{R})})\Big)^{1/2}. \tag{6.30}$$

**Lemma VI.1.** *The operator given component-wise by*

$$\begin{aligned}[\mathbf{KV}]_j(\omega, z) = &-\frac{1_I(z)}{2\beta_j(\omega,z)}\int_{-\infty}^{\infty} d\xi e^{-\beta_j(\omega,z)|\xi|} \\ &\times \left[\begin{pmatrix} 1 \\ \varepsilon\partial_z \end{pmatrix} \sqrt{\varepsilon}\mathcal{C}_j^{(e)}(\omega, z+\varepsilon\xi) - \sigma\mu_j^2(z)\nu\Big(\frac{z}{\varepsilon}+\xi\Big)\boldsymbol{V}_j\right],\end{aligned} \tag{6.31}$$

*is bounded in the space $X$.*

*Proof.* We begin with a use of the triangle inequality

$$\|[\mathbf{KV}]_j^{(2)}\|_{L^2(\mathbb{R})} \le \|[\mathbf{KV}]_j^{(2,1)}\|_{L^2(\mathbb{R})} + \|[\mathbf{KV}]_j^{(2,2)}\|_{L^2(\mathbb{R})} + \|[\mathbf{KV}]_j^{(2,3)}\|_{L^2(\mathbb{R})}. \tag{6.32}$$

We then square both sides and use an elementary inequality to obtain

$$\|[\mathbf{KV}]_j^{(2)}\|^2_{L^2(\mathbb{R})} \le 4\big(\|[\mathbf{KV}]_j^{(2,1)}\|^2_{L^2(\mathbb{R})} + \|[\mathbf{KV}]_j^{(2,2)}\|^2_{L^2(\mathbb{R})} + \|[\mathbf{KV}]_j^{(2,3)}\|^2_{L^2(\mathbb{R})}\big). \tag{6.33}$$

Multiplying by $j^2$ and summing over $j$ we obtain

$$\begin{aligned}\sum_{j\in\mathbb{Z}} j^2\|[\mathbf{KV}]_j^{(2)}\|^2_{L^2(\mathbb{R})} \le 4\big(&\sum_{j\in\mathbb{Z}} j^2\|[\mathbf{KV}]_j^{(2,1)}\|^2_{L^2(\mathbb{R})} \\ &+ \sum_{j\in\mathbb{Z}} j^2\|[\mathbf{KV}]_j^{(2,2)}\|^2_{L^2(\mathbb{R})} + \sum_{j\in\mathbb{Z}} j^2\|[\mathbf{KV}]_j^{(2,3)}\|^2_{L^2(\mathbb{R})}\big).\end{aligned} \tag{6.34}$$

Repeating this argument also gets us

$$\begin{aligned}\sum_{j\in\mathbb{Z}} j^2\|[\mathbf{KV}]_j^{(1)}\|^2_{L^2(\mathbb{R})} \le 4\big(&\sum_{j\in\mathbb{Z}} j^2\|[\mathbf{KV}]_j^{(1,1)}\|^2_{L^2(\mathbb{R})} \\ &+ \sum_{j\in\mathbb{Z}} j^2\|[\mathbf{KV}]_j^{(1,2)}\|^2_{L^2(\mathbb{R})} + \sum_{j\in\mathbb{Z}} j^2\|[\mathbf{KV}]_j^{(1,3)}\|^2_{L^2(\mathbb{R})}\big).\end{aligned} \tag{6.35}$$

We then bound each term on the right-hand side of (6.34) separately. Using (6.29) we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}} j^2\|[\mathbf{KV}]_j^{(2,1)}\|^2_{L^2(\mathbb{R})} &\le \sum_{j\in\mathbb{Z}} j^2\|\frac{\sigma 1_I(\cdot)}{2\varepsilon}(-1)^j[\mathcal{T}(-1)^q\nu''(\cdot/\varepsilon)V_q^{(1)}1_{\{q>\mathcal{N}\}}]_j\|^2_{L^2(\mathbb{R})}\\ &\le C_1\sum_{j\in\mathbb{Z}} j^2\|V_j^{(1)}\|^2_{L^2(\mathbb{R})},\end{aligned} \tag{6.36}$$

where

$$C_1 := \frac{\sigma^2}{\widetilde{C}^2}\Big(\frac{1}{2}+\pi\Big)^2\|\nu\|^2_{W^{2,\infty}(\mathbb{R})}. \tag{6.37}$$

Similarly, again using (6.29) for the second term

$$\begin{aligned}\sum_{j\in\mathbb{Z}} j^2\|[\mathbf{KV}]_j^{(2,2)}\|^2_{L^2(\mathbb{R})} &\le \sum_{j\in\mathbb{Z}} j^2\|\frac{\sigma 1_I(\cdot)}{\varepsilon}(-1)^j[\mathcal{T}(-1)^q\nu'(\cdot/\varepsilon)V_q^{(2)}1_{\{q>\mathcal{N}\}}]_j(z)\|^2_{L^2(\mathbb{R})}\\ &\le C_2\sum_{j\in\mathbb{Z}} j^2\|V_j^{(2)}\|^2_{L^2(\mathbb{R})},\end{aligned} \tag{6.38}$$

where

$$C_2 := \frac{4\sigma^2}{\widetilde{C}^2}\Big(\frac{1}{2}+\pi\Big)^2\|\nu\|^2_{W^{1,\infty}(\mathbb{R})}. \tag{6.39}$$

As for the last term, we use Young's inequality to show

$$\begin{aligned}\|[\mathbf{KV}]_j^{(2,3)}\|_{L^2(\mathbb{R})} &\le \frac{\sigma\pi^2 j^2}{2\varepsilon\widetilde{\beta}_j\widetilde{D}^2}\|e^{-\widetilde{\beta}_j|s|/\varepsilon} * |\nu(\cdot/\varepsilon)V_j^{(2)}|\|_{L^2(\mathbb{R})}\\ &\le \frac{\sigma\pi^2 j^2}{2\varepsilon\widetilde{\beta}_j\widetilde{D}^2}\|e^{-\widetilde{\beta}_j|\cdot|/\varepsilon}\|_{L^1(\mathbb{R})}\|\nu(\cdot/\varepsilon)V_j^{(2)}\|_{L^2(\mathbb{R})}\\ &\le \frac{\sigma\pi^2}{\widetilde{C}^2\widetilde{D}^2}\|\nu(\cdot/\varepsilon)V_j^{(2)}\|_{L^2(\mathbb{R})}\\ &\le \frac{\sigma\pi^2}{\widetilde{C}^2\widetilde{D}^2}\|\nu\|_{L^\infty(\mathbb{R})}\|V_j^{(2)}\|_{L^2(\mathbb{R})}.\end{aligned} \tag{6.40}$$

Then multiplying by $j$, squaring both sides, and summing over $j$ yields

$$\sum_{j\in\mathbb{Z}} j^2\|[\mathbf{KV}]_j^{(2,3)}\|^2_{L^2(\mathbb{R})} \le C_3\sum_{j\in\mathbb{Z}} j^2\|V_j^{(2)}\|^2_{L^2(\mathbb{R})}. \tag{6.41}$$

where

$$C_3 := \frac{\sigma^2\pi^4}{\widetilde{C}^4\widetilde{D}^4}\|\nu\|^2_{L^\infty(\mathbb{R})}. \tag{6.42}$$

We now return to (6.34) and use (6.36), (6.38), and (6.41) to obtain

$$\sum_{j\in\mathbb{Z}} j^2\|[\mathbf{KV}]_j^{(2)}\|_{L^2(\mathbb{R})}^2 \leq 4C_1\sum_{j\in\mathbb{Z}} j^2\|V_j^{(1)}\|_{L^2(\mathbb{R})}^2 + 4(C_2+C_3)\sum_{j\in\mathbb{Z}} j^2\|V_j^{(2)}\|_{L^2(\mathbb{R})}^2 \quad (6.43)$$

We can reuse many of these estimates to bound the terms on the right-hand side of (6.35). We note that

$$\begin{aligned}\|[\mathbf{KV}]_j^{(1,1)}\|_{L^2(\mathbb{R})} &= \|\frac{\sigma 1_I(\cdot)}{2\varepsilon\beta_j(\cdot)}(-1)^j[\mathcal{T}(-1)^q\nu''(\cdot/\varepsilon)V_q^{(1)}1_{\{q>\mathcal{N}\}}]_j\|_{L^2(\mathbb{R})} \\ &\leq \frac{1}{\widetilde{\beta}_j}\|\frac{\sigma 1_I(\cdot)}{2\varepsilon}(-1)^j[\mathcal{T}(-1)^q\nu''(\cdot/\varepsilon)V_q^{(1)}1_{\{q>\mathcal{N}\}}]_j\|_{L^2(\mathbb{R})} \\ &\leq \|\frac{\sigma 1_I(\cdot)}{2\varepsilon}(-1)^j[\mathcal{T}(-1)^q\nu''(\cdot/\varepsilon)V_q^{(1)}1_{\{q>\mathcal{N}\}}]_j\|_{L^2(\mathbb{R})} \qquad (6.44)\end{aligned}$$

Then we have

$$\sum_{j\in\mathbb{Z}} j^2\|[\mathbf{KV}]_j^{(1,1)}\|_{L^2(\mathbb{R})}^2 \leq C_1\sum_{j\in\mathbb{Z}} j^2\|V_j^{(1)}\|_{L^2(\mathbb{R})}^2, \qquad (6.45)$$

and nearly the same argument gets us that

$$\sum_{j\in\mathbb{Z}} j^2\|[\mathbf{KV}]_j^{(1,2)}\|_{L^2(\mathbb{R})}^2 \leq C_2\sum_{j\in\mathbb{Z}} j^2\|V_j^{(2)}\|_{L^2(\mathbb{R})}^2. \qquad (6.46)$$

Finally, we can estimate $[\mathbf{KV}]_j^{(1,3)}$ in the same manner as $[\mathbf{KV}]_j^{(2,3)}$ to obtain

$$\sum_{j\in\mathbb{Z}} j^2\|[\mathbf{KV}]_j^{(1,3)}\|_{L^2(\mathbb{R})}^2 \leq C_3\sum_{j\in\mathbb{Z}} j^2\|V_j^{(1)}\|_{L^2(\mathbb{R})}^2. \qquad (6.47)$$

Returning to (6.35) and using (6.45), (6.46), and (6.47) yields

$$\sum_{j\in\mathbb{Z}} j^2\|[\mathbf{KV}]_j^{(1)}\|_{L^2(\mathbb{R})}^2 \leq 4(C_1+C_3)\sum_{j\in\mathbb{Z}} j^2\|V_j^{(1)}\|_{L^2(\mathbb{R})}^2 + 4C_2\sum_{j\in\mathbb{Z}} j^2\|V_j^{(2)}\|_{L^2(\mathbb{R})}^2 \quad (6.48)$$

We have using (6.43) and (6.48)

$$\begin{aligned}\|\mathbf{KV}\|_X^2 &= \sum_{j\in\mathbb{Z}} j^2(\|[\mathbf{KV}]_j^{(1)}\|_{L^2(\mathbb{R})}^2 + \|[\mathbf{KV}]_j^{(2)}\|_{L^2(\mathbb{R})}^2) \\ &\leq 4(2C_1+C_3)\sum_{j\in\mathbb{Z}} j^2\|V_j^{(1)}\|_{L^2(\mathbb{R})}^2 + 4(2C_2+C_3)\sum_{j\in\mathbb{Z}} j^2\|V_j^{(2)}\|_{L^2(\mathbb{R})}^2 \\ &\leq C^2\sum_{j\in\mathbb{Z}} j^2(\|V_j^{(1)}\|_{L^2(\mathbb{R})}^2 + \|V_j^{(2)}\|_{L^2(\mathbb{R})}^2) \qquad (6.49)\end{aligned}$$

where

$$C := \sqrt{\max\{4(2C_1 + C_3), 4(2C_2 + C_3)\}}. \tag{6.50}$$

Thus, we have

$$\|\mathbf{KV}\|_X \leq C\|\boldsymbol{V}\|_X \tag{6.51}$$

□

# CHAPTER VII

## Conclusion

We considered time-harmonic sound waves emitted by a point source in a two-dimensional random waveguide with turning points. The waveguide has sound-soft boundaries, a slowly bending axis and variable cross-section. The variation consists of a slow and monotone change of the opening $D$ of the waveguide, and small amplitude random fluctuations of the boundary. The slow variations are on a long scale with respect to the wavelength $\lambda$, whereas the random fluctuations are on a scale comparable to $\lambda$. The wavelength $\lambda$ is chosen smaller than $D$, so that the wavefield is a superposition of multiple propagating modes, and infinitely many evanescent modes. The turning points are the locations along the axis of the waveguide where the number of propagating modes decreases by 1 in the direction of decrease of $D$, or increases by 1 in the direction of increase of $D$. The change in the number of propagating modes means that there are modes that transition from propagating to evanescent. Due to energy conservation, the incoming such waves are turned back i.e., they are reflected at the turning points.

We analyzed the transmitted and reflected propagating modes in the waveguide and quantified their interaction with the random boundary. This interaction is called cumulative scattering and it manifests as mode coupling which causes randomization of the wavefield and exchange of power between the modes. We analyzed these effects from first principles, starting from the wave equation, using stochastic asymptotic analysis. We focused attention on the transport of power in the waveguide and showed that cumulative scattering may increase or decrease the transmitted power depending on the source.

One could apply this work to the study of inverse problems in random waveguides. In this application the goal is to determine the waveguide geometry and scatterers from sensor array measurements. Techniques for this have already been developed in [27, 13, 12, 1] for random waveguides with straight walls.

# APPENDICES

# APPENDIX A

# Markov Diffusions and the Kolmogorov Backward Equation

Some of the content of chapters III - V requires a small amount of stochastic analysis which we review in this section. The material as stated here is primarily sourced from [26, Chapter 6]. More detailed accounts are given in [49, 22].

## A.1 Infinitesimal Generators

The infinitesimal generator of a Markov diffusion process $X(z)$ is a partial differential operator defined by

$$\mathcal{L}f(x) := \lim_{h\to 0} \frac{\mathbb{E}[f(X(h)|X(0) = x] - f(x)}{h} \tag{A.1}$$

where $f$ is an appropriate test function for which the limit on the right-hand side is defined. The theorem in chapter V gives us an explicit expression for the infinitesimal generator for the limit complex mode amplitude diffusion processes for the problem in chapter III. The generator encodes statistical information about the process $X$ which one can access through the Kolmogorov backward equations described in the next section.

## A.2 Kolmogorov Backward Equation

Let $X(z)$ be a Markov diffusion process in $\mathbb{R}^n$ with infinitesimal generator $\mathcal{L}$ whose coefficients are smooth and let $Z \in \mathbb{R}$ be given. We have that $u(z, x) :=$

$\mathbb{E}[f(X(Z))|X(z) = x]$ where $f$ is a bounded smooth function will solve

$$\partial_z u(z,x) + \mathcal{L}u(z,x) = 0, \quad z < Z, \tag{A.2}$$

$$u(Z,x) = f(x). \tag{A.3}$$

The equation above is the Kolmogorov backward equation so called because it is solved backward in $z$ from $Z$. The requirement that (A.3) hold is referred to as a terminal condition.

In the case where $X(z)$ is a homogeneous Markov diffusion process we can make a change of variables $z' = Z - z$ and obtain an initial value problem

$$\partial_{z'} u(z',x) = \mathcal{L}u(z',x), \quad z' > 0, \tag{A.4}$$

$$u(0,x) = f(x). \tag{A.5}$$

We will use this version of the backward equation to obtain differential equations for moments of the limit complex mode amplitude diffusion processes in section IV. Though the process we consider in that case is not homogeneous we can instead consider $(z, X(z))$ as a process with state space $\mathbb{R}^{n+1}$, which will be homogeneous.

# APPENDIX B

# Computation of the Infinitesimal Generator

Here we give a detailed computation of the infinitesimal generator of the limit complex mode amplitude process $\boldsymbol{b}(\omega, z) \in \mathbb{C}^{\mathcal{N}}$ in the sector of the guide $z \in (z_-^{(t)}, z_-^{(t-1)})$. We split the generator computation into leading order and second order terms. We then combine them and rewrite the generator in polar coordinates.

## B.1 Real-Valued System for the Mode Amplitudes

Let $F_j$ and $G_j$ are the $j$-th components of the $O(1/\sqrt{\varepsilon})$ and $O(1)$ terms of $\boldsymbol{\Upsilon}^{\varepsilon(bb)}(\omega, z)\boldsymbol{b}^{\varepsilon}(\omega, z)$, respectively. After applying the forward scattering approximation, we have that $b_j^{\varepsilon}(\omega, z)$ satisfies

$$\partial_z b_j^{\varepsilon}(\omega, z) = \frac{1}{\sqrt{\varepsilon}} F_j(\boldsymbol{b}^{\varepsilon}(\omega, z), \nu^{\varepsilon}(z), \boldsymbol{\theta}^{\varepsilon}(z), z) + G_j(\boldsymbol{b}^{\varepsilon}(\omega, z), \nu^{\varepsilon}(z), \boldsymbol{\theta}^{\varepsilon}(z), z), \tag{B.1}$$

for $z \in (z_-^{(t)}, z_-^{(t-1)})$, and $j = 1, \ldots, \mathcal{N}$ where

$$\nu^{\varepsilon}(z) := \nu\left(\frac{z}{\varepsilon}\right), \tag{B.2}$$

$$\theta_j^{\varepsilon}(z) := \frac{1}{\varepsilon}\int_0^z ds \beta_j(\omega, s). \tag{B.3}$$

To apply the theorem of chapter V we have to rewrite the system above in terms of real-valued quantities.

We can rewrite the system above as

$$\partial_z \tilde{\mathbf{b}}^{\varepsilon} = \frac{1}{\sqrt{\varepsilon}} \tilde{\mathbf{F}}\left(\mathbf{b}^{\varepsilon}, \nu^{\varepsilon}, \boldsymbol{\theta}^{\varepsilon}, z\right) + \tilde{\mathbf{G}}\left(\mathbf{b}^{\varepsilon}, \nu^{\varepsilon}, \boldsymbol{\theta}^{\varepsilon}, z\right) \tag{B.4}$$

where

$$\tilde{\mathbf{b}}^{\varepsilon} := \left[\mathbf{b}^{\varepsilon,R}, \mathbf{b}^{\varepsilon,I}\right]^T, \ \tilde{\mathbf{F}} := \left[\mathbf{F}^R, \mathbf{F}^I\right]^T, \ \tilde{\mathbf{G}} := \left[\mathbf{G}^R, \mathbf{G}^I\right]^T, \tag{B.5}$$

and

$$\mathbf{b}^{\varepsilon,R} := \mathrm{Re}(\mathbf{b}^{\varepsilon}), \ \mathbf{F}^R := \mathrm{Re}(\mathbf{F}), \ \mathbf{G}^R := \mathrm{Re}(\mathbf{G}), \tag{B.6}$$

$$\mathbf{b}^{\varepsilon,I} := \mathrm{Im}(\mathbf{b}^{\varepsilon}), \ \mathbf{F}^I := \mathrm{Im}(\mathbf{F}), \ \mathbf{G}^I := \mathrm{Im}(\mathbf{G}). \tag{B.7}$$

We may now apply the theorem of chapter V to (B.4) to compute the infinitesimal generator of $\boldsymbol{b}(\omega, z)$. We will compute the generator in two parts splitting up the first and second order terms which we will denote by $\mathcal{L}_1$, $\mathcal{L}_2$, respectively. We note that in what follows if a sum is written without specifying the starting and final indices one should assume it is from 1 to $\mathcal{N}$.

## B.2 First Order Terms

The first order terms of the generator are given by

$$\mathcal{L}_1 := \sum_{j,j'} \lim_{L\to\infty} \frac{1}{L} \int_0^L \int_0^\infty ds d\zeta \mathbb{E}\Big[\sum_{k=1}^{8} F_{jj'}^{(k)}\Big] \tag{B.8}$$

where

$$F_{jj'}^{(1)} := F_j^R(\boldsymbol{b}, \nu(0), \boldsymbol{\theta} + \boldsymbol{\beta}s, z) F_{j'}^R(\boldsymbol{b}, \nu(\zeta), \boldsymbol{\theta} + \boldsymbol{\beta}(s+\zeta), z)] \partial^2_{b_j^R, b_{j'}^R} \tag{B.9}$$

$$F_{jj'}^{(2)} := F_j^R(\boldsymbol{b}, \nu(0), \boldsymbol{\theta} + \boldsymbol{\beta}s, z) F_{j'}^I(\boldsymbol{b}, \nu(\zeta), \boldsymbol{\theta} + \boldsymbol{\beta}(s+\zeta), z)] \partial^2_{b_j^R, b_{j'}^I} \tag{B.10}$$

$$F_{jj'}^{(3)} := F_j^I(\boldsymbol{b}, \nu(0), \boldsymbol{\theta} + \boldsymbol{\beta}s, z) F_{j'}^R(\boldsymbol{b}, \nu(\zeta), \boldsymbol{\theta} + \boldsymbol{\beta}(s+\zeta), z)] \partial^2_{b_j^I, b_{j'}^R} \tag{B.11}$$

$$F_{jj'}^{(4)} := F_j^I(\boldsymbol{b}, \nu(0), \boldsymbol{\theta} + \boldsymbol{\beta}s, z) F_{j'}^I(\boldsymbol{b}, \nu(\zeta), \boldsymbol{\theta} + \boldsymbol{\beta}(s+\zeta), z)] \partial^2_{b_j^I, b_{j'}^I} \tag{B.12}$$

$$F_{jj'}^{(5)} := F_j^R(\boldsymbol{b}, \nu(0), \boldsymbol{\theta} + \boldsymbol{\beta}s, z) \partial_{b_j^R} F_{j'}^R(\boldsymbol{b}, \nu(\zeta), \boldsymbol{\theta} + \boldsymbol{\beta}(s+\zeta), z) \partial_{b_{j'}^R} \tag{B.13}$$

$$F_{jj'}^{(6)} := F_j^R(\boldsymbol{b}, \nu(0), \boldsymbol{\theta} + \boldsymbol{\beta}s, z) \partial_{b_j^R} F_{j'}^I(\boldsymbol{b}, \nu(\zeta), \boldsymbol{\theta} + \boldsymbol{\beta}(s+\zeta), z) \partial_{b_{j'}^I} \tag{B.14}$$

$$F_{jj'}^{(7)} := F_j^I(\boldsymbol{b}, \nu(0), \boldsymbol{\theta} + \boldsymbol{\beta}s, z) \partial_{b_j^I} F_{j'}^R(\boldsymbol{b}, \nu(\zeta), \boldsymbol{\theta} + \boldsymbol{\beta}(s+\zeta), z) \partial_{b_{j'}^R} \tag{B.15}$$

$$F_{jj'}^{(8)} := F_j^I(\boldsymbol{b}, \nu(0), \boldsymbol{\theta} + \boldsymbol{\beta}s, z) \partial_{b_j^I} F_{j'}^I(\boldsymbol{b}, \nu(\zeta), \boldsymbol{\theta} + \boldsymbol{\beta}(s+\zeta), z) \partial_{b_{j'}^I} \tag{B.16}$$

Towards computing the $F_{jj'}^{(k)}$ we need

$$F_j^R(b,\nu,\theta,z) = \frac{\sigma}{2\beta_j}\mu_j^2\nu b_j^I + \sum_{q\neq j}\frac{1}{2\sqrt{\beta_j\beta_q}}\left(C_{jq}^{R,c}\cos(\theta_{jq}) + C_{jq}^{R,s}\sin(\theta_{jq})\right) \tag{B.17}$$

$$F_j^I(b,\nu,\theta,z) = -\frac{\sigma}{2\beta_j}\mu_j^2\nu b_j^R + \sum_{q\neq j}\frac{1}{2\sqrt{\beta_j\beta_q}}\left(C_{jq}^{I,c}\cos(\theta_{jq}) + C_{jq}^{I,s}\sin(\theta_{jq})\right) \tag{B.18}$$

where

$$C_{jq}^{R,c}(\zeta) := -C_{jq}^R(\zeta)b_q^I - C_{jq}^I(\zeta)b_q^R, \tag{B.19}$$

$$C_{jq}^{R,s}(\zeta) := C_{jq}^R(\zeta)b_q^R - C_{jq}^I(\zeta)b_q^I, \tag{B.20}$$

$$C_{jq}^{I,c}(\zeta) := C_{jq}^R(\zeta)b_q^R - C_{jq}^I(\zeta)b_q^I, \tag{B.21}$$

$$C_{jq}^{I,s}(\zeta) := C_{jq}^R(\zeta)b_q^I + C_{jq}^I(\zeta)b_q^R, \tag{B.22}$$

$$C_{jq}^R(\zeta) := \sigma\Gamma_{jq}\nu''(\zeta), \tag{B.23}$$

$$C_{jq}^I(\zeta) := -\sigma\beta_q\Theta_{jq}\nu'(\zeta), \tag{B.24}$$

and

$$\theta_{jq}(z) := \int_0^z ds\beta_q(\omega,s) - \beta_j(\omega,s). \tag{B.25}$$

We also define the following notation

$$\theta_{jq}^{(1)} := \theta_{jq} + (\beta_q - \beta_j)s \tag{B.26}$$

$$\theta_{jq}^{(2)} := \theta_{jq} + (\beta_q - \beta_j)(s+\zeta) \tag{B.27}$$

which will simply be used to keep our expressions from becoming too long.

We begin with computing $F^{(1)}_{jj'}$ through $F^{(4)}_{jj'}$. We have

$$\begin{aligned} F^{(1)}_{jj'} = \Bigg[ & \frac{\sigma^2}{4\beta_j\beta_{j'}}\mu_j^2\mu_{j'}^2\nu(0)\nu(\zeta)b^I_j b^I_{j'} \\ & + \frac{\sigma}{2\beta_j}\mu_j^2\nu(0)b^I_j \sum_{q'\neq j'} \frac{1}{2\sqrt{\beta_{j'}\beta_{q'}}}\left(C^{R,c}_{j'q'}(\zeta)\cos(\theta^{(2)}_{j'q'}) + C^{R,s}_{j'q'}(\zeta)\sin(\theta^{(2)}_{j'q'})\right) \\ & + \frac{\sigma}{2\beta_{j'}}\mu_{j'}^2\nu(\zeta)b^I_{j'} \sum_{q\neq j} \frac{1}{2\sqrt{\beta_j\beta_q}}\left(C^{R,c}_{jq}(0)\cos(\theta^{(1)}_{jq}) + C^{R,s}_{jq}(0)\sin(\theta^{(1)}_{jq})\right) \\ & + \sum_{q\neq j, q'\neq j'} \frac{1}{4\sqrt{\beta_j\beta_q\beta_{j'}\beta_{q'}}}\Big[C^{R,c}_{jq}(0)C^{R,c}_{j'q'}(\zeta)\cos(\theta^{(1)}_{jq})\cos(\theta^{(2)}_{j'q'}) \\ & + C^{R,s}_{jq}(0)C^{R,s}_{j'q'}(\zeta)\sin(\theta^{(1)}_{jq})\sin(\theta^{(2)}_{j'q'}) + C^{R,s}_{jq}(0)C^{R,c}_{j'q'}(\zeta)\sin(\theta^{(1)}_{jq})\cos(\theta^{(2)}_{j'q'}) \\ & + C^{R,c}_{jq}(0)C^{R,s}_{j'q'}(\zeta)\cos(\theta^{(1)}_{jq})\sin(\theta^{(2)}_{j'q'})\Big]\Bigg]\partial^2_{b^R_j, b^R_{j'}} \end{aligned} \tag{B.28}$$

$$\begin{aligned} F^{(2)}_{jj'} = \Bigg[ & -\frac{\sigma^2}{4\beta_j\beta_{j'}}\mu_j^2\mu_{j'}^2\nu(0)\nu(\zeta)b^R_j b^I_{j'} \\ & + \frac{\sigma}{2\beta_j}\mu_j^2\nu(0)b^I_j \sum_{q'\neq j'} \frac{1}{2\sqrt{\beta_{j'}\beta_{q'}}}\left(C^{I,c}_{j'q'}(\zeta)\cos(\theta^{(2)}_{j'q'}) + C^{I,s}_{j'q'}(\zeta)\sin(\theta^{(2)}_{j'q'})\right) \\ & - \frac{\sigma}{2\beta_{j'}}\mu_{j'}^2\nu(\zeta)b^R_{j'} \sum_{q\neq j} \frac{1}{2\sqrt{\beta_j\beta_q}}\left(C^{R,c}_{jq}(0)\cos(\theta^{(1)}_{jq}) + C^{R,s}_{jq}(0)\sin(\theta^{(1)}_{jq})\right) \\ & + \sum_{q\neq j, q'\neq j'} \frac{1}{4\sqrt{\beta_j\beta_q\beta_{j'}\beta_{q'}}}\Big[C^{R,c}_{jq}(0)C^{I,c}_{j'q'}(\zeta)\cos(\theta^{(1)}_{jq})\cos(\theta^{(2)}_{j'q'}) \\ & + C^{R,s}_{jq}(0)C^{I,s}_{j'q'}(\zeta)\sin(\theta^{(1)}_{jq})\sin(\theta^{(2)}_{j'q'}) + C^{R,s}_{jq}(0)C^{I,c}_{j'q'}(\zeta)\sin(\theta^{(1)}_{jq})\cos(\theta^{(2)}_{j'q'}) \\ & + C^{R,c}_{jq}(0)C^{I,s}_{j'q'}(\zeta)\cos(\theta^{(1)}_{jq})\sin(\theta^{(2)}_{j'q'})\Big]\Bigg]\partial^2_{b^R_j, b^I_{j'}} \end{aligned} \tag{B.29}$$

$$F^{(3)}_{jj'} = \Bigg[ -\frac{\sigma^2}{4\beta_j\beta_{j'}}\mu_j^2\mu_{j'}^2\nu(0)\nu(\zeta)b_j^R b_{j'}^I$$

$$-\frac{\sigma}{2\beta_j}\mu_j^2\nu(0)b_j^R\sum_{q'\neq j'}\frac{1}{2\sqrt{\beta_{j'}\beta_{q'}}}\left(C^{R,c}_{j'q'}(\zeta)\cos(\theta^{(2)}_{j'q'})+C^{R,s}_{j'q'}(\zeta)\sin(\theta^{(2)}_{j'q'})\right)$$

$$+\frac{\sigma}{2\beta_{j'}}\mu_{j'}^2\nu(\zeta)b_{j'}^I\sum_{q\neq j}\frac{1}{2\sqrt{\beta_j\beta_q}}\left(C^{I,c}_{jq}(0)\cos(\theta^{(1)}_{jq})+C^{I,s}_{jq}(0)\sin(\theta^{(1)}_{jq})\right)$$

$$+\sum_{q\neq j,q'\neq j'}\frac{1}{4\sqrt{\beta_j\beta_q\beta_{j'}\beta_{q'}}}\Big[C^{I,c}_{jq}(0)C^{R,c}_{j'q'}(\zeta)\cos(\theta^{(1)}_{jq})\cos(\theta^{(2)}_{j'q'})$$

$$+C^{I,s}_{jq}(0)C^{R,s}_{j'q'}(\zeta)\sin(\theta^{(1)}_{jq})\sin(\theta^{(2)}_{j'q'})+C^{I,s}_{jq}(0)C^{R,c}_{j'q'}(\zeta)\sin(\theta^{(1)}_{jq})\cos(\theta^{(2)}_{j'q'})$$

$$+C^{I,c}_{jq}(0)C^{R,s}_{j'q'}(\zeta)\cos(\theta^{(1)}_{jq})\sin(\theta^{(2)}_{j'q'})\Big]\Bigg]\partial^2_{b_j^I,b_{j'}^R} \tag{B.30}$$

$$F^{(4)}_{jj'} = \Bigg[\frac{\sigma^2}{4\beta_j\beta_{j'}}\mu_j^2\mu_{j'}^2\nu(0)\nu(\zeta)b_j^{\varepsilon,R} b_{j'}^{\varepsilon,R}$$

$$-\frac{\sigma}{2\beta_j}\mu_j^2\nu(0)b_j^R\sum_{q'\neq j'}\frac{1}{2\sqrt{\beta_{j'}\beta_{q'}}}\left(C^{I,c}_{j'q'}(\zeta)\cos(\theta^{(2)}_{j'q'})+C^{I,s}_{j'q'}(\zeta)\sin(\theta^{(2)}_{j'q'})\right)$$

$$-\frac{\sigma}{2\beta_{j'}}\mu_{j'}^2\nu(\zeta)b_{j'}^R\sum_{q\neq j}\frac{1}{2\sqrt{\beta_j\beta_q}}\left(C^{I,c}_{jq}(0)\cos(\theta^{(1)}_{jq})+C^{I,s}_{jq}(0)\sin(\theta^{(1)}_{jq})\right)$$

$$+\sum_{q\neq j,q'\neq j'}\frac{1}{4\sqrt{\beta_j\beta_q\beta_{j'}\beta_{q'}}}\Big[C^{I,c}_{jq}(0)C^{I,c}_{j'q'}(\zeta)\cos(\theta^{(1)}_{jq})\cos(\theta^{(2)}_{j'q'})$$

$$+C^{I,s}_{jq}(0)C^{I,s}_{j'q'}(\zeta)\sin(\theta^{(1)}_{jq})\sin(\theta^{(2)}_{j'q'})+C^{I,s}_{jq}(0)C^{I,c}_{j'q'}(\zeta)\sin(\theta^{(1)}_{jq})\cos(\theta^{(2)}_{j'q'})$$

$$+C^{I,c}_{jq}(0)C^{I,s}_{j'q'}(\zeta)\cos(\theta^{(1)}_{jq})\sin(\theta^{(2)}_{j'q'})\Big]\Bigg]\partial^2_{b_j^I,b_{j'}^I} \tag{B.31}$$

To compute the terms $F^{(5)}_{jj'}$ through $F^{(8)}_{jj'}$, we will need

$$\partial_{b^R_j} F^R_{j'}(b, \nu(\zeta), \theta + \beta(s+\zeta), z) = \frac{1}{2\sqrt{\beta_j \beta'_j}} \Big[ - C^I_{j'j}(\zeta) \cos(\theta^{(2)}_{j'j}) + C^R_{j'j}(\zeta) \sin(\theta^{(2)}_{j'j}) \Big] \tag{B.32}$$

$$\partial_{b^I_j} F^R_{j'}(b, \nu(\zeta), \theta + \beta(s+\zeta), z) = \frac{\sigma}{2\beta_j} \mu_j^2 \nu(\zeta) \delta_{jj'} + \frac{1}{2\sqrt{\beta_j \beta'_j}} \Big[ - C^R_{j'j}(\zeta) \cos(\theta^{(2)}_{j'j}) - C^I_{j'j}(\zeta) \sin(\theta^{(2)}_{j'j}) \Big] \tag{B.33}$$

$$\partial_{b^R_j} F^I_{j'}(b, \nu(\zeta), \theta + \beta(s+\zeta), z) = -\frac{\sigma}{2\beta_j} \mu_j^2 \nu(\zeta) \delta_{jj'} + \frac{1}{2\sqrt{\beta_j \beta'_j}} \Big[ C^R_{j'j}(\zeta) \cos(\theta^{(2)}_{j'j}) + C^I_{j'j}(\zeta) \sin(\theta^{(2)}_{j'j}) \Big] \tag{B.34}$$

$$\partial_{b^I_j} F^I_{j'}(b, \nu(\zeta), \theta + \beta(s+\zeta), z) = \frac{1}{2\sqrt{\beta_j \beta'_j}} \Big[ - C^I_{j'j}(\zeta) \cos(\theta^{(2)}_{j'j}) + C^R_{j'j}(\zeta) \sin(\theta^{(2)}_{j'j}) \Big] \tag{B.35}$$

Then $F^{(5)}_{jj'}$ through $F^{(8)}_{jj'}$ are given by

$$\begin{aligned} F^{(5)}_{jj'} = \Bigg[ & \frac{\sigma}{4\sqrt{\beta_j^3 \beta_{j'}}} \mu_j^2 \nu(0) b^I_j \Big( - C^I_{j'j}(\zeta) \cos(\theta^{(2)}_{j'j}) + C^R_{j'j}(\zeta) \sin(\theta^{(2)}_{j'j}) \Big) \\ & + \sum_{q \neq j} \frac{1}{4\sqrt{\beta_j^2 \beta_q \beta_{j'}}} \Big[ -C^{R,c}_{jq}(0) C^I_{j'j}(\zeta) \cos(\theta^{(1)}_{jq}) \cos(\theta^{(2)}_{j'j}) \\ & + C^{R,s}_{jq}(0) C^R_{j'j}(\zeta) \sin(\theta^{(1)}_{jq}) \sin(\theta^{(2)}_{j'j}) - C^{R,s}_{jq}(0) C^I_{j'j}(\zeta) \sin(\theta^{(1)}_{jq}) \cos(\theta^{(2)}_{j'j}) \\ & + C^{R,c}_{jq}(0) C^R_{j'j}(\zeta) \cos(\theta^{(1)}_{jq}) \sin(\theta^{(2)}_{j'j}) \Big] \Bigg] \partial_{b^R_{j'}} \end{aligned} \tag{B.36}$$

$$F^{(6)}_{jj'} = \Bigg[ -\frac{\sigma^2}{4\beta_j\beta_{j'}}\mu_j^2\mu_{j'}^2\nu(0)\nu(\zeta)b_j^I\delta_{jj'}$$

$$+\frac{\sigma}{4\sqrt{\beta_j^3\beta_j'}}\mu_j^2\nu(0)b_j^I\left(C^R_{j'j}(\zeta)\cos(\theta^{(2)}_{j'j})+C^I_{j'j}(\zeta)\sin(\theta^{(2)}_{j'j})\right)$$

$$-\frac{\sigma}{2\beta_j}\mu_j^2\nu(\zeta)\delta_{j,j'}\sum_{q\neq j}\frac{1}{2\sqrt{\beta_j\beta_q}}\left(C^{R,c}_{jq}(0)\cos(\theta^{(1)}_{jq})+C^{R,s}_{jq}(0)\sin(\theta^{(1)}_{jq})\right)$$

$$+\sum_{q\neq j,}\frac{1}{4\sqrt{\beta_j^2\beta_q\beta_{j'}}}\Big[C^{R,c}_{jq}(0)C^R_{j'j}(\zeta)\cos(\theta^{(1)}_{jq})\cos(\theta^{(2)}_{j'j})$$

$$+C^{R,s}_{jq}(0)C^I_{j'j}(\zeta)\sin(\theta^{(1)}_{jq})\sin(\theta^{(2)}_{j'j})+C^{R,s}_{jq}(0)C^R_{j'j}(\zeta)\sin(\theta^{(1)}_{jq})\cos(\theta^{(2)}_{j'j})$$

$$+C^{R,c}_{jq}(0)C^I_{j'j}(\zeta)\cos(\theta^{(1)}_{jq})\sin(\theta^{(2)}_{j'j})\Big]\Bigg]\partial_{b^I_{j'}} \tag{B.37}$$

$$F^{(7)}_{jj'} = \Bigg[ -\frac{\sigma^2}{4\beta_j\beta_{j'}}\mu_j^2\mu_{j'}^2\nu(0)\nu(\zeta)b_j^R\delta_{jj'}$$

$$+\frac{\sigma}{4\sqrt{\beta_j^3\beta_j'}}\mu_j^2\nu(0)b_j^R\left(C^R_{j'j}(\zeta)\cos(\theta^{(2)}_{j'j})+C^I_{j'j}(\zeta)\sin(\theta^{(2)}_{j'j})\right)$$

$$+\frac{\sigma}{2\beta_j}\mu_j^2\nu(\zeta)\delta_{j,j'}\sum_{q\neq j}\frac{1}{2\sqrt{\beta_j\beta_q}}\left(C^{R,c}_{jq}(0)\cos(\theta^{(1)}_{jq})+C^{R,s}_{jq}(0)\sin(\theta^{(1)}_{jq})\right)$$

$$-\sum_{q\neq j,}\frac{1}{4\sqrt{\beta_j^2\beta_q\beta_{j'}}}\Big[C^{I,c}_{jq}(0)C^R_{j'j}(\zeta)\cos(\theta^{(1)}_{jq})\cos(\theta^{(2)}_{j'j})$$

$$+C^{I,s}_{jq}(0)C^I_{j'j}(\zeta)\sin(\theta^{(1)}_{jq})\sin(\theta^{(2)}_{j'j})+C^{I,s}_{jq}(0)C^R_{j'j}(\zeta)\sin(\theta^{(1)}_{jq})\cos(\theta^{(2)}_{j'j})$$

$$+C^{I,c}_{jq}(0)C^I_{j'j}(\zeta)\cos(\theta^{(1)}_{jq})\sin(\theta^{(2)}_{j'j})\Big]\Bigg]\partial_{b^R_{j'}} \tag{B.38}$$

$$
\begin{aligned}
F^{(8)}_{jj'} = \Bigg[ &- \frac{\sigma}{4\sqrt{\beta_j^3 \beta_{j'}}} \mu_j^2 \nu(0) b_j^R \Big( - C^I_{j'j}(\zeta)\cos(\theta^{(2)}_{j'j}) + C^R_{j'j}(\zeta)\sin(\theta^{(2)}_{j'j})\Big) \\
&+ \sum_{q\neq j} \frac{1}{4\sqrt{\beta_j^2 \beta_q \beta_{j'}}} \Big[ -C^{I,c}_{jq}(0) C^I_{j'j}(\zeta)\cos(\theta^{(1)}_{jq})\cos(\theta^{(2)}_{j'j}) \\
&+ C^{I,s}_{jq}(0) C^R_{j'j}(\zeta)\sin(\theta^{(1)}_{jq})\sin(\theta^{(2)}_{j'j}) - C^{I,s}_{jq}(0) C^I_{j'j}(\zeta)\sin(\theta^{(1)}_{jq})\cos(\theta^{(2)}_{j'j}) \\
&+ C^{I,c}_{jq}(0) C^R_{j'j}(\zeta)\cos(\theta^{(1)}_{jq})\sin(\theta^{(2)}_{j'j}) \Big] \Bigg] \partial_{b^I_{j'}} \qquad \text{(B.39)}
\end{aligned}
$$

We then apply the trigonometric identities

$$2\cos(\alpha_1)\cos(\alpha_2) = \cos(\alpha_1 - \alpha_2) + \cos(\alpha_1 + \alpha_2) \qquad \text{(B.40)}$$

$$2\sin(\alpha_1)\sin(\alpha_2) = \cos(\alpha_1 - \alpha_2) - \cos(\alpha_1 + \alpha_2) \qquad \text{(B.41)}$$

$$2\sin(\alpha_1)\cos(\alpha_2) = \sin(\alpha_1 + \alpha_2) + \sin(\alpha_1 - \alpha_2) \qquad \text{(B.42)}$$

to put the $F^{(k)}_{jj'}$ into a form that we can average with respect to s. We define the notation

$$\theta^{(-)}_{jqj'q'} := \theta^{(1)}_{jq} - \theta^{(2)}_{j'q'} \qquad \text{(B.43)}$$

$$\theta^{(+)}_{jqj'q'} := \theta^{(1)}_{jq} + \theta^{(2)}_{j'q'} \qquad \text{(B.44)}$$

to again keep our expressions from becoming too long. We have

$$
\begin{aligned}
F^{(1)}_{jj'} = \Bigg[ & \frac{\sigma^2}{4\beta_j\beta_{j'}} \mu_j^2 \mu_{j'}^2 \nu(0)\nu(\zeta) b_j^I b_{j'}^I \\
& + \frac{\sigma}{2\beta_j} \mu_j^2 \nu(0) b_j^I \sum_{q' \neq j'} \frac{1}{2\sqrt{\beta_{j'}\beta_{q'}}} \Big( C^{R,c}_{j'q'}(\zeta) \cos(\theta^{(2)}_{j'q'}) + C^{R,s}_{j'q'}(\zeta) \sin(\theta^{(2)}_{j'q'}) \Big) \\
& + \frac{\sigma}{2\beta_{j'}} \mu_{j'}^2 \nu(\zeta) b_{j'}^I \sum_{q \neq j} \frac{1}{2\sqrt{\beta_j\beta_q}} \Big( C^{R,c}_{jq}(0) \cos(\theta^{(1)}_{jq}) + C^{R,s}_{jq}(0) \sin(\theta^{(1)}_{jq}) \Big) \\
& + \sum_{q \neq j, q' \neq j'} \frac{1}{8\sqrt{\beta_j\beta_q\beta_{j'}\beta_{q'}}} \Big[ \Big( C^{R,c}_{jq}(0) C^{R,c}_{j'q'}(\zeta) + C^{R,s}_{jq}(0) C^{R,s}_{j'q'}(\zeta) \Big) \cos(\theta^{(-)}_{jqj'q'}) \\
& + \Big( C^{R,c}_{jq}(0) C^{R,c}_{j'q'}(\zeta) - C^{R,s}_{jq}(0) C^{R,s}_{j'q'}(\zeta) \Big) \cos(\theta^{(+)}_{jqj'q'}) \\
& + \Big( C^{R,s}_{jq}(0) C^{R,c}_{j'q'}(\zeta) + C^{R,c}_{jq}(0) C^{R,s}_{j'q'}(\zeta) \Big) \sin(\theta^{(+)}_{jqj'q'}) \\
& + \Big( C^{R,s}_{jq}(0) C^{R,c}_{j'q'}(\zeta) - C^{R,c}_{jq}(0) C^{R,s}_{j'q'}(\zeta) \Big) \sin(\theta^{(-)}_{jqj'q'}) \Big] \Bigg] \partial^2_{b_j^R, b_{j'}^R}
\end{aligned}
\tag{B.45}
$$

$$
\begin{aligned}
F^{(2)}_{jj'} = \Bigg[ & -\frac{\sigma^2}{4\beta_j\beta_{j'}} \mu_j^2 \mu_{j'}^2 \nu(0)\nu(\zeta) b_j^R b_{j'}^I \\
& + \frac{\sigma}{2\beta_j} \mu_j^2 \nu(0) b_j^I \sum_{q' \neq j'} \frac{1}{2\sqrt{\beta_{j'}\beta_{q'}}} \Big( C^{I,c}_{j'q'}(\zeta) \cos(\theta^{(2)}_{j'q'}) + C^{I,s}_{j'q'}(\zeta) \sin(\theta^{(2)}_{j'q'}) \Big) \\
& - \frac{\sigma}{2\beta_{j'}} \mu_{j'}^2 \nu(\zeta) b_{j'}^R \sum_{q \neq j} \frac{1}{2\sqrt{\beta_j\beta_q}} \Big( C^{R,c}_{jq}(0) \cos(\theta^{(1)}_{jq}) + C^{R,s}_{jq}(0) \sin(\theta^{(1)}_{jq}) \Big) \\
& + \sum_{q \neq j, q' \neq j'} \frac{1}{8\sqrt{\beta_j\beta_q\beta_{j'}\beta_{q'}}} \Big[ \Big( C^{R,c}_{jq}(0) C^{R,c}_{j'q'}(\zeta) + C^{R,s}_{jq}(0) C^{R,s}_{j'q'}(\zeta) \Big) \cos(\theta^{(-)}_{jqj'q'}) \\
& + \Big( C^{R,c}_{jq}(0) C^{I,c}_{j'q'}(\zeta) - C^{R,s}_{jq}(0) C^{I,s}_{j'q'}(\zeta) \Big) \cos(\theta^{(+)}_{jqj'q'}) \\
& + \Big( C^{R,s}_{jq}(0) C^{I,c}_{j'q'}(\zeta) + C^{R,c}_{jq}(0) C^{I,s}_{j'q'}(\zeta) \Big) \sin(\theta^{(+)}_{jqj'q'}) \\
& + \Big( C^{R,s}_{jq}(0) C^{I,c}_{j'q'}(\zeta) - C^{R,c}_{jq}(0) C^{I,s}_{j'q'}(\zeta) \Big) \sin(\theta^{(-)}_{jqj'q'}) \Big] \Bigg] \partial^2_{b_j^R, b_{j'}^I}
\end{aligned}
\tag{B.46}
$$

$$
\begin{aligned}
F^{(3)}_{jj'} = \Bigg[ & -\frac{\sigma^2}{4\beta_j\beta_{j'}}\mu_j^2\mu_{j'}^2\nu(0)\nu(\zeta)b_j^R b_{j'}^I \\
& - \frac{\sigma}{2\beta_j}\mu_j^2\nu(0)b_j^R\sum_{q'\neq j'}\frac{1}{2\sqrt{\beta_{j'}\beta_{q'}}}\Big(C^{R,c}_{j'q'}(\zeta)\cos(\theta^{(2)}_{j'q'}) + C^{R,s}_{j'q'}(\zeta)\sin(\theta^{(2)}_{j'q'})\Big) \\
& + \frac{\sigma}{2\beta_{j'}}\mu_{j'}^2\nu(\zeta)b_{j'}^I\sum_{q\neq j}\frac{1}{2\sqrt{\beta_j\beta_q}}\Big(C^{I,c}_{jq}(0)\cos(\theta^{(1)}_{jq}) + C^{I,s}_{jq}(0)\sin(\theta^{(1)}_{jq})\Big) \\
& + \sum_{q\neq j, q'\neq j'}\frac{1}{8\sqrt{\beta_j\beta_q\beta_{j'}\beta_{q'}}}\Big[\Big(C^{R,c}_{jq}(0)C^{R,c}_{j'q'}(\zeta) + C^{R,s}_{jq}(0)C^{R,s}_{j'q'}(\zeta)\Big)\cos(\theta^{(-)}_{jqj'q'}) \\
& + \Big(C^{I,c}_{jq}(0)C^{R,c}_{j'q'}(\zeta) - C^{I,s}_{jq}(0)C^{R,s}_{j'q'}(\zeta)\Big)\cos(\theta^{(+)}_{jqj'q'}) \\
& + \Big(C^{I,s}_{jq}(0)C^{R,c}_{j'q'}(\zeta) + C^{I,c}_{jq}(0)C^{R,s}_{j'q'}(\zeta)\Big)\sin(\theta^{(+)}_{jqj'q'}) \\
& + \Big(C^{I,s}_{jq}(0)C^{R,c}_{j'q'}(\zeta) - C^{I,c}_{jq}(0)C^{R,s}_{j'q'}(\zeta)\Big)\sin(\theta^{(-)}_{jqj'q'})\Big]\Bigg]\partial^2_{b_j^I, b_{j'}^R}
\end{aligned}
\tag{B.47}
$$

$$
\begin{aligned}
F^{(4)}_{jj'} = \Bigg[ & \frac{\sigma^2}{4\beta_j\beta_{j'}}\mu_j^2\mu_{j'}^2\nu(0)\nu(\zeta)b_j^{\varepsilon,R} b_{j'}^{\varepsilon,R} \\
& - \frac{\sigma}{2\beta_j}\mu_j^2\nu(0)b_j^R\sum_{q'\neq j'}\frac{1}{2\sqrt{\beta_{j'}\beta_{q'}}}\Big(C^{I,c}_{j'q'}(\zeta)\cos(\theta^{(2)}_{j'q'}) + C^{I,s}_{j'q'}(\zeta)\sin(\theta^{(2)}_{j'q'})\Big) \\
& - \frac{\sigma}{2\beta_{j'}}\mu_{j'}^2\nu(\zeta)b_{j'}^R\sum_{q\neq j}\frac{1}{2\sqrt{\beta_j\beta_q}}\Big(C^{I,c}_{jq}(0)\cos(\theta^{(1)}_{jq}) + C^{I,s}_{jq}(0)\sin(\theta^{(1)}_{jq})\Big) \\
& + \sum_{q\neq j, q'\neq j'}\frac{1}{8\sqrt{\beta_j\beta_q\beta_{j'}\beta_{q'}}}\Big[\Big(C^{R,c}_{jq}(0)C^{R,c}_{j'q'}(\zeta) + C^{R,s}_{jq}(0)C^{R,s}_{j'q'}(\zeta)\Big)\cos(\theta^{(-)}_{jqj'q'}) \\
& + \Big(C^{I,c}_{jq}(0)C^{I,c}_{j'q'}(\zeta) - C^{I,s}_{jq}(0)C^{I,s}_{j'q'}(\zeta)\Big)\cos(\theta^{(+)}_{jqj'q'}) \\
& + \Big(C^{I,s}_{jq}(0)C^{I,c}_{j'q'}(\zeta) + C^{I,c}_{jq}(0)C^{I,s}_{j'q'}(\zeta)\Big)\sin(\theta^{(+)}_{jqj'q'}) \\
& + \Big(C^{I,s}_{jq}(0)C^{I,c}_{j'q'}(\zeta) - C^{I,c}_{jq}(0)C^{I,s}_{j'q'}(\zeta)\Big)\sin(\theta^{(-)}_{jqj'q'})\Big]\Bigg]\partial^2_{b_j^I, b_{j'}^I}
\end{aligned}
\tag{B.48}
$$

$$F_{jj'}^{(5)} = \Bigg[\frac{\sigma}{4\sqrt{\beta_j^3\beta_{j'}}}\mu_j^2\nu(0)b_j^I\Big( - C_{j'j}^I(\zeta)\cos(\theta_{j'j}^{(2)}) + C_{j'j}^R(\zeta)\sin(\theta_{j'j}^{(2)})\Big)$$

$$+ \sum_{q\neq j}\frac{1}{8\sqrt{\beta_j^2\beta_q\beta_{j'}}}\Big[\Big(-C_{jq}^{R,c}(0)C_{j'j}^I(\zeta) + C_{jq}^{R,s}(0)C_{j'j}^R(\zeta)\Big)\cos(\theta_{jqj'j}^{(-)})$$

$$+ \Big(-C_{jq}^{R,c}(0)C_{j'j}^I(\zeta) - C_{jq}^{R,s}(0)C_{j'j}^R(\zeta)\Big)\cos(\theta_{jqj'j}^{(+)})$$

$$+ \Big(-C_{jq}^{R,s}(0)C_{j'j}^I(\zeta) + C_{jq}^{R,c}(0)C_{j'j}^R(\zeta)\Big)\sin(\theta_{jqj'j}^{(+)})$$

$$+ \Big(-C_{jq}^{R,s}(0)C_{j'j}^I(\zeta) - C_{jq}^{R,c}(0)C_{j'j}^R(\zeta)\Big)\sin(\theta_{jqj'j}^{(-)})\Big]\Bigg]\partial_{b_{j'}^R} \tag{B.49}$$

$$F_{jj'}^{(6)} = \Bigg[ - \frac{\sigma^2}{4\beta_j\beta_{j'}}\mu_j^2\mu_{j'}^2\nu(0)\nu(\zeta)b_j^I\delta_{jj'}$$

$$+ \frac{\sigma}{4\sqrt{\beta_j^3\beta_j'}}\mu_j^2\nu(0)b_j^I\Big(C_{j'j}^R(\zeta)\cos(\theta_{j'j}^{(2)}) + C_{j'j}^I(\zeta)\sin(\theta_{j'j}^{(2)})\Big)$$

$$- \frac{\sigma}{2\beta_j}\mu_j^2\nu(\zeta)\delta_{j,j'}\sum_{q\neq j}\frac{1}{2\sqrt{\beta_j\beta_q}}\Big(C_{jq}^{R,c}(0)\cos(\theta_{jq}^{(1)}) + C_{jq}^{R,s}(0)\sin(\theta_{jq}^{(1)})\Big)$$

$$+ \sum_{q\neq j,}\frac{1}{8\sqrt{\beta_j^2\beta_q\beta_{j'}}}\Big[\Big(C_{jq}^{R,c}(0)C_{j'j}^R(\zeta) + C_{jq}^{R,s}(0)C_{j'j}^I(\zeta)\Big)\cos(\theta_{jqj'j}^{(-)})$$

$$+ \Big(C_{jq}^{R,c}(0)C_{j'j}^R(\zeta) - C_{jq}^{R,s}(0)C_{j'j}^I(\zeta)\Big)\cos(\theta_{jqj'j}^{(+)})$$

$$+ \Big(C_{jq}^{R,s}(0)C_{j'j}^R(\zeta) + C_{jq}^{R,c}(0)C_{j'j}^I(\zeta)\Big)\sin(\theta_{jqj'j}^{(+)})$$

$$+ \Big(C_{jq}^{R,s}(0)C_{j'j}^R(\zeta) - C_{jq}^{R,c}(0)C_{j'j}^I(\zeta)\Big)\sin(\theta_{jqj'j}^{(-)})\Big]\Bigg]\partial_{b_{j'}^I} \tag{B.50}$$

$$
\begin{aligned}
F^{(7)}_{jj'} = \Bigg[ & - \frac{\sigma^2}{4\beta_j\beta_{j'}} \mu_j^2 \mu_{j'}^2 \nu(0)\nu(\zeta) b_j^R \delta_{jj'} \\
& + \frac{\sigma}{4\sqrt{\beta_j^3 \beta_j'}} \mu_j^2 \nu(0) b_j^R \left( C^R_{j'j}(\zeta)\cos(\theta^{(2)}_{j'j}) + C^I_{j'j}(\zeta)\sin(\theta^{(2)}_{j'j}) \right) \\
& + \frac{\sigma}{2\beta_j} \mu_j^2 \nu(\zeta) \delta_{j,j'} \sum_{q\neq j} \frac{1}{2\sqrt{\beta_j\beta_q}} \left( C^{R,c}_{jq}(0)\cos(\theta^{(1)}_{jq}) + C^{R,s}_{jq}(0)\sin(\theta^{(1)}_{jq}) \right) \\
& - \sum_{q\neq j,} \frac{1}{8\sqrt{\beta_j^2\beta_q\beta_{j'}}} \Big[ \left( C^{I,c}_{jq}(0) C^R_{j'j}(\zeta) + C^{I,s}_{jq}(0) C^I_{j'j}(\zeta) \right) \cos(\theta^{(-)}_{jqj'j}) \\
& + \left( C^{I,c}_{jq}(0) C^R_{j'j}(\zeta) - C^{I,s}_{jq}(0) C^I_{j'j}(\zeta) \right) \cos(\theta^{(+)}_{jqj'j}) \\
& + \left( C^{I,s}_{jq}(0) C^R_{j'j}(\zeta) + C^{I,c}_{jq}(0) C^I_{j'j}(\zeta) \right) \sin(\theta^{(+)}_{jqj'j}) \\
& + \left( C^{I,s}_{jq}(0) C^R_{j'j}(\zeta) - C^{I,c}_{jq}(0) C^I_{j'j}(\zeta) \right) \sin(\theta^{(-)}_{jqj'j}) \Big] \Bigg] \partial_{b^R_{j'}}
\end{aligned}
\tag{B.51}
$$

$$
\begin{aligned}
F^{(8)}_{jj'} = \Bigg[ & - \frac{\sigma}{4\sqrt{\beta_j^3 \beta_{j'}}} \mu_j^2 \nu(0) b_j^R \Big( - C^I_{j'j}(\zeta)\cos(\theta^{(2)}_{j'j}) + C^R_{j'j}(\zeta)\sin(\theta^{(2)}_{j'j}) \Big) \\
& + \sum_{q\neq j} \frac{1}{8\sqrt{\beta_j^2\beta_q\beta_{j'}}} \Big[ \left( -C^{I,c}_{jq}(0) C^I_{j'j}(\zeta) + C^{I,s}_{jq}(0) C^R_{j'j}(\zeta) \right) \cos(\theta^{(-)}_{jqj'j}) \\
& + \left( -C^{I,c}_{jq}(0) C^I_{j'j}(\zeta) - C^{I,s}_{jq}(0) C^R_{j'j}(\zeta) \right) \cos(\theta^{(+)}_{jqj'j}) \\
& + \left( -C^{I,s}_{jq}(0) C^I_{j'j}(\zeta) + C^{I,c}_{jq}(0) C^R_{j'j}(\zeta) \right) \sin(\theta^{(+)}_{jqj'j}) \\
& + \left( -C^{I,s}_{jq}(0) C^I_{j'j}(\zeta) - C^{I,c}_{jq}(0) C^R_{j'j}(\zeta) \right) \sin(\theta^{(-)}_{jqj'j}) \Big] \Bigg] \partial_{b^I_{j'}}
\end{aligned}
\tag{B.52}
$$

Then we average with respect to $s$.

$$
\begin{aligned}
&\sum_{j,j'} \lim_{L\to\infty} \frac{1}{L}\int_0^L ds F^{(1)}_{jj'} = \sum_{j,j'} \frac{\sigma^2}{4\beta_j\beta_{j'}} \mu_j^2\mu_{j'}^2 \nu(0)\nu(\zeta) b_j^I b_{j'}^I \partial^2_{b_j^R b_{j'}^R} \\
&\quad + \sum_{j,q,q\neq j} \frac{1}{8\beta_j\beta_q} \Big[ \Big( C^{R,c}_{jq}(0) C^{R,c}_{jq}(\zeta) + C^{R,s}_{jq}(0) C^{R,s}_{jq}(\zeta) \Big) \cos((\beta_j-\beta_q)\zeta) \\
&\quad + \Big( C^{R,s}_{jq}(0) C^{R,c}_{jq}(\zeta) - C^{R,c}_{jq}(0) C^{R,s}_{jq}(\zeta) \Big) \sin((\beta_j-\beta_q)\zeta) \Big] \partial^2_{b_j^R b_j^R} \\
&\quad + \sum_{j,q,q\neq j} \frac{1}{8\beta_j\beta_q} \Big[ \Big( C^{R,c}_{jq}(0) C^{R,c}_{qj}(\zeta) - C^{R,s}_{jq}(0) C^{R,s}_{qj}(\zeta) \Big) \cos((\beta_j-\beta_q)\zeta) \\
&\quad + \Big( C^{R,s}_{jq}(0) C^{R,c}_{qj}(\zeta) + C^{R,c}_{jq}(0) C^{R,s}_{qj}(\zeta) \Big) \sin((\beta_j-\beta_q)\zeta) \Big] \partial^2_{b_j^R b_q^R} \\
&\quad = \sum_{j,j'} \frac{\sigma^2}{4\beta_j\beta_{j'}} \mu_j^2\mu_{j'}^2 \nu(0)\nu(\zeta) b_j^I b_{j'}^I \partial^2_{b_j^R b_{j'}^R} \\
&\quad + \sum_{j,q,q\neq j} \frac{1}{8\beta_j\beta_q} \big[ \big( C^R_{jq}(0) C^R_{jq}(\zeta) + C^I_{jq}(0) C^I_{jq}(\zeta) \big) \cos((\beta_j-\beta_q)\zeta) \\
&\quad + \big( C^I_{jq}(0) C^R_{jq}(\zeta) - C^R_{jq}(0) C^I_{jq}(\zeta) \big) \sin((\beta_j-\beta_q)\zeta) \big] \Big( \big(b_q^R\big)^2 + \big(b_q^I\big)^2 \Big) \partial^2_{b_j^R b_j^R} \\
&\quad + \sum_{j,q,q\neq j} \frac{1}{8\beta_j\beta_q} \big[ \big( C^R_{jq}(0) C^R_{qj}(\zeta) - C^I_{jq}(0) C^I_{qj}(\zeta) \big) \cos((\beta_j-\beta_q)\zeta) \\
&\quad + \big( C^R_{jq}(0) C^I_{qj}(\zeta) + C^I_{jq}(0) C^R_{qj}(\zeta) \big) \sin((\beta_j-\beta_q)\zeta) \big] \big( b_j^I b_q^I - b_j^R b_q^R \big) \partial^2_{b_j^R b_q^R} \\
&\quad + \sum_{j,q,q\neq j} \frac{1}{8\beta_j\beta_q} \big[ \big( C^R_{jq}(0) C^I_{qj}(\zeta) + C^I_{jq}(0) C^R_{qj}(\zeta) \big) \cos((\beta_j-\beta_q)\zeta) \\
&\quad + \big( C^I_{jq}(0) C^I_{qj}(\zeta) - C^R_{jq}(0) C^R_{qj}(\zeta) \big) \sin((\beta_j-\beta_q)\zeta) \big] \big( b_j^I b_q^R + b_j^R b_q^I \big) \partial^2_{b_j^R b_q^R} \qquad \text{(B.53)}
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{j,j'} \lim_{L\to\infty} \frac{1}{L}\int_0^L ds F^{(2)}_{jj'} = -\sum_{j,j'} \frac{\sigma^2}{4\beta_j\beta_{j'}} \mu_j^2 \mu_{j'}^2 \nu(0)\nu(\zeta) b_j^I b_{j'}^R \partial^2_{b_j^R b_{j'}^I} \\
&+ \sum_{j,q,q\neq j} \frac{1}{8\beta_j\beta_q} \Big[ \Big( C^{R,c}_{jq}(0) C^{I,c}_{jq}(\zeta) + C^{R,s}_{jq}(0) C^{I,s}_{jq}(\zeta) \Big) \cos((\beta_j - \beta_q)\zeta) \\
&+ \Big( C^{R,s}_{jq}(0) C^{I,c}_{jq}(\zeta) - C^{R,c}_{jq}(0) C^{I,s}_{jq}(\zeta) \Big) \sin((\beta_j - \beta_q)\zeta) \Big] \partial^2_{b_j^R b_j^I} \\
&+ \sum_{j,q,q\neq j} \frac{1}{8\beta_j\beta_q} \Big[ \Big( C^{R,c}_{jq}(0) C^{I,c}_{qj}(\zeta) - C^{R,s}_{jq}(0) C^{I,s}_{qj}(\zeta) \Big) \cos((\beta_j - \beta_q)\zeta) \\
&+ \Big( C^{R,s}_{jq}(0) C^{I,c}_{qj}(\zeta) + C^{R,c}_{jq}(0) C^{I,s}_{qj}(\zeta) \Big) \sin((\beta_j - \beta_q)\zeta) \Big] \partial^2_{b_j^R b_q^I} \\
&= \sum_{j,j'} \frac{\sigma^2}{4\beta_j\beta_{j'}} \mu_j^2 \mu_{j'}^2 \nu(0)\nu(\zeta) b_j^I b_{j'}^R \partial^2_{b_j^R b_{j'}^I} \\
&+ \sum_{j,q,q\neq j} \frac{1}{8\beta_j\beta_q} \left[ - \left( C^I_{jq}(0) C^R_{jq}(\zeta) - C^R_{jq}(0) C^I_{jq}(\zeta) \right) \cos((\beta_j - \beta_q)\zeta) \right. \\
&+ \left. \left( C^R_{jq}(0) C^R_{jq}(\zeta) + C^I_{jq}(0) C^I_{jq}(\zeta) \right) \sin((\beta_j - \beta_q)\zeta) \right] \left( \left(b_q^R\right)^2 + \left(b_q^I\right)^2 \right) \partial^2_{b_j^R b_j^I} \\
&+ \sum_{j,q,q\neq j} \frac{1}{8\beta_j\beta_q} \left[ \left( C^R_{jq}(0) C^I_{qj}(\zeta) + C^I_{jq}(0) C^R_{qj}(\zeta) \right) \cos((\beta_j - \beta_q)\zeta) \right. \\
&- \left. \left( C^R_{jq}(0) C^R_{qj}(\zeta) - C^I_{jq}(0) C^I_{qj}(\zeta) \right) \sin((\beta_j - \beta_q)\zeta) \right] \left( b_j^I b_q^I - b_j^R b_q^R \right) \partial^2_{b_j^R b_q^I} \\
&+ \sum_{j,q,q\neq j} \frac{1}{8\beta_j\beta_q} \left[ \left( C^I_{jq}(0) C^I_{qj}(\zeta) - C^R_{jq}(0) C^R_{qj}(\zeta) \right) \cos((\beta_j - \beta_q)\zeta) \right. \\
&- \left. \left( C^R_{jq}(0) C^I_{qj}(\zeta) + C^I_{jq}(0) C^R_{qj}(\zeta) \right) \sin((\beta_j - \beta_q)\zeta) \right] \left( b_j^I b_q^R + b_j^R b_q^I \right) \partial^2_{b_j^R b_q^I} \qquad \text{(B.54)}
\end{aligned}
$$

$$
\begin{aligned}
\sum_{j,j'} \lim_{L\to\infty} \frac{1}{L}\int_0^L ds F^{(3)}_{jj'} = &-\sum_{j,j'} \frac{\sigma^2}{4\beta_j\beta_{j'}}\mu_j^2\mu_{j'}^2\nu(0)\nu(\zeta) b_j^R b_{j'}^I \partial^2_{b_j^I b_{j'}^R} \\
&+ \sum_{j,q,q\neq j} \frac{1}{8\beta_j\beta_q}\Big[\Big(C^{I,c}_{jq}(0)C^{R,c}_{jq}(\zeta) + C^{I,s}_{jq}(0)C^{R,s}_{jq}(\zeta)\Big)\cos((\beta_j-\beta_q)\zeta) \\
&+ \Big(C^{I,s}_{jq}(0)C^{R,c}_{jq}(\zeta) - C^{I,c}_{jq}(0)C^{R,s}_{jq}(\zeta)\Big)\sin((\beta_j-\beta_q)\zeta)\Big]\partial^2_{b_j^I b_j^R} \\
&+ \sum_{j,q,q\neq j} \frac{1}{8\beta_j\beta_q}\Big[\Big(C^{I,c}_{jq}(0)C^{R,c}_{qj}(\zeta) - C^{I,s}_{jq}(0)C^{R,s}_{qj}(\zeta)\Big)\cos((\beta_j-\beta_q)\zeta) \\
&+ \Big(C^{I,s}_{jq}(0)C^{R,c}_{qj}(\zeta) + C^{I,c}_{jq}(0)C^{R,s}_{qj}(\zeta)\Big)\sin((\beta_j-\beta_q)\zeta)\Big]\partial^2_{b_j^I b_q^R} \\
&= \sum_{j,j'} \frac{\sigma^2}{4\beta_j\beta_{j'}}\mu_j^2\mu_{j'}^2\nu(0)\nu(\zeta) b_j^R b_{j'}^I \partial^2_{b_j^I b_{j'}^R} \\
&+ \sum_{j,q,q\neq j} \frac{1}{8\beta_j\beta_q}\big[\big(C^I_{jq}(0)C^R_{jq}(\zeta) - C^R_{jq}(0)C^I_{jq}(\zeta)\big)\cos((\beta_j-\beta_q)\zeta) \\
&- \big(C^R_{jq}(0)C^R_{jq}(\zeta) + C^I_{jq}(0)C^I_{jq}(\zeta)\big)\sin((\beta_j-\beta_q)\zeta)\big]\left(\left(b_q^R\right)^2 + \left(b_q^I\right)^2\right)\partial^2_{b_j^I b_j^R} \\
&+ \sum_{j,q,q\neq j} \frac{1}{8\beta_j\beta_q}\big[\big(C^R_{jq}(0)C^I_{qj}(\zeta) + C^I_{jq}(0)C^R_{qj}(\zeta)\big)\cos((\beta_j-\beta_q)\zeta) \\
&- \big(C^R_{jq}(0)C^R_{qj}(\zeta) - C^I_{jq}(0)C^I_{qj}(\zeta)\big)\sin((\beta_j-\beta_q)\zeta)\big]\left(b_j^I b_q^I - b_j^R b_q^R\right)\partial^2_{b_j^I b_q^R} \\
&+ \sum_{j,q,q\neq j} \frac{1}{8\beta_j\beta_q}\big[\big(C^I_{jq}(0)C^I_{qj}(\zeta) - C^R_{jq}(0)C^R_{qj}(\zeta)\big)\cos((\beta_j-\beta_q)\zeta) \\
&- \big(C^R_{jq}(0)C^I_{qj}(\zeta) + C^I_{jq}(0)C^R_{qj}(\zeta)\big)\sin((\beta_j-\beta_q)\zeta)\big]\left(b_j^I b_q^R + b_j^R b_q^I\right)\partial^2_{b_j^I b_q^R} \qquad \text{(B.55)}
\end{aligned}
$$

$$\begin{aligned}
&\sum_{j,j'} \lim_{L\to\infty} \frac{1}{L} \int_0^L ds F^{(4)}_{jj'} = \sum_{j,j'} \frac{\sigma^2}{4\beta_j\beta_{j'}} \mu_j^2 \mu_{j'}^2 \nu(0)\nu(\zeta) b_j^R b_{j'}^R \partial^2_{b_j^I b_{j'}^I} \\
&+ \sum_{j,q,q\neq j} \frac{1}{8\beta_j\beta_q} \left[ \left( C^{I,c}_{jq}(0) C^{I,c}_{jq}(\zeta) + C^{I,s}_{jq}(0) C^{I,s}_{jq}(\zeta) \right) \cos((\beta_j - \beta_q)\zeta) \right. \\
&+ \left. \left( C^{I,s}_{jq}(0) C^{I,c}_{jq}(\zeta) - C^{I,c}_{jq}(0) C^{I,s}_{jq}(\zeta) \right) \sin((\beta_j - \beta_q)\zeta) \right] \partial^2_{b_j^I b_j^I} \\
&+ \sum_{j,q,q\neq j} \frac{1}{8\beta_j\beta_q} \left[ \left( C^{I,c}_{jq}(0) C^{I,c}_{qj}(\zeta) - C^{I,s}_{jq}(0) C^{I,s}_{qj}(\zeta) \right) \cos((\beta_j - \beta_q)\zeta) \right. \\
&+ \left. \left( C^{I,s}_{jq}(0) C^{I,c}_{qj}(\zeta) + C^{I,c}_{jq}(0) C^{I,s}_{qj}(\zeta) \right) \sin((\beta_j - \beta_q)\zeta) \right] \partial^2_{b_j^I b_q^I} \\
&= \sum_{j,j'} \frac{\sigma^2}{4\beta_j\beta_{j'}} \mu_j^2 \mu_{j'}^2 \nu(0)\nu(\zeta) b_j^R b_{j'}^R \partial^2_{b_j^I b_{j'}^I} \\
&+ \sum_{j,q,q\neq j} \frac{1}{8\beta_j\beta_q} \left[ \left( C^R_{jq}(0) C^R_{jq}(\zeta) + C^I_{jq}(0) C^I_{jq}(\zeta) \right) \cos((\beta_j - \beta_q)\zeta) \right. \\
&+ \left. \left( C^I_{jq}(0) C^R_{jq}(\zeta) - C^R_{jq}(0) C^I_{jq}(\zeta) \right) \sin((\beta_j - \beta_q)\zeta) \right] \left( \left(b_q^R\right)^2 + \left(b_q^I\right)^2 \right) \partial^2_{b_j^I b_j^I} \\
&+ \sum_{j,q,q\neq j} \frac{1}{8\beta_j\beta_q} \left[ - \left( C^R_{jq}(0) C^R_{qj}(\zeta) - C^I_{jq}(0) C^I_{qj}(\zeta) \right) \cos((\beta_j - \beta_q)\zeta) \right. \\
&- \left. \left( C^R_{jq}(0) C^I_{qj}(\zeta) + C^I_{jq}(0) C^R_{qj}(\zeta) \right) \sin((\beta_j - \beta_q)\zeta) \right] \left( b_j^I b_q^I - b_j^R b_q^R \right) \partial^2_{b_j^I b_q^I} \\
&+ \sum_{j,q,q\neq j} \frac{1}{8\beta_j\beta_q} \left[ - \left( C^R_{jq}(0) C^I_{qj}(\zeta) + C^I_{jq}(0) C^R_{qj}(\zeta) \right) \cos((\beta_j - \beta_q)\zeta) \right. \\
&- \left. \left( C^I_{jq}(0) C^I_{qj}(\zeta) - C^R_{jq}(0) C^R_{qj}(\zeta) \right) \sin((\beta_j - \beta_q)\zeta) \right] \left( b_j^I b_q^R + b_j^R b_q^I \right) \partial^2_{b_j^I b_q^I} \qquad \text{(B.56)}
\end{aligned}$$

$$\begin{aligned}
\sum_{j,j'} \lim_{L\to\infty} \frac{1}{L} \int_0^L ds F_{jj'}^{(5)} &= -\sum_j \frac{\sigma}{4\beta_j^2} \mu_j^2 \nu(0) b_j^I C_{jj}^I(\zeta) \partial_{b_j^R} \\
&+ \sum_{j,q,q\neq j} \frac{1}{8\beta_j\beta_q} \left[ \left( -C_{jq}^{R,c}(0) C_{qj}^I(\zeta) - C_{jq}^{R,s}(0) C_{qj}^R(\zeta) \right) \cos((\beta_j - \beta_q)\zeta) \right. \\
&+ \left. \left( -C_{jq}^{R,s}(0) C_{qj}^I(\zeta) + C_{jq}^{R,c}(0) C_{qj}^R(\zeta) \right) \sin((\beta_j - \beta_q)\zeta) \right] \partial_{b_q^R} \\
&= -\sum_j \frac{\sigma}{4\beta_j^2} \mu_j^2 \nu(0) C_{jj}^I(\zeta) b_j^I \partial_{b_j^R} \\
&+ \sum_{j,q,q\neq j} \frac{1}{8\beta_j\beta_q} \left[ \left( C_{jq}^I(0) C_{qj}^I(\zeta) - C_{jq}^R(0) C_{qj}^R(\zeta) \right) \cos((\beta_j - \beta_q)\zeta) \right. \\
&+ \left. \left( -C_{jq}^R(0) C_{qj}^I(\zeta) - C_{jq}^I(0) C_{qj}^R(\zeta) \right) \sin((\beta_j - \beta_q)\zeta) \right] b_q^R \partial_{b_q^R} \\
&+ \sum_{j,q,q\neq j} \frac{1}{8\beta_j\beta_q} \left[ \left( C_{jq}^R(0) C_{qj}^I(\zeta) + C_{jq}^I(0) C_{qj}^R(\zeta) \right) \cos((\beta_j - \beta_q)\zeta) \right. \\
&+ \left. \left( C_{jq}^I(0) C_{qj}^I(\zeta) - C_{jq}^R(0) C_{qj}^R(\zeta) \right) \sin((\beta_j - \beta_q)\zeta) \right] b_q^I \partial_{b_q^R} \qquad \text{(B.57)}
\end{aligned}$$

$$\sum_{j,j'} \lim_{L\to\infty} \frac{1}{L}\int_0^L ds F^{(6)}_{jj'} = -\sum_j \frac{\sigma^2}{4\beta_j^2}\mu_j^4 \nu(0)\nu(\zeta) b_j^I \partial_{b_j^I}$$

$$+\sum_j \frac{\sigma}{4\beta_j^2}\mu_j^2 \nu(0) b_j^I C^R_{jj}(\zeta)\partial_{b_j^I}$$

$$+\sum_{j,q,q\neq j,} \frac{1}{8\beta_j\beta_q}\Big[\Big(C^{R,c}_{jq}(0)C^R_{qj}(\zeta) - C^{R,s}_{jq}(0)C^I_{qj}(\zeta)\Big)\cos((\beta_j-\beta_q)\zeta)$$

$$+\Big(C^{R,s}_{jq}(0)C^R_{qj}(\zeta) + C^{R,c}_{jq}(0)C^I_{qj}(\zeta)\Big)\sin((\beta_j-\beta_q)\zeta)\Big]\partial_{b_q^I}$$

$$= -\sum_j \frac{\sigma^2}{4\beta_j^2}\mu_j^4\nu(0)\nu(\zeta) b_j^I\partial_{b_j^I}$$

$$+\sum_j \frac{\sigma}{4\beta_j^2}\mu_j^2\nu(0)C^R_{jj}(\zeta) b_j^I \partial_{b_j^I}$$

$$+\sum_{j,q,q\neq j,} \frac{1}{8\beta_j\beta_q}\big[\big(-C^I_{jq}(0)C^R_{qj}(\zeta) - C^R_{jq}(0)C^I_{qj}(\zeta)\big)\cos((\beta_j-\beta_q)\zeta)$$

$$+\big(C^R_{jq}(0)C^R_{qj}(\zeta) - C^I_{jq}(0)C^I_{qj}(\zeta)\big)\sin((\beta_j-\beta_q)\zeta)\big]\, b_q^R\partial_{b_q^I}$$

$$+\sum_{j,q,q\neq j,} \frac{1}{8\beta_j\beta_q}\big[\big(C^I_{jq}(0)C^I_{qj}(\zeta) - C^R_{jq}(0)C^R_{qj}(\zeta)\big)\cos((\beta_j-\beta_q)\zeta)$$

$$+\big(-C^I_{jq}(0)C^R_{qj}(\zeta) - C^R_{jq}(0)C^I_{qj}(\zeta)\big)\sin((\beta_j-\beta_q)\zeta)\big]\, b_q^I\partial_{b_q^I} \tag{B.58}$$

$$
\begin{aligned}
\sum_{j,j'} \lim_{L\to\infty} \frac{1}{L}\int_0^L ds F^{(7)}_{jj'} &= -\sum_j \frac{\sigma^2}{4\beta_j^2}\mu_j^4 \nu(0)\nu(\zeta) b_j^R \partial_{b_j^R} \\
&+ \sum_j \frac{\sigma}{4\beta_j^2}\mu_j^2 \nu(0) b_j^R C^R_{jj}(\zeta)\partial_{b_j^R} \\
&- \sum_{j,q,q\neq j,} \frac{1}{8\beta_j\beta_q}\Big[\Big(C^{I,c}_{jq}(0)C^R_{qj}(\zeta) - C^{I,s}_{jq}(0)C^I_{qj}(\zeta)\Big)\cos((\beta_j-\beta_q)\zeta) \\
&+ \Big(C^{I,s}_{jq}(0)C^R_{qj}(\zeta) + C^{I,c}_{jq}(0)C^I_{qj}(\zeta)\Big)\sin((\beta_j-\beta_q)\zeta)\Big]\partial_{b_q^R} \\
&= -\sum_j \frac{\sigma^2}{4\beta_j^2}\mu_j^4\nu(0)\nu(\zeta) b_j^R\partial_{b_j^R} \\
&+ \sum_j \frac{\sigma}{4\beta_j^2}\mu_j^2\nu(0) C^R_{jj}(\zeta) b_j^R \partial_{b_j^R} \\
&- \sum_{j,q,q\neq j,} \frac{1}{8\beta_j\beta_q}\big[\big(C^R_{jq}(0)C^R_{qj}(\zeta) - C^I_{jq}(0)C^I_{qj}(\zeta)\big)\cos((\beta_j-\beta_q)\zeta) \\
&+ \big(C^R_{jq}(0)C^I_{qj}(\zeta) + C^I_{jq}(0)C^R_{qj}(\zeta)\big)\sin((\beta_j-\beta_q)\zeta)\big] b_q^R\partial_{b_q^R} \\
&- \sum_{j,q,q\neq j,} \frac{1}{8\beta_j\beta_q}\big[\big(-C^I_{jq}(0)C^R_{qj}(\zeta) - C^R_{jq}(0)C^I_{qj}(\zeta)\big)\cos((\beta_j-\beta_q)\zeta) \\
&+ \big(C^R_{jq}(0)C^R_{qj}(\zeta) - C^I_{jq}(0)C^I_{qj}(\zeta)\big)\sin((\beta_j-\beta_q)\zeta)\big] b_q^I\partial_{b_q^R} \qquad \text{(B.59)}
\end{aligned}
$$

$$\sum_{j,j'} \lim_{L\to\infty} \frac{1}{L}\int_0^L ds F_{jj'}^{(8)} = \sum_j \frac{\sigma}{4\beta_j^2}\mu_j^2 \nu(0) b_j^R C_{jj}^I(\zeta)\partial_{b_j^I}$$

$$+ \sum_{j,q,q\neq j} \frac{1}{8\beta_j\beta_q}\left[\left(-C_{jq}^{I,c}(0)C_{qj}^I(\zeta) - C_{jq}^{I,s}(0)C_{qj}^R(\zeta)\right)\cos((\beta_j-\beta_q)\zeta)\right.$$

$$\left. + \left(-C_{jq}^{I,s}(0)C_{qj}^I(\zeta) + C_{jq}^{I,c}(0)C_{qj}^R(\zeta)\right)\sin((\beta_j-\beta_q)\zeta)\right]\partial_{b_q^I}$$

$$= \sum_j \frac{\sigma}{4\beta_j^2}\mu_j^2\nu(0)C_{jj}^I(\zeta)b_j^R\partial_{b_j^I}$$

$$+ \sum_{j,q,q\neq j}\frac{1}{8\beta_j\beta_q}\left[\left(-C_{jq}^R(0)C_{qj}^I(\zeta) - C_{jq}^I(0)C_{qj}^R(\zeta)\right)\cos((\beta_j-\beta_q)\zeta)\right.$$

$$\left. + \left(C_{jq}^R(0)C_{qj}^R(\zeta) - C_{jq}^I(0)C_{qj}^I(\zeta)\right)\sin((\beta_j-\beta_q)\zeta)\right] b_q^R\partial_{b_q^I}$$

$$+ \sum_{j,q,q\neq j}\frac{1}{8\beta_j\beta_q}\left[\left(C_{jq}^I(0)C_{qj}^I(\zeta) - C_{jq}^R(0)C_{qj}^R(\zeta)\right)\cos((\beta_j-\beta_q)\zeta)\right.$$

$$\left. + \left(-C_{jq}^I(0)C_{qj}^R(\zeta) - C_{jq}^R(0)C_{qj}^I(\zeta)\right)\sin((\beta_j-\beta_q)\zeta)\right] b_q^R\partial_{b_q^I} \tag{B.60}$$

We can shorten our expression so far by rewriting it using complex derivatives. Let

$$\partial_{b_j} := \frac{1}{2}\left[\partial_{b_j^R} - i\partial_{b_j^I}\right], \tag{B.61}$$

$$\partial_{\bar{b}_j} := \frac{1}{2}\left[\partial_{b_j^R} + i\partial_{b_j^I}\right]. \tag{B.62}$$

In particular, we will make use of the following identities

$$4\,\mathrm{Re}\left(b_j b_q \partial^2_{b_j b_q}\right) = \left(b_j^R b_q^R - b_j^I b_q^I\right)\partial^2_{b_j^R b_q^R} + \left(b_j^R b_q^I + b_j^I b_q^R\right)\partial^2_{b_j^I b_q^R}$$
$$+ \left(b_j^I b_q^R + b_j^R b_q^I\right)\partial^2_{b_j^R b_q^I} + \left(b_j^I b_q^I - b_j^R b_q^R\right)\partial^2_{b_j^I b_q^I}, \tag{B.63}$$
$$4\,\mathrm{Im}\left(b_j b_q \partial^2_{b_j b_q}\right) = \left(b_j^R b_q^I + b_j^I b_q^R\right)\partial^2_{b_j^R b_q^R} + \left(b_j^I b_q^I - b_j^R b_q^R\right)\partial^2_{b_j^I b_q^R}$$
$$+ \left(b_j^I b_q^I - b_j^R b_q^R\right)\partial^2_{b_j^R b_q^I} - \left(b_j^I b_q^R + b_j^R b_q^I\right)\partial^2_{b_j^I b_q^I}, \tag{B.64}$$
$$4\,\mathrm{Re}\left(b_q \bar{b}_q \partial^2_{b_j \bar{b}_j}\right) = \left((b_q^R)^2 + (b_q^I)^2\right)\partial^2_{b_j^R b_j^R} + \left((b_q^R)^2 + (b_q^I)^2\right)\partial^2_{b_j^I b_j^I}, \tag{B.65}$$
$$2\,\mathrm{Re}\left(b_j \partial_{b_j}\right) = b_j^R \partial_{b_j^R} + b_j^I \partial_{b_j^I}, \tag{B.66}$$
$$2\,\mathrm{Im}\left(b_j \partial_{b_j}\right) = b_j^I \partial_{b_j^R} - b_j^R \partial_{b_j^I}, \tag{B.67}$$
$$2\,\mathrm{Re}\left(b_j \bar{b}_{j'} \partial^2_{b_j \bar{b}_{j'}} - b_j b_{j'} \partial^2_{b_j b_{j'}} - b_j \partial_{b_j}\delta_{jj'}\right)$$
$$= b_{j'}^I b_j^I \partial^2_{b_j^R b_{j'}^R} - b_{j'}^I b_j^R \partial^2_{b_j^I b_{j'}^R} - b_{j'}^R b_j^I \partial^2_{b_j^R b_{j'}^I} + b_{j'}^R b_j^R \partial^2_{b_j^I b_{j'}^I}$$
$$- b_j^I \partial_{b_j^I}\delta_{jj'} - b_j^R \partial_{b_j^R}\delta_{jj'}. \tag{B.68}$$

Using the complex derivative identities we have

$$\sum_{j,j'} \lim_{L\to\infty} \frac{1}{L}\int_0^L ds \sum_{k=1}^{8} F_{jj'}^{(k)} = \sum_{j,j'} \frac{\sigma^2 \mu_j^2 \mu_{j'}^2}{2\beta_j \beta_{j'}} \nu(0)\nu(\zeta)\,\mathrm{Re}\left(b_j \bar{b}_{j'} \partial^2_{b_j \bar{b}_{j'}} - b_j b_{j'} \partial^2_{b_j b_{j'}} - b_j \partial_{b_j}\delta_{jj'}\right)$$
$$+ \sum_{j,q,q\neq j} \frac{1}{2\beta_j \beta_q}\Big[\left(C_{jq}^R(0)C_{jq}^R(\zeta) + C_{jq}^I(0)C_{jq}^I(\zeta)\right)\cos((\beta_j - \beta_q)\zeta)$$
$$+ \left(C_{jq}^I(0)C_{jq}^R(\zeta) - C_{jq}^R(0)C_{jq}^I(\zeta)\right)\sin((\beta_j - \beta_q)\zeta)\Big]\,\mathrm{Re}\left(b_q \bar{b}_q \partial^2_{b_j \bar{b}_j}\right)$$
$$- \sum_{j,q,q\neq j} \frac{1}{2\beta_j \beta_q}\Big[\left(C_{jq}^R(0)C_{qj}^R(\zeta) - C_{jq}^I(0)C_{qj}^I(\zeta)\right)\cos((\beta_j - \beta_q)\zeta)$$
$$+ \left(C_{jq}^R(0)C_{qj}^I(\zeta) + C_{jq}^I(0)C_{qj}^R(\zeta)\right)\sin((\beta_j - \beta_q)\zeta)\Big]\,\mathrm{Re}\left(b_j b_q \partial^2_{b_j b_q} + b_j \partial_{b_j}\right)$$
$$+ \sum_{j,q,q\neq j} \frac{1}{2\beta_j \beta_q}\Big[\left(C_{jq}^R(0)C_{qj}^I(\zeta) + C_{jq}^I(0)C_{qj}^R(\zeta)\right)\cos((\beta_j - \beta_q)\zeta)$$
$$+ \left(C_{jq}^I(0)C_{qj}^I(\zeta) - C_{jq}^R(0)C_{qj}^R(\zeta)\right)\sin((\beta_j - \beta_q)\zeta)\Big]\,\mathrm{Im}\left(b_j b_q \partial^2_{b_j b_q} + b_j \partial_{b_j}\right)$$
$$- \sum_j \frac{\sigma}{2\beta_j^2}\mu_j^2 \nu(0) C_{jj}^I(\zeta)\,\mathrm{Im}\left(b_j \partial_{b_j}\right) + \sum_j \frac{\sigma}{2\beta_j^2}\mu_j^2 \nu(0) C_{jj}^R(\zeta)\,\mathrm{Re}\left(b_j \partial_{b_j}\right). \tag{B.69}$$

We then take expectation. We will require the following identities

$$\mathbb{E}\left[C^R_{jq}(0)C^R_{qj}(\zeta) - C^I_{jq}(0)C^I_{qj}(\zeta)\right] = \sigma^2\left(\Gamma_{jq}\Gamma_{qj}\mathcal{R}^{(4)}(\zeta) + \beta_j\beta_q\Theta_{jq}\Theta_{qj}\mathcal{R}''(\zeta)\right) \tag{B.70}$$

$$\mathbb{E}\left[C^R_{jq}(0)C^I_{qj}(\zeta) + C^I_{jq}(0)C^R_{qj}(\zeta)\right] = \sigma^2\left(\beta_q\Theta_{jq}\Gamma_{qj}\mathcal{R}^{(3)}(\zeta) - \beta_j\Gamma_{jq}\Theta_{qj}\mathcal{R}^{(3)}(\zeta)\right) \tag{B.71}$$

$$\mathbb{E}\left[C^R_{jq}(0)C^R_{jq}(\zeta) + C^I_{jq}(0)C^I_{jq}(\zeta)\right] = \sigma^2\left(\Gamma^2_{jq}\mathcal{R}^{(4)}(\zeta) - \beta_q^2\Theta^2_{jq}\mathcal{R}''(\zeta)\right) \tag{B.72}$$

$$\mathbb{E}\left[C^I_{jq}(0)C^R_{jq}(\zeta) - C^R_{jq}(0)C^I_{jq}(\zeta)\right] = 2\sigma^2\beta_q\Gamma_{jq}\Theta_{jq}\mathcal{R}^{(3)}(\zeta) \tag{B.73}$$

where $\mathcal{R}(\zeta) := \mathbb{E}[\nu(0)\nu(\zeta)]$.

Applying these identities we have

$$\begin{aligned}
&\sum_{j,j'}\lim_{L\to\infty}\frac{1}{L}\int_0^L ds\sum_{k=1}^{8}\mathbb{E}[F^{(k)}_{jj'}] = \sum_{j,j'}\frac{\sigma^2\mu_j^2\mu_{j'}^2}{2\beta_j\beta_{j'}}\mathcal{R}(\zeta)\,\mathrm{Re}\left(b_j\bar{b}_{j'}\partial^2_{b_j\bar{b}_{j'}} - b_jb_{j'}\partial^2_{b_jb_{j'}} - b_j\partial_{b_j}\delta_{jj'}\right)\\
&\quad+\sum_{j,q,q\neq j}\frac{\sigma^2}{2\beta_j\beta_q}\left[\left(\Gamma^2_{jq}\mathcal{R}^{(4)}(\zeta) - \beta_q^2\Theta^2_{jq}\mathcal{R}''(\zeta)\right)\cos((\beta_j-\beta_q)\zeta)\right.\\
&\quad\left.+\left(2\beta_q\Gamma_{jq}\Theta_{jq}\mathcal{R}^{(3)}(\zeta)\right)\sin((\beta_j-\beta_q)\zeta)\right]\mathrm{Re}\left(b_q\bar{b}_q\partial^2_{b_j\bar{b}_j}\right)\\
&\quad-\sum_{j,q,q\neq j}\frac{\sigma^2}{2\beta_j\beta_q}\left[\left(\Gamma_{jq}\Gamma_{qj}\mathcal{R}^{(4)}(\zeta) + \beta_j\beta_q\Theta_{jq}\Theta_{qj}\mathcal{R}''(\zeta)\right)\cos((\beta_j-\beta_q)\zeta)\right.\\
&\quad\left.+\left(-\beta_j\Gamma_{jq}\Theta_{qj}\mathcal{R}^{(3)}(\zeta) + \beta_q\Theta_{jq}\Gamma_{qj}\mathcal{R}^{(3)}(\zeta)\right)\sin((\beta_j-\beta_q)\zeta)\right]\mathrm{Re}\left(b_jb_q\partial^2_{b_jb_q} + b_j\partial_{b_j}\right)\\
&\quad+\sum_{j,q,q\neq j}\frac{\sigma^2}{2\beta_j\beta_q}\left[\left(-\beta_j\Gamma_{jq}\Theta_{qj}\mathcal{R}^{(3)}(\zeta) + \beta_q\Theta_{jq}\Gamma_{qj}\mathcal{R}^{(3)}(\zeta)\right)\cos((\beta_j-\beta_q)\zeta)\right.\\
&\quad\left.-\left(\Gamma_{jq}\Gamma_{qj}\mathcal{R}^{(4)}(\zeta) + \beta_j\beta_q\Theta_{jq}\Theta_{qj}\mathcal{R}''(\zeta)\right)\sin((\beta_j-\beta_q)\zeta)\right]\mathrm{Im}\left(b_jb_q\partial^2_{b_jb_q} + b_j\partial_{b_j}\right)\\
&\quad+\sum_j\frac{\sigma^2}{2\beta_j}\mu_j^2\Theta_{jj}\mathcal{R}'(\zeta)\,\mathrm{Im}\left(b_j\partial_{b_j}\right) + \sum_j\frac{\sigma^2}{2\beta_j^2}\mu_j^2\Gamma_{jj}\mathcal{R}''(\zeta)\,\mathrm{Re}\left(b_j\partial_{b_j}\right).
\end{aligned} \tag{B.74}$$

We can simplify using that $\Theta_{jq} = 2\Gamma_{jq}$ and obtain

$$\sum_{j,j'} \lim_{L\to\infty} \frac{1}{L}\int_0^L ds \sum_{k=1}^{8} \mathbb{E}[F_{jj'}^{(k)}] = \sum_{j,j'} \frac{\sigma^2 \mu_j^2 \mu_{j'}^2}{2\beta_j\beta_{j'}} \mathcal{R}(\zeta)\,\mathrm{Re}\left(b_j\bar{b}_{j'}\partial^2_{b_j\bar{b}_{j'}} - b_j b_{j'}\partial^2_{b_j b_{j'}} - b_j\partial_{b_j}\delta_{jj'}\right)$$
$$+ \sum_{j,q,q\neq j} \frac{\sigma^2\Gamma_{jq}^2}{2\beta_j\beta_q}\left[\left(\mathcal{R}^{(4)}(\zeta) - 4\beta_q^2\mathcal{R}''(\zeta)\right)\cos((\beta_j-\beta_q)\zeta)\right.$$
$$\left.+ \left(4\beta_q\mathcal{R}^{(3)}(\zeta)\right)\sin((\beta_j-\beta_q)\zeta)\right]\mathrm{Re}\left(b_q\bar{b}_q\partial^2_{b_j\bar{b}_j}\right)$$
$$- \sum_{j,q,q\neq j} \frac{\sigma^2\Gamma_{jq}\Gamma_{qj}}{2\beta_j\beta_q}\left[\left(\mathcal{R}^{(4)}(\zeta) + 4\beta_j\beta_q\mathcal{R}''(\zeta)\right)\cos((\beta_j-\beta_q)\zeta)\right.$$
$$\left.+ \left(-2\beta_j\mathcal{R}^{(3)}(\zeta) + 2\beta_q\mathcal{R}^{(3)}(\zeta)\right)\sin((\beta_j-\beta_q)\zeta)\right]\mathrm{Re}\left(b_j b_q\partial^2_{b_j b_q} + b_j\partial_{b_j}\right)$$
$$+ \sum_{j,q,q\neq j} \frac{\sigma^2\Gamma_{jq}\Gamma_{qj}}{2\beta_j\beta_q}\left[\left(-2\beta_j\mathcal{R}^{(3)}(\zeta) + 2\beta_q\mathcal{R}^{(3)}(\zeta)\right)\cos((\beta_j-\beta_q)\zeta)\right.$$
$$\left.- \left(\mathcal{R}^{(4)}(\zeta) + 4\beta_j\beta_q\mathcal{R}''(\zeta)\right)\sin((\beta_j-\beta_q)\zeta)\right]\mathrm{Im}\left(b_j b_q\partial^2_{b_j b_q} + b_j\partial_{b_j}\right)$$
$$+ \sum_j \frac{\sigma^2}{\beta_j}\mu_j^2\Gamma_{jj}\mathcal{R}'(\zeta)\,\mathrm{Im}\left(b_j\partial_{b_j}\right) + \sum_j \frac{\sigma^2}{2\beta_j^2}\mu_j^2\Gamma_{jj}\mathcal{R}''(\zeta)\,\mathrm{Re}\left(b_j\partial_{b_j}\right). \tag{B.75}$$

Further, we have $\Gamma_{qj} = -\Gamma_{jq}$ and so we may write

$$\sum_{j,j'} \lim_{L\to\infty} \frac{1}{L}\int_0^L ds \sum_{k=1}^{8} \mathbb{E}[F_{jj'}^{(k)}] = \sum_{j,j'} \frac{\sigma^2 \mu_j^2 \mu_{j'}^2}{2\beta_j\beta_{j'}} \mathcal{R}(\zeta)\,\mathrm{Re}\left(b_j\bar{b}_{j'}\partial^2_{b_j\bar{b}_{j'}} - b_j b_{j'}\partial^2_{b_j b_{j'}} - b_j\partial_{b_j}\delta_{jj'}\right)$$
$$+ \sum_{j,q,q\neq j} \frac{\sigma^2\Gamma_{jq}^2}{2\beta_j\beta_q}\left[\left(\mathcal{R}^{(4)}(\zeta) - 4\beta_q^2\mathcal{R}''(\zeta)\right)\cos((\beta_j-\beta_q)\zeta)\right.$$
$$\left.+4\beta_q\mathcal{R}^{(3)}(\zeta)\sin((\beta_j-\beta_q)\zeta)\right]\mathrm{Re}\left(b_q\bar{b}_q\partial^2_{b_j\bar{b}_j}\right)$$
$$+ \sum_{j,q,q\neq j} \frac{\sigma^2\Gamma_{jq}^2}{2\beta_j\beta_q}\left[\left(\mathcal{R}^{(4)}(\zeta) + 4\beta_j\beta_q\mathcal{R}''(\zeta)\right)\cos((\beta_j-\beta_q)\zeta)\right.$$
$$\left.+2\left(\beta_q-\beta_j\right)\mathcal{R}^{(3)}(\zeta)\sin((\beta_j-\beta_q)\zeta)\right]\mathrm{Re}\left(b_j b_q\partial^2_{b_j b_q} + b_j\partial_{b_j}\right)$$
$$+ \sum_{j,q,q\neq j} \frac{\sigma^2\Gamma_{jq}^2}{2\beta_j\beta_q}\left[2\left(\beta_j-\beta_q\right)\mathcal{R}^{(3)}(\zeta)\cos((\beta_j-\beta_q)\zeta)\right.$$
$$\left.+ \left(\mathcal{R}^{(4)}(\zeta) + 4\beta_j\beta_q\mathcal{R}''(\zeta)\right)\sin((\beta_j-\beta_q)\zeta)\right]\mathrm{Im}\left(b_j b_q\partial^2_{b_j b_q} + b_j\partial_{b_j}\right)$$
$$+ \sum_j \frac{\sigma^2}{\beta_j}\mu_j^2\Gamma_{jj}\mathcal{R}'(\zeta)\,\mathrm{Im}\left(b_j\partial_{b_j}\right) + \sum_j \frac{\sigma^2}{2\beta_j^2}\mu_j^2\Gamma_{jj}\mathcal{R}''(\zeta)\,\mathrm{Re}\left(b_j\partial_{b_j}\right). \tag{B.76}$$

We can eliminate some terms by noting that

$$\sum_{j,q,q\neq j} \frac{\sigma^2\Gamma_{jq}^2}{2\beta_j\beta_q}\Big[2\left(\beta_j-\beta_q\right)\mathcal{R}^{(3)}(\zeta)\cos((\beta_j-\beta_q)\zeta) \\ +\left(\mathcal{R}^{(4)}(\zeta)+4\beta_j\beta_q\mathcal{R}''(\zeta)\right)\sin((\beta_j-\beta_q)\zeta)\Big]\,\mathrm{Im}\left(b_jb_q\partial^2_{b_jb_q}\right)=0 \tag{B.77}$$

after swapping indices and using that sine is odd. We are left with

$$\sum_{j,j'}\lim_{L\to\infty}\frac{1}{L}\int_0^L ds\sum_{k=1}^{8}\mathbb{E}[F_{jj'}^{(k)}]=\sum_{j,j'}\frac{\sigma^2\mu_j^2\mu_{j'}^2}{2\beta_j\beta_{j'}}\mathcal{R}(\zeta)\,\mathrm{Re}\left(b_j\bar{b}_{j'}\partial^2_{b_j\bar{b}_{j'}}-b_jb_{j'}\partial^2_{b_jb_{j'}}-b_j\partial_{b_j}\delta_{jj'}\right) \\ +\sum_{j,q,q\neq j}\frac{\sigma^2\Gamma_{jq}^2}{2\beta_j\beta_q}\Big[\left(\mathcal{R}^{(4)}(\zeta)-4\beta_q^2\mathcal{R}''(\zeta)\right)\cos((\beta_j-\beta_q)\zeta) \\ +4\beta_q\mathcal{R}^{(3)}(\zeta)\sin((\beta_j-\beta_q)\zeta)\Big]\,\mathrm{Re}\left(b_q\bar{b}_q\partial^2_{b_j\bar{b}_j}\right) \\ +\sum_{j,q,q\neq j}\frac{\sigma^2\Gamma_{jq}^2}{2\beta_j\beta_q}\Big[\left(\mathcal{R}^{(4)}(\zeta)+4\beta_j\beta_q\mathcal{R}''(\zeta)\right)\cos((\beta_j-\beta_q)\zeta) \\ +2\left(\beta_q-\beta_j\right)\mathcal{R}^{(3)}(\zeta)\sin((\beta_j-\beta_q)\zeta)\Big]\,\mathrm{Re}\left(b_jb_q\partial^2_{b_jb_q}+b_j\partial_{b_j}\right) \\ +\sum_{j,q,q\neq j}\frac{\sigma^2\Gamma_{jq}^2}{2\beta_j\beta_q}\Big[2\left(\beta_j-\beta_q\right)\mathcal{R}^{(3)}(\zeta)\cos((\beta_j-\beta_q)\zeta) \\ +\left(\mathcal{R}^{(4)}(\zeta)+4\beta_j\beta_q\mathcal{R}''(\zeta)\right)\sin((\beta_j-\beta_q)\zeta)\Big]\,\mathrm{Im}\left(b_j\partial_{b_j}\right) \\ +\sum_j\frac{\sigma^2}{\beta_j}\mu_j^2\Gamma_{jj}\mathcal{R}'(\zeta)\,\mathrm{Im}\left(b_j\partial_{b_j}\right)+\sum_j\frac{\sigma^2}{2\beta_j^2}\mu_j^2\Gamma_{jj}\mathcal{R}''(\zeta)\,\mathrm{Re}\left(b_j\partial_{b_j}\right). \tag{B.78}$$

We now integrate with respect to $\zeta$. We have

$$\int_0^\infty d\zeta\mathcal{R}(\zeta)=\frac{1}{2}\widehat{\mathcal{R}}(0) \tag{B.79}$$

$$\int_0^\infty d\zeta\mathcal{R}'(\zeta)=-\mathcal{R}(0) \tag{B.80}$$

$$\int_0^\infty d\zeta\mathcal{R}''(\zeta)=0 \tag{B.81}$$

We will need the following identities which follow from integration by parts for the remaining terms

$$\int_0^\infty d\zeta \mathcal{R}'(\zeta) \sin\left((\beta_j - \beta_q)\zeta\right) = \frac{\beta_q - \beta_j}{2} \widehat{\mathcal{R}}\left(\beta_j - \beta_q\right) \tag{B.82}$$

$$\int_0^\infty d\zeta \mathcal{R}''(\zeta) \cos\left((\beta_j - \beta_q)\zeta\right) = -\frac{\left(\beta_q - \beta_j\right)^2}{2} \widehat{\mathcal{R}}\left(\beta_j - \beta_q\right) \tag{B.83}$$

$$\int_0^\infty d\zeta \mathcal{R}^{(3)}(\zeta) \sin\left((\beta_j - \beta_q)\zeta\right) = -\frac{\left(\beta_q - \beta_j\right)^3}{2} \widehat{\mathcal{R}}\left(\beta_j - \beta_q\right) \tag{B.84}$$

$$\int_0^\infty d\zeta \mathcal{R}^{(4)}(\zeta) \cos\left((\beta_j - \beta_q)\zeta\right) = \frac{\left(\beta_q - \beta_j\right)^4}{2} \widehat{\mathcal{R}}\left(\beta_j - \beta_q\right) \tag{B.85}$$

$$\begin{aligned}\int_0^\infty d\zeta \mathcal{R}'(\zeta) \cos\left((\beta_j - \beta_q)\zeta\right) &= -\mathcal{R}(0) \\ &\quad - \left(\beta_q - \beta_j\right) \int_0^\infty d\zeta \mathcal{R}(\zeta) \sin\left((\beta_j - \beta_q)\zeta\right)\end{aligned} \tag{B.86}$$

$$\begin{aligned}\int_0^\infty d\zeta \mathcal{R}''(\zeta) \sin\left((\beta_j - \beta_q)\zeta\right) &= -\left(\beta_q - \beta_j\right) \mathcal{R}(0) \\ &\quad - \left(\beta_q - \beta_j\right)^2 \int_0^\infty d\zeta \mathcal{R}(\zeta) \sin\left((\beta_j - \beta_q)\zeta\right)\end{aligned} \tag{B.87}$$

$$\begin{aligned}\int_0^\infty d\zeta \mathcal{R}^{(3)}(\zeta) \cos\left((\beta_j - \beta_q)\zeta\right) &= -\mathcal{R}''(0) + \left(\beta_q - \beta_j\right)^2 \mathcal{R}(0) \\ &\quad + \left(\beta_q - \beta_j\right)^3 \int_0^\infty d\zeta \mathcal{R}(\zeta) \sin\left((\beta_j - \beta_q)\zeta\right)\end{aligned} \tag{B.88}$$

$$\begin{aligned}\int_0^\infty d\zeta \mathcal{R}^{(4)}(\zeta) \sin\left((\beta_j - \beta_q)\zeta\right) &= -\left(\beta_q - \beta_j\right) \mathcal{R}''(0) + \left(\beta_q - \beta_j\right)^3 \mathcal{R}(0) \\ &\quad + \left(\beta_q - \beta_j\right)^4 \int_0^\infty d\zeta \mathcal{R}(\zeta) \sin\left((\beta_j - \beta_q)\zeta\right)\end{aligned} \tag{B.89}$$

Using the identities above we have

$$\int_0^\infty d\zeta\,\left(\mathcal{R}^{(4)}(\zeta)-4\beta_q^2\mathcal{R}''(\zeta)\right)\cos((\beta_j-\beta_q)\zeta)+4\beta_q\mathcal{R}^{(3)}(\zeta)\sin((\beta_j-\beta_q)\zeta)$$
$$=\frac{1}{2}(\beta_q-\beta_j)^2(\beta_q+\beta_j)^2\widehat{\mathcal{R}}(\beta_j-\beta_q) \tag{B.90}$$
$$\int_0^\infty d\zeta\,\left(\mathcal{R}^{(4)}(\zeta)+4\beta_j\beta_q\mathcal{R}''(\zeta)\right)\cos((\beta_j-\beta_q)\zeta)+2\left(\beta_q-\beta_j\right)\mathcal{R}^{(3)}(\zeta)\sin((\beta_j-\beta_q)\zeta)$$
$$=-\frac{1}{2}(\beta_q-\beta_j)^2(\beta_q+\beta_j)^2\widehat{\mathcal{R}}(\beta_j-\beta_q) \tag{B.91}$$
$$\int_0^\infty d\zeta\,\left(2\beta_j-2\beta_q\right)\mathcal{R}^{(3)}(\zeta)\cos((\beta_j-\beta_q)\zeta)+\left(\mathcal{R}^{(4)}(\zeta)+4\beta_j\beta_q\mathcal{R}''(\zeta)\right)\sin((\beta_j-\beta_q)\zeta)$$
$$=(\beta_q-\beta_j)\mathcal{R}''(0)-(\beta_q-\beta_j)(\beta_q+\beta_j)^2\mathcal{R}(0)$$
$$-(\beta_q-\beta_j)^2(\beta_q+\beta_j)^2\int_0^\infty d\zeta\mathcal{R}(\zeta)\sin\left((\beta_j-\beta_q)\zeta\right) \tag{B.92}$$

Then we have

$$\mathcal{L}_1=\sum_{j,j'}\frac{\sigma^2\mu_j^2\mu_{j'}^2}{4\beta_j\beta_{j'}}\widehat{\mathcal{R}}(0)\,\mathrm{Re}\left(b_j\bar{b}_{j'}\partial^2_{b_j\bar{b}_{j'}}-b_jb_{j'}\partial^2_{b_jb_{j'}}-b_j\partial_{b_j}\delta_{jj'}\right)$$
$$+\sum_{j,q,q\neq j}\frac{\sigma^2\Gamma_{jq}^2}{4\beta_j\beta_q}(\beta_q-\beta_j)^2(\beta_q+\beta_j)^2\widehat{\mathcal{R}}(\beta_j-\beta_q)\,\mathrm{Re}\left(b_q\bar{b}_q\partial^2_{b_j\bar{b}_j}\right)$$
$$-\sum_{j,q,q\neq j}\frac{\sigma^2\Gamma_{jq}^2}{4\beta_j\beta_q}(\beta_q-\beta_j)^2(\beta_q+\beta_j)^2\widehat{\mathcal{R}}(\beta_j-\beta_q)\,\mathrm{Re}\left(b_jb_q\partial^2_{b_jb_q}+b_j\partial_{b_j}\right)$$
$$+\sum_{j,q,q\neq j}\frac{\sigma^2\Gamma_{jq}^2}{2\beta_j\beta_q}\left[(\beta_q-\beta_j)\mathcal{R}''(0)-(\beta_q-\beta_j)(\beta_q+\beta_j)^2\mathcal{R}(0)\right]\mathrm{Im}\left(b_j\partial_{b_j}\right)$$
$$-\sum_{j,q,q\neq j}\frac{\sigma^2\Gamma_{jq}^2}{2\beta_j\beta_q}\left[(\beta_q-\beta_j)^2(\beta_q+\beta_j)^2\int_0^\infty d\zeta\mathcal{R}(\zeta)\sin\left((\beta_j-\beta_q)\zeta\right)\right]\mathrm{Im}\left(b_j\partial_{b_j}\right) \tag{B.93}$$

We then use that

$$\frac{\sigma^2\Gamma_{jq}^2}{\beta_j\beta_q}(\beta_q-\beta_j)^2(\beta_q+\beta_j)^2=\frac{\sigma^2\mu_j^2\mu_q^2}{\beta_j\beta_q} \tag{B.94}$$

and write

$$\mathcal{L}_1 = \sum_{j,j'} \frac{\sigma^2 \mu_j^2 \mu_{j'}^2}{4\beta_j \beta_{j'}} \widehat{\mathcal{R}}(0) \operatorname{Re}\left( b_j \bar{b}_{j'} \partial^2_{b_j \bar{b}_{j'}} - b_j b_{j'} \partial^2_{b_j b_{j'}} - b_j \partial_{b_j} \delta_{jj'} \right)$$
$$+ \sum_{j,q,q\neq j} \frac{\sigma^2 \mu_j^2 \mu_q^2}{4\beta_j \beta_q} \widehat{\mathcal{R}}(\beta_j - \beta_q) \operatorname{Re}\left( b_q \bar{b}_q \partial^2_{b_j \bar{b}_j} \right)$$
$$- \sum_{j,q,q\neq j} \frac{\sigma^2 \mu_j^2 \mu_q^2}{4\beta_j \beta_q} \widehat{\mathcal{R}}(\beta_j - \beta_q) \operatorname{Re}\left( b_j b_q \partial^2_{b_j b_q} + b_j \partial_{b_j} \right)$$
$$- \sum_{j,q,q\neq j} \frac{\sigma^2 \mu_j^2 \mu_q^2}{2\beta_j \beta_q (\beta_j - \beta_q)} \left[ -\mathcal{R}(0) + \frac{\mathcal{R}''(0)}{(\beta_j + \beta_q)^2} \right] \operatorname{Im}\left( b_j \partial_{b_j} \right)$$
$$- \sum_{j,q,q\neq j} \frac{\sigma^2 \mu_j^2 \mu_q^2}{2\beta_j \beta_q} \left[ \int_0^\infty d\zeta \mathcal{R}(\zeta) \sin\left( (\beta_j - \beta_q)\zeta \right) \right] \operatorname{Im}\left( b_j \partial_{b_j} \right) \tag{B.95}$$

Using the notation from chapter III and relabeling indices in the first sum we have

$$\mathcal{L}_1 = \sum_{j,q} G^{(0)}_{jq} \operatorname{Re}\left( b_j \bar{b}_q \partial^2_{b_j \bar{b}_q} - b_j b_q \partial^2_{b_j b_q} - b_j \partial_{b_j} \delta_{jq} \right)$$
$$+ \sum_{j,q,q\neq j} G^{(c)}_{jq} \left( \operatorname{Re}\left( b_q \bar{b}_q \partial^2_{b_j \bar{b}_j} \right) - \operatorname{Re}\left( b_j b_q \partial^2_{b_j b_q} + b_j \partial_{b_j} \right) \right)$$
$$- \sum_{j,q,q\neq j} \frac{\sigma^2 \mu_j^2 \mu_q^2}{2\beta_j \beta_q (\beta_j - \beta_q)} \left[ -\mathcal{R}(0) + \frac{\mathcal{R}''(0)}{(\beta_j + \beta_q)^2} \right] \operatorname{Im}\left( b_j \partial_{b_j} \right)$$
$$- \sum_{j,q,q\neq j} G^{(s)}_{jq} \operatorname{Im}\left( b_j \partial_{b_j} \right) \tag{B.96}$$

## B.3 Second Order Terms

Now we compute the second order terms of the generator. These terms will be given by

$$\mathcal{L}_2 := \sum_j \lim_{L\to\infty} \frac{1}{L} \int_0^L ds \mathbb{E}\Big[ \sum_{k=1}^2 G_j^{(k)} \Big] \tag{B.97}$$

where

$$G_j^{(1)} := G_j^R(\boldsymbol{b}, \nu(0), \boldsymbol{\theta} + \boldsymbol{\beta} s, z) \partial_{b_j^R}, \tag{B.98}$$

$$G_j^{(2)} := G_j^I(\boldsymbol{b}, \nu(0), \boldsymbol{\theta} + \boldsymbol{\beta} s, z) \partial_{b_j^I}. \tag{B.99}$$

Due to the averaging in the $s$ variable only "diagonal" terms of $G_j^R$ and $G_j^I$ will remain. More precisely, we will have

$$\lim_{L\to\infty}\frac{1}{L}\int_0^L dsG_j^{(1)} = \left[\frac{\sigma^2}{2\beta_j}\left[-\frac{3}{4}\mu_j^2\nu^2(\zeta)-\left(\frac{(\pi j)^2}{12}+\frac{1}{16}\right)\nu'^2(\zeta)\right]b_j^I\right.$$
$$+\sum_{l>\mathcal{N}}\frac{\sigma^2}{4\beta_j\beta_l}\int_{-\infty}^{\infty}d\xi e^{-\beta_l|\xi|}\left(\cos(\beta_j\xi)\widetilde{C}_{jl}^I(\zeta,\xi)-\sin(\beta_j\xi)\widetilde{C}_{jl}^R(\zeta,\xi)\right)b_j^R$$
$$\left.+\sum_{l>\mathcal{N}}\frac{\sigma^2}{4\beta_j\beta_l}\int_{-\infty}^{\infty}d\xi e^{-\beta_l|\xi|}\left(\cos(\beta_j\xi)\widetilde{C}_{jl}^R(\zeta,\xi)+\sin(\beta_j\xi)\widetilde{C}_{jl}^I(\zeta,\xi)\right)b_j^I\right]\partial_{b_j^R} \quad \text{(B.100)}$$

$$\lim_{L\to\infty}\frac{1}{L}\int_0^L dsG_j^{(2)} = \left[\frac{\sigma^2}{2\beta_j}\left[\frac{3}{4}\mu_j^2\nu^2(\zeta)+\left(\frac{(\pi j)^2}{12}+\frac{1}{16}\right)\nu'^2(\zeta)\right]b_j^R\right.$$
$$+\sum_{l>\mathcal{N}}\frac{\sigma^2}{4\beta_j\beta_l}\int_{-\infty}^{\infty}d\xi e^{-\beta_l|\xi|}\left(\cos(\beta_j\xi)\widetilde{C}_{jl}^I(\zeta,\xi)-\sin(\beta_j\xi)\widetilde{C}_{jl}^R(\zeta,\xi)\right)b_j^I$$
$$\left.-\sum_{l>\mathcal{N}}\frac{\sigma^2}{4\beta_j\beta_l}\int_{-\infty}^{\infty}d\xi e^{-\beta_l|\xi|}\left(\cos(\beta_j\xi)\widetilde{C}_{jl}^R(\zeta,\xi)+\sin(\beta_j\xi)\widetilde{C}_{jl}^I(\zeta,\xi)\right)b_j^R\right]\partial_{b_j^I} \quad \text{(B.101)}$$

where

$$\widetilde{C}_{jl}^R(\zeta,\xi) := \Gamma_{jl}\Gamma_{lj}\nu''(\zeta)\nu''(\zeta+\xi)+\Theta_{jl}\Gamma_{lj}\nu'(\zeta)\nu'''(\zeta+\xi)$$
$$-\beta_j^2\Theta_{jl}\Theta_{lj}\nu'(\zeta)\nu'(\zeta+\xi) \quad \text{(B.102)}$$

$$\widetilde{C}_{jl}^I(\zeta,\xi) := -\beta_j\Gamma_{jl}\Theta_{lj}\nu''(\zeta)\nu'(\zeta+\xi)-\beta_j(\Theta_{jl}\Gamma_{lj}+\Theta_{jl}\Theta_{lj})\nu'(\zeta)\nu''(\zeta+\xi) \quad \text{(B.103)}$$

Then

$$\lim_{L\to\infty}\frac{1}{L}\int_0^L ds\mathbb{E}[G_j^{(1)}] = \left[\frac{\sigma^2}{2\beta_j}\left[-\frac{3}{4}\mu_j^2\mathcal{R}(0)+\left(\frac{(\pi j)^2}{12}+\frac{1}{16}\right)\mathcal{R}''(0)\right]b_j^I\right.$$
$$+\sum_{l>\mathcal{N}}\frac{\sigma^2\Gamma_{jl}^2}{2\beta_j\beta_l}\int_0^{\infty}d\xi e^{-\beta_l\xi}\left[\left(\mathcal{R}^{(4)}(\xi)-4\beta_j^2\mathcal{R}''(\xi)\right)\cos(\beta_j\xi)\right.$$
$$\left.\left.-4\beta_j\mathcal{R}^{(3)}(\xi)\sin(\beta_j\xi)\right]b_j^I\right]\partial_{b_j^R} \quad \text{(B.104)}$$

and

$$\lim_{L\to\infty}\frac{1}{L}\int_0^L ds\mathbb{E}[G_j^{(2)}] = \left[\frac{\sigma^2}{2\beta_j}\left[\frac{3}{4}\mu_j^2\mathcal{R}(0) - \left(\frac{(\pi j)^2}{12}+\frac{1}{16}\right)\mathcal{R}''(0)\right]b_j^R\right.$$
$$-\sum_{l>\mathcal{N}}\frac{\sigma^2\Gamma_{jl}^2}{2\beta_j\beta_l}\int_0^\infty d\xi e^{-\beta_l\xi}\left[\left(\mathcal{R}^{(4)}(\xi)-4\beta_j^2\mathcal{R}''(\xi)\right)\cos(\beta_j\xi)\right.$$
$$\left.\left.-4\beta_j\mathcal{R}^{(3)}(\xi)\sin(\beta_j\xi)\right]b_j^R\right]\partial_{b_j^I} \tag{B.105}$$

where we used

$$\mathbb{E}[\nu^2(0)] = \mathcal{R}(0) \tag{B.106}$$

$$\mathbb{E}[\nu'^2(0)] = -\mathcal{R}''(0) \tag{B.107}$$

$$\begin{aligned}\mathbb{E}[\widetilde{C}_{jl}^R(0,\xi)] &= (\Gamma_{jl}\Gamma_{lj}-\Theta_{jl}\Gamma_{lj})\mathcal{R}^{(4)}(\xi)+\beta_j^2\Theta_{jl}\Theta_{lj}\mathcal{R}''(\xi)\\ &= \Gamma_{jl}^2\left(\mathcal{R}^{(4)}(\xi)-4\beta_j^2\mathcal{R}''(\xi)\right)\end{aligned} \tag{B.108}$$

$$\begin{aligned}\mathbb{E}[\widetilde{C}_{jl}^I(0,\xi)] &= \beta_j(\Theta_{jl}\Gamma_{lj}+\Theta_{jl}\Theta_{lj}-\Gamma_{jl}\Theta_{lj})\mathcal{R}'''(\xi)\\ &= -4\beta_j\Gamma_{jl}^2\mathcal{R}^{(3)}(\xi)\end{aligned} \tag{B.109}$$

as well as properties of even and odd functions.

Then

$$\mathcal{L}_2 = \sum_j\left\{\frac{\sigma^2}{\beta_j}\left[-\frac{3}{4}\mu_j^2\mathcal{R}(0)+\left(\frac{(\pi j)^2}{12}+\frac{1}{16}\right)\mathcal{R}''(0)\right]\right.$$
$$+\sum_{l>\mathcal{N}}\frac{\sigma^2\Gamma_{jl}^2}{\beta_j\beta_l}\int_0^\infty d\xi e^{-\beta_l\xi}\left[\left(\mathcal{R}^{(4)}(\xi)-4\beta_j^2\mathcal{R}''(\xi)\right)\cos(\beta_j\xi)\right.$$
$$\left.\left.-4\beta_j\mathcal{R}^{(3)}(\xi)\sin(\beta_j\xi)\right]\right\}\operatorname{Im}\left(b_j\partial_{b_j}\right). \tag{B.110}$$

We can simplify further using integration by parts and obtain

$$
\begin{aligned}
\mathcal{L}_2 = \sum_j \Bigg\{ & \frac{\sigma^2}{\beta_j}\left[-\frac{3}{4}\mu_j^2\mathcal{R}(0) + \left(\frac{(\pi j)^2}{12} + \frac{1}{16}\right)\mathcal{R}''(0)\right] \\
& + \sum_{l>\mathcal{N}} \frac{\sigma^2\Gamma_{jl}^2}{\beta_j\beta_l}\Big[\int_0^\infty d\xi\mathcal{R}''(\xi)e^{-\beta_l\xi}\left[2\left(\beta_l^2-\beta_j^2\right)\cos(\beta_j\xi)\right. \\
& \left.-4\beta_j\beta_l\sin(\beta_j\xi)\right]\mathrm{Im}\left(b_j\partial_{b_j}\right) - 2\beta_l\mathcal{R}''(0)\Big]\Bigg\}\,\mathrm{Im}\left(b_j\partial_{b_j}\right) \\
= \sum_j \Bigg\{ & \frac{\sigma^2}{\beta_j}\left[-\frac{3}{4}\mu_j^2\mathcal{R}(0) + \left(\frac{(\pi j)^2}{12} + \frac{1}{16}\right)\mathcal{R}''(0)\right] \\
& + \sum_{l>\mathcal{N}} \frac{\sigma^2\mu_j^2\mu_l^2}{\beta_j\beta_l(\beta_j^2+\beta_l^2)^2}\Big[\int_0^\infty d\xi\mathcal{R}''(\xi)e^{-\beta_l\xi}\left[2\left(\beta_l^2-\beta_j^2\right)\cos(\beta_j\xi)\right. \\
& \left.-4\beta_j\beta_l\sin(\beta_j\xi)\right]\mathrm{Im}\left(b_j\partial_{b_j}\right) - 2\beta_l\mathcal{R}''(0)\Big]\Bigg\}\,\mathrm{Im}\left(b_j\partial_{b_j}\right)
\end{aligned} \tag{B.111}
$$

## B.4 Change of Variables

We can express the generator in polar coordinates by writing the backward going mode amplitudes as $b_j = P_j^{\frac{1}{2}}e^{i\psi_j}$. Then

$$
\partial_{b_j} = P_j^{\frac{1}{2}}e^{-i\psi_j}\partial_{P_j} - \frac{ie^{-i\psi_j}}{2P_j^{\frac{1}{2}}}\partial_{\psi_j}, \tag{B.112}
$$

$$
\partial_{\bar{b}_j} = P_j^{\frac{1}{2}}e^{i\psi_j}\partial_{P_j} + \frac{ie^{i\psi_j}}{2P_j^{\frac{1}{2}}}\partial_{\psi_j}. \tag{B.113}
$$

Then we can write

$$
\mathrm{Re}\left(b_jb_q\partial^2_{b_jb_q} + b_j\partial_{b_j}\right) = P_jP_q\partial^2_{P_jP_q} - \frac{1}{4}\partial^2_{\theta_j\theta_q} + P_j\partial_{P_j} \tag{B.114}
$$

$$
\mathrm{Re}\left(b_q\bar{b}_q\partial^2_{b_j\bar{b}_j}\right) = P_jP_q\partial^2_{P_j} + P_q\partial_{P_j} + \frac{P_q}{4P_j}\partial^2_{\theta_j} \tag{B.115}
$$

$$
\mathrm{Re}\left(b_j\bar{b}_q\partial^2_{b_j\bar{b}_q}\right) - \mathrm{Re}\left(b_jb_q\partial^2_{b_jb_q}\right) - \mathrm{Re}\left(b_j\partial_{b_j}\right)\delta_{jq} = \frac{1}{2}\partial^2_{\theta_j\theta_q} \tag{B.116}
$$

$$
\mathrm{Im}\left(b_j\partial_{b_j}\right) = -\frac{1}{2}\partial_{\theta_j} \tag{B.117}
$$

Combining expressions (B.95), (B.111) and using the identites above generator can then be written as in section 3.9.

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