Dynamics on the moduli space of pointed rational curves

by

Rohini Ramadas

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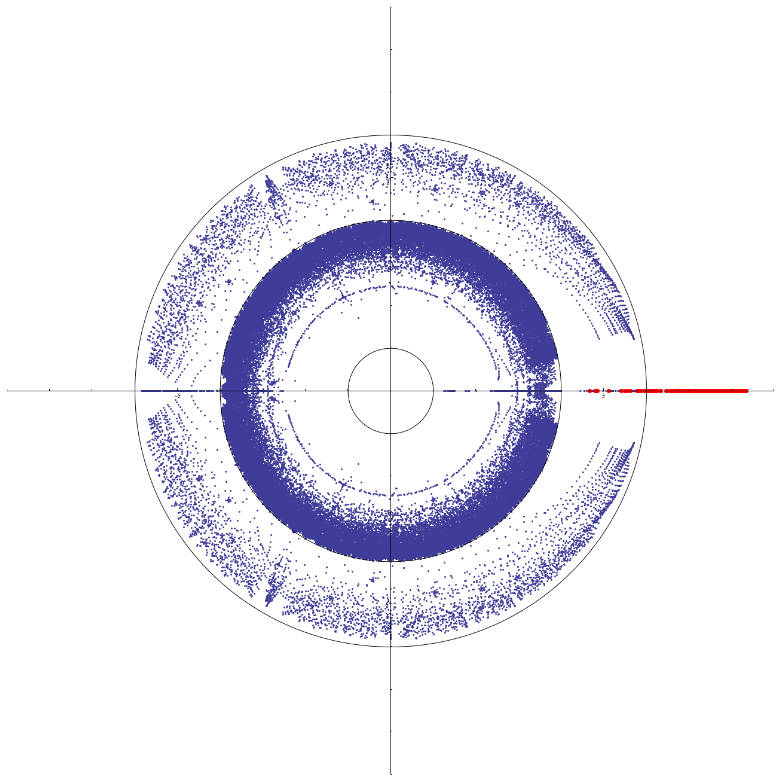
Doctoral Committee:

Professor Sarah Koch, Co-Chair
Professor David Speyer, Co-Chair
Professor William Fulton
Professor Mattias Jonsson
Professor Leopoldo Pando Zayas
Professor Karen Smith



Rohini Ramadas

ramadas@umich.edu

ORCID id: 0000-0001-5974-7115

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ABSTRACT

The moduli space $\mathcal{M}_{0,n}$ parametrizes all ways of labelling n distinct points on the Riemann sphere $\mathbb{P}^1$, up to change of coordinates by Möbius transformations. Hurwitz correspondences are certain multi-valued self-maps of $\mathcal{M}_{0,n}$. They arise in topology and Teichmüller theory by works of Thurston and Koch. In this thesis, we study the dynamics of Hurwitz correspondences via numerical invariants called dynamical degrees.

CHAPTER 1

Introduction

The moduli spaces $\mathcal{M}_{g,n}$ classify smooth genus g algebraic curves/Riemann surfaces with n distinct labelled points. They are non-compact algebraic varieties whose Deligne-Mumford compactifications $\overline{\mathcal{M}}_{g,n}$ have a rich combinatorial structure. There are several tautological maps among the various moduli spaces $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$. It is common to use tautological maps to study all of these spaces together as one system of related spaces, rather than to study each in isolation.

Hurwitz correspondences are a certain family of tautological multivalued maps among moduli spaces of curves. They have been used by algebraic geometers to study the cohomology rings of $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ for $g \geq 1$. In [Koc13], Koch introduced the idea of studying Hurwitz self-correspondences of $\mathcal{M}_{0,n}$ as dynamical systems by establishing a connection to topology and Teichmüller theory. In this work, we systematically investigate Hurwitz correspondences in genus 0. We focus particularly on the dynamics of self-correspondences, studying them via numerical invariants called dynamical degrees.

In this chapter, we give brief introductions to moduli spaces of curves in genus 0, previous studies of Hurwitz correspondences in positive genus, the connection between genus 0 Hurwitz self-correspondences and Teichmüller theory, and dynamical degrees. (Most of these topics will be discussed in more detail in Chapter 2.) We then summarize the results in this thesis.

1.1 Moduli spaces of genus 0 curves

Every smooth genus 0 algebraic curve/Riemann surface is isomorphic to the Riemann sphere $\mathbb{P}^1$. The moduli space $\mathcal{M}_{0,n}$ parametrizes n-marked genus 0 smooth curves, or, equivalently, configurations of n distinct labelled points on $\mathbb{P}^1$ up to automorphisms of $\mathbb{P}^1$. The automorphism group $\mathbb{P}GL_2$ of $\mathbb{P}^1$ is 3-dimensional. Given two ordered triples of distinct points on $\mathbb{P}^1$, there is a unique element of $\mathbb{P}GL_2$ that takes one to the other. This implies that each of the moduli spaces $\mathcal{M}_{0,0}$,

$\mathcal{M}_{0,1}$, $\mathcal{M}_{0,2}$ and $\mathcal{M}_{0,3}$ consists of a single point. For $n \geq 4$, two configurations of n points on $\mathbb{P}^1$ might be truly inequivalent. Given $p_1, \ldots, p_4 \in \mathbb{P}^1$, there is a unique automorphism of $\mathbb{P}^1$ taking the triple (p_1, p_2, p_3) to $(0, 1, \infty)$. The fourth point p_4 gets taken to $\frac{(p_4-p_1)(p_2-p_3)}{(p_4-p_3)(p_2-p_1)}$. This is a complex number that is neither 0 nor 1, and is called the *cross-ratio* of the ordered tuple $(p_1, \ldots, p_4)$. Thus quadruples of points in $\mathbb{P}^1$ are exactly classified up to projective equivalence by the data of their cross-ratio. This identifies the moduli space $\mathcal{M}_{0,4}$ with $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Similarly for $n \geq 4$, the moduli space $\mathcal{M}_{0,n}$ of smooth genus 0 curves with n distinct labelled points is isomorphic to $((\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \text{all diagonals})$.

As it is a parameter space for configurations of n distinct points on $\mathbb{P}^1$, $\mathcal{M}_{0,n}$ is not compact. Its Deligne Mumford compactification $\overline{\mathcal{M}}_{0,n}$ is a smooth projective variety that parametrizes genus 0 stable curves (Definition 2.1.4) with n distinct marked points. The boundary $\overline{\mathcal{M}}_{0,n} \setminus \mathcal{M}_{0,n}$ has codimension 1; points on the boundary correspond to nodal curves whose irreducible components are isomorphic to $\mathbb{P}^1$.

Example 1.1.1. Let $n = 4$. As above, we fix an isomorphism $\mathcal{M}_{0,4} \cong (\mathbb{P}^1 \setminus \{0, 1, \infty\})$ with coordinate z by writing a 4-marked genus 0 curve $(C, p_1, \ldots, p_4)$ as $(\mathbb{P}^1, 0, 1, \infty, z)$. We have $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$ (see Figure 1.1), where

- the point $(z = 0) \in \mathbb{P}^1$ corresponds to a nodal curve with two irreducible components, with smooth points p_1 and p_4 on one component and smooth points p_2 and p_3 on the other,
- the point $(z = 1) \in \mathbb{P}^1$ corresponds to a nodal curve with two irreducible components, with smooth points p_2 and p_4 on one component and smooth points p_1 and p_3 on the other, and
- the point $(z = \infty) \in \mathbb{P}^1$ corresponds to a nodal curve with two irreducible components, with smooth points p_3 and p_4 on one component and smooth points p_1 and p_2 on the other.

Example 1.1.2. Let $n = 5$. The moduli space $\overline{\mathcal{M}}_{0,5}$ is a smooth surface. For each partition of $\{1, \ldots, 5\}$ into disjoint sets $\{i_1, i_2\}$ and $\{i_3, i_4, i_5\}$, the boundary of $\overline{\mathcal{M}}_{0,5}$ contains an irreducible divisor $\mathrm{Div}_{\overline{\mathcal{M}}_{0,5}}(\{\{i_1, i_2\}, \{i_3, i_4, i_5\}\})$ (isomorphic to $\overline{\mathcal{M}}_{0,4}$) that generically parametrizes nodal curves with one irreducible component containing the marked points p_{i_1} and p_{i_2} and a second irreducible component containing the marked points p_{i_3}, p_{i_4} and p_{i_5}. Thus there are 10 irreducible boundary divisors in all.

Two divisors $\mathrm{Div}_{\overline{\mathcal{M}}_{0,5}}(\{\{i_1, i_2\}, \{i_3, i_4, i_5\}\})$ and $\mathrm{Div}_{\overline{\mathcal{M}}_{0,5}}(\{\{i_1, i_2, i_3\}, \{i_4, i_5\}\})$ intersect at a point that corresponds to a chain of three irreducible components, the first containing the marked points p_{i_1} and p_{i_2}, the second containing the marked point p_{i_3}, and the third containing p_{i_4} and p_{i_5}. See Figure 1.2.

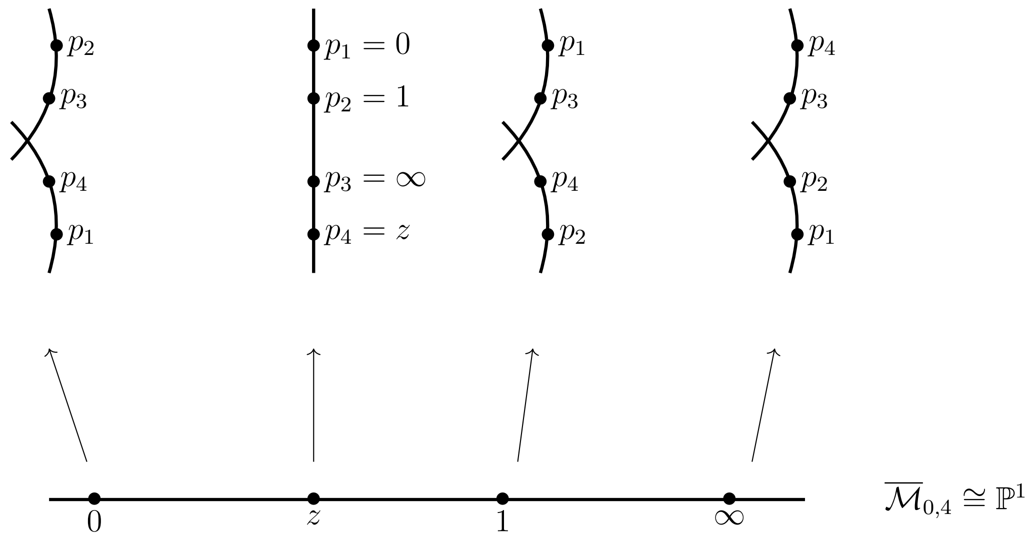


Figure 1.1: The moduli space $\overline{\mathcal{M}}_{0,4}$ is isomorphic to $\mathbb{P}^1$

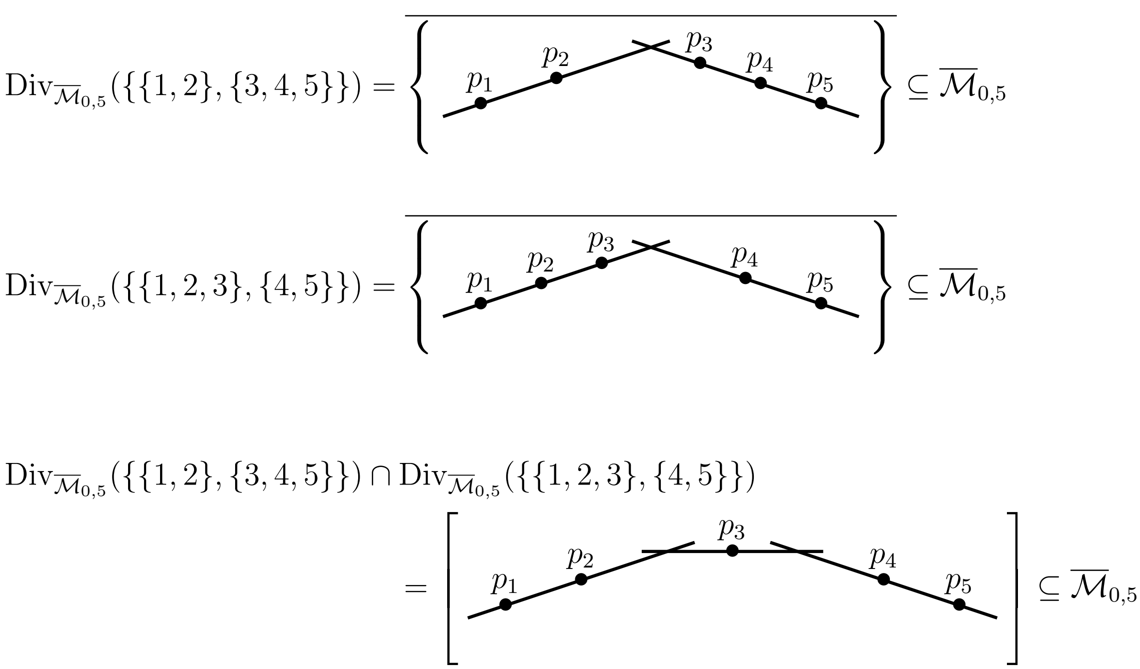


Figure 1.2: Intersecting boundary divisors in $\overline{\mathcal{M}}_{0,5}$

1.2 Tautological maps and Hurwitz correspondences in algebraic geometry

A map from $\mathcal{M}_{0,n}$ to $\mathcal{M}_{0,n'}$ gives an algorithm for producing a configuration of n' distinct points on $\mathbb{P}^1$ given a configuration of n distinct points.

Example 1.2.1. Let $n' < n$. There is a *forgetful* map $\mu : \mathcal{M}_{0,n} \to \mathcal{M}_{0,n'}$ that sends a configuration $[(\mathbb{P}^1, p_1, \ldots, p_n)]$ to $[(\mathbb{P}^1, p_1, \ldots, p_{n'})]$. This map extends to a map $\mu : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,n'}$ (see Section 2.1.1.3).

Example 1.2.2. Let $n' = n$. An element ω of the symmetric group S_n defines a self-map of $\mathcal{M}_{0,n}$ by relabelling the marked points, i.e.

$$[(\mathbb{P}^1, p_1, \ldots, p_n)] \mapsto [(\mathbb{P}^1, p_{\omega(1)}, \ldots, p_{\omega(n)})].$$

This relabelling map extends to a regular self-map of $\overline{\mathcal{M}}_{0,n}$.

Forgetful maps and relabelling maps are referred to as *tautological*, since the algorithms for producing an n'-marked $\mathbb{P}^1$ given an n-marked $\mathbb{P}^1$ are so explicit. In both cases, the coordinates on $\mathcal{M}_{0,n'}$ are degree 1 functions of the coordinates of $\mathcal{M}_{0,n}$. In fact, every map from $\mathcal{M}_{0,n}$ to $\mathcal{M}_{0,n'}$ can be written as a forgetful map followed by a relabelling map ([RS]). The following example illustrates what goes wrong in trying to build a map from $\mathcal{M}_{0,n}$ to $\mathcal{M}_{0,n'}$ out of higher degree (=2) functions.

Example 1.2.3. Let $n' = n = 5$. We fix an identification $\mathcal{M}_{0,5} \cong ((\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})^2 \smallsetminus \text{diagonal})$ by always writing a 5-marked genus 0 curve as $(\mathbb{P}^1, 0, 1, \infty, z_1, z_2)$. The map $sq : \mathcal{M}_{0,5} \dashrightarrow \mathcal{M}_{0,5}$ sending $[(\mathbb{P}^1, 0, 1, \infty, z_1, z_2)]$ to $[(\mathbb{P}^1, 0, 1, \infty, z_1^2, z_2^2)]$ is not defined where $z_i = -1$ or $z_1 = -z_2$: even if the points $0, 1, \infty, z_1, z_2$ are distinct, $0, 1, \infty, z_1^2, z_2^2$ are not.

Although there are no maps from $\mathcal{M}_{0,n}$ to $\mathcal{M}_{0,n'}$ given in coordinates by functions of degree greater than 1, the following example shows that there are multivalued maps given in coordinates by functions of degree < 1. (See Section 2.2 for a precise definition of a multivalued map).

Example 1.2.4. Let $n' = n = 5$. As in Example 1.2.3, we identify $\mathcal{M}_{0,5}$ with $((\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})^2 \smallsetminus \text{diagonal})$. There is a well-defined multivalued map $sqrt : \mathcal{M}_{0,5} \rightrightarrows \mathcal{M}_{0,5}$ sending the configuration $[(\mathbb{P}^1, 0, 1, \infty, z_1, z_2)]$ to the unordered 4-tuple

$$\{[(\mathbb{P}^1, 0, 1, \infty, +\sqrt{z_1}, +\sqrt{z_2})], [(\mathbb{P}^1, 0, 1, \infty, +\sqrt{z_1}, -\sqrt{z_2})],$$
$$[(\mathbb{P}^1, 0, 1, \infty, -\sqrt{z_1}, +\sqrt{z_2})], [(\mathbb{P}^1, 0, 1, \infty, -\sqrt{z_1}, -\sqrt{z_2})]\}$$

The map $sqrt$ has a coordinate-free description as follows. There is a moduli space $\mathcal{H}_{sqrt}$ that parametrizes

- a 5-marked smooth genus 0 curve $(C, p_1, \ldots, p_5)$,
- a 5-marked smooth genus 0 curve $(D, q_1, \ldots, q_5)$, and
- a degree 2 map $f : C \to D$ sending p_i to q_i, and ramified at p_1 and p_3

The space $\mathcal{H}_{sqrt}$ admits two maps to $\mathcal{M}_{0,5}$: a "target" map π_t and a "source" map π_s sending $[f : (C, p_1, \ldots, p_5) \to (D, q_1, \ldots q_5)]$ to $[(C, p_1, \ldots, p_5)]$ and $[(D, q_1, \ldots, q_5)]$ respectively. The map π_t is a $4 : 1$ covering map, so admits a multi- (4-) valued inverse. The multivalued map $sqrt$ is equal to $\pi_s \circ \pi_t^{-1}$, thus we have

$$\begin{aligned} sqrt([(D, q_1, \ldots, q_5)]) = \{[(C, p_1, \ldots, p_5)] | \exists f : C \to D \text{ with} \\ p_i \mapsto q_i, f \text{ is ramified at } p_1 \text{ and } p_3\} \end{aligned} \tag{1.1}$$

Hurwitz correspondences generalize the multivalued map $sqrt$ in Example 1.2.4. Parameter spaces for maps, with prescribed degree and branching, between algebraic curves of prescribed genus are called *Hurwitz spaces*. Let $\mathcal{H}$ be a Hurwitz space parametrizing degree d maps from n'-marked curves of genus g' to n-marked curves of genus g, with prescribed branching at and over the marked points of the source and target curves respectively, and prescribed mapping between the two sets of marked points. $\mathcal{H}$ admits a "target" map π_t to $\mathcal{M}_{g,n}$ and a "source" map π_s to $\mathcal{M}_{g',n'}$. If all branch values on the target curve are marked (as in Example 1.2.4), then π_t is a covering map, so $\pi_s \circ \pi_t^{-1}$ is a multivalued map — a Hurwitz correspondence — from $\mathcal{M}_{g,n}$ to $\mathcal{M}_{g',n'}$ (See Definition 2.3.3 for a precise definition in the case $g = g' = 0$). The Hurwitz correspondence $\mathcal{H}$ has an explicit description as a multivalued map: $[(C, p_1, \ldots, p_{n'})] \in \mathcal{M}_{g',n'}$ is in the image of $[(D, q_1, \ldots, q_n)] \in \mathcal{M}_{g,n}$ if and only if C admits a map to D as specified. For this reason, in analogy with forgetful and relabelling maps, we will refer to Hurwitz correspondences as tautological. In fact, forgetful and relabelling maps are Hurwitz correspondences that happen to be single-valued. Harris and Mumford ([HM82]) construct compactifications of Hurwitz spaces by moduli spaces of admissible covers. It follows from the properties of the spaces of admissible covers that a Hurwitz correspondence from $\mathcal{M}_{g,n}$ to $\mathcal{M}_{g',n'}$ extends continuously to a tautological multivalued regular map from $\overline{\mathcal{M}}_{g,n}$ to $\overline{\mathcal{M}}_{g',n'}$ (see Chapter 3).

Hurwitz correspondences from $\mathcal{M}_{g,n}$ to $\mathcal{M}_{g',n'}$ have been most studied in the setting where $g = 0$ and $g' > 0$. In [HM82], Harris and Mumford use such a Hurwitz correspondence to compute the class of the canonical divisor on $\overline{\mathcal{M}}_{g'}$ for odd g'; they show that $\overline{\mathcal{M}}_{g'}$ is of general type for g' at least 25 and odd. In ([Ion01]), Ionel showed using Hurwitz correspondences that when g' is at least 2, special cohomology classes called tautological classes on $\overline{\mathcal{M}}_{g',n'}$ that have codimension at least g' vanish when restricted to $\mathcal{M}_{g',n'}$.

1.3 Hurwitz correspondences in complex dynamics

There has been a great deal of study of rational functions $f : \mathbb{P}^1 \to \mathbb{P}^1$ from the points of view of complex and topological dynamics. There is a class of rational functions called *postcritically finite* rational functions that has been of particular interest. A rational function $f : \mathbb{P}^1 \to \mathbb{P}^1$ is called postcritically finite if its *post-critical set*

$$\{f^r(x) | \ x \text{ is a critical point of } f \text{ and } r > 0\}$$

is finite. The dynamics of a postcritically finite rational function on all of $\mathbb{P}^1$ are largely governed by its dynamics on its postcritical set. In [DH93], Thurston gave a topological characterization of postcritically finite rational functions as discussed below.

Topologically, $\mathbb{P}^1$ is the sphere S^2. Let $\phi : S^2 \to S^2$ be an orientation-preserving branched covering. A point x on S^2 is a critical point of ϕ if x has no neighbourhood on which ϕ is a covering map; in a small neighbourhood of x, the branched covering ϕ looks like $z \mapsto z^d$ for some $d > 1$. Suppose ϕ is postcritically finite – is it conjugate, up to homotopy, to a postcritically finite rational function on $\mathbb{P}^1$?

There is a non-algebraic complex manifold called Teichmüller space $\mathcal{T}_{0,n}$ parametrizing complex structures on S^2 with n labelled punctures, up to homotopy relative to the punctures. A point in Teichmüller space specifies an identification of an n-punctured topological sphere with an n-punctured copy of $\mathbb{P}^1$, thus in particular specifies a configuration of n points on $\mathbb{P}^1$. This defines a map from $\mathcal{T}_{0,n}$ to $\mathcal{M}_{0,n}$, realizing $\mathcal{T}_{0,n}$ as the universal cover of $\mathcal{M}_{0,n}$.

Denote by $\mathbf{P}$ the finite postcritical set of ϕ. Given one complex structure on $(S^2, \mathbf{P})$, we can pull it back along the branched covering ϕ to obtain another. This defines a holomorphic map called the Thurston pullback map $\mathrm{Thurst}(\phi)$ from $\mathcal{T}_{0,n}$ to itself. An insight of Thurston was to study $\mathrm{Thurst}(\phi)$ as a dynamical system; a fixed point of $\mathrm{Thurst}(\phi)$ is a complex structure under which ϕ is identified with a rational function $f : \mathbb{P}^1 \to \mathbb{P}^1$. Thus the branched covering ϕ is conjugate, up to homotopy, to a rational function on $\mathbb{P}^1$ if and only if $\mathrm{Thurst}(\phi)$ has a fixed point. A major result of Thurston [DH93] gave an explicit topological criterion on ϕ for $\mathrm{Thurst}(\phi)$ to have a fixed point.

The Thurston pullback map $\mathrm{Thurst}(\phi)$ does not in general descend to a self-map of $\mathcal{M}_{0,n}$. However, Koch ([Koc13]) showed that it descends to a Hurwitz correspondence $\mathcal{H}(\phi)$ from $\mathcal{M}_{0,n}$ to itself. Thus the dynamics of $\mathcal{H}(\phi)$ provide an algebraic "shadow" of the holomorphic dynamics of $\mathrm{Thurst}(\phi)$. (See Section 2.3.2 for details.)

$\mathcal{M}_{0,n}$ is birationally equivalent to projective space $\mathbb{P}^{n-3}$, thus a Hurwitz correspondence $\mathcal{H}$ on $\mathcal{M}_{0,n}$ may be regarded as a rational multivalued self-map of $\mathbb{P}^{n-3}$. Koch showed that for certain branched coverings ϕ, the correspondence $\mathcal{H}(\phi)$ has a single-valued inverse. Koch also characterized branched coverings ϕ for which $(\mathcal{H}(\phi))^{-1} : \mathbb{P}^{n-3} \to \mathbb{P}^{n-3}$ is a single-valued regular map that has a property called *critical finiteness*. This discovery yields a large family of self-maps of projective space whose complex dynamical behavior can be studied via a link to topology and Teichmüller theory.

1.4 Dynamical degrees

Dynamical degrees are numerical invariants associated to generalized self-maps — single-valued or multivalued, regular or rational — of a smooth projective variety ([Fri91, RS97]). They extend the notion of topological degree of a continuous self-map of an oriented compact manifold, and are important as measures of dynamical complexity.

Topological entropy is a fundamental numerical invariant associated to a continuous self-map of a compact Hausdorff topological space. It is difficult to compute in most examples. (See [You03] for a definition and survey). Let $g : X \to X$ be a surjective regular self-map of a smooth projective variety. The topological entropy of g is defined independently of its algebraic structure. However Gromov ([Gro03]) and Yomdin ([Yom87]) showed, using the algebraicity of g, that its topological entropy could be computed via its pushforward actions on the singular homology groups of X. Specifically, the topological entropy of g is

$$\max_{k=0,\ldots,\mathbb{C}\dim(X)} \log |\text{dominant eigenvalue of } g_* : H_{2k}(X,\mathbb{R}) \to H_{2k}(X,\mathbb{R})|.$$

The absolute value of the dominant eigenvalue of the action of g_* on $H_{2k}(X,\mathbb{R})$ is called the the *k-th dynamical degree of g*.[1] The top dynamical degree of a map is equal to its topological degree.

The definition of topological entropy can be extended to rational maps and rational multivalued maps of projective varieties. Let $g : X \dashrightarrow X$ or $g : X \overset{-}{\underset{-}{\rightrightarrows}} X$ be a dominant, rational, possibly

[1] The usual, equivalent, definition of dynamical degrees is via pullback actions on cohomology groups, see Section 2.2

multivalued self-map of a smooth projective variety. Although g is not everywhere defined, it is possible to define a linear pushforward action $g_* : H_{2k}(X, \mathbb{R}) \to H_{2k}(X, \mathbb{R})$ (see Section 2.2 for details). However, we may have $(g^r)_* \neq (g_*)^r$, that is, pushforwards do not necessarily respect composition. In these settings, the dominant eigenvalue of g_* is not a useful dynamical invariant; for example it does not provide any information about topological entropy. However, we obtain a useful invariant by considering the asymptotics of the pushforwards $(g^r)_*$. Pick any norm on $H_{2k}(X, \mathbb{R})$ and set the k-th dynamical degree of g to be the non-negative real number:

$$\lim_{r \to \infty} \|(g^r)_* : H_{2k}(X, \mathbb{R}) \to H_{2k}(X, \mathbb{R})\|^{1/r}.$$

Dinh and Sibony ([DS05, DS08]) showed that this number is well-defined and a birational invariant of g. They also showed that the topological entropy of g is at most the logarithm of its largest dynamical degree.

Example 1.4.1. Let $g : \mathbb{P}^N \to \mathbb{P}^N$ be a surjective regular map given in coordinates by homogeneous polynomials of degree d. Then the image under g of a generic k-plane is a subvariety of degree d^k, and so $g_* : H_{2k}(\mathbb{P}^N, \mathbb{R}) \to H_{2k}(\mathbb{P}^N, \mathbb{R})$ is multiplication by d^k. Also, each iterate g^r is a surjective regular map given in coordinates by homogeneous polynomials of degree d^r. Here, the k-th dynamical degree of g is d^k, the topological degree of g is d^N, and the topological entropy of g is $N \log d$.

Now, let $g : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ be a dominant rational map given in coordinates by homogeneous polynomials of degree d. A generic line in $\mathbb{P}^N$ does not meet the codimension ≥ 2 indeterminacy locus of g, and the image under g of such a generic line is a curve of degree d. Thus $g_* : H_2(\mathbb{P}^N, \mathbb{R}) \to H_2(\mathbb{P}^N, \mathbb{R})$ is multiplication by d. However, for $k > 1$, it is possible that every k-plane meets the indeterminacy locus of g, and the image under g of a generic k-plane may not be a subvariety of degree d^k. Thus the pushforward $g_* : H_{2k}(\mathbb{P}^N, \mathbb{R}) \to H_{2k}(\mathbb{P}^N, \mathbb{R})$ is multiplication by some integer d_k, not necessarily equal to d^k.

Also, the r-th iterate of g is a dominant rational map given in coordinates by homogeneous polynomials of some degree, not necessarily d^r. Thus $(g^r)_* : H_2(\mathbb{P}^N, \mathbb{R}) \to H_2(\mathbb{P}^N, \mathbb{R})$ may not be multiplication by d^r, and for $k > 1$, $(g^r)_* : H_{2k}(\mathbb{P}^N, \mathbb{R}) \to H_{2k}(\mathbb{P}^N, \mathbb{R})$ may not be multiplication by $(d_k)^r$. Here, the 1-st dynamical degree of g measures the asymptotic growth of the degrees of the polynomials that give the iterates g^r. The kth dynamical degree measures the asymptotic growth of the degrees of the images of generic k-planes under the iterates g^r. See Section 2.2.4 for an example.

Given $g : X \dashrightarrow X$ or $g : X \rightrightarrows X$, suppose we have $(g^r)_* = (g_*)^r$ on $H_{2k}(X, \mathbb{R})$. That is, g

may or may not be regular, but it does behave like a regular map in terms of its action on $H_{2k}(X, \mathbb{R})$. Then g is called k-stable, and the kth dynamical degree of g is the absolute value of the dominant eigenvalue of $g_* : H_{2k}(X, \mathbb{R}) \to H_{2k}(X, \mathbb{R})$. The k-th dynamical degree of a k-stable map is, up to absolute value, an algebraic integer. Computing the dynamical degrees of a map which is not k-stable involves dealing with the pullbacks along infinitely many iterates. Thus there are only a few examples of rational maps, even single-valued, whose dynamical degrees have been computed. Notably, there are explicit formulas for the dynamical degrees of monomial self-maps of projective space (Jonsson and Wulcan [JW11] and Lin [Lin12]).

Since dynamical degrees are birational invariants, given $g : X \dashrightarrow X$, a common strategy to try to compute the k-th dynamical degree of g is to look for a birational model X' of X such that $g : X' \dashrightarrow X'$ is k-stable. However, there are examples (Favre [F$^+$03]) of $g : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ for which no such birational model exists. Given the difficulty in computing dynamical degrees, there are several open questions about them. For example, there is no dynamical degree that is known not to be an algebraic integer.

Diller and Favre ([DF01]) showed that every birational transformation g of a projective surface X becomes 1-stable on some birational model of X. They use this result to show that the first dynamical degree of g is either 1, a Salem number (a real positive algebraic integer bigger than 1 with all other Galois conjugates at most 1 in modulus, and at least one Galois conjugate whose modulus is exactly 1), or a Pisot number (a real positive algebraic integer bigger than 1 with all other Galois conjugates strictly smaller than 1 in modulus). Blanc and Cantat ([BC16]) describe the set of Salem and Pisot numbers that arise as dynamical degrees of birational surface transformations. For example, they show that the set of all dynamical degrees of all birational transformations of all surfaces that are not geometrically rational is a closed discrete subset of $\mathbb{R}$, and the set of all dynamical degrees of birational transformations of $\mathbb{P}^2$ is a closed and well-ordered subset of $\mathbb{R}$.

1.5 Previous work by Koch and Roeder on the dynamical degrees of Hurwitz correspondences

In [KR15], Koch and Roeder studied Hurwitz correspondences of the form $\mathcal{H}(\phi)$ on $\mathcal{M}_{0,n}$ where

- $\phi : S^2 \to S^2$ is a degree 2 branched covering with finite postcriticical set $\mathbf{P}$,
- $\phi : \mathbf{P} \to \mathbf{P}$ is a bijection, and
- $\mathbf{P}$ contains the two critical points of ϕ

By [Koc13], a correspondence in this class has a single-valued rational inverse $\mathcal{H}(\phi)^{-1}$: $\mathbb{P}^{n-3} \dashrightarrow \mathbb{P}^{n-3}$. Koch and Roeder showed that the map $\mathcal{H}(\phi)^{-1}$ is k-stable for all k on $\overline{\mathcal{M}}_{0,n}$ (thus so is the multivalued map $\mathcal{H}(\phi)$). This implies that

$$\begin{aligned}(k\text{-th dynamical degree of } \mathcal{H}_\phi) &= ((n-3-k)\text{-th dynamical degree of } \mathcal{H}_\phi^{-1}) \\ &= \left|\text{dominant eigenvalue of } \mathcal{H}(\phi)_* : H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{R}) \to H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{R})\right|\end{aligned}$$

Koch and Roeder computed the k-th dynamical degrees of the rational maps $\mathcal{H}(\phi)^{-1}$ for $(k, n) \in \{(1,5), (1,6), (2,6), (1,7), (1,8)\}$ by explicitly computing the pushforward matrices $\mathcal{H}(\phi)_*$.

The dimensions of the homology groups $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{R})$ grow exponentially with n. Thus the computations of $\mathcal{H}(\phi)_* : H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{R}) \to H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{R})$ are prohibitive for larger k and n.

1.6 Summary of this work

We study the behavior of Hurwitz correspondences $\mathcal{H}$ from $\mathcal{M}_{0,n}$ to $\mathcal{M}_{0,n'}$ by investigating their behavior on various compactifications of $\mathcal{M}_{0,n}$ and $\mathcal{M}_{0,n'}$, with special emphasis on the Deligne-Mumford compactifications. We characterize the induced pushforward maps

$$[\mathcal{H}]_*^{k,DM} := [\mathcal{H}]_* : H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{R}) \to H_{2k}(\overline{\mathcal{M}}_{0,n'}, \mathbb{R}),$$

and develop techniques to compute them. (Here, DM stands for "Deligne-Mumford".) We use our characterizations of the pushforwards to study the dynamical degrees of Hurwitz self-correspondences on $\mathcal{M}_{0,n}$. Our work relies heavily on the properties of the Harris-Mumford compactification of the Hurwitz space $\mathcal{H}$ by a space $\overline{\mathcal{H}}$ of admissible covers (see Section 2.4).

Our motivations in undertaking this study are:

1. to understand Hurwitz correspondences as tautological multivalued maps between moduli spaces of genus-zero curves, and to use Hurwitz correspondences to better understand $\mathcal{M}_{0,n}$ and its compactifications,

2. to describe the dynamics of Hurwitz self-correspondences $\mathcal{H}(\phi)$ in the hope that the descriptions may, in the future, help to better understand the Thurston pullback maps $\mathrm{Thurst}(\phi)$, and

3. to introduce tools to compute and study the dynamical degrees of a large class of multivalued and single-valued maps.

We begin by giving, in **Chapter 2**, detailed introductions to $\mathcal{M}_{0,n}$ and its compactifications, multivalued maps (in later chapters we refer to multivalued maps as correspondences), dynamical degrees, Hurwitz correspondences, and moduli spaces of admissible covers.

Chapter 3 is joint work with Sarah Koch and David Speyer. Here we show that a Hurwitz correspondence $\mathcal{H}$ from $\mathcal{M}_{0,n}$ to $\mathcal{M}_{0,n'}$ extends to a regular multivalued map from $\overline{\mathcal{M}}_{0,n}$ to $\overline{\mathcal{M}}_{0,n'}$. This implies that the pushforward maps induced by Hurwitz correspondences between the homology groups of the Deligne-Mumford compactifications respect composition of correspondences, and in particular we obtain:

Theorem 1.6.1. [Koch, Ramadas and Speyer; Corollary 3.0.2 together with Remark 3.0.3] *Let $\mathcal{H} : \mathcal{M}_{0,n} \rightrightarrows \mathcal{M}_{0,n}$ be a Hurwitz self-correspondence. Then $\mathcal{H}$ is k-stable for all k on $\overline{\mathcal{M}}_{0,n}$, and the dominant eigenvalue of $[\mathcal{H}]_*^{k,DM}$ is a real non-negative number equal to the k-th dynamical degree of $\mathcal{H}$.*

The boundary $\overline{\mathcal{M}}_{0,n} \smallsetminus \mathcal{M}_{0,n}$ is stratified; $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Z})$ is generated by the fundamental classes of the k-dimensional strata in the boundary. The linear maps $[\mathcal{H}]_*^{k,DM}$ encode the way in which $\mathcal{H}$ extends from $\mathcal{M}_{0,n}$ to $\overline{\mathcal{M}}_{0,n}$.

The Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,n}$ is "large": the number of k dimensional boundary strata in $\overline{\mathcal{M}}_{0,n}$, and, as mentioned previously, the dimensions of $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{R})$, grow exponentially with n. Thus it is quite difficult to compute the pushforward matrices $[\mathcal{H}]_*^{k,DM}$. In **Chapter 4**, we find a way to compute the dynamical degrees of a Hurwitz self-correspondence $\mathcal{H}$ on $\mathcal{M}_{0,n}$ without having to compute the matrices $[\mathcal{H}]_*^{k,DM}$.

We define a filtration of $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{R})$ by subspaces $\{\Lambda_{\leq\lambda,n}\}_\lambda$ indexed by the partially ordered set $\{$integer partitions λ of $k\}$. (In fact, this filtration is defined over $\mathbb{Z}$.) We show:

Theorem 1.6.2. [Theorem 4.1.6] *Let $\mathcal{H} : \mathcal{M}_{0,n} \rightrightarrows \mathcal{M}_{0,n'}$ be a Hurwitz correspondence. Then for every partition λ of k, $[\mathcal{H}]_*^{k,DM}$ takes the subspace $\Lambda_{\leq\lambda,n}$ to $\Lambda_{\leq\lambda,n'}$. In particular, if $n = n'$ we have that $\{\Lambda_{\leq\lambda,n}\}_\lambda$ is a filtration of $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{R})$ by $[\mathcal{H}]_*^{k,DM}$-invariant subspaces.*

Now, let $\mathcal{H}$ be a Hurwitz self-correspondence on $\mathcal{M}_{0,n}$. Theorem 1.6.2 implies that the filtration $\{\Lambda_{\leq\lambda,n}\}_\lambda$ can be used to write $[\mathcal{H}]_*^{k,DM}$ as a block-lower-triangular matrix; the diagonal blocks encode the actions induced on graded pieces of the filtration. The dominant eigenvalue of $[\mathcal{H}]_*^{k,DM}$ — the k-th dynamical degree of $\mathcal{H}$ — is equal to the dominant eigenvalue of at least one of the diagonal blocks of $[\mathcal{H}]_*^{k,DM}$.

Set $\Omega_{k,n}$ to be the quotient of $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{R})$ by all the *proper* subspaces $\Lambda_{\leq\lambda,n}$; so $\Omega_{k,n}$ is the top graded piece of the filtration $\{\Lambda_{\leq\lambda,n}\}_\lambda$. We show:

Theorem 1.6.3. [Theorem 4.2.7] *The k-th dynamical degree of $\mathcal{H}$ is equal to the dominant eigenvalue of*

$$[\mathcal{H}]_*^{k,\Omega} := [\mathcal{H}]_* : \Omega_{k,n} \to \Omega_{k,n}$$

Thus the dominant eigenvalue of the block-lower-triangular matrix $[\mathcal{H}]_*^{k,DM}$ lies in its topmost block.

Theorem 1.6.3 leads to the question: is $\Omega_{k,n}$ naturally isomorphic to the $2k$-th homology group of some alternate compactification X_n of $\mathcal{M}_{0,n}$ with the property that every Hurwitz correspondence in k-stable on X_n? That is, in our attempts to understand Hurwitz correspondences, is there a "smaller" — in terms of number of boundary strata and ranks of homology groups — compactification that we may restrict our attention to? We give a partial answer in **Chapter 5**. A *modular* compactification of $\mathcal{M}_{0,n}$ is one that is also a moduli space of n marked genus-zero curves (see Definition 2.1.19). We show:

Theorem 1.6.4. [Theorem 5.1.13 together with Proposition 5.1.16]

1. *There is a modular compactification $X_n^\dagger$ of $\mathcal{M}_{0,n}$ such that for $k \geq \frac{\dim(\mathcal{M}_{0,n})}{2}$, there is a natural isomorphism $H_{2k}(X_n^\dagger, \mathbb{R}) \cong \Omega_{k,n}$ and every Hurwitz correspondence on $\mathcal{M}_{0,n}$ is k-stable on $X_n^\dagger$.*
2. *For $k < \frac{\dim(\mathcal{M}_{0,n})}{2}$, there is no modular compactification X_n of $\mathcal{M}_{0,n}$ with a natural isomorphism $H_{2k}(X_n; \mathbb{R}) \cong \Omega_{k,n}$.*

Theorem 1.6.4 suggests that $\overline{\mathcal{M}}_{0,n}$ is possibly the only compactification of $\mathcal{M}_{0,n}$ on which every Hurwitz correspondence is k-stable for all k. We give a non-rigorous interpretation of Theorems 1.6.1, 1.6.2, 1.6.3, and 1.6.4 together as follows. In order to resolve all the indeterminacies of Hurwitz correspondences on some compactification of $\mathcal{M}_{0,n}$, we are forced to add to $\mathcal{M}_{0,n}$ the entire boundary of $\overline{\mathcal{M}}_{0,n}$. The k-th dynamical degree of a Hurwitz self-correspondence $\mathcal{H}$ on $\mathcal{M}_{0,n}$ a priori depends on the way in which $\mathcal{H}$ extends to all the k-dimensional boundary strata of $\overline{\mathcal{M}}_{0,n}$ However, the many boundary strata whose classes lie in one of the proper subspaces $\Lambda_{\leq\lambda,n}$ may in fact be disregarded; Theorem 1.6.3 isolates the dynamically relevant part of the action of $\mathcal{H}$ on the boundary of $\overline{\mathcal{M}}_{0,n}$.

We also address in Chapter 5 the question of how much smaller the dynamically relevant actions of Hurwitz correspondences on $\Omega_{k,n}$ are than their actions on $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{R})$ by comparing the dimensions of these vector spaces. The symmetric group S_n acts on $\mathcal{M}_{0,n}$, thus on $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{R})$; it also acts compatibly on $\Omega_{k,n}$. There are recursive formulas for the dimension of $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{R})$ (Keel, [Kee92]) and for its character as a representation of S_n (Bergström and Minabe, [BM13]).

However, no closed-form expression is known for either. In fact, it is not known if $H_{2k}(\overline{\mathcal{M}}_{0,n},\mathbb{R})$ is a permutation representation. We find that $\Omega_{k,n}$ is a permutation representation of S_n:

Theorem 1.6.5. [See Theorem 5.2.12 for a more precise statement] *$\Omega_{k,n}$ has a S_n-equivariant basis indexed by*

$$\{Q \subseteq \{1,...,n\} \,|\, |Q| \geq (k+3), |Q| = (k+3) \mod 2\}$$

.

Theorem 1.6.3 provides a way to reduce the size of the computation of the k-th dynamical degree of a Hurwitz correspondence $\mathcal{H}$: it suffices to compute a matrix of size $\dim(\Omega_{k,n})$ rather than of size $\dim(H_{2k}(\overline{\mathcal{M}}_{0,n},\mathbb{R}))$. The latter, by Theorem 1.6.5, equals

$$\sum_{\substack{i=(k+3)\\ i=(k+3) \bmod 2}}^{n} \binom{n}{i}.$$

Note that the biggest computational savings are obtained in small codimension: the dimension of $\Omega_{n-3-k,k}$ (fixed codimension k) is a degree k polynomial function of n. The dimension of the divisor class group $H_{2(n-4)}(\overline{\mathcal{M}}_{0,n},\mathbb{R})$ is $\frac{2^n-n^2+n-2}{2}$, whereas the dimension of its $\Omega_{n-4,n}$ is n. In contrast, we have for the class of curves that $H_2(\overline{\mathcal{M}}_{0,n},\mathbb{R}) = \Omega_{1,n}$; by Poncaré duality both vector spaces have dimension $\frac{2^n-n^2+n-2}{2}$.

The sequence of dynamical degrees of a single-valued map is log-concave. However, no analogous result is true for multivalued maps (Truong, [Tru16]). In **Chapter 6** we show:

Theorem 1.6.6. [Theorem 6.0.1] *Let $\mathcal{H} : \mathcal{M}_{0,n} \rightrightarrows \mathcal{M}_{0,n}$ be a Hurwitz self-correspondence, and let Θ_k be the kth dynamical degree of $\mathcal{H}$. Then*

$$\Theta_0 \geq \Theta_1 \geq \cdots \geq \Theta_{n-3}.$$

In fact, Θ_0 is the topological degree of the "target curve" map $\pi_t : \mathcal{H} \to \mathcal{M}_{0,n}$, and the top dynamical degree Θ_{n-3} is the topological degree of the "source curve" map π_s. Thus we obtain:

Corollary 1.6.7. *A generic configuration of n points $\mathbb{P}^1$ is the configuration of marked points on the source $\mathbb{P}^1$ of a map $[f : \mathbb{P}^1 \to \mathbb{P}^1] \in \mathcal{H}$ in fewer ways than it is the configuration of marked points on the target $\mathbb{P}^1$.*

The largest dynamical degree of a multivalued map is the most important, since it provides a bound for topological entropy. Another corollary of Theorem 1.6.6 is:

Corollary 1.6.8. *The topological entropy of $\mathcal{H}$ is less than or equal to* $\log \Theta_0$.

Thus the most important dynamical degree of $\mathcal{H}$ is Θ_0, the topological degree of the "target curve" map $\pi_t : \mathcal{H} \to \mathcal{M}_{0,n}$. In particular it is an integer and a *Hurwitz number*: Hurwitz numbers count covers of $\mathbb{P}^1$ having specified branch locus on $\mathbb{P}^1$ and specified ramification profile. They also count the number of ways to factor the identity in the symmetric group S_d as a product of n permutations with specified cycle types, that collectively generate a transitive subgroup. Thus the dynamically motivated quantity Θ_0 has a purely combinatorial interpretation. We see from Corollary 1.6.8 that the computational savings offered by Theorems 1.6.1, 1.6.2 and 1.6.3 do not help to bound topological entropy.

When $\mathcal{H}^{-1}$ is a single-valued rational function, then by Theorem 1.6.6, its topological degree is greater than or equal to all its other dynamical degrees. When the inequality is strict, which we expect "often," a result of Guedj [Gue05] provides an $\mathcal{H}^{-1}$-invariant measure on $\mathcal{M}_{0,n}$ with positive entropy.

Finally, in **Chapter 7**, we develop tools to compute dynamical degrees of Hurwitz correspondences. Given $\mathcal{H} : \mathcal{M}_{0,n} \rightrightarrows \mathcal{M}_{0,n'}$, we give a technique for using the stratification and local geometry (described in Sections 2.4.3 and 2.4.4 respectively) of the space $\overline{\mathcal{H}}$ of admissible covers to compute the linear maps $[\mathcal{H}]_*^{k,DM}$ and $[\mathcal{H}]_*^{k,\Omega}$. We implement these techniques in a family of examples, and compute dynamical degrees of correspondences in that family.

1.7 Notation and conventions

- All varieties and schemes are over $\mathbb{C}$.
- A partition λ of a positive integer k is a multiset of positive integers whose sum with multiplicity is k. For example, we write $(1, 1, 2)$ for the partition $4 = 1+1+2$. If $\lambda(j)$ is a partition of $k(j)$, then we denote by $\cup_j \lambda(j)$ the multiset union, which is a partition of $\sum_j k(j)$. The partition λ has
$$\prod_{r \in \lambda} (\text{multiplicity of } r \text{ in } \lambda)$$
automorphisms.
- A multiset λ_1 is a ***submultiset*** of λ_2 if for all $r \in \lambda_1$, the multiplicity of occurrence of r in λ_1 is less than or equal to the multiplicity of occurrence of r in λ_2.
- A set partition of a finite set $\mathbf{A}$ is a set of non-empty subsets of $\mathbf{A}$, with empty intersection, whose union is $\mathbf{A}$.
- For a positive integer n, we denote by $[n]$ the set $\{1, \ldots, n\}$.
- For a positive integer n, we denote by S_n the group of automorphisms of $[n]$.

CHAPTER 2

Background

2.1 The moduli space $\mathcal{M}_{0,n}$ and its compactifications

The moduli space $\mathcal{M}_{0,n}$ parametrizes all ways of labeling n distinct points on $\mathbb{P}^1$, up to projective change of coordinates.

Definition 2.1.1. An n-***marked smooth genus zero curve*** is a curve C, isomorphic to $\mathbb{P}^1$, together with distinct labeled points $p_1, \ldots, p_n \in C$. For $\mathbf{P}$ a finite set, an $\mathbf{P}$-***marked smooth genus zero curve*** is a curve C, isomorphic to $\mathbb{P}^1$, together with an injective map $\mathbf{P} \hookrightarrow C$.

Definition 2.1.2. Let $n \geq 3$. There is a smooth quasiprojective variety $\mathcal{M}_{0,n}$ of dimension $n-3$ parametrizing all n-marked smooth genus zero curves up to isomorphism. Likewise, for $\mathbf{P}$ a finite set with $|\mathbf{P}| \geq 3$, there is a moduli space $\mathcal{M}_{0,\mathbf{P}} \cong \mathcal{M}_{0,|\mathbf{P}|}$, parametrizing smooth genus zero $\mathbf{P}$-marked curves up to isomorphism.

Definition 2.1.3. For $n \geq n' \geq 3$, there is a ***forgetful map*** $\mu : \mathcal{M}_{0,n} \to \mathcal{M}_{0,n'}$ sending $(C, p_1, \ldots, p_n)$ to $(C, p_1, \ldots, p_{n'})$. Similarly, for $\mathbf{P}' \hookrightarrow \mathbf{P}$ with $|\mathbf{P}'| \geq 3$, there is a forgetful map $\mu : \mathcal{M}_{0,\mathbf{P}} \to \mathcal{M}_{0,\mathbf{P}'}$.

$\mathcal{M}_{0,n}$ parametrizes smooth curves with n distinct marked points. When $n > 3$ there are 1-parameter families $C(\mathbf{t})_{\mathbf{t}\neq 0}$ of smooth curves with n distinct marked points such that as $\mathbf{t}$ goes to zero, there is no limiting smooth curve where the marked points remain distinct. Thus $\mathcal{M}_{0,n}$ cannot be compact. There are many compactifications of $\mathcal{M}_{0,n}$ that are essentially based on describing what happens when marked points collide. $\overline{\mathcal{M}}_{0,n}$ is the most widely studied of these compactifications. Here, the marked points are always distinct, but the curve may be nodal. Moduli spaces of weighted stable curves are a generalization of $\overline{\mathcal{M}}_{0,n}$ constructed by Hassett in [Has03]. Here, marked points are assigned weights between 0 and 1, and a set of marked points may coincide as long as their total weight is not more than 1. Finally, in [Smy09], Smyth classified all ***modular compactifications*** of $\mathcal{M}_{0,n}$; that is, compactifications that extend the moduli space interpretation of $\mathcal{M}_{0,n}$. Here, we give brief introductions to $\overline{\mathcal{M}}_{0,n}$, spaces of weighted stable curves, and modular compactifications, each of which is a strict generalization of the previous one.

2.1.1 $\overline{\mathcal{M}}_{0,n}$ and its combinatorial structure

The *stable curves* compactification $\overline{\mathcal{M}}_{0,n}$ is the "biggest" modular compactification of $\mathcal{M}_{0,n}$; it admits a birational map to every other modular compactification. We refer the reader to [KV06] for an extended introduction.

Definition 2.1.4. A ***stable*** n***-marked genus zero curve*** is a connected algebraic curve C of arithmetic genus zero whose only singularities are simple nodes, together with n distinct smooth marked points $p_1, \dots, p_n$ on C, such that the set of automorphisms $C \to C$ that fix every marked point p_i is finite.

The irreducible components of such C are all isomorphic to $\mathbb{P}^1$. Points of C that are either marked points or nodes are called ***special points***. The ***stability*** condition that $(C, p_1, \dots, p_n)$ has finitely many automorphisms implies that it has no nontrivial automorphisms, and is equivalent to the condition that every irreducible component of C has at least three special points. Note that every node of C partitions the set $[n]$ into two parts, each of cardinality at least 2.

Theorem 2.1.5 (Deligne-Mumford, Grothendieck, Knudsen)**.** *Let $n \geq 3$. There is a smooth projective variety $\overline{\mathcal{M}}_{0,n}$ of dimension $n-3$ that is a fine moduli space for stable n-marked genus zero curves. It contains $\mathcal{M}_{0,n}$ as a dense open subset.*

For $\mathbf{P}$ a finite set, we analogously define stable genus zero $\mathbf{P}$-marked curves, and their moduli space $\overline{\mathcal{M}}_{0,\mathbf{P}}$ (isomorphic to $\overline{\mathcal{M}}_{0,|\mathbf{P}|}$). It contains $\mathcal{M}_{0,\mathbf{P}}$ as a dense open subset.

2.1.1.1 Dual trees and boundary strata

The ***boundary*** $\overline{\mathcal{M}}_{0,n} \smallsetminus \mathcal{M}_{0,n}$ is a simple normal crossings divisor. Points on the boundary correspond to reducible stable curves. The topological type of a stable curve is captured by the combinatorial information of its dual tree, defined below. This classification of stable curves by topological type gives a stratification of $\overline{\mathcal{M}}_{0,n}$.

Definition 2.1.6. Let $(C, p_1, \dots, p_n)$ be a stable genus zero curve. Its ***dual tree*** σ is the graph defined as follows. The vertices v of σ correspond to the irreducible components C_v of C. Two vertices v_1 and v_2 are connected by an edge if and only if the components C_{v_1} and C_{v_2} meet at a node. For each marked point p_i on C_v, we attach a ***leg*** ℓ_i to the vertex v. Note that edges and legs are distinct from each other. The graph σ is a tree because C has arithmetic genus zero.

For a vertex v on σ, set

$$\mathrm{Flags}_v = \{\text{Legs attached to } v\} \cup \{\text{edges incident to } v\}.$$

We refer to elements of Flags_v as ***flags*** on v. We define the ***valence*** of v, denoted $|v|$, to be the cardinality of Flags_v. For $i \in \{1, \ldots, N\}$, we define $\delta(v \to i)$ to be the unique flag in Flags_v that connects the leg ℓ_i to v, i.e. is part of the unique non-repeating path in σ from v to ℓ_i. If $\ell_i \in \mathrm{Flags}_v$, then $\delta(v \to i) = \ell_i$; otherwise $\delta(v \to i)$ is an edge. There is a canonical injection $\mathrm{Flags}_v \hookrightarrow C_v$ whose image is the set of special points of C_v. Thus C_v is a Flags_v-marked smooth genus zero curve. We denote by $\mathrm{Vertices}(\sigma)$ the set of vertices of σ. We define the ***moduli dimension*** $\mathrm{md}(v)$ of $v \in \mathrm{Vertices}(\sigma)$ to be $|v| - 3$.

Definition 2.1.7. A ***stable*** n***-marked tree*** is a tree σ with marked legs $\ell_1, \ldots, \ell_n$ such that every vertex has valence at least 3.

For fixed n, there are finitely many isomorphism classes of stable n-marked trees, and each of these arises as the dual tree of some stable n-marked genus zero curve.

Definition 2.1.8. Given σ a stable n-marked tree, the closure S_σ of the set $\{[(C, p_1, \ldots, p_n)] : C \text{ has dual graph } \sigma\}$ is an irreducible subvariety of $\overline{\mathcal{M}}_{0,n}$. These special subvarieties are called ***boundary strata***.

Boundary strata on $\overline{\mathcal{M}}_{0,n}$ are in bijection with isomorphism classes of stable n-marked trees.

Boundary strata on $\overline{\mathcal{M}}_{0,n}$ decompose into products of smaller-dimensional spaces of stable curves. Let S_σ be a boundary stratum in $\overline{\mathcal{M}}_{0,n}$. Given stable curves $([C_v] \in \overline{\mathcal{M}}_{0,\mathrm{Flags}_v})_{v \in \mathrm{Vertices}(\sigma)}$, we can glue the curves C_v together to obtain a curve C in S_σ as follows. Whenever there is an edge between v_1 and v_2, glue C_{v_1} to C_{v_2} at the corresponding marked point of each curve. This procedure defines a ***gluing morphism***

$$\prod_{v \in \mathrm{Vertices}(\sigma)} \overline{\mathcal{M}}_{0,\mathrm{Flags}_v} \cong S_\sigma \hookrightarrow \overline{\mathcal{M}}_{0,n}.$$

So

$$\dim S_\sigma = \sum_{v \in \mathrm{Vertices}(\sigma)} (|v| - 3) = \sum_{v \in \mathrm{Vertices}(\sigma)} \mathrm{md}(v).$$

Denote by $\mathrm{Edges}(\sigma)$ the set of edges of σ. Each $e \in \mathrm{Edges}(\sigma)$ partitions the set $[n]$ into two parts, each of cardinality at least 2.

Definition 2.1.9. We say edges e and e' of σ and σ' respectively are *split equivalent* if they yield the same two-part partition of $[n]$. Similarly, we say that nodes η and η' on stable curves C and C' respectively are split equivalent if they yield the same two-part partition of $[n]$.

Definition 2.1.10.

1. For any edge e of an n-marked stable tree, there is a two-vertex stable tree whose edge is split equivalent to e. This stable tree corresponds to a codimension-1 boundary stratum i.e. an irreducible boundary divisor; we denote this divisor $\mathrm{Div}_{\overline{\mathcal{M}}_{0,n}}(e)$.
2. For any node η on an n-marked stable curve, there is an irreducible boundary divisor in $\overline{\mathcal{M}}_{0,n}$ parametrizing all curves containing a node that is split-equivalent to η; we denote this divisor $\mathrm{Div}_{\overline{\mathcal{M}}_{0,n}}(\eta)$.
3. For any set partition of $[n]$ into two parts P_1 and P_2 each with cardinality at least 2, there is a boundary divisor in $\overline{\mathcal{M}}_{0,n}$ consisting of those curves with a node separating all the marks in P_1 from all of those in P_2. We denote this divisor $\mathrm{Div}_{\overline{\mathcal{M}}_{0,n}}(\{P_1, P_2\})$. (The corresponding stable tree has exactly two vertices, one marked by elements of P_1 and the other marked by elements of P_2.)

A dual tree σ' has an edge split equivalent to e if and only if the stratum S'_σ is contained in $\mathrm{Div}_{\overline{\mathcal{M}}_{0,n}}(e)$. We have

$$S_\sigma = \bigcap_{e \in \mathrm{Edges}(\sigma)} \mathrm{Div}_{\overline{\mathcal{M}}_{0,n}}(e),$$

where the above intersection is transverse. Thus the codimension of S_σ in $\overline{\mathcal{M}}_{0,n}$ equals the number of edges of σ.

Suppose there is a containment of boundary strata $S_{\sigma'} \subseteq S_\sigma$. This is equivalent to the existence of a 1-parameter family of stable curves in which the general element has dual tree σ and some special element has dual tree σ'. We write $\sigma \rightsquigarrow \sigma'$, where '$\rightsquigarrow$' denotes "specializes to." The tree σ' can be obtained from σ be a finite sequence of steps

$$\sigma = \sigma(1) \rightsquigarrow \cdots \rightsquigarrow \sigma(j) \rightsquigarrow \cdots \rightsquigarrow \sigma(n) = \sigma',$$

where each $\sigma(j) \rightsquigarrow \sigma(j+1)$ is of the following form: a new edge e is inserted into some vertex $v(j)$ of $\sigma(j)$, turning it into two vertices $v_1(j)$ and $v_2(j)$, and the flags on v are distributed between v_1 and v_2 with each of them receiving at least two flags. The insertion of the edge e into $\sigma(j)$

corresponds to intersecting $S_\sigma(j)$ with the boundary divisor $\mathrm{Div}_{\overline{\mathcal{M}}_{0,n}}(e)$, that is

$$S_{\sigma(j+1)} = S_{\sigma(j)} \cap \mathrm{Div}_{\overline{\mathcal{M}}_{0,n}}(e).$$

At each step we have a surjective map $\mathrm{Vertices}(\sigma(j+1)) \to \mathrm{Vertices}(\sigma(j))$ sending $v_1(j)$ and $v_2(j)$ to $v(j)$. The composite is a surjective map $\mathrm{sp} : \mathrm{Vertices}(\sigma') \to \mathrm{Vertices}(\sigma)$. For $v \in \mathrm{Vertices}(\sigma)$, if $\mathrm{sp}^{-1}(v) = \{v'_1, \ldots, v'_r\}$, we write $(v, \sigma) \rightsquigarrow (v'_1 \cup \cdots \cup v'_r, \sigma')$ is a specialization of vertices. Note that σ may be obtained by collapsing finitely many edges of σ'. We give an example in Figure 2.1.

2.1.1.2 Local geometry of $\overline{\mathcal{M}}_{0,n}$

Let C be a stable n-pointed curve with nodes $\eta_1, \ldots, \eta_k$. Then $[C]$ has an affine neighborhood in $\overline{\mathcal{M}}_{0,n}$ isomorphic to an affine neighborhood of the origin in

$$\mathrm{Spec}\, \mathbb{C}[t_1, \ldots, t_k, (t_{k+1} - \lambda_1), \ldots, (t_{n-3} - \lambda_{n-3-k})].$$

Here, for $j = 1, \ldots, k$, t_j is a smoothing parameter for the node η_j, and each constant λ_j is the cross ratio of four fixed special points on some component of C.

Let σ be the dual tree of C; so $[C]$ is a general point of the boundary stratum S_σ. Each node η_j on C corresponds to an edge e_j of σ. The parameter t_j is a local equation near $[C]$ cutting out the boundary divisor $\mathrm{Div}_{\overline{\mathcal{M}}_{0,n}}(e_j)$. Thus in a neighborhood of $[C]$, the stratum S_σ is cut out by the ideal generated by the t_js.

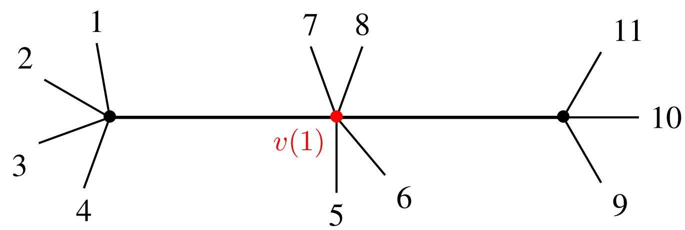
2.1.1.3 Forgetful maps

For $n \geq n' \geq 3$, the forgetful map $\mu : \mathcal{M}_{0,n} \to \mathcal{M}_{0,n'}$ extends to $\mu : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,n'}$. Let $(C, p_1, \ldots, p_n)$ be any n-marked stable curve. The curve $(C, p_1, \ldots, p_{n'})$ obtained by forgetting $p_{n'+1}, \ldots, p_n$ may not be stable. However, we obtain $(C', p_1, \ldots, p_{n'})$ a stable n'-marked curve by *stabilizing*, i.e. successively contracting components of C with fewer than 3 special points. The map μ sends $[(C, p_1, \ldots, p_n)]$ to $[(C', p_1, \ldots, p_{n'})]$.

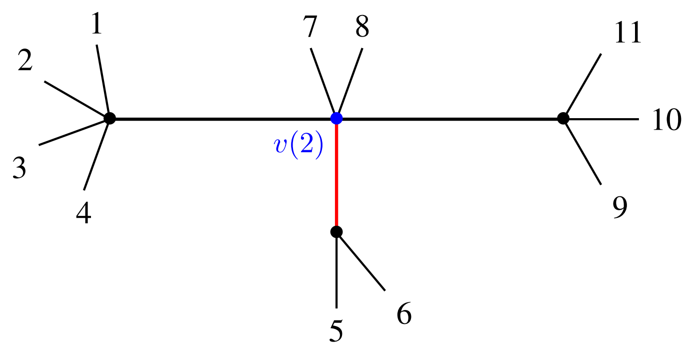
If σ is the dual tree of C, we obtain the dual tree σ' of C' by deleting legs $\ell_{n'+1}, \ldots, \ell_n$ and applying a finite sequence of steps, also called stabilization. Each step is either of the form:

- For a vertex v of valence 1, delete v, together with its incident edge, or
- For a vertex v of valence 2 with edges e_1 and e_2 connecting v to vertices v_1 and v_2, respectively, delete v together with e_1 and e_2, and connect v_1 to v_2 by an edge, or

$\sigma = \sigma(1) =$



$\rightsquigarrow \sigma(2) =$



$\rightsquigarrow \sigma(3) = \sigma' =$

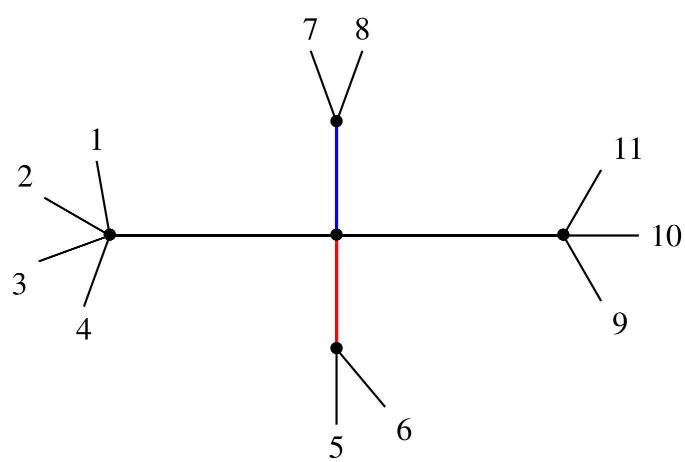


Figure 2.1: Specialization of stable trees

- For a vertex v of valence 2 incident to an edge e connecting v to another vertex v_1, and also incident to a leg ℓ_i, delete v together with e, and attach ℓ_i to v_1.

The deletion of the vertex v on σ corresponds to the contraction of C_v on C.

2.1.1.4 Homology groups of $\overline{\mathcal{M}}_{0,n}$

The homology group $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Z})$ is isomorphic to the Chow group $A_k(\overline{\mathcal{M}}_{0,n})$ and is generated by the classes of k-dimensional boundary strata. $\overline{\mathcal{M}}_{0,n}$ has no odd-dimensional homology (Keel, [Kee92]).

Additive relations among boundary strata are described in [KM94] by Kontsevich and Manin. Let σ be the dual tree of some $(k+1)$-dimensional boundary stratum, let v be a vertex on σ with valence at least four, and let $i_1, i_2, i_3, i_4 \in \{1, \ldots, N\}$ be such that the flags $\delta(v \to i_1), \ldots, \delta(v \to i_4)$ on v are distinct. The data $\sigma, v, i_1, \ldots, i_4$ determine a relation $R(\sigma, v, i_1, \ldots, i_4)$ among k-dimensional boundary strata. For every set partition $\operatorname{Flags}_v \setminus \{\delta(v \to i_1), \ldots, \delta(v \to i_4)\} = \operatorname{Flags}_1 \sqcup \operatorname{Flags}_2$, define a stable tree $\sigma(i_1 i_2 \operatorname{Flags}_1 \,|\, \operatorname{Flags}_2 i_3 i_4)$ as follows.

1. Insert an edge into v, splitting it into two vertices v_1 and v_2.
2. Attach the flags in $\operatorname{Flags}_1 \cup \{\delta(v \to i_1), \delta(v \to i_2)\}$ to v_1.
3. Attach the flags in $\operatorname{Flags}_2 \cup \{\delta(v \to i_3), \delta(v \to i_4)\}$ to v_2.

The tree $\sigma(i_1 i_2 \operatorname{Flags}_1 \,|\, \operatorname{Flags}_2 i_3 i_4)$ has one more edge than σ, and thus it corresponds to a k-dimensional boundary stratum $S(i_1 i_2 \operatorname{Flags}_1 \,|\, \operatorname{Flags}_2 i_3 i_4)$. The tree $\sigma(i_1 i_3 \operatorname{Flags}_1 \,|\, \operatorname{Flags}_2 i_2 i_4)$ and stratum $S(i_1 i_3 \operatorname{Flags}_1 \,|\, \operatorname{Flags}_2 i_2 i_4)$ are defined analogously. Then

$$\begin{aligned} R(\sigma, v, i_1, \ldots, i_4) := & \sum_{(\operatorname{Flags}_1, \operatorname{Flags}_2)} [S(i_1 i_2 \operatorname{Flags}_1 \,|\, \operatorname{Flags}_2 i_3 i_4)] \\ & - \sum_{(\operatorname{Flags}_1, \operatorname{Flags}_2)} [S(i_1 i_3 \operatorname{Flags}_1 \,|\, \operatorname{Flags}_2 i_2 i_4)] \\ & = 0 \in H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Z}) \quad \text{(resp. } A_k(\overline{\mathcal{M}}_{0,n})), \end{aligned} \tag{2.1}$$

where the sum is over set partitions

$$\operatorname{Flags}_v \setminus \{\delta(v \to i_1), \ldots, \delta(v \to i_4)\} = \operatorname{Flags}_1 \cup \operatorname{Flags}_2 .$$

These relations, for the various choices of $\sigma, v, i_1, \ldots, i_4$, generate all additive relations among k-dimensional boundary strata.

2.1.1.5 The tautological ψ-classes

$\overline{\mathcal{M}}_{0,\mathbf{P}}$ has a tautological line bundle $\mathcal{L}_p$ corresponding to each marked point $p \in \mathbf{P}$. This line bundle assigns to the point $[C, \iota]$ the 1-dimensional complex vector space $T^{\vee}_{\iota(p)}C$, namely, the cotangent line to the curve C at the marked point $\iota(p)$. The divisor class associated to $\mathcal{L}_p$ is denoted by ψ_p.

The space $H^0(\overline{\mathcal{M}}_{0,\mathbf{P}}, \mathcal{L}_p)$ is $(|\mathbf{P}| - 2)$-dimensional and basepoint-free. The induced map $\rho : \overline{\mathcal{M}}_{0,\mathbf{P}} \to \mathbb{P}(H^0(\overline{\mathcal{M}}_{0,\mathbf{P}}, \mathcal{L}_p)^{\vee}) \cong \mathbb{P}^{|\mathbf{P}|-3}$ is a birational map onto $\mathbb{P}^{|\mathbf{P}|-3}$ [Kap93].

Consider a forgetful map $\mu : \overline{\mathcal{M}}_{0,\mathbf{P}\cup\{q\}} \to \overline{\mathcal{M}}_{0,\mathbf{P}}$. For $p \in \mathbf{P}$, we have ([AC98])

$$\mu^*\psi_p^{\overline{\mathcal{M}}_{0,\mathbf{P}}} = \psi_p^{\overline{\mathcal{M}}_{0,\mathbf{P}\cup\{q\}}} - \mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{P}}}(\{\{p,q\}, \{p,q\}^C\}).$$

Using induction, we obtain:

Lemma 2.1.11. *For a forgetful map $\mu : \overline{\mathcal{M}}_{0,\mathbf{P}\cup\mathbf{Q}} \to \overline{\mathcal{M}}_{0,\mathbf{P}}$, we have*

$$\mu^*\psi_p^{\overline{\mathcal{M}}_{0,\mathbf{P}}} = \psi_p^{\overline{\mathcal{M}}_{0,\mathbf{P}\cup\mathbf{Q}}} - \sum_{\substack{\mathbf{S}\subseteq\mathbf{Q} \\ \mathbf{S}\ \textit{nonempty}}} \mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{P}}}(\{\{p\} \sqcup \mathbf{S}, (\{p\} \sqcup \mathbf{S})^C\}).$$

2.1.2 Moduli spaces of weighted stable curves

The stable curves compactification $\overline{\mathcal{M}}_{0,n}$ parametrizes nodal curves with n distinct smooth marked points. Moduli spaces of weighted stable curves parametrize nodal curves with n smooth marked points, not necessarily distinct. Each mark is assigned a rational “weight” between 0 and 1; a subset of marked points may coincide as long as their total weight does not exceed 1.

Definition 2.1.12 (Hassett, [Has03])**.** A ***weight datum*** is a tuple $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_n) \in (\mathbb{Q} \cap (0, 1])^n$ such that $\sum_{i=1}^{N} \epsilon_i > 2$.

Definition 2.1.13 (Hassett, [Has03])**.** Let $\boldsymbol{\epsilon}$ be a weight datum. A ***genus zero $\boldsymbol{\epsilon}$-stable curve*** is a connected algebraic curve C of arithmetic genus zero, whose only singularities are simple nodes, together with smooth marked points $p_1, \dots, p_n$, not necessarily distinct, such that

1. If $p_{i_1} = \cdots = p_{i_s}$ then $\epsilon_{i_1} + \cdots + \epsilon_{i_s} \leq 1$, and
2. For any irreducible component C_v,

$$\#\{\text{nodes on } C_v\} + \sum_{p_i \text{ on } C_v} \epsilon_i > 2.$$

Definition-Theorem 2.1.14 (Hassett, [Has03]). Given a weight datum $\boldsymbol{\epsilon}$, there is a smooth projective variety $\overline{\mathcal{M}}_{0,n}(\boldsymbol{\epsilon})$ of dimension $n-3$ that is a fine moduli space for $\boldsymbol{\epsilon}$-stable genus zero curves. It contains $\mathcal{M}_{0,n}$ as a dense open set. There is a ***reduction morphism*** $\rho_{\boldsymbol{\epsilon}} : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,n}(\boldsymbol{\epsilon})$ that respects the open inclusion of $\mathcal{M}_{0,n}$ into both spaces.

Moduli spaces of weighted stable curves have a stratification that comes from the stratification of $\overline{\mathcal{M}}_{0,n}$.

Definition 2.1.15. A ***boundary stratum*** in $\overline{\mathcal{M}}_{0,n}(\boldsymbol{\epsilon})$ is the image, under $\rho_{\boldsymbol{\epsilon}}$, of a boundary stratum in $\overline{\mathcal{M}}_{0,n}$.

The homology group $H_{2k}(\overline{\mathcal{M}}_{0,n}(\boldsymbol{\epsilon}), \mathbb{Z})$ is isomorphic to the Chow group $A_k(\overline{\mathcal{M}}_{0,n}(\boldsymbol{\epsilon}))$ and is generated by the classes of boundary strata ([Cey09]).

Remark 2.1.16. $\overline{\mathcal{M}}_{0,n} = \overline{\mathcal{M}}_{0,n}(\boldsymbol{\epsilon})$ for $\boldsymbol{\epsilon} = (1, \ldots, 1)$.

Definition 2.1.17. Let $\boldsymbol{\epsilon}$ be a weight datum, and σ be an n-marked stable tree. A vertex v on σ is called ***$\boldsymbol{\epsilon}$-stable*** if

$$\sum_{\delta \in \mathrm{Flags}_v} \min \left\{ 1, \sum_{i \mid \delta = \delta(v \to i)} \epsilon_i \right\} > 2.$$

For $[C, p_1, \ldots, p_n] \in \overline{\mathcal{M}}_{0,n}$ with dual tree σ, an $\boldsymbol{\epsilon}$-stable curve representing $\rho_{\boldsymbol{\epsilon}}([C])$ is obtained by contracting to a point every irreducible component C_v corresponding to $v \in \mathrm{Vertices}(\sigma)$ that is not $\boldsymbol{\epsilon}$-stable. It follows that σ has at least one $\boldsymbol{\epsilon}$-stable vertex. Also, the image under $\rho_{\boldsymbol{\epsilon}}$ of the boundary stratum S_σ has dimension

$$\sum_{\substack{v \in \mathrm{Vertices}(\sigma) \\ v\ \boldsymbol{\epsilon}\text{-stable}}} \mathrm{md}(v).$$

Thus, we obtain:

Lemma 2.1.18. *Let S_σ be a boundary stratum in $\overline{\mathcal{M}}_{0,n}$. The pushforward $(\rho_{\boldsymbol{\epsilon}})_*([S_\sigma])$ is nonzero if and only if every vertex $v \in \mathrm{Vertices}(\sigma)$ with positive moduli dimension is $\boldsymbol{\epsilon}$-stable.*

2.1.3 Modular compactifications and extremal assignments

A modular compactification of $\mathcal{M}_{0,n}$, loosely speaking, is a compactification that extends the interpretation of $\mathcal{M}_{0,n}$ as a moduli space of n-pointed rational curves. Thus $\overline{\mathcal{M}}_{0,n}$ and spaces of weighted stable curves are modular compactifications.

Definition 2.1.19 (Smyth, [Smy09])**.** A ***modular compactification*** of $\mathcal{M}_{0,n}$ is an open substack of the stack of all genus zero marked curves that is proper over $\operatorname{Spec}\mathbb{Z}$.

Modular compactifications are fully classified by a certain type of combinatorial datum called an extremal assignment.

Definition 2.1.20 (Smyth, [Smy09])**.** An ***extremal assignment*** $\mathcal{Z}$ on $\mathcal{M}_{0,n}$ is a choice for every stable n-marked tree σ of a proper subset $\mathcal{Z}(\sigma) \subsetneq \{\text{vertices of } \sigma\}$, such that if $(v,\sigma) \rightsquigarrow (v_1' \cup \cdots \cup v_s', \sigma')$, then $v \in \mathcal{Z}(\sigma)$ if and only if $v_1', \ldots, v_s' \in \mathcal{Z}(\sigma')$.

Definition 2.1.21 (Smyth, [Smy09])**.** Let $\mathcal{Z}$ be an extremal assignment on $\mathcal{M}_{0,n}$. For a stable n-marked curve $(C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{0,n}$ with dual tree σ, we define $\mathcal{Z}(C)$ to be the subcurve consisting of irreducible components C_v corresponding to vertices $v \in \mathcal{Z}(\sigma)$.

Definition 2.1.22. A ***rational*** m***-fold point*** is a curve singularity isomorphic to the singularity at the origin on the union of the m coordinate axes in $\mathbb{A}^m$. All singularities on connected reduced genus zero curves are of this form. A curve with a rational m-fold point can always be deformed to a smooth curve.

Definition 2.1.23 ([Smy09])**.** Let $\mathcal{Z}$ be an extremal assignment on $\mathcal{M}_{0,n}$. A reduced marked genus zero algebraic curve $(C^{\mathcal{Z}}, p_1^{\mathcal{Z}}, \ldots, p_n^{\mathcal{Z}})$ is $\mathcal{Z}$***-stable*** if there exists a stable genus zero curve $(C, p_1, \ldots, p_n)$, and a surjective morphism

$$\mathrm{ct} : (C, p_1, \ldots, p_n) \to (C^{\mathcal{Z}}, p_1^{\mathcal{Z}}, \ldots, p_n^{\mathcal{Z}})$$

with connected fibers such that

1. $\mathrm{ct}(p_i) = p_i^{\mathcal{Z}}$,
2. ct maps $C \smallsetminus \mathcal{Z}(C)$ isomorphically onto its image, and
3. If Y is a connected component of $\mathcal{Z}(C)$, then $\mathrm{ct}(Y)$ is a rational m-fold point of $C^{\mathcal{Z}}$, of multiplicity
$$m = \left| Y \cap \overline{(C \smallsetminus Y)} \right|.$$

Definition-Theorem 2.1.24. 1. Let $\mathcal{Z}$ be an extremal assignment on $\mathcal{M}_{0,n}$. Then there is a (complex) algebraic space $\overline{\mathcal{M}}_{0,n}(\mathcal{Z})$, that is a fine moduli space for $\mathcal{Z}$-stable curves. It contains $\mathcal{M}_{0,n}$ as a dense open set.

2. Any modular compactification of $\mathcal{M}_{0,n}$ is isomorphic to $\overline{\mathcal{M}}_{0,n}(\mathcal{Z})$ for some extremal assignment $\mathcal{Z}$.

These algebraic spaces $\overline{\mathcal{M}}_{0,n}(\mathcal{Z})$ are always compact complex analytic spaces, but they are not necessarily smooth, nor are they necessarily projective varieties. Criteria for smoothness and projectivity are given in [MSvAX15]. Any space of weighted stable curves, in particular, $\overline{\mathcal{M}}_{0,n}$, can be canonically realized as a modular compactification. In other words, any weight datum $\boldsymbol{\epsilon}$ determines an extremal assignment $\mathcal{Z}_{\boldsymbol{\epsilon}}$ such that the notions of $\boldsymbol{\epsilon}$-stability and $\mathcal{Z}_{\boldsymbol{\epsilon}}$-stability coincide. As mentioned previously, $\overline{\mathcal{M}}_{0,n}$ is the "biggest" modular compactification of $\mathcal{M}_{0,n}$:

Theorem 2.1.25 ([Smy09],[MSvAX15])**.** *Let $\mathcal{Z}$ be an extremal assignment. There is a* **reduction morphism**

$$\rho_{\mathcal{Z}} : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,n}(\mathcal{Z}),$$

that is a birational contraction with connected fibers, respecting the inclusion of $\mathcal{M}_{0,n}$ into both spaces.

Given a stable curve $(C, p_1, \ldots, p_n)$, we obtain a curve $C^{\mathcal{Z}}$ representing $\rho_{\mathcal{Z}}([C])$ by contracting each connected component Y of $\mathcal{Z}(C, p_1, \ldots, p_n)$ to a point with a rational m-fold point of multiplicity $\left|Y \cap \overline{(C \smallsetminus Y)}\right|$. The induced map $\mathrm{ct} : C \to C^{\mathcal{Z}}$ is a contraction map as described in Definition 2.1.23.

Again, all modular compactifications inherit the stratification of $\overline{\mathcal{M}}_{0,n}$:

Definition 2.1.26. A ***boundary stratum*** in $\overline{\mathcal{M}}_{0,n}(\mathcal{Z})$ is the image under $\rho_{\mathcal{Z}}$ of some boundary stratum in $\overline{\mathcal{M}}_{0,n}$.

2.2 Rational correspondences and dynamical degrees

In this section X, X_1, X_2, and X_3 are smooth irreducible projective varieties.

2.2.1 Rational correspondences

A rational correspondence from X_1 to X_2 is a multivalued map from a dense open set in X_1 to X_2.

Definition 2.2.1. A ***rational correspondence*** $(\Gamma, \pi_1, \pi_2) : X_1 \dashrightarrow X_2$ is a diagram

$$\begin{tikzcd}
 & \Gamma \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
X_1 & & X_2
\end{tikzcd}$$

where Γ is a smooth quasiprojective variety, not necessarily irreducible, and the restriction of π_1 to every irreducible component of Γ is dominant and generically finite.

We sometimes suppress part of the notation and write $\Gamma : X_1 \dashrightarrow X_2$ for the rational correspondence $(\Gamma, \pi_1, \pi_2) : X_1 \dashrightarrow X_2$.

Over a dense open subset $U_1 \subseteq X_1$, π_1 is a finite covering map of some degree d, so $\pi_2 \circ \pi_1^{-1}$ defines a multivalued map from U_1 to X_2 and induces a regular map from U_1 to $\mathrm{Sym}^d(X_2)$. Outside U_1, however, the fibers of π_1 could be empty or positive-dimensional, and it may be impossible to extend this to a multivalued map from X_1 to X_2, respectively a regular map from X_1 to $\mathrm{Sym}^d(X_2)$.

Rational correspondences induce pushforward and pullback maps:

Definition 2.2.2. Let $\overline{\Gamma}$ be a smooth projective compactification of Γ such that Γ is dense in $\overline{\Gamma}$ and π_1 and π_2 extend to maps $\overline{\pi_1}$ and $\overline{\pi_2}$ defined on $\overline{\Gamma}$. The cycle $(\overline{\pi_1} \times \overline{\pi_2})_*([\overline{\Gamma}]) \in H_{2\dim X_1}(X_1 \times X_2, \mathbb{Z})$ is independent of the choice of compactification $\overline{\Gamma}$, so we denote this cycle by $[\Gamma]$. Set

$$[\Gamma]_* := (\overline{\pi_2})_* \circ \overline{\pi_1}^* : H_c(X_1, \mathbb{Z}) \to H_c(X_2, \mathbb{Z})$$

and

$$[\Gamma]^* := (\overline{\pi_1})_* \circ \overline{\pi_2}^* : H^c(X_2, \mathbb{Z}) \to H^c(X_1, \mathbb{Z}).$$

$[\Gamma]_*$ and $[\Gamma]^*$ are well-defined and depend only on $[\Gamma]$. In fact, if pr_1 and pr_2 are the projections from $X_1 \times X_2$ to X_1 and X_2 respectively, we have for $\mathfrak{a} \in H_c(X_1, \mathbb{Z})$ and $\mathfrak{b} \in H^c(X_2, \mathbb{Z})$

$$[\Gamma]_*(\mathfrak{a}) = (\mathrm{pr}_2)_*([\Gamma] \smile \mathrm{pr}_1^*(\mathfrak{a}))$$

and

$$[\Gamma]^*(\mathfrak{b}) = (\mathrm{pr}_1)_*([\Gamma] \smile \mathrm{pr}_2^*(\mathfrak{b})),$$

where $\smile$ denotes cup product in $H^*(X_1 \times X_2, \mathbb{Z})$. The cohomology group $H^c(X_j, \mathbb{Z})$ is dual to the homology group $H_c(X_j, \mathbb{Z})$. By the projection formula, $[\Gamma]_*$ and $[\Gamma]^*$ are dual maps.

Suppose $(\Gamma, \pi_1, \pi_2) : X_1 \dashrightarrow\!\!\!\!\dashrightarrow X_2$ and $(\Gamma', \pi_2', \pi_3') : X_2 \dashrightarrow\!\!\!\!\dashrightarrow X_3$ are rational correspondences such that the image under π_2 of every irreducible component of Γ intersects the domain of definition of the multivalued function $\pi_3' \circ (\pi_2')^{-1}$. The composite $\Gamma' \circ \Gamma$ is a rational correspondence from X_1 to X_3 defined as follows.

Pick dense open sets $U_1 \subseteq X_1$ and $U_2 \subseteq X_2$ such that $\pi_2(\pi_1^{-1}(U_1)) \subseteq U_2$, and $\pi_1|_{\pi_1^{-1}(U_1)}$ and $\pi_2'|_{(\pi_2')^{-1}(U_2)}$ are both covering maps. Set

$$\Gamma' \circ \Gamma := \pi_1^{-1}(U_1) \ {}_{\pi_2}\times_{\pi_2'} \ (\pi_2')^{-1}(U_2),$$

together with its given maps to X_1 and X_3.

Although this definition of $\Gamma' \circ \Gamma$ depends on the choice of open sets U_1 and U_2, the cycle class $[\Gamma' \circ \Gamma]$ is well-defined; in fact, if $[\Gamma_1] = [\Gamma_2]$ and $[\Gamma_1'] = [\Gamma_2']$, then $[\Gamma_1' \circ \Gamma_1] = [\Gamma_2' \circ \Gamma_2]$ ([DS08]). It will be convenient to work with a concrete representative $\Gamma' \circ \Gamma$ in its cycle class. However, composition of rational correspondences is only defined up to equivalence of cycle class.

Unfortunately, it is not necessarily true that $[\Gamma' \circ \Gamma]_* = [\Gamma']_* \circ [\Gamma]_*$ or that $[\Gamma' \circ \Gamma]^* = [\Gamma]^* \circ [\Gamma']^*$. (See Section 2.2.4 for an example.) Suppose

$$[\Gamma' \circ \Gamma]_* = [\Gamma']_* \circ [\Gamma]_* : H_c(X_1, \mathbb{Z}) \to H_c(X_3, \mathbb{Z}).$$

Then we say Γ' and Γ are ***c-homologically composable***. In this case, by duality of pushforward and pullback, Γ' and Γ are also c-cohomologically composable:

$$[\Gamma' \circ \Gamma]^* = [\Gamma]^* \circ [\Gamma']^* : H^c(X_3, \mathbb{Z}) \to H^c(X_1, \mathbb{Z}).$$

Remark 2.2.3. There is a theory of ***correspondences***, as distinct from rational correspondences. A correspondence from X_1 to X_2 is a cycle class in $X_1 \times X_2$. Correspondences also induce maps on (co)homology groups, but these maps are functorial under composition ([Ful98]).

2.2.2 Any rational map is a rational correspondence

Let $g : X_1 \dashrightarrow X_2$ be a rational map. Then $\mathrm{Gr}(g) := \overline{\{(x, g(x))\}} \subseteq X_1 \times X_2$ together with its projections pr_1 and pr_2 to X_1 and X_2 respectively, is a rational correspondence from X_1 to X_2. The pushforward g_* and the pullback g^* by g have been independently defined in the literature to be $[\mathrm{Gr}(g)]_*$ and $[\mathrm{Gr}(g)]^*$, respectively ([Roe13]). If $g : X_1 \dashrightarrow X_2$ and $g' : X_2 \dashrightarrow X_3$ are rational maps such that the image of g intersects the domain of definition of g', then the composite $g' \circ g$ is a rational map $X_1 \dashrightarrow X_3$, and $[\mathrm{Gr}(g' \circ g)] = [\mathrm{Gr}(g') \circ \mathrm{Gr}(g)]$. We may thus identify rational maps with the rational correspondences given by their graphs.

If $(\Gamma, \pi_1, \pi_2) : X_1 \dashrightarrow\!\!\!\!\!\!\dashrightarrow X_2$ is a rational correspondence with π_1 generically one-to-one, then $[\Gamma]$ is the rational correspondence given by the rational map $\pi_2 \circ \pi_1^{-1}$.

2.2.3 Dynamical degrees

We refer the reader to [Roe13] or [Bed11] for more extended discussions of dynamical degrees of rational maps. Let $(\Gamma, \pi_1, \pi_2) : X \dashrightarrow\!\!\!\!\!\!\dashrightarrow X$ be a rational self-correspondence such that the restriction of π_2 to every irreducible component of Γ is dominant.

Definition 2.2.4. In this case we say Γ is a ***dominant*** rational self-correspondence.

Set $\Gamma^r := \Gamma \circ \cdots \circ \Gamma$ (r times). Each $[\Gamma^r]^*$ acts on $H^{2k}(X, \mathbb{C})$, preserving $H^{k,k}(X)$.

Definition 2.2.5. Pick a norm $\|\cdot\|$ on $H^{k,k}(X)$. The kth dynamical degree Θ_k of Γ is defined to be $\lim_{r\to\infty} \|[\Gamma^r]^*\|^{1/r}$.

This limit exists ([DS05]), and is independent of the choice of norm ([CCLG10]). In fact, the dynamical degrees of Γ are determined by the cycle $[\Gamma]$. In [DS05] and [DS08], Dinh and Sibony show that the topological entropy of a rational map or rational correspondence is bounded from above by the logarithm of its largest dynamical degree.

Suppose $H^{k,k}(X) = H^{2k}(X, \mathbb{C})$. Then, since pullback on $H^{2k}(X, \mathbb{C})$ is dual to pushforward on $H_{2k}(X, \mathbb{C})$, we can rewrite

$$(k\text{th dynamical degree of } \Gamma) = \lim_{r\to\infty} \|[\Gamma]_* : H_{2k}(X) \to H_{2k}(X)\|^{1/r}.$$

If $[\Gamma^r]^* = ([\Gamma]^*)^r$ on $H^{k,k}(X)$ for all n, Γ is called ***k-stable***. A rational correspondence that is k-stable for all k is called ***algebraically stable***.

For k-stable Γ, we can rewrite the dynamical degree Θ_k as $\lim_{r\to\infty} \|([\Gamma]^*)^r\|^{1/r^r}$, which is the absolute value of the dominant eigenvalue of $[\Gamma]^*$. The kth dynamical degree is hard to compute for rational maps or correspondences that are not k-stable, except in a few examples.

There is another equivalent definition of k-th dynamical degree, as follows:

Definition 2.2.6. Let Γ be as in Definition 2.2.4. Set $\Gamma^r := \Gamma \circ \cdots \circ \Gamma$ (n times), and pick $\mathfrak{h}$ an ample divisor class on X. The ***kth dynamical degree*** Θ_k of Γ is defined to be

$$\lim_{r\to\infty} \left(([\Gamma^r]^*(\mathfrak{h}^k)) \cdot (\mathfrak{h}^{\dim X - k})\right)^{1/r}.$$

This limit exists and is independent of choice of ample divisor ([DS05, DS08, Tru15]).

We use this second definition in chapter 6.

2.2.4 Example: The Cremona involution on $\mathbb{P}^2$

Let $g : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the rational self-map given in coordinates by

$$[\mathbf{x} : \mathbf{y} : \mathbf{z}] \mapsto [\mathbf{yz} : \mathbf{xz} : \mathbf{xy}].$$

The map g is undefined at the coordinate points $[1:0:0]$, $[0:1:0]$, and $[0:0:1]$. Also, g^2 is the identity where defined — the complement of the coordinate lines. Let $\mathfrak{l} \in H^{1,1}(\mathbb{P}^2)$ be the class of a line. Since g is given by degree two polynomials in the coordinates, it is easy to check that a general line pulls back to a conic. So $g^*\mathfrak{l} = 2\mathfrak{l}$ and g^* acts on $H^{1,1}(\mathbb{P}^2)$ via multiplication by 2. On the other hand, $g^2 = \mathrm{Id}_{\mathbb{P}^2}$, so $(g^2)^*$ acts by the identity on $H^{1,1}(\mathbb{P}^2)$. In particular $(g^2)^* \neq (g^*)^2$.

Although g is not 1-stable, its dynamical degree Θ_1 is easily computable. For n odd, $g^r = g$ on $\mathbb{P}^2$, and $(g^r)^* = g^*$ is multiplication by 2 on $H^{1,1}(\mathbb{P}^2)$. For n even, $g^r = \mathrm{Id}_{\mathbb{P}^2}$ and $(g^r)^*$ is the identity on $H^{1,1}(\mathbb{P}^2)$. For any norm $\|\cdot\|$, the sequence $\|(g^r)^*\|^{1/r}$ goes to 1 as n goes to ∞, so $\Theta_1 = 1$.

We can understand the lack of 1-stability of g by examining its graph $\mathrm{Gr}(g) \subseteq \mathbb{P}^2 \times \mathbb{P}^2$. Let the coordinates on the first $\mathbb{P}^2$ factor be $\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1$, and the coordinates on the second $\mathbb{P}^2$ factor be $\mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2$. Denote by π_1 and π_2 the projections onto the first and second factors, respectively. Then $\mathrm{Gr}(g)$ is given by the equations

$$\mathbf{x}_1\mathbf{x}_2 = \mathbf{y}_1\mathbf{y}_2 = \mathbf{z}_1\mathbf{z}_2.$$

Over the open set $U \subseteq \mathbb{P}^2$ where all coordinates are nonzero, $\mathrm{Gr}(g)$ has equations

$$\mathbf{x}_1 = \frac{1}{\mathbf{x}_2}, \quad \mathbf{y}_1 = \frac{1}{\mathbf{y}_2}, \quad \mathbf{z}_1 = \frac{1}{\mathbf{z}_2},$$

so π_1 and π_2 are equal. The fibered product $V = \mathrm{Gr}(g) \ {}_{\pi_2}\!\times_{\pi_1} \mathrm{Gr}(g)$ embeds naturally in $\mathbb{P}^2 \times \mathbb{P}^2$ and has four irreducible components:

$$\begin{aligned} V_{\mathrm{diag}} &:= \{[\mathbf{x}_1 : \mathbf{y}_1 : \mathbf{z}_1] = [\mathbf{x}_2 : \mathbf{y}_2 : \mathbf{z}_2]\} \\ V_{\mathbf{x}} &:= \{\mathbf{x}_1 = 0\} \times \{\mathbf{x}_2 = 0\} \\ V_{\mathbf{y}} &:= \{\mathbf{y}_1 = 0\} \times \{\mathbf{y}_2 = 0\} \\ V_{\mathbf{z}} &:= \{\mathbf{z}_1 = 0\} \times \{\mathbf{z}_2 = 0\}. \end{aligned}$$

None of $V_{\mathbf{x}}$, $V_{\mathbf{y}}$, or $V_{\mathbf{z}}$ maps dominantly onto either $\mathbb{P}^2$ factor, so V does not define a rational correspondence $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. However, it induces the map $\mathfrak{l} \mapsto 4\mathfrak{l}$ on $H^{1,1}(\mathbb{P}^2)$, which is the same as $(g^*)^2$. On the other hand, the graph of g^2 is V_{diag}, just one of the four irreducible components of V.

2.2.5 The sequence of dynamical degrees of a rational map is log-concave.

Let $g : X \dashrightarrow X$ be a dominant rational map, and let $\mathfrak{h}$ be an ample divisor class on X. For $n > 0$ set $\mathrm{Gr}(g^r)$ to be the graph of g^r in $X \times X$, with its two maps π_1^r and π_2^r to X. If Θ_k denotes the kth dynamical degree of g, we have

$$\begin{aligned}\Theta_k &= \lim_{r\to\infty} \left(((g^r)^*(\mathfrak{h}^k)) \cdot (\mathfrak{h}^{\dim X - k})\right)^{1/r} \\ &= \lim_{r\to\infty} \left(((\pi_2^r)^*(\mathfrak{h}^k)) \cdot ((\pi_1^r)^*(\mathfrak{h}^{\dim X - k}))\right)^{1/r},\end{aligned}$$

by the projection formula. Since $(\pi_2^r)^*(\mathfrak{h})$ and $(\pi_1^r)^*(\mathfrak{h})$ are nef on $\mathrm{Gr}(g^r)$, *and* $\mathrm{Gr}(g^r)$ is irreducible, the sequence of intersection numbers $\{((\pi_2^r)^*(\mathfrak{h}^k)) \cdot ((\pi_1^r)^*(\mathfrak{h}^{\dim X-k}))\}_k$ is log-concave ([Laz04], Example 1.6.4). Thus the sequence $\{\Theta_k\}_k$ is log-concave as well.

This argument breaks down for multivalued maps/rational correspondences since their graphs are not necessarily irreducible.

2.2.6 Birationally conjugate rational correspondences

Let $(\Gamma, \pi_1, \pi_2) : X \dashrightarrow X$ be a dominant rational self-correspondence, and let $\rho : X \dashrightarrow X'$ be a birational equivalence. Then we obtain a dominant rational self-correspondence on X' through conjugation by ρ, as follows. Let U be the domain of definition of ρ. Set $\Gamma' = \pi_1^{-1}(U) \cap \pi_2^{-1}(U)$. Then

$$(\Gamma', \rho \circ \pi_1, \rho \circ \pi_2) : X' \dashrightarrow X'$$

is a dominant rational self-correspondence.

Theorem 2.2.7 ([DS05, DS08, Tru15, Tru16])**.** *The dynamical degrees of* Γ *and* Γ' *are equal.*

Thus we can study the dynamical degrees of $\Gamma : X \dashrightarrow X$ via the action of Γ on the birational model X'. The next two lemmas allow us to compare a rational correspondence on different birational models.

Lemma 2.2.8. *Let X and X' be smooth projective varieties, with $H^{2k}(X) = H^{k,k}(X)$ and $H^{2k}(X') = H^{k,k}(X')$ for some k. Suppose $\Gamma : X \dashrightarrow X$ is a k-stable rational self-correspondence, $\Lambda \subseteq H_{2k}(X)$ is a subspace with $[\Gamma]_*(\Lambda) \subseteq \Lambda$, and X' admits a birational morphism $\rho : X \to X'$ such that $\Lambda \subseteq \ker(\rho_*)$. Set $\Omega = H_{2k}(X)/\Lambda$. Then the kth dynamical degree of Γ is the absolute value of the dominant eigenvalue of the induced map $[\Gamma]_* : \Omega \to \Omega$.*

Proof. Dynamical degrees are birational invariants (Theorem 2.2.7), so the kth dynamical degree of Γ on X is equal to its kth dynamical degree on X'. Denote by $[\Gamma^r]_*^X$ the pushforward induced by the nth iterate of Γ on $H_{2k}(X)$, by $[\Gamma^r]_*^\Omega$ the induced map on Ω, and by $[\Gamma^r]_*^{X'}$ the pushforward on $H_{2k}(X')$. By the k-stability of Γ on X, we have

$$\begin{aligned}[]
[\Gamma^r]_*^X &= ([\Gamma]_*^X)^r \\
[\Gamma^r]_*^\Omega &= ([\Gamma]_*^\Omega)^r.
\end{aligned}$$

Denote by pr the map from $H_{2k}(X)$ to Ω. Pick norms on $H_{2k}(X)$, Ω, and $H_{2k}(X')$. This induces norms on maps among these vector spaces. We abuse notation by using $\|\cdot\|$ to denote all of these norms. Since $\Lambda \subseteq \ker(\rho_*)$, there is a factorization:

$$\begin{tikzcd}
H_{2k}(X) \arrow[rr, "\rho_*"] \arrow[dr, "\mathrm{pr}"'] & & H_{2k}(X') \\
& \Omega \arrow[ur, "\overline{\rho_*}"'] &
\end{tikzcd}$$

For $n > 0$, we have

$$\begin{aligned}[]
[\Gamma^r]_*^{X'} &= \rho_* \circ [\Gamma^r]_*^X \circ \rho^* \\
&= \overline{\rho_*} \circ \mathrm{pr} \circ [\Gamma^r]_*^X \circ \rho^* \\
&= \overline{\rho_*} \circ [\Gamma^r]_*^\Omega \circ \mathrm{pr} \circ \rho^* \\
&= \overline{\rho_*} \circ ([\Gamma]_*^\Omega)^r \circ \mathrm{pr} \circ \rho^*.
\end{aligned}$$

Thus by submultiplicativity of the induced norms:

$$\left\| [\Gamma^r]_*^{X'} \right\| \leq \|\overline{\rho_*}\| \, \|\mathrm{pr}\| \, \|\rho^*\| \left\| ([\Gamma]_*^\Omega)^r \right\|.$$

Taking nth roots and the limit as $r \to \infty$, we obtain:

$$(k\text{th dynamical degree of } \Gamma) \leq \left| \text{dominant eigenvalue of } [\Gamma]_*^\Omega \right|.$$

On the other hand, since Γ is k-stable on X,

$$\begin{aligned}(k\text{th dynamical degree of }\Gamma) &= \left|\text{dominant eigenvalue of } [\Gamma]_*^X\right| \\ &\geq \left|\text{dominant eigenvalue of } [\Gamma]_*^\Omega\right|. \end{aligned}$$ □

Lemma 2.2.9. *Let $\Gamma' : X_2 \dashrightarrow X_3$ be a rational correspondence. For $j \in \{2,3\}$, let X'_j be a smooth projective variety admitting a birational morphism ρ_j from X_j. Suppose for fixed k, $[\Gamma']_* : H_{2k}(X_2) \to H_{2k}(X_3)$ takes $\ker((\rho_2)_*)$ to $\ker((\rho_3)_*)$. Then*

1. *The following diagram commutes:*

$$\begin{tikzcd} H_{2k}(X_2) \arrow[r, "{[\Gamma']_*}"] \arrow[d, "(\rho_2)_*"'] & H_{2k}(X_3) \arrow[d, "(\rho_3)_*"] \\ H_{2k}(X'_2) \arrow[r, "{[\Gamma']_*}"] & H_{2k}(X'_3) \end{tikzcd}$$

   Thus $[\Gamma']_* : H_{2k}(X'_2) \to H_{2k}(X'_3)$ can be identified with the induced map

$$[\Gamma']_* : H_{2k}(X_2)/\ker((\rho_2)_*) \to H_{2k}(X_3)/\ker((\rho_3)_*),$$

2. *If $[\Gamma']_* : H_{2k}(X_2) \to H_{2k}(X_3)$ takes effective classes to effective classes, then so does $[\Gamma']_* : H_{2k}(X'_2) \to H_{2k}(X'_3)$, and*

3. *If $\Gamma : X_1 \dashrightarrow X_2$ is $2k$-homologically composable with $\Gamma' : X_2 \dashrightarrow X_3$, then $\Gamma : X_1 \dashrightarrow X'_2$ is $2k$-homologically composable with $\Gamma' : X'_2 \dashrightarrow X'_3$.*

Proof. Since we discuss the same rational correspondence on different spaces, we include a superscript in the notation for pushforward; e.g. we denote by $[\Gamma']_*^{X_2,X_3}$ the induced map from $H_{2k}(X_2)$ to $H_{2k}(X_3)$, and we denote by $[\Gamma']_*^{X'_2,X'_3}$ the induced map from $H_{2k}(X'_2)$ to $H_{2k}(X'_3)$.

Since the morphism ρ_2 is birational, $(\rho_2)_* \circ \rho_2^*$ is the identity on $H_{2k}(X'_2)$. Thus:

$$\mathrm{Im}(\rho_2^* \circ (\rho_2)_* - \mathrm{Id}) \subseteq \mathrm{Ker}((\rho_2)_*).$$

By assumption $[\Gamma']_*^{X_2,X_3}$ sends $\ker((\rho_2)_*)$ to $\ker((\rho_3)_*)$. Thus

$$\begin{aligned}
[\Gamma']_*^{X_2',X_3'} \circ (\rho_2)_* &= (\rho_3)_* \circ [\Gamma']_*^{X_2,X_3} \circ \rho_2^* \circ (\rho_2)_* \\
&= (\rho_3)_* \circ [\Gamma']_*^{X_2,X_3}.
\end{aligned} \tag{2.2}$$

This proves (1).

For α an effective k-dimensional cycle on X_2', there is an effective cycle $\tilde{\alpha}$ on X_2 satisfying $(\rho_2)_*(\tilde{\alpha}) = \alpha$. Thus $\tilde{\alpha} - \rho_2^*(\alpha) \in \ker((\rho_2)_*)$. Again, since $[\Gamma']_*^{X_2,X_3}$ sends $\ker((\rho_2)_*)$ to $\ker((\rho_3)_*)$,

$$\begin{aligned}
[\Gamma']_*^{X_2',X_3'}(\alpha) &= (\rho_3)_* \circ [\Gamma']_*^{X_2,X_3} \circ (\rho_2^*)(\alpha) \\
&= (\rho_3)_* \circ [\Gamma']_*^{X_2,X_3}(\tilde{\alpha}).
\end{aligned}$$

By assumption, $[\Gamma']_*^{X_2,X_3}$ preserves effectiveness. The pushforward by the regular map ρ_3 preserves effectiveness, so $[\Gamma']_*^{X_2',X_3'}(\alpha)$ is effective, proving (2). Finally, for (3),

$$\begin{aligned}
[\Gamma']_*^{X_2',X_3'} \circ [\Gamma]_*^{X_1,X_2'} &= (\rho_3)_* \circ [\Gamma']_*^{X_2,X_3} \circ \rho_2^* \circ (\rho_2)_* \circ [\Gamma]_*^{X_1,X_2}. \\
&= (\rho_3)_* \circ [\Gamma']_*^{X_2,X_3} \circ [\Gamma]_*^{X_1,X_2} \qquad \text{by (2.2)} \\
&= (\rho_3)_* \circ [\Gamma' \circ \Gamma]_*^{X_1,X_3} \\
&= [\Gamma' \circ \Gamma]_*^{X_1,X_3'}.
\end{aligned}$$

□

2.3 Hurwitz correspondences

Hurwitz spaces are moduli spaces parametrizing finite maps with prescribed ramification between smooth curves of prescribed genus. In this paper, we only deal with Hurwitz spaces of maps between genus zero curves. See [RW06] for a summary and proofs of facts quoted here.

Definition 2.3.1 (*Hurwitz space*, [Ram16], Definition 2.2)**.** Fix discrete data:

- $\mathbf{A}$ and $\mathbf{B}$ finite sets with cardinality at least 3 (marked points on source and target curves, respectively),
- d a positive integer (degree),
- $F : \mathbf{A} \to \mathbf{B}$ a map,
- $\mathrm{br} : \mathbf{B} \to \{\text{partitions of } d\}$ (branching), and

- $\mathrm{rm} : \mathbf{A} \to \mathbb{Z}^{>0}$ (ramification),

such that

- (Condition 1, Riemann-Hurwitz constraint) $\sum_{b \in \mathbf{B}} (d - \text{length of } \mathrm{br}(b)) = 2d - 2$, and
- (Condition 2) for all $b \in \mathbf{B}$, the multiset $(\mathrm{rm}(a))_{a \in F^{-1}(b)}$ is a submultiset of $\mathrm{br}(b)$.

There exists a smooth quasiprojective variety $\mathcal{H} = \mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$, a ***Hurwitz space***, that parametrizes morphisms $f : C \to D$ up to isomorphism, where

- C and D are $\mathbf{A}$-marked and $\mathbf{B}$-marked smooth connected genus zero curves, respectively,
- f is degree d,
- for all $a \in \mathbf{A}$, $f(a) = F(a)$ (via the injections $\mathbf{A} \hookrightarrow C$ and $\mathbf{B} \hookrightarrow D$),
- for all $b \in \mathbf{B}$, the branching of f over b is given by the partition $\mathrm{br}(b)$, and
- for all $a \in \mathbf{A}$, the local degree of f at a is equal to $\mathrm{rm}(a)$.

Remark 2.3.2. One may construct $\mathcal{H}$ as follows ([HM82]). Consider $\mathcal{M}_{0,\mathbf{A}} \times \mathcal{M}_{0,\mathbf{B}}$ with its two universal curves $\mathcal{U}_{0,\mathbf{A}}$ and $\mathcal{U}_{0,\mathbf{B}}$. Let Hilb be the relative Hilbert scheme of degree d morphisms $\mathcal{U}_{0,\mathbf{A}} \to \mathcal{U}_{0,\mathbf{B}}$. The conditions that $a \in \mathbf{A}$ map to $F(a)$ with local degree $\mathrm{rm}(a)$ and that the branching over $b \in \mathbf{B}$ be given by $\mathrm{br}(b)$ are locally closed. Thus $\mathcal{H}$ is a locally closed subvariety of Hilb.

The Hurwitz space $\mathcal{H}$ admits a map $\pi_{\mathbf{A}}$ to $\mathcal{M}_{0,\mathbf{A}}$, sending $[f : C \to D]$ to the marked source curve $[C]$. It similarly admits a map $\pi_{\mathbf{B}}$ to $\mathcal{M}_{0,\mathbf{B}}$, sending $[f : C \to D]$ to the marked target curve $[D]$. The space $\mathcal{H}$ may be empty; if not, the "target curve" map $\pi_{\mathbf{B}}$ is a finite covering map and $\pi_{\mathbf{A}} \circ \pi_{\mathbf{B}}^{-1}$ defines a multivalued map from $\mathcal{M}_{0,\mathbf{A}}$ to $\mathcal{M}_{0,\mathbf{B}}$. If $X_{\mathbf{B}}$ and $X_{\mathbf{A}}$ are projective compactifications of $\mathcal{M}_{0,\mathbf{B}}$ and $\mathcal{M}_{0,\mathbf{A}}$ respectively, $(\mathcal{H}, \pi_{\mathbf{B}}, \pi_{\mathbf{A}}) : X_{\mathbf{B}} \dashrightarrow X_{\mathbf{A}}$ is a rational correspondence. We generalize this as follows.

Definition 2.3.3. [*Hurwitz correspondence*, [Ram16], Definition 2.19] Let $\mathbf{A}'$ be any subset of $\mathbf{A}$ with cardinality at least 3. There is a forgetful morphism $\mu : \mathcal{M}_{0,\mathbf{A}} \to \mathcal{M}_{0,\mathbf{A}'}$. Let Γ be a union of connected components of $\mathcal{H}$. We call the multivalued map $\pi_{\mathbf{A}}|_{\Gamma} \circ \pi_{\mathbf{B}}^{-1}$ from $\mathcal{M}_{0,\mathbf{B}}$ to $\mathcal{M}_{0,\mathbf{A}'}$ a Hurwitz correspondence and we denote it by $\Gamma : \mathcal{M}_{0,\mathbf{B}} \rightrightarrows \mathcal{M}_{0,\mathbf{A}'}$.

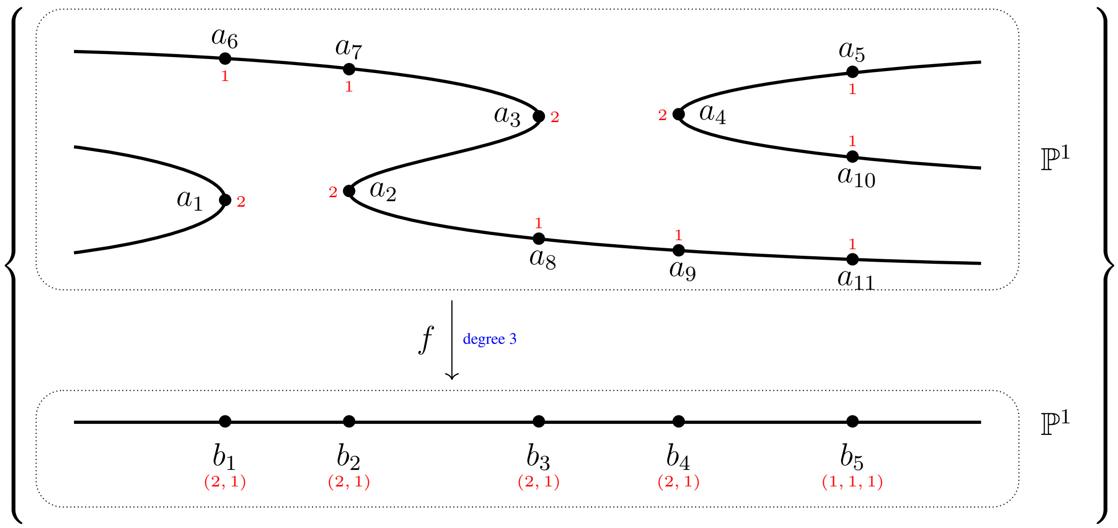


Figure 2.2: The Hurwitz space $\mathcal{H}_0 = \mathcal{H}(\mathbf{A}_0, \mathbf{B}_0, d_0, F_0, \mathrm{br}_0, \mathrm{rm}_0)$, where

- $\mathbf{A}_0 = \{a_1, \ldots, a_{11}\}$,
- $\mathbf{B}_0 = \{b_1, \ldots, b_5\}$,
- $d_0 = 3$
- $F_0(a_i) = b_i$ for $i = 1, \ldots, 5$, $F_0(a_i) = b_{i-5}$ for $i = 6, \ldots, 10$, and $F_0(a_{11}) = b_5$,
- $\mathrm{br}_0(b_i) = (2, 1)$ for $i = 1, \ldots, 4$ and $\mathrm{br}_0(b_5) = (1, 1, 1)$
- $\mathrm{rm}_0(a_i) = 2$ for $i = 1, \ldots, 4$ and $\mathrm{rm}_0(a_i) = 1$ for $i = 6, \ldots, 11$.

2.3.1 Hurwitz correspondences as dynamical systems

Consider a Hurwitz space $\mathcal{H} = \mathcal{H}(\mathbf{P}', \mathbf{P}, d, F, \mathrm{br}, \mathrm{rm})$ together with an injection $\mathbf{P} \hookrightarrow \mathbf{P}'$. This induces a forgetful map $\mu : \mathcal{M}_{0,\mathbf{P}'} \to \mathcal{M}_{0,\mathbf{P}}$. For Γ any union of connected components of $\mathcal{H}$, we have the "target curve" map $\pi_1 = \pi_{\mathbf{P}}|_\Gamma : \Gamma \to \mathcal{M}_{0,\mathbf{P}}$ and the "source curve" map $\pi_2 = \mu \circ \pi_{\mathbf{P}'}|_\Gamma : \Gamma \to \mathcal{M}_{0,\mathbf{P}}$. This defines a Hurwitz self-correspondence $\Gamma : \mathcal{M}_{0,\mathbf{P}} \rightrightarrows \mathcal{M}_{0,\mathbf{P}}$and a rational self-correspondence $(\Gamma, \pi_1, \pi_2) : X_{\mathbf{P}} \dashrightarrow X_{\mathbf{P}}$ on any compactification $X_{\mathbf{P}}$ of $\mathcal{M}_{0,\mathbf{P}}$. Note that by Theorem 2.2.7, the dynamical degrees of the Hurwitz self-correspondence Γ do not depend on the choice of compactification $X_{\mathbf{P}}$.

Definition 2.3.4. For a Hurwitz self-correspondence $\Gamma : \mathcal{M}_{0,\mathbf{P}} \rightrightarrows \mathcal{M}_{0,\mathbf{P}}$ as above, we refer to the discrete data $(d, F : \mathbf{P} \to \mathbf{P}, \mathrm{br}, \mathrm{rm})$ as its *dynamical portrait*.

2.3.2 Connection to Teichmüller theory

Hurwitz self-correspondences were introduced by Koch in [Koc13] in connection with Thurston's work on the topological characterization of postcritically finite rational function on $\mathbb{P}^1$. We summarize the connection in this section. Let S^2 denote an oriented 2-sphere.

Definition 2.3.5. Let $\mathbf{P} \subset S^2$ be a finite set. Define an equivalence relation on orientation-preserving homeomorphisms $S^2 \to \mathbb{P}^1$ as follows: ψ_1 is equivalent to ψ_2 if there exists $\xi \in \mathrm{Aut}(\mathbb{P}^1)$ such that ψ_1 and $\xi \circ \psi_2$ agree on $\mathbf{P}$ and are isotopic relative to $\mathbf{P}$. The ***Teichmüller space*** $\mathcal{T}(S^2, \mathbf{P})$ of $(S^2, \mathbf{P})$ is the set of equivalence classes of homeomorphisms.

Teichmüller space has a natural structure as a noncompact nonalgebraic complex manifold; it is isomorphic to a bounded domain in $\mathbb{C}^{|\mathbf{P}|-3}$. Given an element $[\psi] \in \mathcal{T}(S^2, \mathbf{P})$, the restriction $\psi|_{\mathbf{P}} : \mathbf{P} \to \mathbb{P}^1$ defines an element of $\mathcal{M}_{0,\mathbf{P}}$. This gives rise to a map of complex manifolds

$$\mathcal{T}(S^2, \mathbf{P}) \xrightarrow{\mathrm{cv}} \mathcal{M}_{0,\mathbf{P}}.$$

In fact, this is a covering map, realizing $\mathcal{T}(S^2, \mathbf{P})$ as the universal cover of $\mathcal{M}_{0,\mathbf{P}}$.

Definition 2.3.6. An orientation-preserving branched covering ϕ from S^2 to itself is called ***post-critically finite*** if the post-critical set

$$\mathbf{P} := \{\phi^r(x) | r > 0, x \text{ a critical point of } \phi\}$$

is finite.

We say ϕ is ***combinatorially equivalent*** to an algebraic morphism if there exist orientation-preserving homeomorphisms $\psi_1, \psi_2 : S^2 \to \mathbb{P}^1$ such that

$$\psi_2 \circ \phi \circ \psi_1^{-1} : \mathbb{P}^1 \to \mathbb{P}^1$$

is an algebraic morphism, and ψ_1 and ψ_2 are isotopic relative to the postcritical set $\mathbf{P}$. W. Thurston gave a topological characterization for ϕ to be combinatorially equivalent to an algebraic morphism, in terms of curve systems on $S^2 \smallsetminus \mathbf{P}$. This characterization can also be stated in terms of a self-map on $\mathcal{T}(S^2, \mathbf{P})$ ([DH93]). If $\psi : (S^2, \mathbf{P}) \to (\mathbb{P}^1, \psi(\mathbf{P}))$ is an orientation-preserving homeomorphism then the induced complex structure on $S^2 \smallsetminus \mathbf{P}$ can be pulled back via the covering map $\phi|_{S^2 \smallsetminus \phi^{-1}(\mathbf{P})}$ to obtain another complex structure on $S^2 \smallsetminus \phi^{-1}(\mathbf{P})$. This extends to a complex structure on all of S^2, and yields another homeomorphism $\psi' : (S^2, \mathbf{P}) \to (\mathbb{P}^1, \psi'(\mathbf{P}))$ such that

$$\psi \circ \phi \circ (\psi')^{-1} : (\mathbb{P}^1, \psi'(\mathbf{P})) \to (\mathbb{P}^1, \psi(\mathbf{P}))$$

is an algebraic morphism. The map

$$\begin{aligned} \mathcal{T}(S^2, \mathbf{P}) &\xrightarrow{\mathrm{Thurst}(\phi)} \mathcal{T}(S^2, \mathbf{P}) \\ [\psi] &\mapsto [\psi'] \end{aligned}$$

is a well-defined holomorphic map ([DH93]) called the ***Thurston pullback map***.

This pullback map is a contraction in the hyperbolic metric on $\mathcal{T}(S^2, \mathbf{P})$; it is a theorem of Thurston that ϕ is combinatorially equivalent to an algebraic map if and only if $\mathrm{Thurst}(\phi)$ has a fixed point. The dynamics of $\mathrm{Thurst}(\phi)$ are thus of interest.

The Thurston pullback map is holomorphic, but not algebraic. However, Koch ([Koc13]) showed that it descends to a Hurwitz correspondence on the algebraic variety $\mathcal{M}_{0,\mathbf{P}}$.

Given $\phi : (S^2, \mathbf{P}) \to (S^2, \mathbf{P})$, denote by d the topological degree of ϕ. Define

$$\begin{aligned} \mathrm{br} : \mathbf{P} &\to \{\text{partitions of } d\} \\ p &\mapsto \text{branching of } \phi \text{ over } p \end{aligned}$$

and

$$\begin{aligned} \mathrm{rm} : \mathbf{P} &\to \mathbb{Z}^{>0} \\ p &\mapsto \text{local degree of } \phi \text{ at } p. \end{aligned}$$

The data $(\mathbf{P}, \mathbf{P}, d, \phi|_{\mathbf{P}} : \mathbf{P} \to \mathbf{P}, \mathrm{br}, \mathrm{rm})$ satisfy Conditions 1 and 2 in Definition 2.3.1. Denote by $\mathcal{H}$ the Hurwitz space $\mathcal{H}(\mathbf{P}, \mathbf{P}, d, \phi|_{\mathbf{P}} : \mathbf{P} \to \mathbf{P}, \mathrm{br}, \mathrm{rm})$. Given a homeomorphism $\psi : (S^2, \mathbf{P}) \to (\mathbb{P}^1, \psi(\mathbf{P}))$, there exists a homeomorphism $\psi' : (S^2, \mathbf{P}) \to (\mathbb{P}^1, \psi'(\mathbf{P}))$, with $[\psi'] = \mathrm{Thurst}(\phi)([\psi])$, and such that

$$\psi \circ \phi \circ (\psi')^{-1} : (\mathbb{P}^1, \psi'(\mathbf{P})) \to (\mathbb{P}^1, \psi(\mathbf{P}))$$

is an algebraic morphism. This defines a point in $\mathcal{H}$. By [Koc13], we obtain a holomorphic covering map

$$\begin{aligned} &\mathcal{T}(S^2, \mathbf{P}) \to \mathcal{H} \\ &\qquad [\psi] \mapsto [(\mathbb{P}^1, \psi'(\mathbf{P})) \xrightarrow{\psi \circ \phi \circ (\psi')^{-1}} (\mathbb{P}^1, \psi(\mathbf{P}))] \end{aligned}$$

whose image is a connected component Γ of $\mathcal{H}$. We have the commutative diagram:

$$\begin{tikzcd}
\mathcal{T}(S^2, \mathbf{P}) \arrow[rr, "\mathrm{Thurst}(\phi)"] \arrow[dd, "\mathrm{cv}"'] \arrow[dr] & & \mathcal{T}(S^2, \mathbf{P}) \arrow[dd, "\mathrm{cv}"] \\
& \Gamma \arrow[dl, "\pi_{\mathbf{P}}^{\mathrm{target}}"'] \arrow[dr, "\pi_{\mathbf{P}}^{\mathrm{source}}"] & \\
\mathcal{M}_{0,\mathbf{P}} & & \mathcal{M}_{0,\mathbf{P}}
\end{tikzcd}$$

Our study of the dynamics of Hurwitz self-correspondences is motivated in part by this connection to the dynamics of the Thurston pullback map.

2.4 Moduli spaces of admissible covers

There is a widely used compactification of the Hurwitz space $\mathcal{H}$ by a space $\overline{\mathcal{H}}$ of ***admissible covers***. Moduli spaces of admissible covers were constructed in [HM82] by Harris and Mumford, and parametrize finite maps between possibly nodal curves. In general, they are only coarse moduli spaces, with quotient singularities. For technical ease, we introduce a class of Hurwitz spaces whose admissible covers compactifications are fine moduli spaces. We refer to Hurwitz spaces in this class as *fully marked*.

2.4.1 Fully marked Hurwitz spaces

Definition 2.4.1. Given $(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ as in Definition 2.3.1 with Condition 2 strengthened to:

- (Condition $2'$) For all $b \in \mathbf{B}$, the multiset $(\mathrm{rm}(a))_{a \in F^{-1}(b)}$ is **equal to** $\mathrm{br}(b)$,

we refer to $\mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ as a ***fully marked Hurwitz space***.

Remark 2.4.2. The Hurwitz space depicted in Figure 2.2 is fully marked.

Given any Hurwitz space $\mathcal{H} = \mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$, we can construct a fully marked Hurwitz space $\mathcal{H}^{\mathrm{full}}$ with a finite covering map $\nu : \mathcal{H}^{\mathrm{full}} \to \mathcal{H}$ as follows. We first construct a superset $\mathbf{A}^{\mathrm{full}}$ of $\mathbf{A}$, extending the functions F and rm. For every $b \in \mathbf{B}$, and for every r in the multiset complement $\mathrm{br}(b) \smallsetminus (\mathrm{rm}(a))_{a \in F^{-1}(b) \cap \mathbf{A}}$, add an element $a(b, r)$ to $\mathbf{A}^{\mathrm{full}}$, set $F(a(b, r)) = b$, and set $\mathrm{rm}(a(b, r)) = r$. The data $(\mathbf{A}^{\mathrm{full}}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ satisfy Conditions 1 and $2'$, so $\mathcal{H}^{\mathrm{full}} = \mathcal{H}(\mathbf{A}^{\mathrm{full}}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ is a fully marked Hurwitz space.

Let $\mathrm{Aut}(\mathbf{A}^{\mathrm{full}} \smallsetminus \mathbf{A})$ be the subgroup of permutations of $\mathbf{A}^{\mathrm{full}} \smallsetminus \mathbf{A}$ preserving the functions F and rm. This automorphism group acts freely on $\mathcal{H}^{\mathrm{full}}$ by relabeling points in $\mathbf{A}^{\mathrm{full}} \smallsetminus \mathbf{A}$, and the quotient of this action is $\mathcal{H}$. Denote by ν the quotient map $\mathcal{H}^{\mathrm{full}} \to \mathcal{H}$. The fully marked Hurwitz space $\mathcal{H}^{\mathrm{full}}$ admits a map $\pi_{\mathbf{A}^{\mathrm{full}}}$ to $\mathcal{M}_{0,\mathbf{A}^{\mathrm{full}}}$. Also, the injection $\mathbf{A} \hookrightarrow \mathbf{A}^{\mathrm{full}}$ yields a forgetful map $\mu : \mathcal{M}_{0,\mathbf{A}^{\mathrm{full}}} \to \mathcal{M}_{0,\mathbf{A}}$. The following diagram commutes:

$$\begin{tikzcd}
 & \mathcal{H}^{\mathrm{full}} \arrow[d, "\nu"'] \arrow[dr, "\pi_{\mathbf{A}^{\mathrm{full}}}"] & \\
 & \mathcal{H} \arrow[dl, "\pi_{\mathbf{B}}"'] \arrow[dr, "\pi_{\mathbf{A}}"] & \mathcal{M}_{0,\mathbf{A}^{\mathrm{full}}} \arrow[d, "\mu"] \\
\mathcal{M}_{0,\mathbf{B}} & & \mathcal{M}_{0,\mathbf{A}}
\end{tikzcd}$$

For Γ any union of connected components of $\mathcal{H}$, $\Gamma^{\mathrm{full}} := \nu^{-1}(\Gamma)$ is a union of connected components of $\mathcal{H}^{\mathrm{full}}$. For compactifications $X_{\mathbf{B}}$ and $X_{\mathbf{A}}$ of $\mathcal{M}_{0,\mathbf{B}}$ and $\mathcal{M}_{0,\mathbf{A}}$ respectively, $(\Gamma, \pi_{\mathbf{B}}, \pi_{\mathbf{A}})$ and $(\Gamma^{\mathrm{full}}, \pi_{\mathbf{B}} \circ \nu, \mu \circ \pi^{\mathbf{A}^{\mathrm{full}}})$ are both Hurwitz correspondences from $X_{\mathbf{B}}$ to $X_{\mathbf{A}}$, and $[\Gamma^{\mathrm{full}}] = (\deg \nu)[\Gamma]$ in $H_*(X_{\mathbf{B}} \times X_{\mathbf{A}})$. This observation yields a useful lemma:

Lemma 2.4.3. *Let* $(\Gamma, \pi_{\mathbf{B}}, \pi_{\mathbf{A}}) : X_{\mathbf{B}} \dashrightarrow X_{\mathbf{A}}$ *be any Hurwitz correspondence. Then*

$$[\Gamma] = \frac{1}{\deg \nu}[\Gamma^{\mathrm{full}}],$$

where each Γ^{full} is a union of connected components of a fully marked Hurwitz space $\mathcal{H}^{\text{full}}$ corresponding to a superset $\mathbf{A}^{\text{full}}$ of $\mathbf{A}$, and $\nu : \Gamma^{\text{full}} \to \Gamma$ is a finite covering map.

This means that any Hurwitz correspondence can be written in terms of (connected components of) fully marked Hurwitz spaces. These have convenient compactifications by moduli spaces of admissible covers. We use this Lemma in Proposition 3.0.1 and Theorem 4.1.6.

2.4.2 Admissible Covers

For an introduction see [HM98], Chapter 3G.

Definition 2.4.4 ([HM82]). Fix $(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ as in Definition 2.3.1. An $(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$-admissible cover is a map of curves $f : C \to D$, where

1. D is a $\mathbf{B}$-marked genus zero stable curve,
2. C is a (not necessarily stable) connected nodal genus zero curve, with an injection from $\mathbf{A}$ into the smooth locus of C,
3. $f : C \to D$ is a finite map of degree d, such that
   - for all $a \in \mathbf{A}$, $f(a) = F(a)$ (via the injections $\mathbf{A} \hookrightarrow C$ and $\mathbf{B} \hookrightarrow D$),
   - for all $b \in \mathbf{B}$, the branching of f over b is given by the partition $\mathrm{br}(b)$,
   - for all $a \in \mathbf{A}$, the local degree of f at a is equal to $\mathrm{rm}(a)$,
   - η is a node on C if and only if $f(\eta)$ is a node on D,
   - (balancing condition) for η a node of C between irreducible components C_1 and C_2 of C, $f(C_1)$ and $f(C_2)$ are distinct components of D, and the local degree of $f|_{C_1}$ at η is equal to the local degree of $f|_{C_2}$ at η.

Remark 2.4.5. If $(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ satisfies Conditions 1 and $2'$ as in Definition 2.4.1, then C is an $\mathbf{A}$-marked *stable* curve.

Theorem 2.4.6 ([HM82]). *Given $(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ satisfying Conditions 1 and $2'$ as in Definition 2.4.1, there is a projective variety $\overline{\mathcal{H}} = \overline{\mathcal{H}}(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ that is a fine moduli space for $(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$-admissible covers, and contains $\mathcal{H}$ as a dense open subset. $\overline{\mathcal{H}}$ extends the maps $\pi_{\mathbf{A}}$ and $\pi_{\mathbf{B}}$ to maps $\overline{\pi_{\mathbf{A}}}$ and $\overline{\pi_{\mathbf{B}}}$ to $\overline{\mathcal{M}}_{0,\mathbf{A}}$ and $\overline{\mathcal{M}}_{0,\mathbf{B}}$, respectively, with $\overline{\pi_{\mathbf{B}}} : \overline{\mathcal{H}} \to \overline{\mathcal{M}}_{0,\mathbf{B}}$ a finite flat map. $\overline{\mathcal{H}}$ may not be normal, but its normalization is smooth.*

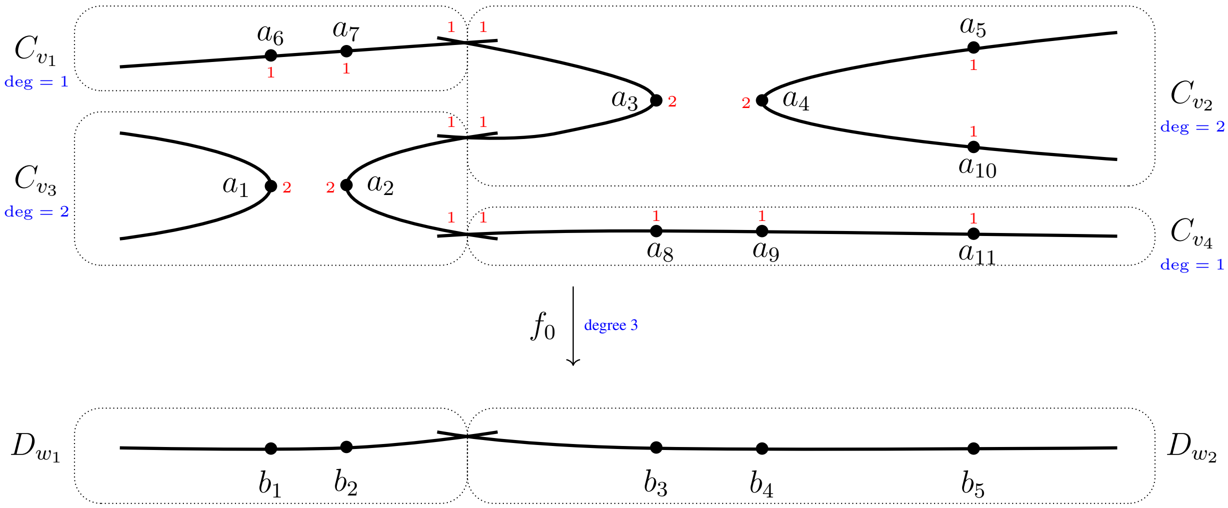


Figure 2.3: An $(\mathbf{A}_0, \mathbf{B}_0, d_0, F_0, \mathrm{br}_0, \mathrm{rm}_0)$-admissible cover $f_0 : C_0 \to D_0$, where the Hurwitz space depicted in Figure 2.2 is $\mathcal{H}_0 = \mathcal{H}(\mathbf{A}_0, \mathbf{B}_0, d_0, F_0, \mathrm{br}_0, \mathrm{rm}_0)$. Each irreducible component of C_0 or D_0 is encircled by a dashed outline.

Remark 2.4.7. The irreducible components of $\overline{\mathcal{H}}$ correspond to the connected components of $\mathcal{H}$.

Remark 2.4.8. The construction in [HM82] is very general, but as stated allows for only simple ramification and no marked points on the source curve, so does not apply to our case. However, it is easily modified: consider $\overline{\mathcal{M}}_{0,\mathbf{A}} \times \overline{\mathcal{M}}_{0,\mathbf{B}}$ with its two universal curves $\overline{\mathcal{U}}_{0,\mathbf{A}}$ and $\overline{\mathcal{U}}_{0,\mathbf{B}}$. Let Hilb be the relative Hilbert scheme of degree d morphisms $\overline{\mathcal{U}}_{0,\mathbf{A}} \to \overline{\mathcal{U}}_{0,\mathbf{B}}$. Then the locus

$$\overline{\mathcal{H}} := \{f : C \to D | f \text{ is an } (\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})\text{-admissible cover}\}$$

is a closed subscheme of Hilb by the proof of Theorem 4 in [HM82].

Given $(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ satisfying Conditions 1 and 2 as in Definition 2.3.1, but not $2'$ as in Definition 2.4.1, there is still a compactification $\overline{\mathcal{H}}$ of $\mathcal{H} = \mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ by admissible covers. Consider the corresponding fully marked Hurwitz space $\mathcal{H}^{\mathrm{full}}$ and its admissible covers compactification $\overline{\mathcal{H}}^{\mathrm{full}}$. The action of $\mathrm{Aut}(\mathbf{A}^{\mathrm{full}} \smallsetminus \mathbf{A})$ on $\mathcal{H}^{\mathrm{full}}$ extends to an action on $\overline{\mathcal{H}}^{\mathrm{full}}$, but this action is no longer free, so the quotient $\overline{\mathcal{H}}$ has *orbifold singularities*, and is only a coarse moduli space.

2.4.3 Stratification of $\overline{\mathcal{H}}$

Moduli spaces of admissible covers have a stratification analogous to and compatible with that of $\overline{\mathcal{M}}_{0,n}$. In this section we fix $(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ satisfying Conditions 1 and $2'$ as in Definition

2.4.1. Let $\mathcal{H} = \mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ be the corresponding fully marked Hurwitz space, and let $\overline{\mathcal{H}} = \overline{\mathcal{H}}(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$.

Definition 2.4.9. Given any admissible cover $[f : C \to D] \in \overline{\mathcal{H}}$, we can extract its ***combinatorial type*** $\gamma = (\sigma, \tau, d_{\mathrm{Vertices}}, f_{\mathrm{Vertices}}, F_{\mathrm{Edges}}, (\mathrm{br}_v)_{v \in \mathrm{Vertices}(\sigma)}, \mathrm{rm}_{\mathrm{Edges}})$, where:

- σ is the dual tree of C,
- τ is the dual tree of D,
- $d_{\mathrm{Vertices}} : \mathrm{Vertices}(\sigma) \to \mathbb{Z}^{>0}$ sends v to $\deg f|_{C_v}$,
- $f_{\mathrm{Vertices}} : \mathrm{Vertices}(\sigma) \to \mathrm{Vertices}(\tau)$, where for C_v an irreducible component of C mapping under f to D_w an irreducible component of D, f_{Vertices} sends v to w,
- $F_{\mathrm{Edges}} : \mathrm{Edges}(\sigma) \to \mathrm{Edges}(\tau)$ is the restriction of f via the inclusions $\mathrm{Edges}(\sigma) \hookrightarrow C$ and $\mathrm{Edges}(\tau) \hookrightarrow D$,
- $\mathrm{br}_v : \mathrm{Flags}_{f_{\mathrm{Vertices}}(v)} \to \{\text{partitions of } d_{\mathrm{Vertices}}(v)\}$ sends $\delta' \in \mathrm{Flags}_{f_{\mathrm{Vertices}}(v)}$ to the branching of $f|_{C_v} : C_v \to D_{f_{\mathrm{Vertices}}(v)}$ over δ', via the inclusion $\mathrm{Flags}_{f_{\mathrm{Vertices}}(v)} \hookrightarrow D_{f_{\mathrm{Vertices}}(v)}$, and
- For $v \in \mathrm{Vertices}(\sigma)$, the map $\mathrm{rm}_{\mathrm{Edges}} : \mathrm{Edges}(\sigma) \to \mathbb{Z}^{>0}$ sends an δ on v to the local degree of $f|_{C_v}$ at δ, via the inclusion $\mathrm{Flags}_v \hookrightarrow C_v$ (This is well-defined by the balancing condition)

Definition 2.4.10. The closure G_γ of $\{f' : C' \to D' | f' \text{ has combinatorial type } \gamma\}$ is a subvariety of $\overline{\mathcal{H}}$. We call such a subvariety a ***boundary stratum*** of $\overline{\mathcal{H}}$.

Example 2.4.11. Let f_0 be the admissible cover depicted in Figure 2.3. Its combinatorial type is $\gamma_0 = (\sigma_0, \tau_0, {d_0}^{\mathrm{vert}}, {f_0}^{\mathrm{vert}}, (F_{0\,\mathrm{Edges}})_{v \in \mathrm{Vertices}(\sigma)}, (\mathrm{br}_{0,v})_{v \in \mathrm{Vertices}(\sigma_0)}, \mathrm{rm}_{0\,\mathrm{Edges}})$, where:

- σ_0 and τ_0 are as depicted in Figure 2.4
- $d_{0\,\mathrm{Vertices}}(v_1) = d_{0\,\mathrm{Vertices}}(v_4) = 1$, and $d_{0\,\mathrm{Vertices}}(v_2) = d_{0\,\mathrm{Vertices}}(v_3) = 2$
- $f_{0\,\mathrm{Vertices}}(v_1) = f_{0\,\mathrm{Vertices}}(v_3) = w_1$, and $f_{0\,\mathrm{Vertices}}(v_2) = f_{0\,\mathrm{Vertices}}(v_4) = w_2$
- $F_{0,v_i}(e_j) = \underline{e}$ for all $(i,j) \in \{(1,1),(2,1),(2,2),(3,2),(3,3),(4,3)$, and $F_{0,v_i}(a_j)$ is as given by F_0 in Figure 2.2
- $\mathrm{br}_{0,v_1}(\underline{e}) = \mathrm{br}_{0,v_4}(\underline{e}) = (1)$ and $\mathrm{br}_{0,v_2}(\underline{e}) = \mathrm{br}_{0,v_3}(\underline{e}) = (1,1)$
- $\mathrm{rm}_{0\,\mathrm{Edges}}(e) = 1$ for all $e \in \mathrm{Edges}(\sigma_0)$.

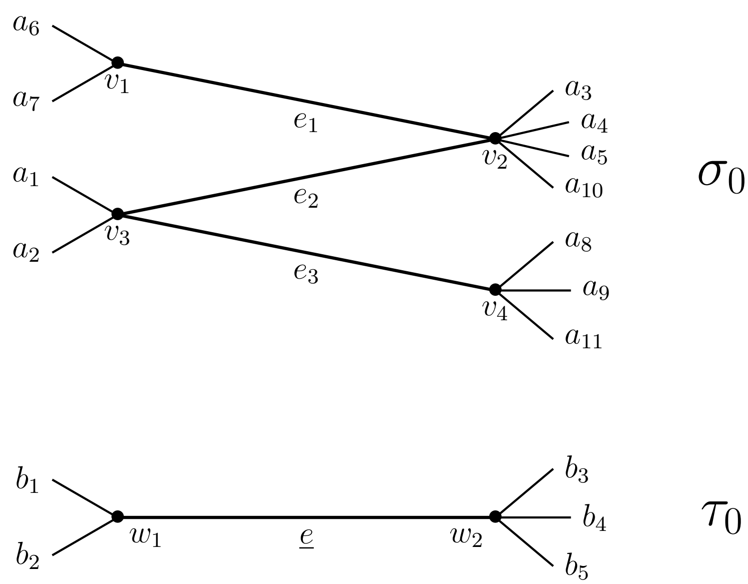


Figure 2.4: The dual trees σ_0 and τ_0 of source and target curves respectively as in Example 2.4.11

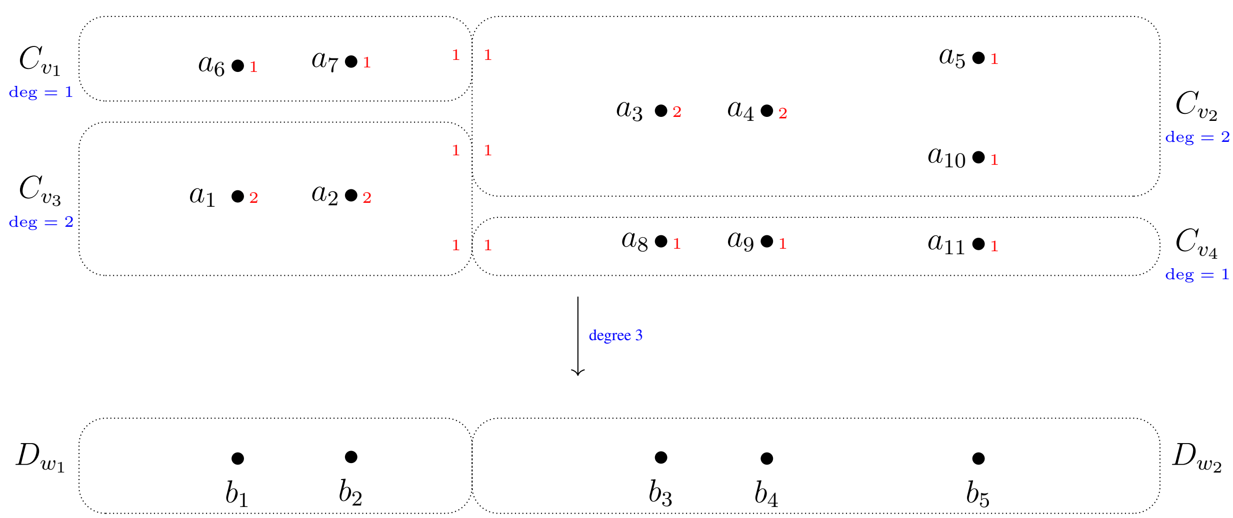


Figure 2.5: Schematic depiction of combinatorial type γ_0 as in Example 2.4.11. Each irreducible component of either the source and target curve is drawn as a bubble. Each component of the source curve is labeled with a degree in blue, and drawn above the component on the target curve it maps to. Each special point of the source curve is drawn directly above the special point on the target curve that it maps to. Ramification at special points is indicated in blue.

Figure 2.5 gives our shorthand for the combinatorial type γ_0.

The boundary stratum G_γ in $\overline{\mathcal{H}}$ can be decomposed into a product of lower-dimensional spaces of admissible covers. For $v \in \mathrm{Vertices}(\sigma)$, the data

$$(\mathrm{Flags}_v, \mathrm{Flags}_{f_{\mathrm{Vertices}}(v)}, d^{\mathrm{vert}}(v), F_{\mathrm{Edges}}, \mathrm{br}_v, \mathrm{rm}_{\mathrm{Edges}})$$

satisfy Conditions 1 and $2'$ as in Definition 2.4.1. Denote by $\overline{\mathcal{H}}_v$ the corresponding space of admissible covers. The space $\overline{\mathcal{H}}_v$ admits maps to the moduli space $\overline{\mathcal{M}}_{0,\mathrm{Flags}_v}$ of source curves and the moduli space $\overline{\mathcal{M}}_{0,\mathrm{Flags}_{f_{\mathrm{Vertices}}(v)}}$ of target curves. For $w \in \mathrm{Vertices}(\tau)$, set $\overline{\mathcal{H}}_w$ to be the *fibered* product

$$\overline{\mathcal{H}}_w := \prod_{v \in (f_{\mathrm{Vertices}})^{-1}(w)} \overline{\mathcal{H}}_v,$$

where the product is fibered over the common moduli space $\overline{\mathcal{M}}_{0,\mathrm{Flags}_w}$ of target curves.

The fibered product $\overline{\mathcal{H}}_w$ is itself a moduli space of possibly disconnected admissible covers, admitting a map $\overline{\pi}_w^{\mathrm{source}}$ to the moduli space $\prod_{v\in(f_{\mathrm{Vertices}})^{-1}(w)} \overline{\mathcal{M}}_{0,\mathrm{Flags}_v}$ of source curves and a finite flat map $\overline{\pi}_w^{\mathrm{target}}$ to the moduli space $\overline{\mathcal{M}}_{0,\mathrm{Flags}_w}$ of target curves. The stratum G_γ is isomorphic to $\prod_{w\in\mathrm{Vertices}(\tau)} \overline{\mathcal{H}}_w$.

Recall that the boundary stratum T_τ in $\overline{\mathcal{M}}_{0,\mathbf{B}}$ is isomorphic to $\prod_{w\in\mathrm{Vertices}(\tau)} \overline{\mathcal{M}}_{0,\mathrm{Flags}_w}$. We have

$$\begin{tikzcd}
\prod_{w\in\mathrm{Vertices}(\tau)} \overline{\mathcal{H}}_w \arrow[r, "\prod_{w\in\mathrm{Vertices}(\tau)} \overline{\pi}_w^{\mathrm{target}}"] \arrow[d, leftrightarrow, "\cong"'] & \prod_{w\in\mathrm{Vertices}(\tau)} \overline{\mathcal{M}}_{0,\mathrm{Flags}_w} \arrow[d, leftrightarrow, "\cong"] \\
G_\gamma \arrow[r, "\overline{\pi_{\mathbf{B}}}|_{G_\gamma}"] \arrow[d, hook] & T_\tau \arrow[d, hook] \\
\overline{\mathcal{H}} \arrow[r, "\overline{\pi_{\mathbf{B}}}"] & \overline{\mathcal{M}}_{0,\mathbf{B}}
\end{tikzcd}$$

All rightwards arrows are finite flat maps to moduli spaces of target curves. Thus the dimension of the boundary stratum G_γ of $\overline{\mathcal{H}}$ is the same as the dimension of its image in $\overline{\mathcal{M}}_{0,\mathbf{B}}$, namely the boundary stratum T_τ.

Similarly, the stratum S_σ in $\overline{\mathcal{M}}_{0,\mathbf{A}}$ is isomorphic to $\prod_{v\in\mathrm{Vertices}(\sigma)} \overline{\mathcal{M}}_{0,\mathrm{Flags}_v}$, and we have

$$\begin{tikzcd}
\prod\limits_{w\in \mathrm{Vertices}(\tau)} \overline{\mathcal{H}}_w \arrow[r, "\prod_{w\in \mathrm{Vertices}(\tau)} \overline{\pi}_w^{\mathrm{source}}"] \arrow[ddd, leftrightarrow, "\cong"'] & \prod\limits_{w\in \mathrm{Vertices}(\tau)} \prod\limits_{v\in (f_{\mathrm{Vertices}})^{-1}(w)} \overline{\mathcal{M}}_{0,\mathrm{Flags}_v} \arrow[d, equal] \\
& \prod\limits_{v\in \mathrm{Vertices}(\sigma)} \overline{\mathcal{M}}_{0,\mathrm{Flags}_v} \arrow[d, leftrightarrow, "\cong"] \\
& \\
G_\gamma \arrow[r, "\overline{\pi_{\mathbf{A}}}|_{G_\gamma}"] \arrow[d, hook] & S_\sigma \arrow[d, hook] \\
\overline{\mathcal{H}} \arrow[r, "\overline{\pi_{\mathbf{A}}}"] & \overline{\mathcal{M}}_{0,\mathbf{A}}
\end{tikzcd}$$

All rightwards arrows are maps to moduli spaces of source curves.

The boundary stratum G_γ is not necessarily irreducible. Its irreducible components are of the form

$$J = \prod_{w\in \mathrm{Vertices}(\tau)} \overline{\mathcal{J}_w},$$

where $\overline{\mathcal{J}_w}$ is an irreducible component of $\overline{\mathcal{H}}_w$.

2.4.4 Local geometry of $\overline{\mathcal{H}}$

Let $\mathcal{H} = \mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ be a fully marked Hurwitz space. Let $\overline{\mathcal{H}}$ be its admissible covers compactification admitting maps $\overline{\pi_{\mathbf{B}}}$ and $\overline{\pi_{\mathbf{A}}}$ to $\overline{\mathcal{M}}_{0,\mathbf{B}}$ and $\overline{\mathcal{M}}_{0,\mathbf{A}}$. Let $f : C \to D$ be an admissible cover, with $\theta_1, \ldots, \theta_k$ the nodes of D, $\eta_{1,j}, \ldots, \eta_{\ell_j,j}$ the nodes of C mapping to θ_j, and $p_{i,j}$ the local degree of f at $\eta_{i,j}$. By [HM82], The completion of the local ring of $[f]$ in $\overline{\mathcal{H}}$ is isomorphic to

$$\mathbb{C}[[t_1, \ldots, t_k, (t_{k+1} - \lambda_1), \ldots, (t_{|\mathbf{B}|-3} - \lambda_{|\mathbf{B}|-3-k}), t_{1,1}, \ldots, t_{1,k_1}, \ldots, t_{k,\ell_k}]]/(t_{i,j}^{p_{i,j}} = t_j) \quad (2.3)$$

Here, the t_js are pulled back from $\overline{\mathcal{M}}_{0,\mathbf{B}}$ and the $t_{i,j}$s are pulled back from $\overline{\mathcal{M}}_{0,\mathbf{A}}$. As described in Section 2.1.1.2, $t_1, \ldots, t_k$ are smoothing parameters for the nodes $\theta_1, \ldots, \theta_k$ respectively, the constants $\lambda_1, \ldots, \lambda_{|\mathbf{B}|-3-k}$ are the cross-ratios of quadruples of special points on D, and $t_{i,j}$ is a smoothing parameter for the node $\eta_{i,j}$. The space $\overline{\mathcal{H}}$ is normal at $[f]$ if and only if for every θ_j, at most one $p_{i,j}$ is bigger than 1. Thus in general $\overline{\mathcal{H}}$ is not normal. However, its normalization $\tilde{n} : \tilde{\overline{\mathcal{H}}} \to \overline{\mathcal{H}}$ is smooth, and easy to describe in coordinates. Set $p_j = \mathrm{lcm}_i\, p_{i,j}$ and $q_j = \prod_i p_{i,j}$. Then in the normalization there are $\prod_j q_j/p_j$ points over $[f]$ (see Remark 2.4.12 below). The local

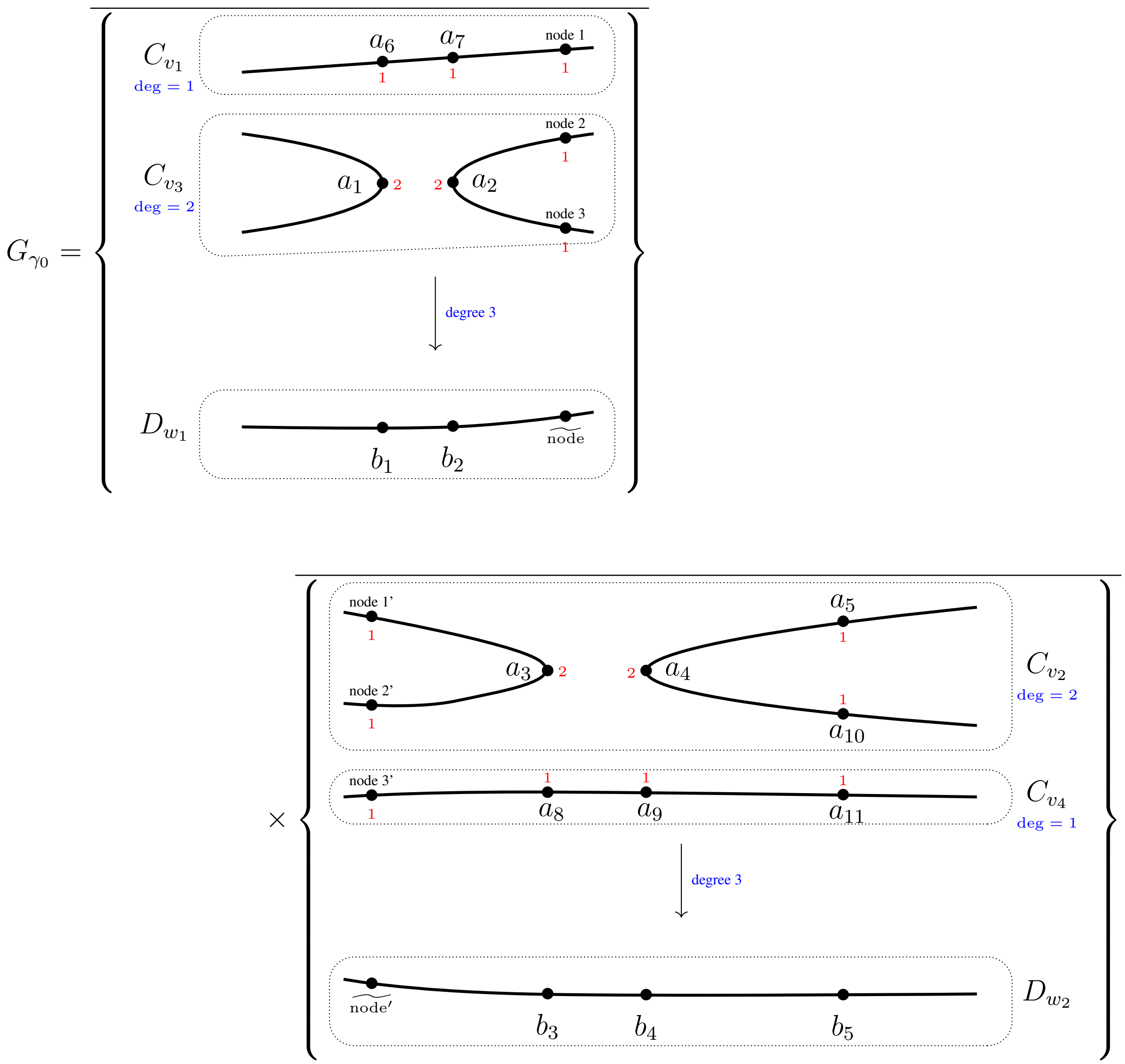


Figure 2.6: Decomposition of the boundary stratum G_{γ_0} with γ_0 as in Example 2.4.11

ring at each of these points in $\overline{\tilde{\mathcal{H}}}$ is isomorphic to

$$\mathbb{C}[[s_1, \ldots, s_k, \lambda_1, \ldots, \lambda_{|B|-3-k}]] \tag{2.4}$$

where $s_j^{p_j} = t_j$ and $s_j^{p_j/p_{i,j}} = t_{i,j}$.

Remark 2.4.12. How many points are in the inverse image of f in $\overline{\tilde{\mathcal{H}}}$? Let U be an analytic neighborhood of $[f]$ in $\overline{\mathcal{H}}$, and let $\tilde{U}$ be its inverse image in $\overline{\tilde{\mathcal{H}}}$. Since $\tilde{n} : \tilde{U} \to U$ is an isomorphism off the singular locus, the two (finite and flat) maps $\overline{\pi_{\mathbf{B}}} \circ \tilde{n} : \tilde{U} \to \overline{\mathcal{M}}_{0,\mathbf{B}}$ and $\overline{\pi_{\mathbf{B}}} : U \to \overline{\mathcal{M}}_{0,\mathbf{B}}$ have the same degree. It is clear from the local coordinates at $[f]$ given above that the latter map has degree $\prod_j q_j$. Similarly, if $[\tilde{f}]$ is any point in $\tilde{U}$ over $[f]$, then by the local coordinates given above, $\overline{\pi_{\mathbf{B}}} \circ \tilde{n}$ has local degree $\prod_j p_j$ at $[\tilde{f}]$. Thus the number of such points must be $\prod_j q_j/p_j$.

Note that in the given coordinates $\overline{\pi_{\mathbf{B}}}$ can be decomposed into a product of flat maps of curves, thus is itself flat. Thus, just as in $\overline{\mathcal{M}}_{0,n}$, a codimension-$k$ boundary stratum in $\overline{\mathcal{H}}$ is the intersection of k codimension-1 boundary strata. Let

$$\gamma = (\sigma, \tau, d_{\text{Vertices}}, f_{\text{Vertices}}, F_{\text{Edges}}, (\text{br}_v)_{v \in \text{Vertices}(\sigma)}, \text{rm}_{\text{Edges}})$$

be the combinatorial type of f; so $[f]$ is a general point of the boundary stratum G_γ of $\overline{\mathcal{H}}$. The dual tree τ of D has edges $\underline{e_1}, \ldots, \underline{e_k}$ corresponding to the nodes $\theta_1, \ldots, \theta_k$ respectively. As described in Section 2.1.1.2, the boundary stratum T_τ of $\overline{\mathcal{M}}_{0,\mathbf{B}}$ is locally cut out by the t_js. For $j = 1, \ldots, k$, $\text{Div}_{\overline{\mathcal{M}}_{0,\mathbf{B}}}(\underline{e_j})$ is the boundary divisor cut out by t_j. Its dual tree is obtained by collapsing all edges of τ except $\underline{e_j}$. There is a boundary divisor $G_\gamma(\underline{e_j})$ in $\overline{\mathcal{H}}$ containing G_γ and mapping onto $\text{Div}_{\overline{\mathcal{M}}_{0,\mathbf{B}}}(\underline{e_j})$. An open subset of this divisor parametrizes admissible covers whose target curves have the dual tree obtained by collapsing all edges of τ except $\underline{e_j}$, and whose source curves have the dual tree obtained by collapsing all the edges of σ except those mapping via some F_{Edges} to $\underline{e_j}$. Just as we have in $\overline{\mathcal{M}}_{0,\mathbf{B}}$:

$$T_\tau = \bigcap_{j=1}^{k} \text{Div}_{\overline{\mathcal{M}}_{0,\mathbf{B}}}(\underline{e_j}),$$

we have in $\overline{\mathcal{H}}$:

$$G_\gamma = \bigcap_{j=1}^{k} G_\gamma(\underline{e_j}).$$

This is another way to see that the codimension of the stratum G_γ is the number of nodes on the target curve of a generic admissible cover.

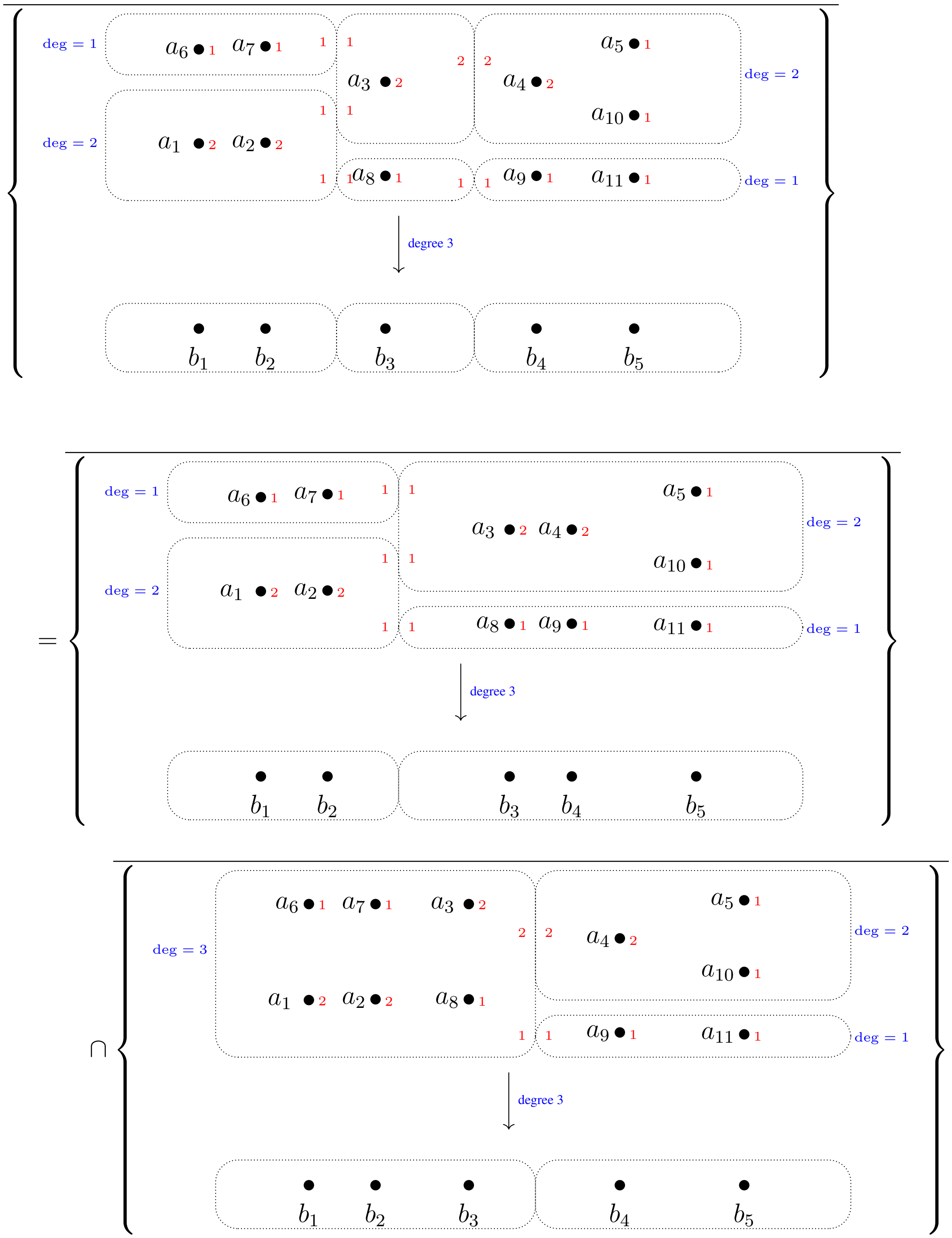


Figure 2.7: A codimension 2 boundary stratum in $\overline{\mathcal{H}}_0 = \overline{\mathcal{H}}(\mathbf{A}_0, \mathbf{B}_0, d_0, f_0, \mathrm{br}_0, \mathrm{rm}_0)$ (as in Figure 2.2), expressed as the intersection of two codimension 1 boundary strata.

Let $\widetilde{G_\gamma}$ be the inverse image of G_γ in $\tilde{\tilde{\mathcal{H}}}$, and let $[\tilde{f}]$ be any point in $\tilde{\tilde{\mathcal{H}}}$ mapping to $[f]$. For $j = 1, \ldots, k$, let $\widetilde{G_\gamma(\underline{e_j})}$ be the inverse image of $G_\gamma(\underline{e_j})$ in $\tilde{\tilde{\mathcal{H}}}$. Then in a formal neighborhood of $[\tilde{f}]$, the parameter s_j cuts out the divisor $\widetilde{G_\gamma(\underline{e_j})}$, and the s_js collectively generate the ideal of $\widetilde{G_\gamma}$.

2.4.5 A remark on cohomology versus Chow groups

Let $\overline{\mathcal{H}} = \overline{\mathcal{H}}(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ be as in the statement of Theorem 2.4.6. Although $\overline{\mathcal{H}}$ is not smooth, there is a pullback map of Chow groups $(\overline{\pi_{\mathbf{B}}})^* : A_k(\overline{\mathcal{M}}_{0,\mathbf{B}}) \to A_k(\overline{\mathcal{H}})$ ([Ful98], Section 6.2). We thus have

$$(\overline{\pi_{\mathbf{A}}})_* \circ (\overline{\pi_{\mathbf{B}}})^* : A_k(\overline{\mathcal{M}}_{0,\mathbf{B}}) \to A_k(\overline{\mathcal{M}}_{0,\mathbf{A}}).$$

For any resolution of singularities $\chi : \tilde{\mathcal{H}} \to \overline{\mathcal{H}}$, $\chi_* \circ (\overline{\pi_{\mathbf{B}}} \circ \chi)^* : A_k(\overline{\mathcal{M}}_{0,\mathbf{B}}) \to A_k(\overline{\mathcal{H}})$ agrees with $(\overline{\pi_{\mathbf{B}}})^*$ ([Ful98], Theorem 6.2a). Thus $(\overline{\pi_{\mathbf{A}}})_* \circ (\overline{\pi_{\mathbf{B}}})^*$ agrees with $(\overline{\pi_{\mathbf{A}}} \circ \chi)_* \circ (\overline{\pi_{\mathbf{B}}} \circ \chi)^*$ as maps from $A_k(\overline{\mathcal{M}}_{0,\mathbf{B}})$ to $A_k(\overline{\mathcal{M}}_{0,\mathbf{A}})$.

By [Kee92], the 'cycle class' map gives canonical isomorphisms

$$A_k(\overline{\mathcal{M}}_{0,n}) \to H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Z}).$$

Under these isomorphisms,

$$(\overline{\pi_{\mathbf{A}}} \circ \chi)_* \circ (\overline{\pi_{\mathbf{B}}} \circ \chi)^* : A_k(\overline{\mathcal{M}}_{0,\mathbf{B}}) \to A_k(\overline{\mathcal{M}}_{0,\mathbf{A}})$$

is identified with

$$(\overline{\pi_{\mathbf{A}}} \circ \chi)_* \circ (\overline{\pi_{\mathbf{B}}} \circ \chi)^* : H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{B}}, \mathbb{Z}) \to H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{A}}, \mathbb{Z}).$$

Thus the pushforward $[\mathcal{H}]_* : H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{B}}, \mathbb{Z}) \to H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{A}}, \mathbb{Z})$ may be identified with $(\overline{\pi_{\mathbf{A}}})_* \circ (\overline{\pi_{\mathbf{B}}})^* : A_k(\overline{\mathcal{M}}_{0,\mathbf{B}}) \to A_k(\overline{\mathcal{M}}_{0,\mathbf{A}})$. An analogous statement holds for $[\Gamma]_*$, where Γ is any union of connected components of $\mathcal{H}$. We use this identification throughout, for instance in Proposition 3.0.1 and Theorem 4.1.6.

CHAPTER 3

Hurwitz correspondences are algebraically stable on $\overline{\mathcal{M}}_{0,n}$

This chapter contains work done in collaboration with Sarah Koch and David Speyer.

The admissible covers compactification $\overline{\mathcal{H}}$ of $\mathcal{H} = \mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ naturally lives over the spaces $\overline{\mathcal{M}}_{0,n}$ of stable curves: it extends the "target curve" map to $\overline{\mathcal{M}}_{0,\mathbf{B}}$ and the "source curve" map to $\overline{\mathcal{M}}_{0,\mathbf{A}}$. We treat the Hurwitz correspondence $\mathcal{H}$ as a multivalued map from $\mathcal{M}_{0,\mathbf{B}}$ to $\mathcal{M}_{0,\mathbf{A}}$. More precisely, $\mathcal{H}$ induces a map from $\mathcal{M}_{0,\mathbf{B}}$ to $\mathrm{Sym}^{(\deg \pi_{\mathbf{B}})} \mathcal{M}_{0,\mathbf{A}}$. Since $\overline{\pi_{\mathbf{B}}} : \overline{\mathcal{H}} \to \overline{\mathcal{M}}_{0,\mathbf{B}}$ is finite and flat, this extends to a map from $\overline{\mathcal{M}}_{0,\mathbf{B}}$ to $\mathrm{Sym}^{(\deg \pi_{\mathbf{B}})} \overline{\mathcal{M}}_{0,\mathbf{A}}$. Thus the rational correspondence $(\mathcal{H}, \pi_{\mathbf{B}}, \pi_{\mathbf{A}}) : \overline{\mathcal{M}}_{0,\mathbf{B}} \dashrightarrow \overline{\mathcal{M}}_{0,\mathbf{A}}$ may be treated as a regular correspondence. This is supported by the fact that Hurwitz correspondences are homologically composable on the stable curves spaces $\overline{\mathcal{M}}_{0,n}$.

Proposition 3.0.1 (Koch, Ramadas, Speyer)**.** *Let $(\Gamma, \pi_1, \pi_2) : \overline{\mathcal{M}}_{0,n_1} \dashrightarrow \overline{\mathcal{M}}_{0,n_2}$ and $(\Gamma', \pi_2', \pi_3') : \overline{\mathcal{M}}_{0,n_2} \dashrightarrow \overline{\mathcal{M}}_{0,n_3}$ be Hurwitz correspondences. Then for all k,*

$$[\Gamma' \circ \Gamma]_* = [\Gamma']_* \circ [\Gamma]_*$$

as maps from $H_{2k}(\overline{\mathcal{M}}_{0,n_1})$ to $H_{2k}(\overline{\mathcal{M}}_{0,n_3})$.

Proof. We prove instead that $[\Gamma' \circ \Gamma]_* = [\Gamma']_* \circ [\Gamma]_*$ as maps from $A_k(\overline{\mathcal{M}}_{0,n_1})$ to $A_k(\overline{\mathcal{M}}_{0,n_3})$ (see Section 2.4.5).

By Lemma 2.4.3, we may reduce to the case in which Γ and Γ' are unions of connected components of fully marked Hurwitz spaces $\mathcal{H}$ and $\mathcal{H}'$ respectively. The maps π_1 and π_2' to the moduli spaces of target curves $\mathcal{M}_{0,n_1}$ and $\mathcal{M}_{0,n_2}$ are finite covering maps. Also we have $\pi_2(\Gamma) \subseteq \mathcal{M}_{0,n_2}$. Applying the definition of composition of rational correspondences given in Section 2.2.1, we may set

$$\Gamma' \circ \Gamma = \Gamma \times_{\mathcal{M}_{0,n_2}} \Gamma',$$

with maps pr and pr' to Γ and Γ' respectively. The map pr is the pullback of a covering map, so is itself a covering map. Let $\overline{\Gamma}$ and $\overline{\Gamma'}$ be the closures of Γ and Γ' in the admissible covers spaces $\overline{\mathcal{H}}$ and $\overline{\mathcal{H}}'$. We have the diagram

$$\begin{tikzcd}
 & & \Gamma \times_{\mathcal{M}_{0,n_2}} \Gamma' \arrow[dl, "\mathrm{pr}"'] \arrow[dr, "\mathrm{pr}'"] & & \\
 & \Gamma \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \Gamma' \arrow[dl, "\pi_2'"'] \arrow[dr, "\pi_3'"] & \\
\mathcal{M}_{0,n_1} & & \mathcal{M}_{0,n_2} & & \mathcal{M}_{0,n_3}
\end{tikzcd}$$

We also have the diagram of compactifications:

$$\begin{tikzcd}
 & & \overline{\Gamma} \times_{\overline{\mathcal{M}}_{0,n_2}} \overline{\Gamma'} \arrow[dl, "\overline{\mathrm{pr}}"'] \arrow[dr, "\overline{\mathrm{pr}'}"] & & \\
 & \overline{\Gamma} \arrow[dl, "\overline{\pi_1}"'] \arrow[dr, "\overline{\pi_2}"] & & \overline{\Gamma'} \arrow[dl, "\overline{\pi_2'}"'] \arrow[dr, "\overline{\pi_3'}"] & \\
\overline{\mathcal{M}}_{0,n_1} & & \overline{\mathcal{M}}_{0,n_2} & & \overline{\mathcal{M}}_{0,n_3}
\end{tikzcd}$$

Since $\overline{\pi_1}$ and $\overline{\pi_2'}$ are finite and flat, so is $\overline{\pi_1} \circ \overline{\mathrm{pr}}$. This means that $\overline{\Gamma} \times_{\overline{\mathcal{M}}_{0,n_2}} \overline{\Gamma'}$ has no irreducible components supported over the boundary $\overline{\mathcal{M}}_{0,N1} \smallsetminus \mathcal{M}_{0,n_1}$. There is an inclusion $\Gamma \times_{\mathcal{M}_{0,n_2}} \Gamma' \hookrightarrow \overline{\Gamma} \times_{\overline{\mathcal{M}}_{0,n_2}} \overline{\Gamma'}$ whose image is $(\overline{\pi_1} \circ \overline{\mathrm{pr}})^{-1}(\mathcal{M}_{0,n_1})$. By the above this is an inclusion as a dense open set. We therefore have

$$\begin{aligned}
[\Gamma' \circ \Gamma]_* &= [\overline{\Gamma} \times_{\overline{\mathcal{M}}_{0,n_2}} \overline{\Gamma'}]_* \\
&= (\overline{\pi_3'} \circ \overline{\mathrm{pr}'})_* \circ (\overline{\pi_1} \circ \overline{\mathrm{pr}})^* \\
&= (\overline{\pi_3'})_* \circ (\overline{\mathrm{pr}'})_* \circ (\overline{\mathrm{pr}})^* \circ (\overline{\pi_1})^*.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
[\Gamma']_* \circ [\Gamma]_* &= [\overline{\Gamma'}]_* \circ [\overline{\Gamma}]_* \\
&= (\overline{\pi_3'})_* \circ (\overline{\pi_2'})^* \circ (\overline{\pi_2})_* \circ (\overline{\pi_1})^* \\
&= (\overline{\pi_3'})_* \circ (\overline{\mathrm{pr}'})_* \circ (\overline{\mathrm{pr}})^* \circ (\overline{\pi_1})^* \\
&= [\Gamma' \circ \Gamma]_*.
\end{aligned}$$

Here, the third equality follows from the fact (Proposition 1.7 in [Ful98]) that for any fibered square of varieties

$$\begin{tikzcd} & \overline{\Gamma} \times_{\mathcal{M}} \overline{\Gamma'} \arrow[dl, "\overline{\mathrm{pr}}"'] \arrow[dr, "\overline{\mathrm{pr}'}"] & \\ \overline{\Gamma} \arrow[dr, "\overline{\pi}"'] & & \overline{\Gamma'} \arrow[dl, "\overline{\pi'}"] \\ & \mathcal{M} & \end{tikzcd}$$

where $\overline{\pi'}$ is a flat map and $\overline{\pi}$ is proper, we have

$$(\overline{\pi'})^* \circ (\overline{\pi})_* = (\overline{\mathrm{pr}'})_* \circ (\overline{\mathrm{pr}})^*. \qquad \square$$

By duality of pushforward and pullback, and the fact (Keel, [Kee92]) that $H^{2k}(\overline{\mathcal{M}}_{0,n}) = H^{k,k}(\overline{\mathcal{M}}_{0,n})$, we obtain:

Corollary 3.0.2 (Koch, Ramadas, Speyer)**.** *Let* $(\Gamma, \pi_1, \pi_2) : \overline{\mathcal{M}}_{0,n} \dashrightarrow \overline{\mathcal{M}}_{0,n}$ *be a dominant Hurwitz self-correspondence. Then* Γ *is algebraically stable, and its* k*th dynamical degree is the absolute value of the dominant eigenvalue of* $[\Gamma]_* : H_{2k}(\overline{\mathcal{M}}_{0,n}) \to H_{2k}(\overline{\mathcal{M}}_{0,n})$

Remark 3.0.3. Let $(\Gamma, \pi_1, \pi_2) : \mathcal{M}_{0,n} \dashrightarrow \mathcal{M}_{0,n}$ be as above, and let $\overline{\Gamma}$ be the admissible covers compactification of Γ, with its maps $\overline{\pi_1}$ and $\overline{\pi_2}$ to $\overline{\mathcal{M}}_{0,n}$. The map $\overline{\pi_1}$ is flat, so $(\overline{\pi_1})^*$ takes effective classes on $\overline{\mathcal{M}}_{0,n}$ to effective classes on $\overline{\Gamma}$. Pushforwards always preserve effectiveness, so $[\Gamma]_* = (\overline{\pi_2})_* \circ (\overline{\pi_1})^* : H_{2k}(\overline{\mathcal{M}}_{0,n}) \to H_{2k}(\overline{\mathcal{M}}_{0,n})$ preserves the cone of effective classes. By continuity, $[\Gamma]_*$ preserves the *pseudoeffective cone*, namely the closure of the cone of effective classes. The pseudoeffective cone of any projective variety is a closed cone with nonempty interior that contains no lines ([BFJ09, FL14]). It follows from the theory of cone-preserving operators ([ST07]) that $[\Gamma]_*$ has a nonnegative real dominant eigenvalue, with a pseudoeffective eigenvector.

CHAPTER 4

The actions of Hurwitz correspondences on the homology groups of $\overline{\mathcal{M}}_{0,n}$

Since Hurwitz correspondences are algebraically stable on $\overline{\mathcal{M}}_{0,n}$, their dynamical degrees are dominant eigenvalues of pushforward maps induced on the homology groups of $\overline{\mathcal{M}}_{0,n}$ (Corollary 3.0.2). Here, we describe these pushforward maps. We show that they preserve a natural combinatorially defined filtration (Theorem 4.1.6) and that the dynamical degrees of Hurwitz correspondences may be recovered from the induced action on the top graded piece of this filtration (Theorem 4.2.7).

4.1 Hurwitz correspondences preserve a natural filtration of $H_{2k}(\overline{\mathcal{M}}_{0,n})$

We recall the notation and terminology introduced in Section 2.1.1.

Definition 4.1.1. Let S_σ be a k-dimensional boundary stratum in $\overline{\mathcal{M}}_{0,n}$. Then

$$\sum_{v \in \text{Vertices}(\sigma)} \text{md}(v) = k.$$

Set λ_σ to be the multiset

$$(\text{md}(v))_{v \,\in\, \text{Vertices}(\sigma) \text{ with } \text{md}(v) \,\neq\, 0}.$$

Then λ_σ is a partition of k; we say λ_σ is the partition ***induced by*** the stratum S_σ.

S_σ is isomorphic to $\prod_{v \in \text{Vertices}(\sigma)} \overline{\mathcal{M}}_{0,\text{Flags}_v}$. Each factor $\overline{\mathcal{M}}_{0,\text{Flags}_v}$ has dimension $\text{md}(v)$, so λ_σ encodes the nonzero dimensions of the factors when we write S_σ as this product.

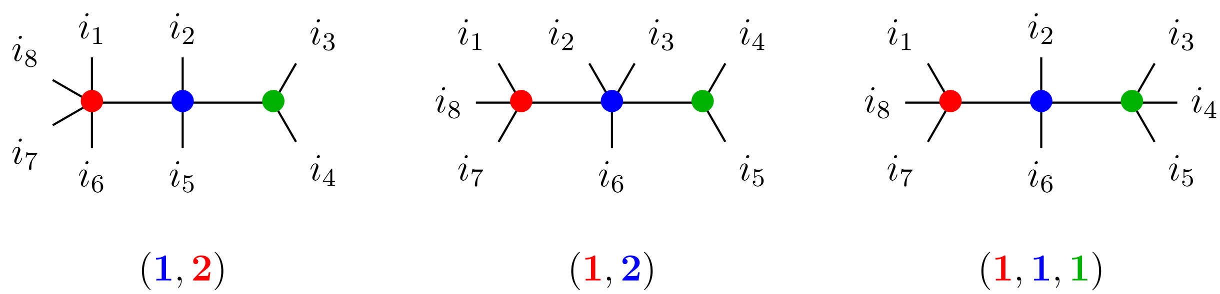


Figure 4.1: Dual trees of boundary strata generating the subspace $\Lambda_8^{\leq(1,2)} \subseteq H_6(\overline{\mathcal{M}}_{0,8})$

Definition 4.1.2. For a partition λ of k let $\Lambda_{\leq\lambda,n}$ be the subspace of $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ generated by k-dimensional boundary strata that induce the partition λ or any refinement. For $\mathbf{P}$ a finite set we define $\Lambda_{\mathbf{P}}^{\leq\lambda} \subseteq H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}}, \mathbb{Q})$ analogously.

Note that $\Lambda_{\leq\lambda,n}$ is defined over $\mathbb{Z}$. If λ_1 is a refinement of λ_2, we write $\lambda_1 \leq \lambda_2$. This is a partial ordering on the set of partitions of k. Clearly if $\lambda_1 \leq \lambda_2$, then $\Lambda_n^{\leq\lambda_1} \subseteq \Lambda_n^{\leq\lambda_2}$.

The dual tree of a k-dimensional boundary stratum S_σ in $\overline{\mathcal{M}}_{0,n}$ has $n-k-2$ vertices. This means that the partition λ_σ has at most $n-k-2$ parts. Conversely, any partition of k with at most $n-k-2$ parts arises from a boundary stratum in $\overline{\mathcal{M}}_{0,n}$. We say that λ is ***realizable*** on $\mathcal{M}_{0,n}$ if λ has at most $n-k-2$ parts.

Lemma 4.1.3. *Let S_{σ_0} be a k-dimensional boundary stratum inducing partition λ_{σ_0} of k. Then there does not exist an equality in $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Z})$ of the form*

$$[S_{\sigma_0}] = \sum_j \beta_j [S_{\sigma(j)}],$$

where every partition $\lambda_{\sigma(j)}$ is distinct from λ_{σ_0}.

Proof. We recall the additive relations $R(\sigma, v, i_1, \ldots, i_4)$ introduced in Section 2.1.1 (Equation 2.1). We can rewrite

$$\begin{aligned} &R(\sigma, v, i_1, \ldots, i_4) \\ &\quad = \sum_{(\mathrm{Flags}_1, \mathrm{Flags}_2)} ([S(i_1 i_2 \,\mathrm{Flags}_1 \,|\, \mathrm{Flags}_2 \, i_3 i_4)] - [S(i_1 i_3 \,\mathrm{Flags}_1 \,|\, \mathrm{Flags}_2 \, i_2 i_4)]). \end{aligned}$$

It is clear from definitions that $S(i_1 i_2 \,\mathrm{Flags}_1 \,|\, \mathrm{Flags}_2 \, i_3 i_4)$ and $S(i_1 i_3 \,\mathrm{Flags}_1 \,|\, \mathrm{Flags}_2 \, i_2 i_4)$ induce the same partition of k, and they appear in $R(\sigma, v, i_1, \ldots, i_4)$ paired up and with opposite signs. Therefore for any partition λ of k, the coefficients in $R(\sigma, v, i_1, \ldots, i_4)$ of boundary strata inducing

λ sum to zero. Since the relations $R(\sigma, v, i_1, \ldots, i_4)$ generate all relations among boundary strata, we conclude that for any additive relation R, the coefficients of boundary strata inducing a fixed partition λ sum to zero. Therefore there is no equality $[S_{\sigma_0}] = \sum_j \beta_j [S_{\sigma(j)}]$, where each partition $\lambda_{\sigma(j)}$ is different from λ_{σ_0}. □

We deduce:

Corollary 4.1.4. *1. For S_σ a k-dimensional boundary stratum and λ a partition of k, $S_\sigma \in \Lambda_{\leq\lambda,n}$ if and only if $\lambda_\sigma \leq \lambda$.*

2. For λ_1 and λ_2 distinct realizable partitions of k, $\Lambda_n^{\leq\lambda_1} \neq \Lambda_n^{\leq\lambda_2}$.

The collection of subspaces $\{\Lambda_{\leq\lambda,n}\}_\lambda$ is a poset-filtration for $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ indexed by the set of partitions of k, with the refinement partial ordering. Denote by (k) the one-part partition of k. Then $\Lambda_n^{\leq(k)} = H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$.

Forgetful maps respect the filtration $\{\Lambda_{\leq\lambda,n}\}_\lambda$:

Lemma 4.1.5. *Let $n \geq n' \geq 3$. Then, if $\mu : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,n'}$ is the forgetful map sending $(C, p_1, \ldots, p_n)$ to $(C, p_1, \ldots, p_{n'})$, the pushforward $\mu_* : H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) \to H_{2k}(\overline{\mathcal{M}}_{0,n'}, \mathbb{Q})$ sends $\Lambda_{\leq\lambda,n}$ to $\Lambda_{n'}^{\leq\lambda}$ for all partitions λ of k.*

Proof. We may assume $n' = N - 1$. Let S_σ be any k-dimensional boundary stratum in $\overline{\mathcal{M}}_{0,n}$, inducing partition λ_σ of k. We show $\mu_*([S_\sigma]) \in \Lambda_{n'}^{\leq\lambda_\sigma}$. The image $\mu(S_\sigma) = S_{\sigma'}$ is a boundary stratum in $\overline{\mathcal{M}}_{0,n'}$, where σ' is obtained from σ by deleting the leg ℓ_n and stabilizing as in Section 2.1.1. Let v_0 on σ be the vertex with the leg ℓ_n.
Case 1. $|v_0| \geq 4$.

The n'-marked tree obtained by deleting ℓ_n is stable, and therefore is σ'. The moduli dimension of v_0 is positive, and the corresponding vertex on σ' has moduli dimension one less. All other vertices of σ' have the same moduli dimension as the corresponding vertices of σ, so $\dim S_{\sigma'} = \dim S_\sigma - 1$. Therefore $\mu_*([S_\sigma]) = 0 \in \Lambda_{n'}^{\leq\lambda_\sigma}$.
Case 2. $|v_0| = 3$ and Flags_{v_0} consists of ℓ_n, an edge e_1 connecting v_0 to v_1, and an edge e_2 connecting v_0 to v_2.

The stabilization σ' is obtained from σ by deleting ℓ_n, v_0, e_1, and e_2, and connecting v_1 and v_2 with a new edge. The tree σ' does not have the vertex v_0, which has moduli dimension zero, however all other vertices on σ' have the same moduli dimension as the corresponding vertices on σ. So $\dim S_{\sigma'} = \dim S_\sigma$ and $\lambda_{\sigma'} = \lambda_\sigma$. Therefore $\mu_*([S_\sigma]) = [S_{\sigma'}] \in \Lambda_{n'}^{\leq\lambda_\sigma}$.
Case 3. $|v_0| = 3$ and Flags_{v_0} consists of ℓ_n, another leg ℓ_i, and an edge e connecting v_0 to v_1.

The stabilization σ' is obtained from σ by deleting ℓ_n, v_0, and e, and attaching the leg ℓ_i to v_1. The tree σ' does not have the vertex v_0, which has moduli dimension zero, however all other vertices on σ' have the same moduli dimension as the corresponding vertices on σ. So $\dim S_{\sigma'} = \dim S_\sigma$ and $\lambda_{\sigma'} = \lambda_\sigma$. Therefore $\mu_*([S_\sigma]) = [S_{\sigma'}] \in \Lambda_{n'}^{\leq \lambda_\sigma}$. $\square$

Our main result in this section is Theorem 4.1.6, showing that the filtration $\{\Lambda_{\leq\lambda,n}\}_\lambda$ is preserved by all Hurwitz correspondences. Our proof uses the stratification of the moduli spaces of admissible covers (Section 2.4.3).

Theorem 4.1.6. *Let $(\Gamma, \pi_{\mathbf{B}}, \pi_{\mathbf{A}}) : \overline{\mathcal{M}}_{0,\mathbf{B}} \dashrightarrow\!\!\!\dashrightarrow \overline{\mathcal{M}}_{0,\mathbf{A}}$ be a Hurwitz correspondence. Then for every partition λ of k, $[\Gamma]_* : H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{B}}, \mathbb{Q}) \to H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{A}}, \mathbb{Q})$ sends $\Lambda_{\mathbf{B}}^{\leq\lambda}$ to $\Lambda_{\mathbf{A}}^{\leq\lambda}$.*

Proof of Theorem 4.1.6. By Lemma 2.4.3, we may assume that Γ is a union of connected components of a fully marked Hurwitz space $\mathcal{H}$, and by Lemma 4.1.5, we may assume that $\mathbf{A} = \mathbf{A}^{\text{full}}$ as in Section 2.4.1.

Let $\overline{\mathcal{H}}$ be the admissible covers compactification of $\mathcal{H}$ and let $\overline{\Gamma}$ be the closure of Γ in $\overline{\mathcal{H}}$. The compactifications $\overline{\mathcal{H}}$ and $\overline{\Gamma}$ both have maps $\overline{\pi_{\mathbf{B}}}$ and $\overline{\pi_{\mathbf{A}}}$ to $\overline{\mathcal{M}}_{0,\mathbf{B}}$ and $\overline{\mathcal{M}}_{0,\mathbf{A}}$. As in Section 2.4.5, we have $[\Gamma]_* = (\overline{\pi_{\mathbf{A}}})_* \circ (\overline{\pi_{\mathbf{B}}})^*$.

Fix T_τ any k-dimensional boundary stratum in $\overline{\mathcal{M}}_{0,\mathbf{B}}$. Let λ_τ be the partition of k induced by T_τ. We show $[\Gamma]_*([T_\tau]) \in \Lambda_{\mathbf{A}}^{\leq\lambda_\tau}$. Since $\overline{\pi_{\mathbf{B}}}$ is flat,

$$(\overline{\pi_{\mathbf{B}}})^*([T_\tau]) = \sum_J m_J[J],$$

where the sum is over irreducible components J of the preimage $(\overline{\pi_{\mathbf{B}}})^{-1}(T_\tau)$ in $\overline{\Gamma}$, and m_J is a positive integer multiplicity.

To show that $(\overline{\pi_{\mathbf{A}}})_* \circ (\overline{\pi_{\mathbf{B}}})^*([T_\tau])$ is in $\Lambda_{\mathbf{A}}^{\leq\lambda_\tau}$, it suffices to show that for J an irreducible component of $(\overline{\pi_{\mathbf{B}}})^{-1}(T_\tau)$, the pushforward $(\overline{\pi_{\mathbf{A}}})_*([J])$ is in $\Lambda_{\mathbf{A}}^{\leq\lambda_\tau}$.

Fix such a J: it is an irreducible component of a boundary stratum G_γ of $\overline{\mathcal{H}}$, with combinatorial type $\gamma = (\sigma, \tau, d_{\text{Vertices}}, f_{\text{Vertices}}, F_{\text{Edges}}, (\text{br}_v)_{v\in\text{Vertices}(\sigma)}, \text{rm}_{\text{Edges}})$ (Section 2.4.3). There is a decomposition $G_\gamma = \prod_{w\in\text{Vertices}(\tau)} \overline{\mathcal{H}}_w$, where $\overline{\mathcal{H}}_w$ is an admissible covers space of dimension $\text{md}(w)$, and a decomposition

$$J = \prod_{w\in\text{Vertices}(t)} \overline{\mathcal{J}_w},$$

where $\overline{\mathcal{J}_w}$ is an irreducible component of $\overline{\mathcal{H}}_w$. Each factor $\overline{\mathcal{J}_w}$ admits a map $\overline{\pi_w^{\text{source}}}$ to

$$\prod_{v\in(f_{\text{Vertices}})^{-1}(w)} \overline{\mathcal{M}}_{0,\text{Flags}_v}$$

and a finite flat map $\overline{\pi_w^{\text{target}}}$ to $\overline{\mathcal{M}}_{0,\text{Flags}_w}$. We conclude that $\dim \overline{\mathcal{J}_w} = \dim \overline{\mathcal{M}}_{0,\text{Flags}_w} = \text{md}(w)$. The key observation of our proof is that this decomposition of J also induces the partition λ_τ.

The cohomology (respectively Chow) ring of $\prod_{v\in(f_{\text{Vertices}})^{-1}(w)} \overline{\mathcal{M}}_{0,\text{Flags}_v}$ is the tensor product of the cohomology (respectively Chow) rings of the factors, which in turn are generated by the classes of boundary strata. Thus we may write

$$(\overline{\pi_w^{\text{source}}})_*([\overline{\mathcal{J}_w}]) = \sum_{q\in Q_w} \beta_q \bigotimes_{v\in(f_{\text{Vertices}})^{-1}(w)} [S^q_{wv}],$$

where Q_w is a finite set, β_q is an integer multiplicity, and S^q_{wv} is a boundary stratum of dimension k^q_{wv} in $\overline{\mathcal{M}}_{0,\text{Flags}_v}$. Denote by λ^q_{wv} the partition of k^q_{wv} induced by S^q_{wv}. For every $q \in Q_w$,

$$\sum_{v\in(f_{\text{Vertices}})^{-1}(w)} k^q_{wv} = \dim \overline{\mathcal{J}_w} = \text{md}(w),$$

so $\bigcup_{v\in(f_{\text{Vertices}})^{-1}(w)} \lambda^q_{wv}$ is a partition of $\text{md}(w)$. The map $\overline{\pi_{\mathbf{A}}}|_J$ decomposes as a product:

$$J = \prod_{w\in\text{Vertices}(\tau)} \overline{\mathcal{J}_w} \xrightarrow{\prod_{w\in\text{Vertices}(\tau)} \overline{\pi_w^{\text{source}}}} \prod_{w\in\text{Vertices}(\tau)} \prod_{v\in(f_{\text{Vertices}})^{-1}(w)} \overline{\mathcal{M}}_{0,\text{Flags}_v} \cong S_\sigma \xrightarrow{\iota} \overline{\mathcal{M}}_{0,\mathbf{A}},$$

so

$$
\begin{aligned}
(\overline{\pi_{\mathbf{A}}}|_J)_*([J]) &= \iota_*\left(\bigotimes_{w\in\mathrm{Vertices}(\tau)} (\overline{\pi_w^{\mathrm{source}}})_*([\overline{\mathcal{J}_w}]\right)\\
&= \iota_*\left(\bigotimes_{w\in\mathrm{Vertices}(\tau)}\left(\sum_{q\in Q_w}\beta_q \bigotimes_{v\in(f_{\mathrm{Vertices}})^{-1}(w)}[S_{wv}^q]\right)\right)\\
&= \iota_*\left(\sum_{(q(w))_w\in\prod_{w\in\mathrm{Vertices}(\tau)}Q_w}\left(\prod_{w\in\mathrm{Vertices}(\tau)}\beta_{q(w)}\right)\bigotimes_{w\in\mathrm{Vertices}(\tau)}\bigotimes_{v\in(f_{\mathrm{Vertices}})^{-1}(w)}[S_{wv}^{q(w)}]\right)\\
&= \sum_{(q(w))_w\in\prod_{w\in\mathrm{Vertices}(\tau)}Q_w}\left(\prod_{w\in\mathrm{Vertices}(\tau)}\beta_{q(w)}\right)\left[\iota\left(\prod_{w\in\mathrm{Vertices}(\tau)}\prod_{v\in(f_{\mathrm{Vertices}})^{-1}(w)}S_{wv}^{q(w)}\right)\right].
\end{aligned}
$$

For fixed $(q(w))_w$, the image

$$
\iota\left(\prod_{w\in\mathrm{Vertices}(\tau)}\prod_{v\in(f_{\mathrm{Vertices}})^{-1}(w)}S_{wv}^{q(w)}\right)
$$

under the gluing morphism is a k-dimensional boundary stratum in $\overline{\mathcal{M}}_{0,\mathbf{A}}$. It induces partition of k:

$$
\bigcup_{w\in\mathrm{Vertices}(\tau)}\bigcup_{v\in(f_{\mathrm{Vertices}})^{-1}(w)}\lambda_{wv}^{q(w)}.
$$

As expressed this is clearly a refinement of

$$
\lambda_\tau = (\mathrm{md}(w))_{w\,\in\,\mathrm{Vertices}(\tau)\text{ with }\mathrm{md}(w)\neq 0}.
$$

Thus $(\overline{\pi_{\mathbf{A}}})_*([J])$ is in $\Lambda_{\mathbf{A}}^{\leq\lambda_\tau}$. $\square$

4.2 The k-th dynamical degree of a Hurwitz correspondence on $\mathcal{M}_{0,n}$ is determined by its action on $\Omega_{k,n}$

Theorem 4.1.6 implies that $[\Gamma]_* : H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{B}},\mathbb{R}) \to H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{A}},\mathbb{R})$ can be written, in multiple different ways, as a block lower triangular matrix. In the case of a Hurwitz self-correspondence, one can ask which of these blocks contains the dynamical degree — the dominant eigenvalue. This section addresses this question.

Definition 4.2.1. Set

$$\Lambda_{<(k),n} := \sum_{\lambda \text{ has } \geq 2 \text{ parts}} \Lambda_{\leq \lambda, n}$$

to be the sum of all the proper subspaces of the filtration, and

$$\Omega_{k,n} := \frac{H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{R})}{\Lambda_{<(k),n}}$$

to be the top graded piece of the filtration.

We prove:

Theorem 4.2.7. *Let $\Gamma : \overline{\mathcal{M}}_{0,n} \dashrightarrow \overline{\mathcal{M}}_{0,n}$ be a dominant Hurwitz correspondence. Then the kth dynamical degree of Γ is the absolute value of the dominant eigenvalue of $[\Gamma]_* : \Omega_{k,n} \otimes \mathbb{R} \to \Omega_{k,n} \otimes \mathbb{R}$.*

4.2.1 Hurwitz correspondences on alternate compactifications of $\mathcal{M}_{0,n}$

We recall weight data and moduli spaces of weighted stable curves, defined in Section 2.1.2.

Definition 4.2.2. A weight datum $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n)$ is called ***minimal*** if, for every subset $P \subseteq \{1, \ldots, N\}$, $\sum_{i \in P} \epsilon_i > 1$ if and only if $\sum_{i \notin P} \epsilon_i < 1$.

Lemma 4.2.3. *Let $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n)$ be a minimal weight datum. Then every n-marked stable tree σ has a unique vertex that is $\boldsymbol{\epsilon}$-stable.*

Proof. Every tree has at least one $\boldsymbol{\epsilon}$-stable vertex (Section 2.1.2). Conversely, let σ be an n-marked stable tree. If σ has a unique vertex, we are done. If not, any edge e disconnects σ into two components σ_1 and σ_2. Since $\boldsymbol{\epsilon}$ is minimal, exactly one of $\sum_{\ell_i \text{ on } \sigma_1} \epsilon_i$ and $\sum_{\ell_i \text{ on } \sigma_2} \epsilon_i$ is greater than 1. If $\sum_{\ell_i \text{ on } \sigma_1} \epsilon_i > 1$, then no vertex on σ_2 is $\boldsymbol{\epsilon}$-stable, and if $\sum_{\ell_i \text{ on } \sigma_2} \epsilon_i > 1$, then no vertex on σ_1 is $\boldsymbol{\epsilon}$-stable. Since e was arbitrary, σ has at most one $\boldsymbol{\epsilon}$-stable vertex. □

Proposition 4.2.4. *Let $\boldsymbol{\epsilon}$ be a minimal weight datum, and let $\rho_{\boldsymbol{\epsilon}} : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,n}(\boldsymbol{\epsilon})$ be the reduction morphism. Then $\ker((\rho_{\boldsymbol{\epsilon}})_*) \subseteq H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ contains $\Lambda_{<(k),n}$.*

Proof. Let S_σ be any boundary stratum in $\Lambda_{<(k),n}$. The partition λ_σ has at least two parts, so σ has at least two vertices with positive moduli dimension. By Lemma 4.2.3, at least one of these is not $\boldsymbol{\epsilon}$-stable, and so by Lemma 2.1.18, $S_\sigma \in \ker((\rho_{\boldsymbol{\epsilon}})_*)$. □

There are many minimal weight data, giving rise to non-isomorphic spaces of weighted stable curves, as follows.

Example 4.2.5. $\epsilon_1 = 1 + \frac{n}{10^n}$, and $\epsilon_i = \frac{1}{n-1} - \frac{1}{10^n}$ for $i = 2, \dots, n$. Then $\overline{\mathcal{M}}_{0,n}(\boldsymbol{\epsilon}) \cong \mathbb{P}^{n-3}$

Example 4.2.6. Let $\boldsymbol{\epsilon}^\dagger$ be the following minimal weight datum:

- n **odd.** $\epsilon_i = \frac{2}{n} + \frac{1}{10^n}$ for all i.
- n **even.** $\epsilon_1 = \frac{2}{n} + \frac{1}{10^n}$, and $\epsilon_2 = \cdots = \epsilon_n = \frac{2}{n} - \frac{1}{n \cdot 10^n}$.

The subsets of $\{\epsilon_1, \dots, \epsilon_n\}$ with sum greater than 1 are those with size more than $n/2$, or those with size $n/2$ containing ϵ_1. Set $X_n^\dagger = \overline{\mathcal{M}}_{0,n}(\boldsymbol{\epsilon}^\dagger)$ and $\rho^\dagger : \overline{\mathcal{M}}_{0,n} \to X_n^\dagger$ the reduction morphism. $X_n^\dagger$ has Picard rank n; in particular, it is not isomorphic to $\mathbb{P}^{n-3}$.

There is a simple description of an $\boldsymbol{\epsilon}^\dagger$-stable curve: C an irreducible genus zero curve, and $p_1, \dots, p_n$ marked points, not necessarily distinct, such that

- at least 3 distinct points on C are marked,
- no point on C has greater than $n/2$ marks, and
- if a point on C has exactly $n/2$ marks, then p_1 is not among them.

By Proposition 4.2.4 and Lemma 2.2.8, we obtain:

Theorem 4.2.7. *Let $\Gamma : \overline{\mathcal{M}}_{0,n} \dashrightarrow \overline{\mathcal{M}}_{0,n}$ be a dominant Hurwitz correspondence. Then the kth dynamical degree of Γ is the absolute value of the dominant eigenvalue of $[\Gamma]_* : \Omega_{k,n} \otimes \mathbb{R} \to \Omega_{k,n} \otimes \mathbb{R}$.*

Thus the dynamical degrees of Γ may be computed without any information about its action on the subspace $\Lambda_{<(k),n}$. This suggests that this action is irrelevant to the dynamics of Γ on $\mathcal{M}_{0,n}$. Relatedly, we have

Proposition 4.2.8. *Any effective cycle $\alpha \in \Lambda_{<(k),n}$ is supported on the boundary of $\overline{\mathcal{M}}_{0,n}$.*

Proof. By Proposition 4.2.4, α is in the kernel of $\rho_*^\dagger : \overline{\mathcal{M}}_{0,n} \to X_n^\dagger$, where $X_n^\dagger$ is as in Example 4.2.6. The interior $\mathcal{M}_{0,n}$ maps isomorphically onto an open set on $X_n^\dagger$. Thus the support of α does not intersect $\mathcal{M}_{0,n}$. □

Corollary 4.2.9. *Let $\Gamma : \overline{\mathcal{M}}_{0,n} \dashrightarrow \overline{\mathcal{M}}_{0,n}$ be a Hurwitz correspondence and $S_\sigma \in \Lambda_{<(k),n}$ be a boundary stratum. Then the cycle $[\Gamma]_*([S_\sigma])$ is supported on the boundary of $\overline{\mathcal{M}}_{0,n}$.*

Our interpretation is as follows: the dynamical degrees of Γ are invariants associated to its action on the non-compact space $\mathcal{M}_{0,n}$. However, to define these invariants, we pick a compactification and examine the action on the boundary. In the case of $\overline{\mathcal{M}}_{0,n}$, it appears as though the action of Γ on the boundary components generating $\Lambda_{<(k),n}$ "stays in the boundary" and thus does not carry enough information about the dynamics on $\mathcal{M}_{0,n}$. This information is instead carried by the components generating the quotient $\Omega_{k,n}$.

CHAPTER 5

The Quotient $\Omega_{k,n}$

In Chapter 4, we show that the k-th dynamical degree of a Hurwitz correspondence is the dominant eigenvalue of the induced action on the quotient $\Omega_{k,n}$ of $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ (Definition 4.2.1). In this chapter, we investigate the significance of $\Omega_{k,n}$. We prove:

Theorem 5.1.13 *Let ${X_n}^\dagger$ be the minimal weighted stable curves compactification of $\mathcal{M}_{0,n}$ in Example 4.2.6. Then for $k \geq \frac{dim(\mathcal{M}_{0,n})}{2}$, the kernel of ${\rho^\dagger}_* : H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) \to H_{2k}({X_n}^\dagger, \mathbb{Q})$ equals $\Lambda_{<k,n}$. It follows that for $\mathcal{H}$ any Hurwitz correspondence on $\mathcal{M}_{0,n}$:*

- *The following diagram commutes*

$$\begin{tikzcd} \Omega_{k,n} \arrow[r, "\mathcal{H}_*"] \arrow[d, "\rho_*"', "\cong"] & \Omega_{k,n} \arrow[d, "\cong"', "\rho_*"] \\ H_{2k}({X_n}^\dagger, \mathbb{Q}) \arrow[r, "\mathcal{H}_*"] & H_{2k}({X_n}^\dagger, \mathbb{Q}) \end{tikzcd}$$

- *$\mathcal{H}$ is k-stable on ${X_n}^\dagger$.*

Thus for $k \geq \frac{dim(\mathcal{M}_{0,n})}{2}$, the quotient $\Omega_{k,n}$ may be interpreted as the k-th homology group of an alternate minimal compactification of $\mathcal{M}_{0,n}$. However, we show that no similar interpretation is possible for $k < \frac{dim(\mathcal{M}_{0,n})}{2}$, in fact:

Proposition 5.1.16 *Fix n and k such that $1 \leq k < \frac{dim(\mathcal{M}_{0,n})}{2}$, and $\lambda_1, \ldots, \lambda_s$ partitions of k. Then there is no extremal assignment $\mathcal{Z}$ on $\mathcal{M}_{0,n}$ with $\overline{\mathcal{M}}_{0,n}(\mathcal{Z})$ either projective or smooth such that the kernel of $(\rho_{\mathcal{Z}})_* : H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) \to H_{2k}(\overline{\mathcal{M}}_{0,n}(\mathcal{Z}), \mathbb{Q})$ is exactly $\Lambda_{\leq \lambda_1,n} + \ldots + \Lambda_{\leq \lambda_s,n}$.*

If $\overline{\mathcal{M}}_{0,n}(\mathcal{Z})$ is any minimal modular compactification of $\mathcal{M}_{0,n}$, then the pushforward $(\rho_{\mathcal{Z}})_*$ factors through $\Omega_{k,n}$. This leads us to conjecture:

Conjecture 5.0.1. *$Q_{k,n}$ is the inverse limit of $H_{2k}(X, \mathbb{Q})$, where X runs over minimal modular compactifications of $\mathcal{M}_{0,n}$.*

Finally, we compute the dimension of $\Omega_{k,n}$ by finding an explicit expression for it as a representation of the symmetric group S_n. We find:

Theorem 5.0.2. *[See Theorem 5.2.12 for a more precise statement] $\Omega_{k,n}$ has a S_n-equivariant basis indexed by*

$$\{Q \subseteq \{1, ..., n\} \mid |Q| \geq (k+3), |Q| = (k+3) \mod 2\}.$$

The dimensions of $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ and thus of $H^{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ grow exponentially with n, and there is a recursive but no closed-form formula to compute them [Kee92]. Theorem 5.2.12 gives a closed-form expression for the dimension of $\Omega_{k,n}$ and implies that the dimension of $\Omega_{n-3-k,n}$ (a quotient of $H^{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$) is a degree k polynomial function of n. It is not known whether the S_n action $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ is a permutation action; by Theorem 5.2.12 the S_n action on $\Omega_{k,n}$ is a permutation action.

5.1 For which k and n is $\Omega_{k,n}$ the homology group of an alternate compactification of $\mathcal{M}_{0,n}$?

5.1.1 Notation and terminology

A vertex (v, σ) determines a set partition of $[n]$: i and j are in the same subset if and only if the unique nonrepeating path on σ between legs ℓ_i and ℓ_j does not contain v. The set of parts of this set partition is in canonical bijection with Flags_v; so the number of parts equals $|v|$ and is at least 3. The one-element subsets $\{i\}$ correspond to legs ℓ_i on v.

Definition 5.1.1. We define an equivalence relation $\sim$ on

$$\bigcup_{\sigma \text{ a stable } n\text{-marked tree}} \{\text{vertices of } \sigma\}$$

as follows: $(v_1, \sigma_1) \sim (v_2, \sigma_2)$ if and only if v_1 on σ_1 and v_2 on σ_2 determine the same set partition of $[n]$.

We denote the equivalence class of (v, σ) by $[v]$ and the corresponding set partition by $\mathrm{SP}_{[v]}$. (See Figure 5.1 for an example.) The set of equivalence classes of vertices is in canonical bijection

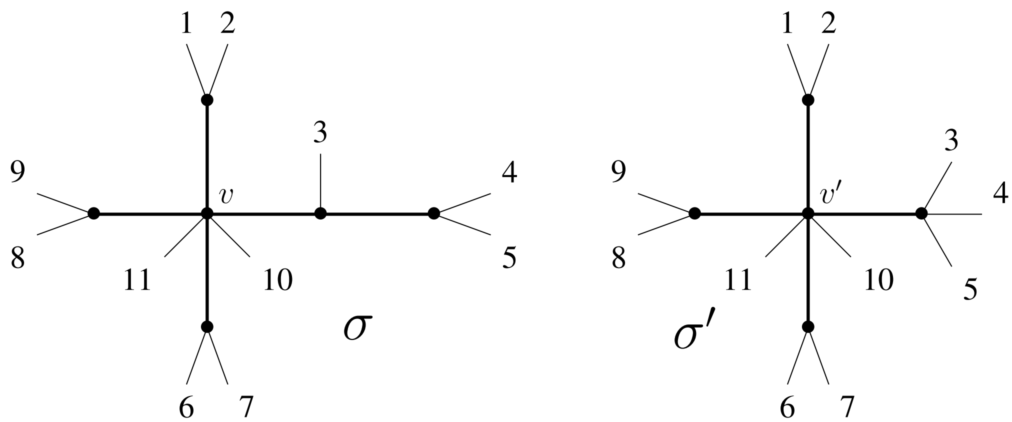


Figure 5.1: The vertices v of σ and v' of σ' induce the same set partition $\mathrm{SP}_v = \mathrm{SP}_{v'}$, namely the set partition $\{\{1,2\},\{3,4,5\},\{6,7\},\{8,9\},\{10\},\{11\}\}$ of $\{1,\dots,11\}$

with the set of set partitions of $[n]$ that have three or more parts. If $(v_1,\sigma_1) \sim (v_2,\sigma_2)$ then $|v_1| = |v_2|$, so we define the moduli dimension of a vertex class $\mathrm{md}([v]) := \mathrm{md}(v)$.

A stable tree σ can have at most one vertex in any equivalence class. If σ contains a vertex in the same class as v, we say $[v]$ is a ***vertex class*** of σ.

The equivalence relation $\sim$ arises naturally when considering specializations of trees and extremal assignments. We have $(v_1,\sigma_1) \sim (v_2,\sigma_2)$ if and only if there exists (v_0,σ_0) with $(v_0,\sigma_0) \rightsquigarrow (v_1,\sigma_1)$ and $(v_0,\sigma_0) \rightsquigarrow (v_2,\sigma_2)$. This tells us that any extremal assignment $\mathcal{Z}$ on $\mathcal{M}_{0,n}$ must respect the equivalence relation $\sim$, i.e. if $(v_1,\sigma_1) \sim (v_2,\sigma_2)$ then $v_1 \in \mathcal{Z}(\sigma_1)$ if and only if $v_2 \in \mathcal{Z}(\sigma_2)$. We can therefore view $\mathcal{Z}$ as a subset of the set of equivalence classes of vertices.

If $(v_1,\sigma_1) \sim (v_2,\sigma_2)$, and there is a specialization $(v_1,\sigma_1) \rightsquigarrow (v'_{1,1} \cup \cdots \cup v'_{1,s}, \sigma'_1)$, then there is also a specialization $(v_2,\sigma_2) \rightsquigarrow (v'_{2,1} \cup \cdots \cup v'_{2,s}, \sigma'_2)$, where $[v'_{1,\alpha}] = [v'_{2,\alpha}]$ for $\alpha = 1,\dots,s$. We therefore say $[v_1] \rightsquigarrow [v'_{1,1}] \cup \cdots \cup [v'_{1,s}]$ is a specialization of vertex classes induced either by the specialization of trees $\sigma_1 \rightsquigarrow \sigma'_1$ or by $\sigma_2 \rightsquigarrow \sigma'_2$. We call each $[v'_{1,\alpha}]$ a ***subvertex*** of $[v_1]$ and write $[v'_{1,\alpha}] \preceq [v_1]$. The relation $\preceq$ is a partial order on the set of equivalence classes of vertices, equivalent to the partial order on set partitions given by 'coarser than'. For an extremal assignment $\mathcal{Z}$ on $\mathcal{M}_{0,n}$, $[v] \in \mathcal{Z}$ if and only if for all subvertices (v',σ') of (v,σ), $[v'] \in \mathcal{Z}$.

Denote by σ^{ir} the one-vertex stable n-marked tree corresponding to irreducible stable curves. Denote by v^{ir} its vertex.

Example 5.1.2. The vertex v^{ir} determines the set partition $\{\{1\},\dots,\{n\}\}$. Every vertex class is a subvertex of $[v^{\mathrm{ir}}]$.

Observation 5.1.3. Given a stable tree σ, distinct vertex classes $[v_1], \ldots, [v_r]$ on σ, and specializations $[v_\alpha] \rightsquigarrow [v'_{\alpha,1}] \cup \cdots \cup [v'_{\alpha,s_\alpha}]$, there is a specialization of trees $\sigma \rightsquigarrow \sigma'$ inducing each of the above specializations, such that for all $[v]$ on σ not one of the $[v_\alpha]$s, $\sigma \rightsquigarrow \sigma'$ induces $[v] \rightsquigarrow [v]$.

Observation 5.1.4. Given a stable tree σ, and distinct vertex classes $[v_1], \ldots, [v_r]$ on σ, there exists a specialization of trees $\sigma \rightsquigarrow \sigma'$ inducing $[v_\alpha] \rightsquigarrow [v_\alpha]$ for $\alpha = 1, \ldots, r$, and so that for all other vertices $[v]$ on σ', $\mathrm{md}([v]) = 0$.

For a vertex $v \in \mathrm{Vertices}(\sigma)$, we denote by Part_v the canonical bijection from Flags_v to the set of parts of $\mathrm{SP}_{[v]}$. If v_1 and v_2 are distinct vertices of σ, then

$$\mathrm{Part}_{v_1}(\delta(v_1 \to v_2)) \supseteq \mathrm{Part}_{v_2}(\delta(v_2 \to v_1))^C$$

and

$$\mathrm{Part}_{v_1}(\delta(v_1 \to v_2))^C \subseteq \mathrm{Part}_{v_2}(\delta(v_2 \to v_1)).$$

Also, $(\mathrm{Part}_{v_1}(\delta(v_1 \to v_2)), \mathrm{Part}_{v_2}(\delta(v_2 \to v_1)))$ is the unique pair

$$(P_1 \text{ a part of } \mathrm{SP}_{[v_1]}, P_2 \text{ a part of } \mathrm{SP}_{[v_2]})$$

satisfying $P_1 \supseteq P_2^C$ and $P_2 \supseteq P_1^C$. Since $\mathrm{Part}_{v_1}(\delta(v_1 \to v_2))$ can be identified using only data from $\mathrm{SP}_{[v_1]}$ and $\mathrm{SP}_{[v_2]}$, we abuse notation, shortening it to $\mathrm{Part}_{[v_1]}([v_2])$.

For $i \in [n]$ we similarly shorten $\mathrm{Part}_v(\delta(v \to i))$ — the part of $\mathrm{SP}_{[v]}$ containing i — to $\mathrm{Part}_{[v]}(i)$.

Observation 5.1.5. Fix $i \in [n]$ and $v_1 \neq v_2$ vertices of σ. Then

1. If the unique path from ℓ_i to v_2 passes through v_1, then $\mathrm{Part}_{[v_1]}(i) \subsetneq \mathrm{Part}_{[v_2]}(i)$.

2. It the unique path from ℓ_i to v_2 does not pass through v_1, then $\mathrm{Part}_{[v_1]}(i) \not\subseteq \mathrm{Part}_{[v_2]}(i)$.

Observation 5.1.6. Let v_1 and v_2 be distinct vertices on σ. Set $P = \mathrm{Part}_{[v_1]}([v_2]) \cap \mathrm{Part}_{[v_2]}([v_1])$. Given any other vertex v' of σ, v' is on the unique non-repeating path between v_1 and v_2 if and only if $\mathrm{SP}_{[v']}$ coarsens (possibly trivially) the set partition

$$\{\mathrm{Part}_{[v_1]}([v_2])^C, \mathrm{Part}_{[v_2]}([v_1])^C, \ldots, \{i\}, \ldots\}_{i \in P}.$$

Example 5.1.7. Let σ, v, v_1, v_2 be as in Figure 5.2. Then

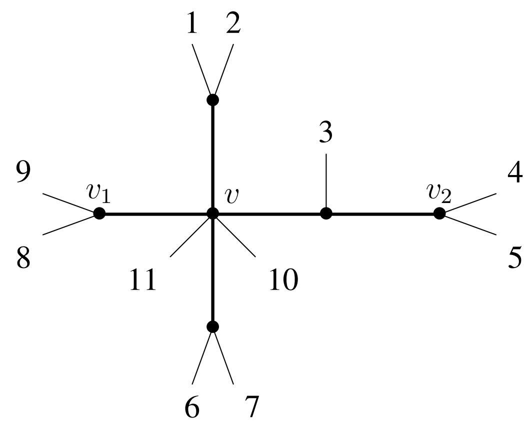


Figure 5.2: The dual tree σ in Example 5.1.7

- $\mathrm{SP}_{v_1} = \{\{8\}, \{9\}, \{1,2,3,4,5,6,7,10,11\}\}$.
- $\mathrm{Part}_{[v_1]}([v_2]) = \{1,2,3,4,5,6,7,10,11\}$.
- $\mathrm{SP}_{v_2} = \{\{4\}, \{5\}, \{1,2,3,6,7,8,9,10,11\}\}$.
- $\mathrm{Part}_{[v_2]}([v_1]) = \{1,2,3,6,7,8,9,10,11\}$.
- Note that the unique path from l_8 to v_2 passes through v_1, and

$$\mathrm{Part}_{[v_1]}(8) = \{8\} \subsetneq \{1,2,3,6,7,8,9,10,11\} = \mathrm{Part}_{[v_2]}(8),$$

  as in Observation 5.1.5,

- Note that the unique path from l_1 to v_2 does not pass through v_1. We have:

$$\mathrm{Part}_{[v_2]}(1) = \{1,2,3,6,7,8,9,10,11\}$$
$$\mathrm{Part}_{[v_1]}(1) = \{1,2,3,4,5,6,7,10,11\}.$$

  As in Observation 5.1.5, $\mathrm{Part}_{[v_2]}(1)$ and $\mathrm{Part}_{[v_1]}(1)$ are not comparable.

- Note that the vertex v is on the unique nonrepeating path between v_1 and v_2, and as in Observation 5.1.6,

$$\mathrm{SP}_v = \{\{1,2\}, \{3,4,5\}, \{6,7\}, \{8,9\}, \{10\}, \{11\}\}$$

nontrivially coarsens the set partition

$$\{\{1\}, \{2\}, \{3\}, \{4,5\}, \{6\}, \{7\}, \{8,9\}, \{10\}, \{11\}\}.$$

5.1.2 Fibers of the reduction morphism $\rho_{\mathcal{Z}}$

Lemma 5.1.8. *Let $(C, p_1, \ldots, p_n)$ and $(C', p'_1, \ldots, p'_n)$ be stable curves with dual trees σ and σ', respectively. Then $\rho_{\mathcal{Z}}(C) = \rho_{\mathcal{Z}}(C')$ if and only if:*

1. *There exists a $\sim$-class-preserving bijection $\mathrm{Vertices}(\sigma) \smallsetminus \mathcal{Z}(\sigma) \leftrightarrow \mathrm{Vertices}(\sigma') \smallsetminus \mathcal{Z}(\sigma')$, and*

2. *For $v \in \mathrm{Vertices}(\sigma) \smallsetminus \mathcal{Z}(\sigma)$, if v' is the corresponding vertex of $\mathrm{Vertices}(\sigma') \smallsetminus \mathcal{Z}(\sigma')$, then the irreducible components C_v and $C_{v'}$ of C and C' respectively determine the same element of $\overline{\mathcal{M}}_{0,\mathrm{Flags}_v}$ (via the canonical isomorphism with $\overline{\mathcal{M}}_{0,\mathrm{Flags}_{v'}}$).*

Proof. Let $C^{\mathcal{Z}}$ be a curve representing $\rho_{\mathcal{Z}}([C])$, and let $\mathrm{ct} : C \to C^{\mathcal{Z}}$ be the contraction morphism as in Definition 2.1.23. $C \smallsetminus \mathcal{Z}(C)$ maps isomorphically onto its image, which is dense in $C^{\mathcal{Z}}$; also special points of C map to special points (marked points or m-fold rational points) of $C^{\mathcal{Z}}$. This means that $C^{\mathcal{Z}}$ has an irreducible component $C_v^{\mathcal{Z}}$ for every $v \in \mathrm{Vertices}(\sigma) \smallsetminus \mathcal{Z}(\sigma)$, and also that $C_v^{\mathcal{Z}}$ with its marked special points is isomorphic to C_v as a Flags_v-marked stable curve. For distinct $v_1, v_2 \in \mathrm{Vertices}(\sigma) \smallsetminus \mathcal{Z}(\sigma)$, $C_{v_1}^{\mathcal{Z}}$ and $C_{v_2}^{\mathcal{Z}}$ meet if and only if every irreducible component of C between C_{v_1} and C_{v_2} is in $\mathcal{Z}(C)$, equivalently if there is no vertex in $\mathrm{Vertices}(\sigma) \smallsetminus \mathcal{Z}(\sigma)$ on the unique path in σ between v_1 and v_2. By Observation 5.1.6, this condition can be checked with just the data of all equivalence classes of vertices in $\mathrm{Vertices}(\sigma) \smallsetminus \mathcal{Z}(\sigma)$. If $C_{v_1}^{\mathcal{Z}}$ and $C_2^{\mathcal{Z}}$ do meet, it must be at the point on $C_{v_1}^{\mathcal{Z}}$ marked by $\delta(v_1 \to v_2)$, and the point on $C_{v_2}^{\mathcal{Z}}$ marked by $\delta(v_2 \to v_1)$. If m components all meet at a point in $C^{\mathcal{Z}}$, it must be at a rational m-fold singularity, whose isomorphism class depends only on m. The marked point $p_i^{\mathcal{Z}} = \rho_{\mathcal{Z}}(p_i)$ is on the irreducible component $C_v^{\mathcal{Z}} = \rho_{\mathcal{Z}}(C_v)$ if and only if all irreducible components of C connecting p_i to C_v are in $\mathcal{Z}(C)$, or equivalently, if all the vertices of σ on the unique non-repeating path from ℓ_i to v are in $\mathcal{Z}(\sigma)$. By Observation 5.1.5 this is equivalent to the non-existence of a vertex $v_0 \in \mathrm{Vertices}(\sigma) \smallsetminus \mathcal{Z}(\sigma)$ such that $\mathrm{Part}_{[v_0]}(i) \subsetneq \mathrm{Part}_{[v]}(i)$. In such a case the point $p_i^{\mathcal{Z}}$ is at the special point of $C_v^{\mathcal{Z}}$ marked by $\delta(v \to i)$. To summarize, a complete description of $(C^{\mathcal{Z}}, p_1^{\mathcal{Z}}, \ldots, p_n^{\mathcal{Z}})$ is possible using just:

1. the equivalence classes (including set partitions) of vertices in $\mathrm{Vertices}(\sigma) \smallsetminus \mathcal{Z}(\sigma)$, and

2. the isomorphism class of C_v as a Flags_v-marked curve, for all $v \in \mathrm{Vertices}(\sigma) \smallsetminus \mathcal{Z}(\sigma)$.

This proves Lemma 5.1.8. $\square$

Corollary 5.1.9. *Let $S_\sigma \subseteq \overline{\mathcal{M}}_{0,n}$ be a boundary stratum. Then*

$$\dim \rho_{\mathcal{Z}}(S_\sigma) = \sum_{v \in \mathrm{Vertices}(\sigma) \smallsetminus \mathcal{Z}(\sigma)} \mathrm{md}(v).$$

5.1.3 The kernels of pushforwards by reduction morphisms

Lemma 5.1.10 is used to prove Lemma 5.1.11, which gives a necessary and sufficient condition for a boundary stratum S_σ in $\overline{\mathcal{M}}_{0,n}$ to be in the kernel of $(\rho_{\mathcal{Z}})_*$ for a given extremal assignment $\mathcal{Z}$. Lemma 5.1.12 states that for a modular compactification that is a space $\overline{\mathcal{M}}_{0,n}(\boldsymbol{\epsilon})$ of weighted stable curves, the kernel of pushforward of the reduction morphism is generated by classes of boundary strata in $\overline{\mathcal{M}}_{0,n}$.

Lemma 5.1.10. *Suppose $\overline{\mathcal{M}}_{0,n}(\mathcal{Z})$ is either projective or smooth. Then no boundary stratum in $\overline{\mathcal{M}}_{0,n}(\mathcal{Z})$ is homologically trivial.*

Proof. (Recall that a boundary stratum in $\overline{\mathcal{M}}_{0,n}(\mathcal{Z})$ is the image of a boundary stratum in $\overline{\mathcal{M}}_{0,n}$.)
Case 1. $\overline{\mathcal{M}}_{0,n}(\mathcal{Z})$ is projective. Then no subvariety is homologically trivial.
Case 2. $\overline{\mathcal{M}}_{0,n}(\mathcal{Z})$ is smooth. Let $S_\sigma \subseteq \overline{\mathcal{M}}_{0,n}$ be any boundary stratum. Consider $\rho_{\mathcal{Z}}(S_\sigma) \subseteq \overline{\mathcal{M}}_{0,n}(\mathcal{Z})$. For a general $[C^{\mathcal{Z}}] \in \rho_{\mathcal{Z}}(S_\sigma)$, the irreducible components $C_v^{\mathcal{Z}}$ of $C^{\mathcal{Z}}$ correspond to vertices $v \in \mathrm{Vertices}(\sigma) \smallsetminus \mathcal{Z}(\sigma)$, and each $C_v^{\mathcal{Z}}$ is a Flags_v-marked smooth curve. For each $v \in \mathrm{Vertices}(\sigma) \smallsetminus \mathcal{Z}(\sigma)$, pick a subset $O_v \subseteq [n]$ such that the map

$$\begin{aligned} O_v &\to \{\text{parts of } \mathrm{SP}_{[v]}\} \\ i &\mapsto \mathrm{Part}_{[v]}(i) \end{aligned}$$

is a bijection. Fix i_v^0, i_v^1, i_v^∞ distinct elements in O_v. For $i \in O_v \smallsetminus \{i_v^0, i_v^1, i_v^\infty\}$, pick $z_v^i \in \mathbb{C} \smallsetminus \{0,1\}$ such that $z_v^i \neq z_v^{i'}$ for $i \neq i'$. Define the divisor Q_v^i to be the closure in $\overline{\mathcal{M}}_{0,n}(\mathcal{Z})$ of the locus of curves on which $(p_{i_v^0}, p_{i_v^1}, p_{i_v^\infty}, p_i)$ have the cross-ratio z_v^i. The number of Q_v^is is exactly $\dim \rho_{\mathcal{Z}}(S_\sigma)$. Set

$$Q := \bigcap_{\substack{v \in \mathrm{Vertices}(\sigma) \smallsetminus \mathcal{Z}(\sigma) \\ i \in O_v \smallsetminus \{i_v^0, i_v^1, i_v^\infty\}}} Q_v^i.$$

The intersection $\rho_{\mathcal{Z}}(S_\sigma) \cap Q$ consists exactly of $[C^{\mathcal{Z}}]$ such that for all $v \in \mathrm{Vertices}(\sigma) \smallsetminus \mathcal{Z}(\sigma)$ and $i \in O_v \smallsetminus \{i_v^0, i_v^1, i_v^\infty\}$, on the component $C_v^{\mathcal{Z}}$, the special points $\delta(v_2 \to i_v^0)$, $\delta(v \to i_v^1)$, $\delta(v \to i_v^\infty)$, and $\delta(v \to i)$ have cross-ratio z_v^i, in particular this intersection consists of one point in $\overline{\mathcal{M}}_{0,n}(\mathcal{Z})$, possibly with multiplicity. The subvarieties $\rho_{\mathcal{Z}}(S_\sigma)$ and Q are of complementary dimension, and their intersection is zero-dimensional. Let $\smile$ denote cup product in the cohomology ring of $\overline{\mathcal{M}}_{0,n}(\mathcal{Z})$. Then:

$$\begin{aligned}[\rho_{\mathcal{Z}}(S_\sigma)] \smile [Q] &= [\rho_{\mathcal{Z}}(S_\sigma) \cap Q] \\ &= m[pt],\end{aligned}$$

where $m \in \mathbb{Z}^{>0}$ is the multiplicity of the intersection. Since $\overline{\mathcal{M}}_{0,n}(\mathcal{Z})$ is a compact complex manifold, $m[pt]$ is not homologically trivial, so neither is $\rho_{\mathcal{Z}}(S_\sigma)$. □

Lemma 5.1.11. *Let $\mathcal{Z}$ be an extremal assignment on $\mathcal{M}_{0,n}$ with $\overline{\mathcal{M}}_{0,n}(\mathcal{Z})$ either projective or smooth, and let S_σ be a boundary stratum in $\overline{\mathcal{M}}_{0,n}$. Then $[S_\sigma]$ is in $\ker(\rho_{\mathcal{Z}})_*$ if and only if there exists $v \in \mathcal{Z}(\sigma)$ with positive moduli dimension.*

Proof. We have

$$\dim S_\sigma = \sum_{v \in \mathrm{Vertices}(\sigma)} \mathrm{md}(v).$$

By corollary 5.1.9,

$$\dim \rho_{\mathcal{Z}}(S_\sigma) = \sum_{v \in \mathrm{Vertices}(\sigma) \smallsetminus \mathcal{Z}(\sigma)} \mathrm{md}(v),$$

so

$$\dim S_\sigma - \dim \rho_{\mathcal{Z}}(S_\sigma) = \sum_{v \in \mathcal{Z}(\sigma)} \mathrm{md}(v).$$

This is greater than zero if and only if there exists $v \in \mathcal{Z}(\sigma)$ with $\mathrm{md}(v) > 0$.

If $\rho_{\mathcal{Z}}(S_\sigma) < \dim(S_\sigma)$, then $(\rho_{\mathcal{Z}})_*([S_\sigma]) = 0$. On the other hand, if $\dim \rho_{\mathcal{Z}}(S_\sigma) = \dim(S_\sigma)$, then $(\rho_{\mathcal{Z}})_*([S_\sigma]) = [\rho_{\mathcal{Z}}(S_\sigma)]$, which by Lemma 5.1.10 is not zero. □

Lemma 5.1.12. *Let $\overline{\mathcal{M}}_{0,n}(\boldsymbol{\epsilon})$ be a moduli space of weighted stable curves, with reduction morphism $\rho_{\boldsymbol{\epsilon}} : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,n}(\boldsymbol{\epsilon})$. Then the kernel of $(\rho_{\boldsymbol{\epsilon}})_* : H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) \to H_{2k}(\overline{\mathcal{M}}_{0,n}(\boldsymbol{\epsilon}), \mathbb{Q})$ is generated by classes of k-dimensional boundary strata.*

Proof. The homology groups of $\overline{\mathcal{M}}_{0,n}$ and $\overline{\mathcal{M}}_{0,n}(\boldsymbol{\epsilon})$ are generated by the classes of boundary strata. Denote by $\mathcal{F}$ the free $\mathbb{Q}$-vector space on the set of k-dimensional boundary strata in $\overline{\mathcal{M}}_{0,n}$, and denote by $\mathcal{F}^{\boldsymbol{\epsilon}}$ the free $\mathbb{Q}$-vector space on the set of k-dimensional boundary strata in $\overline{\mathcal{M}}_{0,n}(\boldsymbol{\epsilon})$. Let $\mathcal{R} \subseteq \mathcal{F}$ and $\mathcal{R}^{\boldsymbol{\epsilon}} \subseteq \mathcal{F}^{\boldsymbol{\epsilon}}$ be the subspaces of relations, i.e. linear combinations of boundary strata that are homologous to zero in $\overline{\mathcal{M}}_{0,n}$ and $\overline{\mathcal{M}}_{0,n}(\boldsymbol{\epsilon})$ respectively. Define a map $\tilde{\rho}_{\boldsymbol{\epsilon}} : \mathcal{F} \to \mathcal{F}^{\boldsymbol{\epsilon}}$ on generators as follows. If $\dim(\rho_{\boldsymbol{\epsilon}}(S_\sigma)) < k$, set $\tilde{\rho}_{\boldsymbol{\epsilon}}(S_\sigma) = 0$, else set $\tilde{\rho}_{\boldsymbol{\epsilon}}(S_\sigma) = \rho_{\boldsymbol{\epsilon}}(S_\sigma)$. We have

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{R} \arrow[r] \arrow[d, "\tilde{\rho}_{\boldsymbol{\epsilon}}"'] & \mathcal{F} \arrow[r] \arrow[d, "\tilde{\rho}_{\boldsymbol{\epsilon}}"'] & H_{2k}(\overline{\mathcal{M}}_{0,n}) \arrow[r] \arrow[d, "(\rho_{\boldsymbol{\epsilon}})_*"'] & 0 \\
0 \arrow[r] & \mathcal{R}^{\boldsymbol{\epsilon}} \arrow[r] & \mathcal{F}^{\boldsymbol{\epsilon}} \arrow[r] & H_{2k}(\overline{\mathcal{M}}_{0,n}(\boldsymbol{\epsilon})) \arrow[r] & 0
\end{tikzcd}$$

Ceyhan [Cey09] gives generators for $\mathcal{R}^{\boldsymbol{\epsilon}}$. Each generator is the image under $\tilde{\rho}_{\boldsymbol{\epsilon}}$ of some generator $\mathcal{R}(\sigma, v, i_1, \ldots, i_4)$ in $\mathcal{R}$ (see Section 2.1.1). Thus $\tilde{\rho}_{\boldsymbol{\epsilon}}(\mathcal{R}) = \mathcal{R}^{\boldsymbol{\epsilon}}$, and the kernel of $\mathcal{F} \to H_{2k}(\overline{\mathcal{M}}_{0,n}(\boldsymbol{\epsilon}))$ is the sum $\mathcal{R} + \ker(\tilde{\rho}_{\boldsymbol{\epsilon}})$.

On the other hand, any two k-dimensional boundary strata in $\overline{\mathcal{M}}_{0,n}$ that have the same k-dimensional image in $\overline{\mathcal{M}}_{0,n}(\boldsymbol{\epsilon})$ are homologous. Thus $\ker(\tilde{\rho}_{\boldsymbol{\epsilon}})/(\ker(\tilde{\rho}_{\boldsymbol{\epsilon}}) \cap \mathcal{R})$ is generated by boundary strata, so $\ker((\rho_{\boldsymbol{\epsilon}})_*)$ is also generated by boundary strata. $\square$

Theorem 5.1.13. *Let ${X_n}^\dagger$ be the minimal weighted stable curves compactification of $\mathcal{M}_{0,n}$ in Example 4.2.6. Then for $k \geq \frac{dim(\mathcal{M}_{0,n})}{2}$, the kernel of ${\rho^\dagger}_* : H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) \to H_{2k}({X_n}^\dagger, \mathbb{Q})$ equals $\Lambda_{<k,n}$. It follows that for $\mathcal{H}$ any Hurwitz correspondence on $\mathcal{M}_{0,n}$:*

- *The following diagram commutes*

$$\begin{tikzcd}
\Omega_{k,n} \arrow[r, "\mathcal{H}_*"] \arrow[d, "\rho_*"', "\cong"] & \Omega_{k,n} \arrow[d, "\rho_*", "\cong"'] \\
H_{2k}({X_n}^\dagger, \mathbb{Q}) \arrow[r, "\mathcal{H}_*"] & H_{2k}({X_n}^\dagger, \mathbb{Q})
\end{tikzcd}$$

- *$\mathcal{H}$ is k-stable on ${X_n}^\dagger$.*

Proof. Fix $k \geq \frac{\overline{\mathcal{M}}_{0,n}}{2}$. By Proposition 4.2.4, we have $\Lambda_{<(k),n} \subseteq \ker(\rho^\dagger_*)$. Suppose S_σ is a k-dimensional boundary stratum not in $\Lambda_{<(k),n}$. Then $\lambda_\sigma = (k)$ and σ has a unique vertex v with positive moduli dimension $\mathrm{md}(v) = k$. So v has valence

$$k + 3 \geq \frac{\dim \mathcal{M}_{0,n}}{2} + 3 = \frac{n+3}{2} > \frac{n}{2} + 1.$$

Thus, for every flag $\delta \in \mathrm{Flags}_v$, the set $\{i|\delta = \delta(v \to i)\}$ has size less than $n/2$, so $\sum_{i|\delta=\delta(v\to i)} \epsilon_i < 1$. We conclude that

$$\sum_{\delta \in \mathrm{Flags}_v} \min\left\{1, \sum_{i|\delta=\delta(v\to i)} \epsilon_i\right\} = \sum_{i=1}^{N} \epsilon_i > 2,$$

so v is $\boldsymbol{\epsilon}$-stable. By Lemma 2.1.18, $S_\sigma \notin \ker(\rho_*^\dagger)$.

Thus a k-dimensional boundary stratum is in $\ker(\rho_*^\dagger)$ exactly if it is in $\Lambda_{<(k),n}$. By Lemma 5.1.12, $\ker(\rho_*^\dagger) = \Lambda_{<(k),n}$. The rest follows by Lemma 2.2.9. $\square$

By Theorem 4.2.7, the dynamical degree of a Hurwitz correspondence $\Gamma : \overline{\mathcal{M}}_{0,n} \dashrightarrow \overline{\mathcal{M}}_{0,n}$ is the absolute value of the dominant eigenvalue of the induced action of Γ on the quotient vector space $\Omega_{k,n}$. By Theorem 5.1.13, for $k \geq \frac{\dim \mathcal{M}_{0,n}}{2}$, this action on $\Omega_{k,n}$ also has an interpretation as the k-stable action of Γ on $H_{2k}(X_n^\dagger)$. We obtain as a corollary:

Corollary 5.1.14. *For $k \geq \frac{\dim \mathcal{M}_{0,n}}{2}$, $[\Gamma]_* : \Omega_{k,n} \to \Omega_{k,n}$ has a nonnegative dominant eigenvalue.*

Proof. By Lemma 2.2.9 2, $[\Gamma]_* : H_{2k}(X_n^\dagger) \to H_{2k}(X_n^\dagger)$ preserves the cone of effective classes, thus as in Remark 3.0.3 has a nonnegative dominant eigenvalue. $\square$

5.1.4 Proof of Proposition 5.1.16

Lemma 5.1.15. *Let $\mathcal{Z}$ be an extremal assignment on $\mathcal{M}_{0,n}$. Then $[v] \in \mathcal{Z}$ if and only if there exists some $r \leq \mathrm{md}[v]$ such that for all $[v'] \preceq [v]$ with moduli dimension r, $[v'] \in \mathcal{Z}$.*

Proof. The forward direction is clear. For the backward direction: Suppose we have $[v''] \preceq [v]$ with moduli dimension zero. We can find $[v'] \preceq [v]$ with moduli dimension r such that $[v''] \preceq [v']$. Since by assumption $[v'] \in \mathcal{Z}$, we must have $[v''] \in \mathcal{Z}$.

On the other hand, there exists a degeneration $[v] \rightsquigarrow [v_1''] \cup \cdots \cup [v_s'']$, where each v_i'' has moduli dimension zero. We have $[v_1''], \ldots, [v_s''] \in \mathcal{Z}$, therefore $[v] \in \mathcal{Z}$. $\square$

Proposition 5.1.16. *Fix n and k such that $1 \leq k < \frac{dim(\mathcal{M}_{0,n})}{2}$, and a non-empty subset $L \subseteq \{$partitions of $k\}$. Then there is no extremal assignment $\mathcal{Z}$ on $\mathcal{M}_{0,n}$ with $\overline{\mathcal{M}}_{0,n}(\mathcal{Z})$ either projective or smooth such that the kernel of $(\rho_\mathcal{Z})_* : H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) \to H_{2k}(\overline{\mathcal{M}}_{0,n}(\mathcal{Z}), \mathbb{Q})$ is exactly $\sum_{\lambda \in L} \Lambda_{\leq\lambda,n}$.*

Proof of Proposition 5.1.16. Assume for contradiction that we have an extremal assignment $\mathcal{Z}$ on $\mathcal{M}_{0,n}$ such that for some fixed $k \leq \frac{n-4}{2}$ and partitions $\lambda_1, \ldots, \lambda_m$ of k, the kernel of $(\rho_\mathcal{Z})_* : H_{2k}(\overline{\mathcal{M}}_{0,n}) \to H_{2k}(\overline{\mathcal{M}}_{0,n}(\mathcal{Z}))$ is exactly $\Lambda_{\leq\lambda_1,n} + \ldots + \Lambda_{\leq\lambda_s,n}$.

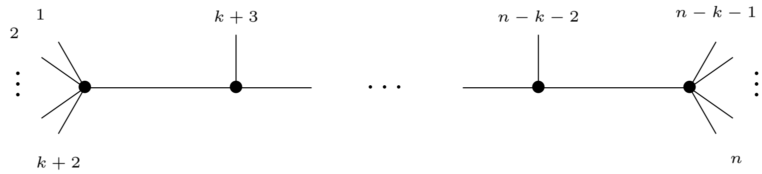


Figure 5.3: The dual tree σ_0

Claim 1. The coarsest partition (k) is not in L.

Proof of Claim 1. Suppose (k) were in L. Then $\ker(\rho_{\mathcal{Z}})_* = \Lambda_n^{\leq(k)}$. Given any $[v]$ with moduli dimension k, we can find $[v]$ as a vertex class of a tree σ corresponding to a k-dimensional boundary stratum S_σ, which must belong to $\mathcal{S}_n^{(k)}$. S_σ is therefore in $\ker(\rho_{\mathcal{Z}})_*$. The vertex v' is the only vertex on σ with positive moduli dimension, and therefore by Lemma 5.1.11 $[v'] \in \mathcal{Z}$. This shows that any k-dimensional subvertex of $[v^{\mathrm{ir}}]$ is in $\mathcal{Z}$. By Lemma 5.1.15, $[v^{\mathrm{ir}}] \in \mathcal{Z}$, but this contradicts the definition of an extremal assignment. This proves the claim.

We conclude that no boundary stratum in $\mathcal{S}_n^{(k)}$ is in $\ker(\rho_{\mathcal{Z}})_*$, and furthermore no $[v]$ with moduli dimension k is in $\mathcal{Z}$.

We now introduce a specific stable tree and vertex classes needed for the rest of the proof. Denote by σ_0 the chain with a leaf v_1 with marked legs $1, \ldots, k+2$, a leaf v_2 with marked legs $n-k-1, \ldots, n$, and a single leg on each intermediate vertex (Figure 5.3). The existence of σ_0 relies on the inequality $k \leq \frac{n-4}{2}$. If $k = \frac{n-4}{2}$, the set of intermediate vertices is empty. The leaves $[v_1]$ and $[v_2]$ have moduli dimension k; we conclude that neither is in $\mathcal{Z}$. (Every other vertex has moduli dimension zero.)

Claim 2. No two-part partition of k is in L.

Proof of Claim 2. Given $r < k$, we produce a stratum S_r inducing the partition $(r, k-r)$ with $S_r \notin \ker(\rho_{\mathcal{Z}})_*$.

By lemma 5.1.15, there exists $[v_1'] \preceq [v_1]$ with $\mathrm{md}[v_1'] = r$ and $[v_1'] \notin \mathcal{Z}$, and $[v_2'] \preceq [v_2]$ with $\mathrm{md}[v_2'] = k-r$ and $[v_2'] \notin \mathcal{Z}$. By Observations 5.1.3 and 5.1.3, there exists a degeneration $\sigma_0 \rightsquigarrow \sigma_r$ inducing $[v_1] \rightsquigarrow [v_1'] \cup \cdots$ and $[v_2] \rightsquigarrow [v_2'] \cup \cdots$, such that every vertex on σ_r other than $[v_1']$ and $[v_2']$ has moduli dimension zero. Then σ_r corresponds to a k-dimensional boundary stratum S_r inducing the partition $(r, k-r)$. Since neither of its two vertices with positive moduli dimension is in $\mathcal{Z}$, by lemma 5.1.11 $S_r \notin \ker(\rho_{\mathcal{Z}})_*$. This proves Claim 2. $\square$

We conclude further that no boundary stratum inducing a two-part partition of k is in $\ker(\rho_{\mathcal{Z}})_*$.

Claim 3. No subvertex of $[v_1]$ or $[v_2]$ with positive moduli dimension is in $\mathcal{Z}$.

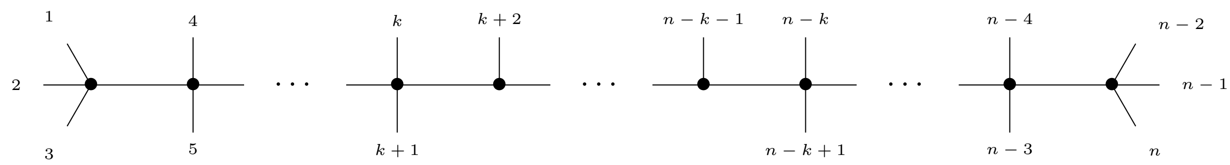


Figure 5.4: The dual tree of S_{ones} when k is even.

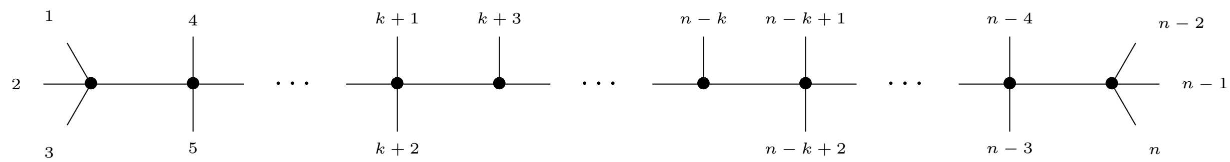


Figure 5.5: The dual tree of S_{ones} when k is odd.

Proof of Claim 3. Let $[v_1'] \preceq [v_1]$ be any subvertex with positive moduli dimension. There exists a subvertex $[v_2'] \preceq [v_2]$ with $\mathrm{md}([v_2']) = k - \mathrm{md}([v_1'])$. Again, by observations 5.1.3 and 5.1.4 there exists a degeneration $\sigma_0 \rightsquigarrow \sigma_0'$ inducing $[v_1] \rightsquigarrow [v_1'] \cup \cdots$ and $[v_2] \rightsquigarrow [v_2'] \cup \cdots$, such that every vertex of σ_0' other than $[v_1']$ and $[v_2']$ has moduli dimension zero. Then σ_0' corresponds to a k-dimensional boundary stratum $S_{\sigma_0'}$ inducing the at most two-part partition $(\mathrm{md}([v_1']), k-\mathrm{md}([v_1']))$. By claims 1 and 2, $S_{\sigma_0'} \notin \ker(\rho_{\mathcal{Z}})_*$, so by Lemma 5.1.11 its positive-moduli-dimensional subvertex $[v_1']$ cannot be in $\mathcal{Z}$. The same argument shows that no positive-moduli-dimensional subvertex of $[v_2]$ is in $\mathcal{Z}$. □

Finally, to prove the proposition, we produce a k-dimensional boundary stratum S_{ones} inducing partition $(1, \ldots, 1)$ that is not in $\ker(\rho_{\mathcal{Z}})_*$. Since $(1, \ldots, 1)$ refines every partition of k, S_{ones} is in every $\Lambda_n^{\leq \lambda}$, therefore by assumption must be in $\ker(\rho_{\mathcal{Z}})_*$. This will give a contradiction.

Case 1: k even. Let S_{ones} be the boundary stratum corresponding to the chain in Figure 5.4. The first $k/2$ vertices are all subvertices of $[v_1]$ with moduli dimension 1, and the last $k/2$ vertices are all subvertices of $[v_2]$ with moduli dimension 1, so by claim 3, none of these is in $\mathcal{Z}$. All vertices in between have moduli dimension zero. By Lemma 5.1.11, $S_{\text{ones}} \notin \ker(\rho_{\mathcal{Z}})_*$.

Case 2: k odd. Let S_{ones} be the boundary stratum corresponding to the chain in Figure 5.5. The first $(k+1)/2$ vertices are all subvertices of $[v_1]$ with moduli dimension 1, and the last $(k-1)/2$ vertices are all subvertices of $[v_2]$ with moduli dimension 1, so by claim 3, none of these is in $\mathcal{Z}$. All vertices in between have moduli dimension zero. By Lemma 5.1.11, $S_{\text{ones}} \notin \ker(\rho_{\mathcal{Z}})_*$.

This proves Proposition 5.1.16. □

5.2 A description of $\Omega_{k,n}$ as a representation of S_n

In this section, we obtain a description of $\Omega_{k,n}$ as a permutation representation of S_n, thus finding a closed-form expression for its dimension.

5.2.1 Set partitions as generators of $\Omega_{k,n}$

We begin by formulating the problem in purely combinatorial terms. For $k \geq 1$, $\Omega_{k,n}$ is a quotient of $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ generated by the fundamental classes of those k-dimensional boundary strata S_σ whose dual trees σ have a unique vertex $v_*(\sigma)$ with positive moduli dimension. For such a stratum σ, the vertex $v_*(\sigma)$ determines a set partition $\Pi_*(\sigma) := \mathrm{SP}_{v_*(\sigma)}$ of $[n]$ with $k+3$ parts, as described in Section 5.1.1. Also, every set partition Π of $[n]$ with $k+3$ parts arises as $\Pi_*(\sigma)$ at least one k-dimensional boundary stratum σ with exactly one vertex with positive moduli dimension.

Lemma 5.2.1. *Let S_σ and $S_{\sigma'}$ be two k-dimensional strata such that σ and σ' each have exactly one vertex with positive moduli dimension and such that $\Pi_*(\sigma) = \Pi_*(\sigma')$. Then $[S_\sigma] = [S_{\sigma'}]$ in $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$.*

Sketch of proof. There is a stable tree σ_* obtained by collapsing all the edges of σ (resp. σ') except those adjacent to $v_*(\sigma)$ (resp. $v_*(\sigma')$). So we have $\sigma_* \rightsquigarrow \sigma$ and $\sigma_* \rightsquigarrow \sigma'$. There is a vertex v_* of σ_* such that $(v_*, \sigma_*) \rightsquigarrow (v_*(\sigma), \sigma)$ and $(v_*, \sigma_*) \rightsquigarrow (v_*(\sigma'), \sigma')$; so $\mathrm{SP}_{v_*} = \Pi_*(\sigma) = \Pi_*(\sigma')$.
Now, both S_σ and $S_{\sigma'}$ are subvarieties of the stratum S_{σ_*}. We have

$$S_{\sigma_*} = \overline{\mathcal{M}}_{0,\mathrm{Flags}_{v_*}} \times \prod_{v \in \mathrm{Vertices}(\sigma_*) \smallsetminus \{v_*\}} \overline{\mathcal{M}}_{0,\mathrm{Flags}_v},$$

and the fundamental classes of both S_σ and $S_{\sigma'}$ as subvarieties of the stratum S_{σ_*} can be obtained as:

$$[\overline{\mathcal{M}}_{0,\mathrm{Flags}_{v_*}}] \otimes \bigotimes_{v \in \mathrm{Vertices}(\sigma_*) \smallsetminus \{v_*\}} [pt].$$

$\square$

As in Lemma 5.1.12, we denote by $\mathcal{F}_{k,n}$ the free $\mathbb{Q}$-vector space on the set of k-dimensional boundary strata in $\overline{\mathcal{M}}_{0,n}$. Denote by $\mathcal{R}_{k,n} \subseteq \mathcal{F}$ the subspace of relations, i.e. linear combinations of boundary strata that are homologous to zero in $\overline{\mathcal{M}}_{0,n}$, and denote by $\widetilde{\Lambda_{k,n}} \subseteq \mathcal{F}_{k,n}$ the subspace generated by boundary strata in $\Lambda_{<(k),n}$. So

$$\Omega_{k,n} = \frac{H_{2k}(\overline{\mathcal{M}}_{0,n},\mathbb{Q})}{\Lambda_{<(k),n}} = \frac{\mathcal{F}_{k,n}}{\mathcal{R}_{k,n}+\widetilde{\Lambda_{k,n}}}.$$

Definition 5.2.2. For $0 \leq r \leq n$, set

$$F_{r,n} = \text{Free } \mathbb{Q}\text{-vector space on } \{\text{set partitions of } [n] \text{ with } r \text{ parts}\}.$$

Definition 5.2.3. For $4 \leq r \leq n-1$, let $R_{r,n}$ be the subspace of $F_{r,n}$ generated by elements of the form:

$$\begin{aligned}&\{P_1 \cup P_2, P_3, P_4, \dots, P_{r+1}\} + \{P_1, P_2, P_3 \cup P_4, \dots, P_{r+1}\}\\ &\quad - \{P_1 \cup P_3, P_2, P_4, \dots, P_{r+1}\} - \{P_1, P_3, P_2 \cup P_4, \dots, P_{r+1}\},\end{aligned}$$

where $\{P_1, P_2, P_3, P_4, \dots, P_{r+1}\}$ is a set partition of $[n]$ with $r+1$ parts.

We have maps $\mathrm{Partition}_{k,n}$ from $\mathcal{F}_{k,n}$ to $F_{k+3,n}$, defined on boundary strata as follows:

$$\mathrm{Partition}_{k,n}(S_\sigma) = \begin{cases}\Pi_*(\sigma) & \sigma \text{ has a a unique vertex with positive moduli dimension}\\ 0 & S_\sigma \in \widetilde{\Lambda_{k,n}}\end{cases}$$

Lemma 5.2.4. *The map* $\mathrm{Partition}_{k,n}$ *descends to an* S_n*-equivariant isomorphism from* $\Omega_{k,n}$ *to* $F_{k+3,n}/R_{k+3,n}$.

Sketch of proof. The proof of this lemma is similar to the proof of Lemma 5.1.12. We recall the generators $R(\sigma, v, i_1, \dots, i_4)$ for $\mathcal{R}_{k,n}$ introduced in Equation 2.1. On can check that, in the language of Sections 2.1.1.4 and 5.1.1, the image $\mathrm{Partition}_{k,n}(R(\sigma, v, i_1, \dots, i_4))$ can be described according to the following cases:

Case 1: v has valance 4. In this case,

$$\mathrm{Partition}_{k,n}(R(\sigma, v, i_1, \dots, i_4)) = 0.$$

Case 2: v has valance bigger than 4, and σ has more than one vertex with positive moduli dimension. In this case, all the terms of $R(\sigma, v, i_1, \dots, i_4)$ are in $\widetilde{\Lambda_{k,n}}$, so again we have

$$\mathrm{Partition}_{k,n}(R(\sigma, v, i_1, \dots, i_4)) = 0.$$

Case 3: v has valance bigger than 4 and is the unique vertex of σ with positive moduli dimension. In this case, we set

$$\mathrm{SP}'_{[v]} = \mathrm{SP}_{[v]} \smallsetminus \{\mathrm{Part}_{[v]}(i_1), \mathrm{Part}_{[v]}(i_2), \mathrm{Part}_{[v]}(i_3), \mathrm{Part}_{[v]}(i_4)\},$$

and then have

$$\begin{aligned}
&\mathrm{Partition}_{k,n}(R(\sigma, v, i_1, \ldots, i_4)) \\
=&\{\mathrm{Part}_{[v]}(i_1) \cup \mathrm{Part}_{[v]}(i_2), \mathrm{Part}_{[v]}(i_3), \mathrm{Part}_{[v]}(i_4)\} \cup \mathrm{SP}'_{[v]} \\
&+ \{\mathrm{Part}_{[v]}(i_1), \mathrm{Part}_{[v]}(i_2), \mathrm{Part}_{[v]}(i_3) \cup \mathrm{Part}_{[v]}(i_4)\} \cup \mathrm{SP}'_{[v]} \\
&- \{\mathrm{Part}_{[v]}(i_1) \cup \mathrm{Part}_{[v]}(i_3), \mathrm{Part}_{[v]}(i_2), \mathrm{Part}_{[v]}(i_4)\} \cup \mathrm{SP}'_{[v]} \\
&- \{\mathrm{Part}_{[v]}(i_1), \mathrm{Part}_{[v]}(i_3), \mathrm{Part}_{[v]}(i_2) \cup \mathrm{Part}_{[v]}(i_4)\} \cup \mathrm{SP}'_{[v]} \\
&\in R_{k+3,n}
\end{aligned}$$

This shows that $\mathrm{Partition}_{k,n}(\mathcal{R}_{k,n}) \subseteq R_{k+3,n}$. Since by definition

$$\mathrm{Partition}_{k,n}(\widetilde{\Lambda_{k,n}}) = 0,$$

there is an induced map

$$\overline{\mathrm{Partition}_{k,n}} : \Omega_{k,n} \to F_{k+3,n}/R_{k+3,n}.$$

Since $\mathrm{Partition}_{k,n}$ is surjective, so is $\overline{\mathrm{Partition}_{k,n}}$. By Lemma 5.2.1, the kernel of $\mathrm{Partition}_{k,n}$ is contained in $(\mathcal{R}_{k,n} + \widetilde{\Lambda_{k,n}})$. It is a straightforward check that in fact $\mathcal{R}_{k,n}$ surjects onto $R_{k+3,n}$, and thus:

$$\begin{aligned}
\ker(\mathcal{F}_{k,n} \to F_{k+3,n}/R_{k+3,n}) &= \ker(\mathrm{Partition}_{k,n}) + \mathcal{R}_{k,n} \\
&= \mathcal{R}_{k,n} + \widetilde{\Lambda_{k,n}}
\end{aligned}$$

The induced map from $\Omega_{k,n}$ onto $F_{k+3,n}/R_{k+3,n}$ is thus injective, and hence an isomorphism. The S_n-equivariance of this identification is immediate. $\square$

In subsequent sections, we use the above identification to work in terms of set partitions of $[n]$ with $(k+3)$ parts rather than in terms of k-dimensional boundary strata.

5.2.2 Pushing forward and pulling back via forgetful maps

The forgetful morphism $\mu : \overline{\mathcal{M}}_{0,n+1} \to \overline{\mathcal{M}}_{0,n}$ induces pushforward and pullback maps of homology groups. The pushforward from $H_{2k}(\overline{\mathcal{M}}_{0,n+1}, \mathbb{Q})$ to $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ descends to a map $\mu_* : \Omega_{k,n+1} \to \Omega_{k,n}$; this in turn lifts to

$$\tilde{\mu}_* : F_{k+3,n+1} \to F_{k+3,n} \tag{5.1}$$

$$\Pi = \{P_1, \ldots, P_{k+3}\} \mapsto \begin{cases} 0 & \{n+1\} \in \Pi \\ \{P_1 \setminus \{n+1\}, \ldots, P_{k+3} \setminus \{n+1\}\} & \text{otherwise} \end{cases} \tag{5.2}$$

Similarly, the pullback from $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ to $H_{2(k+1)}(\overline{\mathcal{M}}_{0,n+1}, \mathbb{Q})$ descends to a map $\mu^* : \Omega_{k,n} \to \Omega_{k+1,n+1}$. This in turn lifts to

$$\tilde{\mu}^* : F_{k+3,n} \to F_{k+4,n+1} \tag{5.3}$$

$$\Pi \mapsto \Pi \cup \{n+1\} \tag{5.4}$$

Since μ has positive relative dimension (equal to one), $\mu_* \circ \mu^* = 0$. In fact, we have:

Lemma 5.2.5. *For $n \geq 4$ and $k \geq 1$,*

$$\Omega_{k,n} \xrightarrow{\mu^*} \Omega_{k+1,n+1} \xrightarrow{\mu_*} \Omega_{k+1,n} \to 0$$

is exact.

Proof. We need to show exactness at $\Omega_{k+1,n+1}$. We use the lifts of μ^* and μ_* to $\tilde{\mu}^* : F_{k,n} \to F_{k+1,n+1}$ and $\tilde{\mu}_* : F_{k+1,n+1} \to F_{k+1,n}$ respectively. We have:

$$\begin{array}{ccccc} R_{k,n} & \xrightarrow{\tilde{\mu}^*} & R_{k+1,n+1} & \xrightarrow{\tilde{\mu}_*} & R_{k+1,n} \\ \downarrow & & \downarrow & & \downarrow \\ F_{k,n} & \xrightarrow{\tilde{\mu}^*} & F_{k+1,n+1} & \xrightarrow{\tilde{\mu}_*} & F_{k+1,n} \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_{k,n} & \xrightarrow{\mu^*} & \Omega_{k+1,n+1} & \xrightarrow{\mu_*} & \Omega_{k+1,n} \end{array}$$

We have:

$$\begin{aligned}\ker \tilde{\mu}_* = \tilde{\mu}^*(F_{k,n}) + \\ + \langle \{P_1 \cup \{n+1\}, P_2, P_3, P_4, \ldots\} - \{P_1, P_2 \cup \{n+1\}, P_3, P_4, \ldots\}\rangle.\end{aligned}$$

Also,

$$\begin{aligned}&\{P_1 \cup \{n+1\}, P_2, P_3, P_4, \ldots\} \\ &\quad -\{P_1, P_2 \cup \{n+1\}, P_3, P_4, \ldots\} = (\{P_1 \cup \{n+1\}, P_2, P_3, P_4, \ldots\} \\ &\qquad + \{P_1, \{n+1\}, P_2 \cup P_3, P_4, \ldots\} \\ &\qquad\quad - \{P_1, P_3, P_2 \cup \{n+1\}, P_4, \ldots\} \\ &\qquad -\{P_1 \cup P_3, P_2, \{n+1\}, P_4, \ldots\}) \\ &\qquad\quad - (\{P_1, \{n+1\}, P_2 \cup P_3, P_4, \ldots\} \\ &\qquad -\{P_1 \cup P_3, P_2, \{n+1\}, P_4, \ldots\}) \\ &\qquad \in R_{k+1,n+1} + \tilde{\mu}^*(F_{k,n}).\end{aligned}$$

Thus $\ker(\tilde{\mu}_*) = \tilde{\mu}^*(F_{k,n}) + R_{k+1,n+1}$. Since $\tilde{\mu}_*(R_{k+1,n+1}) = R_{k+1,n}$, we have

$$\ker(\mu_*) = \mu^*(\Omega_{k,n}).$$

□

In the course of proving Theorem 5.2.12, we will show that in fact μ^* is injective, so

$$0 \to \Omega_{k,n} \xrightarrow{\mu^*} \Omega_{k+1,n+1} \xrightarrow{\mu_*} \Omega_{k+1,n} \to 0$$

is exact.

5.2.3 Towards a dual basis for $\Omega_{k,n}$

Definition 5.2.6. Define a pairing

$$\langle \cdot, \cdot \rangle : \{\text{set partitions of } [n]\} \times \{\text{nonempty subsets of } [n]\} \to \mathbb{Z}.$$

For a set partition Π and subset Q, set

$$\langle \Pi, Q\rangle = \begin{cases} 1 & Q \cap P \neq \emptyset \text{ for all } P \in \Pi \\ 0 & \text{otherwise.} \end{cases}$$

Since $F_{r,n}$ is freely generated by set partitions, we can extend this pairing to obtain linear functionals $\langle \_, Q\rangle$ on $F_{r,n}$. A straightforward check shows that for a generator x of $R_{r,n}$ as in Definition 5.2.3, $\langle x, Q\rangle = 0$ for all $Q \subseteq [n]$. Thus for $k \geq 1$, every $Q \subseteq [n]$ descends to a linear functional $\langle \_, Q\rangle$ on $\Omega_{k,n}$ and thus also defines a functional on $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$.

Remark 5.2.7. Renzo Cavalieri observed that there is a natural and geometric interpretation for the functional $\langle \_, Q\rangle$ as follows. Consider the forgetful map $\mu_Q : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,Q}$. There is a codimension k tautological *kappa* class κ_k on $\overline{\mathcal{M}}_{0,Q}$. The functional $\langle \_, Q\rangle$ is equal to the functional on $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ defined by pairing with $\mu^*(\kappa_k)$.

Definition 5.2.8. For $r \geq 2$, set

$$D_{r,n} = \text{Free } \mathbb{Q}\text{-vector space on } \{Q \subseteq [n] \,|\, |Q| \geq r, |Q| = r \mod 2\}$$

For $r \geq 2$, define maps

$$\begin{aligned} \tilde{\phi}_{r,n} : F_{r,n} &\to D_{r,n} \\ \Pi &\mapsto \sum_{\substack{Q \subseteq [n] \\ |Q| \geq r \\ |Q| = r \mod 2}} \langle \Pi, Q\rangle \cdot Q. \end{aligned}$$

For $r \geq 4$, this descends to maps

$$\phi_{r,n} : \Omega_{r-3,n} = \frac{F_{r,n}}{R_{r,n}} \to D_{r,n}$$

In order to prove Theorem 5.2.12, we will show that $\phi_{r,n}$ is an isomorphism. First, we re-express the pushforward and pullback maps in terms of subsets of n, as follows:

Definition 5.2.9. Define maps

$$\begin{aligned} \alpha : D_{r,n} &\to D_{r,n+1} \\ Q &\mapsto Q \cup \{n+1\} \end{aligned}$$

$$\begin{aligned} \beta : D_{r,n+1} &\to D_{r,n} \\ Q &\mapsto \begin{cases} Q & n+1 \notin Q \\ 0 & n+1 \in Q \end{cases} \end{aligned}$$

The next two lemmas show, respectively, that α is identified with pulling back along the forgetful map μ, and β is identified with pushing forward along the forgetful map.

Lemma 5.2.10.

$$\begin{tikzcd} F_{r,n} \arrow[r, "\tilde{\mu}^*"] \arrow[d, "\tilde{\phi}_{r,n}"] & F_{r+1,n+1} \arrow[d, "\tilde{\phi}_{r+1,n+1}"] \\ D_{r,n} \arrow[r, "\alpha"] & D_{r+1,n+1} \end{tikzcd}$$

commutes, and thus so does

$$\begin{tikzcd} \Omega_{r-3,n} \arrow[r, "\mu^*"] \arrow[d, "\phi_{r,n}"] & \Omega_{r-2,n+1} \arrow[d, "\phi_{r+1,n+1}"] \\ D_{r,n} \arrow[r, "\alpha"] & D_{r+1,n+1} \end{tikzcd}$$

Lemma 5.2.11.

$$\begin{tikzcd} F_{r,n+1} \arrow[r, "\tilde{\mu}_*"] \arrow[d, "\tilde{\phi}_{r,n+1}"] & F_{r,n} \arrow[d, "\tilde{\phi}_{r,n}"] \\ D_{r,n+1} \arrow[r, "\beta"] & D_{r,n} \end{tikzcd}$$

commutes, and thus so does

$$\begin{tikzcd} \Omega_{r-3,n+1} \arrow[r, "\mu_*"] \arrow[d, "\phi_{r,n+1}"] & \Omega_{r-3,n} \arrow[d, "\phi_{r,n}"] \\ D_{r,n+1} \arrow[r, "\beta"] & D_{r,n} \end{tikzcd}$$

5.2.4 An inductive proof of Theorem 5.2.12

The pushforward and pullback maps induced by forgetful maps allow us to to prove, by induction on n:

Theorem 5.2.12. *[Precise version of Theorem 5.0.2] For $n \geq 4$ and $k \geq 1$,*

1. $\phi_{k+3,n} : \Omega_{k,n} \to D_{k+3,n}$ *is an isomorphism.*
2. *the sequence*

$$0 \to \Omega_{k,n} \xrightarrow{\mu^*} \Omega_{k+1,n+1} \xrightarrow{\mu_*} \Omega_{k+1,n} \to 0$$

is exact.

Before we prove Theorem 5.2.12, we give a few more definitions and lemmas that are necessary for the induction to work properly.

Definition 5.2.13. Set:

$$\begin{aligned} E_n &= \text{Free } \mathbb{Q}\text{-vector space on } \{Q \subseteq [n] \,|\, |Q| \text{ even}\} \\ O_n &= \text{Free } \mathbb{Q}\text{-vector space on } \{Q \subseteq [n] \,|\, |Q| \text{ odd}\} \\ F'_{2,n} &= \text{Free } \mathbb{Q}\text{-vector space on } \{(P_1, P_2) | P_1 \cup P_2 = [n], P_1 \cap P_2 = \emptyset, 1 \in P_1\} \end{aligned}$$

Note that $F'_{2,n} \cong \mathbb{Q}\langle([n], \emptyset)\rangle \oplus F_{2,n}$. There are maps $\alpha : E_n \to O_{n+1}$, $\alpha : O_n \to E_{n+1}$, $\beta : E_{n+1} \to E_n$ and $\beta : O_{n+1} \to O_n$, analogous to the maps α and β as in Definition 5.2.9. Define maps:

$$\begin{aligned} \mathrm{odd}_n : F'_{2,n} &\to O_n \\ (P_1, P_2) &\mapsto \sum_{\substack{Q \subseteq P_1 \\ |Q| \text{ odd}}} (-Q) + \sum_{\substack{Q \subseteq P_2 \\ |Q| \text{ odd}}} Q \end{aligned}$$

and

$$\begin{aligned} \mathrm{even}_n : F'_{2,n} &\to E_n \\ (P_1, P_2) &\mapsto \sum_{\substack{Q \subseteq P_1 \\ |Q| \text{ even}}} (-Q) + \sum_{\substack{Q \subseteq P_2 \\ |Q| \text{ even}}} (-Q) \end{aligned}$$

Proposition 5.2.14. *The maps* odd_n *and* even_n *are surjective.*

Proof. We induct on n. Base case: $n = 1$. We have:

$$\begin{aligned} F'_{2,1} &= \mathbb{Q}\langle(\{1\}, \emptyset)\rangle \\ E_1 &= \mathbb{Q}\langle\emptyset\rangle \\ O_1 &= \mathbb{Q}\langle\{1\}\rangle \\ \mathrm{odd}_1(\{1\}, \emptyset) &= -\{1\} \\ \mathrm{even}_1(\{1\}, \emptyset) &= -2\emptyset. \end{aligned}$$

Inductive hypothesis: Suppose the proposition holds up to n.

Inductive step. We have

$$\begin{aligned} \tilde{\mu'}_* : F'_{2,n+1} &\to F'_{2,n} \\ (P_1, P_2) &\mapsto (P_1 \setminus \{n+1\}, P_2 \setminus \{n+1\}). \end{aligned}$$

We also have

$$\begin{aligned} \gamma : F'_{2,n} &\to F'_{2,n+1} \\ (P'_1, P'_2) &\mapsto (P'_1 \cup \{n+1\}, P'_2) - (P'_1, P'_2 \cup \{n+1\}). \end{aligned}$$

We have a diagram

$$\begin{tikzcd} & F'_{2,n} \arrow[r, "\gamma"] \arrow[d, "\mathrm{even}_n"] & F'_{2,n+1} \arrow[r, "\tilde{\mu'}_*"] \arrow[d, "\mathrm{odd}_{n+1}"] & F'_{2,n} \arrow[r] \arrow[d, "\mathrm{odd}_n"] & 0 \\ 0 \arrow[r] & E_n \arrow[r, "\alpha"] & O_{n+1} \arrow[r, "\beta"] & O_n \arrow[r] & 0 \end{tikzcd}$$

with exact rows. We claim that the diagram commutes:

1. Commutativity of the leftmost square. For $(P_1', P_2') \in F_{2,n}'$,

$$
\begin{aligned}
\mathrm{odd}_{n+1}(\gamma(P_1', P_2')) &= \mathrm{odd}_{n+1}(P_1' \cup \{n+1\}, P_2') - \mathrm{odd}_{n+1}(P_1', P_2' \cup \{n+1\}) \\
&= \sum_{\substack{Q \subseteq P_1' \cup \{n+1\} \\ |Q| \text{ odd}}} (-Q) + \sum_{\substack{Q \subseteq P_2' \\ |Q| \text{ odd}}} (Q) \\
&\quad - \left(\sum_{\substack{Q \subseteq P_1' \\ |Q| \text{ odd}}} (-Q) + \sum_{\substack{Q \subseteq P_2' \cup \{n+1\} \\ |Q| \text{ odd}}} (Q) \right) \\
&= \sum_{\substack{Q \subseteq P_1' \cup \{n+1\} \\ |Q| \text{ odd} \\ n+1 \in Q}} (-Q) + \sum_{\substack{Q \subseteq P_2' \cup \{n+1\} \\ |Q| \text{ odd} \\ n+1 \in Q}} (-Q) \\
&= \sum_{\substack{Q' \subseteq P_1' \\ |Q'| \text{ even}}} (-(Q' \cup \{n+1\})) + \sum_{\substack{Q' \subseteq P_2' \\ |Q'| \text{ even}}} (-(Q \cup \{n+1\})) \\
&= \alpha(\mathrm{even}_n(P_1', P_2')).
\end{aligned}
$$

2. Commutativity of the rightmost square. For $(P_1, P_2) \in F_{2,n+1}'$,

$$
\begin{aligned}
\mathrm{odd}_n(\mu_*(P_1, P_2)) &= \mathrm{odd}_n(P_1 \setminus \{n+1\}, P_2 \setminus \{n+1\}) \\
&= \sum_{\substack{Q \subseteq P_1 \setminus \{n+1\} \\ |Q| \text{ odd}}} (-Q) + \sum_{\substack{Q \subseteq P_2 \setminus \{n+1\} \\ |Q| \text{ odd}}} (Q) \\
&= \beta(\mathrm{odd}_{n+1}(P_1, P_2)).
\end{aligned}
$$

This proves the claim. By the inductive hypothesis, even_n and odd_n are surjective, so by the Four Lemma, so is odd_{n+1}.

We likewise have

$$
\begin{array}{ccccccccc}
 & & F_{2,n}' & \xrightarrow{\gamma} & F_{2,n+1}' & \xrightarrow{\tilde{\mu}'_*} & F_{2,n}' & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \mathrm{odd}_n} & & \big\downarrow{\scriptstyle \mathrm{even}_{n+1}} & & \big\downarrow{\scriptstyle \mathrm{even}_n} & & \\
0 & \longrightarrow & O_n & \xrightarrow{\alpha} & E_{n+1} & \xrightarrow{\beta} & E_n & \longrightarrow & 0
\end{array}
$$

with exact rows. We claim that the diagram commutes:

1. Commutativity of the leftmost square. For $(P'_1, P'_2) \in F'_{2,n}$,

$$\begin{aligned}
\mathrm{even}_{n+1}(\gamma(P'_1, P'_2)) &= \mathrm{even}_{n+1}(P'_1 \cup \{n+1\}, P'_2) - \mathrm{even}_{n+1}(P'_1, P'_2 \cup \{n+1\}) \\
&= \sum_{\substack{Q \subseteq P'_1 \cup \{n+1\} \\ |Q| \text{ even}}} (-Q) + \sum_{\substack{Q \subseteq P'_2 \\ |Q| \text{ even}}} (-Q) \\
&\quad + \left(\sum_{\substack{Q \subseteq P'_1 \\ |Q| \text{ even}}} (Q) + \sum_{\substack{Q \subseteq P'_2 \cup \{n+1\} \\ |Q| \text{ even}}} (Q) \right) \\
&= \sum_{\substack{Q \subseteq P'_1 \cup \{n+1\} \\ |Q| \text{ even} \\ n+1 \in Q}} (-Q) + \sum_{\substack{Q \subseteq P'_2 \cup \{n+1\} \\ |Q| \text{ even} \\ n+1 \in Q}} (Q) \\
&= \sum_{\substack{Q' \subseteq P'_1 \\ |Q'| \text{ odd}}} (-(Q' \cup \{n+1\})) + \sum_{\substack{Q' \subseteq P'_2 \\ |Q'| \text{ odd}}} (Q \cup \{n+1\}) \\
&= \alpha(\mathrm{odd}_n(P'_1, P'_2)).
\end{aligned}$$

2. Commutativity of the rightmost square. For $(P_1, P_2) \in F'_{2,n+1}$,

$$\begin{aligned}
\mathrm{even}_n(\mu_*(P_1, P_2)) &= \mathrm{even}_n(P_1 \smallsetminus \{n+1\}, P_2 \smallsetminus \{n+1\}) \\
&= \sum_{\substack{Q \subseteq P_1 \smallsetminus \{n+1\} \\ |Q| \text{ even}}} (-Q) + \sum_{\substack{Q \subseteq P_2 \smallsetminus \{n+1\} \\ |Q| \text{ even}}} (-Q) \\
&= \beta(\mathrm{even}_{n+1}(P_1, P_2)).
\end{aligned}$$

This proves the claim. Again, by the inductive hypothesis odd_n and even_n are surjective, so the Four Lemma shows that even_{n+1} is surjective. □

Lemma 5.2.15. *For all n, the map $\tilde{\phi}_{2,n} : F_{2,n} \to D_{2,n}$ is surjective.*

Proof. We induct on n. Base case: $n = 2$. We have

$$\begin{aligned}
F_{2,n} &= \mathbb{Q}\langle(\{1\}, \{2\})\rangle \\
D_{2,n} &= \mathbb{Q}\langle(\{1, 2\})\rangle.
\end{aligned}$$

Since $\langle(\{1\}, \{2\}), (\{1, 2\})\rangle = 1$, $\tilde{\phi}_{2,2}$ is surjective.

Inductive hypothesis: Suppose the lemma holds up to n. We have a commutative diagram

$$\begin{tikzcd}
 & F'_{2,n} \arrow[r, "\gamma"] \arrow[d, "\mathrm{odd}_n"] & F_{2,n+1} \arrow[r, "\tilde{\mu}_*"] \arrow[d, "\tilde{\phi}_{2,n+1}"] & F_{2,n} \arrow[r] \arrow[d, "\tilde{\phi}_{2,n}"] & 0 \\
0 \arrow[r] & O_n \arrow[r, "\alpha"] & D_{2,n+1} \arrow[r, "\beta"] & D_{2,n} \arrow[r] & 0
\end{tikzcd}$$

Where $\gamma : F'_{2,n} \to F_{2,n+1}$ sends (P'_1, P'_2) to $\{P'_1 \cup \{n+1\}, P'_2\} - \{P'_1, P'_2 \cup \{n+1\}\}$. Here, $\tilde{\mu}_* \circ \gamma = 0$, the bottom row is exact, and $\tilde{\mu}_*$ is surjective.

Claim: the left square commutes. Proof of claim: For $(P'_1, P'_2) \in F'_{2,n}$,

$$\begin{aligned}
\tilde{\phi}_{2,n+1}(\gamma(P'_1, P'_2)) &= \tilde{\phi}_{2,n+1}(\{P'_1 \cup \{n+1\}, P'_2\}) - \tilde{\phi}_{2,n+1}(\{P'_1\}, P'_2 \cup \{n+1\}\}) \\
&= \sum_{\substack{Q \subseteq [n+1] \\ |Q| \geq 2 \\ |Q| \text{ even} \\ Q \cap (P'_1 \cup \{n+1\}) \neq \emptyset \\ Q \cap P'_2 \neq \emptyset}} (Q) - \sum_{\substack{Q \subseteq [n+1] \\ |Q| \geq 2 \\ |Q| \text{ even} \\ Q \cap (P'_1) \neq \emptyset \\ Q \cap P'_2 \cup \{n+1\} \neq \emptyset}} (Q) \\
&= \sum_{\substack{Q \subseteq [n+1] \\ |Q| \geq 2 \\ |Q| \text{ even} \\ n+1 \in Q \\ Q \setminus \{n+1\} \subseteq P'_2}} (Q) - \sum_{\substack{Q \subseteq [n+1] \\ |Q| \geq 2 \\ |Q| \text{ even} \\ n+1 \in Q \\ Q \setminus \{n+1\} \subseteq P'_1}} (Q) \\
&= \sum_{\substack{Q' \subseteq [n] \\ |Q| \text{ odd} \\ Q' \subseteq P'_2}} (Q' \cup \{n+1\}) - \sum_{\substack{Q' \subseteq [n] \\ |Q| \text{ odd} \\ Q' \subseteq P'_1}} (Q' \cup \{n+1\}) \\
&= \alpha(\mathrm{odd}_n(P'_1, P'_2)).
\end{aligned}$$

This proves the claim. Since odd_n is surjective, and by the inductive hypothesis so is $\tilde{\phi}_{2,n}$, by a variant of the Four Lemma, $\tilde{\phi}_{2,n+1}$ is surjective. □

Lemma 5.2.16. $\dim \Omega_{1,n} = \dim D_{4,n}$ *for all* n.

Proof. When $k = 1$, there are no non-trivial integer partitions of k. Thus $\Lambda_{<1,n} = 0$ and $\Omega_{1,n} \cong H_2(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$. By Poincaré duality,

$$\dim H_2(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) = \dim H_{2(n-4)}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$$

Thus

$$\dim \Omega_{1,n} = 2^{n-1} - \binom{n}{2} - 1$$

On the other hand,

$$\begin{aligned}\dim D_{4,n} &= \#\{Q \subseteq [n] \mid |Q| \text{ even}, |Q| \geq 4\} \\ &= 2^{n-1} - \binom{n}{2} - 1.\end{aligned}$$

□

Theorem 5.2.12 [Precise version of Theorem 5.0.2] For $n \geq 4$ and $k \geq 1$,

1. $\phi_{k+3,n} : \Omega_{k,n} \to D_{k+3,n}$ is an isomorphism.
2. the sequence

$$0 \to \Omega_{k,n} \xrightarrow{\mu^*} \Omega_{k+1,n+1} \xrightarrow{\mu_*} \Omega_{k+1,n} \to 0$$

is exact.

Proof of Theorem 5.2.12. First, observe that $\phi_n : \Omega_{n-3,n} \to D_{n,n}$ is always an isomorphism of 1-dimensional vector spaces. We induct on n, with the base case being $n = 4$. We have

$$\begin{aligned}F_{3,4} &= \mathbb{Q}\langle\{\{1,2\},\{3\},\{4\}\},\{\{1,3\},\{2\},\{4\}\},\{\{1,4\},\{2\},\{3\}\},\{\{2,3\},\{1\},\{4\}\}, \\ &\quad \{\{2,4\},\{1\},\{3\}\},\{\{3,4\},\{1\},\{2\}\}\rangle \\ D_{3,4} &= \mathbb{Q}\langle\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\rangle\end{aligned}$$

and

$$\begin{aligned}&\tilde{\phi}_{3,4}(\{\{2,4\},\{1\},\{3\}\} + \{\{3,4\},\{1\},\{2\}\} - \{\{2,3\},\{1\},\{4\}\}) \\ &\quad = \{1,2,3\} + \{1,3,4\} + \{1,2,3\} + \{1,2,4\} - \{1,3,4\} - \{1,2,4\} \\ &\quad = 2\{1,2,3\},\end{aligned}$$

so $\{1,2,3\} \in \mathrm{Im}(\tilde{\phi}_{3,4})$. By symmetry, every other $\{i,j,k\}$ is in $\mathrm{Im}(\tilde{\phi}_{3,4})$, so $\tilde{\phi}_{3,4}$ is surjective.

Inductive hypothesis: For some $n \geq 4$,

1. $\phi_{k+3,n} : \Omega_{k,n} \to D_{k+3,n}$ is an isomorphism for $1 \leq k \leq n-4$, and
2. $\tilde{\phi}_{3,n} : F_{3,n} \to D_{3,n}$ is surjective.

Inductive step: For $1 \leq k \leq (n-4)$, we have

$$\begin{tikzcd}
\ker(\mu^*) \arrow[r] \arrow[d] & \Omega_{k,n} \arrow[r, "\mu^*"] \arrow[d, "\phi_{k+3,n}", "\cong"'] & \Omega_{k+1,n+1} \arrow[r, "\mu_*"] \arrow[d, "\phi_{k+4,n+1}"] & \Omega_{k+1,n} \arrow[r] \arrow[d, "\phi_{k+4,n}", "\cong"'] & 0 \\
0 \arrow[r] & D_{k+3,n} \arrow[r, "\alpha"] & D_{k+4,n+1} \arrow[r, "\beta"] & D_{k+4,n} \arrow[r] & 0
\end{tikzcd}$$

By the Five Lemma, $\phi_{k,n}$ is an isomorphism, so $\ker \mu^* = 0$. Thus

$$0 \to \Omega_{k-1,n-1} \xrightarrow{\mu^*} \Omega_{k,n} \xrightarrow{\mu_*} \Omega_{k,n-1} \to 0$$

is exact. We also have

$$\begin{tikzcd}
 & F_{3,n} \arrow[r] \arrow[d, "\tilde{\phi}_{3,n}"] & \Omega_{1,n+1} \arrow[r, "\mu_*"] \arrow[d, "\phi_{4,n+1}"] & \Omega_{1,n} \arrow[r] \arrow[d, "\phi_4", "\cong"'] & 0 \\
0 \arrow[r] & D_{3,n} \arrow[r] & D_{4,n+1} \arrow[r, "\beta"] & D_{4,n} \arrow[r] & 0
\end{tikzcd}$$

where the bottom row is exact and the top row is a complex. By the inductive hypothesis, $\tilde{\phi}_{3,n}$ is surjective. By the Four Lemma, $\phi_{4,n+1}$ is surjective. But $\dim \Omega_{1,n+1} = \dim D_{4,n+1}$, so $\phi_{4,n+1}$ must be an isomorphism.

Finally, in order to be able to continue the induction, we need to show $\tilde{\phi}_{3,n} : F_{3,n} \to D_{3,n}$ is surjective. For that, consider

$$\begin{tikzcd}
 & F_{2,n} \arrow[r, "\tilde{\mu}^*"] \arrow[d, "\tilde{\phi}_{2,n}"] & F_{3,n+1} \arrow[r, "\tilde{\mu}_*"] \arrow[d, "\tilde{\phi}_{3,n+1}"] & F_{3,n} \arrow[r] \arrow[d, "\tilde{\phi}_{3,n}"] & 0 \\
0 \arrow[r] & D_{2,n} \arrow[r, "\alpha"] & D_{3,n+1} \arrow[r, "\beta"] & D_{3,n} \arrow[r] & 0
\end{tikzcd}$$

By the inductive hypothesis, $\tilde{\phi}_{3,n}$ is surjective. By Lemma 5.2.15, $\tilde{\phi}_{2,n}$ is surjective. Another application of the Four Lemma yields that $\tilde{\phi}_{3,n+1}$ is surjective, as desired. □

5.2.5 The cases of divisors and curves

5.2.5.1 The divisor class group: $k = n - 4$

The homology group $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) = \mathrm{Pic}(\overline{\mathcal{M}}_{0,n}) \otimes \mathbb{Q}$ has dimension $\frac{2^n - n^2 + n - 2}{2}$ and is generated by the classes of boundary divisors. The dual tree of a boundary divisor has exactly two vertices, thus a divisor induces either a one-part or two-part partition of $k = n - 4$. Boundary divisors that induce a two-part partition of k are in $\Lambda_{<k,n}$; these divisors are of the form $\mathrm{Div}_{\overline{\mathcal{M}}_{0,n}}(\{P_1, P_2\})$, where both P_1 and P_2 have cardinality at least 3. The only boundary divisors that are non-zero in

$\Omega_{k,n}$ are those inducing a one-part partition of k; these divisors are of the form

$$\mathrm{Div}_{\overline{\mathcal{M}}_{0,n}}(\{\{i_1, i_2\}, \{i_1, i_2\}^C\}). \tag{5.5}$$

By Theorem 5.2.12, $\Omega_{k,n}$ has a basis consisting of the elements

$$Q_i := [n] \smallsetminus \{i\} \qquad i = 1, \ldots, n \tag{5.6}$$

We have that

$$\mathrm{Div}_{\overline{\mathcal{M}}_{0,n}}(\{\{i_1, i_2\}, \{i_1, i_2\}^C\}) = Q_{i_1} + Q_{i_2} \tag{5.7}$$

By [FG03], we also have

1. The boundary divisors $\{S_\sigma | \lambda_\sigma$ has exactly two parts$\}$ are linearly independent and form a basis for $\Lambda_{<(k),n}$. Thus

$$\dim \Lambda_{<(k),n} = \frac{1}{2}\sum_{j=3}^{n-3}\binom{n}{j} = \frac{1}{2}(2^n - 2 - 2n - n(n-1)).$$

2. The divisor classes $\{\psi_i\}_{i=1,\ldots,n}$ form an alternate basis for $\Omega_{k,n}$.

5.2.5.2 The class of curves: $k = 1$

The homology group $H_2(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ is generated by 1-dimensional boundary strata, and by Poincaré duality has dimension $\frac{2^n - n^2 + n - 2}{2}$. The only partition of 1 is the one-part partition (1), so $\Lambda_{<(k),n} = 0$ and $\Omega_{k,n} = H_2(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ has dimension $\frac{2^n - n^2 + n - 2}{2}$.

CHAPTER 6

Which dynamical degree is the largest?

It is known that the sequence of dynamical degrees of a rational map is log-concave. However, no analogous result holds for multivalued maps since their graphs may be reducible (Section 2.2.3). Here, we prove:

Theorem 6.0.1. *Let $(\mathcal{H}, \pi_1, \pi_2) : \overline{\mathcal{M}}_{0,\mathbf{P}} \rightrightarrows \overline{\mathcal{M}}_{0,\mathbf{P}}$ be a dominant Hurwitz self-correspondence, and let Θ_k be the kth dynamical degree of $\mathcal{H}$. Then*

$$\Theta_0 \geq \Theta_1 \geq \cdots \geq \Theta_{|\mathbf{P}|-3}.$$

Theorem 6.0.1 relies heavily on the moduli space interpretations of $\mathcal{M}_{0,\mathbf{P}}$ and $\mathcal{H}$. In particular, our proof uses the following comparison of tautological line bundles on Hurwitz spaces:

Proposition 6.0.2 (Ionel [Ion01], Lemma 1.17)**.** *Let $\overline{\mathcal{H}} = \overline{\mathcal{H}}(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ be a fully marked space of admissible covers with maps $\overline{\pi_{\mathbf{B}}}$ and $\overline{\pi_{\mathbf{A}}}$ to $\overline{\mathcal{M}}_{0,\mathbf{B}}$ and $\overline{\mathcal{M}}_{0,\mathbf{A}}$ respectively. Suppose we have $a \in \mathbf{A}$ and $b \in \mathbf{B}$ with $F(a) = b$. Then $(\overline{\pi_{\mathbf{B}}})^*(\mathcal{L}_b) = (\overline{\pi_{\mathbf{A}}})^*(\mathcal{L}_a)^{\otimes \mathrm{rm}(a)}$ as line bundles on $\overline{\mathcal{H}}$.*

6.1 Proof of theorem

Theorem 6.0.1 Let $(\Gamma, \pi_1, \pi_2) : \overline{\mathcal{M}}_{0,\mathbf{P}} \rightrightarrows \overline{\mathcal{M}}_{0,\mathbf{P}}$ be a dominant Hurwitz self-correspondence, and let Θ_k be the kth dynamical degree of Γ. Then

$$\Theta_0 \geq \Theta_1 \geq \cdots \geq \Theta_{|\mathbf{P}|-3}.$$

Proof of Theorem 6.0.1. By Lemma 2.4.3, we may assume Γ is a union of connected components of a fully marked Hurwitz space $\mathcal{H} = \mathcal{H}(\mathbf{P}^{\mathrm{full}}, \mathbf{P}, d, F, \mathrm{br}, \mathrm{rm})$ corresponding to a superset $\mathbf{P}^{\mathrm{full}}$ of

$\mathbf{P}$. Let $\overline{\mathcal{H}}$ denote the admissible covers compactification of $\mathcal{H}$, and let $\overline{\Gamma}$ be the closure of Γ in $\overline{\mathcal{H}}$. For $\ell > 0$ set Γ^ℓ to be the ℓth iterate of Γ, that is

$$\Gamma \ {}_{\pi_2}\!\times_{\pi_1} \ \cdots \ {}_{\pi_2}\!\times_{\pi_1} \Gamma \qquad (\ell \text{ times}),$$

Set $\overline{\Gamma^\ell}$ to be its compactification

$$\overline{\Gamma} \ {}_{\overline{\pi_2}}\!\times_{\overline{\pi_1}} \ \cdots \ {}_{\overline{\pi_2}}\!\times_{\overline{\pi_1}} \overline{\Gamma} \qquad (\ell \text{ times}),$$

with $\overline{\pi_1^\ell}$ and $\overline{\pi_2^\ell}$ its two maps to $\overline{\mathcal{M}}_{0,\mathbf{P}}$.

Since $\overline{\pi_1^\ell}$ is a flat map, no irreducible component of $\overline{\Gamma^\ell}$ is supported over the boundary of $\overline{\mathcal{M}}_{0,\mathbf{P}}$. This means that Γ^ℓ is a dense open subset of $\overline{\Gamma^\ell}$. We refer to the complement $\overline{\Gamma^\ell} \smallsetminus \Gamma^\ell$ as the boundary of $\overline{\Gamma^\ell}$. The inverse image under $\overline{\pi_1^\ell}$ of the boundary of $\overline{\mathcal{M}}_{0,\mathbf{P}}$ is exactly the boundary of $\overline{\Gamma^\ell}$. The inverse image under $\overline{\pi_2^\ell}$ of the boundary of $\overline{\mathcal{M}}_{0,\mathbf{P}}$ is contained in the boundary of $\overline{\Gamma^\ell}$.

The compactification $\overline{\Gamma^\ell}$ is singular. However, for Cartier divisors $D_1, \ldots, D_{\dim \overline{\Gamma^\ell}}$, the intersection product $D_1 \cdot \cdots \cdot D_{\dim \overline{\Gamma^\ell}}$ is a well-defined integer as in Section 1.1.C of [Laz04]. For any subscheme Y of dimension k, and Cartier divisors $D_1, \ldots, D_k$, we similarly have the intersection number $D_1 \cdot \cdots \cdot D_k \cdot Y \in \mathbb{Z}$.

Lemma 6.1.1. *For all $p \in \mathbf{P}$ and for all $\ell \geq 0$, there is an equality of Cartier divisors on $\overline{\Gamma^\ell}$ of the form*

$$(\overline{\pi_1^\ell})^*(\psi_{F^\ell(p)}) = R \cdot (\overline{\pi_2^\ell})^*(\psi_p) + E,$$

where R is a positive integer and E is an effective Cartier divisor supported on the boundary of $\overline{\Gamma^\ell}$.

Proof. We induct on ℓ. By convention, $\overline{\Gamma^0}$ is the identity rational correspondence

$$(\overline{\mathcal{M}}_{0,\mathbf{P}},\ \overline{\pi_1^0} = \mathrm{Id},\ \overline{\pi_2^0} = \mathrm{Id}) : \overline{\mathcal{M}}_{0,\mathbf{P}} \dashrightarrow\!\!\!\!\rightarrow \overline{\mathcal{M}}_{0,\mathbf{P}}.$$

For all $p \in \mathbf{P}$, $F^0(p) = p$, so $(\overline{\pi_1^0})^*(\psi_{F^0(p)}) = (\overline{\pi_2^0})^*(\psi_p)$. This gives us the base case $\ell = 0$.

Suppose the Lemma holds for $\ell - 1$. We have

$$\begin{tikzcd}
 & & & \overline{\Gamma^\ell} = \overline{\Gamma} \, {}_{\overline{\pi_2}}\!\times_{\overline{\pi_1^{\ell-1}}} \overline{\Gamma^{\ell-1}} \arrow[dll, "\mathrm{pr}_1"'] \arrow[drr, "\mathrm{pr}_2"] & & & \\
 & \overline{\Gamma} \arrow[ddl, "\overline{\pi_1}"'] \arrow[ddrr, "\overline{\pi_2}"'] \arrow[drr, "\overline{\pi_2^{\mathrm{full}}}"] & & & & \overline{\Gamma^{\ell-1}} \arrow[ddll, "\overline{\pi_1^{\ell-1}}"'] \arrow[ddr, "\overline{\pi_2^{\ell-1}}"] & \\
 & & & \overline{\mathcal{M}}_{0,\mathbf{P}^{\mathrm{full}}} \arrow[d, "\mu"'] & & & \\
\overline{\mathcal{M}}_{0,\mathbf{P}} & & & \overline{\mathcal{M}}_{0,\mathbf{P}} & & & \overline{\mathcal{M}}_{0,\mathbf{P}}
\end{tikzcd}$$

For all $p \in \mathbf{P}$, we have

$$\begin{aligned}
(\overline{\pi_1^\ell})^*(\psi_{F^\ell(p)}) &= \mathrm{pr}_1^*(\overline{\pi_1})^*(\psi_{F^\ell(p)}) \\
&= \mathrm{pr}_1^*\left(\mathrm{rm}(F^{\ell-1}(p)) \cdot (\overline{\pi_2^{\mathrm{full}}})^*(\psi^{\mathbf{P}^{\mathrm{full}}}_{F^{\ell-1}(p)})\right) \qquad \text{(by Proposition 6.0.2).}
\end{aligned}$$

By Lemma 2.1.11,

$$\psi^{\mathbf{P}^{\mathrm{full}}}_{F^{\ell-1}(p)} = \mu^*(\psi_{F^{\ell-1}(p)}) + \sum_{\mathbf{S} \subseteq \mathbf{P}^{\mathrm{full}} \smallsetminus \mathbf{P}} \delta_{\{F^{\ell-1}(p)\} \cup \mathbf{S}}.$$

The inverse image under $\overline{\pi_2^{\mathrm{full}}}$ of the boundary in $\overline{\mathcal{M}}_{0,\mathbf{P}^{\mathrm{full}}}$ is contained in the boundary of $\overline{\Gamma}$ (in fact it is the entire boundary), and the inverse image under pr_1 of the boundary of $\overline{\Gamma}$ is the boundary of $\overline{\Gamma^\ell}$. Thus, the Cartier divisor

$$E_1 := \mathrm{pr}_1^*(\overline{\pi_2^{\mathrm{full}}})^* \sum_{\mathbf{S} \subseteq \mathbf{P}^{\mathrm{full}} \smallsetminus \mathbf{P}} \delta_{\{F^{\ell-1}(p)\} \cup \mathbf{S}}$$

is effective and supported on the boundary of $\overline{\Gamma^\ell}$. Set $R_1 = \mathrm{rm}(F^{\ell-1}(p))$. We continue:

$$\begin{aligned}
(\overline{\pi_1^\ell})^*(\psi_{F^\ell(p)}) &= R_1 \,\mathrm{pr}_1^*(\overline{\pi_2^{\mathrm{full}}})^* \mu^*(\psi_{F^{\ell-1}(p)}) + R_1 E_1 \\
&= R_1 \,\mathrm{pr}_1^*(\overline{\pi_2})^*(\psi_{F^{\ell-1}(p)}) + R_1 E_1 \\
&= R_1 \,\mathrm{pr}_2^*(\overline{\pi_1^{\ell-1}})^*(\psi_{F^{\ell-1}(p)}) + R_1 E_1.
\end{aligned}$$

By the inductive hypothesis, we can rewrite this as

$$R_1 \operatorname{pr}_2^*(R_2(\overline{\pi_2^{\ell-1}})^*(\psi_p) + E_2) + R_1 E_1,$$

where R_2 is a positive integer and E_2 is an effective Cartier divisor supported on the boundary of $\overline{\Gamma^{\ell-1}}$. Since the inverse image under pr_2 of the boundary of $\overline{\Gamma^{\ell-1}}$ is contained in the boundary of $\overline{\Gamma^{\ell}}$, $\operatorname{pr}_2^*(E_2)$ is an effective Cartier divisor supported on the boundary of $\overline{\Gamma^{\ell}}$. Thus we can finally write

$$\begin{aligned}(\overline{\pi_1^{\ell}})^*(\psi_{F^\ell(p)}) &= R_1 R_2 \operatorname{pr}_2^*(\overline{\pi_2^{\ell-1}})^*(\psi_p) + R_1 \operatorname{pr}_2^*(E_2) + R_1 E_1 \\ &= (R_1 R_2)(\overline{\pi_2^{\ell}})^*(\psi_p) + (R_1 \operatorname{pr}_2^*(E_2) + R_1 E_1),\end{aligned}$$

which is as desired. This proves Lemma 6.1.1. □

Now, since $F : \mathbf{P} \to \mathbf{P}$ is a map of finite sets, every point is eventually periodic. Pick $p \in \mathbf{P}$ with $F_0^{\ell}(p) = p$ for some fixed $\ell_0 > 0$. Then for every multiple $m\ell_0$, we have on $\overline{\Gamma^{m\ell_0}}$:

$$(\overline{\pi_1^{m\ell_0}})^*(\psi_p) = R_m(\overline{\pi_2^{m\ell_0}})^*(\psi_p) + E_m,$$

where R_m is a positive integer and E_m is an effective Cartier divisor supported on the boundary of $\overline{\Gamma^{m\ell_0}}$. Thus, if Y is a curve on $\overline{\Gamma^{m\ell_0}}$ with no irreducible component contained in the boundary, we have

$$(\overline{\pi_1^{m\ell_0}})^*(\psi_p) \cdot Y = R_m(\overline{\pi_2^{m\ell_0}})^*(\psi_p) \cdot Y + E_m \cdot Y$$

Since $(\overline{\pi_1^{m\ell_0}})^*(\psi_p)$ and $(\overline{\pi_2^{m\ell_0}})^*(\psi_p)$ are nef on $\overline{\Gamma^{m\ell_0}}$, these intersection numbers are nonnegative and we obtain:

$$(\overline{\pi_1^{m\ell_0}})^*(\psi_p) \cdot Y \geq (\overline{\pi_2^{m\ell_0}})^*(\psi_p) \cdot Y. \tag{6.1}$$

Let $n = |\mathbf{P}|$. Let $\rho : \overline{\mathcal{M}}_{0,\mathbf{P}} \to \mathbb{P}^{n-3}$ be the birational morphism to projective space given by the line bundle $\mathcal{L}_p$. Let $\mathfrak{h}$ be the Cartier divisor class of a hyperplane in $\mathbb{P}^{n-3}$. Then $\rho^*(\mathfrak{h}) = \psi_p$.

The pullback $[\Gamma^r]^*(\mathfrak{h}^k)$ is by definition

$$(\rho \circ \overline{\pi_1^r})_* \circ (\rho \circ \overline{\pi_2^r})^*(\mathfrak{h}^k).$$

So, by the projection formula,

$$([\Gamma^r]^*(\mathfrak{h}^k)) \cdot (\mathfrak{h}^{n-3-k}) = ((\rho \circ \overline{\pi_2^r})^*(\mathfrak{h}^k)) \cdot ((\rho \circ \overline{\pi_1^r})^*(\mathfrak{h}^{n-3-k})).$$

Since dynamical degrees are birational invariants, Θ_k is also the kth dynamical degree of the induced rational correspondence $(\Gamma, \rho \circ \overline{\pi_1}, \rho \circ \overline{\pi_2}) : \mathbb{P}^{n-3} \dashrightarrow \mathbb{P}^{n-3}$. We have

$$\begin{aligned}\Theta_k &= \lim_{r\to\infty}(([\Gamma^r]^*(\mathfrak{h}^k)) \cdot (\mathfrak{h}^{n-3-k}))^{1/r} \\ &= \lim_{r\to\infty}(((\rho \circ \overline{\pi_2^r})^*(\mathfrak{h}^k)) \cdot ((\rho \circ \overline{\pi_1^r})^*(\mathfrak{h}^{n-3-k})))^{1/r} \\ &= \lim_{r\to\infty}(((\overline{\pi_2^r})^*(\psi_p^k)) \cdot ((\overline{\pi_1^r})^*(\psi_p^{n-3-k})))^{1/r}.\end{aligned}$$

Since this sequence converges, we can find its limit using any subsequence, and

$$\begin{aligned}\Theta_k &= \lim_{m\to\infty}(((\overline{\pi_2^{m\ell_0}})^*(\psi_p^k)) \cdot ((\overline{\pi_1^{m\ell_0}})^*(\psi_p^{n-3-k})))^{1/m\ell_0} \\ &= \lim_{m\to\infty}(((\overline{\pi_2^{m\ell_0}})^*(\psi_p))^k \cdot ((\overline{\pi_1^{m\ell_0}})^*(\psi_p))^{n-3-k})^{1/m\ell_0}\end{aligned}$$

Lemma 6.1.2. *Fix $m > 0$. The intersection numbers*

$$\alpha_k := ((\overline{\pi_2^{m\ell_0}})^*(\psi_p))^k \cdot ((\overline{\pi_1^{m\ell_0}})^*(\psi_p))^{n-3-k}$$

on $\overline{\Gamma^{m\ell_0}}$ are a nonincreasing function of k.

Proof of Lemma 6.1.2. We first show that for every irreducible component $\overline{\mathcal{J}}$ of $\overline{\Gamma^{m\ell_0}}$, the intersection numbers

$$\alpha_{\overline{\mathcal{J}},k} := ((\overline{\pi_2^{m\ell_0}})^*(\psi_p))|_{\overline{\mathcal{J}}}^k \cdot ((\overline{\pi_1^{m\ell_0}})^*(\psi_p))|_{\overline{\mathcal{J}}}^{n-3-k}$$

are a nonincreasing function of k.

Fix $\overline{\mathcal{J}}$ such an irreducible component. Since $(\overline{\pi_1^{m\ell_0}})^*(\psi_p)$ and $(\overline{\pi_2^{m\ell_0}})^*(\psi_p)$ are pullbacks of the ample hyperplane class $\mathfrak{h}$, they are nef on $\overline{\Gamma^{m\ell_0}}$ and $\overline{\mathcal{J}}$. So, $\alpha_{\overline{\mathcal{J}},k}$ is a log-concave function of k ([Laz04], Example 1.6.4). It therefore suffices to show that

$$\alpha_{\overline{\mathcal{J}},0} = ((\overline{\pi_1^{m\ell_0}})^*(\psi_p))|_{\overline{\mathcal{J}}}^{n-3} \geq ((\overline{\pi_2^{m\ell_0}})^*(\psi_p))|_{\overline{\mathcal{J}}}^{1} \cdot ((\overline{\pi_1^{m\ell_0}})^*(\psi_p))|_{\overline{\mathcal{J}}}^{n-4} = \alpha_{\overline{\mathcal{J}},1}.$$

Note that $\psi_p^{n-4} = \rho^*(\mathfrak{h}^{n-4})$. The class $\mathfrak{h}^{n-4}$ on $\mathbb{P}^{n-3}$ may be represented by a line L that does not intersect the codimension-two exceptional locus of ρ. Then $\rho^{-1}(L)$ is an irreducible curve in $\overline{\mathcal{M}}_{0,\mathbf{P}}$ not contained in the boundary and $(\overline{\pi_1^{m\ell_0}})^{-1}(\rho^{-1}(L))|_{\overline{\mathcal{J}}}$ is a curve Y none of whose irreducible components lies in the boundary of $\overline{\mathcal{J}}$. Since $\overline{\pi_1^{m\ell_0}}$ is a flat map, and a covering map away from the boundary,

$$((\overline{\pi_1^{m\ell_0}})^*(\psi_p^{n-4}))|_{\overline{\mathcal{J}}} = [Y].$$

Thus,

$$\begin{aligned}
\alpha_{\overline{\mathcal{J}},0} = ((\overline{\pi_1^{m\ell_0}})^*(\psi_p))|_{\overline{\mathcal{J}}}^{n-3} &= ((\overline{\pi_1^{m\ell_0}})^*(\psi_p))|_{\overline{\mathcal{J}}} \cdot ((\overline{\pi_1^{m\ell_0}})^*(\psi_p^{n-4}))|_{\overline{\mathcal{J}}} \\
&= ((\overline{\pi_1^{m\ell_0}})^*(\psi_p))|_{\overline{\mathcal{J}}} \cdot Y \\
&\geq ((\overline{\pi_2^{m\ell_0}})^*(\psi_p))|_{\overline{\mathcal{J}}} \cdot Y \qquad \text{(by (6.1))} \\
&= ((\overline{\pi_2^{m\ell_0}})^*(\psi_p))|_{\overline{\mathcal{J}}} \cdot ((\overline{\pi_1^{m\ell_0}})^*(\psi_p^{n-4}))|_{\overline{\mathcal{J}}} \\
&= ((\overline{\pi_2^{m\ell_0}})^*(\psi_p))|_{\overline{\mathcal{J}}} \cdot ((\overline{\pi_1^{m\ell_0}})^*(\psi_p))|_{\overline{\mathcal{J}}}^{n-4} = \alpha_{\overline{\mathcal{J}},1}.
\end{aligned}$$

We conclude that for fixed $\overline{\mathcal{J}}$, $\alpha_{\overline{\mathcal{J}},k}$ is a nonincreasing function of k.

Thus $\alpha_k = \sum_{\overline{\mathcal{J}}} \alpha_{\overline{\mathcal{J}},k}$ is a nonincreasing function of k. □

We now complete the proof of Theorem 6.0.1. For all m,

$$(((\overline{\pi_2^{m\ell_0}})^*(\psi_p))^k \cdot ((\overline{\pi_1^{m\ell_0}})^*(\psi_p))^{n-3-k})^{1/m\ell_0}$$

is a nonincreasing function of k, so

$$\Theta_k = \lim_{m\to\infty}(((\overline{\pi_2^{m\ell_0}})^*(\psi_p))^k \cdot ((\overline{\pi_1^{m\ell_0}})^*(\psi_p))^{n-3-k})^{1/m\ell_0}$$

is a nonincreasing function of k. □

CHAPTER 7

Computing dynamical degrees of Hurwitz correspondences

7.1 Computing pushforward maps using the combinatorics of admissible covers

Since Hurwitz correspondences are algebraically stable on $\overline{\mathcal{M}}_{0,n}$, their dynamical degrees are dominant eigenvalues of pushforward actions on the homology groups of $\overline{\mathcal{M}}_{0,n}$. In this section we describe techniques for computing these pushforward maps. We treat the non-dynamical case since it is more general. As in Section 2.4.1, we may restrict our attention to fully marked Hurwitz spaces. Let $\mathcal{H} = \mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ be a fully marked Hurwitz space with admissible covers compactification $\overline{\mathcal{H}}$. (Here $\mathbf{A}$ is the full pre-image of $\mathbf{B}$.) Let $\mathbf{A}'$ be any subset of $\mathbf{A}$ with cardinality at least 3, and denote by μ the forgetful map from $\overline{\mathcal{M}}_{0,\mathbf{A}}$ to $\overline{\mathcal{M}}_{0,\mathbf{A}'}$. The Hurwitz space $\mathcal{H}$ induces a correspondence $(\mathcal{H}, \pi_{\mathbf{B}}, \pi'_{\mathbf{A}}) : \overline{\mathcal{M}}_{0,\mathbf{B}} \rightrightarrows \overline{\mathcal{M}}_{0,\mathbf{A}'}$. We would like to be able to compute the pushforwards

$$[\mathcal{H}]_* : H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{B}}, \mathbb{Z}) \to H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{A}'}, \mathbb{Z}).$$

As in Section 2.4.5, we identify $[\mathcal{H}]_*$ with the map $\overline{\pi_{\mathbf{A}'*}} \circ \overline{\pi_{\mathbf{B}}}^* : A_k(\overline{\mathcal{M}}_{0,\mathbf{B}}) \to A_k(\overline{\mathcal{M}}_{0,\mathbf{A}'})$.

Recall that the Chow groups of $\overline{\mathcal{M}}_{0,n}$ are generated by boundary strata, and any boundary stratum is a transverse intersection of boundary divisors. Thus $[\mathcal{H}]_*$ is determined by its values on all boundary strata, or even just a basis of boundary strata. Let T_τ be a k-dimensional boundary stratum in $\overline{\mathcal{M}}_{0,\mathbf{B}}$. By Ponicaré duality its image $[\mathcal{H}]_*(T_\tau)$ is determined by the intersection numbers

$$[\mathcal{H}]_*(T_\tau) \cdot S_\sigma, \tag{7.1}$$

where S_σ ranges over all codimension-k boundary strata in $\overline{\mathcal{M}}_{0,\mathbf{A}'}$, or even just over elements of a basis of codimension-k boundary strata. By the projection formula,

$$[\mathcal{H}]_*(T_\tau) \cdot S_\sigma = \overline{\pi_{\mathbf{A}'}}^*(S_\sigma) \cdot \overline{\pi_{\mathbf{B}}}^*(T_\tau), \tag{7.2}$$

so it suffices to compute the latter intersection numbers.

Since $\overline{\pi_{\mathbf{B}}}$ is finite and flat, $\overline{\pi_{\mathbf{B}}}^*(T_\tau)$ is, up to some positive integer multiplicity, the set-theoretic inverse image $\overline{\pi_{\mathbf{B}}}^{-1}(T_\tau)$. This inverse image, in turn, is the union of k -dimensional boundary strata in $\overline{\mathcal{H}}$, each of which generically parametrizes admissible covers whose target curve has dual tree τ. On the other hand, $\overline{\pi_{\mathbf{A}}}$ and $\overline{\pi_{\mathbf{A}'}}$ might be arbitrarily badly behaved. For example, $\overline{\pi_{\mathbf{A}'}}^{-1}(S_\sigma)$ may not have codimension k in $\overline{\mathcal{H}}$ and even if it does, $\overline{\pi_{\mathbf{B}}}^{-1}(T_\tau) \cap \overline{\pi_{\mathbf{A}'}}^{-1}(S_\sigma)$ may not have the expected dimension. Boundary strata in $\overline{\mathcal{M}}_{0,n}$ can be uniquely written as transverse intersections of boundary divisors, that is:

$$S_\sigma = \bigcap_{e \in \mathrm{Edges}(\sigma)} \mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{A}'}}(e),$$

and:

$$T_\tau = \bigcap_{\underline{e} \in \mathrm{Edges}(\tau)} \mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{B}}}(\underline{e}).$$

We use this to restate our problem again: we would like to be able to compute intersection numbers of the form

$$\overline{\pi_{\mathbf{A}'}}^*(X'_1) \cdots \overline{\pi_{\mathbf{A}'}}^*(X'_k) \cdot \overline{\pi_{\mathbf{B}}}^*(Y_{k+1}) \cdots \overline{\pi_{\mathbf{B}}}^*(Y_{|\mathbf{B}|-3}), \tag{7.3}$$

where the X'_is are boundary divisors in $\overline{\mathcal{M}}_{0,\mathbf{A}'}$ and the Y_js are boundary divisors in $\overline{\mathcal{M}}_{0,\mathbf{B}}$.

Let $\mathbf{A}' = \mathbf{A}'_1 \sqcup \mathbf{A}'_2$ be a set partition of $\mathbf{A}'$ into parts of cardinality at least 2, and let X' be the boundary divisor $\mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{A}'}}(\{\mathbf{A}'_1, \mathbf{A}'_2\})$. Forgetful maps are flat, and so the pullback $\mu^*(X')$ is the same as the inverse image, which in turn is a union of boundary divisors in $\overline{\mathcal{M}}_{0,\mathbf{A}}$. Explicitly, we have:

$$\mu^*(X') = \sum_{\substack{\mathbf{A} = \mathbf{A}_1 \sqcup \mathbf{A}_2 \\ \mathbf{A}_1 \cap \mathbf{A}' = \mathbf{A}'_1 \\ \mathbf{A}_2 \cap \mathbf{A}' = \mathbf{A}'_2}} \mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{A}}}(\{\mathbf{A}_1, \mathbf{A}_2\}).$$

Since $\overline{\pi_{\mathbf{A}'}} = \mu \circ \overline{\pi_{\mathbf{A}}}$, we can use the equation above to restate our problem one last time in order to work with the full preimage $\mathbf{A}$ of $\mathbf{B}$: we want to compute intersection numbers of the form

$$\overline{\pi_{\mathbf{A}}}^*(X_1) \cdots \overline{\pi_{\mathbf{A}}}^*(X_k) \cdot \overline{\pi_{\mathbf{B}}}^*(Y_{k+1}) \cdots \overline{\pi_{\mathbf{B}}}^*(Y_{|\mathbf{B}|-3}), \tag{7.4}$$

where the X_is are boundary divisors in $\overline{\mathcal{M}}_{0,\mathbf{A}}$ and the Y_js are boundary divisors in $\overline{\mathcal{M}}_{0,\mathbf{B}}$.

Once again, because $\overline{\pi_{\mathbf{B}}}$ is flat, different nodes on the target curve of an admissible cover (different edges on the dual tree of the target curve) impose independent codimension-1 condition on $\overline{\mathcal{H}}$. A dense open subset $\mathcal{H} \subset \overline{\mathcal{H}}$ parametrizes admissible covers with smooth source and target curves, so $\overline{\pi_{\mathbf{A}}}$ maps $\mathcal{H}$ to $\mathcal{M}_{0,\mathbf{A}}$. Thus, the inverse image of a boundary divisor in $\overline{\mathcal{M}}_{0,\mathbf{A}}$ in $\overline{\mathcal{H}}$ is a divisor, in fact a union of boundary divisors. The inverse image of $\mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{A}}}(\{\mathbf{A}_1, \mathbf{A}_2\})$ is the set of admissible covers whose source curve contains a node separating all the marks in $\mathbf{A}_1$ from those in $\mathbf{A}_2$. If the source curve of an admissible cover is nodal then the target must be nodal as well, however, different nodes on the source curve might map to the same node on the target curve. Suppose there exists f an admissible cover with combinatorial type

$$\gamma = (\sigma, \tau, d_{\mathrm{Vertices}}, f_{\mathrm{Vertices}}, F_{\mathrm{Edges}}, (\mathrm{br}_v)_{v \in \mathrm{Vertices}(\sigma)}, \mathrm{rm}_{\mathrm{Edges}}),$$

with $e_1, \ldots, e_r$ edges of σ that all map via F_{Edges} to the same edge $\underline{e}$ of τ. Then while the edges $e_1, \ldots, e_r$ do impose independent codimension-1 conditions in $\overline{\mathcal{M}}_{0,\mathbf{A}}$, they do not impose independent conditions in $\overline{\mathcal{H}}$. In particular, the intersection of the r divisors

$$\overline{\pi_{\mathbf{A}}}^{-1}(\mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{A}}}(e_1)), \ldots, \overline{\pi_{\mathbf{A}}}^{-1}(\mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{A}}}(e_r))$$

contains a codimension-1 subscheme, namely the boundary divisor $G_\gamma(\underline{e})$. This observation yields the following lemma.

Lemma 7.1.1. *Let $X_1, \ldots, X_k$ be boundary divisors in $\mathcal{M}_{0,\mathbf{A}}$ and $Y_1, \ldots, Y_\ell$ be boundary divisors in $\mathcal{M}_{0,\mathbf{B}}$. Then the intersection*

$$\bigcap_{i=1}^{k} \overline{\pi_{\mathbf{A}}}^{-1}(X_i) \cap \bigcap_{j=1}^{\ell} \overline{\pi_{\mathbf{B}}}^{-1}(Y_j) \tag{7.5}$$

in $\overline{\mathcal{H}}$ has the expected codimension $k + \ell$ if and only if neither of the following hold:

1. *There exists $[f_1 : C_1 \to D_1] \in \overline{\mathcal{H}}$ with distinct nodes $\eta_1, \eta_1' \in C_1$ such that $X_{i_1} = \mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{A}}}(\eta)$ and $X_{i_1'} = \mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{A}}}(\eta')$ for some i_1, i_1', and $f_1(\eta_1) = f_1(\eta_1')$*
2. *There exists $[f_2 : C_2 \to D_2] \in \overline{\mathcal{H}}$ with a node $\eta_2 \in C_2$ such that $X_{i_2} = \mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{A}}}(\eta_2)$ and $Y_{j_2} = \mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{B}}}(f_2(\eta_2))$ for some i_2, j_2.*

Proof. The intersection (7.5) consists of those admissible covers $f : C \to D$ such that C has nodes $\eta_1, \ldots \eta_k$ with $X_i = \mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{A}}}(\eta_i)$, and D has nodes $\theta_1, \ldots, \theta_\ell$ with $Y_j = \mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{B}}}(\theta_j)$. It has the expected dimension if and only if these nodes impose independent codimension-1 conditions on admissible covers. If 1 holds then as observed above, $\overline{\pi_{\mathbf{A}}}^{-1}(X_{i_1}) \cap \overline{\pi_{\mathbf{A}}}^{-1}(X_{i'_1})$ contains a codimension-1 subscheme. On the other hand, if 2 holds then as observed above, $\overline{\pi_{\mathbf{A}}}^{-1}(X_{i_2}) \cap \overline{\pi_{\mathbf{B}}}^{-1}(Y_{i_2})$ contains a codimension-1 subscheme. If neither holds then all the above divisors impose distinct nodes on the target curves of admissible covers, giving $k+\ell$ independent conditions. $\square$

Lemma 7.1.2. *Let $\mathbf{A} = \mathbf{A}_1 \sqcup \mathbf{A}_2$ be a set partition of $\mathbf{A}$ into two parts each with cardinality at least 2, and suppose $\overline{\pi_{\mathbf{A}}}^{-1}(\mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{A}}}(\{\mathbf{A}_1, \mathbf{A}_2\}))$ is non-empty. Then $\exists$ a partition $\mathbf{B} = \mathbf{B}_1 \sqcup \mathbf{B}_2$ such that $\forall [f : C \to D] \in \overline{\mathcal{H}}$ with a node $\eta \in C$ inducing the set partition $\{\mathbf{A}_1, \mathbf{A}_2\}$, the node $f(\eta) \in D$ induces the set partition $\{\mathbf{B}_1, \mathbf{B}_2\}$. Thus $\overline{\pi_{\mathbf{A}}}^{-1}(\mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{A}}}(\{\mathbf{A}_1, \mathbf{A}_2\})) \subseteq \overline{\pi_{\mathbf{B}}}^{-1}(\mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{B}}}(\{\mathbf{B}_1, \mathbf{B}_2\}))$.*

Proof. Let $[f : C \to D] \in \overline{\mathcal{H}}$ be any admissible cover with a node $\eta \in C$ inducing the set partition $\{\mathbf{A}_1, \mathbf{A}_2\}$. The node η disconnects C into two (not necessarily irreducible) components: C_1 with marked points $\mathbf{A}_1$ and C_2 with marked points $\mathbf{A}_2$. Similarly $f(\eta)$ disconnects D into D_1 and D_2.

We claim that $f|_{C_i}$ has a well-defined degree $d_{i,j}$ over each D_j. To see this, disconnect C_i further at all other nodes that lie over $f(\eta)$. f maps each of these components $C_{i,\alpha}$ to either D_1 or D_2. Since $\overline{\pi_{\mathbf{B}}}$ is flat, we may smooth all the nodes on D except $f(\eta)$. This implies that $f|C_{i,\alpha}$ has a well-defined degree. The claim follows since each C_i is a union of the $C_{i,\alpha}$s.

We next claim that $d_{i,1} \neq d_{i,2}$. To see this, consider the map $f|_{C_i} : C_i \to D$. The points lying over $f(\eta)$ are η and balanced nodes. The irreducible component containing η maps either to D_1 or to D_2, and $|d_{1,1} - d_{1,2}|$ is exactly the local degree of $f|_{C_i}$ at η.

Now, for any point $b \in \mathbf{B}$, let

$$\deg_1(b) = \sum_{a \in \mathbf{A}_1, F(a) = b} \mathrm{rm}(a)$$
$$\deg_2(b) = \sum_{a \in \mathbf{A}_2, F(a) = b} \mathrm{rm}(a).$$

Then $(\deg_1(b), \deg_2(b)) = \begin{cases} (d_{1,1}, d_{2,1}) & b \in D_1 \\ (d_{1,2}, d_{2,2}) & b \in D_2. \end{cases}$

The ordered pairs $(d_{1,1}, d_{2,1})$ and $(d_{1,2}, d_{2,2})$ are distinct, so the subsets $\mathbf{B}_1$ and $\mathbf{B}_2$ of marked points on D_1 and D_2 respectively are determined by the functions $\deg_1$ and $\deg_2$. These in turn depend only on the partition $\mathbf{A} = \mathbf{A}_1 \sqcup \mathbf{A}_2$, and not on any data specific to the admissible cover f. $\square$

We obtain as a corollary:

Corollary 7.1.3. *Let $X_1, \ldots, X_k$ be boundary divisors in $\mathcal{M}_{0,\mathbf{A}}$ and $Y_1, \ldots, Y_\ell$ be boundary divisors in $\mathcal{M}_{0,\mathbf{B}}$. Then the intersection 7.5*

$$\bigcap_{i=1}^{k} \overline{\pi_{\mathbf{A}}}^{-1}(X_i) \cap \bigcap_{j=1}^{\ell} \overline{\pi_{\mathbf{B}}}^{-1}(Y_j)$$

in $\overline{\mathcal{H}}$ is of pure dimension.

Proof. If the intersection 7.5 is empty then the lemma holds. Thus we assume it to be non-empty. For each X_i, by Lemma 7.1.2 there exists a unique boundary divisor $Y(X_i)$ in $\overline{\mathcal{M}}_{0,\mathbf{B}}$ with the property that $\overline{\pi_{\mathbf{A}}}^{-1}(X_i) \subseteq \overline{\pi_{\mathbf{B}}}^{-1}(Y(X_i))$. Then 7.5 has pure codimension equal to the number of distinct elements in $Y(X_1), \ldots, Y(X_k), Y_1, \ldots, Y_\ell$. $\square$

Note that set-theoretically the intersection 7.5 is a union of boundary strata in $\overline{\mathcal{H}}$. The scheme-theoretic intersection is generally non-reduced. We may compute the multiplicity of the intersection along any of the boundary strata using the local coordinates for $\overline{\mathcal{H}}$ given in Section 2.4.4. We obtain using equations 2.3 and 2.4:

Lemma 7.1.4. *Let G_γ be a codimension-r boundary stratum of $\overline{\mathcal{H}}$, where*

$$\gamma = (\sigma, \tau, d_{\text{Vertices}}, f_{\text{Vertices}}, F_{\text{Edges}}, (\text{br}_v)_{v \in \text{Vertices}(\sigma)}, \text{rm}_{\text{Edges}}).$$

Let $e_1, \ldots, e_k$ be edges of σ and $\underline{e_{k+1}}, \ldots, \underline{e_r}$ be edges of τ such that

$$F_{\text{Edges}}(e_1), \ldots, F_{\text{Edges}}(e_k), \underline{e_{k+1}}, \ldots, \underline{e_r}$$

are all distinct edges of τ. Then the subscheme

$$Z := \overline{\pi_{\mathbf{A}}}^{-1}(\text{Div}_{\overline{\mathcal{M}}_{0,\mathbf{A}}}(e_1)) \cap \cdots \cap \overline{\pi_{\mathbf{A}}}^{-1}(\text{Div}_{\overline{\mathcal{M}}_{0,\mathbf{A}}}(e_k)) \tag{7.6}$$

$$\cap\, \overline{\pi_{\mathbf{B}}}^{-1}(\text{Div}_{\overline{\mathcal{M}}_{0,\mathbf{B}}}(e_{k+1})) \cap \cdots \cap \overline{\pi_{\mathbf{B}}}^{-1}(\text{Div}_{\overline{\mathcal{M}}_{0,\mathbf{B}}}(e_r)) \tag{7.7}$$

of $\overline{\mathcal{H}}$ is codimension-r, and its multiplicity along G_γ is:

$$\left(\prod_{i=1}^{k} \frac{\prod_{F_{\text{Edges}}(e) = F_{\text{Edges}}(e_i)} \text{rm}(e)}{\text{rm}(e_i)} \right) \cdot \left(\prod_{j=k+1}^{r} \prod_{F_{\text{Edges}}(e) = \underline{e_j}} \text{rm}(e) \right) \tag{7.8}$$

Proof. As in Section 2.4.4, set $\tilde{n} : \tilde{\overline{\mathcal{H}}} \to \overline{\mathcal{H}}$ be the normalization map, and set $\widetilde{G_\gamma} = \tilde{n}^{-1}(G_\gamma)$ and

$\tilde{Z} = \tilde{n}^{-1}(Z)$. We see from equation 2.4 that The multiplicity of $\tilde{Z}$ along $\widetilde{G_\gamma}$ is

$$\left(\prod_{i=1}^{k} \frac{\mathrm{lcm}_{(F_{\mathrm{Edges}}(e)=F_{\mathrm{Edges}}(e_i))}\,\mathrm{rm}(e)}{\mathrm{rm}(e_i)}\right) \cdot \left(\prod_{j=k+1}^{r} \mathrm{lcm}_{(F_{\mathrm{Edges}}(e)=\underline{e_j})}\,\mathrm{rm}(e)\right) \tag{7.9}$$

On the other hand, by Remark 2.4.12, the map $\tilde{n} : \widetilde{G_\gamma} \to G_\gamma$ has generic degree

$$\prod_{j=1}^{r} \frac{\prod_{(F_{\mathrm{Edges}}(e)=\underline{e_j})}\mathrm{rm}(e)}{\mathrm{lcm}_{(F_{\mathrm{Edges}}(e)=\underline{e_j})}\,\mathrm{rm}(e)},$$

and the lemma follows $\square$

In the special case that $r = \dim(\overline{\mathcal{H}}) = (|\mathbf{B}| - 3)$, we obtain:

Corollary 7.1.5. *Let G_γ be a 0-dimensional boundary stratum of $\overline{\mathcal{H}}$, where*

$$\gamma = (\sigma, \tau, d_{\mathrm{Vertices}}, f_{\mathrm{Vertices}}, F_{\mathrm{Edges}}, (\mathrm{br}_v)_{v\in\mathrm{Vertices}(\sigma)}, \mathrm{rm}_{\mathrm{Edges}}).$$

Let $e_1, \dots, e_k$ be edges of σ and $\underline{e_{k+1}}, \dots, \underline{e_{|\mathbf{B}|-3}}$ be edges of τ such that

$$F_{\mathrm{Edges}}(e_1), \dots, F_{\mathrm{Edges}}(e_k), \underline{e_{k+1}}, \dots, \underline{e_{|\mathbf{B}|-3}}$$

are all distinct edges of τ. Then the subscheme

$$Z := \overline{\pi_{\mathbf{A}}}^{-1}(\mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{A}}}(e_1)) \cap \dots \cap \overline{\pi_{\mathbf{A}}}^{-1}(\mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{A}}}(e_k)) \tag{7.10}$$

$$\cap\, \overline{\pi_{\mathbf{B}}}^{-1}(\mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{B}}}(e_{k+1})) \cap \dots \cap \overline{\pi_{\mathbf{B}}}^{-1}(\mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{B}}}(e_{|\mathbf{B}|-3})) \tag{7.11}$$

of $\overline{\mathcal{H}}$ is 0-dimensional, and its multiplicity along G_γ is:

$$\left(\prod_{i=1}^{k} \frac{\prod_{F_{\mathrm{Edges}}(e)=F_{\mathrm{Edges}}(e_i)}\mathrm{rm}(e)}{\mathrm{rm}(e_i)}\right) \cdot \left(\prod_{j=k+1}^{|\mathbf{B}|-3} \prod_{F_{\mathrm{Edges}}(e)=\underline{e_j}} \mathrm{rm}(e)\right) \tag{7.12}$$

Remark 7.1.6. [**Counting the number of points in a 0-dimensional stratum of $\overline{\mathcal{H}}$**] Let G_γ be a 0-dimensional stratum as in the statement of Corollary 7.1.5. Computing its degree (the number of points) can be difficult. G_γ parametrizes fully degenerate admissible covers, that is, each irreducible component of the target curve has exactly three special points. Thus the admissible cover, when restricted to each component of the source curve, is branched over at most three points. We have

$$G_\gamma \cong \prod_{v \in \mathrm{Vertices}\,\sigma} \mathcal{H}_v,$$

where each $\mathcal{H}_v$ is a 0-dimensional Hurwitz space parametrizing such covers of $\mathbb{P}^1$ with at most three branch points. Thus

$$\deg(G_\gamma) = \prod_{v \in \mathrm{Vertices}\,\sigma} \deg(\mathcal{H}_v).$$

Fix a vertex v of σ. $\mathcal{H}_v$ parametrizes maps of degree $d_{\mathrm{Vertices}}(v)$ from a Flags_v-marked $\mathbb{P}^1$ to a $\mathrm{Flags}_{f_{\mathrm{Vertices}}}(v)$-marked $\mathbb{P}^1$, with branching and specified by br_v. Computing the degree of $\mathcal{H}_v$ comes down to counting factorizations in symmetric group. $\mathrm{Flags}_{f_{\mathrm{Vertices}}(v)}$ has three elements $\underline{e_1}, \underline{e_3}, \underline{e_3}$; branching over $\underline{e_j}$ is given by the integer partition $\mathrm{br}_v(\underline{e_j})$ of $d_{\mathrm{Vertices}}(v)$. Let N be the number of ways of factoring the identity element in the symmetric groups on $d_{\mathrm{Vertices}}(v)$ letters into three elements whose conjugacy classes are given by $\mathrm{br}_v(\underline{e_1}), \mathrm{br}_v(\underline{e_2})$ and $\mathrm{br}_v(\underline{e_3})$. Then

$$\deg(\mathcal{H}_v) = N \cdot \left|\mathrm{Aut}(\mathrm{br}_v(\underline{e_1}))\right| \cdot \left|\mathrm{Aut}(\mathrm{br}_v(\underline{e_2}))\right| \cdot \left|\mathrm{Aut}(\mathrm{br}_v(\underline{e_3}))\right|.$$

In practice, there might be better ways of computing $\deg(\mathcal{H}_v)$, especially when it is small. For example, one can try to solve for the coefficients of a degree $d_{\mathrm{Vertices}}(v)$ rational function on $\mathbb{P}^1$ by imposing the appropriate branching conditions.

Next, we state and prove two lemmas needed to show that there are "enough" 0-dimensional boundary strata in $\overline{\mathcal{H}}$ to determine the pushforward $[\mathcal{H}]_*$.

Lemma 7.1.7. *Let S_σ be a boundary stratum in $\overline{\mathcal{M}}_{0,n}$, and let X be a boundary divisor in $\overline{\mathcal{M}}_{0,n}$ containing S_σ. Then X is linearly equivalent to a (non-effective) sum of boundary divisors, none of which contains S_σ.*

Proof. We must have $X = \mathrm{Div}_{\overline{\mathcal{M}}_{0,n}}(e)$ where $e \in \mathrm{Edges}(\sigma)$. Let v and v' be the two vertices adjacent to e. Pick $i_1, i_2, i_1', i_2' \in [n]$ such that $\{\delta(v \to i_1), \delta(v \to i_2), e\}$ and $\{\delta(v' \to i_1'), \delta(v' \to i_2'), e\}$ are three-element sets. We have the following additive relation among divisors in $\overline{\mathcal{M}}_{0,n}$:

$$\sum_{\substack{P \sqcup Q = [n] \\ i_1, i_2 \in P \\ i_1', i_2' \in Q}} \mathrm{Div}_{\overline{\mathcal{M}}_{0,n}}(\{P, Q\}) = \sum_{\substack{P \sqcup Q = [n] \\ i_1, i_1' \in P \\ i_2, i_2' \in Q}} \mathrm{Div}_{\overline{\mathcal{M}}_{0,n}}(\{P, Q\}) \tag{7.13}$$

$X = \mathrm{Div}_{\overline{\mathcal{M}}_{0,n}}(e)$ is one of the terms on the left side of equation 7.13, and none of the other terms on either side is a divisor containing S_σ. This allows us to re-write X as desired. □

Corollary 7.1.8. *Let G_γ be a boundary stratum in $\overline{\mathcal{H}}$, and let X be a boundary divisor in $\overline{\mathcal{M}}_{0,\mathbf{A}}$ such that $\overline{\pi_{0,\mathbf{A}}}^{-1}(X)$ contains G_γ. Then X is linearly equivalent to a (non-effective) sum of boundary divisors in $\overline{\mathcal{M}}_{0,\mathbf{A}}$, none of whose inverse images in $\overline{\mathcal{H}}$ contains G_γ.*

Proof. Let σ be the dual tree of source curves of generic admissible covers in G_γ. A divisor in $\overline{\mathcal{M}}_{0,\mathbf{A}}$ contains S_σ if and only if its inverse image in $\overline{\mathcal{H}}$ contains G_γ. Thus X contains S_σ; applying Lemma 7.1.7 above yields the desired result. □

Remark 7.1.9. Let G_γ be a k-dimensional boundary stratum in $\overline{\mathcal{H}}$, and let X be a boundary divisor in $\overline{\mathcal{M}}_{0,\mathbf{A}}$ such that $\overline{\pi_{0,\mathbf{A}}}^{-1}(X)$ does not contain G_γ. Set $Z := \overline{\pi_{0,\mathbf{A}}}^{-1}(X) \cap G_\gamma$. Then either Z is empty or Z is supported on the union of $(k-1)$-dimensional boundary strata. In the latter case, given a stratum $G'_\gamma \subseteq \overline{\pi_{0,\mathbf{A}}}^{-1}(X) \cap G_\gamma$ with

$$\gamma' = (\sigma', \tau', d'_{\text{Vertices}}, f'_{\text{Vertices}}, F'_{\text{Edges}}, (\text{br}'_v)_{v \in \text{Vertices}(\sigma')}, \text{rm}'_{\text{Edges}}).$$

Let e be the edge of σ' with $X = \text{Div}_{\overline{\mathcal{M}}_{0,\mathbf{A}}}(e)$. Then the multiplicity of Z along G'_γ is

$$\frac{\prod_{F'_{\text{Edges}}(e')=F'_{\text{Edges}}(e)} \text{rm}'(e')}{\text{rm}'(e)}.$$

Finally, given $X_1, \dots, X_k$ boundary divisors in $\overline{\mathcal{M}}_{0,\mathbf{A}}$ and $Y_{k+1}, \dots, Y_{|\mathbf{B}|-3}$ boundary divisors in $\overline{\mathcal{M}}_{0,\mathbf{B}}$, we would like to compute the intersection number 7.4, that is:

$$*\overline{\pi_{\mathbf{A}}}^*(X_1) \cdots \overline{\pi_{\mathbf{A}}}^*(X_k) \cdot \overline{\pi_{\mathbf{B}}}^*(Y_{k+1}) \cdots \overline{\pi_{\mathbf{B}}}^*(Y_{|\mathbf{B}|-3}) \tag{7.14}$$

There are two possible cases as below.
Case 1 The intersection

$$Z := \overline{\pi_{\mathbf{A}}}^{-1}(X_1) \cap \cdots \cap \overline{\pi_{\mathbf{A}}}^{-1}(X_k) \cap \overline{\pi_{\mathbf{B}}}^{-1}(Y_{k+1}) \cap \cdots \cap \overline{\pi_{\mathbf{B}}}^{-1}(Y_{|\mathbf{B}|-3})$$

is 0-dimensional. In this case we have

$$\overline{\pi_{\mathbf{A}}}^*(X_1) \cdots \overline{\pi_{\mathbf{A}}}^*(X_k) \cdot \overline{\pi_{\mathbf{B}}}^*(Y_{k+1}) \cdots \overline{\pi_{\mathbf{B}}}^*(Y_{|\mathbf{B}|-3}) \tag{7.15}$$

$$= \prod_{\substack{\text{combinatorial types } \gamma \\ \dim(G_\gamma)=0 \\ G_\gamma \in Z}} \binom{\text{multiplicity of } Z}{\text{along } G_\gamma} \cdot (\deg(G_\gamma)) \tag{7.16}$$

Here, the first factor may be evaluated as in Corollary 7.1.5 and the second factor may be evaluated as in Remark 7.1.6.

Case 2 The intersection

$$Z := \overline{\pi_{\mathbf{A}}}^{-1}(X_1) \cap \cdots \cap \overline{\pi_{\mathbf{A}}}^{-1}(X_k) \cap \overline{\pi_{\mathbf{B}}}^{-1}(Y_{k+1}) \cap \cdots \cap \overline{\pi_{\mathbf{B}}}^{-1}(Y_{|\mathbf{B}|-3})$$

is positive-dimensional. The Y_js, when pulled back to to $\overline{\mathcal{H}}$, intersect in the expected dimension; the problem lies with the X_is. We would like to be able to use relations among divisors in $\overline{\mathcal{M}}_{0,\mathbf{A}}$ to re-write and evaluate the product 7.4. Let i_1 be such that if $i_1 \neq k$

$$Z_1 := \overline{\pi_{\mathbf{A}}}^{-1}(X_{i_1+1}) \cap \cdots \cap \overline{\pi_{\mathbf{A}}}^{-1}(X_k) \cap \overline{\pi_{\mathbf{B}}}^{-1}(Y_{k+1}) \cap \cdots \cap \overline{\pi_{\mathbf{B}}}^{-1}(Y_{|\mathbf{B}|-3})$$

or, if $i_1 = k$,

$$Z_1 := \overline{\pi_{\mathbf{B}}}^{-1}(Y_{k+1}) \cap \cdots \cap \overline{\pi_{\mathbf{B}}}^{-1}(Y_{|\mathbf{B}|-3})$$

has the expected dimension but

$$\overline{\pi_{\mathbf{A}}}^{-1}(X_{i_1}) \cap \cdots \cap \overline{\pi_{\mathbf{A}}}^{-1}(X_k) \cap \overline{\pi_{\mathbf{B}}}^{-1}(Y_{k+1}) \cap \cdots \cap \overline{\pi_{\mathbf{B}}}^{-1}(Y_{|\mathbf{B}|-3})$$

has dimension 1 more than expected. Z_1 is supported on a union of boundary strata; we have:

$$\begin{aligned}
&\overline{\pi_{\mathbf{A}}}^{-1}(X_{i_1+1}) \cdots \overline{\pi_{\mathbf{A}}}^{-1}(X_k) \cdot \overline{\pi_{\mathbf{B}}}^{-1}(Y_{k+1}) \cdots \overline{\pi_{\mathbf{B}}}^{-1}(Y_{|\mathbf{B}|-3}) && (7.17)\\
&= [Z_1] && (7.18)\\
&= \sum_{\substack{\text{Combinatorial type}\,\gamma=(\sigma,\tau,\ldots)\\ \sigma \text{ has edges } e_{i_1+1},\ldots,e_k\\ \text{with } X_i=\mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{A}}}(e_i)\\ \tau \text{ has edges } \underline{e_{k+1}},\ldots,\underline{e_{|\mathcal{B}|-3}}\\ \text{with } Y_j=\mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{B}}}(\underline{e_j})}} \begin{pmatrix}\text{multiplicity of } [Z_1]\\ \text{along } G_\gamma\end{pmatrix} [G_\gamma] && (7.19)
\end{aligned}$$

The multiplicities may be evaluated as in Lemma 7.1.4. Now

$$\begin{aligned}
&\overline{\pi_{\mathbf{A}}}^{-1}(X_{i_1}) \cdots \overline{\pi_{\mathbf{A}}}^{-1}(X_k) \cdot \overline{\pi_{\mathbf{B}}}^{-1}(Y_{k+1}) \cdots \overline{\pi_{\mathbf{B}}}^{-1}(Y_{|\mathbf{B}|-3}) && (7.20)\\
&= \overline{\pi_{\mathbf{A}}}^{-1}(X_{i_1}) \cdot [Z_1] && (7.21)\\
&= \sum_{\substack{\text{Combinatorial type}\,\gamma=(\sigma,\tau,\ldots)\\ \sigma \text{ has edges } e_{i_1+1},\ldots,e_k\\ \text{with } X_i=\mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{A}}}(e_i)\\ \tau \text{ has edges } \underline{e_{k+1}},\ldots,\underline{e_{|\mathcal{B}|-3}}\\ \text{with } Y_j=\mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{B}}}(\underline{e_j})}} \begin{pmatrix}\text{multiplicity of } [Z_1]\\ \text{along } G_\gamma\end{pmatrix} \left(\overline{\pi_{\mathbf{A}}}^{*}(X_{i_1}) \cdot [G_\gamma]\right) && (7.22)
\end{aligned}$$

For each term γ in the above sum, we either have:

1. $\overline{\pi_{\mathbf{A}}}^*(X_{i_1}) \cap G_\gamma$ is empty so $\overline{\pi_{\mathbf{A}}}^*(X_{i_1}) \cdot [G_\gamma] = 0$, or

2. $\overline{\pi_{\mathbf{A}}}^*(X_{i_1}) \cap G_\gamma$ is supported on a union of boundary strata of dimension one less than $\dim G_\gamma$; in this case $\overline{\pi_{\mathbf{A}}}^*(X_{i_1}) \cdot [G_\gamma]$ may be computed as in Remark 7.1.9, or

3. $\overline{\pi_{\mathbf{A}}}^*(X_{i_1}) \supseteq G_\gamma$; in this case we use Corollary 7.1.8 to move X_{i_1} to a sum of boundary divisors not containing G_γ and thus reduce to the 2nd case above.

Thus we write the product

$$\overline{\pi_{\mathbf{A}}}^{-1}(X_{i_1}) \cdots \overline{\pi_{\mathbf{A}}}^{-1}(X_k) \cdot \overline{\pi_{\mathbf{B}}}^{-1}(Y_{k+1}) \cdots \overline{\pi_{\mathbf{B}}}^{-1}(Y_{|\mathbf{B}|-3}) \tag{7.23}$$

as a sum of boundary strata in $\overline{\mathcal{H}}$ with multiplicities computed using local coordinates in $\overline{\mathcal{H}}$. We then use an analogous procedure to multiply with the class of $\overline{\pi_{\mathbf{A}}}^{-1}(X_{i_1-1})$ and proceed iteratively, at each step multiplying by a new $\overline{\pi_{\mathbf{A}}}^{-1}(X_i)$. At the final step the intersection number 7.4 is expressed as a sum of 0-dimensional boundary strata weighted with integer multiplicities computed using local coordinates.

Thus computing $[\mathcal{H}]_* : H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{B}}, \mathbb{Z}) \to H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{A}'}, \mathbb{Z})$ involves

1. enumerating the combinatorial types of boundary strata in $\overline{\mathcal{H}}$ that are contained in the inverse images of specified boundary strata in $\overline{\mathcal{M}}_{0,\mathbf{B}}$ and $\overline{\mathcal{M}}_{0,\mathbf{A}}$, and

2. computing the degrees of 0-dimensional boundary strata in $\overline{\mathcal{H}}$ as in Remark 7.1.6.

The combinatorics involved are manageable only when $\mathcal{H}$ parametrizes maps of low degree d, and when $|\mathbf{B}|$ is small. In the next section, we implement the techniques described above in a family of concrete examples of Hurwitz correspondences on $\mathcal{M}_{0,5}$.

7.2 Sample computations in $\mathcal{M}_{0,5}$

In this section, we compute the dynamical degrees of Hurwitz self-correspondences

$$\mathcal{H}_\alpha : \mathcal{M}_{0,\mathbf{P}} \rightrightarrows \mathcal{M}_{0,\mathbf{P}}$$

indexed by permutations $\alpha \in S_5$, where:

- $\mathbf{P} = \{p_1, \ldots, p_5\}$,

- $\mathcal{H}_\alpha = \mathcal{H}(\mathbf{P}, \mathbf{P}, d_0, F_\alpha, \mathrm{br}_0, \mathrm{rm}_\alpha)$,
- $d_0 = 3$,
- $F_\alpha(p_i) = p_{\alpha^{-1}(i)}$,
- $\mathrm{br}_0(p_i) = (2,1)$ for $i = 1, \ldots, 4$ and $\mathrm{br}_0(p_5) = (1,1,1)$, and
- $\mathrm{rm}_\alpha(p_{\alpha(i)}) = 2$ for $i = 1, \ldots, 4$ and $\mathrm{rm}_\alpha(p_{\alpha(5)}) = 1$

Thus the Hurwitz space $\mathcal{H}_\alpha$ parametrizes degree 3 maps from one $\mathbf{P}$-marked $\mathbb{P}^1$ to another, with simple branching over $p_1, \ldots, p_4$, and such that $p_i \mapsto p_{\alpha^{-1}(i)}$. It is convenient to first re-phrase the case where α is the identity permutation as a non-dynamical Hurwitz correspondence.

7.2.1 A non-dynamical computation

Let $\mathcal{H}_0' = \mathcal{H}(\mathbf{A}_0', \mathbf{B}_0, d_0, F_0, \mathrm{br}_0, \mathrm{rm}_0)$, where

- $\mathbf{A}_0' = \{a_1, \ldots, a_5\}$,
- $\mathbf{B}_0 = \{b_1, \ldots, b_5\}$,
- $d_0 = 3$,
- $F_0(a_i) = b_i$ for $i = 1, \ldots, 5$,
- $\mathrm{br}_0(b_i) = (2,1)$ for $i = 1, \ldots, 4$ and $\mathrm{br}_0(b_5) = (1,1,1)$
- $\mathrm{rm}_0(a_i) = 2$ for $i = 1, \ldots, 4$ and $\mathrm{rm}_0(a_5) = 1$

Remark 7.2.1. Under the identifications $a_i = p_i = b_i$ for $i = 1, \ldots, 5$, the Hurwitz correspondence $\mathcal{H}_0' : \mathcal{M}_{0,\mathbf{B}_0} \rightrightarrows \mathcal{M}_{0,\mathbf{A}_0'}$ is identified with $\mathcal{H}_{\text{identity}} : \mathcal{M}_{0,\mathbf{P}} \rightrightarrows \mathcal{M}_{0,\mathbf{P}}$. Under the identifications $a_i = p_{\alpha(i)} = b_{\alpha(i)}$ for $i = 1, \ldots, 5$, the Hurwitz correspondence $\mathcal{H}_0' : \mathcal{M}_{0,\mathbf{B}_0} \rightrightarrows \mathcal{M}_{0,\mathbf{A}_0'}$ is identified with $\mathcal{H}_\alpha : \mathcal{M}_{0,\mathbf{P}} \rightrightarrows \mathcal{M}_{0,\mathbf{P}}$.

Both $\mathcal{M}_{0,\mathbf{A}_0'}$ and $\mathcal{M}_{0,\mathbf{B}}$ are isomorphic to $\mathcal{M}_{0,5}$, thus two-dimensional. The Hurwitz space $\mathcal{H}_0'$ is two-dimensional as well. Let $\pi_{\mathbf{B}_0}$ and $\pi_{\mathbf{A}_0'}$ be the maps from $\mathcal{H}_0'$ to $\mathcal{M}_{0,\mathbf{B}_0}$ and $\mathcal{M}_{0,\mathbf{A}_0'}$ respectively. Let $\overline{\mathcal{H}_0'}$ be the admissible covers compactification of $\mathcal{H}_0'$ with maps $\overline{\pi_{\mathbf{B}_0}}$ and $\overline{\pi_{\mathbf{A}_0'}}$ to $\mathcal{M}_{0,\mathbf{B}_0}$ and $\mathcal{M}_{0,\mathbf{A}_0'}$ respectively. We begin by computing

$$[\mathcal{H}_0']_* = \overline{\pi_{\mathbf{A}_0'}}_* \circ \overline{\pi_{\mathbf{B}_0}}^* : H_2(\overline{\mathcal{M}}_{0,\mathbf{B}_0}, \mathbb{Z}) \to H_2(\overline{\mathcal{M}}_{0,\mathbf{A}_0'}, \mathbb{Z}).$$

The fully marked Hurwitz space corresponding to $\mathcal{H}_0'$ is $\mathcal{H}_0 = \mathcal{H}(\mathbf{A}_0, \mathbf{B}_0, d_0, F_0, \mathrm{br}_0, \mathrm{rm}_0)$, defined previously in Figure 2.2 and referred to in many subsequent examples. Recall that:

- $\mathbf{A}_0 = \{a_1, \ldots, a_{11}\}$,
- $\mathbf{B}_0 = \{b_1, \ldots, b_5\}$,
- $d_0 = 3$
- $F_0(a_i) = b_i$ for $i = 1, \ldots, 5$, $F_0(a_i) = b_{i-5}$ for $i = 6, \ldots, 10$, and $F_0(a_{11}) = b_5$,
- $\mathrm{br}_0(b_i) = (2, 1)$ for $i = 1, \ldots, 4$ and $\mathrm{br}_0(b_5) = (1, 1, 1)$
- $\mathrm{rm}_0(a_i) = 2$ for $i = 1, \ldots, 4$ and $\mathrm{rm}_0(a_i) = 1$ for $i = 6, \ldots, 11$.

There is a $2:1$ covering map $\nu : \mathcal{H}_0 \to \mathcal{H}_0'$; the two points in the fibre over

$$[f : (C, a_1, \ldots, a_5) \to (D, b_1, \ldots, b_5)]$$

correspond to the two bijections between $f^{-1}(b_5) \smallsetminus \{a_5\}$ and the set of labels $\{a_{10}, a_{11}\}$. Denote by μ the forgetful map from $\mathcal{M}_{0,\mathbf{A}_0}$ to $\mathcal{M}_{0,\mathbf{A}_0'}$. The maps ν and μ extend to $\nu : \overline{\mathcal{H}}_0 \to \overline{\mathcal{H}}_0'$ and $\mu : \overline{\mathcal{M}}_{0,\mathbf{A}_0} \to \overline{\mathcal{M}}_{0,\mathbf{A}_0'}$. As in section 2.4.1 we have a commutative diagram:

$$\begin{tikzcd}
 & \overline{\mathcal{H}}_0 \arrow[d, "\nu"'] \arrow[dr, "\overline{\pi_{\mathbf{A}_0}}"] & \\
 & \overline{\mathcal{H}}_0' \arrow[dl, "\overline{\pi_{\mathbf{B}_0}}"'] \arrow[dr, "\overline{\pi_{\mathbf{A}_0'}}"] & \overline{\mathcal{M}}_{0,\mathbf{A}_0} \arrow[d, "\mu"] \\
\mathcal{M}_{0,\mathbf{B}_0} & & \mathcal{M}_{0,\mathbf{A}_0'}
\end{tikzcd}$$

Here, $\overline{\pi_{\mathbf{B}_0}} \circ \nu$ is the "target" map from the admissible covers space $\overline{\mathcal{H}}_0$ to its moduli space of marked target curves. Since ν is degree 2, we have:

$$[\mathcal{H}_0]_* = (\mu \circ \overline{\pi_{\mathbf{A}_0}})_* \circ (\overline{\pi_{\mathbf{B}_0}} \circ \nu)^* = 2[\mathcal{H}_0']_* : H_2(\overline{\mathcal{M}}_{0,\mathbf{B}_0}, \mathbb{Z}) \to H_2(\overline{\mathcal{M}}_{0,\mathbf{A}_0'}, \mathbb{Z}) \tag{7.24}$$

We will compute $[\mathcal{H}_0]_*$. Both $\overline{\mathcal{M}}_{0,\mathbf{B}_0}$ and $\overline{\mathcal{M}}_{0,\mathbf{A}_0'}$ are isomorphic to $\overline{\mathcal{M}}_{0,5}$. Their second homology groups have rank five and are generated by the classes of the ten one-dimensional boundary divisors. For $i, j \in \{1, \dots, 5\}$, denote by $Y_{i,j}$ the boundary divisor

$$\mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{B}_0}}(\{\{b_i, b_j\}, \{b_i, b_j\}^C\})$$

of $\overline{\mathcal{M}}_{0,\mathbf{B}_0}$ and denote by $X_{i,j}$ the boundary divisor

$$\mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{A}_0'}}(\{\{a_i, a_j\}, \{a_i, a_j\}^C\})$$

of $\overline{\mathcal{M}}_{0,\mathbf{A}_0'}$. One can check that $Y_{1,2}$, $Y_{1,3}$, $Y_{1,4}$, $Y_{1,5}$, and $Y_{2,3}$ form a basis of $H_2(\overline{\mathcal{M}}_{0,\mathbf{B}_0}, \mathbb{Z})$ and $X_{1,2}$, $X_{1,3}$, $X_{1,4}$, $X_{1,5}$ and $X_{2,3}$ form a basis of $H_2(\overline{\mathcal{M}}_{0,\mathbf{A}_0'}, \mathbb{Z})$.

Lemma 7.2.2. *We have* $[\mathcal{H}_0]_*([Y_{1,2}]) = 8[X_{1,2}]$.

Proof. In this computation we will refer to the one-dimensional boundary divisors G_1, G_2, G_3 and G_4 in $\overline{\mathcal{H}}_0$, defined in Figures 7.1, 7.2 ,7.3 and 7.4 respectively. We have

$$(\overline{\pi_{\mathbf{B}_0}} \circ \nu)^{-1}(Y_{1,2}) = G_1 \cup G_2 \cup G_3 \cup G_4,$$

and by Lemma 7.1.4,

$$(\overline{\pi_{\mathbf{B}_0}} \circ \nu)^*([Y_{1,2}]) = 3[G_1] + [G_2] + [G_3] + [G_4].$$

The image of G_1 under $\mu \circ \overline{\pi_{\mathbf{A}_0}}$ is the boundary divisor $X_{1,2}$. We claim that $\mu \circ \overline{\pi_{\mathbf{A}_0}} : G_1 \to X_{1,2}$ is generically $2 : 1$. To see this, let $[(C, a_1, \dots, a_5)] \in X_{1,2}$ be a general point. Denote by C_1 the irreducible component of C containing the marked points a_1 and a_2, denote by C_2 the irreducible component of C containing the marked points a_3, a_4 and a_5,and denote by η_1 and η_2 the points on C_1 and C_2 respectively that correspond to the unique node on C. Note that the map $r : \mathbb{P}^1 \to \mathbb{P}^1$ given in coordinates by $z \mapsto (-2z^2 + 3z^3)$ is the only rational function that sends $0 \mapsto 0$ and $1 \mapsto 1$, each with local degree 2, and sends $\infty \mapsto \infty$ with local degree 3. Thus, by identifying (C_1, a_1, a_2, η_1) with the source $(\mathbb{P}^1, 0, 1, \infty)$ of r, and by renaming the target $(\mathbb{P}^1, 0, 1, \infty)$ of r as $(D_1, b_1, b_2, \theta_1)$, we can use the rational function r to specify the unique map $f_1 : (C_1, a_1, a_2, \eta) \to (D_1, b_1, b_2, \theta_1)$ that sends $a_1 \mapsto b_1$ and $a_2 \mapsto b_2$, each with local degree 2, and sends $\eta_1 \mapsto \theta_1$ with local degree 3. Similarly, by identifying (C_2, a_3, a_4, η_2) with the source $(\mathbb{P}^1, 0, 1, \infty)$ of r, and by renaming the target $(\mathbb{P}^1, 0, 1, \infty)$ of r as $(D_2, b_3, b_4, \theta_2)$, we can use the rational function r to specify the unique map $f_2 : (C_2, a_3, a_4, \eta_2) \to (D_2, b_3, b_4, \theta_2)$ that sends $a_3 \mapsto b_3$ and $a_4 \mapsto b_4$, each with local degree 2, and sends $\eta_2 \mapsto \theta_2$ with local degree 3. Denote by $b_5 \in D_2$ the image under f_2 of $a_5 \in C_2$; since $[(C, a_1, \dots, a_5)]$ is assumed to be general, we may assume that b_5 is distinct from the points b_3, b_4, and θ_2. We glue $(D_1, b_1, b_2, \theta_1)$ to $(D_2, b_3, b_4, b_5, \theta_2)$ by identifying θ_1 with θ_2 to obtain a $\mathbf{B}_0$-marked stable curve $(D, b_1, \ldots, b_5)$. The maps f_1 and f_2 glue to form an admissible cover $[f : C \to D] \in \mathcal{H}_0'$. Now, note that for $j = 1, \ldots, 4$ there is a unique point on C that maps via f to b_j with local degree 1; we label that point a_{j+5}. There are 3 points on C that maps via f to b_5; of which one is labelled a_5. The two points in $\mathcal{H}_0$ that lie in the inverse image of $[f] \in \mathcal{H}_0'$ correspond to the two ways of labelling the two elements of $f^{-1}(b_5) \smallsetminus \{a_5\}$ by a_{10} and a_{11}. These are the two points in G_1 that lie above $[(C, a_1, \ldots, a_5)] \in X_{1,2}$.

Thus $(\mu \circ \overline{\pi_{\mathbf{A}_0}})_*([G_1]) = 2[X_{1,2}]$.

The images of G_2 and G_3 under $\mu \circ \overline{\pi_{\mathbf{A}_0}}$ are also both $X_{1,2}$, and a similar check shows that the maps from G_2 and G_3 to $X_{1,2}$ are generically $1:1$. Thus $(\mu \circ \overline{\pi_{\mathbf{A}_0}})_*([G_2]) = (\mu \circ \overline{\pi_{\mathbf{A}_0}})_*([G_3]) = [X_{1,2}]$. On the other hand, the image of G_4 is a single point; thus $(\mu \circ \overline{\pi_{\mathbf{A}_0}})_*([G_4]) = 0$. Putting this together, we obtain the lemma. □

By symmetry, $[\mathcal{H}_0]_*([Y_{1,3}]) = 8[X_{1,3}]$, $[\mathcal{H}_0]_*([Y_{1,4}]) = 8[X_{1,4}]$, and $[\mathcal{H}_0]_*([Y_{2,3}]) = 8[X_{2,3}]$.

Lemma 7.2.3. *We have* $[\mathcal{H}_0]_*([Y_{1,5}]) = -2[X_{1,2}] - 2[X_{1,3}] + 2[X_{1,4}] + 12[X_{1,5}] + 4[X_{2,3}]$.

Proof. In this computation we will refer to the one-dimensional boundary divisors G_5, G_6 and G_7 in and the zero-dimensional boundary strata G_8, G_9, G_{10} and G_{11} in $\overline{\mathcal{H}}_0$, defined in Figures 7.5, 7.6, 7.7, 7.8, 7.9, 7.10 and 7.11 respectively. We have:

$$(\overline{\pi_{\mathbf{B}_0}} \circ \nu)^{-1}(Y_{1,5}) = G_5 \cup G_6 \cup G_7,$$

and by Lemma 7.1.4,

$$(\overline{\pi_{\mathbf{B}_0}} \circ \nu)^*([Y_{1,2}]) = 2[G_5] + 2[G_6] + 2[G_7].$$

The image of G_5 under $\mu \circ \overline{\pi_{\mathbf{A}_0}}$ is the boundary divisor $X_{1,5}$. Thus $(\mu \circ \overline{\pi_{\mathbf{A}_0}})_*([G_5]) = m[X_{1,5}]$, where m is the degree of $\mu \circ \overline{\pi_{\mathbf{A}_0}} : G_5 \to X_{1,5}$. In order to determine m, we observe that $G_5 \cap (\mu \circ \overline{\pi_{\mathbf{A}_0}})^{-1}(X_{2,3})$ is supported on the two zero-dimensional strata G_8 and G_9, and is reduced by Lemma 7.1.4. Also, one can check that G_8 and G_9 each consists of a single point. Thus

$$(\mu \circ \overline{\pi_{\mathbf{A}_0}})_*([G_5]) \cdot [X_{2,3}] = [G_5] \cdot (\mu \circ \overline{\pi_{\mathbf{A}_0}})^*([X_{2,3}]) = 2[pt].$$

Since $X_{1,5} \cdot X_{2,3} = [pt]$, we conclude that $(\mu \circ \overline{\pi_{\mathbf{A}_0}})_*([G_5]) = 2[X_{1,5}]$. By symmetry, $(\mu \circ \overline{\pi_{\mathbf{A}_0}})_*([G_6]) = 2[X_{1,5}]$.

The image of G_7 under $\mu \circ \overline{\pi_{\mathbf{A}_0}}$ is not a boundary divisor. In order to compute its class in $H_2(\overline{\mathcal{M}}_{0,\mathbf{A}'_0}, \mathbb{Z})$, we intersect with each of the divisors $X_{1,2}$, $X_{1,3}$, $X_{1,4}$, $X_{1,5}$ and $X_{2,3}$. First:

$$\begin{aligned}
(\mu \circ \overline{\pi_{\mathbf{A}_0}})_*([G_7]) \cdot [X_{1,4}] &= [G_7] \cdot (\mu \circ \overline{\pi_{\mathbf{A}_0}})^*([X_{1,4}]) \\
&= [G_7] \cdot \sum_{\substack{\mathbf{A}_0 = \mathbf{A}_{0,1} \sqcup \mathbf{A}_{0,2} \\ a_1, a_4 \in \mathbf{A}_{0,1} \\ a_2, a_3, a_5 \in \mathbf{A}_{0,2}}} (\overline{\pi_{\mathbf{A}_0}})^* \operatorname{Div}_{\overline{\mathcal{M}}_{0,\mathbf{A}_0}}(\{\mathbf{A}_{0,1}, \mathbf{A}_{0,2}\}) \\
&= [G_7] \cdot (\overline{\pi_{\mathbf{A}_0}})^* \operatorname{Div}_{\overline{\mathcal{M}}_{0,\mathbf{A}_0}}(\{\{a_1, a_{10}, a_{11}, a_4, a_7, a_8\}, \{a_2, a_3, a_9, a_5, a_6\}\}) \\
&= [G_{11}]
\end{aligned}$$

Here, $[G_{11}]$ is the 0-dimensional stratum defined in Figure 7.11; it appears with multiplicity 1 by Lemma 7.1.4. One can check that G_{11} consists of exactly one point, thus

$$(\mu \circ \overline{\pi_{\mathbf{A}_0}})_*([G_7]) \cdot [X_{1,4}] = [pt].$$

By symmetry,

$$(\mu \circ \overline{\pi_{\mathbf{A}_0}})_*([G_7]) \cdot [X_{1,2}] = \mu \circ \overline{\pi_{\mathbf{A}_0}})_*([G_7]) \cdot [X_{1,3}] = [pt].$$

One can also check that $[G_7] \cap (\mu \circ \overline{\pi_{\mathbf{A}_0}})^{-1}([X_{1,5}])$ is empty, so

$$(\mu \circ \overline{\pi_{\mathbf{A}_0}})_*([G_7]) \cdot [X_{1,5}] = 0.$$

Finally,

$$\begin{aligned}
(\mu \circ \overline{\pi_{\mathbf{A}_0}})_*([G_7]) \cdot [X_{2,3}] &= [G_7] \cdot (\mu \circ \overline{\pi_{\mathbf{A}_0}})^*([X_{2,3}]) \\
&= [G_7] \cdot \sum_{\substack{\mathbf{A}_0 = \mathbf{A}_{0,1} \sqcup \mathbf{A}_{0,2} \\ a_1, a_4, a_5 \in \mathbf{A}_{0,1} \\ a_2, a_3 \in \mathbf{A}_{0,2}}} (\overline{\pi_{\mathbf{A}_0}})^* \operatorname{Div}_{\overline{\mathcal{M}}_{0,\mathbf{A}_0}}(\{\mathbf{A}_{0,1}, \mathbf{A}_{0,2}\}) \\
&= [G_7] \cdot (\overline{\pi_{\mathbf{A}_0}})^* \operatorname{Div}_{\overline{\mathcal{M}}_{0,\mathbf{A}_0}}(\{\{a_1, a_{10}, a_{11}, a_4, a_9, a_6, a_5\}, \{a_2, a_3, a_7, a_8\}\}) \\
&= [G_{10}]
\end{aligned}$$

Here, $[G_{10}]$ is the 0-dimensional stratum defined in Figure 7.10; it appears with multiplicity 1 by Lemma 7.1.4. One can check that G_{10} consists of exactly one point, thus

$$(\mu \circ \overline{\pi_{\mathbf{A}_0}})_*([G_7]) \cdot [X_{2,3}] = [pt].$$

Now, every boundary divisor in $\overline{\mathcal{M}}_{0,5}$ has self intersection -1. Thus the matrix of intersections in $H_2(\overline{\mathcal{M}}_{0,\mathbf{A}_0'}, \mathbb{Z})$ among elements of the basis $X_{1,2}$, $X_{1,3}$, $X_{1,4}$, $X_{1,5}$ and $X_{2,3}$ is

$$IM = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \tag{7.25}$$

Thus, to express $(\mu \circ \overline{\pi_{\mathbf{A}_0}})_*([G_7])$ as a linear combination of $X_{1,2}$, $X_{1,3}$, $X_{1,4}$, $X_{1,5}$ and $X_{2,3}$, we left-multiply $(1, 1, 1, 0, 1)$ by IM^{-1} to obtain

$$(\mu \circ \overline{\pi_{\mathbf{A}_0}})_*([G_7]) = -[X_{1,2}] - [X_{1,3}] + [X_{1,4}] + 2[X_{1,5}] + 2[X_{2,3}].$$

Putting everything together, we obtain the lemma. $\square$

Thus, we can write the pushforward $[\mathcal{H}_0]_*$ as a matrix, with input basis $Y_{1,2}$, $Y_{1,3}$, $Y_{1,4}$, $Y_{1,5}$, and $Y_{2,3}$ of $H_2(\overline{\mathcal{M}}_{0,\mathbf{B}_0}, \mathbb{Z})$ and output basis $X_{1,2}$, $X_{1,3}$, $X_{1,4}$, $X_{1,5}$ and $X_{2,3}$ of $H_2(\overline{\mathcal{M}}_{0,\mathbf{A}_0'}, \mathbb{Z})$:

$$[\mathcal{H}_0]_* = \begin{pmatrix} 8 & 0 & 0 & -2 & 0 \\ 0 & 8 & 0 & -2 & 0 \\ 0 & 0 & 8 & 2 & 0 \\ 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 4 & 8 \end{pmatrix} \tag{7.26}$$

Thus we have:

$$[\mathcal{H}_0']_* = (1/2)[\mathcal{H}_0]_* = \begin{pmatrix} 4 & 0 & 0 & -1 & 0 \\ 0 & 4 & 0 & -1 & 0 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 2 & 4 \end{pmatrix} \tag{7.27}$$

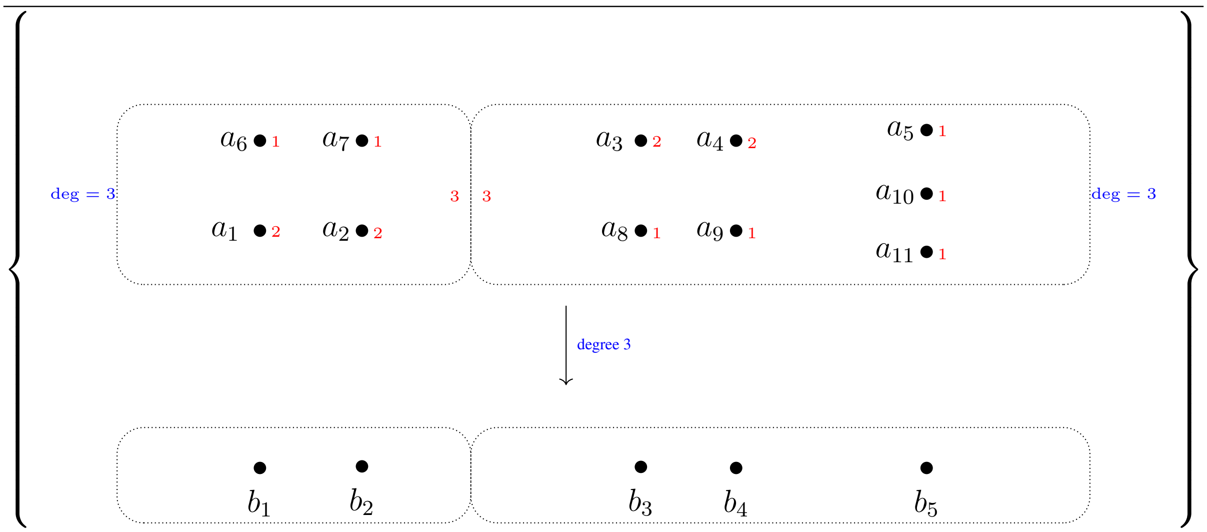


Figure 7.1: The boundary stratum G_1 of $\mathcal{H}_0$

7.2.2 Computing the dynamical degrees of $\mathcal{H}_\alpha$ on $\mathcal{M}_{0,5}$

In this section we use the computation of $[\mathcal{H}_0']_*$ in Section 7.2.1 to compute the first dynamical degrees of all the Hurwitz correspondences $\mathcal{H}_\alpha$ on $\mathcal{M}_{0,\mathbf{P}} \cong \mathcal{M}_{0,5}$. By Corollary 3.0.2, the first dynamical degree of $\mathcal{H}_\alpha$ is the largest eigenvalue of the induced pushforward action on $H_2(\overline{\mathcal{M}}_{0,\mathbf{P}}, \mathbb{R})$. The homology group $H_2(\overline{\mathcal{M}}_{0,\mathbf{P}}, \mathbb{R})$ is canonically isomorphic to $\Omega_{1,\mathbf{P}}$; since the latter has a symmetric basis given by Theorem 5.2.12, we will compute the maps

$$[\mathcal{H}_\alpha]_* : \Omega_{1,\mathbf{P}} \to \Omega_{1,\mathbf{P}}.$$

As in Remark 7.2.1, under the identifications $a_i = p_i = b_i$ for $i = 1, \ldots, 5$, the Hurwitz correspondence $\mathcal{H}_0' : \mathcal{M}_{0,\mathbf{B}_0} \rightrightarrows \mathcal{M}_{0,\mathbf{A}_0'}$ is identified with $\mathcal{H}_{\text{identity}} : \mathcal{M}_{0,\mathbf{P}} \rightrightarrows \mathcal{M}_{0,\mathbf{P}}$. Under these identifications, the divisor $\mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{P}}}(\{\{p_{i_1}, p_{i_2}\}, \{p_{i_3}, p_{i_4}, p_{i_5}\}\})$ on $\overline{\mathcal{M}}_{0,\mathbf{P}}$ is identified with the divisor X_{i_1,i_2} on $\overline{\mathcal{M}}_{0,\mathbf{A}_0'}$ and with the divisor Y_{i_1,i_2} on $\overline{\mathcal{M}}_{0,\mathbf{B}_0}$. Thus, in the basis

$$\begin{aligned} &\mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{P}}}(\{\{p_1,p_2\},\{p_3,p_4,p_5\}\}), \mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{P}}}(\{\{p_1,p_3\},\{p_2,p_4,p_5\}\}), \\ &\quad \mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{P}}}(\{\{p_1,p_4\},\{p_2,p_3,p_5\}\}), \mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{P}}}(\{\{p_1,p_5\},\{p_2,p_3,p_4\}\}), \\ &\quad \mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{P}}}(\{\{p_2,p_3\},\{p_1,p_4,p_5\}\}), \end{aligned} \tag{7.28}$$

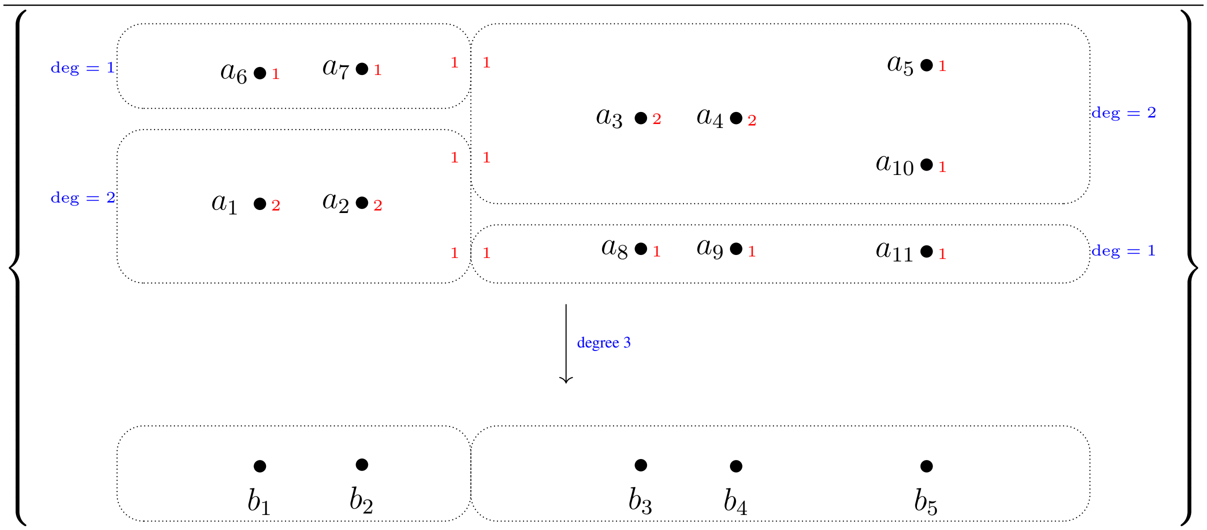


Figure 7.2: The boundary stratum G_2 of $\mathcal{H}_0$

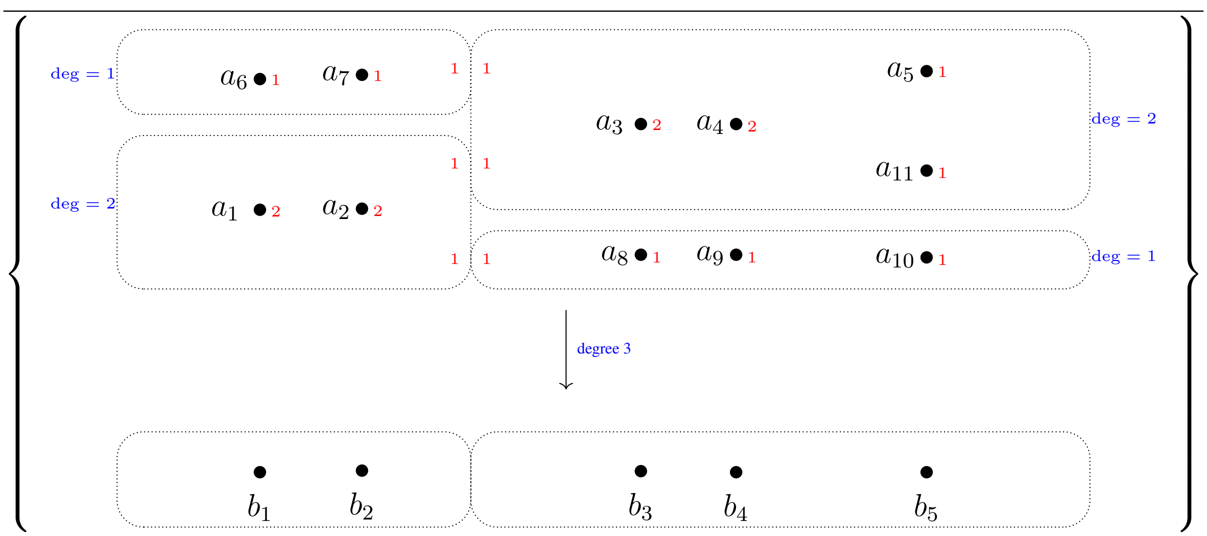


Figure 7.3: The boundary stratum G_3 of $\mathcal{H}_0$.

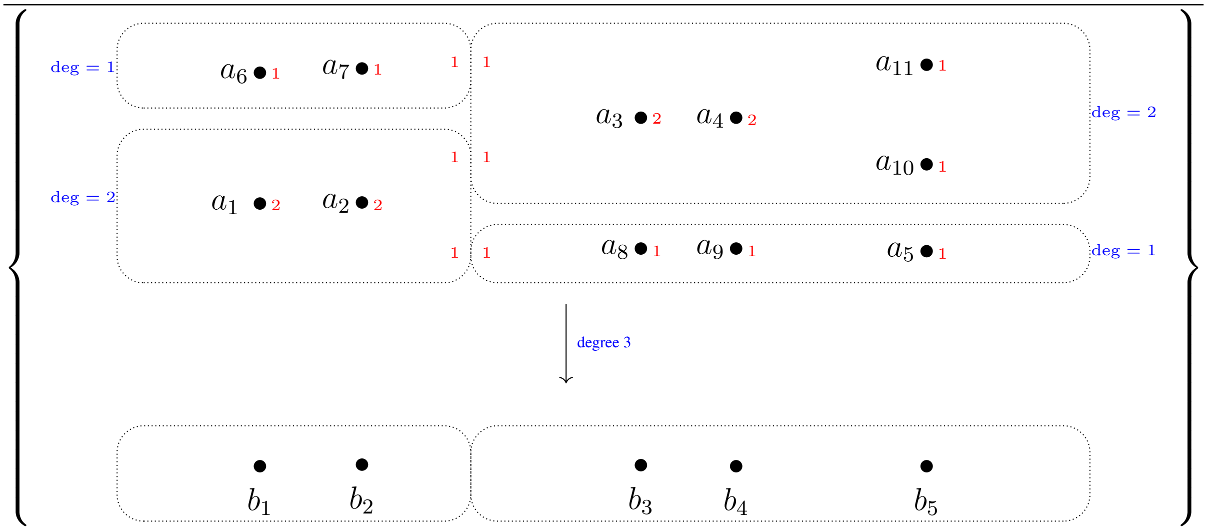


Figure 7.4: The boundary stratum G_4 of $\mathcal{H}_0$.

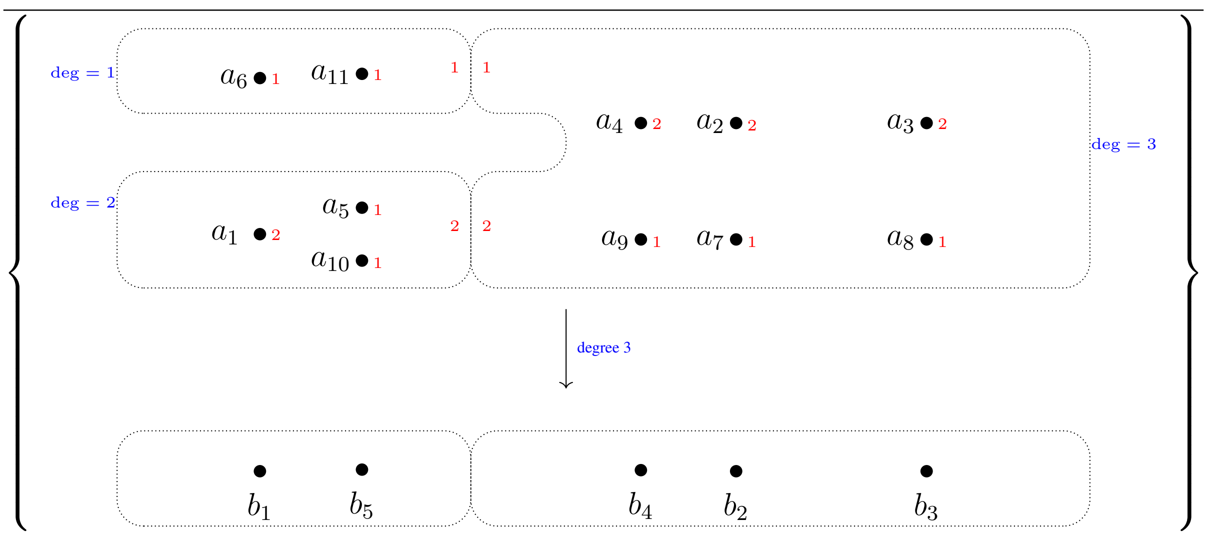


Figure 7.5: The boundary stratum G_5 of $\mathcal{H}_0$.

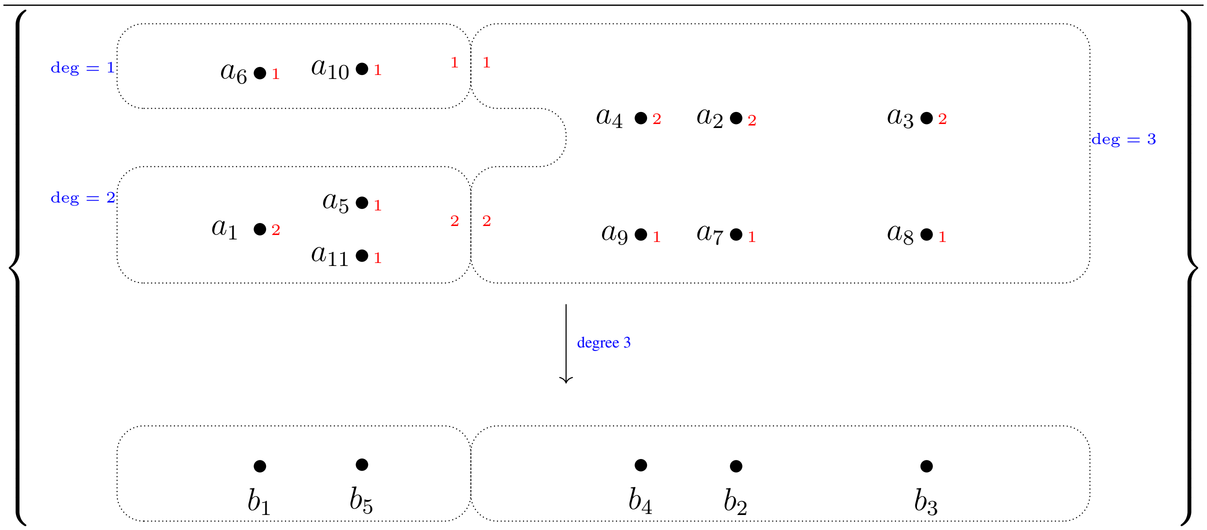


Figure 7.6: The boundary stratum G_6 of $\mathcal{H}_0$.

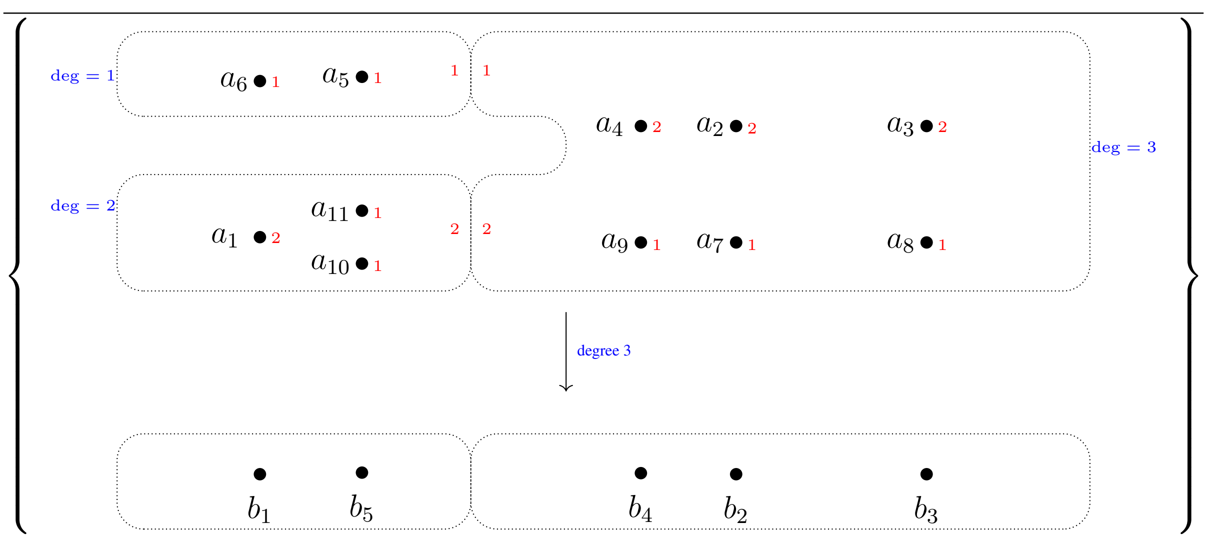


Figure 7.7: The boundary stratum G_7 of $\mathcal{H}_0$.

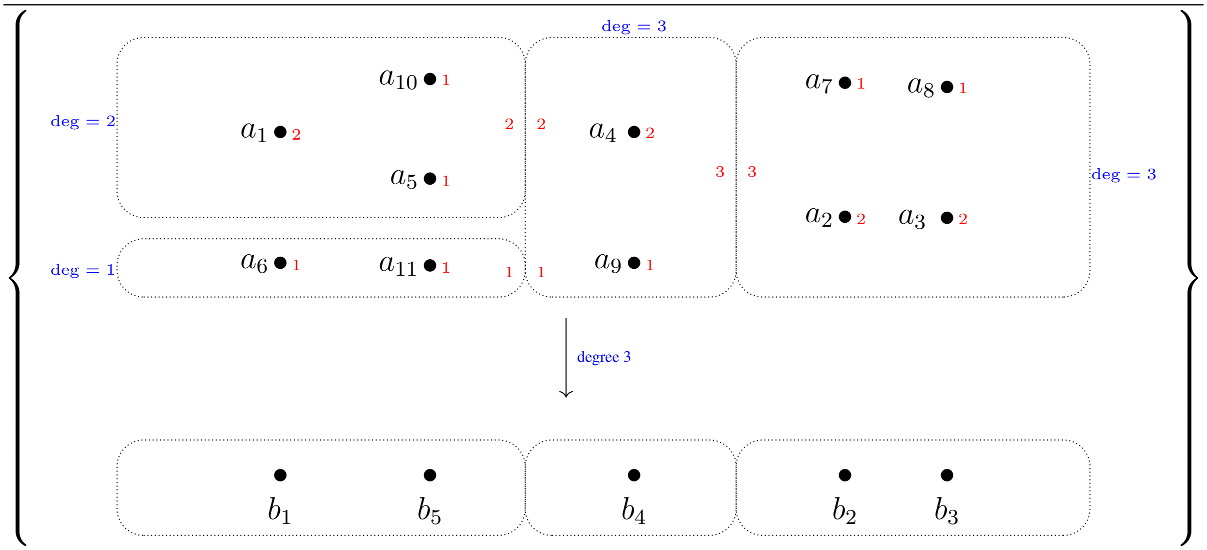


Figure 7.8: The boundary stratum G_8 of $\mathcal{H}_0$.

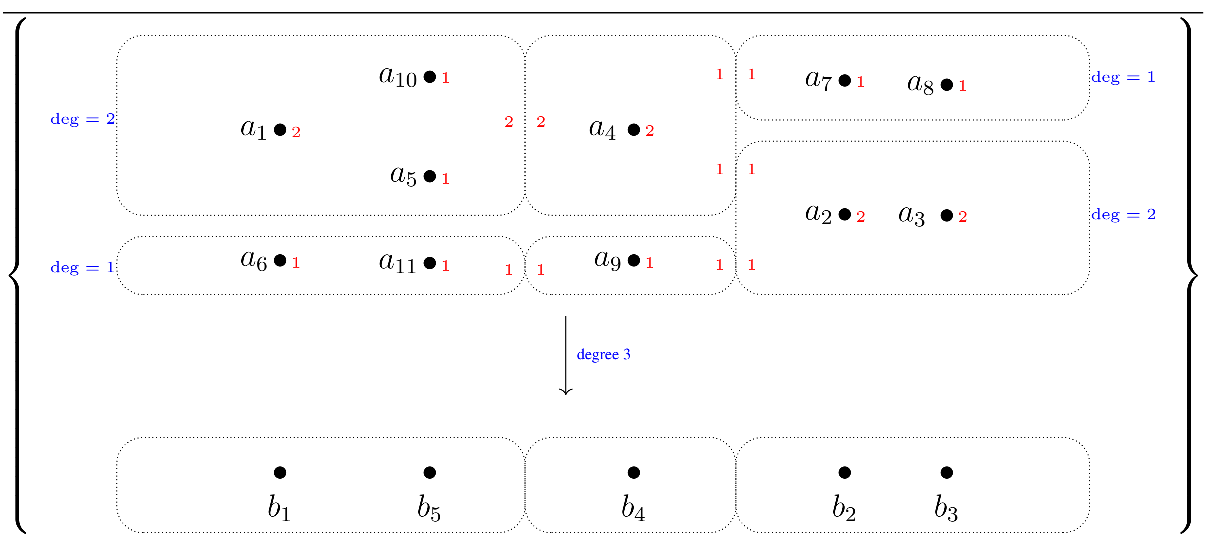


Figure 7.9: The boundary stratum G_9 of $\mathcal{H}_0$.

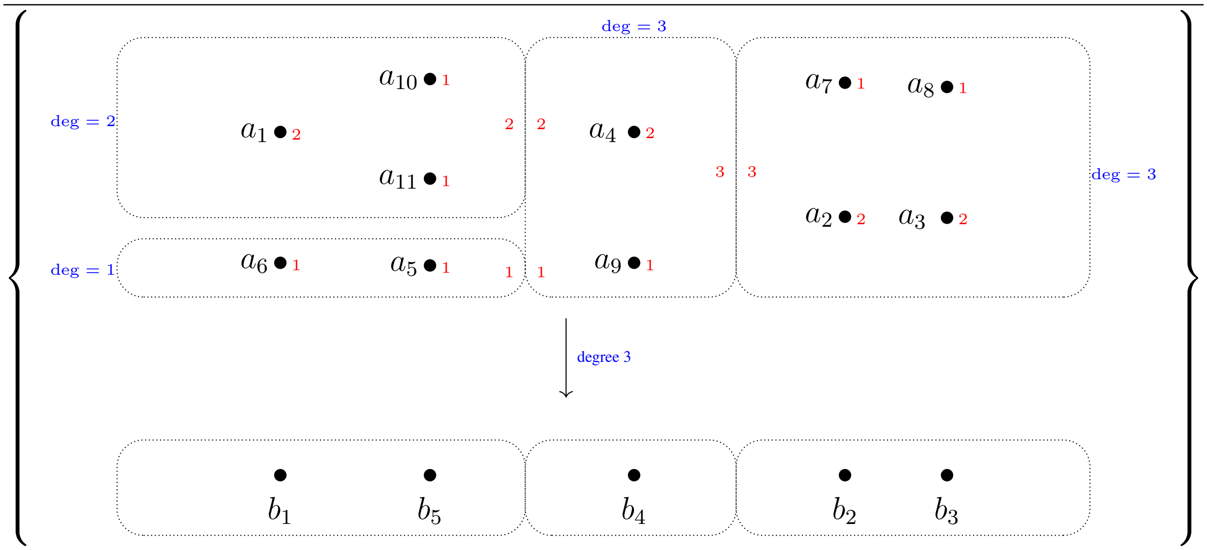


Figure 7.10: The boundary stratum G_{10} of $\mathcal{H}_0$.

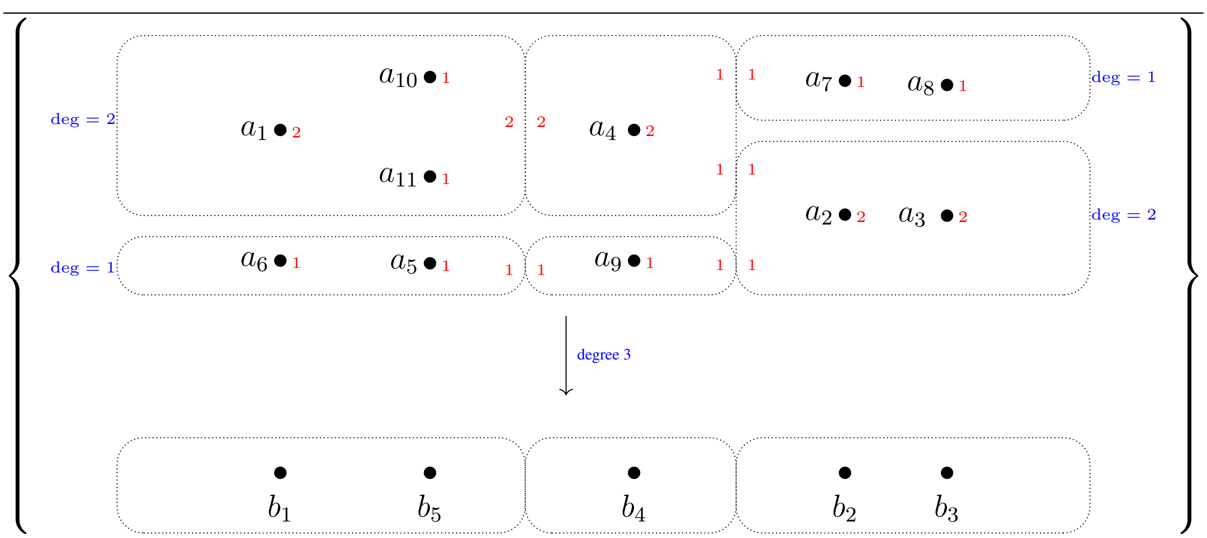


Figure 7.11: The boundary stratum G_{11} of $\mathcal{H}_0$.

the pushforward $[\mathcal{H}_{\text{identity}}]_* : \Omega_{1,\mathbf{P}} \to \Omega_{1,\mathbf{P}}$ is given by the matrix:

$$\begin{pmatrix} 4 & 0 & 0 & -1 & 0 \\ 0 & 4 & 0 & -1 & 0 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 2 & 4 \end{pmatrix} \tag{7.29}$$

As in Section 5.2.5.1, $\Omega_{1,\mathbf{P}}$ has a symmetric basis $Q_1, \ldots Q_5$, where:

$$\mathrm{Div}_{\overline{\mathcal{M}}_{0,\mathbf{P}}}(\{\{p_{i_1}, p_{i_2}\}, \{p_{i_3}, p_{i_4}, p_{i_5}\}\}) = Q_{i_1} + Q_{i_2}.$$

We conjugate the matrix in Equation 7.29 by a change-of-basis matrix to find that in the basis $Q_1, \ldots Q_5$, $[\mathcal{H}_{\text{identity}}]_* : \Omega_{1,\mathbf{P}} \to \Omega_{1,\mathbf{P}}$ is given by the matrix:

$$\begin{pmatrix} 4 & 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & 0 & 1 \\ 0 & 0 & 4 & 0 & 1 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix} \tag{7.30}$$

Any permutation $\alpha \in S_5$ induces an isomorphism $\alpha : \mathcal{M}_{0,\mathbf{P}} \to \mathcal{M}_{0,\mathbf{P}}$ by relabelling the marked point p_i as $p_{\alpha(i)}$. The isomorphism α is also a Hurwitz correspondence, and the pushforward $\alpha_* : \Omega_{1,\mathbf{P}} \to \Omega_{1,\mathbf{P}}$ takes Q_i to $Q_{\alpha(i)}$. The Hurwitz correspondence $\mathcal{H}_\alpha$ factors as $\alpha \circ \mathcal{H}_{\text{identity}}$; by Proposition 3.0.1, We have

$$[\mathcal{H}_\alpha]_* = \alpha_* \circ [\mathcal{H}_{\text{identity}}]_*.$$

Thus to express $[\mathcal{H}_\alpha]_*$ as a matrix using the symmetric basis $Q_1, \ldots, Q_5$, we left-multipy the matrix in 7.30 by the permutation matrix of α. Note that the points $p_1, \ldots, p_4$ play symmetric roles; each of these points has simple branching over it. Thus, if two permutations α and α' are conjugate by a permutation that fixes p_5, then the correspondences $\mathcal{H}_\alpha$ and $\mathcal{H}_{\alpha'}$ are conjugate and have the same dynamical degrees.

We summarize the resulting data in Table 7.1: We give, for several $\alpha \in S_5$,

1. the dynamical portrait of $\mathcal{H}_\alpha$,

2. the characteristic polynomial of $[\mathcal{H}_\alpha]_* : \Omega_{1,\mathbf{P}} \to \Omega_{1,\mathbf{P}}$, factored over $\mathbb{Q}$, and

α	Dynamical portrait	Characteristic polynomial of $[\mathcal{H}_\alpha]_*$	$\Theta(\alpha)$
(1)(2)(3)(4)(5)	$\underset{p_1}{\overset{2}{\circlearrowright}} \quad \underset{p_2}{\overset{2}{\circlearrowright}} \quad \underset{p_3}{\overset{2}{\circlearrowright}} \quad \underset{p_4}{\overset{2}{\circlearrowright}} \quad \underset{p_5}{\overset{1}{\circlearrowright}}$	$-(x-6)(x-4)^4$	6
(12345)	$\begin{tikzcd} p_1 \arrow[r, "1"] & p_5 \arrow[r, "2"] & p_4 \arrow[r, "2"] & p_3 \arrow[r, "2"] & p_2 \arrow[llll, bend right, "2"'] \end{tikzcd}$	$-x^5+x^4+4x^3$ $+16x^2+64x+1536$	≈ 5.115
(12)(345)	$\begin{tikzcd} p_1 \arrow[r, "2"'] & p_2 \arrow[l, bend right, "2"'] \end{tikzcd} \quad \begin{tikzcd} p_3 \arrow[r, "1"] & p_5 \arrow[r, "2"] & p_4 \arrow[ll, bend right, "2"'] \end{tikzcd}$	$-(x-4)(x+4)$ $\cdot(x^3-x^2-4x-96)$	≈ 5.248
(12)(35)(4)	$\begin{tikzcd} p_1 \arrow[r, "2"'] & p_2 \arrow[l, bend right, "2"'] \end{tikzcd} \quad \begin{tikzcd} p_3 \arrow[r, "1"'] & p_5 \arrow[l, bend right, "2"'] \end{tikzcd} \quad \underset{p_4}{\overset{2}{\circlearrowright}}$	$-(x-4)^2(x+4)$ $\cdot(x^2-x-24)$	≈ 5.424
(123)(45)	$\begin{tikzcd} p_1 \arrow[r, "2"] & p_3 \arrow[r, "2"] & p_2 \arrow[ll, bend right, "2"'] \end{tikzcd} \quad \begin{tikzcd} p_4 \arrow[r, "1"'] & p_5 \arrow[l, bend right, "2"'] \end{tikzcd}$	$-(x-4)(x^2-x-24)$ $\cdot(x^2+4x+16)$	≈ 5.424
(1)(2)(345)	$\underset{p_1}{\overset{2}{\circlearrowright}} \quad \underset{p_2}{\overset{2}{\circlearrowright}} \quad \begin{tikzcd} p_3 \arrow[r, "1"] & p_5 \arrow[r, "2"] & p_4 \arrow[ll, bend right, "2"'] \end{tikzcd}$	$-(x-4)^2(x^3-x^2-4x-96)$	≈ 5.248
(1)(2)(3)(45)	$\underset{p_1}{\overset{2}{\circlearrowright}} \quad \underset{p_2}{\overset{2}{\circlearrowright}} \quad \underset{p_3}{\overset{2}{\circlearrowright}} \quad \begin{tikzcd} p_4 \arrow[r, "1"'] & p_5 \arrow[l, bend right, "2"'] \end{tikzcd}$	$-(x-4)^3(x^2-x-24)$	≈ 5.424

Table 7.1: Computations in $\mathcal{M}_{0,5}$

3. the first dynamical degree $\Theta(\alpha)$ of $\mathcal{H}_\alpha$.

7.3 Extrapolating from computations on $\mathcal{M}_{0,5}$ to obtain results on $\mathcal{M}_{0,n}$

In this section, we compute the dynamical degrees of Hurwitz self-correspondences

$$\mathcal{H}_\alpha : \mathcal{M}_{0,\mathbf{P}(n)} \rightrightarrows \mathcal{M}_{0,\mathbf{P}(n)}$$

indexed by permutations $\alpha \in S_n$, where:

- $n \geq 5$

- $\mathbf{P}(n) = \{p_1, \ldots, p_n\}$,
- $\mathcal{H}_\alpha = \mathcal{H}(\mathbf{P}(n), \mathbf{P}(n), d_0, F_\alpha, \mathrm{br}_0, \mathrm{rm}_\alpha)$,
- $d_0 = 3$,
- $F_\alpha(p_i) = p_{\alpha^{-1}(i)}$,
- $\mathrm{br}_0(p_i) = (2, 1)$ for $i = 1, \ldots, 4$ and $\mathrm{br}_0(p_i) = (1, 1, 1)$ for $i = 5, \ldots, n$, and
- $\mathrm{rm}_\alpha(p_{\alpha(i)}) = 2$ for $i = 1, \ldots, 4$ and $\mathrm{rm}_\alpha(p_{\alpha(i)}) = 1$ for $i = 5, \ldots, n$.

In the case that $n = 5$, these Hurwitz correspondences specialize to the correspondences in Section 7.2. As in Section 7.2, the Hurwitz space $\mathcal{H}_\alpha$ parametrizes degree 3 maps from one $\mathbf{P}(n)$-marked $\mathbb{P}^1$ to another, with simple branching over $p_1, \ldots, p_4$, and such that $p_i \mapsto p_{\alpha^{-1}(i)}$. Here, the "new" points p_i for $i > 5$ have simple branching over them, as does the "old" point p_5. As in Section 7.2, we first treat the case where α is the identity permutation. Denote by $\mathrm{identity}(n)$ the identity element of S_n.

The following lemma and its corollary allow us to reduce to the computation in Section 7.2.

Lemma 7.3.1. *Let $\mathcal{H} = \mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ be a Hurwitz space, with $\pi_{\mathbf{A}}$ and $\pi_{\mathbf{B}}$ its maps to $\mathcal{M}_{0,\mathbf{A}}$ and $\mathcal{M}_{0,\mathbf{B}}$. Given*

- $\mathbf{A}^{\mathrm{aug}} \supset \mathbf{A}$,
- $\mathbf{B}^{\mathrm{aug}} \supset \mathbf{B}$*, both finite sets, and*
- $F : \mathbf{A}^{\mathrm{aug}} \to \mathbf{B}^{\mathrm{aug}}$ *extending* $F : \mathbf{A} \to \mathbf{B}$ *and a bijection from* $\mathbf{A}^{\mathrm{aug}} \smallsetminus \mathbf{A}$ *to* $\mathbf{B}^{\mathrm{aug}} \smallsetminus \mathbf{B}$.

Set $N = |\mathbf{A}^{\mathrm{aug}} \smallsetminus \mathbf{A}| = |\mathbf{B}^{\mathrm{aug}} \smallsetminus \mathbf{B}|$. We may extend br to $\mathbf{B}^{\mathrm{aug}}$ by setting $\mathrm{br}(b) = (1, \ldots, 1)$ for all $b \in \mathbf{B}^{\mathrm{aug}} \smallsetminus \mathbf{B}$, and extend rm to $\mathbf{A}^{\mathrm{aug}}$ by setting $\mathrm{rm}(a) = 1$ for all $a \in \mathbf{A}^{\mathrm{aug}} \smallsetminus \mathbf{A}$. Set $\mathcal{H}^{\mathrm{aug}} = \mathcal{H}(\mathbf{A}^{\mathrm{aug}}, \mathbf{B}^{\mathrm{aug}}, d, F, \mathrm{br}, \mathrm{rm})$, with $\pi_{\mathbf{A}^{\mathrm{aug}}}$ and $\pi_{\mathbf{B}^{\mathrm{aug}}}$ its maps to $\mathcal{M}_{0,\mathbf{A}^{\mathrm{aug}}}$ and $\mathcal{M}_{0,\mathbf{B}^{\mathrm{aug}}}$. Denote by $\mu_{\mathbf{A}}$ the forgetful map from $\mathcal{M}_{0,\mathbf{A}^{\mathrm{aug}}}$ to $\mathcal{M}_{0,\mathbf{A}}$, and denote by $\mu_{\mathbf{B}}$ the forgetful map from $\mathcal{M}_{0,\mathbf{B}^{\mathrm{aug}}}$ to $\mathcal{M}_{0,\mathbf{B}}$. Then we have:

1. $[\mathcal{H}^{\mathrm{aug}}]_* \circ \mu_{\mathbf{B}}^* = \mu_{\mathbf{A}}^* \circ [\mathcal{H}]_* : H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{B}}, \mathbb{Z}) \to H_{2k+2N}(\overline{\mathcal{M}}_{0,\mathbf{A}^{\mathrm{aug}}}, \mathbb{Z})$.
2. $\mu_{\mathbf{A}*} \circ [\mathcal{H}^{\mathrm{aug}}]_* = d^N([\mathcal{H}]_* \circ \mu_{\mathbf{B}*}) : H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{B}^{\mathrm{aug}}}, \mathbb{Z}) \to H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{A}}, \mathbb{Z})$.

Proof. Denote by $\overline{\mathcal{H}}$ and $\overline{\mathcal{H}}^{\mathrm{aug}}$ the admissible covers compactifications of $\mathcal{H}$ and $\mathcal{H}^{\mathrm{aug}}$ respectively. There is a forgetful map $\nu : \overline{\mathcal{H}}^{\mathrm{aug}} \to \overline{\mathcal{H}}$. Denote by FP the fibered product

$$\overline{\mathcal{M}}_{0,\mathbf{B}^{\mathrm{aug}}} \ {}_{\mu_{\mathbf{B}}}\!\times_{\overline{\pi_{\mathbf{B}}}} \overline{\mathcal{H}}$$

with its two projections pr_1 and pr_2 to $\overline{\mathcal{M}}_{0,\mathbf{B}^{\mathrm{aug}}}$ and $\overline{\mathcal{H}}$ respectively. Denote by FP' the fibered product

$$\overline{\mathcal{M}}_{0,\mathbf{A}^{\mathrm{aug}}} \ {}_{\mu_{\mathbf{A}}}\!\times_{\overline{\pi_{\mathbf{A}}}} \overline{\mathcal{H}}$$

with its two projections pr_1' and pr_2' to $\overline{\mathcal{M}}_{0,\mathbf{A}^{\mathrm{aug}}}$ and $\overline{\mathcal{H}}$ respectively. There are induced maps q and q' from $\overline{\mathcal{H}}^{\mathrm{aug}}$ to FP and FP' respectively. We have a commutative diagram:

$$\begin{tikzcd}
 & & \overline{\mathcal{H}}^{\mathrm{aug}} \arrow[dl, "q"'] \arrow[dr, "q'"] \arrow[ddd, "\nu"] \arrow[ddll, bend right, "\overline{\pi_{\mathbf{B}^{\mathrm{aug}}}}"'] \arrow[ddrr, bend left, "\overline{\pi_{\mathbf{A}^{\mathrm{aug}}}}"] & & \\
 & FP \arrow[dl, "pr_1"] \arrow[ddr, "pr_2"'] & & FP' \arrow[dr, "pr_1'"'] \arrow[ddl, "pr_2'"] & \\
\overline{\mathcal{M}}_{0,\mathbf{B}^{\mathrm{aug}}} \arrow[dd, "\mu_{\mathbf{B}}"'] & & & & \overline{\mathcal{M}}_{0,\mathbf{A}^{\mathrm{aug}}} \arrow[dd, "\mu_{\mathbf{A}}"] \\
 & & \overline{\mathcal{H}} \arrow[dll, "\overline{\pi_{\mathbf{B}}}"'] \arrow[drr, "\overline{\pi_{\mathbf{A}}}"] & & \\
\overline{\mathcal{M}}_{0,\mathbf{B}} & & & & \overline{\mathcal{M}}_{0,\mathbf{A}}
\end{tikzcd}$$

Claim 1: *The map q' is generically* $1:1$.

Proof of Claim 1: A point in FP' consists of the data of $[f : C \to D] \in \overline{\mathcal{H}}$ where C and D are, respectively, $\mathbf{A}$- and $\mathbf{B}$- marked curves, together with an injection from $(\mathbf{A}^{\mathrm{aug}} \smallsetminus \mathbf{A})$ into the smooth locus of $C \smallsetminus \mathbf{A}$. For a general such point x' in FP', $f(\mathbf{A}^{\mathrm{aug}} \smallsetminus \mathbf{A}) \in D$ is disjoint from the marked points $\mathbf{B}$. To obtain a point lying over x' in $\overline{\mathcal{H}}^{\mathrm{aug}}$, we mark, for every $a \in (\mathbf{A}^{\mathrm{aug}} \smallsetminus \mathbf{A})$, the point $f(a) \in D$ by $F(a) \in \mathbf{B}^{\mathrm{aug}} \smallsetminus \mathbf{B}$. The claim follows since there is a unique way to do this.

Now, we have:

$$[\mathcal{H}^{\mathrm{aug}}]_* \circ \mu_{\mathbf{B}}^* = \overline{\pi_{\mathbf{A}^{\mathrm{aug}}}}_* \circ \overline{\pi_{\mathbf{B}^{\mathrm{aug}}}}^* \circ \mu_{\mathbf{B}}^* \tag{7.31}$$
$$= (pr_1')_* \circ (q')_* \circ (q')^* \circ (pr_2')^* \circ \overline{\pi_{\mathbf{B}}}^* \tag{7.32}$$
$$= (pr_1')_* \circ (pr_2')^* \circ \overline{\pi_{\mathbf{B}}}^* \tag{7.33}$$
$$= \mu_{\mathbf{A}}^* \circ \overline{\pi_{\mathbf{A}}}_* \circ \overline{\pi_{\mathbf{B}}}^* \tag{7.34}$$
$$= \mu_{\mathbf{A}}^* \circ [\mathcal{H}]_* : H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{B}}, \mathbb{Z}) \to H_{2k+2N}(\overline{\mathcal{M}}_{0,\mathbf{A}^{\mathrm{aug}}}, \mathbb{Z}). \tag{7.35}$$

Here, the equality in 7.32 follows from commutativity of the above diagram, the equality in 7.33 follows from Claim 1 and the projection formula, and the equality in 7.34 follows from the facts that the forgetful map $\mu_{\mathbf{A}}$ is flat and that (Proposition 1.7 in [Ful98]) proper pushforward commutes with flat pullback in a fibered square. This proves statement 1 of the lemma.

Claim 2: *The map q is generically $(d^N) : 1$.*

Proof of Claim 2: A point in FP consists of the data of $[f : C \to D] \in \overline{\mathcal{H}}$ where C and D are, respectively, $\mathbf{A}$- and $\mathbf{B}$- marked curves, together with an injection from $(\mathbf{B}^{\mathrm{aug}} \smallsetminus \mathbf{B})$ into the smooth locus of $D \smallsetminus \mathbf{B}$. For a general such point x in FP, $f^{-1}(\mathbf{B}^{\mathrm{aug}} \smallsetminus \mathbf{B}) \in C$ is disjoint from the marked points $\mathbf{A}$. The fiber over x in $\overline{\mathcal{H}}^{\mathrm{aug}}$ consists of ways of choosing, for every $b \in (\mathbf{B}^{\mathrm{aug}} \smallsetminus \mathbf{B})$, exactly one of the d points in $f^{-1}(b) \in C$ to mark by $F^{-1}(b) \in \mathbf{A}^{\mathrm{aug}} \smallsetminus \mathbf{A}$. The claim follows since there are N such choices to be made.

Now, we have:

$$\mu_{\mathbf{A}*} \circ [\mathcal{H}^{\mathrm{aug}}]_* = \mu_{\mathbf{A}*} \circ \overline{\pi_{\mathbf{A}^{\mathrm{aug}}}}_* \circ \overline{\pi_{\mathbf{B}^{\mathrm{aug}}}}^* \tag{7.36}$$
$$= \overline{\pi_{\mathbf{A}}}_* \circ pr_{2*} \circ q_* \circ q^* \circ pr_1^* \tag{7.37}$$
$$= (d^N)(\overline{\pi_{\mathbf{A}}}_* \circ pr_{2*} \circ pr_1^*) \tag{7.38}$$
$$= (d^N)(\overline{\pi_{\mathbf{A}}}_* \circ \overline{\pi_{\mathbf{B}}}^* \circ \mu_{\mathbf{B}*}) \tag{7.39}$$
$$= (d^N)([\mathcal{H}]_* \circ \mu_{\mathbf{B}*}) : H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{B}^{\mathrm{aug}}}, \mathbb{Z}) \to H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{A}}, \mathbb{Z}). \tag{7.40}$$

Here, the equality in 7.37 follows from commutativity of the above diagram, the equality in 7.38 follows from Claim 2 and the projection formula, and the equality in 7.39 follows from the facts that $\overline{\pi_{\mathbf{B}}}$ is flat and that (Proposition 1.7 in [Ful98]) proper pushforward commutes with flat pullback in a fibered square. This proves statement 2 of the lemma.

□

As in Section 5.2, Any forgetful map $\mu : \mathcal{M}_{0,n} \to \mathcal{M}_{0,n+N}$ induces pullback and pushforward maps $\mu^* : \Omega_{k,n} \to \Omega_{k+N,n+N}$ and $\mu_* : \Omega_{k,n} \to \Omega_{k,n+N}$. We obtain as a consequence of Lemma 7.3.1:

Corollary 7.3.2. *Let* $\mathcal{H} = \mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ *and* $\mathcal{H}^{\mathrm{aug}} = \mathcal{H}(\mathbf{A}^{\mathrm{aug}}, \mathbf{B}^{\mathrm{aug}}, d, F, \mathrm{br}, \mathrm{rm})$ *be as in the statement of Lemma 7.3.1. Then:*

1. $[\mathcal{H}^{\mathrm{aug}}]_* \circ \mu^*_{\mathbf{B}} = \mu^*_{\mathbf{A}} \circ [\mathcal{H}]_* : \Omega_{k,\mathbf{B}} \to \Omega_{k+N,\mathbf{A}^{\mathrm{aug}}}$.
2. $\mu_{\mathbf{A}*} \circ [\mathcal{H}^{\mathrm{aug}}]_* = d^N([\mathcal{H}]_* \circ \mu_{\mathbf{B}*}) : \Omega_{k,\mathbf{B}^{\mathrm{aug}}} \to \Omega_{k,\mathbf{A}}$.

Now we have, for $n > n'$, a forgetful map $\mu(n, n') : \overline{\mathcal{M}}_{0,\mathbf{P}(n)} \to \overline{\mathcal{M}}_{0,\mathbf{P}(n')}$ induced by the injection $\{p_1, \ldots, p_{n'}\} \hookrightarrow \{p_1, \ldots, p_n\}$. This induces a pullback $\mu(n, n')^* : \Omega_{k,\mathbf{P}(n')} \to \Omega_{k,\mathbf{P}(n)}$. As mentioned in Section 5.2.5.1, $\Omega_{n-4,\mathbf{P}(n)}$ has an S_n-equivariant basis $Q_1(n), \ldots, Q_n(n)$. In fact, by Lemma 5.2.10 the elements of this basis are stable under pullback, that is $\mu(n, n')^*(Q_i(n')) = Q_i(n)$ for $i \le n'$. Thus, we set Q_i to be $Q_i(n)$ for all $n \ge i$. By Corollary 7.3.2, 1, we have:

$$[\mathcal{H}_{\mathrm{identity}(n)}]_* \circ \mu(n, 5)^* = \mu(n, 5)^* \circ [\mathcal{H}_{\mathrm{identity}(5)}]_* : \Omega_{1,\mathbf{P}(5)} \to \Omega_{n-4,\mathbf{P}(n)}.$$

By Equation 7.30, $[\mathcal{H}_{\mathrm{identity}(5)}]_*(Q_i) = 4Q_i$ for $i = 1, \ldots, 4$, and $[\mathcal{H}_{\mathrm{identity}(5)}]_*(Q_5) = 6Q_5 + Q_1 + Q_2 + Q_3 + Q_4$. Thus, for all $n \ge 5$, $[\mathcal{H}_{\mathrm{identity}(n)}]_*(Q_i) = 4Q_i$ for $i = 1, \ldots, 4$, and $[\mathcal{H}_{\mathrm{identity}(n)}]_*(Q_5) = 6Q_5 + Q_1 + Q_2 + Q_3 + Q_4$. By symmetry of the points $p_5, \ldots, p_n$, we have $[\mathcal{H}_{\mathrm{identity}(n)}]_*(Q_i) = 6Q_i + Q_1 + Q_2 + Q_3 + Q_4$ for $i = 5, \ldots, n$. Thus, in the basis $Q_1, \ldots, Q_n$, the matrix for $[\mathcal{H}_{\mathrm{identity}(n)}]_*$ is the $n \times n$ matrix:

$$\begin{pmatrix} 4 & 0 & 0 & 0 & 1 & 1 & \ldots & 1 & 1 \\ 0 & 4 & 0 & 0 & 1 & 1 & \ldots & 1 & 1 \\ 0 & 0 & 4 & 0 & 1 & 1 & \ldots & 1 & 1 \\ 0 & 0 & 0 & 4 & 1 & 1 & \ldots & 1 & 1 \\ 0 & 0 & 0 & 0 & 6 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 6 \end{pmatrix} \tag{7.41}$$

Any permutation $\alpha \in S_n$ induces an isomorphism $\alpha : \mathcal{M}_{0,\mathbf{P}(n)} \to \mathcal{M}_{0,\mathbf{P}(n)}$ by relabelling the marked point p_i by $p_{\alpha(i)}$. As in Section 7.2, The Hurwitz correspondence $\mathcal{H}_\alpha$ factors as $\alpha \circ \mathcal{H}_{\mathrm{identity}(n)}$. By Proposition 3.0.1, the pushforward $[\mathcal{H}_\alpha]_*$ factors as $\alpha_* \circ [\mathcal{H}_{\mathrm{identity}(n)}]_*$, both on $H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}(n)}, \mathbb{Z})$ and on $\Omega_{k,\mathbf{P}(n)}$. As in Section 7.2, $\alpha_* : \Omega_{n-4,\mathbf{P}(n)} \to \Omega_{n-4,\mathbf{P}(n)}$ takes Q_i to $Q_{\alpha(i)}$. Thus the matrix of $[\mathcal{H}_\alpha]_* : \Omega_{n-4,\mathbf{P}(n)} \to \Omega_{n-4,\mathbf{P}(n)}$, in terms of the basis $Q_1, \ldots, Q_n$, is obtained by left-multiplying the matrix in 7.41 by the permutation matrix of α.

For $n' \leq n$, there is an inclusion $\iota_{n,n'} : S_{n'} \hookrightarrow S_n$:

$$\iota_{n,n'}(\alpha')(i) = \begin{cases} \alpha'(i) & i \leq n' \\ i & i > n' \end{cases}$$

It follows from the block decomposition of the matrix in 7.41 that for $\alpha' \in S_{n'}$:

$$\begin{aligned} &\{\text{eigenvalues of } [\mathcal{H}_{\iota_{n,n'}(\alpha')}]_* : \Omega_{n-4,\mathbf{P}(n)} \to \Omega_{n-4,\mathbf{P}(n)}\} \\ &\quad = \{\text{eigenvalues of } [\mathcal{H}_{\alpha'}]_* : \Omega_{n'-4,\mathbf{P}(n')} \to \Omega_{n'-4,\mathbf{P}(n')}\} \cup \{6\}. \end{aligned}$$

It similarly follows that for $\alpha' \in S_{n'}$ and $\alpha \in S_n$ such that α fixes every $i \leq n'$, we have:

$$\left(\begin{aligned} &\{\text{eigenvalues of } [\mathcal{H}_{\alpha \circ (\iota_{n,n'}(\alpha'))}]_* : \Omega_{n-4,\mathbf{P}(n)} \to \Omega_{n-4,\mathbf{P}(n)}\} \\ &\qquad \setminus \{\text{eigenvalues of } [\mathcal{H}_{\alpha'}]_* : \Omega_{n'-4,\mathbf{P}(n')} \to \Omega_{n'-4,\mathbf{P}(n')}\} \end{aligned} \right) \subseteq \{6 \cdot (\text{root of unity})\}$$

For $\alpha \in S_n$ denote by $\Theta(\alpha)$ the $(n-4)$-th dynamical degree of $\mathcal{H}_\alpha$, that is, the dominant eigenvalue of $[\mathcal{H}_\alpha]_* : \Omega_{n-4,\mathbf{P}(n)} \to \Omega_{n-4,\mathbf{P}(n)}$. We can now compute $\Theta(\alpha)$ for any given α: which numbers arise? Note that by 7.41, $\Theta(\text{identity}(n)) = 6$. In general, $\Theta(\alpha)$ is an algebraic integer whose Galois conjugates are eigenvalues of $[\mathcal{H}_\alpha]_* : \Omega_{n-4,\mathbf{P}(n)} \to \Omega_{n-4,\mathbf{P}(n)}$. In Figure 7.12, we plot the numbers:

$$\begin{aligned} &\{\text{eigenvalues of } [\mathcal{H}_\alpha]_* : \Omega_{n-4,\mathbf{P}(n)} \to \Omega_{n-4,\mathbf{P}(n)}\}_{\substack{\alpha \in S_n \\ n \leq 100}} \\ &\qquad \setminus \{6 \cdot (\text{root of unity})\} \end{aligned} \tag{7.42}$$

as points in the complex plane. Numbers that arise as $\Theta(\alpha)$ for some α are shown in red; all other eigenvalues are shown in blue. The numbers plotted in blue include the Galois conjugates of every $\Theta(\alpha) \neq 6$; we suspect that "most" blue numbers are Galois conjugates of some $\Theta(\alpha)$.

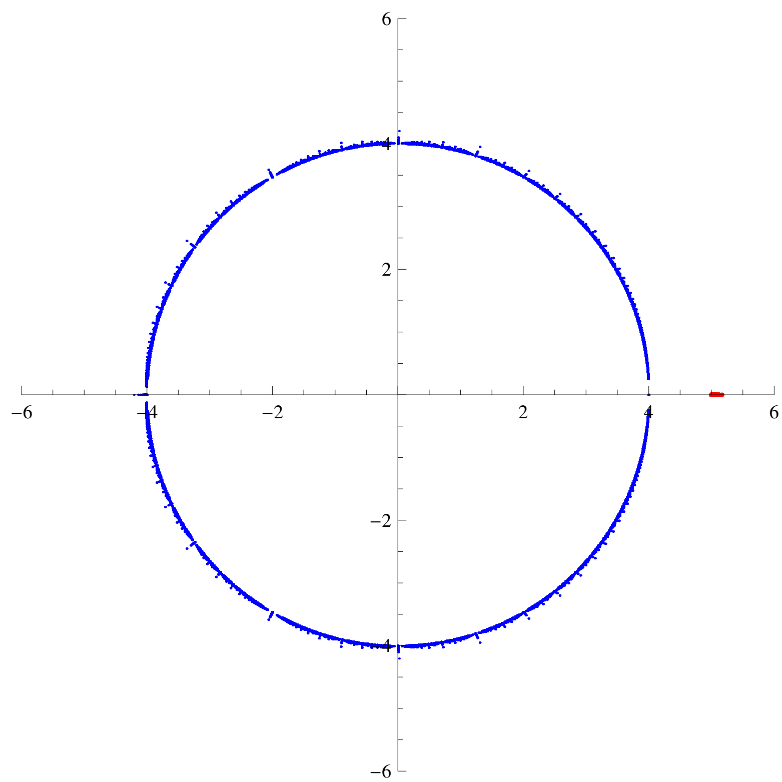


Figure 7.12: Eigenvalues of $[\mathcal{H}_\alpha]_* : \Omega_{n-4,\mathbf{P}(n)} \to \Omega_{n-4,\mathbf{P}(n)}$ for $\alpha \in S_n; n \leq 100$

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