Game-Theoretic Approach for Modeling Market Microstructure

by

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ABSTRACT

This thesis is devoted to investigating possible approaches to endogenous modeling of market microstructure of an auction-based exchange. In chapter II we develop the framework in discrete time and apply it to understanding the economics of trading at high frequency. In chapter III we adapt our modeling approach to continuous time and develop a rich beliefs-driven model of limit-order book evolution between two trades. In the last chapter we introduce discrete admissible prices (i.e. a finite tick size) into our model and investigate the special spatial structures of the equilibria this produces.

Given the novelty of the approach, we have to solve somewhat unusual mathematical problems throughout. We derive a novel estimate of conditional tails of general Ito processes in chapter II, solve a 'non-monotone oblique reflection' RBSDE system and a discontinuous infinite-dimensional fixed point problem in chapter III, and solve a system of control-stopping problems discontinuously coupled through stopping barriers in chapter IV.

We also develop some numerical examples in chapters II, III to illustrate the features of our models and indicate possible applications.

CHAPTER I

Introduction

This thesis is devoted to investigating possible approaches to *endogenous* modeling of market microstructure, that is, dynamics of orders, trades, prices etc., on an auction-based exchange. We refer the reader to the introductions of the individual chapters for a more detailed discussion of the promise and advantages of such models. Our aim is to develop endogenous models which also have *realistic mechanics*, capturing a possibly simplified but sensible view of the actual exchanges' mechanism. The disadvantage of this approach is that the resulting models are rather complex and hard to analyze. We hope to convince the reader that hard in this case does not mean impossible, and demonstrate that the resulting models are rich and interesting and can shed light on various subtle microstructure issues.

This work is comprised of three somewhat independent projects, which are nevertheless closely linked by the commonalities in the framework, modeling approach and sometimes even mathematical issues arising. Below, we first describe how these projects contribute to the development of our microstructure modeling framework and what microstructure issues they deal with. We then describe mathematical challenges and contributions of each project.

In the first project, described in chapter II, we develop a discrete-time continuous-price model of a limit-order book formation at an auction-style exchange, in which a continuum of players can submit both market and limit orders at any prices based on their beliefs about the future order flow. After setting up and motivating the model, we proceed to analyze how the market behavior, in particular the liquidity provision, changes with increasing frequency of trading, arriving at the characterization of efficiency-fragility role of high trading frequency. The results are valid for a very general class of models, and thus should be viewed as concerning the fundamental economics of trading on auction-style exchanges rather than specifics of a particular model.

In the second project, described in chapter III, we focus on the market participants' behavior between two consecutive trades. We extend the modeling framework to continuous-time, and then investigate the trade-off between market and limit orders faced by the agents. The resulting "microscopic" model, in equilibrium, can produce rich dynamics of posted prices and limit-order book shapes. This model allows us to investigate the impact of agents' beliefs (or changes in the signals that affect their beliefs) on the bid-ask spread and on the limit-order book evolution, allowing one to model such beliefs-driven phenomena as for example indirect market impact of both limit and market orders (e.g. the so-called "spoofing" effect).

In the last project, described in chapter IV, we consider a continuous-time and discrete-price framework, by introducing a non-zero tick size. This allows one to explore the interplay between the continuous "true price" estimate and the discrete admissible order price levels, resulting in different behavior depending on whether the "true price" estimate is close to admissible price level or not. Resulting models can be used to explain the clustering of market orders over time and to predict the consequences of changing the tick size.

On the mathematical side, the first project stands somewhat separate from the other two. The main issues there arise from dealing with unusual functionals of general Itô processes under multiple measures, leading us to develop some novel estimates of conditional tails of general Itô processes.

The other two projects are similar in that they deal with (continuum- or two-player) games, coupled through controls and stopping barriers in a way that lacks any of the traditional monotonicity, convexity, contractivity or sometimes even continuity properties, which makes them hard to solve.

In the second project, the difficulties arise from the fact that there are multiple (>2) players whose stopping barriers are given by functionals of all other players' value functions, making the players' interdependence rather discontinuous. Such continuum-player problem appears intractable in general, but under certain monotonicity assumptions on agents' beliefs we manage to split the problem into a 2-agent control-stopping game and an infinite-dimensional fixed point problem. This 2-player game we are then able to reduce to a somewhat unusual 2-dimensional system of RBSDEs with solutions reflected against each other. This system is still not amenable to standard methods, but we manage to solve it by exploiting certain geometric properties of its generator and the specific nature of the reflection. The infinite-dimensional fixed point problem above is also not trivial as its objective is discontinuous. We solve it by exploiting the structure of the problem allowing us to show we can replace that objective by its 'mollification' to which standard fixed-point results can then be applied.

The main challenge of the third project is solving a system of two (control-stopping) optimization problems coupled through controls and barriers which are discontinuous functionals of other agent's value function. Because of the presence of those discontinuous functionals, the fixed point problem corresponding to this system is in a sense even more discontinuous than the one from the second project and doesn't appear tractable. This fixed point problem turns out to be continuous if we can ensure certain monotonicity of individual agents' value functions. We show this monotonicity in the case of a sufficient noise using the recent geometric approach to the optimal stopping of linear diffusions (with irregular barriers) combined with certain special features of the problem.

CHAPTER II

Liquidity Effects of Trading Frequency

2.1 Introduction

This chapter is concerned with liquidity effects of trading frequency on an auction-style exchange, in which the participating agents can post limit or market orders. On the one hand, higher trading frequency provides more opportunities for the market participants to trade, hence, improving the liquidity of the market and increasing the market efficiency. On the other hand, higher trading frequency also provides more opportunities for some participants to manipulate the price and disrupt the market liquidity. Such a manipulation creates a new type of risk, which reveals itself in unusually high price deviations, which cannot be explained by the changes in the present, or projected, fundamental value of the asset. The most famous example of this phenomenon is the "flash crash" of 2010. This example motivates the need for a comprehensive study of the tradeoff between the liquidity providing role of strategic players and the liquidity risk they generate, and its relation to trading frequency. The collective liquidity of the market is captured by the *Limit Order Book (LOB)*, which contains all the limit buy and sell orders.

The goal of the present chapter is two-fold. First, we develop a new framework for modeling market microstructure, in which the shape of the LOB, and its dynamics, arise *endogenously* from the interactions between the agents. Among the many advantages of such approach is the possibility of modeling the market reaction to changes in the rules of the exchange: e.g. limited trading frequency, transaction tax, etc. The second, and most important, goal of the present work is to investigate the *liquidity effects* of *trading frequency*, using the proposed modeling framework. In particular, the main results of this chapter (cf. the discussion in Section 2.3, as well as Theorems II.19, II.21 and Corollary II.20, in Section 3.4) describe the dual effect of high trading frequency. On the one hand, if the agents choose to provide liquidity in equilibrium, higher trading frequency decreases the bid-ask spread and makes the expected profits of all market participants converge to the same (fundamental) value, thus, improving the market efficiency. On the other hand, higher trading frequency also makes the LOB more sensitive to the deviations of the agents from market-neutrality. It is, of course, clear that a strong bullish or bearish signal makes the market participants trade at a higher or lower price. However, the novelty of our observation is in the role that the trading frequency plays in amplifying this effect. Namely, we show that, if the trading frequency is high, even if the agents have plenty of inventory, a very small deviation from market-neutrality may cause the agents to stop providing liquidity, by either withdrawing from the market completely, or by posting the limit orders very far away from the fundamental price. Such actions cause disproportional deviations in the LOB, which cannot be explained by any fundamental reasons: they are much higher than the trading signal (i.e. the expected change in the fundamental price), and they occur without any shortage of supply or demand for the asset. We refer to such a deviation as an *internal* (or, *self-inflicted*) liquidity crisis, because it is due to the trading mechanism (i.e. the rules by which the market participants interact), rather than any fundamental reasons (note the similarity with the flash crash). Our framework allows us to provide more insight into how such liquidity crisis unfolds, connecting it to the so-called *adverse selection* effect. In particular, Section 2.3 constructs an equilibrium model in which an internal liquidity crisis does not occur because of an abnormally large market order, wiping out the liquidity on one side of the LOB, but it occurs because the optimal strategies of the agents require them to stop providing liquidity on one side of the LOB. On the mathematical side, our analysis makes use of the properties of conditional tails of the increments of a general Itô process, with the corresponding result stated in Lemma II.23. This lemma provides a uniform exponential bound on the conditional tails of the increments of a general Itô process. We believe that this result is useful in its own right, and, to the best of our knowledge, it is not available in existing literature.

In recent years, we observed an explosion in the amount of literature devoted to the study of market microstructure. In addition to various empirical studies, a large part of the existing theoretical work focuses on the problem of optimal execution: see, among others, [48], [3], [54], [31], [22], [6], [5],[7], [24], [51], [36], [19], [37], [57], and references therein. In these publications, the dynamics and shape of the LOB are modeled exogenously, or, equivalently, the arrival processes of the limit and market orders are specified exogenously. In particular, none of these works attempt to explain the shape and dynamics of the LOB, arising directly from the interaction between the market participants. A different approach to the analysis of market microstructure has its roots in the economic literature. For example, [50], [29], [34], [17], [44], [52], [27], [9], [10], [12] consider equilibrium models of market microstructure, and they are more closely related to the present work. However, the models proposed in the aforementioned papers do not aim to represent the mechanics of an auction-style exchange with sufficient precision, and, in particular, they are not well suited for analyzing the liquidity effects of trading frequency, which is the main focus of the present chapter. A somewhat related strand of literature focuses on the endogenous formation of LOB in markets with a designated market maker: see e.g. [33], [43], [28], [18], [1]. In these papers, the LOB is not an outcome of a multi-agent equilibrium: instead, it is controlled by a single agent, the market maker. In the present chapter, we model the entire LOB as an output of an equilibrium between a large number of agents, each of whom is allowed to both consume and provide liquidity (in particular, we have no designated market maker). Our setting is related to the literature on *double auctions* (cf. [58], [27]), with the crucial difference that the participants of each auction are allowed to choose two "asymmetric" types of strategies: market or limit orders. In addition, the present framework assumes that, ex ante, all agents have access to the same information, and, in this sense, it is similar to [50], [34], [52]. In particular, the *adverse selection* effect, herein, does not arise from any a priori information asymmetry of the agents, and, instead, it is caused by the *mechanics* of the exchange. We formulate the problem as a *continuum-player game* – this abstraction allows us to obtain computationally tractable results (cf. [4], [56], [15] for more on the concept of a continuum-player game, and [46], [14], [16], [45] for the particular subclass of mean field games).

The chapter is organized as follows. Subsection 3.2 describes the probabilistic setting, along with the execution rules of the exchange and the resulting state processes of the agents. Subsection 2.2.2 defines the equilibrium and introduces the notion of *degeneracy* of the market (which represents an internal liquidity cirsis). In Section 2.3, we construct an equilibrium in a simple model, illustrating how an internal liquidity crisis unfolds, and how it is connected to the adverse selection effect.

Theorems II.19, II.21, and Corollary II.20, in Section 3.4, are the main results of the chapter: they formalize and generalize the conclusions of Section 2.3. In Section 2.5, we prove the key technical results on the (conditional) tails of marginal distributions of Itô processes. Sections 2.6, 2.7 contain the proofs of the main results. We conclude in Section 2.8.

2.2 Modeling framework for a finite-frequency auction-style exchange

2.2.1 Mechanics of the exchange

We consider an exchange in which trading can only occur at discrete times $n = 0, 1, \ldots, N$. We assume that the market participants are split into two groups: the *external investors*, who are "impatient", in the sense that they only submit market orders, and the *strategic players*, who can submit both market and limit orders, and who are willing to optimize their actions over a given (short) time horizon, in order to get a better execution price.[1] In our study, we focus on the strategic players, who are referred to as *agents*, and we model the behavior of the external investors exogenously, via the *exogenous demand.* The interpretation of the external investors is clear: these are the investors who either have a longer-term view on the market, or who simply need to buy or sell the asset for reasons other than the short-term profits. The strategic players (i.e. agents), on contrary, are the short-term traders, who attempt to maximize their objective at a shorter time horizon N. During every time period $[n, n+1)$, all the orders coming to the exchange are split into the *limit* and *market* orders. The limit orders are collected in the so-called *Limit Order Book (LOB)*, and the market orders form the *demand curve.* At time $n+1$, the market orders in the demand curve are executed against the limit orders in the LOB. Then,

[1] We do not distinguish the "aggressive" limit orders, which are posted at the price level of an opposite limit order, and treat them as market orders. This causes no loss of generality, as the market participants in our setting have a perfect observation of the LOB.

the same is repeated in the next time interval. In particular, during a time period $[n, n+1)$ (for simplicity, we say "at time n"), an agent is allowed to submit a market order, post a limit buy or sell order, or wait (i.e. do nothing). If a limit order is not executed in a given time period, it costs nothing to cancel or re-position it for the next time period. Notice that our framework does not allow to model the time-priority of limit orders. However, introducing a time-priority would not change the agents' maximum objective value, as the "tick size" is assumed to be zero (i.e. the set of possible price levels is $\mathbb{R}$), and, hence, an agent can always achieve a priority by posting her order "infinitesimally" above or below a given competing order. Further details on modeling the formation of an LOB and the execution rules are presented below.

The demand curves are modeled exogenously by a random field

$$D = (D_n(p))_{p\in\mathbb{R},n=1,\ldots,N}$$

on a filtered probability space $\left(\Omega, \mathbb{F} = (\mathcal{F}_n)_{n=0}^N, \mathbb{P}\right)$, such that $\mathcal{F}_0$ is a trivial sigma-algebra, completed w.r.t. $\mathbb{P}$. The random variable $D_n^+(p) = \max(D_n(p), 0)$ denotes the amount of asset that the external investors and the agents submitting market orders are willing to purchase at or below the price p, accumulated over the time period $[n-1, n)$, and $D_n^-(p) = -\min(D_n(p), 0)$ denotes the amount of asset that the external investors and the agents submitting market orders are willing to sell at or above the price p, in the same time period. We assume that $D_n(\cdot)$ is a.s. nonincreasing and measurable w.r.t. $\mathcal{F}_n \otimes \mathcal{B}(\mathbb{R})$. We denote by $\mathbb{A}$ a Borel space of *beliefs*, and, for each $\alpha \in \mathbb{A}$, there exists a *subjective probability measure* $\mathbb{P}^\alpha$ on $(\Omega, \mathcal{F}_N)$, which is absolutely continuous with resect to $\mathbb{P}$. We assume that, for any $n = 0, \ldots, N$ and any $\alpha \in \mathbb{A}$, there exists a regular version of the conditional probability

$\mathbb{P}^\alpha$ given $\mathcal{F}_n$, denoted $\mathbb{P}^\alpha_n$.[2] We denote the associated conditional expectations by $\mathbb{E}^\alpha_n$. We also need to assume that, for any $\alpha \in \mathbb{A}$, there exists a modification of the family $\{\mathbb{P}^\alpha_n\}_{n=0}^N$, which satisfies the *tower property with respect to* $\mathbb{P}$, in the following sense: for any $n \leq m$ and any r.v. ξ, such that $\mathbb{E}^\alpha \xi^+ < \infty$, we have

$$\mathbb{E}^\alpha_n \mathbb{E}^\alpha_m \xi = \mathbb{E}^\alpha_n \xi, \qquad \mathbb{P}\text{-a.s.}$$

There exists such a modification, for example, if $\mathbb{P}^\alpha \sim \mathbb{P}$. In any market model, for every α, we fix such a modification of conditional probabilities (up to a set of $\mathbb{P}$-measure zero) and assume that all conditional expectations $\{\mathbb{E}^\alpha_n\}$ are taken under this family of measures. The *Limit Order Book (LOB)* is given by a pair of adapted process $\nu = (\nu^+_n, \nu^-_n)_{n=0}^N$, such that every ν^+_n and ν^-_n is a finite sigma-additive random measure on $\mathbb{R}$ (w.r.t. $\mathcal{F}_n \otimes \mathcal{B}(\mathbb{R})$). Herein, ν^+_n corresponds to the cumulative limit sell orders, and ν^-_n corresponds to the cumulative limit buy orders, posted at time n.The bid and ask prices at any time $n = 0, \ldots, N$ are given by the random variables

$$p^b_n = \sup \operatorname{supp}(\nu^-_n), \qquad p^a_n = \inf \operatorname{supp}(\nu^+_n),$$

respectively. Notice that these extended random variables are always well defined but may take infinite values.

We define the *state space* of an agent as $\mathbb{S} = \mathbb{R} \times \mathbb{A}$, where the first component denotes the *inventory* of an agent (i.e. how much asset she currently holds), and the second component denotes her *beliefs*. Every agent in state (s, α) models the future outcomes using the subjective probability measure $\mathbb{P}^\alpha$. There are infinitely many agents, and their distribution over the state space is given by the *empirical distribution* process $\mu = (\mu_n)_{n=0}^N$, such that every μ is a finite sigma-additive random measure on $\mathbb{S}$ (w.r.t. $\mathcal{F}_n \otimes \mathcal{B}(\mathbb{S})$). In particular, the total mass of agents in the set $S \subset$

[2] This assumption holds, for example, if $\mathcal{F}_N$ is generated by a random element with values in a standard Borel space.

$\mathbb{S}$ at time n is given by $\mu_n(S)$. The inventory level s represents the *number of shares per agent*, held by the agents at state (s,α). In particular, the total number of shares held by all agents in the set $S \subset \mathbb{S}$ is given by $\int_S s\mu_n(ds,d\alpha)$. The interpretation of this definition in a finite-player game is discussed in Remark II.1 below. We refer the reader to [15] for more on the general concept a continuum-player game.

Remark II.1. The continuum-player game defined in this section can be related to a finite-player game as follows. Denote by μ_0 the empirical distribution of the agents' states at a given time. Recall that μ_0 is a measure on $\mathbb{S} = \mathbb{R} \times \mathbb{A}$, and assume that it is a finite linear combination of Dirac measures: $\mu_0 = \frac{1}{M}\sum_{i=1}^M \delta_{(s^i,\alpha^i)}$. In this case, we interpret s^i as the **number of shares per agent** held by the agents in the ith group. Let us explain how this notion is related to the actual inventory levels (i.e. the actual numbers of shares held by the agents) in the associated finite-player game. To this end, consider a collection of M agents, whose states are given by their (actual) inventories and beliefs, denoted (s,α), with the current states being $\{(\tilde{s}^i = s^i/M, \alpha^i)\}$. Define the "unit mass" of agents to be M. In this finite-player collection, the mass of agents (measured relative to the unit mass, M) at any state (Ms,α) is precisely $\mu_0(\{(s,\alpha)\})$, and their total inventory is $Ms\mu_0(\{(s,\alpha)\})$. The number of shares per agent is, then, defined as the total inventory held by these agents divided by their mass, and it is equal to Ms. Choosing $s = \tilde{s}^i$, we conclude that, in the finite-player collection, the number of shares per agent held by the agents at state $(\tilde{s}^i,\alpha^i)$ is given by $M\tilde{s}^i = s^i$, which coincides with our interpretation of s^i in the continuum-player game. It is also easy to show that an equilibrium in the proposed continuum-player game (defined in the next subsection) produces an approximate equilibrium in the associated finite-player game, when the inventory levels $\{\tilde{s}^i\}$ are small (cf. Subsection 2.3 in the extended version of this chapter, [32]) As the parameter α does not change over time, the state process of an agent, denoted (S_n), is an adapted $\mathbb{R}$-valued process, representing her inventory.[3] The control of every agent is given by a triplet of adapted processes $(p,q,r) = (p_n, q_n, r_n)_{n=0}^{N-1}$ on $(\Omega, \mathbb{F})$, with values in $\mathbb{R}^2 \times \{0,1\}$. The first coordinate, p_n, indicates the location of a limit order placed at time n, and q_n indicates the size of the order (measured in shares per agent, and with negative values corresponding to buy orders).[4] The last coordinate r_n shows whether the agent submits a market order (if $r_n = 1$) or a limit order (if $r_n = 0$). Assume that an agent posts a limit sell order at a price level p_n. If the demand to buy the asset at this price level, $D^+_{n+1}(p_n)$, exceeds the amount of all limit sell orders posted below p_n at time n, then (and only then) the limit sell order of the agent is executed. Market orders of the agents are always executed at the bid or ask price available at the time when the order is submitted. We interpret an internal market order (i.e. the one submitted by an agent) as the decision of an agent to join the external investors, in the given time period. Summing up the above, we obtain the following dynamics for the state process of an agent, starting with initial inventory $s \in \mathbb{R}$ at time $m = 0, \dots, N-1$:

$$S^{(p,q,r)}_m(m,s,\nu) = s, \quad \Delta S^{(p,q,r)}_{n+1}(m,s,\nu) = S^{(p,q,r)}_{n+1}(m,s,\nu) - S^{(p,q,r)}_n(m,s,\nu) = -q_n \mathbf{1}_{\{r_n=1\}}$$

$$-\mathbf{1}_{\{r_n=0\}} \left(q^+_n \mathbf{1}_{\left\{D^+_{n+1}(p_n) > \nu^+_n((-\infty,p_n))\right\}} - q^-_n \mathbf{1}_{\left\{D^-_{n+1}(p_n) > \nu^-_n((p_n,\infty))\right\}} \right), \quad n = m, \dots, N-1. \tag{2.1}$$

The above dynamics represent an optimistic view on the execution by the agents. In particular, they imply that all limit orders at the same price level are executed in full,

[3] Note that, although $\mathbb{P}^\alpha$ does not change over time, the conditional distribution of the future demand, as perceived by the agent, changes dynamically, according to the new information received.

[4] Note each agent is only allowed to place her limit order at a single price level, at any given time. However, this results in no loss of optimality. Indeed, using the Dynamic Programming Principle derived in Appendix A, one can show, by induction, that, in equilibrium, an agent does not benefit from posting multiple limit orders at the same time. As shown in [56], this is typical for a continuum-player game.

once the demand reaches them: i.e. each agent believes that her limit order will be executed first among all orders at a given price level. In addition, all agents' market orders are executed at the bid and ask prices: i.e. each agent believes that her market order will be executed first, when the demand curve is cleared against the LOB, at the end of a given time period. These assumptions can be partially justified by the fact that the agents' orders are infinitesimal: q_n is measured in shares per agent, and an individual agent has zero mass. However, if a non-zero mass of agents submit limit orders at the same price level, or execute market orders, at the same time, then, the above state dynamics may violate the market clearance condition: the total size of executed market orders (both in shares and in dollars) may not coincide with the total size of executed limit orders (at least, as viewed by the agents). Nevertheless, this issue is resolved if, at any time, the mass of the agents positing limit orders at the same price level is zero, as well as the mass of the agents posting market orders. In other words, (ν, p, q, r) satisfy, $\mathbb{P}$-a.s.: ν_n is continuous, as a measure on $\mathbb{R}$ (i.e. it has no atoms), and $r_n = 0$. Such an equilibrium is constructed in Section 8 of the extended version of this chapter, [32]. The general definition of a continuum-player game and its connection to a finite-player game can be found, e.g., in [15] and in the references therein (see also Subsection 2.3 in the extended version of this chapter, [32]).

The proposed modeling framework has a close connection to the models of *double auctions*, existing in the economic literature (cf. [27], [58]). The main difference of the present setting is in the non-standard design of the auction. Namely, in the proposed setting, the auction participants may choose different styles of trading, i.e. market or limit orders, which generates an ex-post information asymmetry between the participants: the limit orders have to be submitted before the demand curve is observed, while the market orders are submitted using a complete information about the LOB. This difference is not coincidental – it is, in fact, crucial for a realistic representation of the risks associated with each order type, and it is at the core of the results established herein. A more detailed discussion of the information structure of the proposed framework is provided in the next subsection.

2.2.2 Equilibrium

The objective function of an agent, starting at the initial state $(s, \alpha) \in \mathbb{S}$, at any time $m = 0, \ldots, N$, and using the control (p, q, r), is given by the $\mathcal{F}_m$-measurable random variable:

$$
\begin{aligned}
(2.2) \qquad J^{(p,q,r)}(m, s, \alpha, \nu) = \mathbb{E}^{\alpha}_{m} \Bigg[& \left(S^{(p,q,r)}_{N}(m, s, \nu) \right)^{+} p^{b}_{N} - \left(S^{(p,q,r)}_{N}(m, s, \nu) \right)^{-} p^{a}_{N} \\
& - \sum_{n=m}^{N-1} \left(p_n \mathbf{1}_{\{r_n=0\}} + p^{a}_{n} \mathbf{1}_{\{r_n=1, q_n<0\}} + p^{b}_{n} \mathbf{1}_{\{r_n=1, q_n>0\}} \right) \Delta S^{(p,q,r)}_{n+1}(m, s, \nu) \Bigg]
\end{aligned}
$$

where we assume that $0 \cdot \infty = 0$. In the above expression, we assume that, at the final time $n = N$, each agent is forced to liquidate her position at the bid or ask price available at that time. Alternatively, one can think of it as *marking to market* of the residual inventory, right after the last external market order is executed.

Definition II.2. For a given LOB ν, integer $m = 0, \ldots, N-1$, and state $(s, \alpha) \in \mathbb{S}$, the triplet of adapted processes (p, q, r) is an **admissible control** if the positive part of the expression inside the expectation in (2.2) has a finite expectation under $\mathbb{P}^{\alpha}$.

For a given LOB ν, an initial condition (m, s, α), and a triplet of $\mathbb{F} \times \mathcal{B}(\mathbb{S})$-adapted random fields (p, q, r), we identify the latter (whenever it causes no confusion) with stochastic processes (p, q, r) via:

$$
p_n = p_n \left(S^{(p,q,r)}_{n}(m, s, \nu), \alpha \right), \quad q_n = q_n \left(S^{(p,q,r)}_{n}(m, s, \nu), \alpha \right),
$$

$$
r_n = r_n \left(S^{(p,q,r)}_{n}(m, s, \nu), \alpha \right),
$$

and the state dynamics (2.1), for $n = m, \dots, N$. This system determines (p, q, r) and $S^{(p,q,r)}$ recursively.

Definition II.3. For a given LOB ν, we call the triplet of progressively measurable random fields (p, q, r) an **optimal control** if, for any $m = 0, \dots, N$ and any $(s, \alpha) \in \mathbb{S}$, we have:

- (p, q, r) is admissible,
- $J^{(p,q,r)}(m, s, \alpha, \nu) \geq J^{(p',q',r')}(m, s, \alpha, \nu)$,

  $\mathbb{P}$-a.s., for any admissible control (p', q', r').

In the above, we make the standard simplifying assumption of continuum-player games: each agent is too small to affect the empirical distribution of cumulative controls (reflected in ν) when she changes her control (cf. [15]). Note also that our definition of the optimal control implies that it is time consistent: re-evaluation of the optimality at any future step, using the same terminal criteria, must lead to the same optimal strategy. Next, we discuss the notion of equilibrium in the proposed game. First, we notice that, if p^b_N or p^a_N becomes infinite, the agents with positive or negative inventory may face the objective value of "$-\infty$", for any control they use. In such a case, their optimal controls may be chosen in an arbitrary way, resulting in unrealistic equilibria. To avoid this, we impose the additional regularity condition on ν.

Definition II.4. A given LOB ν is admissible if, for any $m = 0, \dots, N-1$ and any $\alpha \in \mathbb{A}$, we have, $\mathbb{P}$-a.s.:

$$\mathbb{E}^{\alpha}_m |p^a_N| \vee |p^b_N| < \infty.$$

Let us consider the (stochastic) value function of an agent for a fixed (m, s, α, ν):

$$V^{\nu}_m(s, \alpha) = \operatorname{esssup}_{p,q,r} J^{(p,q,r)}\left(m, s, \alpha, \nu\right), \tag{2.3}$$

where the essential supremum is taken under $\mathbb{P}$, over all admissible controls (p,q,r), and $J^{(p,q,r)}$ is given by (2.2). Appendix A shows that, for any admissible ν, $V^{\nu}_m(\cdot,\alpha)$ has a continuous modification under $\mathbb{P}$, which we refer to as the value function of an agent with beliefs α. Using the Dynamic Programming Principle, Appendix A provides an explicit system of recursive equations that characterize optimal strategies and the value function. In particular, the results of Appendix A (cf. Corollary II.31) yield the following proposition.

Proposition II.5. *Assume that, for an admissible LOB ν, there exists an optimal control $(\hat{p},\hat{q},\hat{r})$. Then, for any $(s,\alpha)\in\mathbb{S}$, the following holds $\mathbb{P}$-a.s., for all $n = 0,\ldots,N-1$:*

$$V^{\nu}_n(s,\alpha) = s^+\lambda^a_n(\alpha) - s^-\lambda^b_n(\alpha)$$

with some adapted processes $\lambda^a(\alpha)$ and $\lambda^b(\alpha)$, such that $\lambda^a_N(\alpha) = p^b_N$ and $\lambda^b_N(\alpha) = p^a_N$.

The values of $\lambda^a(\alpha)$ and $\lambda^b(\alpha)$ can be interpreted as the *expected execution prices* of the agents with beliefs α, who are long and short the asset, respectively.

Definition II.6. Consider an empirical distribution process $\mu = (\mu_n)_{n=0}^N$ and a market model, as described in Subsection 3.2. We say that a given LOB process ν and a control (p,q,r) form an **equilibrium**, if there exists a Borel set $\tilde{\mathbb{A}}\subset\mathbb{A}$, called the **support** of the equilibrium, such that:

1. $\mu_n\left(\mathbb{R}\times\left(\mathbb{A}\setminus\tilde{\mathbb{A}}\right)\right) = 0$, $\mathbb{P}$-a.s., for all n,
2. ν is admissible, and (p,q,r) is an optimal control for ν, on the state space $\tilde{\mathbb{S}} = \mathbb{R}\times\tilde{\mathbb{A}}$,

3. and, for any $n = 0, \dots, N-1$, we have, $\mathbb{P}$-a.s.:

$$\nu_n^+((-\infty, x]) = \int_{\tilde{\mathbb{S}}} \mathbf{1}_{\{p_n(s,\alpha)\leq x, r_n(s,\alpha)=0\}}\, q_n^+(s,\alpha)\mu_n(ds, d\alpha), \quad \forall\, x \in \mathbb{R}, \tag{2.4}$$

$$\nu_n^-((-\infty, x]) = \int_{\tilde{\mathbb{S}}} \mathbf{1}_{\{p_n(s,\alpha)\leq x, r_n(s,\alpha)=0\}}\, q_n^-(s,\alpha)\mu_n(ds, d\alpha), \quad \forall\, x \in \mathbb{R}. \tag{2.5}$$

Remark II.7. It follows from Proposition II.5 that, in equilibrium, it is optimal for an agent with zero initial inventory to do nothing. Hence, in equilibrium, roundtrip strategies are impossible. To allow for roundtrip strategies in equilibrium, one can e.g. introduce an upper bound on $|q|$ or on the total inventory of an agent (as it is done, e.g., in [13]). However, we do not believe that such a modification would change the qualitative behavior of market liquidity as a function of trading frequency, which is the main focus of the present chapter.

Notice that, because the optimal controls are required to be time consistent under $\mathbb{P}$, the above definition, in fact, defines a *sub-game perfect equilibrium*. It is also worth mentioning that Definition III.6 defines a *partial equilibrium*, as the empirical distribution process μ is given exogenously. A more traditional version of Nash equilibrium would require μ to be determined by the initial distribution and the values of the state processes:

$$\mu_n = \mu_0 \circ \left((s,\alpha) \mapsto \left(S_n^{(p,q,r)}(0,s,\nu), \alpha\right)\right)^{-1}, \tag{2.6}$$

which must hold $\mathbb{P}$-a.s., for all $n = 0, \dots, N$, with $S_n^{(p,q,r)}(0,s,\nu)$ defined via (2.1), in addition to the other conditions in Definition III.6. Nevertheless, we choose not to enforce the condition (2.6) in the definition of equilibrium, in order to allow new agents to enter the game, which, in effect, amounts to modeling μ exogenously. If one assumes that no new agents arrive to the market, then, the fixed-point condition

(2.6) has to be enforced. Note also that our interpretation of the demand curve $D_n(\cdot)$ implies that it consists of both the external (i.e. due external investors) and internal (i.e. due to the agents) market orders. Therefore, it may be reasonable to consider an additional consistency condition for an equilibrium. A part of this condition is to ensure that a non-zero mass of agents submit market buy orders only if the fundamental price rises above the ask price (i.e. only if a market buy order is actually executed), and, similarly, a non-zero mass of agents submit market sell orders only if the fundamental price falls below the bid price. We assume that the agents' market orders enter into the demand curve with the highest level of priority: e.g. their market buy orders enter the demand curve at the price level infinitesimally close to, but below, the fundamental price, in order to guarantee that they are the first ones to be executed. Thus, another part of the aforementioned consistency condition is to ensure that the absolute value of the demand curve to the left or to the right of the fundamental price is sufficiently large to account for all internal market orders. Mathematically, such consistency condition can be formulated as follows:

$$\begin{aligned} (2.7)\quad d_n^b := \mu_n\left(\{(s,\alpha)\,:\,q_n(s,\alpha)<0,\, r_n(s,\alpha)=1\}\right) > 0 \\ \Rightarrow\; p^0_{n+1} > p^a_n,\; \lim_{p\uparrow p^0_{n+1}} D^+_{n+1}(p) \geq d^b_n, \end{aligned}$$

$$\begin{aligned} (2.8)\quad d_n^a := \mu_n\left(\{(s,\alpha)\,:\,q_n(s,\alpha)>0,\, r_n(s,\alpha)=1\}\right) > 0 \\ \Rightarrow\; p^0_{n+1} < p^b_n,\; \lim_{p\downarrow p^0_{n+1}} D^-_{n+1}(p) \geq d^a_n. \end{aligned}$$

The above conditions become redundant if the agents never submit market orders in equilibrium. Section 8 of the extended version of this chapter, [32], shows how to construct an equilibrium which satisfies condition (2.6), and in which the agents never submit market orders (hence, (2.7) and (2.8) are also satisfied). However, it is important to emphasize that the main results of the present work (cf. Section 3.4) provide necessary conditions for **all** equilibria: for those satisfying the conditions (2.6), (2.7), (2.8) and for the ones that do not.

Remark II.8. Let us comment on the information structure of the game. In the present setting, all agents observe the same information, given by the filtration $\mathbb{F}$. We consider an open-loop Nash equilibrium, in which the agent's strategy is viewed as an adapted stochastic process (rather than a function of the states and controls of other players), and the definition of optimality is chosen accordingly. In addition, as μ is adapted to $\mathbb{F}$, each agent has a complete information about the present and past states of other agents, and their beliefs. However, as the agents use different (subjective) measures $\{\mathbb{P}^{\alpha}\}$, their views on the future values of μ may be different. Of course, it would be more realistic to assume that the agents do not have a complete information about each other's current states, but this would make the problem significantly more complicated. In the present setting, the agents also have a complete information about the current location of the fundamental price. In the next chapter we relax this assumption, which allows us to develop a more realistic model for the "local" behavior of an individual agent. However, such a relaxation does not seem necessary for the questions analyzed herein.

As all agents use the same information, the present work belongs to the strand of literature that attempts to explain microstructure phenomena without information asymmetry (cf. [34], [52], [50], [29]). Nevertheless, it is important to mention that certain information asymmetry arises ex-post, between the market participants submitting market and limit orders. This asymmetry is not due to superior information a priori available to any of the agents. Instead, it stems from the very nature of limit orders, which are "passive" by design (cf. the discussion on the last paragraph of Subsection 3.2). Similar observation is made in [34].

Next, we need to add another condition to the notion of equilibrium. Notice that equations (3.7)–(3.8) should serve as the fixed-point constraints that allow one to obtain the optimal controls (p, q, r), along with the LOB ν. However, these equations only hold for $n = 0, \ldots, N-1$: indeed, the agents do not need to choose their controls at time $n = N$, as the game is over and their residual inventory is marked to the bid and ask prices. However, the terminal bid and ask prices are determined by the LOB ν_N, which, in turn, can be chosen arbitrarily. To avoid such ambiguity, we impose an additional constraint on the equilibria studied herein. First, we introduce the notion of a *fundamental price*.

Definition II.9. Assume that $\mathbb{P}$-a.s., for any $n = 1, \ldots, N$, there exists a unique p_n^0 satisfying $D_n\left(p_n^0\right) = 0$. Then, the adapted process $(p_n^0)_{n=1}^N$ is called the **fundamental price process**.

Whenever the notion of a fundamental price is invoked, we assume that it is well defined. The intuition behind p^0 is clear: it is a price level at which the immediate demand is balanced. However, it is important to stress that we do not assume that the asset can be traded at the fundamental price level. Rather, p^0 is a feature of the abstract immediate demand curve, whereas all actual trading happens on the exchange, against the current LOB. This makes our setting different from many other approaches existing in the literature.

Definition II.10. Assume that the fundamental price is well defined and denote $\xi_N = p_N^0 - p_{N-1}^0$. Then, an equilibrium with LOB ν is **linear at terminal crossing**

(LTC) if

$$\nu_N = \nu_{N-1} \circ (x \mapsto x + \xi_N)^{-1}, \qquad \mathbb{P}\text{-a.s.} \tag{2.9}$$

The above definition assumes that the terminal LOB ν_N is obtained from ν_{N-1} by a simple shift, with the size of the shift equal to the increment of the fundamental price. This definition connects the LOB at the terminal time with the demand process, ruling out many unnatural equilibria. In particular, the question of existence of an equilibrium becomes non-trivial. However, the mere existence of an equilibrium is not the main focus of the present work: the existence results, established herein, are limited to Section 2.3, which constructs an LTC equilibrium in a specific Gaussian random walk model (a slightly more general existence result is given in Section 8 of the extended version of this chapter, [32]). What is central to the present investigation is the observation that the agents may reach an equilibrium in which one side of the LOB becomes empty (as demonstrated by the example of Section 2.3). We call such LOB, and the associated equilibrium, *degenerate*.

Definition II.11. We say that an equilibrium with LOB ν is **non-degenerate** if $\nu_n^+(\mathbb{R}) > 0$ and $\nu_n^-(\mathbb{R}) > 0$, for all $n = 0, \ldots, N-1$, $\mathbb{P}$-a.s..

Intuitively, the degeneracy of LOB refers to a situation where, with positive probability, one side of the LOB disappears from the market: i.e. $\nu_n^+(\mathbb{R})$ or $\nu_n^-(\mathbb{R})$ becomes zero. Clearly, this happens when the agents who are supposed to provide liquidity choose to post market orders (i.e. consume liquidity) or wait (neither provide nor consume liquidity). Such a degeneracy can be interpreted as the internal (or, self-inflicted) *liquidity crisis* – the one that arises purely from the interaction between the agents, and cannot be justified by any fundamental economic reasons (e.g. the external demand for the asset may still be high, on both sides). Taking an optimistic point of view, we assume that the agents choose a non-degenerate equilibrium, whenever one is available. However, if a non-degenerate equilibrium does not exist, an internal liquidity crisis may occur with positive probability. One of the main goals of this chapter is to provide insights into the occurrence of an internal liquidity crisis and its relation to trading frequency.

2.3 Example: a Gaussian random walk model

In this section, we consider a specific market model for the external demand D to construct a non-degenerate LTC equilibrium. More importantly, using this model, we illustrate the liquidity effects of trading frequency, which, as mentioned in the introduction, is the main goal of the present work. The present example, albeit very simplistic, allows us to identify certain important phenomena that occur to the optimal strategies of the agents (and, hence, to the LOB) as the trading frequency increases. In particular, we demonstrate how the *adverse selection* effect may be amplified disproportionally by the high trading frequency and may cause a liquidity crisis. Note that the adverse selection phenomenon, in the present setting, is not a consequence of any ex-ante information asymmetry but is due to the mechanics of the exchange (i.e. the nature of limit orders), which is similar to the phenomena documented in [34], [29]. In the rest of the chapter, we show that the conclusions of this section are not due to the particular choice of a model made in the present section and, in fact, persist in a much more general setting.

On a complete stochastic basis $(\Omega, \tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t\in[0,T]}, \mathbb{P})$, we consider a continuous time process $\tilde{p}_0$:

$$\tilde{p}^0_t = p^0_0 + \alpha t + \sigma W_t, \quad p^0_0 \in \mathbb{R}, \quad t \in [0, T], \tag{2.10}$$

where $\alpha \in \mathbb{R}$ and $\sigma > 0$ are constants, and W is a Brownian motion. We also consider an arbitrary progressively measurable random field $(\tilde{D}_t(p))$, s.t., $\mathbb{P}$-a.s., the function $\tilde{D}_t(\cdot) - \tilde{D}_s(\cdot)$ is strictly decreasing and vanishing at zero, for any $0 \leq s < t \leq T$. Finally, we introduce the empirical distribution process $(\tilde{\mu}_t)$, with values in the space of finite sigma-additive measures on $\mathbb{S}$. We partition the time interval $[0, T]$ into N subintervals of size $\Delta t = T/N$. A discrete time model is obtained by discretizing the continuous time one:[5]

$$\mathcal{F}_n = \tilde{\mathcal{F}}_{n\Delta t}, \quad p^0_n = \tilde{p}^0_{n\Delta t}, \quad D_n(p) = (\tilde{D}_{n\Delta t} - \tilde{D}_{(n-1)\Delta t})(p - p^0_n), \quad \mu_n = \tilde{\mu}_{n\Delta t}.$$

In this section, for simplicity, we assume that the set of agents' beliefs is a singleton: $\mathbb{A} = \{\alpha\}$ and $\mathbb{P}^\alpha = \mathbb{P}$. We also assume that (at least, from the agents' point of view) there are always some long and short agents present in the market: $\mu_n\left((0,\infty) \times \mathbb{A}\right), \mu_n\left((-\infty, 0) \times \mathbb{A}\right) > 0$, $\mathbb{P}$-a.s., for all n. Clearly, N represents the trading frequency, and the continuous time model represents the "limiting model", which the agents use as a benchmark, in order to make consistent predictions in the markets with different trading frequencies. We assume that the benchmark model is fixed, and N is allowed to vary. In the remainder of this section, we propose a method for constructing a non-degenerate LTC equilibrium in the above discrete time model. We show that the method succeeds for any (N, σ) if $\alpha = 0$. However, for $\alpha \neq 0$, we demonstrate numerically that the method fails as N becomes large enough. We show why, precisely, the proposed construction fails, providing an economic interpretation of this phenomenon. Moreover, we analyze the market close to the moment when a non-degenerate equilibrium fails to exist and demonstrate that the agents' behavior at this time follows the pattern typical for an internal (or, self-inflicted) liquidity crisis.

[5] In order to ensure the existence of regular conditional probabilities for the discrete time model, we can, for example, assume that $\tilde{\mathcal{F}}_T$ is generated by a random element with values in a standard Borel space.

In view of Proposition II.5, in order to construct a non-degenerate LTC equilibrium, we need to find a control $(\hat{p}, \hat{q}, \hat{r})$, and the expected execution prices $(\hat{\lambda}^a, \hat{\lambda}^b)$, s.t. the value function of an agent with inventory s is given by $V_n(s) = s^+\hat{\lambda}^a_n - s^-\hat{\lambda}^b_n$, and it is attained by the strategy $(\hat{p}, \hat{q}, \hat{r})$. In addition, we need to find a non-degenerate LOB ν, s.t. (3.7), (3.8) and (2.9) hold. Our ansatz is as follows:

$$\nu_n = \left(h^a_n \delta_{p^a_n}, h^b_n \delta_{p^b_n}\right), \quad p^a_n = \hat{p}^a_n + p^0_n, \quad p^b_n = \hat{p}^b_n + p^0_n, \quad -\infty < \hat{p}^b_n, \hat{p}^a_n < \infty,$$

$$\hat{p}_n(s) = p^a_n \mathbf{1}_{\{s>0\}} + p^b_n \mathbf{1}_{\{s<0\}}, \quad \hat{q}_n(s) = s, \quad \hat{r}_n(s) = 0, \quad \lambda^a_n = \hat{\lambda}^a_n + p^0_n, \quad \lambda^b_n = \hat{\lambda}^b_n + p^0_n,$$

where δ is the Dirac measure, $(\hat{p}^a, \hat{p}^b, \hat{\lambda}^a, \hat{\lambda}^b)$ are deterministic processes, and $h^a_n = \int_0^\infty s\mu_n(ds) > 0$, $h^b_n = \int_{-\infty}^0 |s|\mu_n(ds) > 0$. With such an ansatz, the conditions (3.7), (3.8) are satisfied automatically. Thus, we only need to choose finite deterministic processes $(\hat{p}^a, \hat{p}^b, \hat{\lambda}^a, \hat{\lambda}^b)$ s.t.: $\hat{p}^a_N = \hat{p}^a_{N-1}$, $\hat{p}^b_N = \hat{p}^b_{N-1}$ (so that the equilibrium is LTC) and the associated $(\hat{p}, \hat{q}, 0)$ form an optimal control, producing the value function $V_n(s) = s^+\lambda^a_n - s^-\lambda^b_n$. Appendix A contains necessary and sufficient conditions for characterizing such families $(p^a, p^b, \lambda^a, \lambda^b)$. In particular, we deduce from Corollaries II.31 and II.32 that $(\hat{p}^a_{N-1}, \hat{p}^b_{N-1}, \hat{\lambda}^a_{N-1}, \hat{\lambda}^b_{N-1})$ form a suitable family in a single-period case, $[N-1, N]$, if they solve the following system:

$$\begin{cases} \hat{p}^a_{N-1} \in \arg\max_{p\in\mathbb{R}} \mathbb{E}\left((p - \hat{p}^b_{N-1} - \xi)\mathbf{1}_{\{\xi>p\}}\right), \quad \hat{p}^b_{N-1} < 0, \\ \hat{p}^b_{N-1} \in \arg\max_{p\in\mathbb{R}} \mathbb{E}\left((\hat{p}^a_{N-1} - p + \xi)\mathbf{1}_{\{\xi<p\}}\right), \quad \hat{p}^a_{N-1} > 0, \\ \hat{\lambda}^a_{N-1} = \hat{p}^b_{N-1} + \alpha\Delta t + \mathbb{E}\left((\hat{p}^a_{N-1} - \hat{p}^b_{N-1} - \xi)\mathbf{1}_{\left\{\xi>\hat{p}^a_{N-1}\right\}}\right), \\ \hat{\lambda}^b_{N-1} = \hat{p}^a_{N-1} + \alpha\Delta t - \mathbb{E}\left((\hat{p}^a_{N-1} - \hat{p}^b_{N-1} + \xi)\mathbf{1}_{\left\{\xi<\hat{p}^b_{N-1}\right\}}\right), \\ \hat{p}^b_{N-1} \leq \hat{\lambda}^a_{N-1}, \quad \hat{\lambda}^b_{N-1} \leq \hat{p}^a_{N-1}, \quad \hat{p}^a_{N-1} \geq \hat{p}^b_{N-1} + |\alpha|\Delta t, \end{cases} \tag{2.11}$$

where $\xi = \Delta p^0_N \sim \mathcal{N}(\alpha\Delta t, \sigma^2\Delta t)$. Let us comment on the economic meaning of the equations in (2.11). The expectations in the first two lines represent the *relative expected profit* from executing a limit order at time N, at the chosen price level

$p+p^0_{N-1}$, versus marking the inventory to market at time N, at the best price available on the other side of the book: i.e. $p^b_N = \hat{p}^b_{N-1} + \xi + p^0_{N-1}$ or $p^a_N = \hat{p}^a_{N-1} + \xi + p^0_{N-1}$. Notice that a limit order is executed if and only if the fundamental price at time N is above or below the chosen level of agent's limit order: i.e. if $p^0_{N-1} + \xi > p + p^0_{N-1}$ or $p^0_{N-1} + \xi < p + p^0_{N-1}$.[6] Clearly, it is only optimal for an agent to post a limit order if the relative expected profit is nonnegative, which is the case if and only if $\hat{p}^b_{N-1} < 0 < \hat{p}^a_{N-1}$. The third and fourth lines in (2.11) represent the expected execution prices of the agents at time $N-1$, assuming they use the controls given by $(\hat{p}^a_{N-1}, \hat{p}^b_{N-1})$. Each of the right hand sides is a sum of two components: the relative expected profit from posting a limit order and the expected value of marking to market at time N, measured relative to p^0_{N-1}. Let us analyze the inequalities in the last line of (2.11). If the bid price at time $N-1$ exceeds the expected execution price of a long agent, i.e. $\hat{p}^b_{N-1} + p^0_{N-1} > \hat{\lambda}^a_{N-1} + p^0_{N-1}$, then every agent with positive inventory prefers to submit a market order, rather than a limit order, at time $N-1$, which causes the ask side of the LOB to degenerate. Similarly, we establish $\hat{\lambda}^b_{N-1} \leq \hat{p}^a_{N-1}$. Finally, if $\alpha > 0$ and $\hat{p}^a_{N-1} < \hat{p}^b_{N-1} + \alpha\Delta t$, an agent may buy the asset using a market order at time $N-1$, at the price $\hat{p}^a_{N-1} + p^0_{N-1}$, and sell it at time N, at the expected price $\hat{p}^b_{N-1} + p^0_{N-1} + \alpha\Delta t > \hat{p}^a_{N-1} + p^0_{N-1}$ (a reverse strategy works for $\alpha < 0$). This strategy can be scaled to generate infinite expected profit and, hence, is excluded by the last inequality in the last line of (2.11).

We construct a solution to (2.11) by solving a fixed-point problem given by the first two lines of (2.11) and verifying that the desired inequalities hold.[7] We implement this computation in MatLab, and the results can be seen as the right-most points

[6]The execution of limit orders simplifies in the chosen ansatz, because the agents on each side of the book (i.e. long or short) post orders at the same prices level.

[7]In fact, it is not difficult to prove rigorously that, for any (α, σ), there exists a unique solution to such system, provided Δt is small enough. We omit this result for the sake of brevity.

on the graphs in Figure 3.2. From the numerical solution, we see that, whenever Δt is small enough, the conditions $\hat{p}^b_{N-1} \leq \hat{\lambda}^a_{N-1}$ and $\hat{\lambda}^b_{N-1} \leq \hat{p}^a_{N-1}$ are satisfied (cf. the right part of Figure 3.2).[8] In addition, for $\alpha \geq 0$, we have

$$0 < \mathbb{E}\left(\hat{p}^a_{N-1} - \hat{p}^b_{N-1} - \xi \,|\, \xi > \hat{p}^a_{N-1}\right) =$$

$$\hat{p}^a_{N-1} - \hat{p}^b_{N-1} - \mathbb{E}\left(\xi \,|\, \xi > \hat{p}^a_{N-1}\right) \leq \hat{p}^a_{N-1} - \hat{p}^b_{N-1} - \alpha\Delta t,$$

which yields the last inequality in (2.11). The case of $\alpha < 0$ is treated similarly. Notice that $\hat{\lambda}^a_N = \hat{p}^b_N = \hat{p}^b_{N-1}$ and $\hat{p}^a_{N-1} = \hat{p}^a_N = \hat{\lambda}^b_N$. Thus, the single-period equilibrium we have constructed satisfies:

$$\hat{p}^b_n \leq \hat{\lambda}^a_n, \quad \hat{\lambda}^b_n \leq \hat{p}^a_n, \quad \hat{\lambda}^a_{n+1} < 0, \quad \hat{\lambda}^b_{n+1} > 0, \tag{2.12}$$

for $n = N - 1$. If one of the first two inequalities in (2.12) fails, the agents choose to submit market orders, as opposed to limit orders, which leads to *degeneracy* of the LOB – one side of it disappears. If one of the last two inequalities fails, the execution of a limit order, at any price level, yields a negative relative expected profit for the agents on one side of the book (given by the expectation in the first or second line of (2.11)). As a result, it becomes optimal for all such agents to not post any limit orders, and the LOB degenerates. The latter is interpreted as the *adverse selection* effect. For example, if the third inequality in (2.12) fails, then, every long agent believes that, no matter at which price level her limit order is posted, if it is executed in the next time period, her expected execution price at the next time step will be higher than the price at which the limit order is executed. Hence, it does not make sense to post a limit order at all.

In a single period $[N - 1, N]$, by choosing small enough Δt, we can ensure that the inequalities in (2.12) are satisfied. However, it turns out that, as we progress

[8] This is easy to explain intuitively, as the optimal objective values in the first two lines of (2.11) are of the form $C\sqrt{\Delta t} + \alpha \underline{\underline{O}}(\Delta t)$.

recursively backwards, constructing an equilibrium, we may encounter a time step at which one of the inequalities in (2.12) fails, implying that a non-degenerate LTC equilibrium cannot be constructed for the given time period (at least, using the proposed method). To see this, consider the recursive equations for $(\hat{p}^a, \hat{\lambda}^a)$ (which are chosen to satisfy the conditions of Corollary II.31, in Appendix A, given our ansatz):

$$\begin{cases} \hat{p}^a_n \in \arg\max_{p\in\mathbb{R}} \mathbb{E}\left(\left(p - \hat{\lambda}^a_{n+1} - \xi\right) \mathbf{1}_{\{\xi > p\}}\right), \\ \hat{\lambda}^a_n = \hat{\lambda}^a_{n+1} + \alpha\Delta t + \mathbb{E}\left(\left(\hat{p}^a_n - \hat{\lambda}^a_{n+1} - \xi\right) \mathbf{1}_{\{\xi > \hat{p}^a_n)\}}\right) < 0, \end{cases} \tag{2.13}$$

and similarly for $(\hat{p}^b, \hat{\lambda}^b)$. Using the properties of Gaussian distribution, it is easy to see that, if $\hat{\lambda}^a_{n+1} < 0$, we have $\hat{p}^a_n > 0$. Similar conclusion holds for $(\hat{\lambda}^b, \hat{p}^b)$. Thus, if $\hat{\lambda}^a_k < 0 < \hat{\lambda}^b_k$, for $k = n+1, \ldots, N$, our method allows us to construct a non-degenerate LTC equilibrium on the time interval $[n, N]$, with $\hat{p}^b < 0 < \hat{p}^a$. Such a construction always succeeds if the agents are market-neutral: i.e. $\alpha = 0$. Indeed, in this case, assuming $\hat{\lambda}^a_{n+1} < 0 < \hat{\lambda}^b_{n+1}$, we have: $\hat{p}^b_n < 0 < \hat{p}^a_n$ and

$$\hat{\lambda}^a_{n+1} + \left(\mathbb{E}\left(\left(\hat{p}^a_n - \hat{\lambda}^a_{n+1} - \xi\right) \mathbf{1}_{\{\xi > \hat{p}^a_n)\}}\right)\right)^+ = \mathbb{E}\left(\hat{\lambda}^a_{n+1} \mathbf{1}_{\{\xi > \hat{p}^a_n)\}}\right) + \mathbb{E}\left(\left(\hat{p}^a_n - \xi\right) \mathbf{1}_{\{\xi > \hat{p}^a_n)\}}\right) < 0.$$

Hence, $\hat{\lambda}^a_n < 0$, and, similarly, we deduce that $\hat{\lambda}^b_n > 0$. By induction, we obtain a non-degenerate LTC equilibrium on $[0, N]$, for any (N, σ), as long as $\alpha = 0$. Corollary II.20 shows that, as $N \to \infty$, the processes $(\hat{\lambda}^a, \hat{\lambda}^b)$ converge to zero, which means that the expected execution prices converge to the fundamental price. The latter is interpreted as *market efficiency* in the high-frequency trading regime: any market participant expects to buy or sell the asset at the fundamental price. The left hand side of Figure 3.3 shows that the bid and ask prices also converge to the fundamental price if $\alpha = 0$. This can be interpreted as a *positive liquidity effect* of increasing the trading frequency.

However, the situation is quite different if $\alpha \neq 0$. Assume, for example, that $\alpha > 0$. Then, the second line of (2.13) implies that $\hat{\lambda}^a$ increases by, at least, $\alpha \Delta t$ at each step of the (backward) recursion. Recall that the number of steps is $N = T/\Delta t$, hence, $\hat{\lambda}^a_0 \geq \hat{\lambda}^a_N + \alpha T$. If $|\hat{\lambda}^a_N|$ is small (which is typically the case if N is large), then, we may obtain $\hat{\lambda}^a_{n+1} \geq 0$, at some time n, which violates the third inequality in (2.12), or, equivalently, implies that the objective in the first line of (2.13) is strictly negative for all p. The latter implies that it is suboptimal for the agents with positive inventory to post limit orders, and the proposed method fails to produce a non-degenerate LTC equilibrium in the interval $[n, N]$. Figure 3.2 shows that this does, indeed, occur. Figures 3.2 and 3.3 also show that, for a given (finite) frequency N, if $|\alpha|$ is small enough, a non-degenerate equilibrium may still be constructed. Nevertheless, for any $|\alpha| \neq 0$, however small it is, there exists a large enough N, s.t. the non-degenerate LTC equilibrium fails to exist (at least, within the class defined by the proposed method). This is illustrated in Figure 3.3.

It is important to provide an economic interpretation of why such degeneracy occurs. A careful examination of Figure 3.2 reveals that, around the time when $\hat{\lambda}^a$ becomes nonnegative, the ask price $\hat{p}^a$ explodes. This means that the agents who want to sell the asset are only willing to sell it at a very high price. Notice also that this price is several magnitudes larger than the expected change in the fundamental price (represented by the black dashed line in the left hand side of Figure 3.2). Hence, such a behavior cannot be justified by the fundamental reasons. Indeed, this is precisely what is called an *internal* (or, *self-inflicted*) *liquidity crisis*. So, what causes such a liquidity crisis? Recall that there are two potential reasons for the market to degenerate: the agents may choose to submit market orders (if $\hat{p}^b_n > \hat{\lambda}^a_n$ or $\hat{p}^a_n < \hat{\lambda}^b_n$), or they may choose to wait and do nothing (if $\hat{\lambda}^a_{n+1} \geq 0$ or $\hat{\lambda}^b_{n+1} \leq 0$).

The right hand side of Figure 3.2 shows that the degeneracy is caused by the second scenario. This means that the naive explanation of the internal liquidity crisis, based on the claim that, in a bullish market, those who need to buy the asset will submit market orders wiping out liquidity on the sell side of the book, is wrong. Instead, if the agents on the sell side of the book have the same beliefs, they will increase the ask price so that it is no longer profitable for the agents who want to buy the asset to submit market buy orders. In fact, the ask price may increase disproportionally to the expected change in the fundamental price (i.e. the signal), and this is what causes an internal liquidity crisis. The size of the resulting change in the bid or ask price depends not only on the signal, but also on the trading frequency, which demonstrates the *negative liquidity effect* of increasing the trading frequency: it makes the market more fragile with respect to deviations of the agents from market-neutrality. The latter, in turn, is explained by the fact that higher trading frequency makes the *adverse selection* effect more pronounced. To see this, consider e.g. an agent who is trying to sell one share of the asset. Increasing the trading frequency increases the expected execution value of this agent, bringing it closer to the fundamental price: this corresponds to $\hat{\lambda}^a$ approaching zero (from below). Assume that the agent posts a limit sell order at a price level p. If this order is executed in the next period, then, the agent receives p, but, for this to happen, the fundamental price value at the next time step, p^0_{n+1}, has to be above p. On the other hand, the expected execution price of the agent at the next time step is $p^0_{n+1} + \hat{\lambda}^a_{n+1}$. Thus, the expected relative profit of the agent, given the execution of her limit order, is $\mathbb{E}_n(p - p^0_{n+1} - \hat{\lambda}^a_{n+1} \,|\, p^0_{n+1} > p)$. The latter expression cannot be positive, unless $\hat{\lambda}^a_{n+1} < 0$ and $|\hat{\lambda}^a_{n+1}|$ is sufficiently large. Therefore, if $|\hat{\lambda}^a_{n+1}|$ is small relative to $\mathbb{E}_n(p^0_{n+1} - p \,|\, p^0_{n+1} > p)$, the agent is reluctant to post a limit order at the price level p. Hence, p needs to be sufficiently large, to ensure that $\mathbb{E}_n(p^0_{n+1} - p \,|\, p^0_{n+1} > p)$ is smaller than $|\hat{\lambda}^a_{n+1}|$ (in the Gaussian model of this section, the latter expectation vanishes as $p \to \infty$) – and smallest such level of p determines the effect of adverse selection. It turns out that, if the agents are market-neutral (i.e. $\alpha = 0$), as the frequency N increases, the quantity $\mathbb{E}_n(p^0_{n+1} - p \,|\, p^0_{n+1} > p)$, for any fixed p, converges to zero at the same rate as $|\hat{\lambda}^a_{n+1}|$, hence, the above adverse selection effect does not get amplified. On contrary, if the agents are not market-neutral, $\hat{\lambda}^a_{n+1}$ reaches zero at some high enough (but finite) frequency, while $\mathbb{E}_n(p^0_{n+1} - p \,|\, p^0_{n+1} > p)$ remains strictly positive, for any p, which amplifies the adverse selection effect infinitely and causes the market to degenerate. Of course, so far, these conclusions are based on a very specific example and on a particular method of constructing an equilibrium. The next section shows that all these conclusions remain valid in any model (with, possibly, heterogeneous beliefs) in which the fundamental price is given by an Itô process.

It is worth mentioning that a similar adverse selection effect arises in [34], and it is referred to as the "winner's curse" in [29]. However, the latter papers do not investigate the nature of this phenomenon and focus on other questions instead. In the literature on double auctions (cf. [27], [58]), a similar effect arises when the auction participants choose to decrease their trading activity in a given auction, because they expect many more opportunities to trade in the future. The latter is similar to the agents choosing not to post limit orders and wait, in the present example.

2.4 Main results

In this section, we generalize the conclusions made in the previous section, so that they hold in a general model and for any choice of an equilibrium. As before, we begin with the "limiting" continuous time model. Consider a terminal time horizon $T > 0$ and a complete stochastic basis $(\Omega, \tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t\in[0,T]}, \mathbb{P})$, with a Brownian motion W on it.[9] We define the adapted process $\tilde{p}^0$ as a continuous modification of

$$\tilde{p}^0_t = p^0_0 + \int_0^t \sigma_s dW_s, \qquad p^0_0 \in \mathbb{R}, \tag{2.14}$$

where σ is a progressively measurable locally square integrable process.

Assumption II.12. *There exists a constant $C > 1$, such that, $1/C \leq \sigma_t \leq C$, for all $t \in [0, T]$, $\mathbb{P}$-a.s..*

Consider a Borel set of beliefs $\mathbb{A}$ and the associated family of measures $\{\mathbb{P}^\alpha\}_{\alpha\in\mathbb{A}}$ on $(\Omega, \tilde{\mathcal{F}}_T)$, absolutely continuous with respect to $\mathbb{P}$. Then, for any $\alpha \in \mathbb{A}$, we have

$$\tilde{p}^0_t = p^0_0 + A^\alpha_t + \int_0^t \sigma_s dW^\alpha_s, \quad p^0_0 \in \mathbb{R}, \quad \mathbb{P}^\alpha\text{-a.s.}, \ \forall t \in [0, T],$$

where W^α is a Brownian motion under $\mathbb{P}^\alpha$, and A^α is a process of finite variation. We assume that A^α is absolutely continuous: i.e. for any $\alpha \in \mathbb{A}$, there exists a locally integrable process μ^α, such that:

$$A^\alpha_t = \int_0^t \mu^\alpha_s ds, \quad \mathbb{P}^\alpha\text{-a.s.}, \ \forall t \in [0, T].$$

Assumption II.13. *For any $\alpha \in \mathbb{A}$, the process μ^α is $\mathbb{P}$-a.s. right-continuous, and there exists a constant $C > 0$, such that $|\mu^\alpha_t| \leq C$, for all $t \in [0, T]$, $\mathbb{P}$-a.s..*

Thus, we can rewrite the dynamics of $\tilde{p}^0$, under each $\mathbb{P}^\alpha$, as follows: $\mathbb{P}^\alpha$-a.s., the following holds for all $t \in [0, T]$:

$$\tilde{p}^0_t = p^0_0 + \int_0^t \mu^\alpha_s ds + \int_0^t \sigma_s dW^\alpha_s, \quad p^0_0 \in \mathbb{R}. \tag{2.15}$$

In addition, we modify the above stochastic integral on a set of $\mathbb{P}^\alpha$-measure zero, so that (2.15) holds for *all* (t, ω). In what follows, we often need to analyze the future

[9] In order to ensure the existence of regular conditional probabilities for the discrete time model, we can, for example, assume that $\tilde{\mathcal{F}}_T$ is generated by a random element with values in a standard Borel space.

dynamics of $\tilde{p}^0$ under $\mathbb{P}^\alpha$, conditional on $\tilde{\mathcal{F}}_t$, for various (t, α) simultaneously. This is why we need the following joint regularity assumption.

Assumption II.14. *There exists a modification of regular conditional probabilities*

$$\left\{ \tilde{\mathbb{P}}_t^\alpha = \mathbb{P}^\alpha \left(\cdot \,|\, \tilde{\mathcal{F}}_t \right) \right\}_{t \in [0,T],\, \alpha \in \mathbb{A},}$$

such that it satisfies the tower property with respect to $\mathbb{P}$ *(as described in Section 3.2).*

Assumption II.14 is satisfied, for example, if $\mathbb{P}^\alpha \sim \mathbb{P}$, for all $\alpha \in \mathbb{A}$, or if the set $\mathbb{A}$ is countable. Throughout the rest of the chapter, $\tilde{\mathbb{P}}_t^\alpha$ refers to a member of the family appearing in Assumption II.14. All conditional expectations $\tilde{\mathbb{E}}_t^\alpha$ are taken under such $\tilde{\mathbb{P}}_t^\alpha$.

The main results of this section require additional continuity assumptions on σ and μ^α. The following assumption can be viewed as a stronger version of $\mathbb{L}^2$-continuity of σ.

Assumption II.15. *There exists a function* $\varepsilon(\cdot) \geq 0$*, such that* $\varepsilon(\Delta t) \to 0$*, as* $\Delta t \to 0$*, and,* $\mathbb{P}$*-a.s.,*

$$\tilde{\mathbb{P}}_t^\alpha \left(\mathbb{E}^\alpha \left((\sigma_{s \vee \tau} - \sigma_\tau)^2 \,|\, \mathcal{F}_\tau \right) \leq \varepsilon(\Delta t) \right) = 1$$

holds for all $t \in [0, T - \Delta t]$*, all* $s \in [t, t + \Delta t]$*, all stopping times* $t \leq \tau \leq s$*, and all* $\alpha \in \mathbb{A}$*.*

The above assumption is satisfied, for example, if σ is an Itô process with bounded drift and diffusion coefficients. Next, we state a continuity assumption on the drift, which can be interpreted as a uniform right-continuity in probability of the martingale $\tilde{\mathbb{E}}_t^\alpha \mu_s^\alpha$.

Assumption II.16. *For any $\alpha \in \mathbb{A}$ and any $t \in [0,T)$, there exists a deterministic function $\varepsilon(\cdot) \geq 0$, such that $\varepsilon(\Delta t) \to 0$, as $\Delta t \to 0$, and, $\mathbb{P}^\alpha$-a.s.,*

$$\tilde{\mathbb{P}}^\alpha_{t'} \left(\left| \int_t^T \left(\tilde{\mathbb{E}}^\alpha_{t''} \mu^\alpha_s - \tilde{\mathbb{E}}^\alpha_{t'} \mu^\alpha_s \right) ds \right| \geq \varepsilon(\Delta t) \right) \leq \varepsilon(\Delta t)$$

holds for all $t \leq t' \leq t'' \leq t + \Delta t \leq T$.

Notice that Assumptions II.14, II.15, and II.16 are not quite standard. Therefore, below, we describe a more specific (although, still, rather general) diffusion-based framework, in which the Assumptions II.12–II.16 reduce to standard regularity conditions on the diffusion coefficients, and are easily verified. To this end, consider a model in which $\mu^\alpha_t = \bar{\mu}^\alpha(t, Y_t)$, $\sigma_t = \bar{\sigma}(t, Y_t)$, and, under $\mathbb{P}$, the process Y is a diffusion taking values in $\mathbb{R}^d$:

$$dY_t = \Gamma(t, Y_t)dt + \Sigma(t, Y_t)d\bar{B}_t,$$

where $\Gamma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\Sigma = (\Sigma^{i,j})$ is a mapping on $[0,T] \times \mathbb{R}^d$ with values in the space of $d \times m$ matrices, and $\bar{B}$ is m-dimensional Brownian motion under $\mathbb{P}$ (on the original stochastic basis). We assume that Γ and Σ possess enough regularity to conclude that Y is a strongly Markov process. Notice that Assumptions II.12 and II.13 reduce to the upper and lower bounds on the functions $\bar{\mu}^\alpha$ and $\bar{\sigma}$. Assumption II.14 is satisfied if we assume that $\mathbb{P}^\alpha \sim \mathbb{P}$, for all $\alpha \in \mathbb{A}$. Let us further assume that the Radon-Nikodym derivative of each measure is in Girsanov form:

$$\frac{d\mathbb{P}^\alpha}{d\mathbb{P}} = \exp\left(-\frac{1}{2} \int_0^t \|\gamma^\alpha(s, Y_s)\|^2 ds + \int_0^t \gamma^\alpha(s, Y_s) d\bar{B}_s \right),$$

with an $\mathbb{R}^d$-valued function γ^α, for each $\alpha \in \mathbb{A}$. Let us assume that all entries of Γ, γ^α and Σ are absolutely bounded by a constant (uniformly over $\alpha \in \mathbb{A}$). Assuming, in addition, that $\bar{\sigma}$ is globally Lipschitz, we easily verify Assumption II.15. In order to verify Assumption II.16, we assume that the quadratic form generated by $A(t,y) := \Sigma(t,y)\Sigma^T(t,y)$ is bounded away from zero, uniformly over all (t,y), and that the entries of Γ, γ^α and Σ are continuously differentiable with absolutely bounded derivatives (uniformly over $\alpha \in \mathbb{A}$). Then, the Feynman-Kac formula implies that, for any $t \leq s$,

$$\tilde{\mathbb{E}}^\alpha_t \mu^\alpha_s = u^{s,\alpha}(t, Y_t),$$

where $u^{s,\alpha}$ is the unique solution to the associated partial differential equation (PDE):

$$\partial_t u^{s,\alpha} + \sum_{i=1}^d \Gamma^{\alpha,i} \partial_{y_i} u^{s,\alpha} + \frac{1}{2} \sum_{i,j=1}^d A^{i,j} \partial^2_{y_i y_j} u^{s,\alpha} = \\ 0, \quad (t,y) \in (0,s) \times \mathbb{R}^d, \quad u^{s,\alpha}(s,y) = \bar{\mu}^\alpha(s,y),$$

and $\Gamma^\alpha = \Gamma + \Sigma\gamma^\alpha$. Assume that, for each $s \in [0,T]$, the function $\bar{\mu}^\alpha(s,\cdot)$ is continuously differentiable with absolutely bounded derivatives, uniformly over all (s,α). Then, the standard Gaussian estimates for derivatives of the fundamental solution to the above PDE (cf. Theorem 9.4.2 in [30]) imply that every $\partial_{y_i} u^{s,\alpha}$ is absolutely bounded, uniformly over all (s,α). Then, Itô's formula and Itô's isometry yield, for all $t' \leq t''$ and $s \geq t'$:

$$\tilde{\mathbb{E}}^\alpha_{t'} \left(\tilde{\mathbb{E}}^\alpha_{t''} \mu^\alpha_s - \tilde{\mathbb{E}}^\alpha_{t'} \mu^\alpha_s \right)^2 = \\ \sum_{j=1}^m \int_{t'}^{t'' \wedge s} \tilde{\mathbb{E}}^\alpha_{t'} \left(\sum_{i=1}^d \partial_{y_i} u^{s,\alpha}(v, Y_v) \Sigma^{i,j}(v, Y_v) \right)^2 dv \leq C_1(t'' \wedge s \, - \, t'),$$

with some constant $C_1 > 0$. The above estimate and Jensen's inequality imply the statement of Assumption II.16 and complete the description of the diffusion-based setting.

As in Section 2.3, we also consider a progressively measurable random field $\tilde{D}$, s.t. $\mathbb{P}$-a.s. the function $\tilde{D}_t(\cdot) - \tilde{D}_s(\cdot)$ is strictly decreasing and vanishing at zero, for any $0 \leq s < t \leq T$. We assume that the demand curve, $\tilde{D}_t(\cdot) - \tilde{D}_s(\cdot)$, cannot be "too flat".

Assumption II.17. *There exists $\varepsilon > 0$, s.t., for any $0 \leq t - \varepsilon \leq s < t \leq T$, there exists a $\tilde{\mathcal{F}}_s \otimes \mathcal{B}(\mathbb{R})$-measurable random function $\kappa_s(\cdot)$, s.t., $\mathbb{P}$-a.s., $\kappa_s(\cdot)$ is strictly decreasing and $\left|\tilde{D}_t(p) - \tilde{D}_s(p)\right| \geq |\kappa_s(p)|$, for all $p \in \mathbb{R}$.*

Finally, we introduce the empirical distribution process $(\tilde{\mu}_t)$, with values in the space of finite sigma-additive measures on $\mathbb{S}$. The next assumption states that every $\tilde{\mu}_t$ is dominated by a deterministic measure.

Assumption II.18. *For any $t \in [0, T]$, there exists a finite sigma-additive measure μ_t^0 on $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$, s.t., $\mathbb{P}$-a.s., $\tilde{\mu}_t$ is absolutely continuous w.r.t. μ_t^0.*

We partition the time interval $[0, T]$ into N subintervals of size $\Delta t = T/N$. A discrete time model is obtained by discretizing the continuous time one:

$$\mathcal{F}_n = \tilde{\mathcal{F}}_{n\Delta t}, \quad p_n^0 = \tilde{p}_{n\Delta t}^0, \quad D_n(p) = (\tilde{D}_{n\Delta t} - \tilde{D}_{(n-1)\Delta t})(p - p_n^0), \quad \mu_n = \tilde{\mu}_{n\Delta t}.$$

Before we present the main results, let us comment on the above assumptions. These assumptions are important from a technical point of view, however, some of them have economic interpretation that may provide (partial) intuitive explanations of the results that follow. In particular, Assumption II.12 ensures that the fundamental price remains "noisy", which implies that an agent can execute a limit order very quickly by posting it close to the present value of p^0, if there are no other orders posted there. In combination with Assumption II.17, the latter implies that, when the frequency, N, is high, an agent has a lot of opportunities to execute her limit order at a price close to the fundamental price (at least, if no other orders are posted too close to the fundamental price). Intuitively, this means that the agent's execution value should improve as the frequency increases. Assumption II.16 ensures that, if an agent has a signal about the direction of the fundamental price, this signal is persistent – i.e. it is continuous in the appropriate sense. When the trading frequency N is large, such persistency means that an agent has a large number of opportunities to exploit the signal, implying that she is in no rush to have her order executed immediately. The main results of this work, presented below, along with their proofs, confirm that these heuristic conclusions are, indeed, correct.

As mentioned in the preceding sections, our main goal is to analyze the liquidity effects of increasing the trading frequency. Therefore, we fix a limiting continuous time model, and consider a sequence of discrete time models, obtained from the limiting one as described above, for $N \to \infty$. This can be interpreted as observing the same population of agents, each of whom has a fixed continuous time model for future demand, in various exchanges that allow for different trading frequencies. We begin with the following theorem, which shows that, if every market model in a given sequence admits a non-degenerate equilibrium, then, the terminal bid and ask prices converge to the fundamental price, as the trading frequency goes to infinity.

Theorem II.19. *Let Assumptions II.12, II.13, II.14, II.15, II.17, II.18 hold. Consider a family of uniform partitions of a given time interval* $[0, T]$*, with diameters* $\{\Delta t = T/N > 0\}$ *and with the associated family of discrete time models, and denote the associated fundamental price process by* $p^{0,\Delta t}$*. Assume that every such model admits a non-degenerate LTC equilibrium, and denote the associated bid and ask prices by* $p^{b,\Delta t}$ *and* $p^{a,\Delta t}$ *respectively. Then, there exists a deterministic function* $\varepsilon(\cdot)$*, s.t.* $\varepsilon(\Delta t) \to 0$*, as* $\Delta t \to 0$*, and, for all small enough* $\Delta t > 0$*, the following holds* $\mathbb{P}$*-a.s.:*

$$\left|p_N^{a,\Delta t} - p_N^{0,\Delta t}\right| + \left|p_N^{b,\Delta t} - p_N^{0,\Delta t}\right| \leq \varepsilon(\Delta t)$$

The above theorem has a useful corollary, which can be interpreted as follows: *if the market does not degenerate as the frequency increases, then, such an increase improves market efficiency.* Here, we understand the "improving efficiency" in the sense that the expected execution price (i.e. the price per share that an agent expects to receive or pay by the end of the game) of every agent converges to the fundamental price.

Corollary II.20. *Under the assumptions of Theorem II.19, denote the support of every equilibrium by* $\tilde{\mathbb{A}}^{\Delta t}$ *and the associated expected execution prices by* $\lambda^{a,\Delta t}$ *and* $\lambda^{b,\Delta t}$*. Then, there exists a deterministic function* $\varepsilon(\cdot)$*, such that* $\varepsilon(\Delta t) \to 0$*, as* $\Delta t \to 0$*, and,* $\mathbb{P}$*-a.s.,*

$$\sup_{n=0,\ldots,N,\,\alpha\in\tilde{\mathbb{A}}^{\Delta t}} \left(\left|\lambda^{a,\Delta t}_n(\alpha) - p^{0,\Delta t}_n\right| + \left|\lambda^{b,\Delta t}_n(\alpha) - p^{0,\Delta t}_n\right|\right) \leq \varepsilon(\Delta t),$$

for all small enough $\Delta t > 0$.

Proof: Denote $\mathbb{E}^\alpha_n = \tilde{\mathbb{E}}^\alpha_{n\Delta t}$. It follows from Corollary II.31, in Appendix A, and the definition of LTC equilibrium that $\lambda^{a,\Delta t}_N(\alpha) = p^{b,\Delta t}_N$ and $\lambda^{b,\Delta t}_N(\alpha) = p^{a,\Delta t}_N$. It also follows from Corollary II.31 (or, more generally, from the definition of a value function) that $\lambda^{a,\Delta t}(\alpha)$ is a supermartingale, and $\lambda^{b,\Delta t}(\alpha)$ is a submartingale, under $\mathbb{P}^\alpha$. Thus, we have: $\lambda^{a,\Delta t}_n(\alpha) \geq \mathbb{E}^\alpha_n p^{b,\Delta t}_N$ and $\lambda^{b,\Delta t}_n(\alpha) \leq \mathbb{E}^\alpha_n p^{a,\Delta t}_N$. On the other hand, notice that we must have: $\lambda^{a,\Delta t}_n(\alpha) \leq \mathbb{E}^\alpha_n p^{a,\Delta t}_N$ and $\lambda^{b,\Delta t}_n(\alpha) \geq \mathbb{E}^\alpha_n p^{b,\Delta t}_N$. Assume, for example, that $\lambda^{a,\Delta t}_n(\alpha) > \mathbb{E}^\alpha_n p^{a,\Delta t}_N$ on the event Ω' of positive $\mathbb{P}^\alpha$-probability. Consider an agent at state $(0,\alpha)$, who follows the optimal strategy of an agent at state $(1,\alpha)$, starting from time n and onward, on the event Ω' (otherwise, she does not do anything). It is easy to see that the objective value of this strategy is

$$\mathbb{E}^\alpha\left(\mathbf{1}_{\Omega'}\left(\lambda^{a,\Delta t}_n(\alpha) - \mathbb{E}^\alpha_n p^{a,\Delta t}_N\right)\right) > 0,$$

which contradicts Corollary II.31. The second inequality is shown similarly. Thus, we conclude that, for any $n = 0,\ldots,N-1$, both $\lambda^{a,\Delta}_n(\alpha)$ and $\lambda^{b,\Delta}_n(\alpha)$ belong to the interval

$$\left[\mathbb{E}^\alpha_n p^{b,\Delta t}_N,\, \mathbb{E}^\alpha_n p^{a,\Delta t}_N\right],$$

which, in turn, converges to zero, as $\Delta t \to 0$, due to the deterministic bounds obtained in the proof of Proposition II.19. ∎

The results of Theorem II.19 and Corollary II.20 can be viewed as a specific case of a more general observation: markets become more efficient as the frictions become smaller. In the present setting, the limited trading frequency is viewed as friction, and the market efficiency is measured by the difference between the bid and ask prices, or between the expected execution prices. Many more instances of analogous results can be found in the literature, depending on the choice of a friction type. For example, the markets become efficient in [33] and [43] as the number of insiders vanishes. Similarly, the markets become efficient in [27] as the trading frequency increases and the size of private signals vanishes. It is also mentioned in [13] that the market would become efficient if there was no restriction on the size of agents' inventories therein.

The above results demonstrate the positive role of high trading frequency. However, they are based on the assumption that the market does not degenerate as frequency increases. In the context of Section 2.3, we saw that the markets do not degenerate only if the agents are market-neutral (i.e. $\alpha = 0$). If this condition is violated and the frequency N is sufficiently high, the market admits no non-degenerate equilibrium (i.e. there exists no safe regime, in which the liquidity crisis would never occur). It turns out that this conclusion still holds in the general setting considered herein.

Theorem II.21. *Let Assumptions II.12, II.13, II.14, II.15, II.16, II.17, II.18 hold. Consider a family of uniform partitions of a given time interval* $[0, T]$*, with diameters* $\{\Delta t = T/N > 0\}$*, containing arbitrarily small* Δt*, and with the associated family of discrete time models. Assume that every such model admits a non-degenerate*

LTC equilibrium, with the same support $\tilde{\mathbb{A}}$. *Then, for all* $\alpha \in \tilde{\mathbb{A}}$, *we have:* $\tilde{p}^0$ *is a* **martingale** *under* $\mathbb{P}^\alpha$.

The above theorem shows that the market degenerates even if the signal μ^α is very small (but non-zero), provided the trading frequency N is large enough. Therefore, as discussed at the end of Section 2.3, such degeneracy cannot be attributed to any fundamental reasons, and we refer to it as the *internal* (or, *self-inflicted*) *liquidity crisis*. Let us provide an intuitive (heuristic) argument for why the statement of Theorem II.21 holds. Assume, first, that all long agents (i.e. those having positive inventory) are bullish about the asset (i.e. have a positive drift μ^α). Then, similar to Section 2.3, the higher trading frequency amplifies the *adverse selection effect*, forcing the long agents to withdraw liquidity from the market (i.e. they prefer to do nothing and wait for a higher fundamental price level). Note that, in the present setting, the agents may have different beliefs, the LOB may have a complicated shape and dynamics, and the expected execution prices are no longer deterministic. All this makes it difficult to provide a simple description of how the high frequency amplifies the adverse selection. Nevertheless, the general analysis of this case is still based on the idea discussed at the end of Section 2.3: it has to do with how fast $\tilde{\mathbb{E}}^\alpha_{n\Delta t}(p^0_{n+1} - p \,|\, p^0_{n+1} > p)$ vanishes (as the frequency increases), relative to the rate at which the expected execution prices approach the fundamental price. Thus, there must be a non-zero mass of long agents who are market-neutral or bearish. As the trading frequency grows, these agents will post their limit orders at lower levels. Next, assume that there exists a bullish agent (long, short, or with zero inventory). Then, at a sufficiently high trading frequency, the agent's expected value of a long position in a single share of the asset will exceed the ask prices posted by the market-neutral and bearish long agents. In this case, the bullish agent prefers to buy more asset at the posted ask price, in order to sell it later. As the agents are small and their objectives are linear, the bullish agent can scale up her strategy to generate infinite expected profits. This contradicts the definition of optimality and implies that an equilibrium fails to exist. Thus, all agents have to be either market-neutral or bearish. Applying a symmetric argument, we conclude that all agents must be market-neutral.[10] A rigorous formulation of the above arguments, which constitutes the proof of Theorem II.21, is given in Section 2.7.

It is worth mentioning that the possible degeneracy of LOB is also documented in [33], and is referred to as a "market shut down". The setting used in the latter paper is very different: it analyzes a quote-driven exchange (i.e. the one with a designated market maker) and assumes the existence of insiders with superior information. Nevertheless, it is possible to draw a parallel with the LOB degeneracy in the present setting. Namely, the degeneracy in [33] occurs when the number of insiders increases, which implies that the signal, generated by the insiders' trading, becomes sufficiently large. The latter is similar to the deviation from martingality of the fundamental price in the present setting. However, an increase in the number of insiders in [33] also implies an increase in frictions (since the insiders can be interpreted as friction in [33]). Theorem II.21, on the other hand, states that a market degeneracy will occur when the frictions are sufficiently small. Perhaps, this dual role of the number of insiders did not allow for a detailed analysis of market shut downs in [33]. Many other models of market microstructure (cf. [34], [52], [50], [29], [27]) are not well suited for the analysis of market degeneracy, either because the agents in these models pursue "one-shot" strategies (i.e. they cannot choose to wait and post a limit order later) or because the fundamental price (or its analogue) is

[10] This argument, along with the fact that Definition II.3 requires an optimal control to be optimal for *all* α, explains why the statement of Theorem II.21 holds for *all*, as opposed to μ_n-a.e., $\alpha \in \tilde{\mathbb{A}}$.

restricted to be a martingale.

2.5 Conditional tails of the marginal distributions of Itô processes

As follows form the discussion in the preceding sections, in order to prove the main results of the chapter, we need to investigate the properties of marginal distributions of the fundamental price $\tilde{p}^0$ (more precisely, the distributions of its increments). In order to prove Theorem II.19, we need to show that the difference between the fundamental price and the bid or ask price converges to zero, as the frequency N increases to infinity. It turns out that, for this purpose, it suffices to show that the distribution of a normalized increment of $\tilde{p}^0$ converges to the standard normal distribution. The following lemma summarizes these results. It is rather simple, but technical, hence, its proof is postponed to Appendix B. In order to formulate the result (and to facilitate the derivations in subsequent sections), we introduce addiitonal notation. For convenience, we drop the superscript Δt in many variables which do, in fact, depend on Δt, hoping it causes no confusion (we emphasize this dependence whenever it is important). For any market model on the time interval $[0,T]$, associated with a uniform partition with diameter $\Delta t = T/N > 0$, and having a fundamental price process p^0, we define

(2.16)

$$\xi_n = p^0_n - p^0_{n-1} = \tilde{p}^0_{t_n} - \tilde{p}^0_{t_{n-1}}, \quad \mathbb{E}^\alpha_n = \tilde{\mathbb{E}}^\alpha_{t_n}, \quad \mathbb{P}^\alpha_n = \tilde{\mathbb{P}}^\alpha_{t_n}, \quad t_n = n\Delta t, \quad n = 1, \dots, NT/\Delta t.$$

We denote by η_0 a standard normal random variable (on a, possibly, extended probability space), which is independent of $\mathcal{F}_N$ under every $\mathbb{P}^\alpha$.

Lemma II.22. *Let Assumptions II.12, II.13, II.14, II.15 hold. Then, there exists a function $\varepsilon(\cdot) \geq 0$, s.t. $\varepsilon(\Delta t) \to 0$, as $\Delta t \to 0$, and the following holds $\mathbb{P}$-a.s., for all $p \in \mathbb{R}$, all $\alpha \in \mathbb{A}$, and all $n = 1, \dots, N$:*

(i) $(|p| \vee 1) \left| \mathbb{P}^{\alpha}_{n-1} \left(\frac{\xi_n}{\sqrt{\Delta t}} > p \right) - \mathbb{P}^{\alpha}_{n-1} \left(\sigma_{t_{n-1}} \eta_0 > p \right) \right| \leq \varepsilon(\Delta t)$

(ii) $\left| \mathbb{E}^{\alpha}_{n-1} \left(\frac{\xi_n}{\sqrt{\Delta t}} \mathbf{1}_{\{\xi_n/\sqrt{\Delta t} > p\}} \right) - \mathbb{E}^{\alpha}_{n-1} \left(\sigma_{t_{n-1}} \eta_0 \mathbf{1}_{\{\sigma_{t_{n-1}} \eta_0 > p\}} \right) \right| \leq \varepsilon(\Delta t)$

In addition, the above estimates hold if we replace (ξ_n, η_0, p) *by* $(-\xi_n, -\eta_0, -p)$.

In order to prove Theorem II.21 we need to compare the rates at which the conditional expectations $\mathbb{E}^{\alpha}_n(p^0_{n+1} - p \,|\, p^0_{n+1} > p)$ vanish (as the frequency N goes to infinity) to the rate at which the expected execution prices converge to the fundamental price. This requires a more delicate analysis – in particular, the mere proximity of the distribution of a (normalized) fundamental price increment to the Gaussian distribution is no longer sufficient. In fact, what we need is a precise uniform estimate of the conditional tail of the distribution of a fundamental price increment. The desired property is formulated in the following lemma, which, we believe, is valuable in its own right. This result allows us to estimate the tails of the conditional marginal distribution of an Itô process X uniformly by an exponential. To the best of our knowledge, this result is new. The main difficulties in establishing the desired estimates are: (a) the fact that we estimate the *conditional*, as opposed to the regular, tail, and (b) the fact that the estimates need to be uniform over the values of the argument. Note that, even in the case of a diffusion process X, the classical Gaussian-type bounds for the tails of the marginal distributions of X are not sufficient to establish the desired estimates. The reason is that, in general, the Gaussian estimates of the regular tails from above and from below have different orders of decay, for the large values of the argument, which makes them useless for estimating the conditional tail (which is a ratio of two regular tails).

Lemma II.23. *Consider the following continuous semimartingale on a stochastic* *basis* $(\hat{\Omega}, (\hat{\mathcal{F}}_t)_{t\in[0,1]}, \hat{\mathbb{P}})$*:*

$$X_t = \int_0^t \hat{\mu}_u du + \int_0^t \hat{\sigma}_u dB_u, \qquad t \in [0,1],$$

where B *is a Brownian motion (with respect to the given stochastic basis),* $\hat{\mu}$ *and* $\hat{\sigma}$ *are progressively measurable processes, such that the above integrals are well defined. Assume that, for any stopping time* τ *with values in* $[0,1]$*,* $c \le |\hat{\sigma}_\tau| \le C$ *holds a.s. with some constants* $c, C > 0$*. Then, there exists* $\varepsilon > 0$*, depending only on* (c, C)*, s.t., if*

$$\hat{\mu}_\tau^2 \le \varepsilon, \quad \hat{\mathbb{E}}\left((\hat{\sigma}_{s\vee\tau} - \hat{\sigma}_\tau)^2 \,|\, \hat{\mathcal{F}}_\tau\right) \le \varepsilon \ \ a.s.,$$

for all $s \in [0,1]$ *and all stopping time* τ*, with values in* $[0,1]$*, then, for any* $c_1 > 0$*, there exists* $C_1 > 0$*, depending only on* (c, C, ε, c_1)*, s.t. the following holds:*

$$\hat{\mathbb{P}}(X_1 > x + z \,|\, X_1 > x) \le C_1 e^{-c_1 z}, \quad \forall x, z \ge 0.$$

Proof: In the course of this proof, we will use the shorthand notation, $\hat{\mathbb{E}}_\tau$ and $\hat{\mathbb{P}}_\tau$, to denote the conditional expectation and the conditional probability w.r.t $\hat{\mathcal{F}}_\tau$. We also denote

$$A_t = \int_0^t \hat{\mu}_u du, \qquad G_t = \int_0^t \hat{\sigma}_u dB_u.$$

For any $x \ge 0$, let us introduce $\tau_x = 1 \wedge \inf\{t \in [0,1] \,:\, X_t = x\}$. Then

$$\hat{\mathbb{P}}(X_1 > x + z) \le \hat{\mathbb{P}}(\sup_{t\in[0,1]} X_t > x + z) = \hat{\mathbb{E}}\left(\mathbf{1}_{\{\tau_x<1\}}\hat{\mathbb{P}}_{\tau_x}\left(\sup_{s\in[\tau_x,1]} (X_s - x) > z\right)\right)$$

Notice that, on $\{\tau_x \le s\}$, we have: $X_s - x = A_{s\vee\tau_x} - A_{\tau_x} + G_{s\vee\tau_x} - G_{\tau_x}$. In addition, the process $(Y)_{s\in[0,1]}$, with $Y_s = A_{s\vee\tau_x} - A_{\tau_x}$, is adapted to the filtration $(\hat{\mathcal{F}}_{\tau_x\vee s})$, while the process $(Z)_{s\in[0,1]}$, with $Z_s = G_{s\vee\tau_x} - G_{\tau_x}$, is a martingale with respect to it. Next, on $\{\tau_x < 1\}$, we have:

$$\hat{\mathbb{P}}_{\tau_x}\left(\sup_{s\in[\tau_x,1]} (X_s - x) > z\right) = \hat{\mathbb{P}}_{\tau_x}\left(\sup_{s\in[0,1]} (Y_s + Z_s) > z\right)$$

$$\leq \hat{\mathbb{P}}_{\tau_x}\left(\sup_{s\in[0,1]} \exp\left(c_1 Z_s - \frac{1}{2}c_1^2\langle Z\rangle_s\right) > \exp\left(c_1 z - c_1\sqrt{\varepsilon} - \frac{1}{2}c_1^2 C^2\right)\right),$$

where we used the fact that $\langle Z\rangle_s \leq \langle X\rangle_1 \leq C^2$, for all $s \in [0,1]$. Using the Novikov's condition, it is easy to check that

$$M_s = \exp\left(c_1 Z_s - \frac{1}{2}c_1^2\langle Z\rangle_s\right), \qquad s \in [0,1],$$

is a true martingale, and, hence, we can apply the Doob's martingale inequality to obtain, on $\{\tau_x < 1\}$:

$$\hat{\mathbb{P}}_{\tau_x}\Big(\sup_{s\in[0,1]} \exp\left(c_1 Z_s - \frac{1}{2}c_1^2\langle Z\rangle_s\right) > \exp\left(c_1 z - c_1\sqrt{\varepsilon} - \frac{1}{2}c_1^2 C^2\right)\Big) \leq \exp\left(-c_1 z + c_1\sqrt{\varepsilon} + \frac{1}{2}c_1^2 C^2\right).$$

Collecting the above inequalities, we obtain

(2.17)

$$\hat{\mathbb{P}}(X_1 > x+z) \leq \hat{\mathbb{P}}(\sup_{t\in[0,1]} X_t > x+z) \leq C_2(\varepsilon)e^{-c_1 z}\hat{\mathbb{P}}(\tau_x < 1) = C_2(\varepsilon)e^{-c_1 z}\hat{\mathbb{P}}(\sup_{t\in[0,1]} X_t > x).$$

The next step is to estimate the distribution tails of a running maximum via the tails of the distribution of X_1. To do this, we proceed as before:

$$\hat{\mathbb{P}}(X_1 > x) = \hat{\mathbb{E}}\left(\mathbf{1}_{\{\tau_x<1\}}\hat{\mathbb{P}}_{\tau_x}\left(Y_1 + Z_1 > 0\right)\right), \tag{2.18}$$

with Y and Z defined above. Notice that, on $\{\tau_x < 1\}$,

$$\hat{\mathbb{P}}_{\tau_x}\left(Y_1 + Z_1 > 0\right) = \hat{\mathbb{P}}_{\tau_x}\left(\hat{\sigma}_{\tau_x}\frac{B_1 - B_{\tau_x}}{\sqrt{1-\tau_x}} + \frac{1}{\sqrt{1-\tau_x}}\int_{\tau_x}^1 \hat{\mu}_u du + \frac{1}{\sqrt{1-\tau_x}}\int_0^1 (\hat{\sigma}_{u\vee\tau_x} - \hat{\sigma}_{\tau_x})dB_u^x > 0\right),$$

where $B_s^x = B_{s\vee\tau_x}$ is a continuous square-integrable martingale with respect to $(\hat{\mathcal{F}}_{s\vee\tau_x})$. Denote

$$R_s = \int_0^s (\hat{\sigma}_{u\vee\tau_x} - \hat{\sigma}_{\tau_x})dB_u^x, \quad s \in [0,1],$$

and notice that it is a square-integrable martingale with respect to $(\hat{\mathcal{F}}_{s\vee\tau_x})$. Then, on $\{\tau_x < 1\}$ (possibly, without a set of measure zero), we have:

$$\hat{\mathbb{E}}_{\tau_x}\left(\frac{1}{\sqrt{1-\tau_x}}R_1\right)^2 = \frac{1}{1-\tau_x}\hat{\mathbb{E}}_{\tau_x}R_1^2 \leq \frac{1}{1-\tau_x}\int_{\tau_x}^1 \hat{\mathbb{E}}_{\tau_x}(\hat{\sigma}_{u\vee\tau_x} - \hat{\sigma}_{\tau_x})^2 du \leq \varepsilon.$$

In addition,

$$\hat{\mathbb{E}}_{\tau_x}\left(\frac{1}{\sqrt{1-\tau_x}}\int_{\tau_x}^1 \hat{\mu}_u du\right)^2 \leq \varepsilon.$$

Collecting the above and using Chebyshev's inequality, we obtain, on $\{\tau_x < 1\}$:

$$\left|\hat{\mathbb{P}}_{\tau_x}\left(Y_1 + Z_1 > 0\right) - \hat{\mathbb{P}}_{\tau_x}\left(\hat{\sigma}_{\tau_x}\frac{B_1 - B_{\tau_x}}{\sqrt{1-\tau_x}} \leq -\varepsilon^{1/3}\right)\right| \leq 2\varepsilon^{1/6}.$$

On the other hand, due to the strong Markov property of Brownian motion, on $\{\tau_x < 1\}$, we have, a.s.:

$$\hat{\mathbb{P}}_{\tau_x}\left(\hat{\sigma}_{\tau_x}\frac{B_1 - B_{\tau_x}}{\sqrt{1-\tau_x}} \leq -\varepsilon^{1/3}\right) = \hat{\mathbb{P}}\left(\xi \leq -\frac{\varepsilon^{1/3}}{\sigma}\right)\bigg|_{\sigma=\hat{\sigma}_{\tau_x}},$$

where ξ is a standard normal. As $\hat{\sigma}_{\tau_x} \in [c, C]$, we conclude that the right hand side of the above converges to $1/2$, as $\varepsilon \to 0$, uniformly over almost all random outcomes in $\{\tau_x < 1\}$. In particular, for all small enough $\varepsilon > 0$, we have:

$$\mathbf{1}_{\{\tau_x<1\}}\left|\hat{\mathbb{P}}_{\tau_x}\left(Y_1 + Z_1 \leq 0\right) - \hat{\mathbb{P}}_{\tau_x}\left(Y_1 + Z_1 > 0\right)\right| \leq \mathbf{1}_{\{\tau_x<1\}}\delta(\varepsilon) < 1,$$

and, in view of (2.18),

$$\hat{\mathbb{P}}(X_1 > x) \geq \hat{\mathbb{E}}\left(\mathbf{1}_{\{\tau_x<1\}}\hat{\mathbb{P}}_{\tau_x}\left(Y_1 + Z_1 \leq 0\right)\right) - \delta(\varepsilon)\hat{\mathbb{P}}(\tau_x < 1)$$

Summing up the above inequality and (2.18), we obtain

$$2\hat{\mathbb{P}}(X_1 > x) \geq (1 - \delta(\varepsilon))\hat{\mathbb{P}}(\tau_x < 1) = (1 - \delta(\varepsilon))\hat{\mathbb{P}}(\sup_{t\in[0,1]} X_t > x),$$

which, along with (2.17), yields the statement of the lemma. ∎

2.6 Proof of Theorem II.19

Within the scope of this proof, we adopt the notation introduced in (2.16) and use the following convention.

Notation II.24. The LOB, the bid and ask prices, the expected execution prices, and the demand, are all measured relative to p^0. Namely, we use ν_n to denote $\nu_n \circ (x \mapsto x + p^0_n)^{-1}$, p^a_n to denote $p^a_n - p^0_n$, p^b_n to denote $p^b_n - p^0_n$, λ^a_n to denote $\lambda^a_n - p^0_n$, λ^b_n to denote $\lambda^b_n - p^0_n$, and $D_n(p)$ to denote $D_n(p^0_n + p)$.

Herein, we are only concerned with what happens in the last trading period – at time $(N-1)$, where $N = T/\Delta t$. Hence, we omit the subscript $N-1$ whenever it is clear from the context. In particular, we write p^a and p^b for p^a_{N-1} and p^b_{N-1}, ν for ν_{N-1}, and ξ for ξ_N. Note also that, in an LTC equilibrium, we have: $p^a = p^a_N = p^a_{N-1}$, with similar equalities for p^b and ν. For convenience, we also drop the superscript Δt in the LOB and the associated bid and ask prices. Finally, we denote by $\tilde{\mathbb{A}}$ the support of a given equilibrium. As the roles of p^a and p^b in our model are symmetric, we will only prove the statement of the proposition for p^b. We are going to show that, under the assumptions of the theorem, there exists a constant $C_0 > 0$, depending only on the constant C in Assumptions II.12 and II.13, such that, for all small enough Δt, we have, $\mathbb{P}$-a.s.:

$$-C_0 \leq p^b/\sqrt{\Delta t} < 0 \tag{2.19}$$

First, we introduce $\hat{A}^\alpha(p;x)$, which we refer to as the simplified objective:

$$\hat{A}^\alpha(p;x) = \mathbb{E}^\alpha_{N-1}\left((p - x - \xi)\mathbf{1}_{\{\xi > p\}}\right). \tag{2.20}$$

Recall that the expected relative profit from posting a limit sell order at price level

p, in the last time period,[11] is given by $A^\alpha(p; p^b_N)$, where

$$A^\alpha(p;x) = \mathbb{E}^\alpha_{N-1}\left((p-x-\xi)\mathbf{1}_{\{D^+_N(p-\xi)>\nu^+((-\infty,p))\}}\right). \tag{2.21}$$

The simplified objective is similar to A^α, but it assumes that there are no orders posted at better prices than the one posted by the agent. In particular, $\hat{A}^\alpha(p;x) = A^\alpha(p;x)$ for $p \leq p^a$. Corollary II.31, in Appendix A, states that, in equilibrium, $\mathbb{P}$-a.s., if the agents in the state (s,α) post limit sell orders, then they post them at a price level p that maximizes the true objective $A^\alpha(p;p^b)$. The following lemma shows that the value of the modified objective becomes close to the value of the true objective, for the agents posting limit sell orders close to the ask price.

Lemma II.25. *$\mathbb{P}$-a.s., either $\nu^+(\{p^a\}) > 0$ or we have:*

$$\left|A^\alpha(p;p^b) - \hat{A}^\alpha(p^a;p^b)\right| \to 0,$$

as $p \downarrow p^a$, uniformly over all $\alpha \in \tilde{\mathbb{A}}$.

Proof: If $\nu^+(\{p^a\}) = 0$, then ν^+ is continuous at p^a, and $\nu^+((-\infty,p]) \to 0$, as $p \downarrow p^a$. Then, we have

$$\begin{aligned}
&\left|A^\alpha(p;p^b) - \hat{A}^\alpha(p^a;p^b)\right| \\
&= \left|\mathbb{E}^\alpha_{N-1}\left((p-p^b-\xi)\mathbf{1}_{\{D^+_N(p-\xi)>\nu^+((-\infty,p))\}}\right) - \mathbb{E}^\alpha_{N-1}\left((p^a-p^b-\xi)\mathbf{1}_{\{\xi>p^a\}}\right)\right| \\
&\leq |p-p^a| + \left\|p^a-p^b-\xi\right\|_{\mathbb{L}^2\left(\mathbb{P}^\alpha_{N-1}\right)} \mathbb{P}^\alpha_{N-1}\left(\xi > p^a,\, D^+_N(p-\xi) \leq \nu^+((-\infty,p))\right)
\end{aligned}$$

Thus, it suffices to show that: (i) $\left\|p^a-p^b-\xi\right\|_{\mathbb{L}^2(\mathbb{P}^\alpha_{N-1})}$ is bounded by a finite random variable independent of α, and (ii)

$$\mathbb{P}^\alpha_{N-1}\left(\xi_N > p^a,\, D^+_N(p-\xi) \leq \nu^+((-\infty,p))\right) \to 0, \quad \mathbb{P}\text{-a.s.},$$

[11] Recall that everything is measured relative to the fundamental price, according to the Notational Convention II.24

as $p \downarrow p^a$, uniformly over α. For (i), we have:

$$\left\|p^a - p^b - \xi\right\|_{\mathbb{L}^2(\mathbb{P}^{\alpha}_{N-1})} \leq |p^a - p^b| + \|\xi\|_{\mathbb{L}^2(\mathbb{P}^{\alpha}_{N-1})} \leq |p^a - p^b| + 2C\sqrt{\Delta t},$$

where the constant C appears in Assumptions II.12 and II.13. For (ii), we note that

$$\{\xi_N > p^a,\, D^+_N(p-\xi) \leq \nu^+((-\infty,p))\} = \{\xi_N > p^a,\, \xi \leq p - D^{-1}_N\left(\nu^+((-\infty,p))\right)\},$$

as $D_N(\cdot)$ is strictly decreasing, with $D_N(0) = 0$. Assumption II.17 implies that

$$\kappa^{-1}(\nu^+((-\infty,p))) \leq D^{-1}_N(\nu^+((-\infty,p))) < 0,$$

where κ is known at time $N-1$. Therefore,

$$\mathbb{P}^{\alpha}_{N-1}\left(\xi > p^a,\, D^+_N(p-\xi) \leq \nu^+((-\infty,p))\right) \leq \mathbb{P}^{\alpha}_{N-1}\left(\xi \in \left(p^a, p - \kappa^{-1}(\nu^+((-\infty,p)))\right]\right).$$

It remains to show that, $\mathbb{P}$-a.s., the right hand side of the above converges to zero, uniformly over all α. Assume that it does not hold. Then, with positive probability $\mathbb{P}$, there exists $\varepsilon > 0$ and a sequence of (p_k, α_k), such that $p_k \downarrow p^a$ and

$$\mathbb{P}^{\alpha_k}_{N-1}\left(\xi \in (p^a, p_k - \kappa^{-1}(\nu^+((-\infty,p_k)))]\right) \geq \varepsilon.$$

Notice that, $\mathbb{P}$-a.s., the family of measures $\left\{\hat{\mu}_k = \mathbb{P}^{\alpha_k}_{N-1} \circ \xi^{-1}\right\}_k$ is tight. The latter follows, for example, from the fact that, $\mathbb{P}$-a.s., the conditional second moments of ξ are bounded uniformly over all α (which, in turn, is a standard exercise in stochastic calculus). Prokhorov's theorem, then, implies that there is a subsequence of these measures that converges weakly to some measure $\hat{\mu}$ on $\mathbb{R}$. Next, notice that, for any fixed k in the chosen subsequence, there exists a large enough k', such that

$$\left|\hat{\mu}\left(\left(p^a, p_k - \kappa^{-1}(\nu^+((-\infty,p_k)))\right]\right) - \mu_{k'}\left(\left(p^a, p_k - \kappa^{-1}(\nu^+((-\infty,p_k)))\right]\right)\right| \leq \varepsilon/2.$$

Thus, for any k in the subsequence, we have

$$\hat{\mu}\left(\left(p^a, p_k - \kappa^{-1}(\nu^+((-\infty,p_k)))\right]\right) \geq \varepsilon/2.$$

The above is a contradiction, as the intersection of the corresponding intervals, $(p^a, p_k - \kappa^{-1}(\nu^+((-\infty, p_k)))]$, over all k is empty. ∎

Now we are ready to prove the upper bound in (2.19).

Lemma II.26. *In any non-degenerate LTC equilibrium, $p^b < 0 < p^a$, $\mathbb{P}$-a.s..*

Proof: We only show that $p^b < 0$ hold, the other inequality being very similar. Assume that $p^b \geq 0$ on some positive $\mathbb{P}$-probability set $\Omega' \in \mathcal{F}_{N-1}$. We are going to show that this results in a contradiction. First, Corollary II.31, in Appendix A, implies that, $\mathbb{P}$-a.s., if the agents in state (s, α) post a limit sell order, then we must have: $\sup_{p \in \mathbb{R}} A^\alpha(p; p^b) \geq 0$. In addition, on Ω', we have: $\hat{A}^\alpha(p^a; p^b) < 0$ for all $\alpha \in \tilde{\mathbb{A}}$, as ξ has full support in $\mathbb{R}$ under every $\mathbb{P}^\alpha_{N-1}$ (which, in turn, follows from the fact that σ is bounded uniformly away from zero). Then, Lemma II.25 implies that there exists a $\mathcal{F}_{N-1}$-measurable $\bar{p} \geq p^a$, such that, on Ω', the following holds a.s.: if $\nu^+(\{p^a\}) = 0$ then $\bar{p} > p^a$, and, in all cases,

$$A^\alpha(p; p^b) < 0, \qquad \forall p \in [p^a, \bar{p}], \quad \forall \alpha \in \tilde{\mathbb{A}} \tag{2.22}$$

Clearly, it is suboptimal for an agent to post a limit sell order below $\bar{p}$. However, an agent's strategy only needs to be optimal up to a set of $\mathbb{P}$-measure zero, and these sets can be different for different (s, α). Therefore, a little more work is required to obtain the desired contradiction. Consider the set $B \subset \Omega' \times \mathbb{R} \times \tilde{\mathbb{A}}$:

$$B = \{(\omega, s, \alpha) \,|\, \hat{q}(s, \alpha) > 0, \; \hat{p}(s, \alpha) \leq \bar{p}\}.$$

This set is measurable with respect to $\mathcal{F}_{N-1} \otimes \mathcal{B}\left(\mathbb{R} \times \tilde{\mathbb{A}}\right)$, due to the measurability properties of $\hat{q}$ and $\hat{p}$. Notice that, due to the above discussion and the optimality of agents' actions (cf. Corollary II.31, in Appendix A), for any $(s, \alpha) \in \mathbb{R} \times \tilde{\mathbb{A}}$, we have:

$$\mathbb{P}(\{\omega \,|\, (\omega, s, \alpha) \in B\}) = 0,$$

and hence

$$\mathbb{E}_{N-1}\int_{\mathbb{R}\times\tilde{\mathbb{A}}} \mathbf{1}_B(\omega,s,\alpha)\mu_{N-1}(ds,d\alpha) = \int_{\mathbb{R}\times\tilde{\mathbb{A}}} \mathbb{E}_{N-1}\left(\mathbf{1}_B(\omega,s,\alpha)\rho_{N-1}(\omega,s,\alpha)\right)\mu^0_{N-1}(ds,d\alpha) = 0,$$

where ρ_{N-1} is the Radon-Nikodym density of μ_{N-1} w.r.t. to the deterministic measure μ^0_{N-1} (cf. Assumption II.18).

The above implies that, $\mathbb{P}_{N-1}$-a.s., $\mathbf{1}_B(\omega,s,\alpha)\rho_{N-1}(\omega,s,\alpha) = 0$, for μ^0_{N-1}-a.e. (s,α). Notice also that, for all $(\omega,s,\alpha)\in\Omega'\times\mathbb{R}\times\tilde{\mathbb{A}}$,

$$\mathbf{1}_{\{\hat{p}(s,\alpha)\leq\bar{p}\}}\hat{q}^+(s,\alpha)\mathbf{1}_{B^c} = 0.$$

From the above observations and the condition (3.7) in the definition of equilibrium (cf. Definition III.6), we conclude that, on Ω', the following holds a.s.:

$$\nu^+([p^a,\bar{p}]) = 0,$$

where $\bar{p}\geq p^a$, and, if $\nu^+(\{p^a\}) = 0$, then $\bar{p} > p^a$. This contradicts the definition of p^a (recall that p^a is $\mathbb{P}$-a.s. finite, due to non-degeneracy of the LOB). ∎

It only remains to prove the lower bound on p^b in (2.19). Assume that it does not hold. That is, assume that there exists a family of equilibria, with arbitrary small Δt, and positive $\mathbb{P}$-probability $\mathcal{F}_{N-1}$-measurable sets $\Omega^{\Delta t}$, such that $p^b < -C_0\sqrt{\Delta t}$ on $\Omega^{\Delta t}$. We are going to show that this leads to a contradiction with $p^a > 0$. To this end, assume that the agents maximize the simplified objective function, $\hat{A}^\alpha$, instead of the true one, A^α. Then, it turns out that, if p^b is negative enough, the optimal price levels become negative for all α. The precise formulation of this is given by the following lemma.

Lemma II.27. *There exists a constant $C_0 > 0$, s.t., for any small enough Δt, there exist constants $\epsilon, \delta > 0$, s.t., $\mathbb{P}$-a.s., we have:*

$$\hat{A}^\alpha(-\delta; x) \geq \epsilon + \sup_{y \geq 0} \hat{A}^\alpha(y; x),$$

for all $\alpha \in \tilde{\mathbb{A}}$ and all $x \leq -C_0\sqrt{\Delta t}$.

Proof: Denote $\bar{\xi} = \xi/\sqrt{\Delta t}$ and consider the random function

$$\bar{A}^\alpha(p; x) = \mathbb{E}^\alpha_{N-1}\left((p - x - \bar{\xi})\mathbf{1}_{\{\bar{\xi} > p\}}\right).$$

Notice that

$$\hat{A}^\alpha(p; x) = \sqrt{\Delta t}\bar{A}^\alpha\left(p/\sqrt{\Delta t}; x/\sqrt{\Delta t}\right),$$

and, hence, we can reformulate the statement of the lemma as follows: there exists a constant $C_0 > 0$, s.t., for any small enough Δt, there exist constants $\epsilon, \delta > 0$, s.t., $\mathbb{P}$-a.s., we have:

$$\bar{A}^\alpha(-\delta; x) \geq \epsilon + \sup_{y \geq 0} \bar{A}^\alpha(y; x),$$

for all $\alpha \in \tilde{\mathbb{A}}$ and all $x \leq -C_0$. Notice that

$$\begin{aligned} &\bar{A}^\alpha(-\delta; x) - \bar{A}^\alpha(y; x) = \\ &- x\mathbb{E}^\alpha_{N-1}\left(\mathbf{1}_{\{-\delta < \bar{\xi} \leq y\}}\right) - \mathbb{E}^\alpha_{N-1}\left(\xi\mathbf{1}_{\{-\delta < \bar{\xi} \leq y\}}\right) - \delta\mathbb{E}^\alpha_{N-1}\left(\mathbf{1}_{\{\bar{\xi} > -\delta\}}\right) - y\mathbb{E}^\alpha_{N-1}\left(\mathbf{1}_{\{\bar{\xi} > y\}}\right) \end{aligned}$$

is non-increasing in x, and, hence, such is $\bar{A}^\alpha(-\delta; x) - \sup_{y \geq 0} \bar{A}^\alpha(y; x)$. Hence, it suffices to prove the above statement for $x = -C_0$. Next, consider the deterministic function $A_\sigma(p; x)$, defined via

$$A_\sigma(p; x) = \hat{\mathbb{E}}\left((p - x - \sigma\eta_0)\mathbf{1}_{\{\sigma\eta_0 > p\}}\right), \tag{2.23}$$

where η_0 is a standard normal random variable on some auxiliary probability space $(\hat{\Omega}, \hat{\mathbb{P}})$. It follows from Lemma II.22 that there exists a function $\varepsilon_2(\cdot) \geq 0$, s.t.

$\varepsilon_2(\Delta t) \to 0$, as $\Delta t \to 0$, and, $\mathbb{P}$-a.s., we have:

$$\left|\bar{A}^{\alpha}(p; -C_0) - A_{\sigma_{t_{N-1}}}(p; -C_0)\right| \leq \varepsilon_2(\Delta t),$$

for all $\alpha \in \tilde{\mathbb{A}}$ and all $p \in \mathbb{R}$. Then, as we can always choose Δt small enough, so that $\varepsilon_2(\Delta t) < \epsilon$, the statements of the lemma would follow if we can show that there exist constants $\epsilon, \delta, C_0 > 0$, s.t., $\mathbb{P}$-a.s.,

$$A_{\sigma_{t_{N-1}}}(-\delta; -C_0) \geq 3\epsilon + \sup_{y \geq 0} A_{\sigma_{t_{N-1}}}(y; -C_0)$$

As $\sigma_{t_{N-1}}(\omega) \in [1/C, C]$, $\mathbb{P}$-a.s., it suffices to find $\epsilon, \delta, C_0 > 0$, s.t.

$$A_{\sigma}(-\delta; -C_0) \geq 3\epsilon + \sup_{y \geq 0} A_{\sigma}(y; -C_0), \quad \forall\, \sigma \in [1/C, C].$$

Note that the above inequality does not involve ω or ξ, and it is simply a property of a deterministic function. Notice also that $A_{\sigma}(p; x) = \sigma A_1\left(p/\sigma; x/\sigma\right)$, with A_1 given in (2.23). Then, if we denote by $F(x)$ and $f(x)$, respectively, the cdf and pdf of a standard normal, we obtain:

$$A_1(p; x) = (p - x)(1 - F(p)) - \int_p^{\infty} t f(t)\mathrm{d}t.$$

A straightforward calculation gives us the following useful properties of A_1 and A_{σ}:

(i) For any $\sigma > 0$ and any $x < 0$, the function $p \mapsto A_{\sigma}(p; x)$ has a unique maximizer $p_{\sigma}(x)$, in particular, it is increasing in $p \leq p_{\sigma}(x)$ and decreasing in $p \geq p_{\sigma}(x)$.

(ii) The function

$$x \mapsto p_{\sigma}(x) = \sigma p_1(x/\sigma) = \sigma\left((1 - F)/f\right)^{-1}(-x/\sigma)$$

is increasing in $x < 0$ and converges to $-\infty$, as $x \to -\infty$.

Then, choosing C_0 large enough, so that $p_1(-C_0/C) < 0$, ensures $p_{\sigma}(-C_0) < 0$, for all $\sigma \in [1/C, C]$. Setting $\delta = -p_1(-C_0/C)/C$ guarantees that $p_{\sigma}(-C_0) \leq -\delta$, for all $\sigma \in [1/C, C]$. Then, by property (i) above, we have, for all $\sigma \in [1/C, C]$:

$$A_\sigma(-\delta; -C_0) > A_\sigma(0; -C_0) = \sup_{y \geq 0} A_\sigma(y; -C_0).$$

Finally, as $A_\sigma(-\delta; -C_0) - A_\sigma(0; -C_0)$ is a continuous function of $\sigma \in [1/C, C]$, we can find ϵ, such that

$$A_\sigma(-\delta; -C_0) \geq 3\epsilon + \sup_{y \geq 0} A_\sigma(y; -C_0), \quad \forall\, \sigma \in [1/C, C].$$

∎

Recall that our assumption is that $p^b < -C_0\sqrt{\Delta t}$ holds on a set $\Omega^{\Delta t}$ of positive $\mathbb{P}$-measure. Recall also that $p^a > 0$, $\mathbb{P}$-a.s., due to Lemma II.26. Then, Lemmas II.25 and II.27 imply that there exists $\mathcal{F}_{N-1}$-measurable $\bar{p} \geq p^a$, s.t., on $\Omega^{\Delta t}$, we have a.s.: if $\nu^+(\{p^a\}) = 0$ then $\bar{p} > p^a$, and, in all cases,

$$A^\alpha(p; p^b) < \sup_{p' \in \mathbb{R}} A^\alpha(p'; p^b), \qquad \forall p \in [p^a, \bar{p}], \quad \forall \alpha \in \tilde{\mathbb{A}}.$$

It is intuitively clear that posting limit sell orders at the above price levels p must be suboptimal for the agents. However, the above inequality, on its own, does not yield a contradiction, as the agents' strategies are only optimal up to a set of $\mathbb{P}$-probability zero, and these sets may be different for different states (s, α). To obtain a contradiction with the definition of p^a, we simply repeat the last part of the proof of Lemma II.26 (following equation (2.22)). This ensures that (2.19) holds and completes the proof of the theorem.

2.7 Proof of Theorem II.21

Within the scope of this proof, we adopt the notation introduced in (2.16) and use Notational Convention II.24 (i.e. we measure the LOB, the expected execution prices, and the demand, relative to p^0, but keep the same variables' names). Assume that the statement of the theorem does not hold: i.e. there exists $\alpha_0 \in \tilde{\mathbb{A}}$, such that $\tilde{p}^0$ is not a martingale under $\mathbb{P}^{\alpha_0}$. Then, there exists $s \in [0, T)$, s.t., with positive probability $\mathbb{P}^{\alpha_0}$, we have:

$$\tilde{\mathbb{E}}_s^{\alpha_0} \tilde{p}_T^0 \neq \tilde{p}_s^0.$$

Without loss of generality, we assume that there exists a constant $\delta > 0$ and a set $\Omega' \in \mathcal{F}_s$, having positive probability $\mathbb{P}^{\alpha_0}$ (and hence $\mathbb{P}$), s.t., for all random outcomes in Ω', we have:

$$\tilde{\mathbb{E}}_s^{\alpha_0} (\tilde{p}_T^0 - \tilde{p}_s^0) \geq \delta \tag{2.24}$$

(the case of negative values is analogous). Next, we fix an arbitrary Δt from a given family and consider the associated non-degenerate LTC equilibrium.

Lemma II.28. *There exists a deterministic function $\varepsilon(\cdot) \geq 0$, s.t. $\varepsilon(\Delta t) \to 0$, as $\Delta t \to 0$, and, for any small enough $\Delta t > 0$, there exists $n = 0, \ldots, N-3$ and $\Omega'' \in \mathcal{F}_n$, s.t. $\mathbb{P}_n^{\alpha_0}(\Omega'') > 0$ and the following holds on Ω'':*

$$\mathbb{P}_{n+2}^{\alpha_0} \left(\mathbb{E}_{n+3}^{\alpha_0} \left(p_N^0 - p_{n+3}^0 \right) \leq \delta/2 \right) \leq \varepsilon(\Delta t).$$

Proof: The proof follows from Assumption II.16. Consider $t = t' = s$ and $t'' = t_{n+2}$. Then, Assumption II.16 implies

$$\tilde{\mathbb{P}}_s^{\alpha_0} \left(\left| \tilde{\mathbb{E}}_{t_{n+2}}^{\alpha_0} \int_s^T \mu_u^{\alpha_0} du - \tilde{\mathbb{E}}_s^{\alpha_0} \int_s^T \mu_u^{\alpha_0} du \right| \geq \varepsilon(\Delta t) \right) \leq \varepsilon(\Delta t)$$

on Ω', a.s.. Notice also that

$$\tilde{\mathbb{E}}_s^{\alpha_0} (\tilde{p}_T^0 - \tilde{p}_s^0) = \tilde{\mathbb{E}}_s^{\alpha_0} \int_s^T \mu_u^{\alpha_0} \mathrm{d}u.$$

Then, assuming that $\varepsilon(\Delta t)$ is small enough and recalling (2.24), we obtain

$$\tilde{\mathbb{P}}_s^{\alpha_0} \left(\tilde{\mathbb{E}}_{t_{n+2}}^{\alpha_0} \int_s^T \mu_u^{\alpha_0} du \leq 3\delta/4 \right) \leq \varepsilon(\Delta t),$$

on Ω'. Therefore, there exists a set $\Omega'' \in \mathcal{F}_s \subset \mathcal{F}_{t_n}$, s.t. $\tilde{\mathbb{P}}^{\alpha_0}_{t_n}(\Omega'') > 0$ and

$$\tilde{\mathbb{E}}^{\alpha_0}_{t_{n+2}} \int_s^T \mu^{\alpha_0}_u du \geq 3\delta/4,$$

on Ω''. Next, we choose $t = s$, $t' = t_{n+2}$, $t'' = t_{n+3}$, and use Assumption II.16, to obtain:

$$\tilde{\mathbb{P}}^{\alpha_0}_{t_{n+2}} \left(\left| \tilde{\mathbb{E}}^{\alpha_0}_{t_{n+3}} \int_s^T \mu^{\alpha_0}_u du - \tilde{\mathbb{E}}^{\alpha_0}_{t_{n+2}} \int_s^T \mu^{\alpha_0}_u du \right| \geq \varepsilon(\Delta t) \right) \leq \varepsilon(\Delta t),$$

on Ω'', a.s.. Assuming that $\varepsilon(\Delta t)$ is small enough and using the last two inequalities, we obtain

$$\tilde{\mathbb{P}}^{\alpha_0}_{t_{n+2}} \left(\tilde{\mathbb{E}}^{\alpha_0}_{t_{n+3}} \int_s^T \mu^{\alpha_0}_u du \leq \delta/2 \right) \leq \varepsilon(\Delta t).$$

Finally, due to Assumption II.13, and as Δt is small, we can replace $\int_s^T \mu^{\alpha_0}_u du$ by $\int_{t_{n+3}}^T \mu^{\alpha_0}_u du$, and $\delta/2$ by $\delta/4$, in the above equation. This completes the proof of the lemma. ∎

Using the strategy at which the agent in state $(1, \alpha_0)$ waits until the last moment $n = N$, we conclude that the process $(\lambda^a_n(\alpha_0) + p^0_n)$ must be a supermartingale under $\mathbb{P}^{\alpha_0}$. More precisely, due to the definition of an optimal strategy, we have, $\mathbb{P}$-a.s.:

$$\lambda^a_{n+2}(\alpha_0) \geq \mathbb{E}^{\alpha_0}_{n+2} \lambda^a_N(\alpha_0) + \mathbb{E}^{\alpha_0}_{n+2} \left(\mathbb{E}^{\alpha_0}_{n+3}(p^0_N - p^0_{n+3}) + \xi_{n+3} \right).$$

Recall that $\lambda^a_N(\alpha_0) = p^b_N$ and, due to Theorem II.19 (more precisely, it follows from the proof of the theorem), there exists a constant $C_0 > 0$, s.t., for all small enough $\Delta t > 0$, the following holds $\mathbb{P}$-a.s.:

$$-C_0\sqrt{\Delta t} \leq p^b_N < 0 < p^a_N \leq C_0\sqrt{\Delta t}.$$

Thus, we have, $\mathbb{P}$-a.s.:

$$\lambda^a_{n+2}(\alpha_0) \geq -C_0\sqrt{\Delta t} + \mathbb{E}^{\alpha_0}_{n+2} \left(\mathbb{E}^{\alpha_0}_{n+3}(p^0_N - p^0_{n+3}) \right) + \mathbb{E}^{\alpha_0}_{n+2} \xi_{n+3}. \tag{2.25}$$

Due to Assumption II.13, we have, $\mathbb{P}$-a.s.:

$$\mathbb{E}^{\alpha_0}_{n+2}\xi_{n+3} \leq C\Delta t, \qquad \left|\mathbb{E}^{\alpha_0}_{n+3}(p^0_N - p^0_{n+3})\right| \leq CT,$$

and, hence,

$$\lambda^a_{n+2}(\alpha_0) \geq -C_0\sqrt{\Delta t} + CT + C\Delta t.$$

In addition, making use of Lemma II.28, we conclude that, for any small enough Δt, there exist $n = 0, \dots, N-2$ and $\Omega'' \in \mathcal{F}_n$, s.t. $\mathbb{P}^{\alpha_0}_n(\Omega'') > 0$ and

$$\mathbb{P}^{\alpha}_{n+2}\left(\mathbb{E}^{\alpha}_{n+3}\left(p^0_N - p^0_{n+3}\right) \leq \delta/2\right) \leq \varepsilon(\Delta t), \quad \text{on } \Omega''.$$

Using (2.25) and assuming that Δt is small enough, we obtain:

$$\lambda^a_{n+2}(\alpha_0) \geq \delta/4, \quad \text{on } \Omega''.$$

Next, Corollary II.31, in Appendix A, implies that, $\mathbb{P}$-a.s.,

$$p^b_{n+1} \geq \mathbb{E}^{\alpha_0}_{n+1}\left(\lambda^a_{n+2}(\alpha_0) + \xi_{n+2}\middle|\xi_{n+2} < p^b_{n+1}\right).$$

Thus, on Ω'', we obtain:

$$p^b_{n+1} - \mathbb{E}^{\alpha_0}_{n+1}\left(\xi_{n+2}\middle|\xi_{n+2} < p^b_{n+1}\right) \geq \delta/4. \tag{2.26}$$

The following lemma shows that, for any number p, the conditional expectation of the fundamental price increment, $\mathbb{E}^{\alpha_0}_{n+1}(\xi_{n+2}|\xi_{n+2} < p)$, approaches p as the size of the time interval vanishes. This result follows from Lemma II.23.

Lemma II.29. *There exists a constant $C_3 > 0$, s.t., for all small enough $\Delta t > 0$, and for any $t \in [0, T - \Delta t]$, the following holds $\mathbb{P}$-a.s.:*

$$\sup_{p\leq 0}\left|p - \tilde{\mathbb{E}}^{\alpha_0}_t\left(\tilde{p}^0_{t+\Delta t} - \tilde{p}^0_t \,\middle|\, \tilde{p}^0_{t+\Delta t} - \tilde{p}^0_t < p\right)\right| \leq C_3\sqrt{\Delta t}.$$

Proof: Fix t and $\Delta t > 0$ and consider the evolution of $\tilde{p}^0_s$, for $s \in [t, t + \Delta t]$, under $\mathbb{P}^{\alpha_0}_t$:

$$\tilde{p}^0_s - \tilde{p}^0_t = \int_t^s \mu^{\alpha_0}_u du + \int_t^s \sigma_u dW^{\alpha_0}_u,$$

where W^{α_0} is a Brownian motion under $\mathbb{P}^{\alpha_0}$. Rescaling by $\sqrt{\Delta t}$, we obtain

$$(\tilde{p}^0_s - \tilde{p}^0_t)/\sqrt{\Delta t} = X_{(s-t)/\Delta t}, \quad X_s = \int_0^s \hat{\mu}_u du + \int_0^s \hat{\sigma}_u d\hat{W}_u, \quad s \in [0,1],$$

with

$$\hat{\mu}_s = \sqrt{\Delta t}\, \mu^{\alpha_0}_{t+s\Delta t}, \quad \hat{\sigma}_s = \sigma_{t+s\Delta t}, \quad \hat{W}_s = \frac{1}{\sqrt{\Delta t}} \left(W^{\alpha_0}_{t+s\Delta t} - W^{\alpha_0}_t \right), \quad s \in [0,1].$$

Notice that the above processes are adapted to the new filtration $\hat{\mathbb{F}}$, with $\hat{\mathcal{F}}_s = \tilde{\mathcal{F}}_{t+s\Delta t}$, and, $\mathbb{P}$-a.s., under $\tilde{\mathbb{P}}^{\alpha_0}_t$, $\hat{W}$ is a Brownian motion with respect to $\hat{\mathbb{F}}$. Next, due to Assumptions II.12 and II.15, for any small enough $\Delta t > 0$, $\mathbb{P}$-a.s., the dynamics of $(-X_s)$, under $\tilde{\mathbb{P}}^{\alpha_0}_t$, satisfy all the assumptions of Lemma II.23. As a result, we obtain:

$$\tilde{\mathbb{P}}^{\alpha_0}_t(X_1 < -x - z) \leq C_1 e^{-z} \tilde{\mathbb{P}}^{\alpha_0}_t(X_1 < -x), \qquad \forall x, z \geq 0.$$

Finally, we notice that

$$\sup_{p \leq 0} \left| p - \tilde{\mathbb{E}}^{\alpha_0}_t \left(\tilde{p}^0_{t+\Delta t} - \tilde{p}^0_t \middle| \tilde{p}^0_{t+\Delta t} - \tilde{p}^0_t < p \right) \right| = \sqrt{\Delta t} \sup_{p \leq 0} \left| p - \tilde{\mathbb{E}}^{\alpha_0}_t \left(X_1 \middle| X_1 < p \right) \right|$$

$$= \sqrt{\Delta t} \sup_{p \leq 0} \left| p - \frac{\int_{-p}^{\infty} x \, d\, \tilde{\mathbb{P}}^{\alpha_0}_t(X_1 < -x)}{\tilde{\mathbb{P}}^{\alpha_0}_t(X_1 < p)} \right| = \sqrt{\Delta t} \sup_{p \leq 0} \left| \frac{\int_0^{\infty} \tilde{\mathbb{P}}^{\alpha_0}_t(X_1 < p - z) dz}{\tilde{\mathbb{P}}^{\alpha_0}_t(X_1 < p)} \right| \leq C_1 \sqrt{\Delta t},$$

which completes the proof of the lemma. ∎

Using (2.26) and Lemma II.29, we conclude that, for all small enough Δt, we have: $p^b_{n+1} > 0$ on Ω'', $\mathbb{P}$-a.s.. In addition, Corollary II.31, in Appendix A, implies that, for any $\alpha \in \tilde{\mathbb{A}}$, the following holds $\mathbb{P}$-a.s.:

$$\lambda^a_{n+1}(\alpha) \geq p^b_{n+1}.$$

Next, with a slight abuse of notation (similar notation was introduced in the proof of Proposition II.19), we consider the simplified objective of an agent who posts a limit sell order at the ask price p_n^a:

$$\hat{A}^\alpha(p_n^a;\lambda_{n+1}^a) = \mathbb{E}_n^\alpha\left(p_n^a - \lambda_{n+1}^a - \xi_{n+1}\,|\,\xi_{n+1} > p_n^a\right)$$

The above estimates imply that, on Ω'', we have, $\mathbb{P}$-a.s.:

(2.27)

$$\hat{A}^\alpha(p_n^a;\lambda_{n+1}^a) \leq \mathbb{E}_n^\alpha\left(p_n^a - \xi_{n+1}\,|\,\xi_{n+1} > p_n^a\right) - \mathbb{E}_n^\alpha\left(p_{n+1}^b \mathbf{1}_{\Omega''}\,|\,\xi_{n+1} > p_n^a\right) < 0, \qquad \forall\alpha\in\tilde{\mathbb{A}}.$$

To obtain the last inequality in the above, we recall that $\Omega''\in\mathcal{F}_n$ and, $\mathbb{P}$-a.s., $\mathbf{1}_{\Omega''}\mathbb{P}_n(\Omega\setminus\Omega'') = 0$, $p_{n+1}^b > 0$ on Ω'', and $\mathbb{P}_n^\alpha(\xi_{n+1} > p_n^a) > 0$, for all $\alpha\in\tilde{\mathbb{A}}$. Next, repeating the proof of Lemma II.25 (and using the fact that λ_{n+1}^a is absolutely bounded, as shown in Corollary II.20), we conclude that, $\mathbb{P}$-a.s., either $\nu_n^+(\{p_n^a\}) > 0$, or we have:

$$\left|A^\alpha(p;\lambda_{n+1}^a) - \hat{A}^\alpha(p_n^a;\lambda_{n+1}^a)\right| \to 0,$$

as $p\downarrow p^a$, uniformly over all $\alpha\in\tilde{\mathbb{A}}$, where we introduce the true objective,

$$A^\alpha(p;\lambda_{n+1}^a) = \mathbb{E}_n^\alpha\left(\left(p-\lambda_{n+1}^a-\xi_{n+1}\right)\mathbf{1}_{\{D_{n+1}^+(p-\xi_{n+1})>\nu_n^+((-\infty,p))\}}\right).$$

This convergence, along with (2.27), implies that there exists a $\mathcal{F}_n$-measurable $\bar{p}\geq p_n^a$, such that, on Ω'', the following holds $\mathbb{P}$-a.s.: if $\nu_n^+(\{p_n^a\}) = 0$ then $\bar{p} > p_n^a$, and, in all cases,

$$A^\alpha(p;\lambda_{n+1}^a) < 0, \qquad \forall p\in[p_n^a,\bar{p}],\quad \forall\alpha\in\tilde{\mathbb{A}}.$$

Finally, we repeat the last part of the proof of Lemma II.26 (following equation (2.22)), to obtain a contradiction with the definition of p_n^a, and complete the proof of the theorem. The last argument also shows that, when Δt is small enough, it becomes suboptimal for the agents to post limit sell orders, as the expected relative profit from this action becomes negative, causing the market to degenerate.

2.8 Summary and future work

In this chapter, we present a new framework for modeling market microstructure, which does not require the existence of a designate market maker, and in which the LOB arises endogenously, as a result of equilibrium between multiple strategic players (aka agents). This framework is based on a continuum-player game. It reproduces the mechanics of an auction-style exchange very closely, so that, in particular, it can be used to analyze the liquidity effects of changes in the rules of the exchange. We use the proposed modeling framework to study the liquidity effects of high trading frequency. In particular, we demonstrate the dual nature of high trading frequency. On the one hand, in the absence of a bullish or bearish signal about the asset, the higher trading frequency makes market more efficient. On the other hand, at a sufficiently high trading frequency, even a very small trading signal may amplify the adverse selection effect, creating a disproportionally large change in the LOB, which is interpreted as an internal (or, self-inflicted) liquidity crisis.

The present work raises many questions for further research. Notice that the main results of the present work are of a qualitative nature: they demonstrate the general behavior of LOB, as a function of trading frequency, but do not immediately allow for any computations. It would also be interesting to establish quantitative results. In particular, we would like to construct an equilibrium in a more realistic, and more concrete, model than the one used in Section 2.3. Such a model would allow for heterogeneous beliefs, and it would prescribe the specific sources of information (i.e. relevant market indicators) used by the agents to form their beliefs. A model of this type could be calibrated to market data and used to study the effects of changes in relevant market parameters on the LOB. Finally, it is interesting to develop a

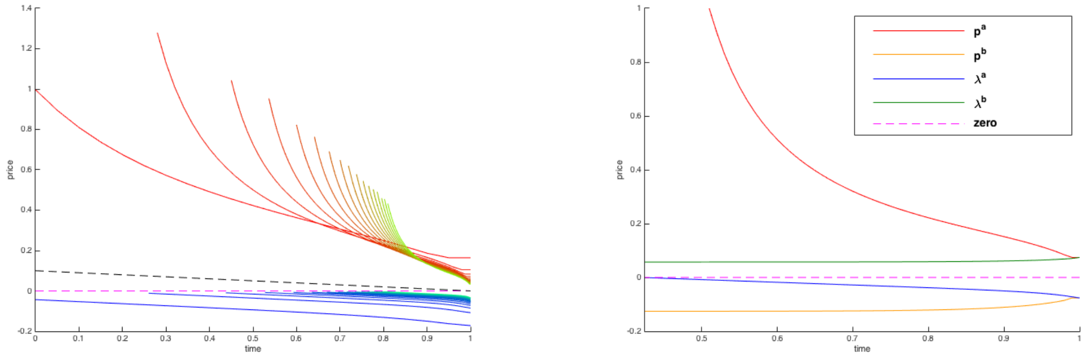


Figure 2.1: On the left: ask price $\hat{p}^a$ (in red) and the associated expected execution prices $\hat{\lambda}^a$ (in blue); different curves correspond to different trading frequencies ($N = 20, \ldots, 500$); black dashed line is the expected change in the fundamental price $\alpha(T-t)$. On the right: ask price $\hat{p}^a$ (in red) and the associated expected execution price $\hat{\lambda}^a$ (in blue), bid price $\hat{p}^b$ (in orange) and the associated expected execution price $\hat{\lambda}^b$ (in green), for $N = 100$. Non-degenerate equilibrium exists only on a time interval where $\hat{\lambda}^a < 0$. All prices are measured relative to the fundamental price and are plotted as functions of time. Positive drift: $\alpha = 0.1$, $\sigma = 1$, $T = 1$.

continuous time version of the proposed framework, in order to better capture the present state of the markets, where the trading frequency is not restricted. All these questions are the subject of the next chapter.

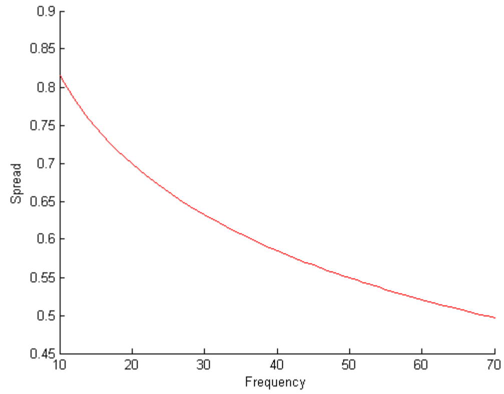
2.9 Appendix A

This section contains several useful technical results on the representation of the value function of an agent in the proposed game. Notice that (2.1) and (2.2) imply that, if ν is admissible, then, for any (α, m, p, q, r), we have, $\mathbb{P}$-a.s.:

$$\left| J^{(p,q,r)}(m, s, \alpha, \nu) - J^{(p,q,r)}(m, s', \alpha, \nu) \right| \leq |s - s'| \, \mathbb{E}^{\alpha}_m |p^a_N| \vee |p^b_N|, \qquad \forall s, s' \in \mathbb{R}$$

This implies that every $J^{(p,q,r)}(m, \cdot, \alpha, \nu)$ and $V^{\nu}_m(\cdot, \alpha)$ has a continuous modification under $\mathbb{P}$. Thus, whenever ν is admissible, we define the value function of an agent as the aforementioned continuous modification of the left hand side of (2.3).

Lemma II.30. *Assume that an optimal control exists for an admissible LOB ν.*



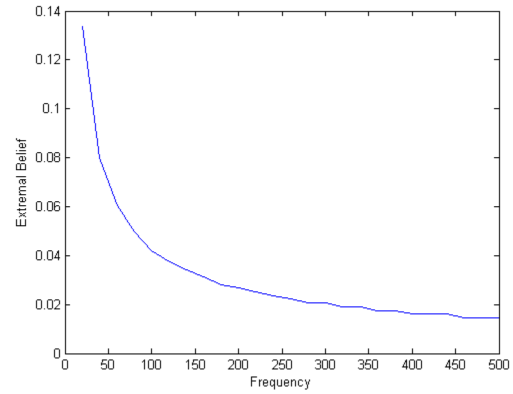


Figure 2.2: The horizontal axis represents trading frequency, measured in the number of steps N. Left: time-zero bid-ask spread in the zero-drift case ($\alpha = 0$). Right: the maximum value of drift α for which a non-degenerate equilibrium exists on the entire time interval. Parameters: $\sigma = 1$, $T = 1$.

Assume also that, for any $\alpha \in \mathbb{A}$, the associated value function $V_n^\nu(\cdot, \alpha)$, defined in (2.3), is measurable with respect to $\mathcal{F}_n \otimes \mathcal{B}(\mathbb{R})$. Then, it satisfies the following Dynamic Programming Principle.

- *For $n = N$ and all $(s, \alpha) \in \mathbb{S}$, we have, $\mathbb{P}$-a.s.:*

$$V_N^\nu(s, \alpha) = s^+ p_N^b - s^- p_N^a \tag{2.28}$$

- *For all $n = N-1, \dots, 0$ and all $(s, \alpha) \in \mathbb{S}$, we have:*

$$\begin{aligned} & V_n^\nu(s,\alpha) = \\ & esssup_{p,q,r} \Big\{ \mathbf{1}_{\{r_n=0\}} \mathbb{E}_n^\alpha \Big(V_{n+1}^\nu(s,\alpha) + \big(q_n p_n + V_{n+1}^\nu(s - q_n, \alpha) - V_{n+1}^\nu(s,\alpha) \big) \cdot \\ & \cdot \Big(\mathbf{1}_{\left\{ q_n \geq 0,\, D_{n+1}^+(p_n) > \nu_n^+((-\infty, p_n)) \right\}} + \mathbf{1}_{\left\{ q_n < 0,\, D_{n+1}^-(p_n) > \nu_n^-((p_n, \infty)) \right\}} \Big) \Big) \\ & + \mathbf{1}_{\{r_n=1\}} \big(q_n^+ p_n^b - q_n^- p_n^a + \mathbb{E}_n^\alpha V_{n+1}^\nu(s - q_n, \alpha) \big) \Big\}, \end{aligned} \tag{2.29}$$

where the essential supremum is taken under $\mathbb{P}$, over all admissible controls (p, q, r).

Proof: The most important step is to show that, for all $n = 0, \dots N-1$ and $(s, \alpha) \in \mathbb{S}$,

$$V_n^\nu(s,\alpha) = \text{esssup}_{p,q,r} \mathbb{E}_n^\alpha \left(V_{n+1}^\nu \left(S_{n+1}^{n,s,(p,q,r)}, \alpha \right) - g_n^\nu \left(p_n, q_n, r_n, D_{n+1} \right) \right), \tag{2.30}$$

where the essential supremum is taken under $\mathbb{P}$, over all admissible controls (p, q, r), and

$$g_n^\nu \left(p_n, q_n, r_n, D_{n+1} \right) = \left(p_n \mathbf{1}_{\{r_n=0\}} + p_n^a \mathbf{1}_{\{r_n=1, q_n<0\}} + p_n^b \mathbf{1}_{\{r_n=1, q_n>0\}} \right) \Delta S_{n+1}^{n,s,(p,q,r)}$$

does not depend on s. Assume that $J^{(p,q,r)}(n, \cdot, \alpha, \nu)$ is a continuous modification of the objective function. Notice that, for all $m \le k \le n$, we have, $\mathbb{P}$-a.s.:

$$\mathbb{E}_k^\alpha J^{(p,q,r)} \left(n, S_n^{m,s,(p,q,r)}, \alpha, \nu \right) = J^{(p,q,r)} \left(k, S_k^{m,s,(p,q,r)}, \alpha, \nu \right) + \mathbb{E}_k^\alpha \sum_{j=k}^{n-1} g_j^\nu \left(p_j, q_j, r_j, D_{j+1} \right)$$

Notice also that, for any (p, q, r) we have, $\mathbb{P}$-a.s.: $J^{(p,q,r)}(m, s, \alpha, \nu) \le V_m^\nu(s, \alpha)$, for all $s \in \mathbb{S}$. Let us show that the left hand side of (2.30) is less than its right hand side:

$$V_m^\nu(s,\alpha) = \text{essup}_{p,q,r} J^{(p,q,r)} \left(m, S_m^{m,s,(p,q,r)}, \alpha, \nu \right)$$

$$= \text{essup}_{p,q,r} \mathbb{E}_m^\alpha \left(J^{(p,q,r)} \left(m+1, S_{m+1}^{m,s,(p,q,r)}, \alpha, \nu \right) - g_m^\nu \left(p_m, q_m, r_m, D_{m+1} \right) \right)$$

$$\le \text{essup}_{p,q,r} \mathbb{E}_m^\alpha \left(V_{m+1}^\nu \left(S_{m+1}^{m,s,(p,q,r)}, \alpha \right) - g_m^\nu \left(p_m, q_m, r_m, D_{m+1} \right) \right)$$

Next, we show that the right hand side of (2.30) is less than its left hand side. For any (p, q, r), we have, $\mathbb{P}$-a.s.:

$$\mathbb{E}_m^\alpha \left(V_{m+1}^\nu \left(S_{m+1}^{m,s,(p,q,r)}, \alpha \right) - g_m^\nu \left(p_m, q_m, r_m, D_{m+1} \right) \right)$$

$$= \mathbb{E}_m^\alpha \left(J^{(\hat{p},\hat{q},\hat{r})} \left(m+1, S_{m+1}^{m,s,(p,q,r)}, \alpha, \nu \right) - g_m^\nu \left(p_m, q_m, r_m, D_{m+1} \right) \right) =$$

$$J^{(\tilde{p},\tilde{q},\tilde{r})}(m, s, \alpha, \nu) \le V_m^\nu(s, \alpha),$$

where $(\tilde{p}_n, \tilde{q}_n, \tilde{r}_n)$ coincide with $(\hat{p}_n, \hat{q}_n, \hat{r}_n)$, for $n \ge m+1$, while they are equal to (p_m, q_m, r_m), for $n = m$. The proof is completed easily by plugging the dynamics of the state process, (2.1), into (2.30). ∎

The following corollary provides a more explicit recursive formula for the value function and optimal control. In particular, it states that the value function of an agent at any time remains linear in s, in both positive and negative half lines (with possibly different slopes).

Corollary II.31. *Assume that an admissible LOB ν has an optimal control $(\hat{p}, \hat{q}, \hat{r})$. Then, for any $(s, \alpha) \in \mathbb{S}$, the following holds $\mathbb{P}$-a.s., for all $n = 0, \ldots, N-1$:*

1. *$V_n^\nu(s,\alpha) = s^+ \lambda_n^a(\alpha) - s^- \lambda_n^b(\alpha)$, with some adapted processes $\lambda^a(\alpha)$ and $\lambda^b(\alpha)$, such that $\lambda_N^a(\alpha) = p_N^b$ and $\lambda_N^b(\alpha) = p_N^a$;*

2. *$p_n^a \geq \mathbb{E}_n^\alpha \left(\lambda_{n+1}^a(\alpha)\right)$ and $p_n^b \leq \mathbb{E}_n^\alpha \left(\lambda_{n+1}^b(\alpha)\right)$;*

3. *if, for some $p \in \mathbb{R}$, $\mathbb{P}_n^\alpha \left(D_{n+1}^+(p) > \nu_n^+((-\infty, p))\right) > 0$, then*
$$p \leq \mathbb{E}_n^\alpha \left(\lambda_{n+1}^b(\alpha) \,|\, D_{n+1}^+(p) > \nu_n^+((-\infty,p))\right);$$

4. *if, for some $p \in \mathbb{R}$, $\mathbb{P}_n^\alpha \left(D_{n+1}^-(p) > \nu_n^-((p, \infty))\right) > 0$, then*
$$p \geq \mathbb{E}_n^\alpha \left(\lambda_{n+1}^a(\alpha) \,|\, D_{n+1}^-(p) > \nu_n^-((p,\infty))\right);$$

5. *for all $s > 0$,*
   - $\lambda_n^a(\alpha) =$
     $\max\left\{ p_n^b, \mathbb{E}_n^\alpha \lambda_{n+1}^a(\alpha) + \left(\sup_{p\in\mathbb{R}} \mathbb{E}_n^\alpha \left(\left(p - \lambda_{n+1}^a(\alpha)\right) \mathbf{1}_{\left\{D_{n+1}^+(p) > \nu_n^+((-\infty,p))\right\}}\right)\right)^+ \right\}$,
   - *if $\hat{q}_n(s,\alpha) \neq 0$ and $\hat{r}_n(s,\alpha) = 0$, then*
     $$\lambda_n^a(\alpha) = \mathbb{E}_n^\alpha \lambda_{n+1}^a(\alpha) + \sup_{p\in\mathbb{R}} \mathbb{E}_n^\alpha \left(\left(p - \lambda_{n+1}^a(\alpha)\right) \mathbf{1}_{\left\{D_{n+1}^+(p) > \nu_n^+((-\infty,p))\right\}}\right),$$
     and $p = \hat{p}_n(s,\alpha)$ attains the above supremum,
   - *if $\hat{q}_n(s,\alpha) = 0$ and $\hat{r}_n(s,\alpha) = 0$, then $\lambda_n^a(\alpha) = \mathbb{E}_n^\alpha \lambda_{n+1}^a(\alpha)$,*
   - *if $\hat{r}_n(s,\alpha) = 1$, then $\lambda_n^a(\alpha) = p_n^b$;*

6. for all $s < 0$,

- $\lambda^b_n(\alpha) =$
$\min\left\{ p^a_n, \mathbb{E}^\alpha_n \lambda^b_{n+1}(\alpha) - \left(\sup_{p\in\mathbb{R}} \mathbb{E}^\alpha_n \left(\left(\lambda^b_{n+1}(\alpha) - p\right) \mathbf{1}_{\left\{D^-_n(p) > \nu^-_{n-1}((p,\infty))\right\}}\right)\right)^+ \right\},$

- *if $\hat{q}_n(s,\alpha) \neq 0$ and $\hat{r}_n(s,\alpha) = 0$, then*

$$\lambda^b_n(\alpha) = \mathbb{E}^\alpha_n \lambda^b_{n+1}(\alpha) - \sup_{p\in\mathbb{R}} \mathbb{E}^\alpha_n \left(\left(\lambda^b_{n+1}(\alpha) - p\right) \mathbf{1}_{\left\{D^-_n(p) > \nu^-_{n-1}((p,\infty))\right\}}\right),$$

and $p = \hat{p}_n(s,\alpha)$ attains the above supremum,

- *if $\hat{q}_n(s,\alpha) = 0$ and $\hat{r}_n(s,\alpha) = 0$, then $\lambda^b_n(\alpha) = \mathbb{E}^\alpha_n \lambda^b_{n+1}(\alpha)$,*

- *if $\hat{r}_n(s,\alpha) = 1$, then $\lambda^b_n(\alpha) = p^a_n$.*

Proof:

Let us plug the piecewise-linear form of the value function into (2.29):

$$V^\nu_n(s,\alpha) = \text{esssup}_{p,q,r} \left\{ \mathbf{1}_{\{r_n=0\}} \left(s^+ \mathbb{E}^\alpha_n \lambda^a_{n+1}(\alpha) - s^- \mathbb{E}^\alpha_n \lambda^b_{n+1}(\alpha) \right.\right.$$

$$+\mathbb{E}^\alpha_n \left(\left(q_n p_n + (s-q_n)^+ \lambda^a_{n+1}(\alpha) - (s-q_n)^- \lambda^b_{n+1}(\alpha) - s^+ \lambda^a_{n+1}(\alpha) + s^- \lambda^b_{n+1}(\alpha) \right) \cdot \right.$$

$$\left.\left.\left. \left(\mathbf{1}_{\left\{q_n \geq 0,\, D^+_{n+1}(p_n) > \nu^+_n((-\infty,p_n))\right\}} + \mathbf{1}_{\left\{q_n < 0,\, D^-_{n+1}(p_n) > \nu^-_n((p_n,\infty))\right\}} \right) \right) \right)$$

$$\left. +\mathbf{1}_{\{r=1\}} \left(q^+_n p^b_n - q^-_n p^a_n + (s-q_n)^+ \mathbb{E}^\alpha_n \lambda^a_{n+1}(\alpha) - (s-q_n)^- \mathbb{E}^\alpha_n \lambda^b_{n+1}(\alpha) \right) \right\}$$

First, notice that it suffices to consider the essential supremum over all random variables (p_n, q_n, r_n).[12] Moreover, the essential supremum can be replaced by the supremum over all deterministic $(p_n, q_n, r_n) \in \mathbb{R}^2 \times \{0,1\}$. To see the latter, it suffices to assume that the supremum is not attained by the optimal strategy (with positive probability), and construct a superior strategy via the standard measurable selection argument (cf. Corollary 18.27 and Theorem 18.26 in [2]), which results in

[12] The admissibility constraint does not cause any difficulties here, as, in the case where (p_n, q_n, r_n) do not attain the supremum, they can be improved, so that (p_n, q_n) increase by no more than a fixed constant.

a contradiction. It is easy to see that, for any fixed (p_n, s, r_n), the above function is piece-wise linear in q_n, with the slope changing at $q_n = 0$ and $q_n = s$. Hence, for a finite maximum to exists, the slope of this function must be nonnegative, at $q_n \to -\infty$, and non-positive, at $q_n \to \infty$. This must hold for any (p_n, r_n, s), to ensure that the value function of an agent is finite: otherwise, an agent can scale up her position to increase the value function arbitrarily. Considering $r_n = 1$, we obtain condition 2 of the corollary. The case $r_n = 0$ yields conditions 3 and 4. Notice also that the maximum of the aforementioned function is always attained at $q_n = 0$ or $q_n = s$. Considering all possible cases: $r_n = 0, 1$, $q_n = 0, s$, $s = 0$, $s > 0$ and $s < 0$ – we obtain the recursive formulas for λ_n^a and λ_n^b (i.e. conditions 5 and 6 of the corollary). In addition, as the optimal q_n takes values 0 and s, it is easy to see that the piece-wise linear structure of the value function in s is propagated backwards, and, hence, condition 1 of the corollary holds. ∎

It is also useful to have a converse statement.

Corollary II.32. *Consider an admissible LOB ν and admissible control $(\hat{p}, \hat{q}, \hat{r})$, such that $\hat{q}_n(s, \alpha) \in \{0, s\}$. Assume that, for any $\alpha \in \mathbb{A}$ and any $n = 0, \ldots, N$, there exists a progressively measurable random function $V_\cdot^\nu(\cdot, \alpha)$, such that, for any $s \in \mathbb{R}$, $\mathbb{P}$-a.s., $(\hat{p}, \hat{q}, \hat{r}, V^\nu)$ satisfy the conditions 1–6 of Corollary II.31. Then, $(\hat{p}, \hat{q}, \hat{r})$ is an optimal control for the LOB ν.*

Proof: It suffices to revert the arguments in the proof of Corollary II.31, and recall that $\hat{q}$ can always be chosen to be equal to 0 or s, without compromising the optimality. ∎

2.10 Appendix B

Proof of Lemma II.22. The following lemma shows that the normalized price increments are close to Gaussian in the conditional $\mathbb{L}^2$ norm.

Lemma II.33. *Let Assumptions II.12, II.13, II.14, II.15 hold. Then, there exists a deterministic function $\epsilon(\cdot) \geq 0$, such that $\epsilon(\Delta t) \to 0$, as $\Delta t \to 0$, and, $\mathbb{P}$-a.s., for all $\alpha \in \mathbb{A}$ and all $n = 1, \ldots, N$, we have:*

$$\mathbb{E}^{\alpha}_{n-1}\left(\left(\xi_n/\sqrt{\Delta t} - \sigma_{t_{n-1}}(W^{\alpha}_{t_n} - W^{\alpha}_{t_{n-1}})/\sqrt{\Delta t}\right)^2\right) \leq \epsilon(\Delta t).$$

Proof: Notice: $\xi_n/\sqrt{\Delta t} - \sigma_{t_{n-1}}(W^{\alpha}_{t_n} - W^{\alpha}_{t_{n-1}})/\sqrt{\Delta t} = \frac{1}{\sqrt{\Delta t}}\int\limits_{t_{n-1}}^{t_n} \mu^{\alpha}_s \mathrm{d}s + \frac{1}{\sqrt{\Delta t}}\int\limits_{t_{n-1}}^{t_n} (\sigma_s - \sigma_{t_{n-1}})\mathrm{d}W^{\alpha}_s$. Then, using Assumptions II.13, II.15, and Itô's isometry, we obtain the statement of the lemma. ∎

The next lemma connects the proximity in terms of $\mathbb{L}^2$ norm and the proximity of expectations of certain functions of random variables. This result would follow trivially from the classical theory, but, in the present case, we require additional uniformity – hence, a separate lemma is needed (whose proof is, nevertheless, quite simple).

Lemma II.34. *For any constant $C > 1$, there exists a deterministic function $\gamma(\cdot) \geq 0$, s.t. $\gamma(\varepsilon) \to 0$, as $\varepsilon \to 0$, and, for any $\varepsilon > 0$, $\sigma \in [1/C, C]$, and any random variables $\eta \sim \mathcal{N}(0, \sigma^2)$ and ξ (the latter is not necessarily Gaussian), satisfying $\mathbb{E}(\xi - \eta)^2 \leq \varepsilon$, the following holds for all $p \in \mathbb{R}$:*

(i) $(|p| \vee 1)\,|\mathbb{P}(\xi > p) - \mathbb{P}(\eta > p)| \leq \gamma(\varepsilon)$,

(ii) $\left|\mathbb{E}(\xi \mathbf{1}_{\{\xi > p\}}) - \mathbb{E}(\eta \mathbf{1}_{\{\eta > p\}})\right| \leq \gamma(\varepsilon)$.

Proof: (ii) Note that

$$\left|\mathbb{E}(\xi \mathbf{1}_{\{\xi > p\}}) - \mathbb{E}(\eta \mathbf{1}_{\{\eta > p\}})\right| \leq \left|\mathbb{E}\left((\xi - \eta)\mathbf{1}_{\{\xi > p\}}\right)\right| + \left|\mathbb{E}\left(\eta(\mathbf{1}_{\{\xi > p\}} - \mathbf{1}_{\{\eta > p\}})\right)\right|$$

$$\leq \sqrt{\varepsilon} + \|\eta\|_2 \sqrt{\mathbb{P}(\xi > p, \eta \leq p) + \mathbb{P}(\xi \leq p, \eta > p)},$$

and

$$\mathbb{P}(\xi > p, \eta \leq p) \leq \mathbb{P}(p \geq \eta \geq p - \sqrt[3]{\varepsilon}) + \mathbb{P}(|\xi - \eta| > \sqrt[3]{\varepsilon}) \leq$$
$$M\sqrt[3]{\varepsilon} + \frac{\mathbb{E}(\xi - \eta)^2}{(\sqrt[3]{\varepsilon})^2} \leq (M+1)\sqrt[3]{\varepsilon},$$

where we used the fact that η has a density bounded by a fixed constant M. We can similarly show that $\mathbb{P}[\xi \leq p, \eta > p] \leq (M+1)\sqrt[3]{\varepsilon}$. The resulting estimates yield the statement of the lemma. ∎

Taking $\varepsilon(\Delta t) = \gamma(\epsilon(\Delta t))$ and applying the above lemmas, we get the statement of Lemma II.22, with $(W^\alpha_{t_n} - W^\alpha_{t_{n-1}})/\sqrt{\Delta t}$ in place of η_0. Finally, we note that the laws of the two random variables coincide under $\mathbb{P}^\alpha_{n-1}$, and the statement depends only on these laws. The last statement of Lemma II.22 follows from the fact that Lemma II.34 is stable under analogous substitution. ∎ ∎

CHAPTER III

Dynamics Between Trades

3.1 Introduction

In this chapter, we continue the development of an equilibrium-based modeling framework for market microstructure, initiated in the previous chapter. As in that chapter, we analyze the market microstructure in the context of an auction-style exchange (as most modern exchanges are), in which the participating agents can post limit or market orders. A crucial component of such a market is the *Limit Order Book (LOB)*, which contains all the limit buy and sell orders, and whose shape and dynamics represent the liquidity of the market. We are interested in developing a modeling framework in which the shape of the LOB, and its dynamics, arise *endogenously* from the interactions between the agents. This is in contrast to many of the existing results on market microstructure, which assume that the shape and dynamics of the LOB are given exogenously. Among the many advantages of our approach is the possibility of modeling the reaction of the LOB to the changes in a relevant market indicator or in the rules of the exchange.[1]

Herein, we extend the discrete time modeling framework proposed in chapter II to continuous time. We analyze the dynamics of the LOB between two consecutive

[1]We refer the reader to chapter II, whose introduction contains a more detailed explanation of the problems of market microstructure and a motivation for our study.

trades. The latter simplifies the problem and is justified by the well known empirical fact that most changes in the market microstructure are not due to trades. We manage to establish the existence, and obtain a numerically tractable representation, of an equilibrium in a general continuous time framework, in which the competing agents have different beliefs about the future demand for the asset. These beliefs determine the future distribution of the demand, given the (common) information observed thus far. The latter may, e.g., be generated by a relevant signal (or, market indicator). One can view such conditional distributions as the "models" that the market participants use to predict future demand, and which are based on the (commonly observed) relevant market indicators. Given the beliefs, the agents choose their optimal trading strategies (i.e. limit and market orders), aiming to maximize their expected profits. and find an equilibrium Herein, we obtain the desired "quantitative" description of an equilibrium in such a game, which constitutes an endogenous model of market microstructure. Such a result can be used for modeling the reaction of a LOB to various changes in the relevant market indicators. In particular, if the relevant market indicator depends on the LOB itself, our framework allows one to model the indirect market impact: i.e. how an initial change to the LOB may cause further changes to it. Note that the initial change may or may not be due to a trade. Indeed, a trade (i.e. a market order) eliminates a part of the LOB, thus, making a direct impact on it. However, it is well known that, even after a trade is completed, the LOB keeps changing, representing the reaction of the market participants to the information revealed by the initial change. Similarly, a large incoming limit buy or sell order does not constitute a trade, nevertheless, it also affects the LOB, which causes further changes to it, due to the same reason. In fact an extreme example of the latter activity is called "spoofing", and it is an illegal activity aimed at manipulating the market. Our model can be used to quantify such indirect market impact, and it can be, ultimately, used to improve the optimal execution algorithms or to test the consequences of "spoofing" activity. We provide a simplistic example illustrating the use of our model in Section 3.5.

On the mathematical side, the problem we analyze is the construction of an equilibrium in a control-stopping game with a continuum of players (cf. [4], [56], [15], for more on the general theory of continuum-player games). The main mathematical challenges stem from three sources: the complicated dependence between the individual state dynamics and the controls of other players (which lacks the standard convexity and continuity properties), the presence of multiple participants (as opposed to analyzing a two-player game) and the mixed control-stopping nature of the game. Equilibria in the games with any number of players can often be constructed directly, by means of a system of Partial Differentia Equations or a system of (Forward-) Backward Stochastic Differential Equations (BSDEs). However, as the number of players grows, solving such systems numerically becomes very challenging. In such cases, the description of an equilibrium is, typically, limited to the proof of its existence, which, in turn, is obtained by an abstract fixed-point argument. However, even the latter method presents a challenge in the game considered herein. Namely, the complicated dependence structure between the players' controls and state processes, along with the mixed-control stopping nature of the game, make it very challenging (or even impossible) to (a) find the right space for the controls and states, on which the compactness of the range of the objective function can be established, and (b) establish the sufficient regularity of the objective function (e.g. even its continuity may be lacking). In order to overcome these challenges, we make certain "monotonicity" assumptions on the space of agents' beliefs (which is a part of the state space), which allow us to split the problem into two parts: a control-stopping game with two players, and a pure control game (without stopping) with a continuum of players. Such a split simplifies our task dramatically, but both resulting problems remain challenging. The first one, concerned with the construction of an equilibrium in a two-player game, leads to a non-standard system of Reflected BS-DEs (RBSDEs), whose components reflect against each other, and whose generator lacks to desired regularity. In Subsection 3.3.2, we prove the existence of a solution to this system, and, in Section 3.5, we show how it can be computed in a simple example. The second problem, concerned with the equilibrium in a continuum-player game (without stopping), reduces to the maximization of an instantaneous reward function, which depends both on the individual controls and on the joint controls of all players. The latter is formulated as a fixed-point problem, and is solved in Subsection 3.4.1. One of the computational benefits of the solution method proposed herein is that the aforementioned fixed-point problem can be solved separately for each (t, ω). In particular, it is not necessary to solve a forward-backward system at each step of the iteration, as it is, for example, done in a typical mean field game (cf. [46], [14], [16], [45], for more on mean field games). On the other hand, the local nature of the fixed-point problem causes additional measurability issues, in the proof of the existence result (i.e. choosing a measurable selector from the set of fixed points requires more work than choosing it from the set of maxima points). In addition, the objective function of the aforementioned fixed-point problem lacks the desired continuity properties, and, hence, it has to be "mollified" before an abstract fixed-point argument can be applied. All these issues are addressed in Subsection 3.4.1, and the main existence result is stated in Theorem III.29, in Section 3.4. Of course, the results of Section 3.3 and Subsection 3.4.1 are also needed to construct an equilibrium (which is demonstrated in Section 3.5).

The literature on market microstructure is vast. However, most of the theoretical work is concerned with the problem of optimal execution, in which an investor needs to liquidate her position in the asset within a given time horizon, by submitting smaller (limit or market) orders and aiming to maximize the profits. The relevant publications, include, among others, [48], [3], [53], [54], [31], [47], [22], [6], [5],[7], [24], [51], [36], [19], [37], [57], and references therein. In these publications, the dynamics and shape of the LOB are modeled exogenously, or, equivalently, the arrival processes of the limit and market orders of other agents are specified exogenously. In particular, none of these works attempt to explain the shape and dynamics of the LOB, arising directly from the interaction between market participants. Finally, several recent papers have applied an equilibrium-based approach to the problem of optimal execution (cf. [55], [40]). These papers describe an equilibrium between several agents solving an optimal execution problem, with the LOB (or, the market), against which these agents trade, being specified exogenously, rather than being modeled as an output of the equilibrium. The endogenous formation of the LOB is investigated, e.g., in [50], [29], [34], [17], [44], [52], [9], [10], [11], [12]. However, the models proposed in the aforementioned papers do not aim to represent the mechanics of an auction-style exchange with sufficient precision, which is needed to address the questions we investigate herein.

The chapter is organized as follows. Section 3.2 describes the proposed continuum-player game and defines the associated equilibrium. Section 3.3 introduces an auxiliary two-player game. This game can be interesting in its own right, but its main purpose is to facilitate the construction of an equilibrium in the continuum-player game. The equilibria in the two-player game can be described by a system of RBS-

DEs, whose generator does not satisfy the desired regularity conditions. Proposition III.20, in Subsection 3.3.2, provides the existence and uniqueness result for such a system, which, to the best of our knowledge, is not available in the existing literature. Section 3.4 completes the construction of an equilibrium in the continuum-player game, stating the main result of the chapter, Theorem III.29. Finally, in Section 3.5, we implement an example of the proposed continuum-player game and show how it can be used to study certain empirically observed phenomena.

3.2 Modeling framework in continuous time

3.2.1 Preliminary constructions

We consider an auction-style exchange in which the trades may occur, and the limit orders may be posted, at any time $t \in [0, T]$. The market participants are split into two groups: the external investors, who are "impatient", in the sense that they only submit market orders and need to execute immediately, and the strategic players, who can submit both market and limit orders, and who are willing to spend time doing so, in order to get a better execution price. In our model, we focus on the strategic players, who we refer to as *agents*, and we model the behavior of the external investors exogenously, via the *external demand*. The external demand for the asset is modeled using three components: the arrival times of the *potential external market orders*, the value of the *potential fundamental price* at these times, and the *elasticity* of the demand. In our previous investigation in chapter II, we have considered a general family of discrete time games for an auction-style exchange, with the exogenous demand process given by a discretization of a (very general) continuous time demand process, over a chosen partition of $[0, T]$. One of the main conclusions of that chapter can be, roughly, interpreted as follows: in order for a non-degenerate equilibrium[2] to exist in a high-frequency limit (i.e. as the diameter of the partition vanishes), the agents have to be market-neutral – i.e. they should not expect the future fundamental price of the asset to increase or decrease. In other words, the results of chapter II seem to imply that it is hopeless to search for an equilibrium in a continuous time game (i.e. with unlimited trading frequency) in which the agents have non-trivial trading signals about the direction of the future moves of the asset price. This may sound very discouraging, however, there is a subtle feature hidden in the setting considered in chapter II. Namely, the assumptions of that chapter imply that, in the limiting high-frequency regime, the (potential) external market orders arrive with an infinite frequency, while the beliefs of the agents (i.e. their trading signals) satisfy certain continuity properties. In other words, the agents' signals are assumed to be persistent relative to the trades – they cannot change on the same time scale on which the market orders arrive. It turns out that this assumption is crucial, and, allowing the (potential) external market orders to arrive at a finite frequency, and making the agents' beliefs be short-lived (i.e. only lasting until the next market order is executed), we can obtain a non-degenerate equilibrium in the continuous time (i.e. unlimited trading frequency) regime. Thus, herein, we model the arrival of the (potential) external market orders via a (rather general) point process, and we assume that the game ends after the first trade occurs.

Let $(\Omega, \tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t\in[0,T]}, \mathbb{P})$ be a stochastic basis, satisfying the usual conditions, and supporting a (multidimensional) Brownian motion W and a Poisson random measure N. We assume that the compensator of N is finite on $[0,T] \times \mathbb{R}$ (i.e. N is the jump measure of a compound Poisson process) and that it is absolutely continuous w.r.t. Lebesgue measure in time and space. We denote by $\mathbb{F}^W$ the usual

[2]Degeneracy of an equilibrium is defined formally in chapter II. For the discussion presented herein, it suffices to know that degeneracy is an extremal state of the market, and the present work is concerned with the description of the typical (or, normal) states.

augmented filtration generated by W. We assume that W and N are independent under $\mathbb{P}$. The arrival times of the potential external market orders and the values of the potential fundamental price at these times are described by a counting random measure M on $[0,T]\times(\mathbb{R}\setminus\{0\})$, defined as

$$M(A) = \int_0^T \int_{\mathbb{R}} \mathbf{1}_A\left(t, J_t(x)\right) N(dt, dx),$$

where $J : (t,x) \mapsto J_t(x)$ is a predictable random function (as defined in [39]). We assume that J is adapted to $\mathbb{F}^W$ (in particular, it is independent of N). It is clear that the compensator of M is finite on $[0,T]\times\mathbb{R}$, it is absolutely continuous w.r.t. Lebesgue measure in time and space, and it is adapted to $\mathbb{F}^W$. Then, it can be represented as $\lambda_t f_t(x)\, dt\, dx$, with an $\mathbb{R}$-valued process $\lambda \geq 0$ and a random function $f : (t,x) \mapsto f_t(x) \geq 0$, progressively measurable and adapted to $\mathbb{F}^W$, and s.t. $\int_{\mathbb{R}} f_t(x)dx = 1$. Notice that, conditional on $\mathcal{F}^W_T$, M is a Poisson random measure with the compensator $\lambda_t f_t(x)\, dt\, dx$. The t-components of the atoms of M are the arrival times of the potential external market orders, and their x-components represent the values of the potential fundamental price at these times. A positive value of x corresponds to the arrival time of a potential external buy order, and a negative value corresponds to the arrival time of a potential external sell order. More precisely, we define the *fundamental price* process X as the jump process of M:

$$X_t = \int_{\mathbb{R}} x M(\{t\}\times dx).$$

The process λ describes the intensity of arrival of the potential external market orders (both buy and sell). The function f_t is the probability density of the value of the potential fundamental price at time t. We refer to f as the density process of the jump sizes. When the jump size of the fundamental price (along with the demand elasticity, defined below) is not enough to trigger a trade, the jump remains “unregistered” by the agents, and the fundamental price returns to zero. The *elasticity* of the external demand for the asset is described by the progressively measurable random field $D : (t,p) \mapsto D_t(p)$, adapted to $\mathbb{F}^W$. We assume that, a.s., $D_t(\cdot)$ is a strictly decreasing continuous function taking value zero at zero. Then, the total external demand to buy and sell the asset at time t, at the price level p and at all more favorable prices, is equal to

(3.1)

$$D_t^+(p) = \max\left(0, D_t(p - X_t)\mathbf{1}_{\{X_t>0\}}\right), \quad D_t^-(p) = -\min\left(0, D_t(p - X_t)\mathbf{1}_{\{X_t<0\}}\right),$$

respectively.

At any time t, every agent (i.e. strategic player) is allowed to submit a *market order* or a *limit order*. The assumptions made further in the chapter make it possible to submit a limit order at such a level that it may never get executed – this, effectively, allows the agents to wait (i.e. do nothing). We do not allow for any time-priority in the limit orders. Instead, we assume that the tick size is zero (the set of possible price levels is $\mathbb{R}$), and, hence an agent can achieve a priority by posting her order slightly above or below the competing ones (and arbitrarily close to them). The game stops at the terminal time T or at the time when the first trade occurs – whichever one is the earliest. The mechanics of order execution are explained in the next subsection. There is an infinite number of agents, and the inventory of an agent is measured in "shares per unit mass of agents" (see a discussion of this assumption in chapter II). We assume that the agents are split into two groups: the ones whose initial inventory s is positive (the *long* agents, typically, indicated with a superscript "a"), and those whose initial inventory s is negative (the *short* agents, indicated with a superscript "b"). We assume that the absolute size of each agent's inventory is the same, $s \in \{-1, 1\}$, and that an agent with inventory s posts orders of size s. These assumptions are motivated by the results of our previous investigation in chapter II, which demonstrate that, in equilibrium, the absolute value of the agent's inventory only scales the size of her orders proportionally, but does not change their type and location (which, ultimately, is due to the fact that each agent is infinitesimally small and has no risk aversion). We also assume that we are given a pair of measurable spaces of *beliefs*, $\mathbb{A}$ and $\mathbb{B}$, and, for each $\alpha \in \mathbb{A} \cup \mathbb{B}$, there exists a *subjective probability measure* $\mathbb{P}^\alpha$ on $\left(\Omega, \tilde{\mathbb{F}}\right)$, which is dominated by $\mathbb{P}$. An agent with beliefs α models the external demand under measure $\mathbb{P}^\alpha$. The empirical distribution of the agents across beliefs is given by a pair of countably additive finite measures $\mu = (\mu^a, \mu^b)$, on $\mathbb{A}$ and $\mathbb{B}$, respectively. Note that, because the game stops right after the first market order is executed, the empirical distribution μ remains constant throughout the game. We make the following assumption on the measures $\{\mathbb{P}^\alpha\}$.

Assumption III.1. *Under every $\mathbb{P}^\alpha$, W remains a Brownian motion, and the jump process of N is a process with conditionally independent increments w.r.t. $\mathcal{F}_T^W$ (in the sense of [39]).*

The above assumption holds throughout the chapter. It implies that, under every $\mathbb{P}^\alpha$, X is a process with conditionally independent increments w.r.t. $\mathcal{F}_T^W$. Using this observation and the absolute continuity of $\mathbb{P}^\alpha$ w.r.t. $\mathbb{P}$, it is easy to deduce that, under every $\mathbb{P}^\alpha$, the compensator of the jump measure of X, i.e. of the measure M, is given by

$$\lambda_t^\alpha f_t^\alpha(x) dt dx,$$

with some nonnegative $\mathbb{F}^W$-adapted λ and $\mathbb{F}^W$-progressively measurable f^α, s.t. $\int_{\mathbb{R}} f_t^\alpha(x)dx = 1$. The interpretation of λ^α and f^α is the same as the interpretation of λ and f, but under the measure $\mathbb{P}^\alpha$. It is clear that the above assumption is satisfied if $Z_T^\alpha = d\mathbb{P}^\alpha/d\mathbb{P}$ is given by a stochastic exponential of a process that is an integral of $\mathbb{F}^W$-adapted random function w.r.t. compensated N. Namely,

$$dZ_t^\alpha = Z_{t-}^\alpha \int_{\mathbb{R}} \Gamma_t^\alpha(x)\,[N(dt,dx) - \lambda_t f_t(x) dt dx],$$

where $\Gamma^\alpha \geq -1$ is $\mathbb{F}^W$-progressively measurable. The compensator of N under $\mathbb{P}^\alpha$ is obtained by multiplying its compensator under $\mathbb{P}$ by $1+\Gamma^\alpha$, hence, Assumption III.1 is clearly satisfied in this case (cf. [39]). In Section 3.5, we provide an example of a family of probability measures $\{\mathbb{P}^\alpha\}$ in the above form.

3.2.2 The continuum-player game

Throughout the rest of this chapter we, mostly, work with the filtration $\mathbb{F}^W$, hence, we denote $\mathbb{F} = \mathbb{F}^W$. The state of an agent is $(s,\alpha) \in (\{1\}\times\mathbb{A}) \cup (\{-1\}\times\mathbb{B}) =: \mathbb{S}$. Let us now discuss the controls of the agents and the order execution rules. First, we assume that α, representing the agent's beliefs, does not change over time.[3] Therefore, the state process of an agent represents only her inventory, which can only change once (because the game ends after the first trade). The control of every agent is given by a pair of processes $(p,v) = (p_t, v_t)_{t\in[0,T]}$, progressively measurable with respect to $\mathbb{F}$ (note that the controls are, in particular, predictable). The process p takes values in $\mathcal{P}(\mathbb{R})$, the space of probability measures on $\mathbb{R}$, equipped with the weak topology, while v takes values in $\mathbb{R}$. The first coordinate, p_t, indicates the time-t distribution of the agent's limit orders across the price levels. For example, if p_t is a Dirac measure located at x, then, at time t, the agent posts all her limit orders at the price level x. The collection of all limit orders is described by the *Limit Order Book (LOB)*, which is a pair of process $\nu = (\nu_t^a, \nu_t^b)_{t\in[0,T]}$, with values in the finite sigma-additive measures on $\mathbb{R}$, adapted to $\mathbb{F}$. Herein, ν_t^a corresponds to the cumulative limit sell orders, and ν_t^b corresponds to the cumulative limit buy orders,

[3] Note that the conditional distribution of the future demand can change dynamically, according the new information revealed.

posted at time t.[4] The bid and ask prices at any time $t \in [0,T]$ are given by the random variables

$$p_t^b = Q^+(\nu_t^b), \qquad p_t^a = Q^-(\nu_t^a),$$

respectively, where the functions Q^- and Q^+ act on sigma-additive measures κ on $\mathbb{R}$ via

$$Q^+(\kappa) = \sup \operatorname{supp}(\kappa), \qquad Q^-(\kappa) = \inf \operatorname{supp}(\kappa). \tag{3.2}$$

Notice that p_t^b and p_t^a are always well defined as extended random variables, but may take infinite values. Assume that, at time t, an agent posts a limit sell order at a price level p'. If the demand to buy the asset at or below the price level p', $D_t^+(p')$, exceeds the amount of all limit sell orders posted below p' at time t, i.e. $D_t^+(p') > \nu_t^a((-\infty, p'))$, then the limit sell order of the agent is executed. Analogous execution rules hold for the limit buy orders. Thus, for an agent following strategy (p, v), her limit order is (partially) executed by an external market order at the time

$$T^{p,a} = \inf\{t \in [0,T] \,:\, D_t^+\left(Q^-(p_t)\right) > \nu_t^a\left((-\infty, Q^-(p_t))\right)\},$$

$$T^{p,b} = \inf\{t \in [0,T] \,:\, D_t^-\left(Q^+(p_t)\right) > \nu_t^b\left((Q^+(p_t), \infty)\right)\},$$

for the long and short agents, respectively. The value of v_t indicates the critical level of the bid or ask price (i.e. a threshold), at which the agent decides to submit a market order. We assume that the size of the agent's market order is equal to her inventory, and it is executed at the bid or ask price available at the time when the order is submitted. Thus, the agent will submit her own market order at the time

$$\tau^{v,a} = \inf\{t \in [0,T] \,:\, v_t \leq p_t^b\}, \qquad \tau^{v,b} = \inf\{t \in [0,T] \,:\, v_t \geq p_t^a\},$$

[4]For convenience, we sometimes refer to ν_t as a "measure", rather than a "pair of measures".

for the long and short agents, respectively.[5] The collection of all thresholds v is described by the pair of processes $\theta = (\theta^a_t, \theta^b_t)_{t\in[0,T]}$, with values in the finite sigma-additive measures on $\mathbb{R}$, adapted to $\mathbb{F}$. We define

$$v^b_t = Q^+(\theta^b_t), \qquad v^a_t = Q^-(\theta^a_t)$$

Remark III.2. The above definitions of the execution times make use of the assumption that each agent is infinitesimally small, and, hence, her order is necessarily executed once the demand reaches it. They also use the following two implicit assumptions: each agent believes that her limit order will be executed first among all orders at the same price level, and her market order will be executed at the best price available. These assumptions and their connection to a finite-player game are discussed in chapter II.

Consider the first "significant" execution times of external market orders:

$$\begin{aligned} T^a &= \inf\{t \in [0,T] \,:\, D^+_t(p^a_t) > \nu^a_t((-\infty, p^a_t))\}, \\ T^b &= \inf\{t \in [0,T] \,:\, D^-_t(p^b_t) > \nu^b_t((p^b_t, \infty))\}, \end{aligned} \tag{3.3}$$

Similarly, we define the first "significant" execution times of internal market orders:

$$\tau^a = \inf\{t \in [0,T] \,:\, v^a_t \le p^b_t\}, \qquad \tau^b = \inf\{t \in [0,T] \,:\, v^b_t \ge p^a_t\}. \tag{3.4}$$

For a long agent with strategy (p, v), the game ends at the time $T^{p,a} \wedge \tau^{v,a} \wedge T \wedge T^a \wedge T^b \wedge \tau^a \wedge \tau^b$ (and similarly for the short agents). If an agent has any inventory left at the end of the game, then it is marked to market. The precise rules for computing an agent's payoff are described below. For a given market (ν, θ, M, D), let us introduce the clearing prices:

$$\tilde{p}^{c,a}_t = \sup\{p < Q^+(\nu^a_t) \,:\, D^+_t(p) > \nu^a_t((-\infty, p))\}, \quad p^{c,a}_t = \tilde{p}^{c,a}_t \mathbf{1}_{\{\tilde{p}^{c,a}_t \ge p^a_t\}},$$

[5] It is clear that, for every stopping time $\tau^{v,a/b}$ with respect to $\mathbb{F}$, there exists a process v_t, adapted to $\mathbb{F}$, such that $\tau^{v,a/b}$ has the above representation.

$$\tilde{p}_t^{c,b} = \inf\{p > Q^-(\nu_t^b) \,:\, D_t^-(p) > \nu_t^b((p,\infty))\}, \quad p_t^{c,b} = \tilde{p}_t^{c,b}\mathbf{1}_{\{\tilde{p}_t^{c,b} \leq p_t^b\}}.$$

Consider a long agent who follows a strategy (p, v). Then, the payoff function of the agent is described by the following list of rules.

- Assume that $T^{p,a} \wedge T^a \wedge T^b < T \wedge \tau^a \wedge \tau^b$ (note that equality is impossible, as the right hand side is predictable and the left hand side is totally inaccessible).
  - If $T^{p,a} \wedge T^a < T^b$ (equality is impossible), then the payoff is
$$\int_{-\infty}^{\tilde{p}_t^{c,a}} z p_t(dz) + \int_{\tilde{p}_t^{c,a}}^{\infty} (p_t^{c,a} + p_t^b) p_t(dz), \qquad \text{with} \quad t = T^{p,a} \wedge T^a.$$
  - If $T^b < T^{p,a} \wedge T^a$, then the payoff is $p_{T^b}^b + p_{T^b}^{c,b}$.
- Assume that $T \wedge \tau^a \wedge \tau^b < T^{p,a} \wedge T^a \wedge T^b$.
  - If $\tau^a \wedge T \leq \tau^b$ then the payoff is $p_{\tau^a}^b$.
  - If $\tau^b < \tau^a \wedge T$ then the payoff is $p_{\tau^b}^a$.

Remark III.3. The above choice of the payoff is motivated by the desire to project the agent's view on what happens after the game is over. In particular, if an external market order arrives but does not fully execute the agent's limit orders, then the residual is marked to the bid price shifted by the clearing price (i.e. the new projected value of the bid price, after the game is over). If an internal market order is executed first, then, depending on which side of the book initiates this order, the agent's position is marked to the bid or to the ask price.

Similar rules apply to the short agents. Thus, the objective function of an agent in the market (ν, θ, M, D), starting at the initial state $(1, \alpha)$ and using the control

(p,v), is given by:

$$(3.5)\quad J^{(\nu,\theta),(p,v)}(1,\alpha) =$$
$$\mathbb{E}^{\alpha}\Big[\int_{\mathbb{R}}\left(z\mathbf{1}_{\{z\leq\tilde{p}^{c,a}_{\hat{T}^{p,a}}\}}+\left(p^{b}_{\hat{T}^{p,a}}+p^{c,a}_{\hat{T}^{p,a}}\right)\mathbf{1}_{\{z>\tilde{p}^{c,a}_{\hat{T}^{p,a}}\}}\right)p_{\hat{T}^{p,a}}(dz)\mathbf{1}_{\{\hat{T}^{p,a}<T^{b}\wedge\hat{\tau}^{v,a}\wedge\tau^{b}\}}+$$
$$\left(p^{b}_{T^{b}}+p^{c,b}_{T^{b}}\right)\mathbf{1}_{\left\{T^{b}<\hat{T}^{p,a}\wedge\hat{\tau}^{v,a}\wedge\tau^{b}\right\}}+\left(p^{a}_{\tau^{b}}\mathbf{1}_{\{\tau^{b}<\hat{\tau}^{v,a}\}}+p^{b}_{\hat{\tau}^{v,a}}\mathbf{1}_{\{\tau^{b}\geq\hat{\tau}^{v,a}\}}\right)\mathbf{1}_{\{\hat{T}^{p,a}\wedge T^{b}>\hat{\tau}^{v,a}\wedge\tau^{b}\}}\Big]$$

where $\hat{T}^{p,a}=T\wedge T^{p,a}\wedge T^{a}$, $\hat{\tau}^{v,a}=T\wedge\tau^{v,a}\wedge\tau^{a}$, and we assume that $0\cdot\infty=0$. Similarly,

$$(3.6)\quad J^{(\nu,\theta),(p,v)}(-1,\alpha) =$$
$$\mathbb{E}^{\alpha}\Big[-\int_{\mathbb{R}}\left(z\mathbf{1}_{\{z\geq\tilde{p}^{c,b}_{\hat{T}^{p,b}}\}}+\left(p^{a}_{\hat{T}^{p,b}}+p^{c,a}_{\hat{T}^{p,b}}\right)\mathbf{1}_{\left\{z<\tilde{p}^{c,b}_{\hat{T}^{p,b}}\right\}}\right)p_{\hat{T}^{p,b}}(dz)\mathbf{1}_{\{\hat{T}^{p,b}<T^{a}\wedge\hat{\tau}^{v,b}\wedge\tau^{a}\}}$$
$$-\left(p^{a}_{T^{b}}+p^{c,a}_{T^{a}}\right)\mathbf{1}_{\{T^{a}<\hat{T}^{p,b}\wedge\hat{\tau}^{v,b}\wedge\tau^{a}\}}-\left(p^{b}_{\tau^{a}}\mathbf{1}_{\{\tau^{a}<\hat{\tau}^{v,b}\}}+p^{a}_{\hat{\tau}^{v,b}}\mathbf{1}_{\{\tau^{a}\geq\hat{\tau}^{v,b}\}}\right)\mathbf{1}_{\{\hat{T}^{p,b}\wedge T^{a}>\hat{\tau}^{v,b}\wedge\tau^{a}\}}\Big]$$

where $\hat{T}^{p,b}=T\wedge T^{p,b}\wedge T^{b}$, $\hat{\tau}^{v,b}=T\wedge\tau^{v,b}\wedge\tau^{b}$. Every agents aims to maximize the objective. In the following definitions, we assume that a stochastic basis, a Brownian motion W, a random measure M, a random field D, spaces $\mathbb{A}$ and $\mathbb{B}$, an associated set of measures $\{\mathbb{P}^{\alpha}\}_{\alpha\in\mathbb{A}\cup\mathbb{B}}$, and the empirical distribution μ, are fixed and satisfy the assumptions made earlier in this section. The above objectives may seem very convoluted – this is because they are meant to provide a close approximation to the real-world execution rules and marking to market. In the next subsection, we establish a more transparent representation of the objectives.

Definition III.4. For a given market (ν,θ) and a state $(s,\alpha)\in\mathbb{S}$, a pair of $\mathbb{F}$-progressively measurable processes (p,v) is an **admissible** control, if the positive part of the expression inside the expectation in (3.5) (if $s=1$) or (3.6) (if $s=-1$) has a finite expectation under $\mathbb{P}^{\alpha}$.

Definition III.5. For a given market (ν,θ) and state $(s,\alpha)\in\mathbb{S}$, we call an admissible control (p, v) **optimal** if

$$J^{(\nu,\theta),(p,v)}(s,\alpha) \geq J^{(\nu,\theta),(p',v')}(s,\alpha)$$

$\mathbb{P}$-a.s., for any admissible control (p', v').

In the above, we make the standard assumption of games with *a continuum of players*: each agent is too small to affect the distribution of cumulative controls (described by ν) when she changes her control. Next, we define the notion of equilibrium in the proposed game.

Definition III.6. A given market (ν, θ) and a pair of $\mathbb{F}$-progressively measurable random fields $(p, v) : \Omega \times [0, T] \times \mathbb{S} \to \mathcal{P}(\mathbb{R}) \times \mathbb{R}$ form an **equilibrium**, if

1. for μ-a.e. $(s, \alpha) \in \mathbb{S}$, $(p(s, \alpha), v(s, \alpha))$ is an optimal control for (ν, θ) and (s, α),
2. and the following holds $\mathbb{P}$-a.s., for any $t < \bar{T} := T \wedge T^a \wedge T^b \wedge \tau^a \wedge \tau^b)$ and any $x \in \mathbb{R}$:

$$\begin{aligned} \nu^a_t((-\infty, x]) &= \int_{\mathbb{A}} p_t\left(1, \alpha; (-\infty, x]\right) \mu^a(d\alpha), \\ \nu^b_t((-\infty, x]) &= \int_{\mathbb{B}} p_t\left(-1, \alpha; (-\infty, x]\right) \mu^b(d\alpha), \end{aligned} \tag{3.7}$$

(3.8)
$$\theta^a_t((-\infty, x]) = \int_{\mathbb{A}} \mathbf{1}_{\{v_t(1,\alpha) \leq x\}} \mu^a(d\alpha), \quad \theta^b_t((-\infty, x]) = \int_{\mathbb{B}} \mathbf{1}_{\{v_t(-1,\alpha) \leq x\}} \mu^b(d\alpha).$$

Remark III.7. In the above definition, it is implicitly assumed that the empirical measure of the agents' states remains constant in time until the game is over for all players. This is, indeed, the case, if the equilibrium is such that, $\mathbb{P}$-a.s., for all $t < \bar{T}$, we have:

$$\mu \circ \left((s, \alpha) \mapsto S_t(s, \alpha)\right)^{-1} = \mu, \tag{3.9}$$

with

$$S_t(1,\alpha) = \mathbf{1}_{\left[0,T^{p(1,\alpha),a}\wedge\tau^{v(1,\alpha),a}\right)}(t), \text{ and } S_t(-1,\alpha) = -\mathbf{1}_{\left[0,T^{p(-1,\alpha),b}\wedge\tau^{v(-1,\alpha),b}\right)}(t).$$

The condition (3.9) may fail if a non-zero mass of agents manages to execute their orders strictly before $\bar{T}$: i.e. if $T^{p(1,\alpha),a} \wedge \tau^{v(1,\alpha),a} < \bar{T}$ for a set of α with a positive μ^a-measure, or $T^{p(-1,\alpha),b} \wedge \tau^{v(-1,\alpha),b} < \bar{T}$ for a set of α with a positive μ^b-measure. However, the first scenario is impossible, because the external market orders only arrive at a finite number of times and, before $T^a \wedge T^b \geq \bar{T}$, only a zero mass of agents can execute their limit orders against any such market order (cf. (3.3)). It is also true that, at any time t, before $\tau^a \wedge \tau^b \geq \bar{T}$, only a zero mass of agents can execute their internal market orders (cf. (3.4)). However, the set of such times t may be uncountable. Therefore, to ensure that μ remains constant and, hence, (3.9) holds, it suffices to consider only the equilibria satisfying, $\mathbb{P}$-a.s., for all t, except, possibly, a countable set:

$$v_t(1,\alpha) \geq v_t^a, \qquad v_t(-1,\alpha) \leq v_t^b, \qquad \forall \alpha \in \mathbb{A} \cup \mathbb{B}.$$

In the subsequent sections, we construct such an equilibrium.

3.2.3 Representation of the objective

In this section, we provide an equivalent representation of the objective of the agents, which makes it more tractable and more convenient for the analysis that follows. This representation is derived following standard arguments, making use of the independence of the driving Poisson measure and W. First, we introduce new notation that will be used throughout the chapter. For any $\alpha \in \mathbb{A} \cup \mathbb{B}$, $t \in [0,T]$,

$p, x, y \in \mathbb{R}$ and $\kappa \in \mathcal{P}(\mathbb{R})$, we define:

$$F_t^{+,\alpha}(p) = \int_{p\vee 0}^{\infty} f_t^{\alpha}(u)\mathrm{d}u, \qquad F_t^{-,\alpha}(p) = \int_{-\infty}^{p\wedge 0} f_t^{\alpha}(u)\mathrm{d}u,$$
$$c_t^{\alpha}(x,y) = \lambda_t^{\alpha}\left(F_t^{-,\alpha}(y) + F_t^{+,\alpha}(x)\right), \tag{3.10}$$

$$l_t^{c,a}(x) = \sup\left\{p < Q^+(\nu_t^a)\ :\ D_t(p-x) > \nu_t^a((-\infty,p))\right\}, \tag{3.11}$$

$$l_t^{c,b}(x) = \inf\left\{p > Q^-(\nu_t^b)\ :\ -D_t(p-x) > \nu_t^b((p,\infty))\right\}, \tag{3.12}$$

(3.13) $h_t^{\alpha,a}(\kappa,x,y) =$

$$\lambda_t^{\alpha}\int_{(Q^-(\kappa)\wedge x)\vee 0}^{\infty} f_t^{\alpha}(u)\left[\int_{-\infty}^{l_t^{c,a}(u)} z\kappa(dz) + \left(y + l_t^{c,a}(u)\mathbf{1}_{\{l_t^{c,a}(u)\geq x\}}\right)\kappa\left((l_t^{c,a}(u),\infty)\right)\right]du +$$
$$\lambda_t^{\alpha}\int_{-\infty}^{y\wedge 0} f_t^{\alpha}(u)\left(y + l_t^{c,b}(u)\right)du,$$

(3.14) $h_t^{\alpha,b}(\kappa,x,y) =$

$$\lambda_t^{\alpha}\int_{-\infty}^{(Q^+(\kappa)\vee y)\wedge 0} f_t^{\alpha}(u)\left[\int_{l_t^{c,b}(u)}^{\infty} z\kappa(dz) + \left(x + l_t^{c,b}(u)\mathbf{1}_{\{l_t^{c,b}(u)\leq y\}}\right)\kappa\left((-\infty, l_t^{c,b}(u))\right)\right]du +$$
$$\lambda_t^{\alpha}\int_{x\vee 0}^{\infty} f_t^{\alpha}(u)\left(x + l_t^{c,a}(u)\right)du.$$

Notice that, if X has a positive jump at time t, then the clearing price at time t is given by $\tilde{p}_t^{c,a} = l_t^{c,a}(X_t)$. Similarly, if X has a negative jump at time t, then $\tilde{p}_t^{c,b} = l_t^{c,b}(X_t)$. Using the above notation, we can obtain the following simplified expression for the objective.

Lemma III.8. *Let Assumption III.1 hold. Given a market (ν,θ), for any $\alpha \in \mathbb{A}\cup\mathbb{B}$ and any admissible strategy (p,v), we have:*

(3.15)
$$J^{(\nu,\theta),(p,v)}(1,\alpha) = \mathbb{E}\Big[\int_0^{\hat{\tau}^{v,a}\wedge\tau^b} \exp\left(-\int_0^s c_u^{\alpha}\left(p_u^a \wedge Q^-(p_u), p_u^b\right)du\right) h_s^{\alpha,a}(p_s, p_s^a, p_s^b)ds$$

$$+\exp\left(-\int_0^{\hat{\tau}^{v,a}\wedge\tau^b} c_u^\alpha\left(p_u^a\wedge Q^-(p_u),p_u^b\right)du\right)\left(p_{\tau^b}^a\mathbf{1}_{\{\tau^b<\hat{\tau}^{v,a}\}}+p_{\hat{\tau}^{v,a}}^b\mathbf{1}_{\{\tau^b\geq\hat{\tau}^{v,a}\}}\right)\Big],$$

(3.16) $J^{(\nu,\theta),(p,v)}(-1,\alpha) =$

$$-\mathbb{E}\Big[\int_0^{\hat{\tau}^{v,b}\wedge\tau^a}\exp\left(-\int_0^s c_u^\alpha\left(p_u^a,p_u^b\vee Q^+(p_u)\right)du\right)h_s^{\alpha,b}(p_s,p_s^a,p_s^b)ds+$$
$$\exp\left(-\int_0^{\hat{\tau}^{v,a}\wedge\tau^b} c_u^\alpha\left(p_u^a,p_u^b\vee Q^+(p_u)\right)du\right)\left(p_{\tau^a}^b\mathbf{1}_{\{\tau^a<\hat{\tau}^{v,b}\}}+p_{\hat{\tau}^{v,b}}^a\mathbf{1}_{\{\tau^a\geq\hat{\tau}^{v,b}\}}\right)\Big],$$

where $\hat{\tau}^{v,a}=T\wedge\tau^{v,a}\wedge\tau^a$, $\hat{\tau}^{v,b}=T\wedge\tau^{v,b}\wedge\tau^b$ *and the expectations are taken under* $\mathbb{P}$.

Proof: The proof follows easily by conditioning on W. Notice that, conditional on $\mathcal{F}_T$, M is a Poisson random measure, with the deterministic compensator $\lambda_t^\alpha f_t^\alpha(x)\,dt\,dx$, which is finite on $[0,T]\times\mathbb{R}$. Recall also that D, ν, θ, p, v, p^a, p^b, $\tau^{v,a}$, $\tau^{v,b}$, τ^a, τ^b, and all the random functions defined above the lemma, are adapted to $\mathbb{F}$. Conditional on $\mathcal{F}_T$, they become deterministic functions of time. Recall the fundamental price process, $X_t=\int_{\mathbb{R}}xM(\{t\}\times dx)$, and introduce

$$Y_t=X_t\left(\mathbf{1}_{\{X_t>(p_t^a\wedge Q^-(p_t))\vee 0\}}+\mathbf{1}_{\{X_t<p_t^b\wedge 0\}}\right).$$

Notice that $\hat{T}^{p,a}$ is the time of the first positive jump of Y_t, and T^b is the time of its first negative jump. Notice also that, conditional on $\mathcal{F}_T$, the clearing price $\tilde{p}_t^{c,a}$ becomes a deterministic function of t and Y_t: $\tilde{p}_t^{c,a}=l_t^{c,a}(Y_t)$. Thus, conditional on $\mathcal{F}_T$, the expression inside the expectation in (3.5) becomes a function of the time and size of the first jump of Y. Conditional on $\mathcal{F}_T$, X is the jump process of a Poisson random measure with the compensator $\lambda_t^\alpha f_t^\alpha(u)dudt$. It is also clear that, conditional on $\mathcal{F}_T$, Y is the jump process of a non-homogeneous compound Poisson process with intensity $c_t^\alpha\left(p_t^a\wedge Q^-(p_t),p_t^b\right)$, and with the distribution of jump sizes at time t given by

$$\frac{\lambda_t^\alpha f^\alpha(x)}{c_t^\alpha\left(p_t^a \wedge Q^-(p_t), p_t^b\right)}\left(\mathbf{1}_{\{x\le p_t^b\wedge 0\}} + \mathbf{1}_{\{x\ge (p_t^a\wedge Q^-(p_t))\vee 0\}}\right) dx.$$

A standard computation, then, yields (3.15). The equation (3.16) is derived similarly. The expectations in (3.15) and (3.16) are taken under $\mathbb{P}$, because the expressions inside the expectations are adapted to $\mathbb{F} = \mathbb{F}^W$, and W has the same distribution under $\mathbb{P}$ and $\mathbb{P}^\alpha$. ∎

3.3 A two-player game

In this section, we consider an auxiliary two-player game. It is related to the continuum-player game, but the precise connection will be established in the subsequent sections. Assume that all the probabilistic constructions made in Subsection 3.2.1 are in place. Namely, we are given a stochastic basis, with a Brownian motion W, a Poisson measure N, a counting random measure M, a family of probability measures $\{\mathbb{P}^\alpha\}$, and with the demand elasticity process D, as described in Section 3.2. We assume that Assumption III.1 holds. Assume, in addition, that $\mathbb{A} = \{\alpha^0\}$ and $\mathbb{B} = \{\beta^0\}$. Consider a two-player game, in which the first (long) player starts with the initial inventory 1 and has beliefs α^0, and the second (short) player starts with the initial inventory -1 and has beliefs β^0. The game proceeds according to the rules similar to those described in the previous section: each agent can post limit orders on the respective side of the book, or can terminate the game by submitting a market order. The execution of limit orders against the external market orders occurs in exactly the same way as described in the previous section. However, herein, at any given time, each agent is only allowed to post limit orders at a single location (i.e. the control p_t is a Dirac measure). In addition, the main difference between the present game and the one defined in the previous section is that, herein, each player has a non-zero mass and, hence, can affect the LOB. In fact, since there is only one player on each side of the book, the LOB is given by a combination of two Dirac measures: $\nu^a_t = \delta_{p^a_t}$, $\nu^b_t = \delta_{p^b_t}$, controlled by the locations of the players' limit orders: p^a for the long agent, and p^b for the short one. Clearly, p^a also coincides with the ask price, and p^b is the bid price. Note that each of these prices is now controlled by a single agent, which is not the case in the original game described in the previous section. The same is true for the stopping thresholds: θ^a and θ^b are given by Dirac measures, and the locations of these measures correspond to the thresholds v^a and v^b used by the long and short agents, respectively. In this new game (due to its simplicity), it turns out to be more convenient to work with the associated stopping times τ^a and τ^b. In fact, we will further constraint the agents' controls, so that $\tau^a = \tau^b =: \tau$ and $p^a_\tau = p^b_\tau = \bar{p}_\tau$. The meaning behind these constraints is clear: every agent assumes that the counterparty will execute a market order at exactly the same time as she does, and that these orders are executed at the same price. Taking into account the above considerations, we transform (3.5) into the objective of a long player:

$$(3.17)\quad \tilde{J}^{a,(p^b,\bar{p}),(p,\tau)} = \mathbb{E}^{\alpha^0}\left[p_{T^{p,a}}\mathbf{1}_{\{T^{p,a}<T^b\wedge\tau\}} + 2p^b_{T^b}\mathbf{1}_{\left\{T^b<T^{p,a}\wedge\tau\right\}} + \bar{p}_\tau\mathbf{1}_{\{T^{p,a}\wedge T^b>\tau\}}\right],$$

where p^b, $\bar{p}$ and p are $\mathbb{R}$-valued $\mathbb{F}$-adapted processes, τ is a stopping time with values in $[0,T]$, and

$$T^b = \inf\{t\in[0,T]\,:\,X_t<p^b_t\},\quad T^{p,a} = \inf\{t\in[0,T]\,:\,X_t>p_t\},\quad X_t = M(\{t\}\times\mathbb{R}).$$

Similarly, for the short agents,

$$(3.18)\quad \tilde{J}^{b,(p^a,\bar{p}),(p,\tau)} = -\mathbb{E}^{\alpha^0}\left[p_{T^{p,b}}\mathbf{1}_{\{T^{p,b}<T^a\wedge\tau\}} + 2p^a_{T^a}\mathbf{1}_{\left\{T^a<T^{p,b}\wedge\tau\right\}} + \bar{p}_\tau\mathbf{1}_{\{T^{p,b}\wedge T^a>\tau\}}\right],$$

where p^a, $\bar{p}$ and p are $\mathbb{R}$-valued $\mathbb{F}$-adapted processes, τ is a stopping time with values in $[0,T]$, and

$$T^a = \inf\{t \in [0,T] \,:\, X_t > p^a_t\}, \qquad T^{p,b} = \inf\{t \in [0,T] \,:\, X_t < p_t\}.$$

Using Lemma III.8, we deduce the following form of the objective functions

$$\begin{aligned}(3.19) \qquad \tilde{J}^{a,(p^b,\bar{p}),(p,\tau)} = \mathbb{E}\Big[& \int_0^\tau \exp\left(-\int_0^s c^{\alpha^0}_u(p_u,p^b_u)du\right) g^a_s(p_s,p^b_s)ds \\ & + \exp\left(-\int_0^\tau c^{\alpha^0}_u(p_u,p^b_u)du\right)\bar{p}_\tau\Big],\end{aligned}$$

where c^α is defined in (3.10) and

$$(3.20) \qquad g^a_t(x,y) = \lambda^{\alpha^0}_t\left(2yF^{\alpha^0,-}_t(y) + xF^{\alpha^0,+}_t(x)\right).$$

Similarly,

$$\begin{aligned}(3.21) \qquad \tilde{J}^{b,(p^a,\bar{p}),(p,\tau)} = -\mathbb{E}\Big[& \int_0^\tau \exp\left(-\int_0^s c^{\beta^0}_u(p^a_u,p_u)du\right) g^b_s(p^a_s,p_s)ds \\ & + \exp\left(-\int_0^\tau c^{\beta^0}_u(p^a_u,p_u)du\right)\bar{p}_\tau\Big],\end{aligned}$$

where

$$(3.22) \qquad g^b_t(x,y) = \lambda^{\beta^0}_t\left(yF^{\beta^0,-}_t(y) + 2xF^{\beta^0,+}_t(x)\right).$$

To ensure that the above expressions are well defined, and to analyze the equilibrium in a two-player game, we need to make the following assumptions.

Assumption III.9. *There exists a constant $C' > 0$, s.t., $\mathbb{P}$-a.s., $|\lambda^\alpha_t|, |f^\alpha_t(x)| \leq C'$, for all $\alpha \in \mathbb{A}\cup\mathbb{B}$, $t \in [0,T]$ and $x \in \mathbb{R}$.*

We also assume that the possible price jump sizes are bounded.

Assumption III.10. *There exists a constant $C_p > 0$, s.t., $\mathbb{P}$-a.s., supp$(f^\alpha_t) \subseteq [-C_p, C_p]$, for all $\alpha \in \mathbb{A}\cup\mathbb{B}$ and $t \in [0,T]$.*

Denote by $\mathbb{S}^2$ the set of continuous $\mathbb{F}$-adapted processes Y, such that $\sup_{0\leq t\leq T}|Y_t| \in \mathbb{L}^2$. We say that the terminal execution price $\bar{p}$ is admissible if $\bar{p} \in \mathbb{S}^2$. A control (p,τ) is admissible if p is $\mathbb{F}$-progressively measurable, satisfying, $\mathbb{P}$-a.s., $|p_t| \leq C_p$ for all $t \in [0,T]$, and τ is $\mathbb{F}$-stopping time. Next, we introduce the notions of optimality and equilibrium in the two-player game – they are analogous to Definitions III.5–III.6.

Definition III.11. For a given admissible $(p^b,\bar{p})$, we call an admissible control (p,τ) *optimal for the long agent* if

$$\tilde{J}^{a,(p^b,\bar{p}),(p,\tau)} \geq \tilde{J}^{a,(p^b,\bar{p}),(p',\tau')},$$

for any admissible control (p',τ'). Similarly, for a given admissible $(p^a,\bar{p})$, we call an admissible control (p,τ) *optimal for the short agent* if

$$\tilde{J}^{b,(p^a,\bar{p}),(p,\tau)} \geq \tilde{J}^{b,(p^a,\bar{p}),(p',\tau')},$$

for any admissible control (p',τ').

Definition III.12. A combination $(p^a,p^b,\tau,\bar{p})$ is an *equilibrium in the two-player game*, if it is admissible and, given $(p^b,\bar{p})$, the control (p^a,τ) is optimal for the long agent, while, given $(p^a,\bar{p})$, the control (p^b,τ) is optimal for the short agent.

In the next subsection, we characterize the equilibrium strategies via a system of Reflected Backward Stochastic Differential Equations (RBSDEs).

3.3.1 Characterizing the equilibria via a system of RBSDEs

The next assumptions are used to guarantee the uniqueness and regularity of the optimal control of an agent.

Assumption III.13. *$\mathbb{P}$-a.s., for any $\alpha \in \mathbb{A}\cup\mathbb{B}$ and $t \in [0,T]$, $f^\alpha_t(\cdot)$ is continuous in the interior of its support, $f^\alpha_t(0) = 0$, and $0 < F^{+,\alpha}_t(0) < 1$.*

Assumption III.14. *$\mathbb{P}$-a.s., for any $\alpha \in \mathbb{A}$ and $t \in [0,T]$, $F_t^{+,\alpha}(\cdot)/f_t^{\alpha}(\cdot)$ is a decreasing function in the interior of supp$(f_t^{\alpha}) \cap \mathbb{R}_+$, vanishing at the right end of the interval. Similarly, $\mathbb{P}$-a.s., for any $\beta \in \mathbb{B}$ and $t \in [0,T]$, $F_t^{-,\beta}(\cdot)/f_t^{\beta}(\cdot)$ is an increasing function in the interior of supp$(f_t^{\beta}) \cap \mathbb{R}_-$, vanishing at the left end of the interval.*

Remark III.15. The monotonicity of $F_t^{+,\alpha}(\cdot)/f_t^{\alpha}(\cdot)$, for example, is implied by the log-concavity of the distribution of positive jumps (similarly, for the negative jumps). Instead of requiring that $F_t^{+,\alpha}(\cdot)/f_t^{\alpha}(\cdot)$ is decreasing, it suffices to assume that its growth rate is bounded from above by $1-\varepsilon$, for a constant $\varepsilon > 0$ independent of (t,ω).

To prove the existence of a solution to a system of RBSDEs characterizing the equilibria in a two-player game, we also need to assume that "the range of beliefs is relatively bounded".

Assumption III.16. *There exists a constant $C > 0$, s.t., $\mathbb{P}$-a.s.:*

$$\frac{1}{C} \leq \left| \frac{\lambda_t^{\alpha^0}}{\lambda_t^{\beta^0}} \right| \leq C, \quad \frac{1}{C} \leq \left| \frac{f_t^{\alpha^0}(x)}{f_t^{\beta^0}(x)} \right| \leq C, \quad \forall\, x \in \mathbb{R}\ t \in [0,T].$$

First we analyze the individual optimization problem of an agent, taking the actions of the counterparty as given. Assume that we are given a process $\bar{p} \in \mathbb{S}^2$ and progressively measurable (p^a, p^b), such that $\mathbb{P}$-a.s., $|p_t^a|, |p_t^b| \leq C_p$, $\forall t \in [0,T]$. Let us introduce the value functions of the agents:

$$V_t^a = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,p}} \mathbb{E}\Big[\int_t^{\tau} \exp\left(- \int_t^s c_u^{\alpha^0}(p_u, p_u^b) du \right) g_s^a(p_s, p_s^b) ds + \exp\left(- \int_t^{\tau} c_u^{\alpha^0}(p_u, p_u^b) du \right) \bar{p}_{\tau} \Big| \mathcal{F}_t \Big], \tag{3.23}$$

$$V_t^b = \operatorname{ess\,inf}_{\tau \in \mathcal{T}_{t,p}} \mathbb{E}\Big[\int_t^{\tau} \exp\left(- \int_t^s c_u^{\beta^0}(p_u^a, p_u) du \right) g_s^b(p_s^a, p_s) ds \tag{3.24}$$

$$+\exp\left(-\int_t^\tau c_u^{\beta^0}(p_u^a, p_u)du\right)\bar{p}_\tau\Big|\mathcal{F}_t\Big],$$

where $\mathcal{T}_t$ is the set of $\mathbb{F}$-stopping times with values in $[t,T]$, p is any $\mathbb{F}$-progressively measurable process, with $|p| \leq C_p$, and c^α, g^a and g^b are defined in (3.10), (3.20) and (3.22). In addition, we introduce the following random functions:

$$\mathcal{G}_t^{a,x}(y,z) = -c_t^{\alpha^0}(x,z)y + g_t^a(x,z), \qquad x, y \in \mathbb{R},$$

$$\mathcal{G}_t^a(y,z) = \sup_{x\in\mathbb{R}} \mathcal{G}_t^{a,x}(y,z) = -c_t^{\alpha^0}\left(P_t^a(y), z\right)y + g_t^a\left(P_t^a(y), z\right), \qquad y \in \mathbb{R},$$

where P_t^a provides the optimal price location at the ask side, given in a feedback form:

$$P_t^a(y) = \inf \arg\max_{p\in\mathbb{R}} (p-y)F_t^{+,\alpha^0}(p), \qquad y \in \mathbb{R}. \tag{3.25}$$

Similarly, for any admissible p^a, we define

$$P_t^b(y) = \sup \arg\max_{p\in\mathbb{R}} (y-p)F_t^{-,\beta^0}(p), \qquad y \in \mathbb{R}, \tag{3.26}$$

$$\mathcal{G}_t^b(z,y) = -c_t^{\beta^0}\left(z, P_t^b(y)\right)y + g_t^b\left(z, P_t^b(y)\right), \qquad y \in \mathbb{R}.$$

The value of $P_t^a(y)$ can be described as the unique nonnegative solution p of

$$p - y = F_t^{+,\alpha^0}(p)/f_t^{\alpha^0}(p), \tag{3.27}$$

unless y is too large, in which case $P_t^a(y)$ is the upper boundary of the support of $f_t^{\alpha^0}$, or too small, in which case $P_t^a(y) = 0$. Similarly, $P_t^b(y)$ is the unique non-positive solution p of

$$y - p = F_t^{-,\beta^0}(p)/f_t^{\beta^0}(p), \tag{3.28}$$

or the lower boundary of the support of $f_t^{\beta^0}$, if y is too small, or zero, if y is too large.

Lemma III.17. *Let Assumptions III.1–III.14 hold. Then, the random functions P^a and P^b are progressively measurable and satisfy, $\mathbb{P}$-a.s., for all $t \in [0,T]$:*

$$0 \leq P_t^a(y) \leq C_p, \quad -C_p \leq P_t^b(y) \leq 0, \quad P_t^a(y) \geq y, \quad P_t^b(y) \leq y, \quad \forall y \in \mathbb{R},$$

and, in addition, $P_t^a(\cdot)$ and $P_t^b(\cdot)$ are non-decreasing and 1-Lipschitz.

Proof: The progressive measurability property and the above inequalities follow directly from Assumptions III.9–III.13. The monotonicity and 1-Lipschitz property follow from Assumption III.14 and the representations (3.27)–(3.28). ∎

The above lemma, along with Assumptions III.9–III.13, implies that, for any admissible $(p, p^b, \bar{p})$, $\mathcal{G}_t^a(0, p_t^b)$ and $\mathcal{G}_t^{a,p_t}(0, p_t^b)$ are bounded processes, and that $\mathcal{G}_t^a(y, p_t^b)$ and $\mathcal{G}_t^{a,p_t}(y, p_t^b)$ are Lipschitz in y, uniformly over a.e. (t, ω). This allows us to use Proposition 7.1 from [42], to show that, for any admissible $(p, p^b, \bar{p})$, the process Y, which is a continuous modification of

$$Y_t := \hat{J}_t^{a,(p^b,\bar{p}),p} = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\Big[\int_t^\tau \exp\left(-\int_t^s c_u^{\alpha^0}(p_u, p_u^b) du \right) g_s^a(p_s, p_s^b) ds$$
$$+ \exp\left(-\int_t^\tau c_u^{\alpha^0}(p_u, p_u^b) du \right) \bar{p}_\tau \Big| \mathcal{F}_t \Big],$$

is the unique $\mathbb{S}^2$ solution of the affine RBSDE,

$$-dY_t = \mathcal{G}_t^{a,p_t}(Y_t, p_t^b)dt - Z_t \mathrm{d}W_t + \mathrm{d}K_t \quad 0 \leq t \leq T \tag{3.29}$$

$$Y_t \geq \bar{p}_t \quad 0 \leq t \leq T, \quad \int_0^T (Y_t - \bar{p}_t)\mathrm{d}K_t = 0 \tag{3.30}$$

$$Y_T = \bar{p}_T, \tag{3.31}$$

where Z is a progressively measurable square-integrable (multidimensional) process, $K \in \mathbb{S}^2$ is increasing and satisfies $K_0 = 0$. Similarly, the existence results from [42] imply that

$$-dY_t = \mathcal{G}^a_t(Y_t, p^b_t)dt - Z_t \mathrm{d}W_t + \mathrm{d}K_t \quad 0 \le t \le T \tag{3.32}$$

$$Y_t \ge \bar{p}_t \quad 0 \le t \le T \quad \int_0^T (Y_t - \bar{p}_t)\mathrm{d}K_t = 0 \tag{3.33}$$

$$Y_T = \bar{p}_T \tag{3.34}$$

has a unique solution (Y_t, Z_t, K_t). Then, Theorem 7.2 in [42] implies that Y is a continuous modification of V^a, and that $p^a_t = P^a_t(Y_t)$ and $\tau^a = \inf\{s \ge 0 \colon Y_s = \bar{p}_s\}$ form an optimal control for the long agent. Similarly, for a given admissible $(p^a, \bar{p})$, there exists a unique solution (Y_t, Z_t, K_t) to

$$-dY_t = \mathcal{G}^b_t(p^a_t, Y_t)dt - Z_t \mathrm{d}W_t - \mathrm{d}K_t \quad 0 \le t \le T \tag{3.35}$$

$$Y_t \le \bar{p}_t \quad 0 \le t \le T, \quad \int_0^T (\bar{p}_t - Y_t)\mathrm{d}K_t = 0 \tag{3.36}$$

$$Y_T = \bar{p}_T, \tag{3.37}$$

Y is a continuous modification of V^b, and $p^b_t = P^b_t(Y_t)$ and $\tau^b = \inf\{s \ge 0 \colon Y_s = \bar{p}_s\}$ form an optimal control for the short agent. It turns out that, because the optimal stopping time has to be the same for both agents in equilibrium, we can formulate a system of equations for V^a and V^b without $\bar{p}$. In order to state this result formally, we need to introduce the following random functions

(3.38)

$$\tilde{\mathcal{G}}^a_t(y,z) = \mathcal{G}^a_t(y, P^b_t(z)) = -c^{\alpha^0}_t \left(P^a_t(y), P^b_t(z)\right) y + g^a_t \left(P^a_t(y), P^b_t(z)\right), \quad y, z \in \mathbb{R},$$

(3.39)

$$\tilde{\mathcal{G}}^b_t(y,z) = \mathcal{G}^b_t(P^a_t(y), z) = -c^{\beta^0}_t \left(P^a_t(y), P^b_t(z)\right) z + g^b_t \left(P^a_t(y), P^b_t(z)\right), \quad y, z \in \mathbb{R},$$

where c^α, g^a and g^b are defined, respectively, in (3.10), (3.20) and (3.22), and P^a and P^b are given by (3.25) and (3.26).

Lemma III.18. *Let Assumptions III.1–III.16 hold. For any equilibrium* $(p^a, p^b, \tau, \bar{p})$ *in the two-player game (in the sense of Definition III.12), the value functions of the agents,* $V^a, V^b \in \mathbb{S}^2$*, satisfy*

$$
\begin{cases}
-dV_t^a = \tilde{\mathcal{G}}_t^a(V_t^a, V_t^b)dt - Z_t^a dW_t + dK_t^a \\
-dV_t^b = \tilde{\mathcal{G}}_t^b(V_t^a, V_t^b)dt - Z_t^b dW_t - dK_t^b \\
V_t^a \geq V_t^b \quad \forall t \in [0,T], \quad \int_0^T (V_t^a - V_t^b) d(K_t^a + K_t^b) = 0 \\
V_T^a = V_T^b,
\end{cases}
\tag{3.40}
$$

with some increasing processes $K^a, K^b \in \mathbb{S}^2$*, starting at zero, and with progressively measurable square-integrable* (Z^a, Z^b)*. Moreover,* $(\hat{p}^a, \hat{p}^b, \hat{\tau}, \bar{p})$ *also form an equilibrium, with the same value functions, where:* $\hat{p}_t^a = P_t^a(V_t^a)$*,* $\hat{p}_t^b = P_t^b(V_t^b)$ *and* $\hat{\tau} = \inf\{s \geq 0 \colon V_s^a = V_s^b\}$*. Conversely, given a solution to (3.40), we can define the optimal controls* $(\hat{p}^a, \hat{p}^b, \hat{\tau})$ *as above, and choose* $\bar{p} = (1-\eta)V^a + \eta V^b$*, with any progressively measurable process* η *taking values in* $(0,1)$*, to obtain an equilibrium* $(\hat{p}^a, \hat{p}^b, \hat{\tau}, \bar{p})$*.*

Proof: Consider an equilibrium $(p^a, p^b, \tau, \bar{p})$. As discussed earlier, the standard results on BSDEs (cf. [42]) imply that (V^a, Z^a, K^a) solves (3.32)–(3.34), and (V^b, Z^b, K^b) solves (3.35)–(3.37) (both systems are considered with the same $\bar{p}$). It follows from the optimality of τ, via the standard theory, that $V_\tau^b = \bar{p}_\tau = V_\tau^a$. Consider the long agent. It is clear that the objective of the long agent cannot increase if we replace $\bar{p}$ by V^b in its definition (cf. (3.19)). On the other hand, τ is optimal and $\bar{p}_\tau = V_\tau^b$, hence, the value function V^a remains the same if we replace $\bar{p}$ by V^b in its definition (cf. (3.23)). Therefore, (V^a, Z^a, K^a) solves (3.32)–(3.34) with $\bar{p}$ replaced by V^b. Similar argument applies to the short agent, and yields that (V^b, Z^b, K^b) solves (3.35)–(3.37) with $\bar{p}$ replaced by V^a. Next, using the optimality of p^a and the comparison principle for the BSDE (3.29), we easily deduce that, for a.e. (t, ω),

p^a_t coincides with $\hat{p}^a_t = P^a_t(V^a_t)$ whenever $\lambda^{\alpha^0}_t > 0$ and $V^a_t < \sup\operatorname{supp}(f^{\alpha^0}_t)$. On the other hand, Assumption III.16 implies that, if $\lambda^{\alpha^0}_t = 0$ or $V^a_t \geq \sup\operatorname{supp}(f^{\alpha^0}_t)$, then $\lambda^{\beta^0}_t = 0$ or $V^a_t \geq \sup\operatorname{supp} f^{\beta^0}_t$, and, in turn, $\mathcal{G}^b_t\left(p^a_t, V^b_t\right) = \mathcal{G}^b_t\left(\hat{p}^a_t, V^b_t\right)$. Thus, we conclude that V^b satisfies (3.35)–(3.37) with p^a replaced by $\hat{p}^a$. Similarly, we conclude that V^a satisfies (3.32)–(3.34) with p^b replaced by $\hat{p}^b$. Thus, (V^a, V^b) satisfy (3.40).

Next, consider a solution to (3.40). Choosing $\bar{p}$ as shown in the statement of the lemma, we conclude that (V^a, Z^a, K^a) solves (3.32)–(3.34), with p^b replaced by $\hat{p}^b$. Then, the standard results (cf. [42]) imply that, given $\hat{p}^b$ and $\bar{p}$, V^a is the value function of the long agent, and her optimal control is given by $\hat{p}^a$ and

$$\inf\{s \geq 0 \colon V^a_s \leq \bar{p}_s\} = \inf\{s \geq 0 \colon V^a_s = V^b_s\} = \hat{\tau}.$$

Similar argument applies to the short agent, completing the proof. ∎

3.3.2 Existence of a solution

In this subsection, we address the question of existence of a solution to the RBSDE (3.40). The main difficulty in analyzing (3.40) is the non-standard form of the reflection: the components of the solution reflect against each other, as opposed to reflecting against a given boundary. Related equations have been analyzed in the literature on BSDEs constrained by oblique reflections (e.g. in [20]): indeed, our problem can be viewed as a limit of oblique reflection problems (see [20], for further details). However, the present problem is new, and, in particular, the question of existence of its solution has not been addressed in the existing literature. Before we analyze the existence, it is convenient to consider the question of uniqueness. Note that the arbitrary choice of η in Lemma III.18 indicates that there may be multiple solutions to (3.40). Indeed, a different choice of η produces a different $\bar{p}$, which results in a different pair of value functions (V^a, V^b), which, nevertheless, have to solve the same system (3.40). This heuristic observation turns out to be correct and, in fact, allows us to construct a solution to (3.40). Consider a solution $(V^a, V^b, K^a, K^b, Z^a, Z^b)$ to (3.40). Introducing $K_t = K_t^a + K_t^b$, we notice that there must exist a process η, with values in $[0,1]$, such that $\mathrm{d}K_t^a = \eta_t \mathrm{d}K_t$, $\mathrm{d}K_t^b = (1-\eta_t)\mathrm{d}K_t$. Then, we introduce $\tilde{Y}_t^1 = V_t^a - V_t^b$ and $\tilde{Y}_t^2 = (1-\eta_t)V_t^a + \eta_t V_t^b$ as the new variables, replacing V^a and V^b. Assuming that η is sufficiently regular, one can obtain a system of RBSDEs for $(\tilde{Y}^1, \tilde{Y}^2)$, in which only the first component reflects against zero, and $\tilde{Y}_T^1 = 0$. Conversely, we can start by prescribing η and a terminal condition for $\tilde{Y}^2$, solving the associated system of RBSDEs for $(\tilde{Y}^1, \tilde{Y}^2)$, and, then, recover (V^a, V^b) from $(\tilde{Y}^1, \tilde{Y}^2, \eta)$ via the above formulas. Naturally, the resulting (V^a, V^b) are expected to satisfy (3.40). This method seems to describe all solutions to (3.40), however, herein, we are only interested in constructing a particular one.[6] Hence, we choose $\eta \equiv 1/2$ and $\tilde{Y}_T^2 = 0$, to obtain $Y^1 = \tilde{Y}^1 = V^a - V^b$ and $Y^2 = 2\tilde{Y}^2 = V^a + V^b$, which are expected to satisfy:

$$
\begin{cases}
-\mathrm{d}Y_t^1 = \mathcal{G}_t^1(Y_t^1, Y_t^2)\mathrm{d}t - Z_t^1 \mathrm{d}W_t + \mathrm{d}K_t \\
Y_t^1 \geq 0, \qquad \int_0^T Y_t^1 \mathrm{d}K_t = 0, \quad Y_T^1 = 0 \\
-\mathrm{d}Y_t^2 = \mathcal{G}_t^2(Y_t^1, Y_t^2)\mathrm{d}t - Z_t^2 \mathrm{d}W_t, \quad Y_T^2 = 0
\end{cases}
\tag{3.41}
$$

where $Y^1, Y^2 \in \mathbb{S}^2$, the processes Z^1, Z^2 are progressively measurable and square-integrable, $K \in \mathbb{S}^2$ is increasing and satisfies $K_0 = 0$. In addition, we denote

$$
\mathcal{G}_t^1(y^1, y^2) = \tilde{\mathcal{G}}_t^a\left((y^1+y^2)/2, (y^2-y^1)/2\right) - \tilde{\mathcal{G}}_t^b\left((y^1+y^2)/2, (y^2-y^1)/2\right),
$$

$$
\mathcal{G}_t^2(y^1, y^2) = \tilde{\mathcal{G}}_t^a\left((y^1+y^2)/2, (y^2-y^1)/2\right) + \tilde{\mathcal{G}}_t^b\left((y^1+y^2)/2, (y^2-y^1)/2\right)
$$

[6] It is an interesting topic for future research, to describe rigorously all solutions of (3.40)

where $\tilde{\mathcal{G}}^a$ and $\tilde{\mathcal{G}}^b$ are defined in (3.38) and (3.39). The following lemma formalizes the connection between (3.41) and (3.40).

Lemma III.19. *Let* (Y^1, Y^2, Z^1, Z^2, K) *be a solution to (3.41). Then*

$$V^a = \frac{1}{2}Y^1 + \frac{1}{2}Y^2,\ V^b = \frac{1}{2}Y^2 - \frac{1}{2}Y^1,$$
$$Z^a = \frac{1}{2}Z^1 + \frac{1}{2}Z^2,\ Z^b = \frac{1}{2}Z^2 - \frac{1}{2}Z^1,\ K^a = \frac{1}{2}K, K^b = \frac{1}{2}K$$

form a solution to (3.40).

Proof: Follows easily by a direct verification. ∎

However, the existence of a solution to (3.41) is far from obvious. Indeed, the generator of this system can be written as

$$\mathcal{G}^1_t(y^1, y^2) = -c^1_t(y^1, y^2)y^1 + c^2_t(y^1, y^2)y^2 + g^1_t(y^1, y^2), \tag{3.42}$$

$$\mathcal{G}^2_t(y^1, y^2) = -c^2_t(y^1, y^2)y^1 - c^1_t(y^1, y^2)y^2 + g^2_t(y^1, y^2), \tag{3.43}$$

where

$$c^1_t(y^1, y^2) =$$
$$\frac{1}{2}c^{\alpha^0}_t\left(P^a_t\left((y^1+y^2)/2\right), P^b_t\left((y^2-y^1)/2\right)\right) + \frac{1}{2}c^{\beta^0}_t\left(P^a_t\left((y^1+y^2)/2\right), P^b_t\left((y^2-y^1)/2\right)\right)$$

$$c^2_t(y^1, y^2) =$$
$$\frac{1}{2}c^{\beta^0}_t\left(P^a_t\left((y^1+y^2)/2\right), P^b_t\left((y^2-y^1)/2\right)\right) - \frac{1}{2}c^{\alpha^0}_t\left(P^a_t\left((y^1+y^2)/2\right), P^b_t\left((y^2-y^1)/2\right)\right),$$

$$g^1_t(y^1, y^2) =$$
$$g^a_t\left(P^a_t\left((y^1+y^2)/2\right), P^b\left((y^2-y^1)/2\right)\right) - g^b_t\left(P^a\left((y^1+y^2)/2\right), P^b\left((y^2-y^1)/2\right)\right),$$

$$g_t^2(y^1, y^2) =$$

$$g_t^a\left(P_t^a\left((y^1+y^2)/2\right), P^b\left((y^2-y^1)/2\right)\right) + g_t^b\left(P^a\left((y^1+y^2)/2\right), P^b\left((y^2-y^1)/2\right)\right),$$

with c^α, P^a, P^b, g^a and g^b defined in (3.10), (3.25), (3.26), (3.20) and (3.22). It is easy to see that every $c_t^i(\cdot,\cdot)$ and $g_t^i(\cdot,\cdot)$ is bounded and globally Lipschitz, uniformly over a.e. (t,ω). However, due to the presence of the multipliers y^1 and y^2, $\mathcal{G}_t^i(\cdot,\cdot)$ is unbounded and does not possess the global Lipschitz property. There exists an existence theory for one-dimensional BSDEs with linear growth. However, the present equation is multidimensional, and it cannot be reduced to the one-dimensional case: event if we restrict one coordinate y^i to a bounded range, the corresponding generator $\mathcal{G}^i$ is not bounded or Lipschitz in y^i, as the other component can take arbitrarily large values. Nevertheless, we can make use of the fact that the generator of (3.41) has the "correct" asymptotic behavior, to prove the existence of a solution. In particular, we exploit the fact that, due to the assumptions made earlier in this section, whenever $\|(Y_t^1, Y_t^2)\|$ becomes large, the generator $(\mathcal{G}_t^1, \mathcal{G}_t^2)$ pushes (Y_t^1, Y_t^2) in the direction in which the largest $|Y_t^i|$ decreases.

Proposition III.20. *Let Assumptions III.9–III.16 hold. Then, there exists a solution to (3.41), s.t. its components Y^1 and Y^2 are absolutely bounded by a constant. Such a solution is unique.*

Proof: Step 1: Existence for the fully capped system. For any constant $C > 0$, denote $\Psi_C(y) = (-C \vee y) \wedge C$. Clearly, this function is 1-Lipschitz in y and absolutely bounded by C. We fix arbitrary constants $\{C_i^j > 0\}$ and consider the fully capped system:

(3.44)
$$
\begin{cases}
-\mathrm{d}Y_t^1 = \\
\Big(-c_t^1(Y_t^1,Y_t^2)\Psi_{C_1^1}(Y_t^1)+c_t^2(Y_t^1,Y_t^2)\Psi_{C_1^2}(Y_t^2)+g_t^1(Y_t^1,Y_t^2)\Big)\,\mathrm{d}t - Z_t^1\mathrm{d}W_t + \mathrm{d}K_t, \\
-\mathrm{d}Y_t^2 = \Big(-c_t^2(Y_t^1,Y_t^2)\Psi_{C_2^1}(Y_t^1)-c_t^1(Y_t^1,Y_t^2)\Psi_{C_2^2}(Y_t^2)+g_t^2(Y_t^1,Y_t^2)\Big)\,\mathrm{d}t - Z_t^2\mathrm{d}W_t
\end{cases}
$$

Here, and in some expressions that follow, we omit the terminal condition, barrier, and the minimality condition for K_t, as they remain unchanged throughout. Assumptions III.9–III.16 imply that $c_t^1(y^1,y^2)$, $c_t^2(y^1,y^2)$, $g_t^1(y^1,y^2)$ and $g_t^2(y^1,y^2)$ are bounded and globally Lipschitz in (y^1,y^2), uniformly over a.e. (t,ω). Hence, the generator of (3.44) is globally Lipschitz in (y^1,y^2) (and independent of (Z^1,Z^2)), and the standard existence results for Lipschitz BSDEs (cf. for example, Theorem 2.2 in [59]) yield the existence (and uniqueness) of a solution to (3.44). Denote the Y-component of this solution (Y_t^{1c},Y_t^{2c}).

Step 2: Bounds on solution components via partial uncapping. We want to bound the components $(Y_t^{1c},\ Y_t^{2c})$, of the solution to the capped system, by using the control-stopping interpretation of the individual (R)BSDEs comprising our system. Consider the associated equation for Y^1, with Y_t^{2c} being given:

(3.45)
$$
\begin{cases}
-\mathrm{d}Y_t^1 = \\
\Big(-c_t^1\left(Y_t^1,Y_t^{2c}\right)Y_t^1+c_t^2\left(Y_t^1,Y_t^{2c}\right)\Psi_{C_1^2}\left(Y_t^{2c}\right)+g_t^1\left(Y_t^1,Y_t^{2c}\right)\Big)\,\mathrm{d}t - Z_t^1\mathrm{d}W_t + \mathrm{d}K_t, \\
Y_t^1 \geq 0, \quad \int_0^T Y_t^1\mathrm{d}K_t = 0, \quad Y_T^1 = 0
\end{cases}
$$

Note that, as c_t^1,c_t^2, g^1 and $\Psi_{C_1^2}$ are bounded, this one-dimensional RBSDE has a continuous generator with linear growth in Y^1, and, for example, by Theorem 4.1 in [59], it has a solution, which we denote Y_t^1. Next, for Y^1 and Y^{2c} constructed above, we introduce the processes

$$\tilde{c}^1_t = c^1_t(Y^1_t, Y^{2c}_t), \quad \tilde{c}^2_t = c^2_t(Y^1_t, Y^{2c}_t), \quad \tilde{g}^1_t = g^1_t(Y^1_t, Y^{2c}_t), \quad \tilde{g}^2_t = g^2_t(Y^1_t, Y^{2c}_t),$$

and consider the one-dimensional RBSDE (for $\tilde{Y}$), obtained from (3.45) by pretending that the coefficients should depend on the solution itself:

$$\begin{cases} -\mathrm{d}\tilde{Y}^1_t = \left(-\tilde{c}^1_t \tilde{Y}^1_t + \tilde{c}^2_t \Psi_{C^2_1}(Y^{2c}_t) + \tilde{g}^1_t\right) \mathrm{d}t - Z^1_t \mathrm{d}W_t + \mathrm{d}K_t \\ \tilde{Y}^1_t \geq 0, \quad \int_0^T \tilde{Y}^1_t \mathrm{d}K_t = 0, \quad \tilde{Y}^1_T = 0 \end{cases} \tag{3.46}$$

Note that $\tilde{Y} = Y^1$ is the unique solution of this equation. On the other hand, the above RBSDE is affine in $\tilde{Y}$, and, for example, by Theorem 7.1 in [42], its unique solution admits the following interpretation, as the value function of an optimal stopping problem:

$$Y^1_t = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\int_t^\tau \exp\left(-\int_t^s \tilde{c}^1_u \mathrm{d}u\right) \left(\tilde{c}^2_s \Psi_{C^2_1}(Y^{2c}_s) + \tilde{g}^1_s\right) \mathrm{d}s \middle| \mathcal{F}_t\right]$$

We will use this representation to establish a bound on $|Y^1|$. First, note that, under our assumptions, there exist constants $C_0 > 0$ and $\lambda \in (0,1)$, such that, for all t, y^1, y^2, and a.e. ω, we have:

$$\left|\frac{g^i_t(y^1,y^2)}{c^1_t(y^1,y^2)}\right| = 2\left|\frac{g^a_t\left(P^a_t\left(\frac{y^1+y^2}{2}\right), P^b_t\left(\frac{y^2-y^1}{2}\right)\right) \pm g^b_t\left(P^a_t\left(\frac{y^1+y^2}{2}\right), P^b_t\left(\frac{y^2-y^1}{2}\right)\right)}{c^{\alpha^0}_t\left(P^a_t\left(\frac{y^1+y^2}{2}\right), P^b_t\left(\frac{y^2-y^1}{2}\right)\right) + c^{\beta^0}_t\left(P^a_t\left(\frac{y^1+y^2}{2}\right), P^b_t\left(\frac{y^2-y^1}{2}\right)\right)}\right| \leq C_0,$$

$$\left|\frac{c^2_t(y^1,y^2)}{c^1_t(y^1,y^2)}\right| = \left|\frac{c^{\alpha^0}_t\left(P^a_t\left(\frac{y^1+y^2}{2}\right), P^b_t\left(\frac{y^2-y^1}{2}\right)\right) - c^{\beta^0}_t\left(P^a_t\left(\frac{y^1+y^2}{2}\right), P^b_t\left(\frac{y^2-y^1}{2}\right)\right)}{c^{\alpha^0}_t\left(P^a_t\left(\frac{y^1+y^2}{2}\right), P^b_t\left(\frac{y^2-y^1}{2}\right)\right) + c^{\beta^0}_t\left(P^a_t\left(\frac{y^1+y^2}{2}\right), P^b_t\left(\frac{y^2-y^1}{2}\right)\right)}\right| \leq \lambda < 1,$$

with c^α, P^a, P^b, g^a and g^b defined in (3.10), (3.25), (3.26), (3.20) and (3.22). The first inequality holds with $C_0 = 5C_p$, and it follows from the boundedness of P^a,

P^b and the jump sizes. The second one follows from Assumption III.16. The above inequalities imply:

$$\left| \frac{\tilde{c}_t^2 \Psi_{C_1^2}(Y_t^{2c}) + \tilde{g}_t^1}{\tilde{c}_t^1} \right| \leq \lambda C_1^2 + C_0,$$

for all t and a.e. ω. The latter estimate, together with the following lemma, imply the desired upper bound:

$$|Y_t^1| \leq \lambda C_1^2 + C_0$$

for all t and a.e. ω.

Lemma III.21. *Consider any constant $C > 0$, any continuous function $S : [0,T] \to \mathbb{R}$, absolutely bounded by C, any nonnegative continuous function c on $[0,T]$, and any continuous function g on $[0,T]$, satisfying $|g| \leq C|c|$. For any $0 \leq t \leq \tau \leq T$, denote:*

$$Y_{t,\tau} = \int_t^\tau \exp\left(-\int_t^s c(u)\,du\right) g(s)\,ds + \exp\left(-\int_t^\tau c(u)\,du\right) S(\tau).$$

Then

$$|Y_{t,\tau}| \leq C, \quad \forall\, 0 \leq t \leq \tau \leq T.$$

Proof: For any $0 \leq t \leq \tau \leq T$, we have

$$\begin{aligned} &\left| \int_t^\tau \exp\left(-\int_t^s c(u)du\right) g(s)ds + \exp\left(-\int_t^\tau c(u)du\right) S(\tau) \right| \\ &\qquad \leq -\int_t^\tau C d\left(\exp\left(-\int_t^s c(u)du\right)\right) + \exp\left(-\int_t^\tau c(u)du\right) C = C \end{aligned}$$

∎

Thus, we have a solution Y^1 of (3.45) which satisfies $|Y_t^1| \leq \lambda C_1^2 + C_0$, $\mathbb{P}$-a.s., for all t. Then, for $C_1^1 \geq \lambda C_1^2 + C_0$, we have $\Psi_{C_1^1}(Y_t^1) = Y_t^1$, and, hence, Y^1 also solves

$$\begin{cases} -\mathrm{d}Y_t^1 = \left(-c_t^1(Y_t^1, Y_t^{2c})\Psi_{C_1^1}(Y_t^1) + c_t^2 \Psi_{C_1^2}(Y_t^{2c}) + g_t^1(Y_t^1, Y_t^{2c})\right) \mathrm{d}t - Z_t^1 \mathrm{d}W_t + \mathrm{d}K_t \\ Y_t^1 \geq 0, \quad \int_0^T Y_t^1 \mathrm{d}K_t = 0, \quad Y_T^1 = 0 \end{cases}$$

Note that the above RBSDE coincides with the Y^1-equation in (3.44). This one-dimensional RBSDE has a globally Lipschitz generator and, thus, a unique solution. This implies that $Y^1 = Y^{1c}$, and we obtain the desired bound on Y^{1c}:

$$|Y_t^{1c}| \leq \lambda C_1^2 + C_0,$$

$\mathbb{P}$-a.s. for all t, provided $C_1^1 \geq \lambda C_1^2 + C_0$. Similarly, considering the Y^2 part of the capped system (3.44), with Y^{1c} fixed, we obtain

$$|Y_t^{2c}| \leq \lambda C_2^1 + C_0,$$

$\mathbb{P}$-a.s. for all t, provided $C_2^2 \geq \lambda C_2^1 + C_0$.

Step 3: Solution of the appropriately capped system solves the original system. To show that the solution (Y_t^{1c}, Y_t^{2c}) of (3.44) also solves the original system (3.41), we only need to show that, given the bounds on (Y^{1c}, Y^{2c}), the capped system's generator coincides with the original generator, which translates into

$$\Psi_{C_1^1}(Y_t^{1c}) = Y_t^{1c}, \quad \Psi_{C_2^2}(Y_t^{2c}) = Y_t^{2c}, \quad \Psi_{C_1^2}(Y_t^{2c}) = Y_t^{2c}, \quad \Psi_{C_2^1}(Y_t^{1c}) = Y_t^{1c}.$$

The first two equalities are satisfied if

$$C_1^1 \geq \lambda C_1^2 + C_0, \quad C_2^2 \geq \lambda C_2^1 + C_0,$$

while the last two require

$$\lambda C_2^1 + C_0 \leq C_1^2, \quad \lambda C_1^2 + C_0 \leq C_2^1.$$

One can check these inequalities have a solution, as long as $\lambda < 1$. The "minimal" solution being

$$C_1^1 = C_2^1 = C_2^2 = C_1^2 = \frac{C_0}{1-\lambda}.$$

With the above choice of capping, the solution to (3.44) also solves (3.41), thus, showing the existence of a solution of (3.41). This solution is bounded by construction. The uniqueness of a bounded solution follows from the fact that, when (y^1, y^2) vary over a bounded set, the generator of (3.41) is Lipschitz, hence, the standard results yield uniqueness. ■ ■

Remark III.22. The above proof provides an existence result for any system (3.41), whose generator is given by (3.42)–(3.43), with arbitrary (bounded and Lipschitz) progressively measurable random functions $\{c^i, g^i\}$, as long as the following holds for a.e. (t, ω) and all $(y^1, y^2) \in \mathbb{R}^2$:

$$\sum_{i=1}^{2} \left| g_t^i(y^1, y^2) \right| \le C_0 c_t^1(y^1, y^2), \quad \left| c_t^2(y^1, y^2) \right| \le \lambda c_t^1(y^1, y^2),$$

with some constants $C_0 > 0$ and $\lambda \in (0, 1)$.

3.4 Equilibrium in the continuum-player game.

In this section we construct an equilibrium for the continuum-player game described in Section 3.2, in the sense of Definition III.6. The main difficulty in constructing the equilibrium stems from the mixed control-stopping nature of the game (and, of course, the fact there are multiple participants). Therefore, we attempt to break the problem into two parts - isolating the “stopping” part of the game. In order to do this, it is convenient to make assumptions that guarantee the existence of the so-called “extremal” agents on each side of the book. These agents are called “extremal”, because their beliefs dominate the beliefs of the other agents on the same side of the book, in the appropriate sense. We denote the extremal beliefs on the long side by α^0, and, on the short side, by β^0. It is worth mentioning that the “extremal” beliefs are, in fact, the mildest ones: i.e. the agents with beliefs α^0 are the least bullish among the long ones, and the agents with beliefs β^0 are the least bearish among the short ones. There are two economic interpretations of the extremal agents. First, they can be viewed as market-makers, as they are closer to being market-neutral, than any other agent on the same side of the book (although, we do not have any designated market maker in this game). Second, they can be viewed as the fastest traders. Indeed, instead of thinking of these beliefs in terms of bullishness or bearishness, it is possible to interpret the extremal beliefs as the beliefs that the jumps of the fundamental price are small (relative to the beliefs of other players). This, in turn, can be interpreted as the ability of extremal agents to predict, and react to, the smallest changes in the price (whereas the other agents are too slow for this, and, hence, they only take into account larger jumps). In this section, we construct an equilibrium in which the time of the first internal market order and the bid and ask prices are determined by the extremal agents, while the rest of the shape of the LOB is due to the other agents' actions. The construction of an equilibrium, thus, splits into two parts. In the first part, the extremal agents find an equilibrium among themselves, using the results of the auxiliary two-player game, and determining the time of the first internal market order τ and the bid and ask prices p^a and p^b. In the second part, the other agents, taking (p^a, p^b, τ) as given, determine their optimal actions. Of course, we, ultimately, prove that the strategy of every agent is optimal in the overall market, consisting of both extremal and non-extremal agents. The resulting LOB ν has two atoms – at the bid and ask prices – comprised of the limit orders of the extremal and some of the non-extremal agents. The rest of the LOB contains limit orders of the non-extremal agents only.

In order to implement the above program, we assume that $\mathbb{A} = \{\alpha^0\} \cup \hat{\mathbb{A}}$ and $\mathbb{B} = \{\beta^0\} \cup \hat{\mathbb{B}}$. We assume that Assumptions III.1–III.16 hold throughout this section. In addition, we make the following assumptions.

Assumption III.23. *For any $\alpha \in \hat{\mathbb{A}}$, $\beta \in \hat{\mathbb{B}}$ and a.e. (t,ω), we have:*

$$\lambda_t^{\alpha} F_t^{+,\alpha}(p) \geq \lambda_t^{\alpha^0} F_t^{+,\alpha^0}(p), \quad \lambda_t^{\beta} F_t^{+,\beta}(p) \leq \lambda_t^{\beta^0} F_t^{+,\beta^0}(p), \quad \forall p \geq 0,$$

$$\lambda_t^{\alpha} F_t^{-,\alpha}(p) \leq \lambda_t^{\alpha^0} F_t^{-,\alpha^0}(p), \quad \lambda_t^{\beta} F_t^{-,\beta}(p) \geq \lambda_t^{\beta^0} F_t^{-,\beta^0}(p), \quad \forall p \leq 0.$$

Assumption III.24. *For any $\alpha \in \hat{\mathbb{A}}$, $\beta \in \hat{\mathbb{B}}$ and a.e. (t,ω), we have:*

$$\frac{F_t^{+,\alpha_0}(p)}{f_t^{\alpha_0}(p)} \leq \frac{F_t^{+,\alpha}(p)}{f_t^{\alpha}(p)}, \quad \frac{F_t^{-,\beta_0}(-p)}{f_t^{\beta_0}(-p)} \leq \frac{F_t^{-,\beta}(-p)}{f_t^{\beta}(-p)} \quad \forall\, p \geq 0.$$

Assumption III.23 ensures that the distribution of the fundamental price at any time t, from an α-agent's perspective, dominates stochastically the respective distribution from the α^0-agent's perspective. The opposite relation holds for the short agents. The first inequality in Assumption III.24 ensures that $\left|\log F_t^{+,\alpha_0}(\cdot)\right|$ decays faster than $\left|\log F_t^{+,\alpha}(\cdot)\right|$, which is also consistent with the interpretation that α^0-agents assign smaller probabilities to the large jumps of the fundamental price, and larger probabilities to the small jumps, as opposed to the α-agents. Analogous interpretation holds for the second inequality in Assumption III.24. Assumption III.24 ensures that, in an empty LOB, the non-extremal agents would prefer to post their limit order further away from zero than the extremal ones do.

Lemma III.25. *Let Assumptions III.1–III.24 hold. Fix any $\alpha \in \hat{\mathbb{A}}$ and $\beta \in \hat{\mathbb{B}}$. Then, for a.e. (t,ω), the following holds for all $y \in \mathbb{R}$: $p \mapsto (p-y)F_t^{+,\alpha}(p)$ is non-decreasing in $p \in [y, P_t^a(y)]$, and $p \mapsto (y-p)F_t^{-,\beta}(p)$ is non-increasing in $p \in [P_t^b(y), y]$.*

Proof: The statement follows easily by differentiating the target functions, recalling (3.27)–(3.28), and making use of Assumption III.24. ∎

We also need to make an assumption that limits the maximum possible demand size, as viewed by the extremal agents. Namely, the extremal agents believe that the external demand can never exceed the inventory held by these agents.

Assumption III.26. *For Leb $\otimes \mathbb{P}$-a.e. (t,ω), we have:*

$$D_t\left(-Q^+\left(f_t^{\alpha^0}(x)dx\right)\right) \leq \mu^a\left(\{\alpha^0\}\right), \quad -D_t\left(-Q^-\left(f_t^{\alpha^0}(x)dx\right)\right) \leq \mu^b\left(\{\beta^0\}\right),$$
$$-D_t\left(-Q^-\left(f_t^{\beta^0}(x)dx\right)\right) \leq \mu^b\left(\{\beta^0\}\right), \quad D_t\left(-Q^+\left(f_t^{\beta^0}(x)dx\right)\right) \leq \mu^a\left(\{\alpha^0\}\right).$$

where Q^+ and Q^- are defined in (3.2).

In order to construct an equilibrium, we need to impose certain topological conditions on the space of beliefs and on the mapping $\alpha \mapsto f^\alpha$.

Assumption III.27. *The spaces $\hat{\mathbb{A}}$ and $\hat{\mathbb{B}}$ are compact metric spaces, with the Borel sigma-algebras on them (i.e. μ^a and μ^b are measures with respect to the Borel sigma-algebras). In addition, for a.e. (t,ω), the mapping $\alpha \mapsto f_t^\alpha$ is continuous as a mapping $\hat{\mathbb{A}} \to \mathbb{L}^1[0,C_p]$ and as a mapping $\hat{\mathbb{B}} \to \mathbb{L}^1[-C_p,0]$.*

Finally, we need to ensure that the demand size curve is "not too flat".

Assumption III.28. *There exists an increasing continuous (deterministic) function $\epsilon\colon [0,\infty) \to [0,\infty)$, s.t. $\epsilon(0)=0$ and, for a.e. (t,ω), $|D_t^{-1}(x) - D_t^{-1}(y)| \leq \epsilon(|x-y|)$, for all $x,y \in \mathbb{R}$.*

Now, we proceed to construct a special class of equilibria in the continuum-player game. As announced earlier, the equilibrium is constructed by, first, solving the auxiliary two-player game, as described in Section 3.3. In the two-player game, we assume that the two agents have beliefs α^0 and β^0. Thus, we consider the unique bounded solution (Y^1,Y^2) to (3.41) and construct the associated (V^a,V^b), which solve (3.40), according to Lemma III.19. Then, Lemma III.18 implies that (V^a,V^b) are the value functions of the two-player equilibrium $(\hat{p}^a,\hat{p}^b,\hat{\tau},\bar{p})$, where

$$\hat{p}_t^a = P_t^a(V_t^a), \quad \hat{p}_t^b = P_t^b(V_t^b), \quad \hat{\tau} = \inf\{t\in[0,T]\,:\,V_t^a = V_t^b\}, \quad \bar{p}_t = \frac{1}{2}V_t^a + \frac{1}{2}V_t^b.$$

Let us introduce

$$p^a_t = \hat{p}^a_t \mathbf{1}_{\{t<\hat{\tau}\}} + \bar{p}_{\hat{\tau}} \mathbf{1}_{\{t\geq\hat{\tau}\}}, \quad p^b_t = \hat{p}^b_t \mathbf{1}_{\{t<\hat{\tau}\}} + \bar{p}_{\hat{\tau}} \mathbf{1}_{\{t\geq\hat{\tau}\}}.$$

Using these auxiliary quantities, we aim to construct an equilibrium for the continuum-player game, in which (ν, θ) satisfy the following two conditions. First,

$$\nu^a_t = \mu^a(\{\alpha^0\})\delta_{p^a_t} + \bar{\nu}^a_t, \quad \nu^b_t = \mu^b(\{\beta^0\})\delta_{p^b_t} + \bar{\nu}^b_t, \tag{3.47}$$

with progressively measurable $\bar{\nu}^a$ and $\bar{\nu}^b$ taking values in the space of sigma-additive measures on $\mathbb{R}$, such that, $\mathbb{P}$-a.s., for all $t \in [0, T]$, $\bar{\nu}^a_t$ is supported on $[p^a_t, C_p]$ and $\bar{\nu}^b_t$ is supported on $[-C_p, p^b_t]$.[7] Second,

$$\theta^a_t = \mu^a(\mathbb{A})\delta_{V^a_t}, \quad \theta^b_t = \mu^b(\mathbb{B})\delta_{V^b_t}. \tag{3.48}$$

Note that, in such a market, we have

$$\tau^a = \tau^b = \hat{\tau}$$

The following theorem is the main result of this chapter.

Theorem III.29. *Let Assumptions III.1–III.28 hold. Then, there exist progressively measurable measure-valued processes (ν, θ), satisfying (3.47)–(3.48), and progressively measurable random fields $p, v : \Omega \times [0, T] \times \mathbb{S} \to \mathcal{P}(\mathbb{R}) \times \mathbb{R}$, which form an equilibrium, in the sense of Definition III.6. Moreover, an equilibrium can be constructed so that $v_t(1, \alpha) = V^a_t$, $v_t(-1, \alpha) = V^b_t$, for all (t, ω, α), and the optimal limit order strategies of extremal agents are as follows: $p_t(1, \alpha^0) = p^a_t$, $p_t(-1, \beta^0) = p^b_t$, for all (t, ω).*

The remainder of this section is devoted to the proof of Theorem III.29. First, we show that, in a market (ν, θ), as described by (3.47)–(3.48), it is never (strictly)

[7] The components $\bar{\nu}^a$ and $\bar{\nu}^b$ are introduced for convenience, in order to indicate that $\nu^a_t(\{p^a_t\}) \geq \mu^a(\{\alpha^0\})$ and $\nu^b(\{p^b_t\}) \geq \mu^b(\{\beta^0\})$.

optimal for the agents to post limit sell orders below the ask price or to post limit buy orders above the bid price. In addition, it is never (strictly) optimal for the agents to submit a market order before $\hat{\tau}$. To achieve this, we need to compare the value functions of the agents to V^a and V^b, making use of Assumptions III.23, III.24.

Lemma III.30. *Let Assumptions III.1–III.24 hold, and let (ν, θ) satisfy (3.47)–(3.48). Given any $\alpha \in \mathbb{A}$ and any admissible control (p, τ), for a long agent with beliefs α, there exists an admissible control p', s.t., $\mathbb{P}$-a.s., supp$(p'_t) \subset [p^a_t, \infty)$, for all $t \in [0, T]$, and $(p', \hat{\tau})$ does not decrease the objective value, i.e.*

$$J^{(\nu,\theta),(p,\tau)}(1, \alpha) \leq J^{(\nu,\theta),(p',\hat{\tau})}(1, \alpha).$$

Similarly, given any $\beta \in \mathbb{B}$ and any admissible control (p, τ), for a short agent with beliefs β, there exists an admissible control p', s.t., $\mathbb{P}$-a.s., supp$(p'_t) \subset (-\infty, p^b_t]$, for all $t \in [0, T]$, and $(p', \hat{\tau})$ does not decrease the objective value, i.e.

$$J^{(\nu,\theta),(p,\tau)}(-1, \beta) \leq J^{(\nu,\theta),(p',\hat{\tau})}(-1, \beta).$$

Proof: We consider a long agent with beliefs α and introduce

$$\begin{aligned}\bar{J}^{\alpha,(p,\tau)}_t = \mathbb{E}\Big[\int_t^\tau \exp\left(- \int_t^s \bar{c}^\alpha_u \left(p^a_u \wedge Q^-(p_u), p^b_u \right) du \right) \bar{h}^{\alpha,a}_s(p_s, p^a_s, p^b_s) ds \\ + \exp\left(- \int_t^\tau \bar{c}^\alpha_u \left(p^a_u \wedge Q^-(p_u), p^b_u \right) du \right) p^b_{\tau\wedge\hat{\tau}} | \mathcal{F}_t \Big],\end{aligned}$$

where

$$\bar{c}^\alpha_t(x, y) = c^\alpha_t(x, y)\mathbf{1}_{\{t \leq \hat{\tau}\}}, \quad \bar{h}^{\alpha,a}_t(\kappa, x, y) = h^{\alpha,a}_t(\kappa, x, y)\mathbf{1}_{\{t \leq \hat{\tau}\}}, \quad x, y \in \mathbb{R}\ \kappa \in \mathcal{P}(\mathbb{R}),$$

with c^α and $h^{\alpha,a}$ defined in (3.10) and (3.13). Next, for any $t \in [0, T]$, any $\alpha \in \mathbb{A}$, and any admissible p, we introduce

$$Y^{\alpha,p}_t = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_t} \bar{J}^{\alpha,(p,\tau)}_t, \tag{3.49}$$

The standard results on RBSDEs imply that $Y^{\alpha,p}$ is the unique $\mathbb{S}^2$ solution of the affine RBSDE,

$$-dY_t^{\alpha,p} = \bar{\mathcal{G}}_t^{\alpha,p}(Y_t^{\alpha,p})dt - Z_t \mathrm{d}W_t + \mathrm{d}K_t \quad 0 \le t \le T \tag{3.50}$$

$$Y_t^{\alpha,p} \ge p^b_{t\wedge\hat{\tau}} \quad 0 \le t \le T, \qquad \int_0^T (Y_t^{\alpha,p} - p^b_{t\wedge\hat{\tau}})\mathrm{d}K_t = 0 \tag{3.51}$$

$$Y_T^{\alpha,p} = p^b_{\hat{\tau}}, \tag{3.52}$$

where

$$\bar{\mathcal{G}}_t^{\alpha,p}(y) = -\bar{c}_t^\alpha\left(p_t^a \wedge Q^-(p_t), p_t^b\right) y + \bar{h}_t^{\alpha,a}(p_t, p_t^a, p_t^b) =$$
$$\left[-c_t^\alpha\left(p_t^a \wedge Q^-(p_t), p_t^b\right) y + h_t^{\alpha,a}(p_t, p_t^a, p_t^b)\right] \mathbf{1}_{\{t<\hat{\tau}\}},$$

with c^α and $h^{\alpha,a}$ defined in (3.10) and (3.13). Recall that V^a satisfies (3.40), with the generator

$$\mathcal{G}_t^a(y, p_t^b) = 2\lambda_t^{\alpha^0} p_t^b F_t^{\alpha^0,-}(p_t^b) - \lambda_t^{\alpha^0} F_t^{-,\alpha^0}(p_t^b)y + \lambda_t^{\alpha^0} P_t^a(y) F_t^{\alpha^0,+}(P_t^a(y)) - \lambda_t^{\alpha^0} F_t^{+,\alpha^0}(P_t^a(y))y.$$

It is easy to deduce that

$$\bar{\mathcal{G}}_t^{\alpha^0,p^a}(V_t^a) = \mathcal{G}_t^a(V_t^a, p_t^b)\mathbf{1}_{\{t<\hat{\tau}\}}.$$

Hence, $(V^a_{t\wedge\hat{\tau}})$ satisfies the same RBSDE as $(Y_t^{\alpha^0,p^a})$. From the comparison principle, we conclude that $Y_t^{\alpha^0,p^a} = V^a_{t\wedge\hat{\tau}}$. On the other hand, for any $\alpha \in \mathbb{A}$, let us choose $p_t = \delta_{p_t^a}$, to obtain:

$$\bar{\mathcal{G}}_t^{\alpha,p^a}\left(Y_t^{\alpha^0,p^a}\right) = \bar{\mathcal{G}}_t^{\alpha,p^a}(V_t^a) = \Bigg[\lambda_t^\alpha p_t^b F_t^{-,\alpha}(p_t^b) + \lambda_t^\alpha \int_{-\infty}^{p_t^b} f_t^\alpha(u) l_t^{c,b}(u)du - \lambda_t^\alpha F_t^{-,\alpha}(p_t^b)V_t^a$$
$$+\lambda_t^\alpha p_t^a F_t^{+,\alpha}(p_t^a) - \lambda_t^\alpha F_t^{+,\alpha}(p_t^a)V_t^a\Bigg] \mathbf{1}_{\{t<\hat{\tau}\}}$$
$$\ge \left[\lambda_t^\alpha F_t^{-,\alpha}(p_t^b)(p_t^b - V_t^a) + \lambda_t^\alpha F_t^{+,\alpha}(p_t^a)(p_t^a - V_t^a) + \lambda_t^{\alpha^0} p_t^b F_t^{-,\alpha^0}(p_t^b)\right] \mathbf{1}_{\{t<\hat{\tau}\}},$$

where $l^{c,b}$ is defined in (3.12), and the last inequality is based on the Assumptions III.23, III.26, and on the monotonicity of $l^{c,b}_t(\cdot)$, which imply

$$\lambda^\alpha_t \int_{-\infty}^{p^b_t} f^\alpha_t(u) l^{c,b}_t(u) du \geq \lambda^{\alpha^0}_t \int_{-\infty}^{p^b_t} f^{\alpha^0}_t(u) l^{c,b}_t(u) du = \lambda^{\alpha^0}_t p^b_t F^{-,\alpha^0}_t(p^b_t).$$

Notice that, by construction, $p^b_t \leq V^b_t \leq V^a_t \leq p^a_t$. Then, Assumption III.23 implies

$$\lambda^\alpha_t F^{-,\alpha}_t(p^b_t)(2p^b_t - V^a_t) \geq \lambda^{\alpha^0}_t F^{-,\alpha^0}_t(p^b_t)(2p^b_t - V^a_t),$$
$$\lambda^\alpha_t F^{+,\alpha}_t(p^a_t)(p^a_t - V^a_t) \geq \lambda^{\alpha^0}_t F^{+,\alpha^0}_t(p^a_t)(p^a_t - V^a_t).$$

Thus, we obtain:

$$\bar{\mathcal{G}}^{\alpha^0,p^a}_t\left(Y^{\alpha^0,p^a}_t\right) \leq \bar{\mathcal{G}}^{\alpha,p^a}_t\left(Y^{\alpha^0,p^a}_t\right).$$

Using the comparison principle for RBSDEs, we conclude that $Y^{\alpha,p^a}_t \geq Y^{\alpha^0,p^a}_t = V^a_{t\wedge\hat{\tau}}$. Consider an arbitrary strategy (p,τ). By switching between p^a and p, we can construct a new strategy p', such that $Y^{\alpha,p'}_t \geq V^a_{t\wedge\hat{\tau}} \vee Y^{\alpha,p}_t$, for all t. More precisely, we define

$$\bar{\mathcal{G}}^{\alpha,p'}_t(y) = \bar{\mathcal{G}}^{\alpha,p^a}_t(y) \vee \bar{\mathcal{G}}^{\alpha,p}_t(y),$$

and solve the RBSDE (3.50)–(3.52). By the standard argument, the Y-component of the solution is $Y^{\alpha,p'}$, where p'_t is defined to be equal to $\delta_{p^a_t}$ if the maximum in the above equation is achieved at $\bar{\mathcal{G}}^{\alpha,p^a}_t(Y^{\alpha,p'}_t)$, and it is equal to p_t otherwise. The comparison principle implies that $Y^{\alpha,p'}_t \geq Y^{\alpha,p^a}_t \vee Y^{\alpha,p}_t \geq V^a_{t\wedge\hat{\tau}} \vee Y^{\alpha,p}_t$. Then, the standard results on RBSDEs imply that the optimal stopping time associated with $Y^{\alpha,p'}$ is

$$\inf\{t \in [0,T] \,:\, Y^{\alpha,p'}_t \leq p^b_{t\wedge\hat{\tau}}\} = \inf\{t \in [0,T] \,:\, V^a_t \leq p^b_{t\wedge\hat{\tau}}\} = \hat{\tau}.$$

Thus,

$$J^{(\nu,\theta),(p,\tau)}_0(1,\alpha) = \bar{J}^{\alpha,(p,\tau)}_0 \leq Y^{\alpha,p}_0 \leq Y^{\alpha,p'}_0 = \bar{J}^{\alpha,(p',\hat{\tau})}_0 = J^{(\nu,\theta),(p',\hat{\tau})}_0(1,\alpha).$$

Next, we show that the control p can be chosen so that, $\mathbb{P}$-a.s., for all t, $\text{supp}(p_t) \subset [p^a_t, \infty)$. Consider any control p. By switching, if necessary, between p^a and p, we can ensure that $Y^{\alpha,p}_t \geq V^a_{t\wedge\hat{\tau}}$. Then, for $t < \hat{\tau}$, the generator of $Y^{\alpha,p}$ is given by

$$\begin{aligned}
\bar{\mathcal{G}}^{\alpha,p}_t(y) &= -c^{\alpha}_t\left(p^a_t \wedge Q^-(p_t), p^b_t\right) y + h^{\alpha,a}_t(p_t, p^a_t, p^b_t) \\
&= -\lambda^{\alpha}_t F^{+,\alpha}_t(p^a_t \wedge Q^-(p_t))y + \lambda^{\alpha}_t p^b_t F^{+,\alpha}_t(Q^-(p_t)\wedge p^a_t) p_t\left((p^a_t,\infty)\right) + \\
&\lambda^{\alpha}_t \int_{(Q^-(p_t)\wedge p^a_t)\vee 0}^{p^a_t} f^{\alpha}_t(u) \int_{-\infty}^{p^a_t} \left[\left(z\wedge u + \left(p^b_t - u\right)\mathbf{1}_{\{z>u\}}\right)\right] p_t(dz)du + \\
&\lambda^{\alpha}_t F^{+,\alpha}_t(p^a_t)\int_{-\infty}^{p^a_t} z p_t(dz) - \lambda^{\alpha}_t F^{-,\alpha}_t(p^b_t) y + 2\lambda^{\alpha}_t p^b_t F^{-,\alpha}_t(p^b_t) - \lambda^{\alpha}_t p^b_t F^{+,\alpha}_t(p^a_t) p_t\left((p^a_t,\infty)\right) \\
&+ \lambda^{\alpha}_t \int_{p^a_t}^{\infty} f^{\alpha}_t(u)\int_{p^a_t}^{\infty} \left[\left(z\wedge l^{c,a}_t(u) + p^b_t \mathbf{1}_{\{z>l^{c,a}_t(u)\}}\right)\right] p_t(dz)du
\end{aligned}$$

Let us estimate the first four terms in the right hand side of the above (i.e. the ones that depend on $p_t(dx)$ restricted to $x < p^a_t$):

$$\begin{aligned}
&- \lambda^{\alpha}_t F^{+,\alpha}_t(p^a_t \wedge Q^-(p_t))y + \lambda^{\alpha}_t p^b_t F^{+,\alpha}_t(Q^-(p_t)\wedge p^a_t) p_t\left((p^a_t,\infty)\right) \\
&\quad + \lambda^{\alpha}_t \int_{(Q^-(p_t)\wedge p^a_t)\vee 0}^{p^a_t} f^{\alpha}_t(u) \int_{-\infty}^{p^a_t} \left[\left(z\wedge u + \left(p^b_t - u\right)\mathbf{1}_{\{z>u\}}\right)\right] p_t(dz)du + \\
&\qquad \lambda^{\alpha}_t F^{+,\alpha}_t(p^a_t)\int_{-\infty}^{p^a_t} z p_t(dz) \\
&\leq \lambda^{\alpha}_t \sup_{x\leq p^a_t}\Big[\left(-y + p^b_t\right) F^{+,\alpha}_t(x) + p_t\left((-\infty, p^a_t]\right) \sup_{z\in[x,p^a_t]}\Big[(z - p^b_t)F^{+,\alpha}_t(z)\Big]\Big].
\end{aligned}$$

Notice that, for $t < \hat{\tau}$ and $y = Y^{\alpha,p}_t$, we have $p^b_t - y \leq 0$, and, hence,

$$\begin{aligned}
&\sup_{x\leq p^a_t}\Big[\left(-y + p^b_t\right) F^{+,\alpha}_t(x) + p_t\left((-\infty, p^a_t]\right) \sup_{z\in[x,p^a_t]}\Big[(z - p^b_t)F^{+,\alpha}_t(z)\Big]\Big] \\
&\qquad = \sup_{z\leq p^a_t}\Big[(z-y)F^{+,\alpha}_t(z) + p^b_t p_t\left((p^a_t,\infty)\right) F^{+,\alpha}_t(z)\Big].
\end{aligned}$$

Due to Lemma III.25, the function $z \mapsto (z-y)F^{+,\alpha}_t(z)$ is nondecreasing in $z \leq P^a_t(y)$. As $p^b_t \leq 0$, the function $z \mapsto p^b_t\, p_t\left((p^a_t,\infty)\right) F^{+,\alpha}_t(z)$ is also nondecreasing, and, hence, the above supremum is attained at $z = p^a_t$, provided $P^a_t(y) \geq p^a_t$. The latter does hold for $t < \hat{\tau}$ and $y = Y_t^{\alpha,p}$, as $P_t^a(\cdot)$ is non-decreasing, $p_t^a = P_t^a(V_t^a)$ and $Y_t^{\alpha,p} \geq V_t^a$. Thus, the generator $\bar{\mathcal{G}}_t^{\alpha,p}(Y_t^{\alpha,p})$ does not decrease if we replace p by

$$p'_t(dx) = p_t(dx)\mathbf{1}_{[p_t^a,\infty)} + p_t((-\infty, p_t^a))\delta_{p_t^a}(dx).$$

In other words,

$$\bar{\mathcal{G}}_t^{\alpha,p}(Y_t^{\alpha,p}) \leq \bar{\mathcal{G}}_t^{\alpha,p'}(Y_t^{\alpha,p}).$$

The comparison principle, then, yields $Y_t^{\alpha,p} \leq Y_t^{\alpha,p'}$. Moreover, the optimal stopping strategy associated with $Y^{\alpha,p'}$ is $\hat{\tau}$. Repeating the argument used earlier in this proof, we conclude that any strategy $(p, \hat{\tau})$ can be modified to $(p', \hat{\tau})$, satisfying the properties stated in the lemma, so that the objective value does not decrease. The case of short agents is treated similarly. ∎

The above lemma has a straight-forward but useful corollary.

Corollary III.31. *Let Assumptions III.1–III.24 hold, and let (ν, θ) satisfy (3.47)–(3.48). Given any $\alpha \in \mathbb{A}$, let (p, τ) be an optimal strategy for the long agents with beliefs α, in the class of all admissible strategies satisfying: $\mathbb{P}$-a.s. supp$(p_t) \subset [p_t^a, \infty)$, for all $t \in [0, T]$, and $\tau = \hat{\tau}$. Then (p, τ) is optimal in the class of all admissible strategies, in the sense of Definition III.5. Similarly, given any $\beta \in \mathbb{B}$, let (p, τ) be an optimal strategy for the short agents with beliefs β, in the class of all admissible strategies satisfying: $\mathbb{P}$-a.s. supp$(p_t) \subset (-\infty, p_t^b]$, for all $t \in [0, T]$, and $\tau = \hat{\tau}$. Then (p, τ) is optimal in the class of all admissible strategies, in the sense of Definition III.5.*

Thus, no matter which limit order strategy p an agent is using, it is optimal for her to choose the following stopping threshold:

$$\hat{v}(s) = V^a \mathbf{1}_{\{s>0\}} + V^b \mathbf{1}_{\{s<0\}}.$$

This implies that, given a LOB ν in the form (3.47) and the stopping strategy $\hat{v}$ as above, if an optimal limit order strategy $\hat{p}(s,\alpha)$ exists for any state (s,α), then $(\hat{p}(s,\alpha),\hat{v})$ form an optimal control for the agents in state (s,α), in the sense of Definition III.5. Moreover, in such a case, θ, given by (3.48), satisfies the condition (3.8). Next, we need to construct a LOB ν, in the form (3.47), and the associated optimal limit order strategies for all agents, s.t. (3.7) is satisfied. We begin by showing that, for any ν in the form (3.47), the strategies (δ_{p^a}, V^a) and (δ_{p^b}, V^b) are optimal for the extremal agents.

Lemma III.32. *Let Assumptions III.1–III.26 hold, and let (ν,θ) satisfy (3.47)–(3.48). Then, given (ν,θ), the strategy (δ_{p^a}, V^a) is optimal for a long agent with beliefs α^0, and the strategy (δ_{p^b}, V^b) is optimal for a short agent with beliefs β^0, in the sense of Definition III.5.*

Proof: Consider a long agent with beliefs α^0. In view of Corollary III.31, it suffices to show the optimality in the class of strategies $(p,\hat{\tau})$, with $\text{supp}(p_t) \subset [p^a_t,\infty)$. Notice that Assumption III.26 implies:

$$l^{c,a}_t(x) = x \wedge p^a_t, \quad \forall x \in \text{supp}\left(f^{\alpha^0}_t\right).$$

Using the above observation, we recall the constructions from the proof of Lemma III.30, to obtain, for any strategy p and all $t<\hat{\tau}$:

$$\bar{\mathcal{G}}^{\alpha^0,p}_t(y) = -\lambda^{\alpha^0}_t F^{+,\alpha^0}_t(p^a_t)y - \lambda^{\alpha^0}_t F^{-,\alpha^0}_t(p^b_t)y + 2\lambda^{\alpha^0}_t p^b_t F^{-,\alpha^0}_t(p^b_t)$$

$$+\lambda^{\alpha^0}_t F^{+,\alpha^0}_t(p^a_t)\left(p^a_t p_t(\{p^a_t\}) + (p^a_t+p^b_t)p_t\left((p^a_t,\infty)\right)\right).$$

As $p^b_t \le 0$, the above expression is maximized at $p_t = \delta_{p^a_t}$. Using the comparison principle for the RBSDE satisfied by $Y^{\alpha^0,p}$, we conclude that $p=\delta_{p^a}$ produces the largest $Y^{\alpha^0,p}$ and, hence, the largest objective value for the long agents with beliefs α^0. The case of short agents is treated similarly. ∎

3.4.1 Equilibrium strategies of the non-extremal agents

In this subsection we construct the measure-valued processes (ν^a, ν^b), in the form (3.47), and a progressively measurable random field $(\hat{p}_t(s,\alpha))$, such that the controls $(\hat{p}(1,\alpha), V^a)$ and $(\hat{p}(-1,\alpha), V^b)$ are optimal for the non-extremal agents with beliefs α, long and short, respectively (recall that the optimal strategies for the extremal agents are constructed in Lemma III.32), and the fixed-point constraint (3.7) is satisfied. In view of Lemma III.30, we can restrict the possible controls p to the those satisfying: $\mathrm{supp}(p_t) \subset [p^a_t, \infty)$, for all $t \in [0,T]$. It is also obvious that we can restrict the support of p_t to be in $[-C_p, C_p]$. As the stopping strategy is fixed, for any $\alpha \in \hat{\mathbb{A}}$, the objective of a long player reduces to $\bar{J}^{\alpha,(p)}_0$, where

$$\bar{J}^{\alpha,(p)}_t =$$

$$\mathbb{E}\Big[\int_t^T \exp\left(-\int_t^s \bar{c}^{\alpha}_u\left(p^a_u, p^b_u\right) du\right) \bar{h}^{\alpha,a}_s(p_s, p^a_s, p^b_s) ds + \exp\left(-\int_t^T \bar{c}^{\alpha}_u\left(p^a_u, p^b_u\right) du\right) p^b_{\hat{\tau}} | \mathcal{F}_t\Big],$$

$$\bar{c}^{\alpha}_t(p^a_t, p^b_t) = c^{\alpha}_t(p^a_t, p^b_t)\mathbf{1}_{\{t\leq\hat{\tau}\}}, \quad \bar{h}^{\alpha,a}_t(p_t, p^a_t, p^b_t) = h^{\alpha,a}_t(p_t, p^a_t, p^b_t)\mathbf{1}_{\{t\leq\hat{\tau}\}},$$

and c^α and $h^{\alpha,a}$ defined in (3.10) and (3.13). Due to Assumptions III.23 and III.26, we have

$$l^{c,b}_t(x) = \inf\left\{p > Q^-(\nu^b_t) \,:\, -D_t(p-x) > \nu^b_t((p,\infty))\right\} = p^b_t \vee x, \quad \forall x \in \mathrm{supp}(f^\alpha_t).$$

In addition, for any $z \geq p^a_t$,

$$\{u > 0 \,:\, l^{c,a}_t(u) \geq z\} = \{u > 0 \,:\, u \geq z - D^{-1}_t\left(\nu^a_t([p^a_t, z))\right)\},$$

and, hence, for any $B \geq p^a_t$,

$$\int_{p^a_t}^B f^\alpha_t(u)(l^{c,a}_t(u) - p^a_t) du = \int_0^{l^{c,a}_t(B)-p^a_t} \int_{u+p^a_t-D^{-1}_t(\nu^a_t([p^a_t, p^a_t+u)))}^{C_p} f^\alpha_t(y) dy du.$$

The above observations allow us to simplify the objective:

$$h_t^{\alpha,a}(p_t, p_t^a, p_t^b) = \lambda_t^\alpha \int_{p_t^a}^\infty \Big[(z - p_t^b) F_t^{+,\alpha}\left(z - D_t^{-1}\left(\nu_t^a([p_t^a, z))\right)\right) + p_t^b F_t^{+,\alpha}(p_t^a)$$

$$+ \int_{p_t^a}^{z - D_t^{-1}(\nu_t^a([p_t^a, z)))} f_t^\alpha(u) l_t^{c,a}(u) du\Big] p_t(dz) + 2\lambda_t^\alpha p_t^b F_t^{-,\alpha}(p_t^b) =$$

$$\lambda_t^\alpha \int_{p_t^a}^{C_p} \Big[(z - p_t^b) F_t^{+,\alpha}\left(z - D_t^{-1}\left(\nu_t^a([p_t^a, z))\right)\right) +$$

$$\int_0^{z - p_t^a} F_t^{+,\alpha}\left(u + p_t^a - D_t^{-1}\left(\nu_t^a([p_t^a, p_t^a + u))\right)\right) du\Big] p_t(dz) +$$

$$2\lambda_t^\alpha p_t^b F_t^{-,\alpha}(p_t^b) + \lambda_t^\alpha p_t^b F_t^{+,\alpha}(p_t^a).$$

Notice that the above objective does not depend on ν^b (for a given p^b), hence, we can separate the equilibrium problems of the long and short agents (this is only true for the non-extremal agents, of course). For simplicity, we only consider the problem of the long agents – the short agents can be treated similarly. Denote by κ_t and $\hat{\nu}_t^a$ the push-forward measures of p_t and ν_t^a, under the mapping $x \mapsto x - p_t^a$. Clearly, the measurability property is preserved by this transformation, hence, we can reformulate the equilibrium problem as a search for κ and $\hat{\nu}^a$, with the values in the space of measures with support in $[0, C_p]$. In the new variables, the objective takes a more convenient form. In particular, $h_t^{\alpha,a}(p_t, p_t^a, p_t^b) = \hat{h}_t^{\alpha,a}(\kappa_t, p_t^a, p_t^b)$, where

$$\hat{h}_t^{\alpha,a}(\kappa_t, p_t^a, p_t^b) = \lambda_t^\alpha \int_0^{C_p} \Big[(z + p_t^a - p_t^b) F_t^{+,\alpha}\left(z + p_t^a - D_t^{-1}\left(\hat{\nu}_t^a([0, z))\right)\right)$$

$$+ \int_0^z F_t^{+,\alpha}\left(u + p_t^a - D_t^{-1}\left(\hat{\nu}_t^a([0, u))\right)\right) du\Big] \kappa_t(dz) + 2\lambda_t^\alpha p_t^b F_t^{-,\alpha}(p_t^b) + \lambda_t^\alpha p_t^b F_t^{+,\alpha}(p_t^a).$$

Note that $\bar{J}^{\alpha,(p)}$ solves a BSDE with the affine generator

$$\hat{\mathcal{G}}_t^\alpha(y) = \bar{c}_u^\alpha\left(p_u^a, p_u^b\right) y + \hat{h}_t^{\alpha,a}(\kappa_t, p_t^a, p_t^b).$$

In order to maximize $\bar{J}^{\alpha,(p)}$, it suffices to find a strategy κ which maximizes the above generator. The latter is, in turn, equivalent to maximizing $\hat{h}_t^{\alpha,a}(\cdot, p_t^a, p_t^b)$. Thus, we need to find a progressively measurable random field $(\kappa_t(\alpha))$, with values in $\mathcal{P}(\mathbb{R})$ (with the weak topology on it), s.t., for μ^a-a.e. $\alpha \in \hat{\mathbb{A}}$,

$$\kappa_t(\alpha) \in \operatorname{argmax}_{\kappa' \in \psi} \hat{h}_t^{\alpha,a}(\kappa', p_t^a, p_t^b) \tag{3.53}$$

holds for $dt \times \mathbb{P}$-a.e. (t, ω), where $\psi = \{p \in \mathcal{P}(\Pi) \colon \operatorname{supp}(p) \subseteq \Pi\}$ and $\Pi = [0, C_p]$. The standard BSDE results, then, imply that $\kappa(\alpha)$ is optimal for the agents in state $(1, \alpha)$, for μ^a-a.e. $\alpha \in \hat{\mathbb{A}}$. If, in addition, we ensure that the fixed-point constraint (3.7) is satisfied (and a similar construction holds for the short agents), we obtain an equilibrium in the continuum-player game, in the sense of Definition III.6. Notice that we can rewrite

$$\hat{h}_t^{\alpha,a}(\kappa', p_t^a, p_t^b) = \lambda_t^\alpha \int_{\mathbb{R}} F_t(\alpha, p, \hat{\nu}_t^a) \kappa'(dp) + 2\lambda_t^\alpha p_t^b F_t^{-,\alpha}(p_t^b) + \lambda_t^\alpha p_t^b F_t^{+,\alpha}(p_t^a),$$

$$\begin{aligned} F_t(\alpha, p, \hat{\nu}_t^a) = (p + p_t^a - p_t^b) F_t^{+,\alpha} \left(p + p_t^a - D_t^{-1}\left(\hat{\nu}_t^a([0,p))\right)\right) + \\ \int_0^p F_t^{+,\alpha} \left(u + p_t^a - D_t^{-1}\left(\hat{\nu}_t^a([0,u))\right)\right) du. \end{aligned} \tag{3.54}$$

Assuming the extremal long agents post limit orders at p^a, the fixed-point constraint (3.7) (more precisely, the part of (3.7) that corresponds to the long agents) becomes:

$$\hat{\nu}_t^a([0,x]) = \mu^a(\{\alpha_0\}) + \int_{\hat{\mathbb{A}}} \kappa_t(\alpha; [0,x]) \mu^a(\mathrm{d}\alpha), \quad \forall x \geq 0. \tag{3.55}$$

The above equations can be solved separately for different (t, ω), hence, to this end, we fix (t, ω) and omit the t subscript whenever it causes no ambiguity. The statements that follow hold for a.e. (t, ω). It turns out that it is more convenient to search for a measure

$$K(d\alpha, dx) = \kappa(\alpha; dx) \mu^a(d\alpha),$$

which is an element of $\mathcal{M}_{\mu^a}\left(\hat{\mathbb{A}} \times \Pi\right)$, the space of finite sigma-additive measures on $\hat{\mathbb{A}} \times \Pi$, with the first marginal μ^a. Transition from K to κ is accomplished via the usual disintegration. Thus, for a.e. (t,ω), we need to find $(K,\nu) \in \mathcal{M}_{\mu^a}\left(\hat{\mathbb{A}} \times \Pi\right) \times \mathcal{M}_{\mu^a(\mathbb{A})}(\Pi)$ solving the following system

$$\begin{cases} K \in \operatorname{argmax}_{K\in\mathcal{M}_{\mu^a}\left(\hat{\mathbb{A}}\times\Pi\right)} \int F(\alpha,p,\nu)K(d\alpha,dp), \\ \nu(dx) = \mu^a(\{\alpha_0\})\delta_0(dx) + K\left(\hat{\mathbb{A}} \times dx\right), \end{cases} \tag{3.56}$$

where $\mathcal{M}_{\mu^a(\mathbb{A})}(\Pi)$ is the space of finite sigma-additive measures on Π, with the total mass $\mu^a(\mathbb{A}) = \mu^a(\{\alpha_0\}) + \mu^a(\hat{\mathbb{A}})$. The above system can be formulated as a fixed-point problem, in an obvious way. However, the main challenge in solving this problem stems from the fact that $F(\alpha,\cdot,\cdot)$ is not continuous: e.g. it may be discontinuous in p, if ν has atoms. Therefore, we replace F by its "mollified" version:

$$\hat{F}(\alpha,p,\nu) = \sup_{p'\in\Pi} F(\alpha,p',\nu) - |p'-p|\,.$$

The following lemma shows that we can replace F by $\hat{F}$ in (3.56), and any solution to the new problem will solve the original one.

Lemma III.33. *For any $\alpha \in \hat{\mathbb{A}}$ and $\nu \in \mathcal{M}_{\mu^a(\mathbb{A})}(\Pi)$, the function $p \mapsto \hat{F}(\alpha,p,\nu)$ is 1-Lipschitz in $p \in \Pi$, and*

$$\operatorname{argmax}_{p\in\Pi} \hat{F}(\alpha,p,\nu) = \operatorname{argmax}_{p\in\Pi} F(\alpha,p,\nu).$$

Proof: For convenience, we drop the dependence on (α,ν). The first statement is clear from the definition. It is also clear that $\sup_{p\in\Pi}\hat{F}(p) = \sup_{p\in\Pi} F(p)$, and we denote this supremum by S. As $\hat{F}$ is continuous in Π, it achieves its supremum, hence, it suffices to show that $F(p_0) = S$, for every p_0 such that $\hat{F}(p_0) = S$ (note that the opposite implication is obvious). Assume the contrary, then $F(p) \leq S - \varepsilon$, for some $\varepsilon > 0$ and all $p \in \Pi \cap (p_0 - \varepsilon, p_0 + \varepsilon)$ by the upper semi-continuity of F. Then, we obtain $\hat{F}(p_0) \leq S - \varepsilon$, which is a contradiction. To see that F is upper semi-continuous, notice that it is left-continuous, with only downward jumps, which follows directly from (3.54). ∎

Summarizing the above discussion, to find a solution to (3.56), it suffices to find a fixed point of the following correspondence

$$\mathcal{M}_{\mu^a}\left(\hat{\mathbb{A}}\times\Pi\right)\ni K\mapsto\tilde{K}\left(\tilde{\nu}(K)\right),$$

where

$$\tilde{\nu}(K;dx)=\mu(\{\alpha_0\})\delta_{p^a}(dx)+K(\hat{\mathbb{A}}\times dx)\in\mathcal{M}_{\mu^a(\mathbb{A})}\left(\Pi\right) \tag{3.57}$$

is single-valued, and

$$\tilde{K}(\nu)=\operatorname{argmax}_{K\in\mathcal{M}_{\mu^a}\left(\hat{\mathbb{A}}\times\Pi\right)}\int\hat{F}(\alpha,p,\nu)K(d\alpha,dp)\subset\mathcal{M}_{\mu^a}\left(\hat{\mathbb{A}}\times\Pi\right). \tag{3.58}$$

Proposition III.34. *Let Assumptions III.27, III.28 hold. Then, the correspondence* $\mathcal{K}\colon K\mapsto\tilde{K}\left(\tilde{\nu}(K)\right)$*, defined by (3.57)–(3.58), has a fixed point.*

Proof: To prove the proposition, we use the Kakutani's theorem for correspondences (cf. Definition $II.7.8.1$ and Theorem $II.7.8.6$ in [35]). Note that $\mathcal{M}_{\mu^a}\left(\hat{\mathbb{A}}\times\Pi\right)$, equipped with the weak topology, is convex and compact (by Prokhorov's theorem). In addition, it can be viewed as a subspace of the dual of the space of continuous functions on $\hat{\mathbb{A}}\times\Pi$, which is semi-normed. Thus, in order to apply the Kakutani's theorem, it only remains to show that $\mathcal{K}$ is upper hemi-continuous (uhc), with nonempty compact convex values. Notice also that $\tilde{K}(\nu)$ is convex by definition (as an argmax of a linear functional on a convex set), hence, $\mathcal{K}$ is convex-valued, and we only need to show that it is uhc, with non-empty compact values. As $p\mapsto\tilde{\nu}(p)$ is a continuous function, and a composition of a continuous function and a uhc correspondence is a uhc correspondence, it suffices to verify that $\nu\mapsto\tilde{K}(\nu)$ is a uhc non-empty compact valued correspondence. To achieve this, we use the classical Berge's theorem (cf. [49], section E.3), which reduces to problem to the continuity of the functon

$$(K,\nu)\mapsto\phi(K,\nu)=\int\hat{F}(\alpha,p,\nu)K(d\alpha,dp), \tag{3.59}$$

on $\mathcal{M}_{\mu^a}\left(\hat{\mathbb{A}}\times\Pi\right)\times\mathcal{M}_{\mu^a(\mathbb{A})}(\Pi)$, metrized via the Lévy-Prokhorov metric. In the remainder of the proof, we show that $\phi(K,\nu)$ is jointly continuous in (K,ν). More precisely, $\phi(K,\nu)$ is continuous in K, and it is continuous in ν (with respect to Lévy-Prokhorov metric), uniformly over K.

First, we show that $\phi(K,\nu)$ is continuous in K. By the definition of weak topology, the desired continuity would follow from the joint continuity of $\hat{F}(\alpha,p,\nu)$ with respect to (α,p). Due to Lemma III.33, $\hat{F}(\alpha,p,\nu)$ is 1-Lipschitz in p (uniformly over $\alpha\in\hat{\mathbb{A}}$), hence, it suffices to check that $\hat{F}(\alpha,p,\nu)$ is continuous in α. The latter follows from the fact that $F(\alpha,p,\nu)$ is continuous in α, uniformly over $p\in\Pi$. Indeed, notice that, if, for some $\alpha'\in U(\alpha)$, we have $|F(\alpha',p,\nu)-F(\alpha,p,\nu)|\leq\varepsilon\ \forall p\in\Pi$, then

$$\hat{F}(\alpha',p,\nu)=F(\alpha',p',\nu)-|p'-p|\leq F(\alpha,p',\nu)-|p'-p|+\varepsilon\leq\hat{F}(\alpha,p,\nu)+\varepsilon,$$

which, together with the analogous symmetric inequality, shows that

$$\left|\hat{F}(\alpha',p,\nu)-\hat{F}(\alpha,p,\nu)\right|\leq\varepsilon$$

The first equality in the above follows from the fact that F is upper semi-continuous in p (and bounded from above by $2C_p$), which is shown in the proof of Lemma III.33, and, hence, the supremum in the definition of $\hat{F}$ is achieved at some p'. To show that $F(\alpha,p,\nu)$ is continuous in α, uniformly over $p\in\Pi$, we recall (3.54), and the desired continuity follows directly from Assumption III.27.

It remains to show that $\phi(K,\nu)$ is continuous in $\nu\in\mathcal{M}_{\mu^a(\mathbb{A})}(\Pi)$, uniformly over $K\in\mathcal{M}_{\mu^a}\left(\hat{\mathbb{A}}\times\Pi\right)$. As every such K has a fixed finite total mass, due to the definition of ϕ, the desired continuity follows from the fact that $\hat{F}(\alpha,p,\nu)$ is continuous in ν, uniformly over $(\alpha,p)\in\hat{\mathbb{A}}\times\Pi$. To prove the latter, fix $\varepsilon>0$, and let d_0 be Lévy-Prokhorov metric on $\mathcal{M}_{\mu^a(\mathbb{A})}(\Pi)$. Let us show that there exists an increasing continuous deterministic function $C_0\colon [0,\infty) \to [0,\infty)$, s.t. $C_0(0) = 0$ and

$$\left|\hat{F}(\alpha, p, \nu_1) - \hat{F}(\alpha, p, \nu_2)\right| \leq C_0(\varepsilon), \quad \forall\, p \in \Pi,\ \alpha \in \hat{\mathbb{A}},\ d_0(\nu_1, \nu_2) \leq \varepsilon.$$

If we manage to show that there exists an increasing continuous deterministic function $B\colon [0,\infty) \to [0,\infty)$, s.t. $B(0) = 0$ and

$$F(\alpha, p, \nu_1) \leq F(\alpha, (p-\varepsilon) \vee 0, \nu_2) + B(\varepsilon), \tag{3.60}$$

then

$$\begin{aligned} &\hat{F}(\alpha, p, \nu_1) = F(\alpha, p', \nu_1) - |p' - p| \leq F(\alpha, (p'-\varepsilon) \vee 0, \nu_2) - |p' - p| + B(\varepsilon) \\ &\leq F(\alpha, (p'-\varepsilon) \vee 0, \nu_2) - |(p'-\varepsilon) \vee 0 - p| + B(\varepsilon) + \varepsilon \leq \hat{F}(\alpha, p, \nu_2) + B(\varepsilon) + \varepsilon. \end{aligned}$$

The latter, together with the analogous inequality in which ν_1 and ν_2 are switched, yields the desired uniform continuity of $\hat{F}$ in ν. Thus, it is only left to prove (3.60). For any $p \in \Pi$, by the definition of the Lévy-Prokhorov metric, we have:

$$\nu_1([0,p)) \geq \nu_2([0,(p-\varepsilon) \vee 0)) - \varepsilon$$

and, hence, by Assumption III.28,

$$-D^{-1}(\nu_1([0,p))) \geq -D^{-1}\left(\nu_2([0,(p-\varepsilon) \vee 0))\right) - \epsilon(\varepsilon).$$

Then, for any $p \in \Pi$,

$$p + p^a - D^{-1}(\nu_1([0,p))) \geq (p-\varepsilon) \vee 0 + p^a - D^{-1}\left(\nu_2([0,(p-\varepsilon) \vee 0))\right) - \epsilon(\varepsilon),$$

which implies

$$F^{+,\alpha}\left(p + p^a - D^{-1}(\nu_1^+(p))\right) \leq F^{+,\alpha}\left((p-\varepsilon) \vee 0 + p^a - D^{-1}(\nu_2^+((p-\varepsilon) \vee 0))\right) + M_f \epsilon(\varepsilon),$$

where we used the fact that f^α is bounded by some constant M_f. The above estimate, along with the boundedness of p^a, p^b and $F^{+,\alpha}$, yields the desired inequality (3.60) for the first term in (3.54). Integrating the above estimate, we obtain the analogous inequality for the last term in the right hand side of (3.54), thus, completing the proof. ∎

Proposition III.34 implies that, for a.e. (t,ω), we can find $K_{t,\omega} \in \mathcal{M}_{\mu^a}\left(\hat{\mathbb{A}} \times \Pi\right)$, s.t.

$$K_{t,\omega} \in \tilde{K}\left(\tilde{\nu}(K_{t,\omega})\right),$$

and, hence, $(K_{t,\omega}, \tilde{\nu}(K_{t,\omega}))$ satisfies (3.56). Next, we need to establish the measurability of $K_{t,\omega}$ with respect to (t,ω). Namely, we need to show that there exists a progressively measurable mapping $(t,\omega) \mapsto K_{t,\omega} \in \mathcal{M}_{\mu^a}\left(\hat{\mathbb{A}} \times \Pi\right)$, such that

$$K_{t,\omega} \in \operatorname{argmax}_{K' \in \mathcal{M}_{\mu^a}\left(\hat{\mathbb{A}}\times\Pi\right)} \phi_{t,\omega}\left(K', \tilde{\nu}(K_{t,\omega})\right), \tag{3.61}$$

for Leb $\otimes\, \mathbb{P}$-a.e. (t,ω), where ϕ and $\tilde{\nu}$ are defined in (3.59) and (3.57). We denote $S = [0,T]\times\Omega$, and let $\mathcal{S}$ be the progressive sigma-algebra (defined w.r.t. the filtration $\mathbb{F}$) on S. We also denote $\mathbb{X} = \mathcal{M}_{\mu^a}\left(\hat{\mathbb{A}} \times \Pi\right)$ and introduce the correspondence $g_1\colon S \times \mathbb{X} \to \mathbb{X}$, given by

$$(t,\omega,K) \mapsto \operatorname{argmax}_{K' \in \mathbb{X}} \phi_{t,\omega}(K', \tilde{\nu}(K)).$$

Notice that $\mathbb{X}$ is separable and metrizable, and consider the function $(t,\omega,K,K') \mapsto \phi_{t,\omega}(K', \tilde{\nu}(K))$, defined on $(S \times \mathbb{X}^2, \mathcal{S} \otimes \mathcal{B}(\mathbb{X}^2))$. Note that this function is continuous in K' (as shown in the proof of Proposition III.34) and measurable in (t,ω,K) (as it is continuous in K and measurable in (t,ω), as shown in the proof of Proposition III.34), hence, it is a Carathéodory function. Then, the Measurable Maximum theorem (cf. Theorem 18.18 in [2]) implies that g_1 is a $(\mathcal{S} \otimes \mathcal{B}(\mathbb{X}))$-measurable correspondence with nonempty and compact values. Consider another correspondence $g_2\colon S \to \mathbb{X}$, given by

$$(t,\omega) \mapsto \left\{K \in \mathbb{X}\colon\, K \in \operatorname{argmax}_{K'} \phi_{t,\omega}(K', \tilde{\nu}(K))\right\}.$$

Let us show how to measurably select from g_2, for $\mathrm{Leb}\otimes\mathbb{P}$-a.e. (t,ω). The standard measurable selection results (cf. Corollary 18.27 and Theorem 18.26 in [2]) imply that such a selection is possible if g_2 has $\mathcal{S}\otimes\mathcal{B}(\mathbb{X})$-measurable graph and non-empty values. The latter follows from Proposition III.34, and the former is guaranteed by the following lemma.

Lemma III.35. *The correspondence g_2 has a $\mathcal{S}\otimes\mathcal{B}(\mathbb{X})$-measurable graph.*

Proof: Denote this graph by Γ_{g_2}. Let $I_{\mathbb{X}}\colon \mathbb{X}\to\mathbb{X}\times\mathbb{X}$ be given by $I_{\mathbb{X}}(K)=(K,K)$. Then, $\Gamma_{g_2}=(\mathrm{id}\times I_{\mathbb{X}})^{-1}(\Gamma)$, where $\Gamma\subset S\times\mathbb{X}\times\mathbb{X}$ is given by

$$\Gamma=\{(t,\omega,K,K'|(t,\omega)\in S,\ K\in\mathbb{X},\ K'\in\mathrm{argmax}_{K''\in\mathbb{X}}\phi_{t,\omega}(K'',\tilde{\nu}(K)))\}$$

$$\cap\{(t,\omega,K,K)|(t,\omega)\in S,\ K\in\mathbb{X}\}.$$

Clearly, $\mathrm{id}\times I_{\mathbb{X}}$ is a measurable map, and the set $\{(t,\omega,K,K)|(t,\omega)\in S,\ K\in\mathbb{X}\}$ is measurable. Therefore, we only need to check that

$$\{(t,\omega,K,K'|(t,\omega)\in S,\ K\in\mathbb{X},\ K'\in\mathrm{argmax}_{K''\in\mathbb{X}}\phi_{t,\omega}(K'',\tilde{\nu}(K)))\}$$

is $\mathcal{S}\otimes\mathcal{B}(\mathbb{X}^2)$-measurable. The latter set is precisely the graph of g_1, and it is measurable as the correspondence g_1 is measurable (cf. Theorem 18.6 in [2]). ∎

Thus, we conclude that there exists a progressively measurable K, with values in $\mathcal{M}_{\mu^a}\left(\hat{\mathbb{A}}\times\Pi\right)$, satisfying (3.61) for $\mathrm{Leb}\otimes\mathbb{P}$-a.e. (t,ω). It only remains to construct κ from K, by disintegration. Let us introduce $A=S\times\hat{\mathbb{A}}$, equipped with the sigma-algebra $\mathcal{S}\otimes\mathcal{B}\left(\hat{\mathbb{A}}\right)$, and the measure $\mathbb{Q}$ on $A\times\Pi$, defined via $\mathbb{Q}(dt,d\omega,d\alpha,dp)=K_{t,\omega}(d\alpha,dp)dt\mathbb{P}(d\omega)$. Note that the marginal distribution of $\mathbb{Q}$ on A is $\mu^a(d\alpha)dt\mathbb{P}(d\omega)$. Then, as the natural projection from $A\times\Pi$ to Π has a Borel range, Theorems 5.3 and 5.4 from [41] imply that there exists a kernel $\kappa\colon A\ni(t,\omega,\alpha)\mapsto\kappa_{t,\omega}(\alpha)\in\mathcal{P}(\Pi)$, which is a regular conditional distribution of the natural projection from $A \times \Pi$ to Π, given the natural projection from $A \times \Pi$ to $\mathbb{A}$, under $\mathbb{Q}$. Namely, for every absolutely bounded measurable $f \colon A \times \Pi \to \mathbb{R}$, we have (3.62)

$$\int_{A\times\Pi} f(t,\omega,\alpha,p)K_{t,\omega}(d\alpha,dp)dt\mathbb{P}(d\omega) = \int_{A\times\Pi} f(t,\omega,\alpha,p)\kappa_{t,\omega}(\alpha;dp)\mu^a(d\alpha)dt\mathbb{P}(d\omega).$$

The above property yields that $\hat{\nu}^a_{t,\omega} = \tilde{\nu}(K_{t,\omega})$ and $\kappa_{t,\omega}$ satisfy the fixed-point constraint (3.55). It only remains to show that κ satisfies (3.53), for $\mathrm{Leb} \otimes \mathbb{P} \otimes \mu^a$-a.e. (t,ω,α). Assume that this is not the case, then, there exists a measurable set $B \subset [0,T] \times \Omega$, with positive measure, s.t. for any fixed $(t,\omega) \in B$, there exists a measurable set $C \subset \hat{\mathbb{A}}$, s.t. $\mu^a(C) > 0$ and, for all $\alpha \in C$,

$$\int_{\mathbb{R}} \hat{F}_{t,\omega}(\alpha,p,\tilde{\nu}(K_{t,\omega}))\kappa_{t,\omega}(\alpha;dp) \le \int_{\mathbb{R}} F_{t,\omega}(\alpha,p,\tilde{\nu}(K_{t,\omega}))\kappa_{t,\omega}(\alpha;dp)$$
$$< \sup_{\kappa'\in\psi} \int_{\mathbb{R}} F_{t,\omega}(\alpha,p,\tilde{\nu}(K_{t,\omega}))\kappa'(dp) = \sup_{\kappa'\in\psi} \int_{\mathbb{R}} \hat{F}_{t,\omega}(\alpha,p,\tilde{\nu}(K_{t,\omega}))\kappa'(dp).$$

The above inequality becomes non-strict for all $\alpha \in \hat{\mathbb{A}} \setminus C$. Then, for a fixed $(t,\omega) \in B$, we can choose a measurable $\tilde{\kappa} \colon \hat{\mathbb{A}} \to \mathcal{P}(\Pi)$ (in the same way as we chose a measurable K, except that, in this case, the measurability is required in the α-variable), s.t.

$$\sup_{\kappa'\in\psi} \int_{\mathbb{R}} \hat{F}_{t,\omega}(\alpha,p,\tilde{\nu}(K_{t,\omega}))\kappa'(dp) = \int_{\mathbb{R}} \hat{F}_{t,\omega}(\alpha,p,\tilde{\nu}(K_{t,\omega}))\tilde{\kappa}(\alpha;dp), \quad \mu^a\text{-a.e. } \alpha \in \hat{\mathbb{A}}.$$

Thus, we obtain

$$\int_{\mathbb{R}} \hat{F}_{t,\omega}(\alpha,p,\tilde{\nu}(K_{t,\omega}))\kappa_{t,\omega}(\alpha;dp) < \int_{\mathbb{R}} \hat{F}_{t,\omega}(\alpha,p,\tilde{\nu}(K_{t,\omega}))\tilde{\kappa}(\alpha;dp),$$

for all $\alpha \in C$, and the non-strict inequality holds for all $\alpha \in \hat{\mathbb{A}}$. Integrating with respect to μ^a, and using (3.62) with $f(t,\omega,\alpha,p) = \hat{F}\left(t,\omega,\alpha,p,\tilde{\nu}(K_{t,\omega})\right)$, we obtain a contradiction with (3.61) on the set B (which has a positive measure). Thus, for μ^a-a.e. $\alpha \in \hat{\mathbb{A}}$, (3.53) holds for $\mathrm{Leb} \otimes \mathbb{P}$-a.e. (t,ω). This means that, if we define

$\hat{p}_t(\alpha)$ as the push-forward of $\kappa_t(\alpha)$, under the mapping $x \mapsto x + p^a_t$, the resulting strategy $\hat{p}(\alpha)$ maximizes the generator $\hat{\mathcal{G}}^\alpha_t(y)$, for any y and a.e. (t, ω). Then, we define ν^a_t to be the push-forward of $\hat{\nu}^a_t$, under the mapping $x \mapsto x + p^a_t$, and use the standard BSDE results to conclude that, for μ^a-a.e. $\alpha \in \hat{\mathbb{A}}$,

$$J^{(\nu,\theta),(\hat{p}(\alpha),V^a)}(1,\alpha) = \bar{J}^{\alpha,(\hat{p}(\alpha))}_0 \geq \bar{J}^{\alpha,(p')}_0 = J^{(\nu,\theta),(p',V^a)}(1,\alpha)$$

holds for all admissible strategies p', which means that $\hat{p}(\alpha)$ is optimal for the long agents with beliefs α. With such a choice of ν^a and $\hat{p}$, the fixed-point condition on ν^a, given in (3.7), is satisfied, as it is equivalent to (3.55) (assuming the extremal long agents post limit orders at p^a, which is optimal for them). This, along with Corollary III.31, implies that $(\hat{p}(\alpha), V^a)$ is an optimal strategy for the long agents with beliefs $\alpha \in \hat{\mathbb{A}}$. The short agents are treated similarly. Thus, we complete the proof of Theorem III.29.

Remark III.36. Notice that, as announced in Remark III.7, we have constructed an equilibrium, satisfying

$$v_t(1,\alpha) = v^a_t = V^a_t, \qquad v_t(-1,\alpha) = v^b_t = V^b_t, \qquad \forall\, \alpha \in \mathbb{A} \cup \mathbb{B},\ (t,\omega) \in [0,T] \times \Omega.$$

Therefore, in such an equilibrium, no agents execute market orders before the end of the game $\hat{\tau}$, and, hence, the empirical distribution μ remains constant and (3.9) holds.

3.5 Example

In this section, we consider the simplest concrete example of our model and show how it can be used. Consider a stochastic basis $(\Omega, \tilde{\mathbb{F}} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, with a Poisson random measure N, whose compensator is $\lambda_t f_t(x)\mathrm{d}x\mathrm{d}t$, as described in Subsection 3.2.1. We assume that $J_t(x) = x$ (i.e. $M \equiv N$), so that N is the jump measure of the (potential) fundamental price process X. We also assume that $T = 20$, $\lambda_t \equiv 1$ and f_t is the density of a uniform distribution on $[-C_0, C_0]$, where the constant C_0 is chosen to be sufficiently large, so that this interval contains the supports of all f^α described below. We take $\mathbb{A} = \{\alpha_0\} \cup \hat{\mathbb{A}}$, $\mathbb{B} = \{\beta_0\} \cup \hat{\mathbb{B}}$, where

$$\hat{\mathbb{A}} = \left\{ \frac{i}{K} | 0 \leq i < K \right\}, \quad \hat{\mathbb{B}} = \left\{ -\frac{i}{K} | 0 \leq i < K \right\}$$

are the uniform partitions of unit intervals, and $K = 500$ is used for most of the computations herein. The restrictions of μ^a (resp. μ^b) on $\hat{\mathbb{A}}$ (resp. $\hat{\mathbb{B}}$) assign a mass of $1/K$ to every point of the corresponding discrete space. Note that this implies $\mu^a(\hat{\mathbb{A}}) = \mu^b(\hat{\mathbb{B}}) = 1$. We also define $\mu^a(\{\alpha^0\}) = \mu^b(\{\beta^0\}) = 0.1$.

Next, we consider a collection of positive numbers $\{\lambda^{+,\alpha}, \lambda^{-,\alpha}, C^{+,\alpha}, C^{-,\alpha}\}_{\alpha \in \mathbb{A} \cup \mathbb{B}}$, and define

$$f^\alpha(x) = \frac{\lambda^{+,\alpha}}{(\lambda^{+,\alpha} + \lambda^{-,\alpha}) C^{+,\alpha}} \mathbf{1}_{[0, C^{+,\alpha}]}(x) + \frac{\lambda^{-,\alpha}}{(\lambda^{+,\alpha} + \lambda^{-,\alpha}) C^{-,\alpha}} \mathbf{1}_{[-C^{-,\alpha}, 0]}(x),$$
$$\lambda^\alpha = \lambda^{+,\alpha} + \lambda^{-,\alpha}.$$

Herein, we use $C^{+,\alpha_0} = C^{-,\alpha_0} = C^{+,\beta_0} = C^{-,\beta_0} = 0.5$ and

$$C^{+,\alpha} = a + b\alpha, \quad C^{-,\alpha} = C^{-,\alpha_0}, \quad \forall\, \alpha \in \hat{\mathbb{A}},$$

$$C^{-,\beta} = a - b\beta, \quad C^{+,\beta} = C^{+,\beta_0}, \quad \forall\, \beta \in \hat{\mathbb{B}},$$

with $a = 0.5$ and $b = 10$. Finally, for any $\alpha \in \mathbb{A} \cup \mathbb{B}$, we introduce

$$\Gamma^\alpha(x) = \frac{\lambda^\alpha}{\lambda} \frac{f^\alpha(x)}{f(x)} - 1, \quad \mathrm{d}Z^\alpha_t = Z^\alpha_{t-} \int_{\mathbb{R}} \Gamma^\alpha(x) \left[N(\mathrm{d}t, \mathrm{d}x) - \lambda f(x) dt dx \right],$$

and define $\mathbb{P}^\alpha << \mathbb{P}$ by its Radon-Nikodym density Z^α_T. One can easily check, using the general results in [39] (or in [23], for the deterministic case, used herein) that, under such $\mathbb{P}^\alpha$, N is a Poisson random measure with the compensator $\lambda^\alpha f^\alpha(x) \mathrm{d}x \mathrm{d}t$.

We assume that the demand elasticity is deterministic, constant in time, and linear in price:

$$D_t(p) = -kp,$$

with the elasticity parameter $k = 0.2$.

With the above choice of $\{C^{\pm,\alpha_0}, C^{\pm,\beta_0}, \mu^a(\{\alpha^0\}), \mu^b(\{\beta^0\}), k\}$, it is easy to see that Assumption III.26 is satisfied. Notice that the choice of $\lambda^{\pm,\alpha}$, for $\alpha \in \hat{\mathbb{A}} \cup \hat{\mathbb{B}}$, does not affect the equilibrium, as long as Assumptions III.23 and III.24 are satisfied. This is, clearly, the case if we choose $\lambda^{\pm,\alpha} = \lambda^{\pm,\alpha^0}$ and $\lambda^{\pm,\beta} = \lambda^{\pm,\beta^0}$, for $\alpha \in \mathbb{A}$ and $\beta \in \mathbb{B}$. Herein, we consider several different sets of values for $\{\lambda^{\pm,\alpha^0}, \lambda^{\pm,\beta^0}\}$.

Let us construct an equilibrium in this example. Notice that, in the present case, the Brownian motion W does not affect the jump intensities and, in turn, the agents' objectives, hence, the RBSDE system (3.41) becomes a system of reflected ODEs. We can solve it easily, using a simple Euler scheme, then, recover the value functions (V^a, V^b), as shown in Lemma III.19, and construct the bid and ask prices, (p^a, p^b), in the feedback form, as shown in Lemma III.18. We implement this strategy with the parameters chosen above, and with $\lambda^{+,\alpha_0} = 2.5$, $\lambda^{-,\alpha_0} = 1$, $\lambda^{+,\beta_0} = 1$, $\lambda^{-,\beta_0} = 2.5$ (so that the extremal ask agents are bullish whereas the extremal bid agents are bearish). The results are shown in the left part of Figure 3.1. Using the same parameters, we consider the book beyond the best bid and ask prices. In order to construct it, we solve the fixed-point problem (3.56) numerically. The latter is achieved by limiting the set of possible price levels for the limit orders to a finite set (i.e. to a partition of a large interval), which reduces (3.56) to a finite-dimensional fixed-point problem. In addition, we allow each agent to post a limit order at a single price level only, which further simplifies the problem.[8] Thus, we find a solution by the standard

[8]Note that this restriction does not compromise the optimality of the agents' actions, provided a fixed point can be found. Indeed, it is a well known phenomenon that, in a continuum-player game, an equilibrium with pure

recursive iteration, maximizing, at each step, the objective over a finite set. The resulting optimal limit order strategies of the agents (at time zero) are plotted in the right part of Figure 3.1, as a function of the agents' beliefs $\alpha \in \hat{\mathbb{A}} \cup \hat{\mathbb{B}}$. Notice that the optimal limit order strategy $p(\cdot)$ is piece-wise constant. It is worth mentioning that this discreteness seems to be inherent in the model and not just an artifact of the discretization of prices or beliefs that we chose herein. Indeed, Figure 3.1 was obtained with 500 different values of beliefs (i.e. $K = 500$) and with 1000 possible price levels, while the number of jumps of $p(\cdot)$ is clearly much smaller than any one of these numbers. In fact, we have repeated the computations, increasing both K and the number of possible price levels, and the results do not change. Naturally, the associated LOB is given by a finite combination of Dirac measures – it is shown in the left part of Figure 3.2.

Finally, we demonstrate how the proposed framework can be used to model the indirect market impact: i.e. how a change to the LOB may create a "feedback" effect and cause further changes to it. Note that the initial change may be triggered by a trade (which is the case in the classical literature on optimal execution), or by a new limit order. An extreme example of the latter is the so-called "spoofing" – i.e. posting a large limit order with the goal to make the price of the asset move in the opposite direction.[9] To the best of our knowledge, to date, there exists no model capable of explaining how exactly this activity causes the LOB (and, in particular, the price) to change. To model this process within the present example, we assume that $\{\lambda^{\pm,\alpha^0}, \lambda^{\pm,\beta^0}\}$ are, in fact, functions of a relevant market indicator, which we

controls also provides an equilibrium for a setting with distributed controls. This is, in fact, one of the advantages of the continuum-player games. We consider distributed controls only to prove that the equilibrium does exist, which is much harder (if at all possible) to show for a setting with pure controls.

[9] We stress that intentional spoofing is an illegal activity.

denote by I:

$$\lambda^{+,\alpha^0} = 2.3 \exp\left(Is\right), \quad \lambda^{-,\alpha^0} = 1 \exp\left(-Is\right),$$

$$\lambda^{+,\beta^0} = 1 \exp\left(Is\right), \quad \lambda^{-,\beta^0} = 2.3 \exp\left(-Is\right),$$

where $s = 2.6$ is the sensitivity. We further assume that I is the so-called *market imbalance*: the ratio of the size of all orders at the best bid over the size of all orders at the best ask, less one. It is a well known empirical fact (cf. [21]) that such an indicator has a predictive power for the direction of the next price move. Note that I is a function of the LOB, which, in turn, is an outcome of an equilibrium, in which I is the input. Strictly speaking, our results do not guarantee the existence of an equilibrium with this additional fixed-point constraint. Nevertheless, we can try to compute it numerically. For example, choosing the initial factor value $I = .0984456$, we obtain the equilibrium LOB (consistent with the chosen level of market imbalance I) shown in the right part of Figure 3.2 (at the top). Next, we add an extra limit buy order of size 0.05, located at the best bid price, to this LOB – as shown in the bottom graph in the right part of Figure 3.2. Clearly, the new LOB is no longer in equilibrium, hence, the agents will adjust their controls to reach a new equilibrium. Of course, in theory, this process happens instantaneously, and we simply observe the outcome of the new equilibrium. However, it is also very insightfull to see how the new equilibrium is reached, in a sequence of steps. At each step, we fix the value of I given by the imbalance of the LOB at the previous step, and compute the new LOB from the equilibrium, and so on. Figure 3.3 shows what happens to the LOB and to the functions (V^a, V^b) in the first five steps. We can see that the change in the factor makes the agents more bullish about the asset, and they tend to move their limit orders higher. In particular, the size of the best bid increases, while the size of the best ask decreases, further increasing the market imbalance. The left part of Figure 3.3 also shows that, starting from step three, the value functions V^a and V^b coincide at time zero, which means that the agents, in fact, choose to submit an internal market order, terminating the game. Thus, our model, in particular, shows how the market imbalance can be interpreted as a "self-fulfilling prophecy": the fact that the agents base their beliefs on the market imbalance, itself, implies that a sufficient increase in the market imbalance will, indeed, trigger a market buy order.

Of course, the analysis provided in this section is merely an example, which is meant to illustrate how our results can be used to model the changes in a LOB resulting from the changes in a relevant market indicator. In general, we do not need to limit ourselves to the market imbalance, but may consider other indicators: e.g. choosing the size and direction of the last trade as the relevant indicator, would allow one to model the "indirect" impact of a market order on the LOB (in addition to the obvious, direct, impact resulting from immediate execution of the limit orders). In our future research, we plan to find an appropriate model specification (including the choice of the most appropriate market factors), that is consistent with empirical findings, and to test the predictions of our model against the real market data.

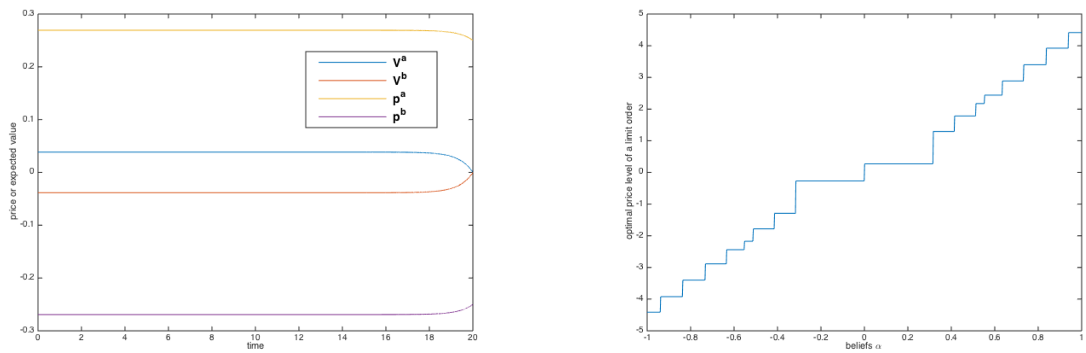


Figure 3.1: On the left: value functions (V^b, V^a) (red and blue), and the bid and ask prices (p^b, p^a) (purple and orange), as functions of time. On the right: the optimal price level of a limit order, as a function of the beliefs $\alpha \in \hat{\mathbb{A}} \cup \hat{\mathbb{B}}$. Parameters: $\lambda^{+,\alpha_0} = 2.5$, $\lambda^{-,\alpha_0} = 1$, $\lambda^{+,\beta_0} = 1$, $\lambda^{-,\beta_0} = 2.5$.

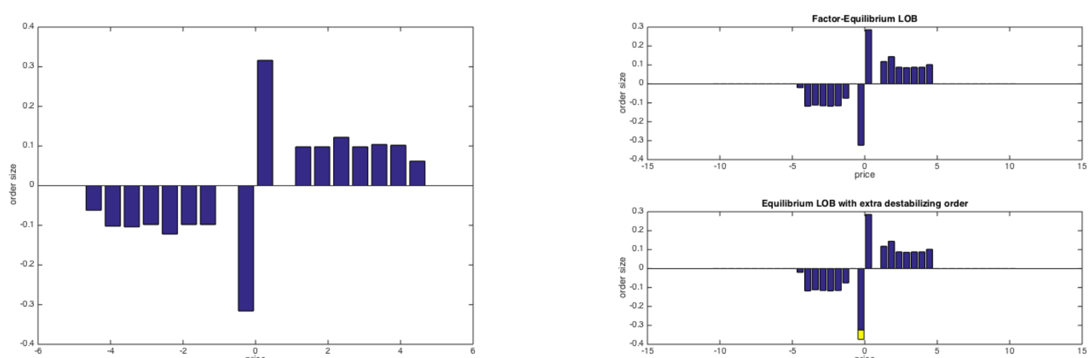


Figure 3.2: Left: LOB at time zero, with $\lambda^{+,\alpha_0} = 2.5$, $\lambda^{-,\alpha_0} = 1$, $\lambda^{+,\beta_0} = 1$, $\lambda^{-,\beta_0} = 2.5$. Right: equilibrium LOB at time zero, with the parameters depending on the market imbalance I (top), and the same LOB, with an additional (yellow) limit order (bottom).

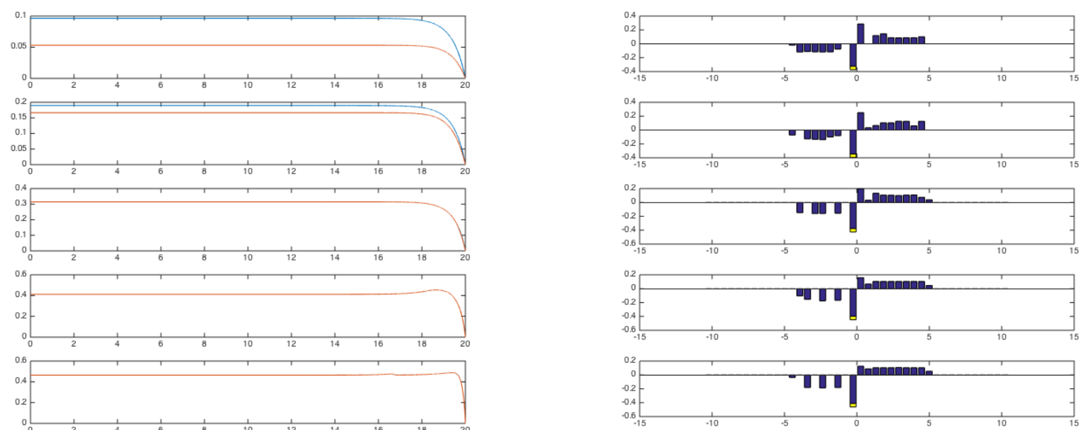


Figure 3.3: On the left: value functions (V^b, V^a) (red and blue), as functions of time. On the right: LOB at each step of the convergence to a new equilibrium.

CHAPTER IV

Effects of Tick Size

4.1 Introduction.

In this chapter we continue our investigation of agent-based approaches to modeling various phenomena in market microstructure initiated in chapters II, III. This time we focus on a continuous-time case with *discrete* prices, to model more realistically the actual markets which have a finite tick size. Unlike in those two chapters, we only consider a two-agent game with one long and one short agent. It should be understood that the equilibrium we construct for this game can be extended to an equilibrium for the continuum-player game with two clusters of equivalent agents on buy and sell sides, similarly to chapter III.

What we focus on here is instead the relevance of the location of the agents' estimate of the 'true price' relative to multiples of the tick size: this spatial inhomogeneity is important for understanding the clustering of market orders and, more generally, the non-uniformity of market impact across time. Consider the following as an example of a possible practical relevance of this spatial structure: if we think of the true price as an imbalance-weighted bid-ask price, it is well-known that for many 1-tick stocks the trade volume is higher on average when this true price is close to bid or ask. But this is exactly what our model would imply, as with realistic parameters equilibria it produces are such that the agents would join external investors in submitting market orders when the true price is close to a multiple of the tick size. Moreover, the spatial structure implied by the model depends on the potentially measurable model parameters (volatility, adverse selection level), suggesting some avenues for empirical investigations we plan to pursue in the future work.

The main challenges of this work are on the mathematical side. The game we consider reduces to a system of two Markovian control-stopping optimization problems, coupled through controls and stopping barriers which are discontinuous functionals (actually, floors and ceilings) of the other agent's value function. A fixed point problem which solving such system reduces to lacks continuity, contractivity or monotonicity properties, rendering it intractable by conventional methods. We are able to solve it for the case of a 1-dimensional Brownian motion true price, which allows us to utilize the geometric approach to (possibly rather irregular) linear diffusion stopping problems developed in [26],[25]. This approach, combined with the quasi-periodic structure of the problem arising from the discreteness of admissible order price levels, allows us to establish a sufficiently strong monotonicity property of agents' value functions in the case when the true price volatility is sufficiently high. This monotonicity implies the agents' value processes are sufficiently noisy, which means that the discontinuity of the coupling between the agents gets smoothed out. Along with a few other special properties of agents' value functions, this allows us to restrict the fixed point problem to a subset on which its continuity can actually be established.

The chapter is organized as follows. In section 4.2 we describe the game mechanics and introduce the main ingredients of the model. In section 4.3 we investigate the properties of the individual agent's value function given that the price controls of both agents and the other agent's value function are fixed. We first show its basic relative boundedness and quasi-periodicity properties. We then introduce the analytical machinery of second order ODEs related to linear diffusions, which together with the geometric approach to the linear diffusion optimal stopping mentioned above allows us to establish a sufficiently strong monotonicity of an agent's value function for all admissible price controls. The section ends with a proposition establishing the continuity of one agent's value function in the other agent's value function provided the latter is appropriately monotonous, relatively bounded and quasi-periodic. In section 4.4 we first address the control parts of agents' optimization problems: we introduce response control operators and show the price controls they produce are indeed optimal. Our situation is somewhat less regular than the one treated in standard references, so we have to exploit the special structure of the problem and introduce some additional tricks to show this optimality. We then show these response control operators are continuous in the appropriate topology and also show how the coupled optimization problem we are solving reduces to a certain fixed point problem. Finally we show the continuity of this fixed point problem and the existence of a fixed point of a special type, which implies the existence of a solution to our coupled optimization system.

4.2 Buyers-sellers game

Consider two agents, long and short, aiming to sell and buy, respectively, one unit of the asset.

Every agent can post a limit order (of the respective type – buy or sell) at a chosen price level, or submit a market order (of the same type). We denote the location of a limit order posted by the long agent at time t by p_t^a. Similarly, we denote the location of a limit order posted by the short agent at time t by p^b_t. Both processes take values in he set of integers $\mathbb{Z}$. We interpret their values as prices in multiples of tick size for the asset.

We denote the stopping time at which the long agent executes a market order by τ^a. Similarly, we denote the stopping time at which the long agent executes a market order by τ^b.

Each agent believes that the external investors arrive to the market according to a Poisson process N with intensity λ.

The **current estimated mean value** of the fundamental price, from the point of view of each agent, is given by the process X, with

$$X_t = X_0 + \sigma B_t,$$

where B is a Brownian motion, independent of N.

The processes (p^a, p^b) and the stopping times (τ^a, τ^b) are adapted to $\mathbb{F}^B$.

At any arrival time t of N, the value of the **fundamental price**, p^0_t, is determined by

$$p^0_t = X_t + \xi$$

where ξ is a random variable, independent of B, with mean 0 and cdf F.

An **external market buy** order is executed at time t if and only if t is an arrival time of N and $p^0_t \geq p^a_t$. Similarly, an **external market sell** order is executed at time t if and only if t is an arrival time of N and $p^0_t \leq p^b_t$.

If a long (short) agent submits an internal market sell (buy) order at time t, it is executed at $\lfloor V^b_t \rfloor$ (resp. $\lceil V^a_t \rceil$), where V^b, V^a are value processes for short and long agents respectively. Given our perfect information/perfect rationality setup, a sell agent would run away from a predictable buy market order if it doesn't improve on her value function, and would take it if it does, so the smallest price (which also should be an integer) at which the long agent would agree to trade is $\lceil V_t^a \rceil$. Hence, a short agent can execute her market buy order at this price level.

The game ends at the time of the first trade: when the first market order is executed. The agent, or agents, who participated in that trade receive/pay the price the trade was struck at. If the trade ended via an external market buy (resp. sell) order, so that only one of the agents actually traded, at some time τ, and $\xi_\tau = p^0_\tau - X_\tau$, then the short (resp. long) agent's remaining inventory is marked to $\lceil X_\tau + \alpha\xi_\tau \rceil$ (resp. $\lfloor X_\tau + \alpha\xi_\tau \rfloor$). The choice of $X_\tau + \alpha\xi_\tau$ (where $0 < \alpha < 1$ is the adverse selection parameter) is based on the following. The observed difference between the actual fundamental price p^0 and the mean fundamental price X is informative (to a degree controlled by α) for the subsequent estimate of the mean fundamental price, or, in other words, the external order flow affects linearly the agents' estimate of the mean fundamental price. Taking the ceiling and floors is meant to approximate the 'next round' p^a and p^b.

In this work, we consider a game with infinite time horizon and only allow for Markovian strategies, so that p^a_t, p^b_t are (time-independent) functions of X_t, and τ^a, τ^b are hitting times of X_t.

Then agents value processes are given by functions of X_t as well. It's not hard to show that if one of the agents is using such Markovian controls, it is possible for the other agent to choose optimal controls which are Markovian as well. (We do not claim that all equilibria of the game are Markovian.)

Similarly to chapter III, one can show that an equilibrium in the control-stopping game described above can be constructed by solving the following fixed point problem for the functions $(p^{a*}, p^{b*}, \bar{V}^a, \bar{V}^b)$:

$$\tag{4.1} \begin{cases} \bar{V}^a(x) = \sup_{p^a \in \mathcal{A}^a} \sup_\tau J^a\left(x, \tau, p^a, p^{b*}, \bar{V}^b\right), \\ \bar{V}^b(x) = \inf_{p^b \in \mathcal{A}^b} \inf_\tau J^b\left(x, \tau, p^{a*}, p^b, \bar{V}^a\right), \end{cases}$$

where the classes of admissible control functions, $\mathcal{A}^a$, $\mathcal{A}^b$, are defined below, in Assumption IV.2, and the optimal $p^{a*} \in \mathcal{A}^a$ and $p^{b*} \in \mathcal{A}^b$ satisfy

$$\sup_{p^a \in \mathcal{A}^a} \sup_\tau J^a\left(x, \tau, p^a, p^{b*}, \bar{V}^b\right) = \sup_\tau J^a\left(x, \tau, p^{a*}, p^{b*}, \bar{V}^b\right),$$
$$\inf_{p^b \in \mathcal{A}^b} \inf_\tau J^b\left(x, \tau, p^{a*}, p^b, \bar{V}^a\right) = \inf_\tau J^b\left(x, \tau, p^{a*}, p^{b*}, \bar{V}^a\right),$$

and

(4.2)

$$\begin{aligned} J^a\left(x, \tau, p^a, p^b, v\right) = \mathbb{E}^x\Big[\int_0^\tau \exp\left(-\int_0^t c(p^a(X_s), p^b(X_s), X_s)\mathrm{d}s\right) g^a(p^a(X_t), p^b(X_t), X_t)\mathrm{d}t \\ + \exp\left(-\int_0^\tau c(p^a(X_s), p^b(X_s), X_s)\mathrm{d}s\right) \lfloor v(X_\tau)\rfloor \Big], \end{aligned}$$

(4.3)

$$\begin{aligned} J^b\left(x, \tau, p^a, p^b, v\right) = \mathbb{E}^x\Big[\int_0^\tau \exp\left(-\int_0^t c(p^a(X_s), p^b(X_s), X_s)\mathrm{d}s\right) g^b(p^a(X_t), p^b(X_t), X_t)\mathrm{d}t \\ + \exp\left(-\int_0^\tau c(p^a(X_s), p^b(X_s), X_s)\mathrm{d}s\right) \lceil v(X_\tau)\rceil \Big] \end{aligned}$$

with $\mathbb{E}^x[\cdot] = \mathbb{E}\left[\cdot|X_0 = x\right]$ and

$$\tag{4.4} c(p^a, p^b, x) = \lambda\left((1 - F\left(p^a - x\right)) + F\left(p^b - x\right)\right),$$

$$\tag{4.5} g^a(p^a, p^b, x) = \lambda\left(p^a\left(1 - F\left(p^a - x\right)\right) + \mathcal{F}^b(p^b, x)\right),$$

$$\tag{4.6} g^b(p^a, p^b, x) = \lambda\left(p^b F\left(p^b - x\right) + \mathcal{F}^a(p^a, x),\right)$$

$$\tag{4.7} \mathcal{F}^b(p^b, x) = \int_{-\infty}^{p^b - x} \lfloor x + \alpha y\rfloor \mathrm{d}F(y),$$

$$\tag{4.8} \mathcal{F}^a(p^a, x) = \int_{p^a - x}^{\infty} \lceil x + \alpha y\rceil \mathrm{d}F(y).$$

4.3 Agents' optimal stopping problems

In this section, we introduce the assumptions and derive a number of properties of the agents' objective functions. In the subsequent section, we use these results to establish the existence of a solution to the target fixed-point problem, and, in turn, construct an equilibrium of the associated control-stopping game.

First we introduce the notion of admissible prices. Unfortunately, we have to constrain agents' choice a bit a priori, namely, we have to assume agents bid and ask prices are always such that the order execution rate at either side is bounded away from zero. It is necessary to have this undegenerate discounting for our analytical machinery to work, and we couldn't find a way to obtain this property endogenously in sufficient generality. In all the realistic examples we investigated this constraint is not binding. More specifically, define

$$\mathcal{A}^a_0 = \{x|1 - F(x) \geq \frac{c_l}{2\lambda}\}, \quad \mathcal{A}^b_0 = \{x|F(x) \geq \frac{c_l}{2\lambda}\}$$

$$\mathcal{A}^a(x) = x + \mathcal{A}^a_0, \quad \mathcal{A}^b(x) = x + \mathcal{A}^b_0$$

For any $c_l > 0$ we define

Definition IV.1. The control $p^a(x)$ $(p^b(x))$ is admissible if it is a measurable function and

$$p^a(x) \in \mathcal{A}^a(x) \cap \mathbb{Z},\ \forall x \quad \big(\text{resp.} p^b(x) \in \mathcal{A}^b(x) \cap \mathbb{Z},\ \forall x\big)$$

and there exists at least one p^b (resp. p^a) with the above property such that $p^a \geq p^b$.

One can check admissibility of p^a, p^b implies in particular that

$$\|p^a(x) - x\|_\infty \leq C, \quad \left\|p^b(x) - x\right\|_\infty \leq C$$

for some $C > 0$, where $\|\cdot\|_\infty$ denotes $\mathbb{L}^\infty$ norm. It also implies

$$c(p^a(x), p^b(x), x) \geq c_l, \quad \forall x$$

Also, from the definition of c, and as λ is fixed throughout the chapter, we get

$$c(p^a(x), p^b(x), x) \leq c_u = 2\lambda > 0 \tag{4.9}$$

We denote by $\mathcal{A}^a$ ($\mathcal{A}^b$) the sets of all admissible p^a (resp. p^b). These sets might be empty for all $c_l > 0$ if the support of ξ is too narrow, so we need to make the following

Assumption IV.2. *There exists $c_l > 0$ so that the corresponding $\mathcal{A}^a$, $\mathcal{A}^b$ are non-empty.*

We fix such c_l for the rest of the chapter.

Another property we are going to use a lot is the following

Definition IV.3. We say a measurable function f is C-*close to* x, where $C > 0$ is a constant, if

$$\|f(x) - x\|_\infty \leq C$$

We call a barrier $v(x)$ admissible if it is measurable and is C-close to x.

Consider

$$V^a\left(x, p^a, p^b, v\right) := \sup_\tau J^a\left(x, \tau, p^a, p^b, v\right), \tag{4.10}$$

$$V^b\left(x, p^a, p^b, v\right) := \inf_\tau J^b\left(x, \tau, p^a, p^b, v\right). \tag{4.11}$$

It's easy to see these are well-defined and C-close to x for some C for all admissible controls p^a, p^b and admissible barriers v.

Note that p^a, p^b, $\bar{V}^a$, $\bar{V}^b$ and x are measured in ticks, and only the relative measurements are interpretable, not the absolute numbers, so the sensible equilibria should satisfy

(4.12)

$$\bar{V}^a(x+1) = \bar{V}^a(x)+1,\ \bar{V}^b(x+1) = \bar{V}^b(x)+1,\ p^{a*}(x+1) = p^{a*}(x)+1,\ p^{b*}(x+1) = p^{b*}(x)+1$$

We will use this property a lot in what follows so we introduce a term for it:

Definition IV.4. We say a function $f(x)$ has 1-shift property if

$$f(x+1) = f(x) + 1, \quad \forall x \in \mathbb{R}$$

An important first step toward establishing the existence of an equilibrium of (4.1) in the class of functions which have 1-shift property and are C-close to x would be to check these properties are preserved by individual agents' optimization functionals $V^a(\cdots)$, $V^b(\cdots)$:

Lemma IV.5. *If admissible barriers v^a, v^b and admissible controls p^a, p^b have 1-shift property then so do $V^a(x, p^a, p^b, v^b)$ and $V^b(x, p^a, p^b, v^a)$. Additionally, we have*

$$\begin{aligned}
c(x+1, p^a+1, p^b+1) &= c(x, p^a, p^b) \\
\frac{g^a}{c}(x+1, p^a+1, p^b+1) &= \frac{g^a}{c}(x, p^a, p^b) + 1, \\
\frac{g^b}{c}(x+1, p^a+1, p^b+1) &= \frac{g^b}{c}(x, p^a, p^b) + 1,
\end{aligned}$$

for all x, p^a, p^b, which means that $c(x, p^a(x), p^b(x))$ is 1-periodic and $\frac{g^a}{c}(x, p^a(x), p^b(x))$, $\frac{g^b}{c}(x, p^a(x), p^b(x))$ have the 1-shift property.

Proof: Immediate after rewriting the objective in the form (4.14). ∎

In what follows we will often suppress the dependence on $p^a(x)$, $p^b(x)$ from notation to avoid clutter, and denote

$$\begin{aligned}
c(x) &= c_p(x) = c(p^a(x), p^b(x), x) \\
g^a(x) &= g^a_p(x) = g^a(p^a(x), p^b(x), x) \\
g^b(x) &= g^b_p(x) = g^b(p^a(x), p^b(x), x) \\
\mathcal{F}^b(x) &= \mathcal{F}^b_p(x) = \int_{-\infty}^{p^b(x)-x} \lfloor x + \alpha y \rfloor \mathrm{d}F(y) \\
\mathcal{F}^a(x) &= \mathcal{F}^a_p(x) = \int_{p^a(x)-x}^{\infty} \lceil x + \alpha y \rceil \mathrm{d}F(y)
\end{aligned} \tag{4.13}$$

where we use the subscript p when we want to emphasize the dependence of certain coefficients on $p^a(x)$, $p^b(x)$.

Lemma IV.6. *If admissible barriers v^a, v^b, admissible controls p^a, p^b are such that v^a, v^b, $\frac{g^a}{c}$, $\frac{g^b}{c}$ are C-close to x then so are $V^a(x, p^a, p^b, v^b)$ and $V^b(x, p^a, p^b, v^a)$.*

Proof: We will now show the claim for $V^a(\cdots)$, the one for $V^b(\cdots)$ being analogous. From the definitions (4.10) and (4.2) we get

$$
\begin{aligned}
(4.14)\quad & V^a(x, p^a, p^b, v^b) - x = \\
& \sup_\tau \mathbb{E}^x \left[\int_0^\tau \exp\left(-\int_0^t c(X_s)\mathrm{d}s \right) g^a(X_t)\mathrm{d}t + \exp\left(-\int_0^\tau c(X_s)\mathrm{d}s \right) \lfloor v^b \rfloor \right] - x = \\
& \qquad \sup_\tau \mathbb{E}^x \Big[\int_0^\tau \frac{g^a}{c}(x) - x \,\mathrm{d}\left(-\exp\left(-\int_0^t c(X_s)\mathrm{d}s \right) \right) + \\
& \qquad\qquad \exp\left(-\int_0^\tau c(X_s)\mathrm{d}s \right) \left(\lfloor v^b \rfloor - x \rfloor \right) \Big]
\end{aligned}
$$

To get the upper bound, note the last expression in the above is

$$
\leq \sup_\tau \mathbb{E}^x \left[\int_0^\tau C\mathrm{d}\left(-\exp\left(-\int_0^t c(X_s)\mathrm{d}s \right) \right) + \exp\left(-\int_0^\tau c(X_s)\mathrm{d}s \right) C \right] = C,
$$

where makes use of $\frac{g^a}{c} - x \leq C$, $\lfloor v^b \rfloor - x \leq v^b - x \leq C$, by assumption. To get the lower bound, note the same expression also is

$$
\begin{aligned}
& \geq \mathbb{E}^x \left[\int_0^\infty \frac{g^a}{c}(x) - x\mathrm{d}\left(-\exp\left(-\int_0^t c(X_s)\mathrm{d}s \right) \right) \right] \geq \\
& \qquad \mathbb{E}^x \left[\int_0^\infty -C\mathrm{d}\left(-\exp\left(-\int_0^t c(X_s)\mathrm{d}s \right) \right) \right] \geq -C
\end{aligned}
$$

where we used $\tau = \infty$. ∎

In what follows, we analyze $V^a(x, p^a, p^b, v^b)$ ($V^b(x, p^a, p^b, v^a$ being analogous) more closely, in particular establishing its monotonicity in x and continuity in v^b (resp. v^a), under appropriate conditions. Throughout this analysis, we think of p^a and p^b as fixed functions of x, while we vary x and $v^b(v^a)$. We are going to make heavy use of the well-known connection between linear diffusions and certain second-order ODEs. Our discounting and running cost functions are a bit less regular (measurable and locally bounded, but not continuous) than is commonly assumed in the literature, so a modicum of care is required in making this connection rigorous.

First, as in [38], define, for given $p^a(x)$, $p^b(x)$

$$\psi(x) = \psi_p(x) = \begin{cases} \mathbb{E}^x \left[\exp\left(-\int_0^{\tau_0} c_p(X_s)\mathrm{d}s\right)\right], \ x \leq 0 \\ \mathbb{E}^0 \left[\exp\left(-\int_0^{\tau_x} c_p(X_s)\mathrm{d}s\right)\right]^{-1}, \ x > 0 \end{cases} \tag{4.15}$$

and

$$\phi(x) = \phi_p(x) = \begin{cases} \mathbb{E}^x \left[\exp\left(-\int_0^{\tau_0} c_p(X_s)\mathrm{d}s\right)\right], \ x > 0 \\ \mathbb{E}^0 \left[\exp\left(-\int_0^{\tau_x} c_p(X_s)\mathrm{d}s\right)\right]^{-1}, \ x \leq 0 \end{cases} \tag{4.16}$$

Clearly $\psi(0) = \phi(0) = 1$, ψ is strictly increasing, ϕ is strictly decreasing. The results from [38] (and the absolute continuity of the killing measure for the diffusion corresponding to our discounted problem) imply $f = \phi$ or $f = \psi$ has right derivative, f^+, everywhere and satisfies

$$\frac{2}{\sigma^2} \int_{(a,b]} c(x) f(x) \mathrm{d}x = f^+(b) - f^+(a)$$

for all $b > a$. Passing to the limit $b \downarrow a$ or $a \uparrow b$ shows that f^+ is continuous. One can also show the following.

Lemma IV.7. *If f is continuous and has continuous right derivative on $[a, b]$ then $f \in C^1(a, b)$.*

The (elementary) proof of this fact can be found in the appendix. Thus, the equation for $f = \phi$ or $f = \psi$ can be rewritten as

$$\frac{2}{\sigma^2} \int_{(a,b]} c(x) f(x) \mathrm{d}x = f'(b) - f'(a) \tag{4.17}$$

As $c \in \mathbb{L}^\infty(\mathbb{R})$ and $f \in C(\mathbb{R})$, f_{xx} exists a.e. and satisfies

$$\frac{\sigma^2}{2} f_{xx} - cf = 0, \text{ a.e.}$$

for $f = \psi$ or $f = \phi$, and, in particular, $f \in \mathbb{W}^{2,loc}$.

Next, define

$$f^a(x) = f^a_p(x) = \frac{2}{\sigma^2 W}\left(\phi(x)\int_{-\infty}^x \psi(y)g^a(y)\mathrm{d}y + \psi(x)\int_x^\infty \phi(y)g^a(y)\mathrm{d}y\right) \tag{4.18}$$

$$f^b(x) = f^b_p(x) = -\frac{2}{\sigma^2 W}\left(\phi(x)\int_{-\infty}^x \psi(y)g^b(y)\mathrm{d}y + \psi(x)\int_x^\infty \phi(y)g^b(y)\mathrm{d}y\right) \tag{4.19}$$

where the Wronskian $W = \psi'(x)\phi(x) - \phi'(x)\psi(x)$ is actually independent of x and positive. Using the fact that $\phi(x)\int_{-\infty}^x \psi(y)g^a(y)\mathrm{d}y$ (and other similar terms) is equal to the integral of its (a.e. defined) derivative as a product of two absolutely-continuous (on bounded intervals) functions, we get that f^a is an integral of a continuous function

$$\frac{2}{\sigma^2 W}\left(\phi'(x)\int_{-\infty}^x \psi(y)g^a(y)\mathrm{d}y + \psi'(x)\int_x^\infty \phi(y)g^a(y)\mathrm{d}y\right)$$

and so has a continuous derivative, which is moreover a.e. differentiable (as ϕ', ψ' and the integrals terms are), and furthermore

$$\frac{\sigma^2}{2} f^a_{xx} - cf^a = -g^a, \text{ a.e.} \tag{4.20}$$

Similar claims hold for f^b. This implies in particular that f^a, f^b are in $\mathbb{W}^{2,loc}$. Applying Dynkin's formula together with some easy to get asymptotics to $\exp\left(-\int_0^t c(X_s)\mathrm{d}s\right) f^a(X_t)$ and passing to the limit over a sequence of increasing to infinity stopping times one can further get the following probabilistic representation

$$f^a(x) = \mathbb{E}^x\left[\int_0^\infty \exp\left(-\int_0^t c(X_s)\mathrm{d}s\right) g^a(X_t)\mathrm{d}t\right] \tag{4.21}$$

and similarly for f^b.

We'll also make use of the following elementary bounds on ϕ, ψ:

Lemma IV.8. *Let $c_l > 0$ ($c_u > 0$) be the lower (resp. upper) bound of $c(x)$. Then for all x*

$$\psi(x) \leq \exp\left(\sqrt{\frac{2c_l}{\sigma^2}}x\right) \vee \exp\left(\sqrt{\frac{2c_u}{\sigma^2}}x\right)$$

and

$$\phi(x) \leq \exp\left(-\sqrt{\frac{2c_l}{\sigma^2}}x\right) \vee \exp\left(-\sqrt{\frac{2c_u}{\sigma^2}}x\right)$$

See appendix for the proof.

Next, these f^a, f^b inherit the C-closeness to x and 1-shift property from prices:

Lemma IV.9. *If admissible p^a, p^b are such that $\frac{g^a}{c}$, $\frac{g^b}{c}$ are C-close to x then so are $f^a(x)$, $f^b(x)$.*

Proof: The claim about f^a follows from

$$f^a(x) - x = \mathbb{E}^x\left[\int_0^\infty \left(\frac{g^a}{c}(X_t) - X_t\right) \mathrm{d}\left(\exp\left(-\int_0^t c(X_s)\mathrm{d}s\right)\right)\right] \tag{4.22}$$

and similarly for the one about f^b. ∎

Lemma IV.10. *If admissible p^a, p^b have 1-shift property then so do f^a, f^b.*

Proof: Follows from the representation (4.22) and lemma IV.5. ∎

More interestingly, the 1-shift property of p^a, p^b implies f^a, f^b approach $f_0(x) = x$ *in* C^1 in fast-diffusion limit, and admit two-sided derivative bounds as long as the diffusion coefficient is nonzero:

Proposition IV.11. *If admissible p^a, p^b have 1-shift property, and are such that $\frac{g^a}{c}$, $\frac{g^b}{c}$ are C-close to x, then for $f = f^a, f^b$ we have*

$$1 - w \leq f'(x) \leq 1 + w \tag{4.23}$$

where the constant w depends only on C, c_l, c_u, σ and satisfies

$$w(\sigma) \to 0 \ \mathit{as}\, \sigma \to \infty. \tag{4.24}$$

Proof: We'll only show the upper bound on the derivative for f^a only, the proof of other parts being analogous. Differentiating the representation (4.18) we get

$$(f^a)'(x) = \frac{|\phi'(x)|\psi'(x)}{W}\Big(\int_x^\infty \frac{g^a}{c}(y)\phi(y)c(y)\frac{2}{\sigma^2|\phi'(x)|}\mathrm{d}y - \int_{-\infty}^x \frac{g^a}{c}(y)\psi(y)c(y)\frac{2}{\sigma^2\psi'(x)}\mathrm{d}y\Big) \tag{4.25}$$

As our diffusion killed at the rate $c(x) - c_l$ has $\pm\infty$ as natural boundary points, results of [38] imply $\phi'(-\infty) = 0$, $\psi'(\infty) = 0$, so passing to the appropriate limits in (4.17) we get

$$\psi'(x) = \frac{2}{\sigma^2}\int_{-\infty}^x \psi(y)c(y)\mathrm{d}y \tag{4.26}$$

and

$$\phi'(x) = -\frac{2}{\sigma^2}\int_x^\infty \phi(y)c(y)\mathrm{d}y \tag{4.27}$$

From the representations above we see $\phi(y)c(y)\frac{2}{\sigma^2|\phi'(x)|}$ is a density on $[x,\infty)$ and $\psi(y)c(y)\frac{2}{\sigma^2\psi'(x)}$ is a density on $(-\infty, x]$. Using this and $\frac{g^a}{c}(x) \leq x + C$ we get

$$(f^a)'(x) \leq 2C\frac{|\phi'(x)|\psi'(x)}{W} + (f_0^a)'(x)$$

where $\tilde{f}^a(x)$ is 'the f^a' corresponding to $\tilde{g}^a(x) = xc(x)$, which is the unique solution of

$$\frac{\sigma^2}{2}f''(x) - c(x)f(x) = -xc(x)$$

which is easily seen to be given by $\tilde{f}^a(x) = x$, hence $\left(\tilde{f}^a\right)' = 1$ and we get

$$(f^a)'(x) \leq 1 + 2C\frac{|\phi'(x)|\psi'(x)}{W}$$

Thus to show the claim we only need to establish the appropriate bound on the last summand above.

To do this, first note $c(x+1) = c(x)$ implies $\phi(x+1) = \gamma\phi(x)$,$\psi(x+1) = \frac{1}{\gamma}\psi(x)$, where $0 < \gamma = \phi(1) < 1$ and so $|\phi'(x)|\psi'(x)$ is 1-periodic, so it is sufficient to bound $2C\frac{|\phi'(x)|\psi'(x)}{W}$ only for $x \in [0,1]$. As

$$|\phi'(x)| = \frac{2}{\sigma^2}\int_x^\infty c(y)\phi(y)\mathrm{d}y$$

For $x \geq 0$, using lemma IV.8 and the bounds on $c(x)$, we further get that the above expression is

$$\leq \frac{2c_u}{\sigma^2}\int_x^\infty \exp\left(-\sqrt{\frac{2c_l}{\sigma^2}}y\right)\mathrm{d}y = \frac{2c_u}{\sigma\sqrt{2c_l}}\exp\left(-\sqrt{\frac{2c_l}{\sigma^2}}x\right)$$

Combined with a similar estimate from below this gives

$$\frac{2c_l}{\sigma\sqrt{2c_u}}\exp\left(-\sqrt{\frac{2c_u}{\sigma^2}}x\right) \leq |\phi'(x)| \leq \frac{2c_u}{\sigma\sqrt{2c_l}}\exp\left(-\sqrt{\frac{2c_l}{\sigma^2}}x\right) \tag{4.28}$$

for $x \geq 0$. Similarly, for $x \leq 0$ we get

$$\frac{2c_l}{\sigma\sqrt{2c_u}}\exp\left(\sqrt{\frac{2c_u}{\sigma^2}}x\right) \leq |\psi'(x)| \leq \frac{2c_u}{\sigma\sqrt{2c_l}}\exp\left(\sqrt{\frac{2c_l}{\sigma^2}}x\right)$$

which using $\psi'(x+1) = \frac{1}{\gamma}\psi'(x)$ gives

$$\frac{1}{\gamma}\frac{2c_l}{\sigma\sqrt{2c_u}}\exp\left(\sqrt{\frac{2c_u}{\sigma^2}}(x-1)\right) \leq |\psi'(x)| \leq \frac{1}{\gamma}\frac{2c_u}{\sigma\sqrt{2c_l}}\exp\left(\sqrt{\frac{2c_l}{\sigma^2}}(x-1)\right) \tag{4.29}$$

for $x \in [0,1]$. Combining the above and replacing exp-terms with their worst-case bounds we get

$$|\phi'(x)|\psi'(x) \leq \frac{1}{\gamma}\left(\frac{2c_u}{\sigma\sqrt{2c_l}}\right)^2$$

Note on $[0,1]$ $1 \geq \phi(x) \geq \gamma$, $\frac{1}{\gamma} \geq \psi(x) \geq 1$, which together with $|\phi'|$, ψ' estimates from above gives

$$W = \phi\psi' + \psi|\phi'| \geq \gamma\cdot\left(\frac{1}{\gamma}\frac{2c_l}{\sigma\sqrt{2c_u}}\exp\left(\sqrt{\frac{2c_u}{\sigma^2}}(x-1)\right)\right) + 1\cdot\frac{2c_l}{\sigma\sqrt{2c_u}}\exp\left(-\sqrt{\frac{2c_u}{\sigma^2}}x\right) \geq \frac{4c_l}{\sigma\sqrt{2c_u}}\exp\left(-\sqrt{\frac{2c_u}{\sigma^2}}\right) \tag{4.30}$$

and so

$$\frac{|\phi'(x)|\psi'(x)}{W} \leq \frac{1}{\sigma}\frac{1}{\gamma}\left(\frac{2c_u}{\sqrt{2c_l}}\right)^2 \frac{\sqrt{2c_u}}{4c_l} \exp\left(\frac{\sqrt{2c_u}-\sqrt{2c_l}}{\sigma}\right)$$

Note also that as $\frac{1}{\gamma} = \psi(1)$ the bound on ψ from lemma IV.8 implies

$$\frac{1}{\gamma} \leq \exp\left(\sqrt{\frac{2c_u}{\sigma^2}}\right)$$

This together with the previous expression implies the existence of the desired upper bound on $2C\frac{|\phi'(x)|\psi'(x)}{W}$ and hence on $(f^a)'(x)$, which goes to 0 as $\frac{1}{\sigma}$ when $\sigma \to \infty$. ∎

Define

$$\mathrm{F}(x) = \mathrm{F}_p(x) = \frac{\psi(x)}{\phi(x)} \tag{4.31}$$

and

$$\widehat{\cdot}\colon h \to \widehat{h}(y) = \frac{h}{\phi}(\mathrm{F}^{-1}(y)) \tag{4.32}$$

where $\widehat{h}$ is defined on $(0,\infty)$. Then for given p^a, p^b, v^a, v^b we have the following description of the value function of the individual stopping problem.

Proposition IV.12. *For any admissible controls p^a, p^b and admissible barriers v^a, v^b, the individual agents' value functions $V^a(x) = V^a(x,p^a,p^b,v^b)$, $V^b(x) = V^b(x,p^a,p^b,v^a)$ are uniquely determined by*

$$\begin{aligned} \hat{V}^a(y) &= \mathrm{mcm}\,(\widehat{J^b - f^a})(y) + \hat{f}^a(y) \\ \hat{V}^b(y) &= -\,\mathrm{mcm}\,(\widehat{f^b - J^a})(y) + \hat{f}^b(y) \end{aligned} \tag{4.33}$$

where $\mathrm{mcm}(f)$ *denotes the smallest nonnegative concave majorant of a function f.*

Proof: We only prove the claim for V^a. First, note

$$
\begin{aligned}
&J^a(\tau, x, p^a, p^b, v^b) = \\
&\quad \mathbb{E}^x \left[\int_0^\tau \exp\left(- \int_0^t c(X_s)\mathrm{d}s \right) g^a(X_t)\mathrm{d}t + \exp\left(- \int_0^\tau c(X_s)\mathrm{d}s \right) \lfloor v^b(X_\tau) \rfloor \right] = \\
&\qquad \mathbb{E}^x \Big[\int_0^\infty \exp\left(- \int_0^t c(X_s)\mathrm{d}s \right) g^a(X_t)\mathrm{d}t + \\
&\exp\left(- \int_0^\tau c(X_s)\mathrm{d}s \right) \left(\lfloor v^b(X_\tau) \rfloor - \int_0^\infty \exp\left(- \int_0^t c(X_{\tau+s})\mathrm{d}s \right) g^a(X_{\tau+t})\mathrm{d}t \right) \Big] = \\
&\qquad f^a(x) + \mathbb{E}^x \left[\exp\left(- \int_0^\tau c(X_s)\mathrm{d}s \right) \left(\lfloor v^b(X_\tau) \rfloor - f^a(X_\tau) \right) \right]
\end{aligned}
$$

hence

$$
V^a(x, p^a, p^b, v^b) = f^a(x) + \sup_\tau \mathbb{E}^x \left[\exp\left(- \int_0^\tau c(X_s)\mathrm{d}s \right) \left(\lfloor v^b(X_\tau) \rfloor - f^a(X_\tau) \right) \right]
$$

Given $\lfloor v^b \rfloor(x) - f^a(x)$ is measurable and locally bounded, the last term above (i.e., the value function of a pure stopping problem (with discounting)) has the claimed mcm-characterization by Proposition 3.4 from [25]. ∎

For the use in the proofs below, we'd also need a y-domain version of the shift-properties from Lemma IV.10.

Lemma IV.13. *Given admissible p^a, p^b which have 1-shift property, and ϕ, ψ defined as in (4.16,4.15), we get*

$$
\phi(x+1) = \gamma\phi(x), \;\; \psi(x+1) = \frac{1}{\gamma}\psi(x) \tag{4.34}
$$

where $\gamma = \phi(1)$, $0 < \gamma < 1$.

Furthermore

$$
\mathrm{F}(x+1) = \frac{1}{\gamma^2}\mathrm{F}(x), \;\; \hat{H}\left(\frac{y}{\gamma^2}\right) = \frac{1}{\gamma}\hat{H} \tag{4.35}
$$

for any 1-periodic $H(x)$. In particular this can be applied to $H = v^a - f^a, f^b - v^b$, if v^a, v^b have 1-shift property, by Lemma IV.10.

The next proposition is the most important one in this subsection, and it is the key to our proof of existence of a Markovian equilibrium for sufficiently large σ. Without the monotonicity established therein, the fixed point problem we need to solve appears to be too discontinuous to be tractable.

Proposition IV.14. *Consider admissible controls p^a, p^b, admissible barriers v^a, v^b which have 1-shift property and such that $\lceil v^a \rceil$, $\lfloor v^b \rfloor$, $\frac{g^a}{c}$, $\frac{g^b}{c}$ are C-close to x. Then, $V(x) = V^a(x, p^a, p^b, v^b), V^b(x, p^a, p^b, v^a)$ is absolutely continuous and its derivative satisfies:*

$$|V'(x) - 1| \leq w, \;\; a.e. \tag{4.36}$$

with $w(\sigma) \to 0$, as $\sigma \to \infty$, uniformly in p^a, p^b, v^a, v^b with the properties above. In particular, there exists $\epsilon > 0$, s.t.

$$V'(x) \geq \epsilon, \;\; a.e.$$

for all σ large enough.

Proof: We only prove the lower bound on the derivative of V^a, other parts being similar. Note that Proposition IV.12 implies

$$V^a(x) = f^a(x) + \phi(x)\,\mathrm{mcm}\left(\widehat{\lfloor v^b \rfloor - f^a}\right)(\mathrm{F}(x))$$

As $f^a, \phi, \mathrm{F} \in C^1(\mathbb{R})$, and the mcm above is absolutely continuous as it is concave, then so is V^a, and for its (a.e. defined) derivative we have

$$(V^a)'(x) = (f^a)'(x) + \phi'(x)\,\mathrm{mcm}\left(\widehat{\lfloor v^b \rfloor - f^a}\right)(\mathrm{F}(x)) + \phi(x)\mathrm{F}'(x)\,\mathrm{mcm}\left(\widehat{\lfloor v^b \rfloor - f^a}\right)'(\mathrm{F}(x))$$

From Proposition IV.11, we get $(f^a)'(x) \geq 1 - w$ with w as in the statement of the proposition, so we only need to show

$$(V^a)'(x) - (f^a)'(x) = \phi'(x)\,\mathrm{mcm}\left(\widehat{\lfloor v^b \rfloor - f^a}\right)(\mathrm{F}(x)) + \phi(x)\mathrm{F}'(x)\,\mathrm{mcm}\left(\widehat{\lfloor v^b \rfloor - f^a}\right)'(\mathrm{F}(x)) \geq -\tilde{w}(\sigma) \tag{4.37}$$

with $\tilde{w}$ having appropriate asymptotic properties. As, by Lemma IV.5, V^a is 1-periodic, it is sufficient to only consider x on any bounded interval of length at least 1 and not on the entire real line. To simplify the notation a bit, we denote

$$h(y) = \mathrm{mcm}\left(\widehat{\lfloor v^b \rfloor - f^a}\right)(y)$$

Note that the assumed 1-shift property of v^b and f^a, via Lemma IV.10, imply that $\lfloor v^b \rfloor - f^a$ is 1-periodic, which by Lemma IV.13 implies that

$$h_0(y) = \left(\widehat{\lfloor v^b \rfloor - f^a}\right)(y)$$

satisfies $h_0(\frac{y}{\gamma^2}) = \frac{1}{\gamma} h_0(y)$. It can be easily checked that this property passes on to its minimal concave majorant $h(y)$. Define

$$\bar{\phi}(y) = \widehat{\phi^2}(y) = \phi(\mathrm{F}^{-1}(y))$$

It's easy to check $\left(\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} - c(x)\right)(\phi^2) > 0$ and so $\bar{\phi}(y)$ is convex by the following lemma which can be proven by a straightforward calculation.

Lemma IV.15. *Let $H \in \mathbb{W}^{2,loc}$ then $\widehat{H}(y)$ is convex (resp. concave) on (y_1, y_2), $y_i = \mathrm{F}(x_i)$, if*

$$\frac{\sigma^2}{2} H_{xx} - cH > 0, \ (\textit{resp.} \ < 0) \ \textit{a.e. on} \ (x_1, x_2)$$

Furthermore, $\bar{\phi}$ is decreasing and satisfies $\bar{\phi}(\frac{y}{\gamma^2}) = \gamma\bar{\phi}(y)$. Let us define

$$\tilde{c} = \sup_{y \in [1, \frac{1}{\gamma^2}]} h_0(y)\bar{\phi}(y)$$

Note that, as $(h_0\bar{\phi})(\frac{1}{\gamma^2}y) = (h_0\bar{\phi})(y)$, we get

$$h_0(y) \leq \frac{\tilde{c}}{\bar{\phi}(y)},$$

for all y (and not just $y \in [1, \frac{1}{\gamma^2}]$), as follows from the definition of $\tilde{c}$.

Next, suppose that $\tilde{c} \leq 0$. We will show that, in this case, $h(y) \equiv 0$, and $\tilde{w} = 0$ gives the desired lower bound for (4.37), and thus there is nothing left to prove. Indeed, the constant function 0 is a concave majorant of h_0 in this case. If the actual mcm of h_0 was different we'd be able to find $y > 0$ s.t. $h(y) = z < 0$. But then as $h(\frac{y}{\gamma^2}) = \frac{1}{\gamma}h(y)$ all the points $(\frac{y}{\gamma^{2k}}, \frac{z}{\gamma^k})$ for all integer k will also lie on the graph of h. But one can check the slope between two consecutive such points increases if $z < 0$ contradicting the concavity of h.

Having dealt with the simpler $\tilde{c} \leq 0$ case we assume $\tilde{c} > 0$ for the rest of the proof. As

$$h_0(y)\bar{\phi}(y) = \left(\lfloor v^b \rfloor - f^a\right)\left(\mathrm{F}^{-1}(y)\right)$$

and $\lfloor v^b \rfloor$, f^a (see Lemma IV.9) are $(C+1)$-close to x , we get $\tilde{c} \leq 2C+2$. . Moreover $1/\bar{\phi}(y) = \hat{1}$ (that is, the $\hat{\cdot}$-transform applied to a constant function 1) is concave by the previous lemma as

$$\left(\frac{\sigma^2}{2}\frac{d^2}{dx^2} - c(x)\right)(1) < 0$$

and so $\tilde{c}/\bar{\phi}(y)$ is actually a concave majorant of $h_0(y)$ which implies

$$h(y) \leq \frac{\tilde{c}}{\bar{\phi}(y)}, \quad \forall y \in (0, \infty)$$

From the definition of $\tilde{c}$ we can find a sequence of points $\{y_i\}$ on $[1, \frac{1}{\gamma^2}]$ s.t. $(h_0\bar{\phi})(y_i) \to \tilde{c}$. Let y_* be any concentration point of that sequence. Then, from the continuity of the concave majorant $h(y)$ and by $h(y) \leq \tilde{c}/\bar{\phi}(y)$, we get

$$h(y_*) = \frac{\tilde{c}}{\bar{\phi}(y_*)}$$

Recall that we only need to establish (4.37) on some x-interval of length ≥ 1. It would be convenient to use the x-interval corresponding (via F^{-1}) to the y-interval $[y_*, \frac{y_*}{\gamma^2}]$. Note that, as $y_* \in [1, \frac{1}{\gamma^2}]$, this x-interval necessarily lies inside $[0, 2]$. Note also that

$$\phi(x) = \bar{\phi}(F(x)), \;\; \phi'(x) = \bar{\phi}'(F(x))F'(x)$$

and so the left-hand side of (4.37) can be rewritten as

$$F'(x)\left(\bar{\phi}'(F(x))h(F(x)) + \bar{\phi}(F(x))h'(F(x))\right)$$

To estimate the above, it would suffice to get an estimate of $F'(x)$, for $x \in [0, 2]$, and of

$$\bar{\phi}'(y)h(y) + \bar{\phi}(y)h'(y),$$

for $y \in [y_*, \frac{y_*}{\gamma^2}]$.

For F', we have

$$F'(x) = \frac{\psi'\phi - \phi'\psi}{\phi^2}(x) = \frac{W}{\phi^2(x)}$$

so we need to estimate W:

$$W = \psi'(0)\phi(0) + |\phi'(0)|\psi(0) = \psi'(0) + |\phi'(0)|$$

Each of these derivatives can be estimated using their integral representation, as in the proof of Proposition IV.11, using the asymptotic properties of ϕ and ψ from Lemma IV.8. This yields:

$$\frac{4c_u}{\sigma\sqrt{2c_l}} \geq W \geq \frac{4c_l}{\sigma\sqrt{2c_u}}$$

And, as $\phi(x)$ is between 1 and γ^2, for $x \in [0, 2]$, we further get

$$\frac{1}{\gamma^2}\frac{4c_u}{\sigma\sqrt{2c_l}} \geq F'(x) \geq \frac{4c_l}{\sigma\sqrt{2c_u}}$$

for $x \in [0,2]$. So it remains to estimate

$$\bar{\phi}'(y)h(y) + \bar{\phi}(y)h'(y)$$

for $y \in [y_*, \frac{y_*}{\gamma^2}] \subset [1, \frac{1}{\gamma^4}]$. As $\bar{\phi}' < 0$, $h > 0$, and, as shown below, $h' > 0$, to estimate the above expression from below we need the estimates on $\bar{\phi}'$ from below, on h from above, on $\bar{\phi}$ from below, and on h' from below. Clearly, $\bar{\phi}(y) \geq \gamma^2$, for the chosen range of y. In addition, as h is dominated by $\frac{\tilde{c}}{\bar{\phi}}$, we get $h(y) \leq \frac{\tilde{c}}{\gamma^2}$ on the y-range we consider.

To estimate $h'(y)$ on $[y_*, \frac{y_*}{\gamma^2}]$, note that $h(y)$ coincides with $\tilde{c}/\bar{\phi}(y)$ at the endpoints of this interval, and $h \leq \tilde{c}/\bar{\phi}$ on the entire interval. Then, as $\tilde{c}/\bar{\phi}$ is differentiable, we must have

$$\left(\frac{\tilde{c}}{\bar{\phi}}\right)'\left(\frac{y_*}{\gamma^2}\right) \leq h'\left(\frac{y_*}{\gamma^2}\right),$$

as, otherwise, we get a contradiction with the domination relationship between these two functions in the left neighborhood of $\frac{y_*}{\gamma^2}$. In the above, and in the rest of the argument, $h'(y)$ is understood as the left derivative at $y = y_*/\gamma^2$, as the right derivative at $y = y_*$, and as any element in the superdifferential for $y \in (y_*, \frac{y_*}{\gamma^2})$.

The last inequality together with the concavity of h implies that for all $y \in [y_*, \frac{y_*}{\gamma^2}]$

$$h'(y) \geq \left(\frac{\tilde{c}}{\bar{\phi}}\right)'\left(\frac{y_*}{\gamma^2}\right) = -\frac{\tilde{c}}{\bar{\phi}^2(\frac{y_*}{\gamma^2})}\bar{\phi}'(\frac{y_*}{\gamma^2}) \geq -\frac{\tilde{c}}{\gamma^2}\bar{\phi}'(\frac{y_*}{\gamma^2})$$

Note further that as $\frac{y_*}{\gamma^2} \leq \frac{y}{\gamma^2}$ for any $y \in [y_*, \frac{y_*}{\gamma^2}]$ and $-\bar{\phi}'$ is positive decreasing we get

$$-\frac{\tilde{c}}{\gamma^2}\bar{\phi}'(\frac{y_*}{\gamma^2}) \geq -\frac{\tilde{c}}{\gamma^2}\bar{\phi}'(\frac{y}{\gamma^2}) = -\frac{\tilde{c}}{\gamma}\bar{\phi}'(y)$$

as $\bar{\phi}'(\frac{y}{\gamma^2}) = \gamma\bar{\phi}'(y)$. Combinig the estimates above we get for $x \in [\mathrm{F}^{-1}(y_*), \mathrm{F}^{-1}(y_*)+1]$

$$F'(x)\left(\bar{\phi}'(F(x))h(F(x)) + \bar{\phi}(F(x))h'(F(x))\right) \geq F'(x)\left(\bar{\phi}'(F(x))\frac{\tilde{c}}{\gamma^2} - \tilde{c}\gamma\bar{\phi}'(F(x))\right) =$$
$$F'(x)\bar{\phi}'(F(x))\tilde{c}\left(\frac{1}{\gamma^2} - \gamma\right)$$

As $F'(x)\bar{\phi}'(F(x)) = \phi'(x) \geq \phi'(0)$, $\tilde{c} \leq 2C+2$, the last expression above is furthermore

$$\geq \phi'(0)(2C+2)\left(\frac{1}{\gamma^2} - \gamma\right)$$

Finally, as $\phi'(0) \geq -\frac{2c_u}{\sigma\sqrt{2c_l}}$ and $\exp\left(\sqrt{\frac{2c_l}{\sigma^2}}\right) \leq \frac{1}{\gamma} \leq \exp\left(\sqrt{\frac{2c_u}{\sigma^2}}\right)$, we have $\left(\frac{1}{\gamma^2} - \gamma\right) \to 0$, as $\sigma \to \infty$, at a rate depending only on c_l, c_u. This, in turn, yields (4.37) and completes the proof of the proposition. ∎

If we assume the existence of a positive lower bound on derivatives of the barrier functions of the agents' stopping problems, as suggested by the last proposition, it becomes relatively easy to prove the continuity of the value function with respect to the barrier.

Proposition IV.16. *Assume admissible controls p^a, p^b and admissible barriers J^1, J^2 have 1-shift property, and $J'(x) \geq \epsilon > 0$ for $J = J^1, J^2$, and J^1, J^2 are C-close to x, $\frac{g^a}{c}, \frac{g^b}{c}$ are C-close to x . Then, if $|J^1(x) - J^2(x)| \leq \delta$ for all x in $\mathbb{R}$, we get*

$$\left|V^a(x, p^a, p^b, J^1) - V^a(x, p^a, p^b, J^2)\right| \leq \varepsilon(\delta), \quad \forall\, x \in \mathbb{R} \tag{4.38}$$

for some $\varepsilon(\delta) \to 0$, as $\delta \to 0$. Analogous statement holds for V^b.

Proof: We will show that $V^a(x, p^a, p^b, J^1) \geq V^a(x, p^a, p^b, J^2) - \epsilon(\delta)$, which, together with the symmetric inequality (proved analogously), yields the statement of the proposition. For any $\delta > 0$, consider an almost-optimal τ_2, such that

$$J^a(\tau_2, x, p^a, p^b, J^2) \geq V^a(x, p^a, p^b, J^2) - \delta$$

Then it suffices to find τ_1 such that

$$J^a(\tau_1, x, p^a, p^b, J^1) \geq J^a(\tau_2, x, p^a, p^b, J^2) - \epsilon(\delta)$$

We construct $\tau_1 \geq \tau_2$ separately on two different $\mathcal{F}_{\tau_2}$-measurable sets. On the event

$$\Omega_1 = \left\{\omega \colon \lfloor J^1(X_{\tau_2}) \rfloor \geq \lfloor J^2(X_{\tau_2}) \rfloor\right\}$$

we set $\tau_1 = \tau_2$. If $\lfloor J^1(X_{\tau_2})\rfloor < \lfloor J^2(X_{\tau_2})\rfloor$ we still have

$$J^1(X_{\tau_2}) \geq J^2(X_{\tau_2}) - \delta$$

and so, by the assumption on J',

$$J^1\left(X_{\tau_2} + \frac{\delta}{\epsilon}\right) \geq J^2(X_{\tau_2})$$

The above implies

$$\lfloor J^1\left(X_{\tau_2} + \frac{\delta}{\epsilon}\right)\rfloor \geq \lfloor J^2(X_{\tau_2})\rfloor \tag{4.39}$$

On

$$\Omega_2 = \Omega_1^c = \left\{\omega \colon \lfloor J^1(X_{\tau_2})\rfloor < \lfloor J^2(X_{\tau_2})\rfloor\right\}$$

we define

$$\tau_{10} = \inf\left\{t \geq \tau_2 \colon X_t \geq X_{\tau_2} + \frac{\delta}{\epsilon}\right\}, \quad \tau_{11} = \inf\left\{t \geq \tau_2 \colon X_t \leq X_{\tau_2} - 1\right\}, \quad \tau_1 = \tau_{10} \wedge \tau_{11}.$$

In the subsequent derivations, we express various quantities in terms of the following expression, which can be interpreted as the 'relative to x' objective, and which is more convenient than its 'absolute' version.

$$\begin{aligned} J^a(\tau, x, p^a, p^b, J) - x = \\ \mathbb{E}^x\Big[\int_0^\tau \exp\left(-\int_0^t c(X_s)\mathrm{d}s\right)(g^a(X_t) - c(X_t)X_t)\,\mathrm{d}t + \\ \exp\left(-\int_0^\tau c(X_s)\mathrm{d}s\right)(\lfloor J(X_\tau)\rfloor - X_\tau)\Big] \end{aligned} \tag{4.40}$$

where $|\lfloor J(x)\rfloor - x| \leq C + 1$ and $|g^a(x) - c(x)x| \leq c_u C$ by the assumption of the proposition. Using the above expression, we get

$$
\begin{aligned}
&J^a(\tau_1, x, p^a, p^b, J^1) - J^a(\tau_2, x, p^a, p^b, J^2) = \\
&\qquad \mathbb{E}^x\Big[\mathbf{1}_{\Omega_1} \exp\left(-\int_0^{\tau_2} c(X_s)\mathrm{d}s\right)\left(\lfloor J^1(X_{\tau_2})\rfloor - \lfloor J^2(X_{\tau_2})\rfloor\right) + \\
&\qquad \mathbf{1}_{\Omega_2}\int_{\tau_2}^{\tau_1} \exp\left(-\int_0^{t} c(X_s)\mathrm{d}s\right)\left(g^a(X_t) - c(X_t)X_t\right)\mathrm{d}t + \\
&\qquad \mathbf{1}_{\Omega_2}\Big(\exp\left(-\int_0^{\tau_1} c(X_s)\mathrm{d}s\right)\left(\lfloor J^1\left(X_{\tau_1}\right)\rfloor - X_{\tau_1}\right) - \\
&\qquad\qquad \exp\left(-\int_0^{\tau_2} c(X_s)\mathrm{d}s\right)\left(\lfloor J^2\left(X_{\tau_2}\right)\rfloor - X_{\tau_2}\right)\Big)\Big]
\end{aligned}
$$

Note that the first one of the three summands inside the expectation above is non-negative for every ω, by the definition of Ω_1. Note also that, as $|g^a(x) - c(x)x| \leq c_u C$, we have the following bound for the second summand:

$$
\begin{aligned}
&\left|\mathbb{E}^x\left[\mathbf{1}_{\Omega_2}\int_{\tau_2}^{\tau_1} \exp\left(-\int_0^{t} c(X_s)\mathrm{d}s\right)\left(g^a(X_t) - c(X_t)X_t\right)\mathrm{d}t\right]\right| \leq \\
&\qquad\qquad c_u C\mathbb{E}^x|\tau_1 - \tau_2| = c_u C\mathbb{E}^0\tau' \leq \epsilon(\delta)
\end{aligned}
$$

where

$$\tau' = \inf\left\{t \geq 0 \colon X_t \notin (-1, \delta/\epsilon)\right\}$$

and $\mathbb{E}^0\tau'$ is easily seen to go to 0 as $O(\delta)$ for $\delta \to 0$. So it only remains to deal with the last term. We have

$$\mathbb{E}^x\Big[\mathbf{1}_{\Omega_2}\Big(\exp\left(-\int_0^{\tau_1} c(X_s)\mathrm{d}s\right)\left(\lfloor J^1\left(X_{\tau_1}\right)\rfloor - X_{\tau_1}\right) - \exp\left(-\int_0^{\tau_2} c(X_s)\mathrm{d}s\right)\left(\lfloor J^2\left(X_{\tau_2}\right)\rfloor - X_{\tau_2}\right)\Big)\Big] =$$
$$\mathbb{E}^x\Big[\mathbf{1}_{\Omega_2}\mathbf{1}_{\{\tau_1=\tau_{10}\}}\Big(\exp\left(-\int_0^{\tau_{10}} c(X_s)\mathrm{d}s\right)\left(\lfloor J^1\left(X_{\tau_{10}}\right)\rfloor - X_{\tau_{10}}\right) - \exp\left(-\int_0^{\tau_2} c(X_s)\mathrm{d}s\right)\left(\lfloor J^2\left(X_{\tau_2}\right)\rfloor - X_{\tau_2}\right)\Big) +$$
$$\mathbf{1}_{\Omega_2}\mathbf{1}_{\{\tau_1=\tau_{11}\}}\Big(\exp\left(-\int_0^{\tau_{11}} c(X_s)\mathrm{d}s\right)\left(\lfloor J^1\left(X_{\tau_{11}}\right)\rfloor - X_{\tau_{11}}\right) - \exp\left(-\int_0^{\tau_2} c(X_s)\mathrm{d}s\right)\left(\lfloor J^2\left(X_{\tau_2}\right)\rfloor - X_{\tau_2}\right)\Big)\Big]$$

As $|J(x)-x| \leq C$, for $J = J^1, J^2$, and as

$$\mathbb{P}^x\left(\tau_1 = \tau_{11}\right) = \mathbb{P}^0\left(\inf\left\{t \geq 0\,:\, X_t = \frac{\delta}{\epsilon}\right\} > \inf\left\{t \geq 0\,:\, X_t = -1\right\}\right) = \frac{\frac{\delta}{\epsilon}}{1+\frac{\delta}{\epsilon}} = O(\delta)$$

for $\delta \to 0$, we get

$$\Big|\mathbb{E}^x\Big[\mathbf{1}_{\Omega_2}\mathbf{1}_{\{\tau_1=\tau_{11}\}}\Big(\exp\left(-\int_0^{\tau_{11}} c(X_s)\mathrm{d}s\right)\left(\lfloor J^1\left(X_{\tau_{11}}\right)\rfloor - X_{\tau_{11}}\right) - \exp\left(-\int_0^{\tau_2} c(X_s)\mathrm{d}s\right)\left(\lfloor J^2\left(X_{\tau_2}\right)\rfloor - X_{\tau_2}\right)\Big)\Big]\Big| \leq (2C+2)\mathbb{P}^x(\tau_1=\tau_{11}) = \epsilon(\delta)$$

Finally, we estimate the remaining term from below:

$$\mathbb{E}^x\Big[\mathbf{1}_{\Omega_2}\mathbf{1}_{\{\tau_1=\tau_{10}\}}\Big(\exp\Big(-\int_0^{\tau_{10}} c(X_s)\mathrm{d}s\Big)\left(\lfloor J^1\left(X_{\tau_{10}}\right)\rfloor - X_{\tau_{10}}\right) -$$
$$\exp\Big(-\int_0^{\tau_2} c(X_s)\mathrm{d}s\Big)\left(\lfloor J^2\left(X_{\tau_2}\right)\rfloor - X_{\tau_2}\right)\Big)\Big] \geq$$
$$\mathbb{E}^x\Big[\mathbf{1}_{\Omega_2}\mathbf{1}_{\{\tau_1=\tau_{10}\}}\Big(\exp\Big(-\int_0^{\tau_{10}} c(X_s)\mathrm{d}s\Big)\left(\lfloor J^2\left(X_{\tau_2}\right)\rfloor - X_{\tau_{10}}\right) -$$
$$\exp\Big(-\int_0^{\tau_2} c(X_s)\mathrm{d}s\Big)\left(\lfloor J^2\left(X_{\tau_2}\right)\rfloor - X_{\tau_2}\right)\Big)\Big] =$$
$$\mathbb{E}^x\Big[\mathbf{1}_{\Omega_2}\mathbf{1}_{\{\tau_1=\tau_{10}\}}\Big(\left(\lfloor J^2(X_{\tau_2})\rfloor - X_{\tau_2}\right)\Big(\exp\Big(-\int_0^{\tau_{10}} c(X_s)\mathrm{d}s\Big) -$$
$$\exp\Big(-\int_0^{\tau_2} c(X_s)\mathrm{d}s\Big)\Big) + (X_{\tau_2} - X_{\tau_{10}})\exp\Big(-\int_0^{\tau_{10}} c(X_s)\mathrm{d}s\Big)\Big)\Big] \geq$$
$$(-C-1)\mathbb{E}^x\left[\exp\Big(-\int_0^{\tau_2} c(X_s)\mathrm{d}s\Big) - \exp\Big(-\int_0^{\tau_1} c(X_s)\mathrm{d}s\Big)\right] - \frac{\delta}{\epsilon}$$

where the first inequality follows from $X_{\tau_{10}} = X_{\tau_2} + \delta/\epsilon$ and

$$\lfloor J^1(X_{\tau_{10}})\rfloor = \lfloor J^1(X_{\tau_2} + \frac{\delta}{\epsilon})\rfloor \geq \lfloor J^2(X_{\tau_2})\rfloor,$$

by (4.39); the second inequality follows from $X_{\tau_2} - X_{\tau_{10}} = -\delta/\epsilon$ and $\lfloor J^2(x)\rfloor - x \geq -C-1$ together with $\tau_1 \geq \tau_2$. It only remains to notice that

$$\left|\mathbb{E}^x\left[\exp\Big(-\int_0^{\tau_2} c(X_s)\mathrm{d}s\Big) - \exp\Big(-\int_0^{\tau_1} c(X_s)\mathrm{d}s\Big)\right]\right| \leq$$
$$\mathbb{E}^x\left|\int_{\tau_2}^{\tau_1} c(X_s)\mathrm{d}s\right| \leq c_u\mathbb{E}^x|\tau_1 - \tau_2| = O(\delta),$$

which concludes the proof. ∎

4.4 Optimization over prices and existence of equilibrium.

For any admissible v^a, v^b, define

$$P^a(v^a)(x) = \min\operatorname{argmax}_{p\in\mathcal{A}^a(x)\cap\mathbb{Z}}\left(p - v^a(x)\right)F^+\left(p-x\right)$$
$$(4.41)\qquad P^b(v^b)(x) = \max\operatorname{argmax}_{p\in\mathcal{A}^b(x)\cap\mathbb{Z}}\left(v^b(x) - p\right)F\left(p-x\right)$$

where we denote

$$F^{+}(x) = 1 - F(x)$$

for a cdf F.

The following fundamental proposition allows us to reduce the control-stopping problem to just a stopping problem and a fixed point problem for response controls.

Proposition IV.17. *If*

$$\overline{V}^{a}(x) = \sup_{\tau} J^{a}(x, \tau, P^{a}(\overline{V}^{a}), p^{b}, v^{b})$$

for some admissible p^b, with p^b and v^b with 1-shift property and v^b, $\frac{g^a}{c}$ C-close to x, and with $\left(v^b\right)' \geq 1 - w > 0$, then

$$\overline{V}^{a}(x) = \sup_{p^{a} \in \mathcal{A}^{a}(x) \cap \mathbb{Z}} \sup_{\tau} J^{a}(x, \tau, p^{a}, p^{b}, v^{b})$$

and similarly for $\overline{V}^{b}$, $P^{b}(\overline{V}^{b})$.

Proof: Subtracting x from all $\overline{V}^{a}$ and J^{a} as in (4.43), (4.42) below, we see it's sufficient to show the claim for these relative versions.

What we need to do here is the verification that our response-form control $p^{a} = P^{a}(\overline{V}^{a})$ is optimal, which requires some sort of differential characterization of the objective

$$V^{a}(x, p, p^{b}, v^{b}) = \sup_{\tau} J^{a}(x, \tau, p, p^{b}, v^{b})$$

for any control p, together with a comparison principle allowing us to claim our response-form control constructed in a way to maximize the appropriate generator would indeed result in maximal objective. We use the theory of variational inequalities (VIs) to implement this program. Unfortunately, we could not locate any VI results to deal with our case (unbounded domain, $\mathbb{L}^{\infty}$ discount factor and running costs, discontinuous obstacle) directly: we have VI existence, uniqueness and comparison results, but are lacking stopping problem to VI solution connection under this combination of circumstances, so we'll need to do additional approximation steps.

More specifically, in *step 1* we show we can replace our discontinuous barrier $\lfloor v^b \rfloor$ by its continuous majorating approximation $s_\epsilon = s_\epsilon(\lfloor v^b \rfloor)$, without affecting the value of the associated optimal stopping problem, no matter which admissible (p, p^b) are chosen. In *step 2*, we use a sequence of smooth approximating functions $J^b_n \downarrow s_\epsilon$ and continuity of VI solutions in the obstacle from [8] to show that the value function corresponding to s_ϵ satisfies an appropriate VI. Finally, in *step 3* we use the comparison results for that VI to show $P^a(\overline{V}^a)$ is indeed the optimal control.

As we're now going to use obstacles which aren't floors of some other function, we need to redo the definitions (4.2), (4.10), and also to put them in a relative to x form to get better boundedness properties for the coefficients, so we define for any admissible controls p, p^b, admissible barrier J^b,

$$\begin{aligned}(4.42)\quad & J^a_0(x,\tau,p,p^b,J^b) = \\ & \mathbb{E}^x\Big[\int_0^\tau \exp\left(-\int_0^t c(p(X_s),p^b(X_s),X_s)\mathrm{d}s\right)\big(g^a(p(X_t),p^b(X_t),X_t) - c(p(X_t),p^b(X_t),X_t)X_t\big)\,\mathrm{d}t \\ & \qquad + \exp\left(-\int_0^\tau c(p(X_s),p^b(X_s),X_s)\mathrm{d}s\right)\big(J^b(X_\tau) - X_\tau\big)\Big]\end{aligned}$$

$$V^a_0(x,p,p^b,J^b) = \sup_\tau J^a_0(x,\tau,p,J^b) \tag{4.43}$$

$$\begin{aligned}(4.44)\quad & f^a_0(x,p,p^b) = \\ & \mathbb{E}^x\left[\int_0^\infty \exp\left(-\int_0^t c(X_s,p(X_s),p^b(X_s))\mathrm{d}s\right)\big(g^a(p(X_t),p^b(X_t),X_t) - c(p(X_t),p^b(X_t),X_t)X_t\big)\,\mathrm{d}t\right]\end{aligned}$$

Notice V^a_0 and f^a_0 are 1-periodic and C-close to 0 if J^b, $\frac{g^a}{c}$ are such. Also $f = V^a_0, f^a_0$ satisfies $|f'| \le w$ by applying slight modifications of Lemmas IV.6, IV.5 and Proposition IV.14 for either $J^b = \lfloor v^b \rfloor$ or $J^b = s_\epsilon\left(\lfloor v^b \rfloor\right)$.

Step 1. This step is taken care of by the following lemma, whose (geometric) proof can be found in the appendix.

Lemma IV.18. *If p, p^b are admissible controls, v^b an admissible barrier, p, p^b, v^b have 1-shift property, v^b, $\frac{g^a}{c}$, $\frac{g^b}{c}$ are C-close to x, and $\left(v^b\right)' \geq 1 - w > 0$, then there exists continuous piecewise linear $s_\epsilon \geq \lfloor v^b \rfloor$ independent of p^b, p, satisfying 1-shift property and C-close to x, and such that*

$$V_0^a(\cdot, p, p^b, \lfloor v^b \rfloor) = V_0^a(\cdot, p, p^b, s_\epsilon)$$

for all p, p^b satisfying the properties above.

Step 2. First we need to introduce some notation from [8]. Let $\mu > 0$. We introduce the weight function

$$m_\mu(x) = \exp(-\mu|x|)$$

Denote by $H_\mu = \mathbb{W}^{0,2,\mu}$, $V_\mu = \mathbb{W}^{1,2,\mu}$ appropriate m_μ-weighted Sobolev spaces on $\mathbb{R}$ (we need weighted spaces as our coefficients are bounded and periodic while we want to make them integrable over the whole unbounded domain).

For any $u, v \in V_\mu$ define

$$a(u, v) = \int \frac{\sigma^2}{2} m_\mu^2 u_x v_x - 2\mu \operatorname{sgn}(x) \frac{\sigma^2}{2} m_\mu v_x m_\mu v + c_p m_\mu^2 uv$$

where the integral is over $\mathbb{R}$,

Let

$$f_p(x) = g_p^a(x) - c_p(x)x \in H_\mu$$

be the running cots of our relative-to-x stopping problem, and

$$K_\mu(J^b) = \left\{ v \in V_\mu | v \geq J^b - x \text{ a.e.} \right\}$$

the appropriate set of test functions.

We will call $\mathrm{VI}(p, J^b)$ the following VI (in the weak form)

$$a(u, v-u) = a_p(u, v-u) \leq (f_p, v-u), \ \forall v \in K_\mu(J^b) \tag{4.45}$$

where

$$(u,v) = (u,v)_\mu = \int m_\mu^2 uv$$

We say u is a solution of the above VI if $u \in K_\mu(J^b)$ and u satisfies (4.45).

As all our coefficients c, $\frac{\sigma^2}{2}$, f_p, $J^b - x$ are in $\mathbb{L}^\infty(\mathbb{R})$, and as the form $a(\cdot,\cdot)$ is coercive for μ sufficiently small, we get that (for such μ) the VI (4.45) has a unique solution in $K_\mu(J^b)$ for any admissible p, any J^b C-close to x, by Theorem 1.13, [8] p. 217.

Let J_n^b be a C^∞-approximation from above of s_ϵ, associated with v^b as in Lemma IV.18, which is $1/n$ close to s_ϵ in sup-norm. Then by Theorem 3.19, [8] p. 387 , $u_n = V_0^a(\cdot, p, p^b, J_n^b)$ is the unique solution (for sufficiently small μ) of $\mathrm{VI}(p, J_n^b)$. Denote also by u_0 the unique solution of $\mathrm{VI}(p, s_\epsilon)$. Rewriting these VIs as unweighted VIs for $m_\mu u$ and restricting to a bounded domain, one can generalize Theorem 1.10, [8] p. 207, to get $u_n \to u_0$ in $\mathbb{L}^\infty(\mathbb{R})$. The latter fact, together with the easy to check convergence of value functions

$$V_0^a(\cdot, p, p^b, J_n^b) \to V_0^a(\cdot, p, p^b, s_\epsilon) = V_0^a(\cdot, p, p^b, \lfloor v^b \rfloor),$$

implies that the latter value function is the unique solution of $\mathrm{VI}(p, s_\epsilon)$.

Step 3. By Theorem 1.4, [8] p. 198 , extended to the unbounded domain as in Remark 1.21, p. 219, the unique solutions $u, \tilde{u} \in K_\mu(J^b)$ of VIs

$$a(u, v-u) \leq (h, v-u), \ \forall v \in K_\mu(J^b)$$

$$\text{resp. } a(\tilde{u}, v-\tilde{u}) \leq (\tilde{h}, v-\tilde{u}), \ \forall v \in K_\mu(J^b)$$

sharing the obstacle J^b and the form a, but with different right-hand sides h, $\tilde{h}$, satisfy $\tilde{u} \geq u$ if $\tilde{h} \geq h$.

Now let $\overline{V}^a_0(x) = \overline{V}^a(x) - x$, $\bar{p} = P^a(\overline{V}^a)$.

We need to show that, in this case,

$$\bar{u} := \overline{V}^a_0 = V^a_0(\cdot, \bar{p}, p^b, \lfloor v^b \rfloor) = V^a_0(\cdot, \bar{p}, p^b, s_\epsilon) \geq$$
$$V^a_0(\cdot, p_0, p^b, s_\epsilon) = V^a_0(\cdot, p_0, p^b, \lfloor v^b \rfloor) =: \bar{u}_0 = \bar{u}_0(p_0)$$

for any admissible p_0. It is shown in step 2 that $\bar{u}$ satisfies a version of (4.45) with running costs $f_{\bar{p}}$ and the quadratic form $a_{\bar{p}}$, which after some algebraic manipulations turns out to be equivalent to

$$a_{p_0}(u, v-u) \leq (\tilde{f}, v-u),\ \forall v \in K_\mu(J^b) \tag{4.46}$$

where $\tilde{f} = f_{p_0} + a$,

$$a = g_{\bar{p}} - c_{\bar{p}}(u+x) - (g_p - c_p(u+x)) \geq 0,$$

which follows from the definition of $\bar{p} = P^a(\overline{V}^a)$ and the fact that $\bar{u}(x) + x = \overline{V}^a(x)$. As $\bar{u}_0$ satisfies the above equation with running costs function f_{p_0} instead of $\tilde{f}$, and $f_{p_0} \leq \tilde{f}$, we can apply the comparison principle stated at the beginning of this step, which completes the proof. ∎

Proposition IV.17 allows us to sidestep the optimization over p^a or p^b in the definitions of $\bar{V}^a$ and $\bar{V}^b$, respectively, by using the feedback controls P^a and P^b throughout. To ensure that these feedback controls are well-behaved, we make the following assumption (compare to chapter III, Assumptions 2, 5).

Assumption IV.19. *The distribution of ξ has density f, which is bounded so that $f(x) \leq C_f$, $\forall x$, with some constant $C_f > 0$, and which is supported inside $[-C_0, C_0]$*

for some constant $C_0 > 0$*. Furthermore,* f *is continuous in the interior of its support, and is such that*

$$\frac{F^+}{f}(x) \text{ is decreasing,} \quad \frac{F}{f}(x) \text{ is increasing,} \quad \forall x \in \operatorname{int} suppf$$

where int *denotes the interior of a set.*

See also Remark 4 in chapter III for sufficient conditions for Assumption IV.19. This assumption, in particular, implies that the optimal feedback prices will always be $C_0' = C_0 + 1$ close to x, and also inherit the 1-shift property from the value functions they correspond to:

Lemma IV.20. *If*

$$p^a(x) = P^a(v^a)(x), \; p^b(x) = P^b(v^b)(x)$$

for some admissible barriers v^a*,* v^b*, then*

$$|p^a(x) - x| \leq C_0', \; |p^b(x) - x| \leq C_0'$$

and

$$\left|\frac{g^a}{c}(x)\right| \leq C_0', \quad \left|\frac{g^a}{c}(x)\right| \leq C_0'$$

If in addition v^a *(*v^b*) has 1-shift property, then so does* p^a *(resp.* p^b*).*

Proof: From definition (4.41) and $\operatorname{supp} \xi \subset [-C_0, C_0]$, it's easy to see that $p(x) - x$ must be no smaller than the largest integer $\leq -C_0$ and no larger than the smallest integer $\geq C_0$ so that

$$p(x) \geq x - C_0', \quad p(x) \leq x + C_0'$$

Similar conclusion holds for $p^b(x)$.

From (4.4,4.5) we get

$$\left|\frac{g^a}{c}(x) - x\right| = \left|\frac{(p^a(x) - x)\,(1 - F(p^a(x) - x)) + \mathcal{F}^b(p^b(x), x) - xF(p^b(x) - x)}{(1 - F(p^a(x) - x)) + F(p^b(x) - x)}\right|$$

A similar representation holds for g^b. Thus, to prove the claim, it suffices to show

$$|p^a(x) - x| \leq C_0', \quad |\mathcal{F}^b(p^b(x), x) - x| \leq C_0' F(p^b(x) - x)$$

The first inequality has already been established. For the second one, we have

$$\mathcal{F}^b(p^b(x), x) - x = \int_{-\infty}^{p^b(x)-x} \lfloor x + \alpha y \rfloor - x \mathrm{d}F(y)$$

To finish the proof it's enough to notice that

$$|\lfloor x + \alpha y \rfloor - x| \leq C_0'$$

when $y \in \operatorname{supp} \xi$ (as $\mathrm{d}F(y) = 0$ otherwise). The claim for $\frac{g^b}{c}$ can be proven analogously.

1-shift property for p^a, p^b given that of v^a, v^b is immediate from (4.41). ∎

Next, for any admissible barriers v^a, v^b, define

$$\overline{\Phi}(v^a, v^b) = \left(V^a\left(\cdot, P^a(v^a), P^b(v^b), v^b\right), V^b(\cdot, P^a(v^a), P^b(v^b), v^a)\right) \tag{4.47}$$

It's easy to see the components of the right-hand side of the equation above are themselves admissible barriers, so we can iterate this mapping. We will actually be only interested in the restriction of $\overline{\Phi}$ to either A_0 or $A_0(w)$, where

Definition IV.21. We say $v \in A_0$ if $v \in C(\mathbb{R})$, has 1-shift property and is C_0'-close to x. We say $v \in A_0(w)$ if $v \in A_0$, v absolutely continuous and $1 - w \leq v' \leq 1 + w$ a.e.

Note by lemma IV.20 and the results from the previous section $\overline{\Phi}$ maps $A_0 \times A_0$ into itself, or more precisely into $A_0(w) \times A_0(w)$ for w sufficiently large (see proposition IV.14).

Using Proposition IV.17, we will show, below, that a fixed point of this mapping in the appropriate subset gives a solution to the system (4.1). But first we need to establish the existence of such a fixed point.

The first step is to show that $\overline{\Phi}$ is continuous on $A_0(w)$ for $w < 1$. To this end, we first choose an appropriate topology and the space for intermediate price controls $P^a(v)$, $P^b(v)$, and show that $P^a(v)$, $P^b(v)$ are continuous operators in $v^a, v^b \in A_0(w)$. Then, we show that $V^a\left(\cdot, p^a, p^b, v^b\right)$ and $V^b\left(\cdot, p^a, p^b, v^a\right)$ are each continuous (as operators) in (functions) p^a, p^b jointly, with respect to the chosen topology, uniformly in $v^a, v^b \in A_0$. This, together with the continuity of $V^a\left(\cdot, p^a, p^b, v^b\right)$ and $V^b\left(\cdot, p^a, p^b, v^a\right)$ in v^a and v^b, established in Proposition IV.16, yields the continuity of $\overline{\Phi}$.

First we define the space for the intermediate price controls:

Definition IV.22. Denote by B_0 the space of functions which are admissible prices and have 1-shift property.

This definition also implies the functions in B_0 are C_0-close to x, by the definition of admissibility IV.1 and assumption IV.19. We equip B_0 with a topology induced by its natural restriction mapping into $\mathbb{L}^1([0,1])$. Note $P^a(v), P^b(v) \in B_0$ for any admissible barrier v with 1-shift property.

The following, somewhat tricky, lemma is the first result we need in order to establish the continuity of $\overline{\Phi}$.

Lemma IV.23. *The mappings*

$$v^a \mapsto P^a(v^a), \quad v^b \mapsto P^b(v^b)$$

from $A_0(w)$, $w < 1$, (with uniform topology) to B_0 (with the topology described above) are continuous.

Proof: We only show the P^a version, the P^b one being analogous. We do it in two steps. First we show that, given v^a, with the properties described in the statement (in particular, increasing), $P^a(v^a)(x)$ is also an increasing function of x. Then, we use this monotonicity property to show the desired continuity of P^a.

Step 1. For a fixed v^a, we denote $p_x = P^a(v^a)(x)$. Assume, to the contrary, that for some $x_1 > x_2$ we have $p_{x_1} < p_{x_2}$. Note the admissible control set $\mathcal{A}^a(x)$ shifts upward with x, and so if p_{x_2} was admissible at lower x_2, and $p_{x_2} > p_{x_1}$ with the latter being admissible at x_1, then p_{x_2} is admissible at x_1 as well, and similarly p_{x_1} is admissible at x_2. Then to obtain a contradiction it is sufficient to just show p_{x_2} gives better objective value at x_1 than p_{x_1}:

$$(p_{x_2} - v^a(x_1))F^+(p_{x_2} - x_1) > (p_{x_1} - v^a(x_1))F^+(p_{x_1} - x_1) \tag{4.48}$$

This is clearly true if $p_{x_1} \leq v^a(x_1)$. Hence, without loss of generality, we assume $p_{x_1} > v^a(x_1)$. Then, the above inequality is equivalent to:

$$\frac{p_{x_2} - v^a(x_1)}{p_{x_1} - v^a(x_1)} > \frac{F^+(p_{x_1} - x_1)}{F^+(p_{x_2} - x_1)} \tag{4.49}$$

Note

$$(p_{x_2} - v^a(x_2))F^+(p_{x_2} - x_2) \geq (p_{x_1} - v^a(x_2))F^+(p_{x_1} - x_2) \tag{4.50}$$

from the fact that p_{x_2} is the optimal price at x_2 and is thus no worse than another admissible at x_2 price p_{x_1}. The assumption $p_{x_1} > v^a(x_1)$ above implies also $p_{x_2} > v^a(x_2)$, as $v^a(x_2) < v^a(x_1)$, so the inequality (4.50) is equivalent to

$$\frac{p_{x_2} - v^a(x_2)}{p_{x_1} - v^a(x_2)} \geq \frac{F^+(p_{x_1} - x_2)}{F^+(p_{x_2} - x_2)} \tag{4.51}$$

To get the desired contradiction it will thus suffice to notice that

$$\frac{p_{x_2} - v}{p_{x_1} - v} = 1 + \frac{p_{x_2} - p_{x_1}}{p_{x_1} - v}$$

is strictly increasing in $v \in \mathbb{R}$, for $v < p_{x_1}$, and that

$$\frac{F^+(p_{x_1} - x)}{F^+(p_{x_2} - x)}$$

is decreasing in x. The former is obvious, while the latter follows from

$$\frac{\partial}{\partial x}\left(\frac{F^+(p_{x_1}-x)}{F^+(p_{x_2}-x)}\right) = \frac{f(p_{x_1}-x)F^+(p_{x_2}-x) - f(p_{x_2}-x)F^+(p_{x_1}-x)}{F^+\left(p_{x_2}-x\right)^2} =$$
$$\frac{f(p_{x_1}-x)f(p_{x_2}-x)}{F^+\left(p_{x_2}-x\right)^2}\left(\frac{F^+}{f}(p_{x_2}-x) - \frac{F^+}{f}(p_{x_1}-x)\right) \leq 0,$$

which, in turn, follows from the fact that F^+/f is decreasing, by Assumption IV.19.

Given the above monotonicity properties of the terms in (4.51), we deduce (4.49), thus, obtaining the desired contradiction and proving the monotonicity of $P^a(v^a)(\cdot)$.

Step 2. One can easily check $P^a(v_1^a) \geq P^a(v_2^a)$, if $v_1^a(x) \geq V_2^a(x)$ for all x. To show that $P^a(v_1^a)$ and $P^a(v_2^a)$ are close in the topology of B_0, it suffices to show that

$$\int_0^1 |P^a(v_1^a) - P^a(v_2^a)|\,\mathrm{d}x$$

is small. Notice that

$$\int_0^1 |P^a(v_1^a) - P^a(v_2^a)|\,\mathrm{d}x = \int_0^1 |P^a(v_1^a \wedge v_2^a) - P^a(v_1^a \vee v_2^a)|\,\mathrm{d}x$$

as $P^a(v)(x)$ is uniquely determined by the value of v at x, and $\{v_1^a(x), v_2^a(x)\} = \{v_1^a(x) \vee v_2^a(x), v_1^a(x) \wedge v_2^a(x)\}$ for all x.

Thus, without loss of generality, we can assume that $v_2^a \geq v_1^a$. Assume v_1^a and v_2^a are also δ-close in sup-norm, so we have $v_2^a \leq v_1^a + \delta$. Then, we need to show that

$$\int_0^1 |P^a(V_2^a) - P^a(V_1^a)|\,\mathrm{d}x \to 0$$

as $\delta \to 0$.

Note that the monotonicity and the 1-shift property of the integer-valued function $P^a(v_2^a)$ imply that it coincides with $\lfloor x - \alpha_2 \rfloor$ (except, possibly, on a countable number of jump points of the latter), for some α_2. Similar conclusion holds for $P^a(v_1^a)$, with some $\alpha_1 \geq \alpha_2$, as $v_2^a \geq v_1^a$. If we can further show that

$$\alpha_1 \leq \alpha_2 + \frac{\delta}{1-w} \tag{4.52}$$

then a straightforward calculation would yield

$$\int_0^1 |P^a(v_1^a) - P^a(v_2^a)| \,\mathrm{d}x = O(\delta)$$

Thus, it remains to show (4.52). To this end, we note that, under our assumptions on v^a-s, for every $x \in \mathbb{R}$, there exists $x \leq x_* \leq x + \frac{\delta}{1-w}$ such that $v_1^a(x_*) = v_2^a(x)$. Assuming that

$$P^a(v_1^a)(x_*) < P^a(v_2^a)(x)$$

and recalling that $x_* \geq x$ and $v_1^a(x_*) = v_2^a(x)$, we follow the arguments in step 1 to obtain a contradiction. Thus,

$$P^a(v_1^a)(x_*) \geq P^a(v_2^a)(x)$$

which implies (4.52). ∎

The following lemma provides the last result we need in order to prove the continuity of $\overline{\Phi}$.

Lemma IV.24. *The operators $(p^a, p^b) \mapsto V_0^a(\cdot, p^a, p^b, J^b), V_0^b(\cdot, p^a, p^b, J^a)$, from $B_0 \times B_0$ to A_0, are continuous, uniformly over $J^a, J^b \in A_0$.*

Proof: We'll only show the continuity of V_0^a in p^a, p^b, uniformly over x and in $J^b \in A_0$, the other part being analogous. Recall that

$$V_0^a(x, p^a, p^b, J^b) = \sup_{\tau} J_0^a(x, \tau, p^a, p^b, J^b)$$

Thus, it suffices to show that J_0^a-s corresponding to two close pairs prices (p_1^a, p_1^b) and (p_2^a, p_2^b), with the same τ, are also close, uniformly in τ. To this end we can write, recalling the formula (4.42)

$$
\begin{aligned}
&J^a(x,\tau,p_1^a,p_1^b,J^b) - J^a(x,\tau,p_2^a,p_2^b,J^b) = \\
&\qquad \mathbb{E}^x\Big[\int_0^\tau \exp\left(-\int_0^t c_1(X_s)\mathrm{d}s\right)\left(g_1^a(X_t) - c_1(X_t)X_t\right)\mathrm{d}t - \\
&\qquad \int_0^\tau \exp\left(-\int_0^t c_2(X_s)\mathrm{d}s\right)\left(g_2^a(X_t) - c_2(X_t)X_t\right)\mathrm{d}t + \\
&\qquad \left(\exp\left(-\int_0^\tau c_1(X_s)\mathrm{d}s\right) - \exp\left(-\int_0^\tau c_2(X_s)\mathrm{d}s\right)\right)\left(J^b(X_\tau) - X_\tau\right)\Big]
\end{aligned}
$$

where we denote

$$c_1(x) = c(p_1^a(x), p_1^b(x), x),\ \ c_2(x) = c(p_2^a(x), p_2^b(x), x)$$

$$g_1^a(x) = c(p_1^a(x), p_1^b(x), x),\ \ g_2^a(x) = c(p_2^a(x), p_2^b(x), x)$$

The previous expression for the difference of objectives is furthermore equal to

(4.53)

$$
\begin{aligned}
&\mathbb{E}^x\Big[\int_0^\tau \left(\exp\left(-\int_0^t c_1(X_s)\mathrm{d}s\right) - \exp\left(-\int_0^t c_2(X_s)\mathrm{d}s\right)\right)\left(g_1^a(X_t) - c_1(X_t)X_t\right)\mathrm{d}t + \\
&\quad \int_0^\tau \exp\left(-\int_0^t c_2(X_s)\mathrm{d}s\right)\left(g_1^a(X_t) - c_1(X_t)X_t - (g_2^a(X_t) - c_2(X_t)X_t)\right)\mathrm{d}t + \\
&\qquad \left(\exp\left(-\int_0^\tau c_1(X_s)\mathrm{d}s\right) - \exp\left(-\int_0^\tau c_2(X_s)\mathrm{d}s\right)\right)\left(J^b(X_\tau) - X_\tau\right)\Big]
\end{aligned}
$$

So to complete the proof it is sufficient to show the expectations of the absolute values of each of the three terms on different lines inside the expectation above are small when (p_1^a, p_1^b) and (p_2^a, p_2^b) are close in their topology.

For the third term, note that as $|\exp(-x) - \exp(-y)| \leq \max\left(\exp(-x), \exp(-y)\right)|x-$

$y|$, and as $c_i(x) \geq c_l > 0$ for all x, $i = 1, 2$, we get

$$\begin{aligned}\left|\left(\exp\left(-\int_0^\tau c_1(X_s)\mathrm{d}s\right) - \exp\left(-\int_0^\tau c_2(X_s)\mathrm{d}s\right)\right)\left(J^b(X_\tau) - X_\tau\right)\right| &\leq \\ \exp(-c_l\tau)\left(\int_0^\tau |c_1(X_s) - c_2(X_s)|\,\mathrm{d}s\right)\left|J^b(X_\tau) - X_\tau\right| &\leq \\ C\int_0^\tau \exp(-c_l s)\,|c_1(X_s) - c_2(X_s)|\,\mathrm{d}s &\leq \\ C\int_0^\tau \exp(-c_l s)\left(|p_1^a(X_s) - p_2^a(X_s)| + \left|p_1^b(X_s) - p_2^b(X_s)\right|\right)\mathrm{d}s\end{aligned}$$

where the positive constants C can differ between the lines. The second inequality in the above follows from the closeness to x of J^b, and the last follows from the fact that

$$c(p^a(x), p^b(x), x) = \lambda\left(F^+(p^a(x) - x) + F(p^b(x) - x)\right)$$

is Lipschitz in $p^a(x)$ and $p^b(x)$, as the density of ξ is bounded by assumption IV.19.

Similarly, it's not hard to show that

$$|(g_1^a(x) - c_1(x)x - (g_2^a(x) - c_2(x)x)| \leq C\left(|p_1^a(x) - p_2^a(x)| + \left|p_1^b(x) - p_2^b(x)\right|\right)$$

using the boundedness of the density of ξ and the uniform closeness of all admissible p^a, p^b to x. This allows us to estimate the second term in (4.53):

$$\begin{aligned}\left|\int_0^\tau \exp\left(-\int_0^t c_2(X_s)\mathrm{d}s\right)(g_1^a(X_t) - c_1(X_t)X_t - (g_2^a(X_t) - c_2(X_t)X_t))\,\mathrm{d}t\right| &\leq \\ C\int_0^\tau \exp(-c_l t)\left(|p_1^a(X_t) - p_2^a(X_t)| + \left|p_1^b(X_t) - p_2^b(X_t)\right|\right)\mathrm{d}t\end{aligned}$$

Finally, we notice that $|g_1^a(X_t) - c_1(X_t)X_t| \leq C$, which follows from the fact that g_1^a/c_1 is C_0'-close to x, and recall that $c_1 \leq c_u$. This allows us to estimate the first term in (4.53)

$$\left|\int_0^\tau \left(\exp\left(-\int_0^t c_1(X_s)\mathrm{d}s\right) - \exp\left(-\int_0^t c_2(X_s)\mathrm{d}s\right)\right)\left(g_1^a(X_t) - c_1(X_t)X_t\right)\mathrm{d}t\right| \leq$$
$$C\int_0^\tau \exp(-c_l t)\left(\int_0^t \left(|p_1^a(X_s) - p_2^a(X_s)| + \left|p_1^b(X_s) - p_2^b(X_s)\right|\right)\mathrm{d}s\right)\mathrm{d}t \leq$$
$$C\int_0^\tau \exp(-c_l t)\left(|p_1^a(X_t) - p_2^a(X_t)| + \left|p_1^b(X_t) - p_2^b(X_t)\right|\right)\mathrm{d}t$$

where the second inequality follows from integration by parts, after discarding some negative terms.

Thus, the absolute values of all terms in (4.53) are estimated from above via

$$\int_0^\tau \exp(-c_l t)\left(|p_1^a(X_t) - p_2^a(X_t)| + \left|p_1^b(X_t) - p_2^b(X_t)\right|\right)\mathrm{d}t \leq$$
$$\int_0^\infty \exp(-c_l t)\left(|p_1^a(X_t) - p_2^a(X_t)| + \left|p_1^b(X_t) - p_2^b(X_t)\right|\right)\mathrm{d}t$$

which implies

$$\left|V_0^a(x, p_1^a, p_1^b, J^b) - V_0^a(x, p_2^a, p_2^b, J^b)\right| \leq$$
$$C\mathbb{E}^x\left[\int_0^\infty \exp(-c_l t)\left(|p_1^a(X_t) - p_2^a(X_t)| + \left|p_1^b(X_t) - p_2^b(X_t)\right|\right)\mathrm{d}t\right]$$

It only remains to estimate the latter expectation in terms of $\mathbb{L}^1([0,1])$ norms of $p_1^a - p_2^a$, $p_1^b - p_2^b$. The latter follows easily by passing the expectation inside the integral and using the standard estimates of a Gaussian kernel.

∎

Now we state our main result.

Theorem IV.25. *Let σ be sufficiently large so that w, defined in Proposition IV.14, is < 1.*

Let $A = A_0(w) \times A_0(w)$. It is a closed convex subset of $C(\mathbb{R})^2$.

Then $\overline{\Phi}$, defined in (4.47), is a continuous mapping of A into itself.

In particular, as A is also compact, $\overline{\Phi}$ has a fixed point.

Proof: Given our choice of w, $\overline{\Phi}$ maps A into itself, see definition IV.21. The closeness and convexity of A are also clear.

A is compact because $A_0(w)$ is. $A_0(w)$ is compact because (by 1-shift property) the restriction mapping $C(\mathbb{R}) \to C([0,1])$ induces an isomorphism between $A_0(w)$ and a closed subset of the compact set of functions in $C[0,1]$ which are bounded by $C_0' + 1$ and are Lipschitz with constant $1 + w$.

Finally, $\overline{\Phi}$ is continuous because it can be written as a composition of

$$e\colon (v^a, v^b) \mapsto \big(v^a, v^b, P^a(v^a), P^b(v^b)\big)$$

and

$$V\colon (v^a, v^b, p^a, p^b) \mapsto \big(V^a(\cdot, p^a, p^b, v^b), V^b(\cdot, p^a, p^b, v^a)\big)$$

In the above, $e\colon A \to A \times (B_0)^2$ is continuous by Lemma IV.23. The operator $V\colon A \times (B_0)^2 \to A$ is continuous as it is continuous in $(p^a, p^b) \in (B_0)^2$, uniformly over $v^a, v^b \in A_0(w)$, by Lemma IV.24, and it is continuous in $v^a, v^b \in A_0(w)$ by Proposition IV.16.

The existence of a fixed point for $\overline{\Phi}$ follows from the Schauder fixed point theorem.

∎

Combining the last theorem with proposition IV.17 we immediately get

Corollary IV.26. *There exists a solution* $(V^a, V^b, P^a(V^a), P^b(V^b))$ *of the problem (4.1).*

Remark IV.27. Note for any solution $(V^a, V^b, P^a(V^a), P^b(V^b))$ of (4.1) we have

$$P^a(V^a)(x) = p^a(x) \geq p^b(x) = P^b(V^b)(x), \quad \forall x \in \mathbb{R} \tag{4.54}$$

Indeed, let $p^a_{max}(x)$ ($p^b_{min}(x)$) be the largest (smallest) integer in $\mathcal{A}^a(x)$ ($\mathcal{A}^b(x)$). By definition IV.1 of admissibility, admissible $p^a(x) \geq p^b_{min}(x)$, admissible $p^b(x) \leq$

$p^a_{max}(x)$ and $p^a_{max}(x) \geq p^b_{min}(x)$. By definition (4.41), $p^a(x) = P^a(V^a)(x) \geq V^a(x)$ if $V^a(x) \leq p^a_{max}(x)$, $p^a(x) = p^a_{max}(x)$ otherwise, and similarly for $p^b(x)$. Thus (4.54) is clear if $V^a(x) > p^a_{max}(x)$ or $V^b(x) < p^b_{min}(x)$. If neither of these hold, then

$$p^a(x) \geq \lceil V^a(x) \rceil \geq \lfloor V^b(x) \rfloor \geq p^b(x)$$

4.5 Appendix.

4.5.1 Proof of lemma IV.7

We need to show that if f is continuous and has continuous right derivative on $[a,b]$ then f is continuously differentiable on (a,b). If for any $x < y$ in (a,b) we can find $\xi \in [x,y]$ s.t.

$$\frac{f(y) - f(x)}{y - x} = f^+(\xi) \tag{4.55}$$

then for any $x \in (a,b)$ we can take $y \downarrow x$ and $y \uparrow x$ limits in $\frac{f(y)-f(x)}{y-x}$ using the above and the continuity of f^+ to get that $f^-(x) = f^+(x)$ hence f is differentiable for every x, and its derivative coincides with f^+ and hence continuous. So it only remains to show (4.55) holds.

Consider

$$h(u) = f(u) - f(x) - (u - x)\frac{f(y) - f(x)}{y - x}$$

Clearly h has continuous right derivative on $[x,y]$, $h(x) = h(y) = 0$, and (4.55) is equivalent to the existence of $\xi \in [x,y]$ such that $h^+(\xi) = 0$. Assume the contrary. Then as h^+ is continuous it must have values of only one sign over $[x,y]$. Note also h is not identically 0 in this case, and so replacing it with $-h$ if necessary we can assume it has maximum > 0 achieved at some $x_{max} \in (x,y)$. Then from the definition of right derivative we must have $h^+(x_{max}) \leq 0$, hence < 0, hence h^+ must be < 0 on $[x,y]$. But if $h^+(x) < 0$, $h(x) = 0$ then h also achieves minimum < 0 at some $x_{min} \in (x,y)$ which immediately leads to contradiction with $h^+(x_{min}) < 0$. ∎

4.5.2 Proof of lemma IV.18

The idea is that if we take $\lfloor x \rfloor$ or a similar stepping function, and modify it around jump points by replacing jumps by steep line segments, this wouldn't affect its mcm. Recall our value function can be represented as

$$V_0^a(x,p,p^b,J^b) = f_0^a(x) + \phi(x)\,\mathrm{mcm}\left(\widehat{J^b - f_0^a} - x\right)(\mathrm{F}(x))$$

Where the only dependence on the obstacle is inside the mcm, and f_0^a, V_0^a were defined in (4.44,4.43). So it is sufficient to show that this y-domain mcm doesn't change if we replace $J^b = \lfloor V^b \rfloor$ by $J^b = s_\epsilon$.

First we need to define s_ϵ. We know $\lfloor V^b \rfloor$ has 1-shift property, jumps up by 1 at a sequence of points $\{x_0 + n\}_{n\in\mathbb{Z}}$, and is constant in between two consecutive points from that sequence. We define $s_\epsilon(x)$ to coincide with $\lfloor V^b \rfloor$ outside the intervals $(x_0 + n - \epsilon, x_0 + n]$, and to coincide with the line segment connecting $(x_0 + n - \epsilon, \lfloor V^b \rfloor (x_0 + n - \epsilon))$ and $(x_0 + n, \lfloor V^b \rfloor (x_0 + n+))$ on those intervals; note s_ϵ is a line segment with slope $1/\epsilon$ in the left ϵ-neighborhood of $\lfloor V^b \rfloor$ jump point, and coincides with it (and locally constant) elsewhere. Notice also $s_\epsilon \geq \lfloor V^b \rfloor$ by construction, and so $\mathrm{mcm}\left(\widehat{s_\epsilon - f_0^a} - x\right) \geq \mathrm{mcm}\left(\widehat{\lfloor V^b \rfloor - f_0^a} - x\right)$ and it only remains to prove the opposite inequality. Note also, under our running assumption of sufficiently large σ, $f_0^a + x$ is strictly increasing, and so $\lfloor V^b \rfloor - f_0^a - x$ achieves its maximum exactly at points $\{x_0 + n\}$; if this maximum is non-positive, we know from the proof of proposition IV.14 that the corresponding mcm is $\equiv 0$, and so the claimed inequality is clear as $s_\epsilon - f_0^a - x$ has the same supremum as $\lfloor V^b \rfloor - f_0^a - x$. So from now on we deal with the case where that supremum is positive, which implies $\mathrm{mcm}\left(\widehat{\lfloor V^b \rfloor - f_0^a} - x\right) > 0$ everywhere.

Similarly to the mcm of the function dominating its modification by replacing it by line segment connecting any two points of its graph over appropriate interval, $\phi(x)$ mcm $\left(\widehat{f}\right)$ (F(x)) dominates the function obtained from f by replacing its values on some $[x_0, x_1]$ by the values of $a\phi(x)+b\psi(x)$ function coinciding with f at x_0,x_1 (as $\widehat{}$ operation converts $a\phi + b\psi$ into a line segment). So to show mcm $\left(s_\epsilon \widehat{- f_0^a} - x\right) \leq$ mcm $\left(\lfloor V^b \widehat{\rfloor - f_0^a} - x\right)$ it suffices to show $s_\epsilon - f_0^a - x$ is dominated on $[x_0, x_0+1]$ by $a\phi(x) + b\psi(x)$ interpolation between (x_0, y) and (x_0+1, y), where $y = \lfloor V^b \rfloor(x_0) - f_0^a(x_0) - x_0 = \lfloor V^b \rfloor(x_0+1) - f_0^a(x_0+1) - (x_0+1)$.

So, consider $h = a\phi + b\psi$, $h(x_0) = y > 0$, $h(x_1) = y > 0$, $x_1 = x_0 + 1$. As h satisfies

$$\frac{\sigma^2}{2} h_{xx} - ch = 0, \text{ a.e.}$$

and is continuously differentiable, we get a contradiction if we assume h achieves maximum on $[x_0, x_1]$ in the interior of this interval, as then by the equation above h and so h_{xx} are positive (as $y > 0$) in the neighborhood of that maximum, which is not possible. So $h(x) \leq y$ on $[x_0, x_1]$. But then as $h(x) \leq y$ the equation above implies

$$h_{xx} \leq \frac{2c}{\sigma^2} y \leq \frac{2c_u}{\sigma^2} y$$

and as average slope of h over the length 1 interval $[x_0, x_1]$ is 0 this implies also

$$h_x \leq \frac{2c_u}{\sigma^2} y$$

As $y = \sup \lfloor V^b \rfloor - x - f_0^a$, the $(C+1)$-closeness to x (0) of V^b (resp. f_x^0) implies $y \leq 2C+2$. But this means h_x is bounded by a constant independent of the choice of p, p^b, V^b with properties as in the statement of the lemma. As $h(x_1) = s_\epsilon(x_1) - x_1 - f_0^a(x_1)$, and $s_\epsilon(x) - f_0^a - x$ has slope $\geq 1/\epsilon - 1 - w$ on $[x_1 - \epsilon, x_1]$, if ϵ is small enough so that this last expression is above the constant bounding h_x, we get

$$s_\epsilon(x) - f_0^a - x \leq h, \; x \in [x_1 - \epsilon, x_1]$$

hence also elsewhere on $[x_0, x_1]$, and on other intervals $[x_0+n, x_0+n+1]$ by the same argument. This shows

$$\mathrm{mcm}\left(\widehat{s_\epsilon - f_0^a} - x\right) \leq \mathrm{mcm}\left(\widehat{\lfloor V^b \rfloor - f_0^a} - x\right)$$

and so also

$$\mathrm{mcm}\left(\widehat{s_\epsilon - f_0^a} - x\right) = \mathrm{mcm}\left(\widehat{\lfloor V^b \rfloor - f_0^a} - x\right)$$

as claimed. ∎

4.5.3 Proof of lemma IV.8

We'll prove the claim about ψ, the one about ϕ being analogous. As ψ satisfies

$$\psi(x) = \mathbb{E}^x\left[\exp\left(-\int_0^{\tau_0} c(X_s)\mathrm{d}s\right)\right]$$

for $x < 0$ and as $c(X_s) \geq c_l > 0$ we have

$$\psi(x) = \mathbb{E}^x\left[\exp\left(-\int_0^{\tau_0} c(X_s)\mathrm{d}s\right)\right] \leq \mathbb{E}^x\left[\exp\left(-c_l\tau_0\right)\right] = \psi_0(x)$$

where ψ_0 is the ψ corresponding to $c(x) \equiv c_l$ and so is a unique positive increasing solution with $\psi_0(0) = 1$ of

$$\frac{\sigma^2}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} - c_l f = 0$$

and hence $\psi_0 = \exp\left(\sqrt{\frac{2c_l}{\sigma^2}}x\right)$. Similarly for $x \geq 0$ ψ satisfies

$$\psi(x) = \mathbb{E}^0\left[\exp\left(-\int_0^{\tau_x} c(X_s)\mathrm{d}s\right)\right]^{-1} \leq \mathbb{E}^0\left[\exp\left(-c_u\tau_x\right)\right]^{-1} = \psi_1(x)$$

where ψ_1 is the positive increasing solution with $\psi_1(0) = 1$ of

$$\frac{\sigma^2}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} - c_u f = 0$$

and hence $\psi_1 = \exp\left(\sqrt{\frac{2c_u}{\sigma^2}}x\right)$. This gives the required bound by observing $c_l < c_u$ which allows to resolve max there for $x \geq 0$ or $x \leq 0$ to ψ_0 or ψ_1 as appropriate. ∎

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