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Engineering Notes

Linear-Matrix-Inequality-Based Solution to Wahba's Problem

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I. Introduction

WAHBA'S problem was introduced in 1965 by Grace Wahba [1] and is an important problem in aerospace engineering that typically involves finding an optimal rotation to fit a series of vector measurements. There have been many different methods developed to solve Wahba's problem, both directly in terms of the rotation matrix [2,3] and in terms of the unit quaternion [4], the most famous method being QUEST [5]. A good survey of the different methods may be found in [6] and the references therein. Recently, a useful generalization of Wahba's problem has been made, allowing the determination of both attitude and body rate using a time history of vector measurements (see [7–9]). However, this Note does not examine this problem and treats only Wahba's original problem.

This Note presents a new characterization of the solution to Wahba's problem, directly in terms of the rotation matrix. It is shown that, under a mild condition (that is satisfied in many practical applications), Wahba's problem may be recast as a convex linear matrix inequality (LMI) optimization problem. This opens the door to a whole new class of solvers for Wahba's problem. This is accomplished by relaxing the nonconvex special orthogonal group [SO(3)] constraint on the rotation matrix to a convex LMI constraint. This constraint relaxation approach has applications beyond the solution of Wahba's problem and can potentially be useful for other optimization problems involving vehicle attitude, such as guidance and control problems.

The remainder of the Note is organized as follows. Section II presents an overview of Wahba's problem and its solution in terms of the singular value decomposition (SVD) [3]. Section III demonstrates that, under a mild condition, the Wahba problem may be recast as a LMI problem, leading to an identical solution, and conditions under which this mild condition is satisfied are investigated. Section IV presents a pair of numerical examples comparing the LMI-based solution to existing well-established solutions to Wahba's problem. Section V contains concluding remarks. The appendix contains a technical mathematical result, which is used in the Note.

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II. Wahba's Problem and Solution

Wahba's problem was originally posed by Grace Wahba in 1965 [1]. This section presents a brief overview of Wahba's problem, a well-known reformulation, and its solution based upon the SVD. The SVD solution is reviewed because it will be referenced when the LMI-based solution is derived in the next section.

Problem 1: (Wahba's problem) Given N vectors, $\boldsymbol{s}^m_{b,k} \in \mathbb{R}^3$, with corresponding vectors $\boldsymbol{s}_{I,k} \in \mathbb{R}^3$, Wahba's Problem is to find the matrix $C \in \mathrm{SO}(3)$, where

$$\mathrm{SO}(3) = \{C \in \mathbb{R}^{3\times3}\colon C^T C = I, \det C = +1\}$$

to minimize the cost function

$$J = \sum_{k=1}^{N} w_k (\boldsymbol{s}^m_{b,k} - C\boldsymbol{s}_{I,k})^T (\boldsymbol{s}^m_{b,k} - C\boldsymbol{s}_{I,k}), \tag{1}$$

where $0 < w_k < \infty$ are positive weights for $k = 1, \ldots, N$.

Problem 1 is readily shown to be equivalent to solving the minimization problem [6],

$$\text{minimize } \hat{J} = -tr[CB^T] \text{ subject to } C \in \mathrm{SO}(3) \tag{2}$$

where

$$B^T = \sum_{k=1}^{N} w_k \boldsymbol{s}_{I,k} \boldsymbol{s}^{mT}_{b,k} \tag{3}$$

In some instances, it may be desirable to orthonormalize a given matrix $D \in \mathbb{R}^{3\times3}$. For example, when $C \in \mathrm{SO}(3)$, its kinematics satisfy Poisson's equation $\dot{C} = -\omega^\times C$ [10, chapter 2], with initial condition $C(t_0) \in \mathrm{SO}(3)$. Direct numerical integration will result in a solution $\hat{C}(t)$ that is no longer in SO(3) due to numerical inaccuracies. It is therefore desirable to orthonormalize $\hat{C}(t)$ after each numerical integration step. Orthonormalization of a rotation matrix estimate $\hat{C}(t)$ can be thought of in the same manner as normalization of a quaternion estimate.

Considering the matrix D to be an approximation of a matrix $C \in \mathrm{SO}(3)$, it is reasonable to expect $\det[D] > 0$ (otherwise, it would be a very poor approximation). It would be desirable to orthonormalize the matrix $D \in \mathbb{R}^{3\times3}$ in an optimal manner, as stated in the next problem. Note that this is very similar to the orthogonal Procrustes problem [11], in which C is only required to be orthonormal, without any restriction on the sign of its determinant.

Problem 2: (Matrix orthonormalization) Let $D \in \mathbb{R}^{3\times3}$, with $\det D > 0$,

$$\text{minimize } J = tr[(D - C)^T (D - C)] \tag{4}$$

subject to $C \in \mathrm{SO}(3)$.

In fact, since $\det D > 0$, the solution of problem 2 is identical to the orthogonal Procrustes problem.

Analogously to Wahba's problem (problem 1), it is readily shown that the minimization problem in Eq. (4) is equivalent to the minimization problem

$$\text{minimize } \hat{J} = -tr[CD^T] \text{ subject to } C \in \mathrm{SO}(3) \tag{5}$$

Comparing Eqs. (2) and (5), it can be seen that problems 1 and 2 are identical in form. As such, only problem 1 shall be considered from this point on.

It is well known that the set of all solutions to problem 1 is given by [3]

$$C = V\,\mathrm{diag}\{1, 1, \det V \det U\}U^T \tag{6}$$

where V and U are obtained from a SVD of B such that

$$B = V\Sigma U^T \tag{7}$$

where $V^TV = I$, $U^TU = I$ and $\Sigma = \mathrm{diag}\{\sigma_1, \sigma_2, \sigma_3\}$ with $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq 0$. Furthermore, the solution is unique when $\det B > 0$, or $\mathrm{rank}[B] = 2$.

From Eq. (6), it can be seen that

$$C = VU^T, \quad \text{if } \det B > 0 \tag{8}$$

and

$$C = V\,\mathrm{diag}\{1, 1, -1\}U^T, \quad \text{if } \det B < 0 \tag{9}$$

III. LMI-Based Solution

Relax the constraint in Eq. (2) (which is equivalent to problem 1) to obtain the new problem.

Problem 3: Let $B \in \mathbb{R}^{3\times3}$,

$$\text{minimize } \hat{J} = -tr[CB^T] \text{ subject to } \|C\| \leq 1 \tag{10}$$

where $\|A\| = \sigma_{\max}(A)$ is the induced 2-norm of the matrix A, and $\sigma_{\max}$ is the maximum singular value.

Note that $\|C\| = 1$ for all $C \in \mathrm{SO}(3)$, so the constraint set in problem 3 includes the constraint set in Eq. (2).

The constraint set in problem 3 is compact. As such, a global minimizing solution exists. Furthermore, since the cost function $\hat{J}$ is linear in C, the minimizing solution must lie on the boundary of the constraint set. That is, the minimizing solution must satisfy $\|C\| = 1$.

Next, consider any C_1 and C_2 satisfying $\|C_1\| \leq 1$ and $\|C_2\| \leq 1$, and form the convex combination $\alpha C_1 + (1-\alpha)C_2$ for some $\alpha \in [0, 1]$. Then,

$$\|\alpha C_1 + (1-\alpha)C_2\| \leq \alpha\|C_1\| + (1-\alpha)\|C_2\| \leq 1$$

which shows that the constraint set is convex. Consequently, any local minimizing solution of Eq. (10) must be a global minimizing solution. The minimizing solutions are now found in the cases $\mathrm{rank}[B] = 3$ and $\mathrm{rank}[B] = 2$.

Consider a SVD of B, as given in Eq. (7). Since V and U are nonsingular, without loss of generality, one may write

$$C = VSU^T \tag{11}$$

for some $S \in \mathbb{R}^{3\times3}$ (e.g., simply set $S = V^TCU$ given C). Then, problem 3 is equivalent to

$$\text{minimize } \hat{J} = -tr[S\Sigma] \text{ subject to } \|S\| \leq 1 \tag{12}$$

Denote the ijth term of S by s_{ij}. Then, the cost function in Eq. (12) becomes

$$\hat{J} = -\sum_{i=1}^{3} s_{ii}\sigma_i \tag{13}$$

By corollary 1 (see the Appendix), if $\|S\| \leq 1$, it must be that $|s_{ii}| \leq 1$. As such, one has

$$\hat{J}(S) \geq -\sum_{i=1}^{3}\sigma_i, \quad \forall\ S \text{ satisfying } \|S\| \leq 1 \tag{14}$$

Noting that $S = I$ is a member of the constraint set, the lower bound in Eq. (14) is in fact the global minimum for Eq. (12). Therefore, all minimizing S of Eq. (12) must have

$$s_{ii} = 1 \quad \text{if } \sigma_i > 0$$

Correspondingly, by proposition 1 (see the Appendix), they must have

$$s_{ij} = s_{ji} = 0 \quad \text{for } j \neq i \quad \text{if } \sigma_i > 0$$

Hence, if $\mathrm{rank}[B] \geq 2$, any minimizing S of Eq. (12) takes the form

$$S = \mathrm{diag}\{1, 1, s_{33}\} \tag{15}$$

where

$$s_{33} = 1, \quad \text{if } \mathrm{rank}[B] = 3$$

and

$$|s_{33}| \leq 1, \quad \text{if } \mathrm{rank}[B] = 2$$

When $\mathrm{rank}[B] = 2$ (and thus $\sigma_{33} = 0$), then s_{33} has no effect on the cost function given in Eq. (13), and, as such, s_{33} may arbitrarily be chosen subject to the norm constraint $\|S\| \leq 1$.

A. Case 1: rank[B] = 3

When B has full rank, from Eqs. (16) and (11), the minimizing C for problem 3 is unique and is given by

$$C = VU^T \tag{16}$$

Comparing this to Eq. (8), it can be seen that it coincides with the solution to Wahba's problem (problem 1) when $\det B > 0$. However, it does not coincide when $\det B < 0$ [compare Eq. (16) to Eq. (9)].

B. Case 2: *rank*[B] = 2

When $\mathrm{rank}[B] = 2$, from Eqs. (16) and (11), the minimizing solutions of problem 3 are nonunique and are given by

$$C = V\,\mathrm{diag}\{1, 1, s_{33}\}U^T \tag{17}$$

where

$$|s_{33}| \leq 1, \quad \text{if } \mathrm{rank}[B] = 2$$

Consequently, problem 3 is only equivalent to Wahba's problem (problem 1) when $\det B > 0$.

Problem 3 shall now be reformulated as a convex optimization problem with a LMI constraint. To this end, note that the norm constraint in problem 3 is equivalent to

$$C^TC \leq I \tag{18}$$

Recall the Schur complement [12, chapter 2]: For $\Phi_{11} = \Phi_{11}^T \in \mathbb{R}^{p\times p}$, $\Phi_{12} \in \mathbb{R}^{p\times q}$, $\Phi_{21} \in \mathbb{R}^{q\times p}$, $\Phi_{22} = \Phi_{22}^T \in \mathbb{R}^{q\times q}$, where $\Phi_{22} > 0$, then

$$\Phi_{11} - \Phi_{12}\Phi_{22}^{-1}\Phi_{21} \geq 0 \Leftrightarrow \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \geq 0$$

Therefore, setting $\Phi_{11} = I$, $\Phi_{12} = C^T$, $\Phi_{21} = C$, $\Phi_{22} = I$ and using the Schur complement, Eq. (18) is in turn equivalent to the LMI,

$$\begin{bmatrix} I & C^T \\ C & I \end{bmatrix} \geq 0$$

Therefore, problem 3 may be recast as the following LMI problem.

Problem 4: Let $B \in \mathbb{R}^{3\times3}$. Find $C \in \mathbb{R}^{3\times3}$ to

$$\text{minimize } \hat{J} = -tr[CB^T] \tag{19}$$

subject to

$$\begin{bmatrix} I & C^T \\ C & I \end{bmatrix} \geq 0 \tag{20}$$

The previous analysis is now summarized in the following theorem.

Theorem 1: Problem 4 with $\det B > 0$ has a unique global minimum, with no other local minima. In this case, the solution of problem 4 is equal to the unique solution of problem 1.

Problem 4 may be easily solved using existing LMI solvers.

The limitation of requiring $\det[B] > 0$ for the LMI-based solution to Wahba's problem is now examined. Suppose that the measured vectors $\boldsymbol{s}^m_{b,k} \in \mathbb{R}^3$ are generated according to

$$\boldsymbol{s}^m_{b,k} = \boldsymbol{s}_{b,k} + \boldsymbol{v}_k \tag{21}$$

where

$$\boldsymbol{s}_{b,k} = C\boldsymbol{s}_{I,k}, \tag{22}$$

and $\boldsymbol{v}_k \in \mathbb{R}^3$ is a measurement error. Then, Eq. (3) becomes

$$B^T = \bar{B}^T + \Delta B^T \tag{23}$$

where

$$\bar{B}^T = \sum_{k=1}^{N} w_k \boldsymbol{s}_{I,k} \boldsymbol{s}^T_{b,k}, \qquad \Delta B^T = \sum_{k=1}^{N} w_k \boldsymbol{s}_{I,k} \boldsymbol{v}^T_k \tag{24}$$

Rewrite $\bar{B}^T$ and ΔB^T in Eq. (24) as

$$\bar{B}^T = S_I \bar{W} S^T_b, \qquad \Delta B^T = S_I \bar{W} \bar{V}^T \tag{25}$$

where

$$S_I = [\boldsymbol{s}_{I,1} \quad \cdots \quad \boldsymbol{s}_{I,N}], \qquad \bar{W} = \text{diag}\{w_1, \ldots, w_N\},$$
$$S_b = [\boldsymbol{s}_{b,1} \quad \cdots \quad \boldsymbol{s}_{b,N}]$$

and

$$\bar{V} = [\boldsymbol{v}_1 \quad \cdots \quad \boldsymbol{v}_N]$$

From Eq. (22), one obtains

$$S_b = CS_I$$

Therefore,

$$\bar{B}^T = S_I W S^T_I C^T \tag{26}$$

Consequently, one has

$$\det \bar{B} = \det(S_I \bar{W} S^T_I) \det C$$

When $\bar{B}$ has full rank, the matrix $S_I \bar{W} S^T_I$ is positive definite, and

$$\text{sign}[\det \bar{B}] = \text{sign}[\det C] \tag{27}$$

Finally, by continuity of the determinant and Eq. (23) together with Eq. (25), it is concluded that for a given S_I and weight $\bar{W}$, there exists $\delta > 0$ such that

$$\text{sign}[\det B] = \text{sign}[\det C], \quad \forall\ \bar{V} \in \mathbb{R}^{3\times N} \text{ such that } \|\bar{V}\| < \delta \tag{28}$$

That is, if the collection of vectors $\boldsymbol{s}_{i,k}$ is geometrically rich enough, and the measurement errors $\boldsymbol{v}_k$ are small enough, $\det B$ will have the same sign as $\det C$. Clearly, there must therefore be at least three vector measurements. On the other hand, if the objective is to solve problem 2, then the determinant condition is automatically satisfied.

IV. Numerical Examples

The LMI-based solution to Wahba's problem is now demonstrated with a pair of numerical examples. In the first example, a noise-free set of measurement vectors is used, demonstrating that the LMI-based method returns the original rotation matrix, which in this case is the known optimal solution to Wahba's problem. In the second example, a set of noise-corrupted measurement vectors is used. In this case, the LMI-based solution is compared to the SVD-based solution in Eq. (6), as well as other well-established solution methods including the q-method [13, chapter 12], QUEST [5], and ESOQ2 [14], each of which returns the same solution. All numerical work is done on a MacBook Pro with a 2.3 GHz Intel Core i5 processor and 4 GB of RAM running MATLAB® 7.12.0 (R2011a).

Both examples use the vectors and weights

$$\boldsymbol{s}_{I,1} = \frac{1}{\sqrt{5}}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \qquad \boldsymbol{s}_{I,2} = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix},$$
$$\boldsymbol{s}_{I,3} = \frac{1}{\sqrt{26}}\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{s}_{I,4} = \frac{1}{\sqrt{18}}\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \qquad \boldsymbol{s}_{I,5} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and

$$w_k = \frac{1}{\sigma_i^2}, \qquad k = 1 \ldots 5$$

where

$$\sigma_1 = 0.0100, \qquad \sigma_2 = 0.0325, \qquad \sigma_3 = 0.0550,$$
$$\sigma_4 = 0.0775, \qquad \sigma_5 = 0.1000$$

Assume the true attitude is given by

$$C = C_3(60^\circ)C_2(-30^\circ)C_1(45^\circ) = \begin{bmatrix} 0.4330 & 0.4356 & 0.7891 \\ -0.7500 & 0.6597 & 0.0474 \\ -0.5000 & -0.6124 & 0.6124 \end{bmatrix}$$

where $C_1(\cdot)$, $C_2(\cdot)$, and $C_3(\cdot)$ are principal rotations about the 1, 2, and 3 axes, respectively [10, chapter 2].

A. Case 1: Noise-Free Measurements

Consider the case in which the vector measurements are not corrupted by any noise, that is, $\boldsymbol{s}^m_{b,k} = C\boldsymbol{s}_{I,k}$. In this case, it is simple to show that $\text{rank}[B] = 3$ and $\det[B] > 0$, and, as such, the solution to problem 4, equals the unique solution to problem 1. Using the software YALMIP [15] and SeDuMi [16] to solve problem 4, C is exactly recovered, as expected.

B. Case 2: Noisy Measurements

Next, consider the case in which the vector measurements are corrupted by noise. Specifically, consider the following vector measurements:

$$\boldsymbol{s}^m_{b,1} = \begin{bmatrix} 0.9082 \\ 0.3185 \\ 0.2715 \end{bmatrix}, \qquad \boldsymbol{s}^m_{b,2} = \begin{bmatrix} 0.5670 \\ 0.3732 \\ -0.7343 \end{bmatrix},$$
$$\boldsymbol{s}^m_{b,3} = \begin{bmatrix} -0.2821 \\ 0.7163 \\ 0.6382 \end{bmatrix}, \quad \boldsymbol{s}^m_{b,4} = \begin{bmatrix} 0.7510 \\ -0.3303 \\ 0.5718 \end{bmatrix},$$
$$\boldsymbol{s}^m_{b,5} = \begin{bmatrix} 0.9261 \\ -0.2053 \\ -0.3166 \end{bmatrix}$$

Again, it is straightforward to verify that $\text{rank}[B] = 3$ and $\det[B] > 0$. As such, the solution to problem 4 equals the unique solution to problem 1. Using the software YALMIP and SeDuMi to solve problem 4, the best estimate of C, called $\hat{C}$, is found to be

$$\hat{C} = \begin{bmatrix} 0.4153 & 0.4472 & 0.7921 \\ -0.7562 & 0.6537 & 0.0274 \\ -0.5056 & -0.6104 & 0.6097 \end{bmatrix}$$

Define the error between C and $\hat{C}$ to be $C_e = \hat{C}C^T$. Recall that any element of SO(3) can be expressed in terms of a Euler axis and a Euler angle [10, chapter 2]. To assess how close $\hat{C}$ is to C, the Euler angle associated with C_e will be computed. The Euler angle of C_e is $\phi_e = 1.27^\circ$, where $\cos\phi_e = \cos(0.5(\mathrm{trace}\, C_e - 1))$, indicating that $\hat{C}$ is a good estimate of C.

Using either the q-method [13, chapter 12], QUEST [5], the SVD method [3], or ESOQ2 [14] to find $\hat{C}$ yields the same result as the LMI solution presented. This is expected, as they are each a solution to the same problem, that being problem 1. The execution time is computed using MATLAB®'s tic and toc functions. As expected, ESOQ2 is the fastest algorithm (0.000607732 s) [6], followed by QUEST (0.00077290 s), the q-method (0.00143210 s), the SVD method (0.00307701 s), and finally the proposed LMI method (0.0662991 s).

The fact that the LMI method is the slowest is due to the fact that the YALMIP and SeDuMi have been used; these are general tools used to solve LMI problems. ESOQ2, the q-method, and QUEST are taylored to find the quaternion representing the attitude, and, as such, their execution time is much faster. Additionally, the q-method uses MATLAB®'s custom eigenvalue solver, and the SVD method uses MATLAB®'s custom SVD solver, both of which are highly optimized. Future work will focus on designing a custom computer code to solve problem 4, exploiting the specific structure of the objective function and LMI constraint. A custom computer code is expected to decrease the execution time significantly. As discussed in [17,18], a custom computer code can be executed much faster than a standard code, depending on the application.

Remark: Although the LMI form of the constraint given in Eq. (20) has been used to solve for the attitude in Wahba's problem, Eq. (20) can be used as a constraint in other optimization problems, such as guidance and control problems involving attitude. For example, in the domain of Mars powered-decent guidance, convex optimization methods are often employed [19]. Future work will investigate the use of Eq. (20) in guidance and control problems.

V. Conclusions

This Note has presented a new characterization of the solution to the famous Wahba problem. It has been shown that, when a mild condition is satisfied (which is demonstrated to hold for many practical problems of interest), the Wahba problem can be recast as a LMI optimization problem. This opens the door to a whole new class of solution methods for these types of Wahba problems. Equivalence between the Wahba problem and the LMI problem is accomplished by relaxing the nonconvex constraint on the rotation matrix $C \in \mathrm{SO}(3)$, creating instead a convex constraint of the form $\|C\| \leq 1$, which is equivalently represented in LMI form. While this approach has been demonstrated for the Wahba problem, it has applicability to other optimization problems involving attitude, such as guidance and navigation problems.

Appendix: Useful Matrix Results

Proposition 1: Consider any matrix $A \in \mathbb{R}^{n\times m}$, with $\|A\| = \ell$, for some $\ell \geq 0$. Denote the ijth term of A as a_{ij}. Then,

$$\sqrt{\sum_{i=1}^{n} a_{ij}^2} \leq \ell, \qquad j = 1, \ldots, m \tag{29}$$

and

$$\sqrt{\sum_{j=1}^{m} a_{ij}^2} \leq \ell, \quad i = 1, \ldots, n \tag{30}$$

Proof: By definition,

$$\|A\| = \max_{\|x\|_2=1} \|Ax\|_2$$

where $\|x\|_2 = \sqrt{x^T x}$ denotes the vector 2-norm. Let us now set $x = e_j = [e_{j,1}, \ldots, e_{j,m}]^T$, for some $j \in \{1, \ldots, m\}$, where

$$e_{j,k} = \begin{cases} 1, & k = j, \\ 0, & \text{otherwise} \end{cases}$$

Clearly, $\|x\|_2 = 1$, and

$$Ax = [a_{1,j}, \ldots, a_{n,j}]^T$$

Then, by definition of the matrix norm previously mentioned, one must have

$$\|Ax\|_2 = \sqrt{\sum_{i=1}^{n} a_{ij}^2} \leq \|A\| = \ell$$

which is Eq. (29). Since $\|A\| = \|A^T\|$, repeating the previous argument for A^T yields Eq. (30).

The following corollary is an immediate consequence of proposition 1.

Corollary 1: Consider any matrix $A \in \mathbb{R}^{n\times m}$, with $\|A\| = \ell$, for some $\ell \geq 0$. Denote the ijth term of A as a_{ij}. Then,

$$|a_{ij}| \leq \ell.$$

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