International Perspectives on the Teaching and Learning of Geometry in Secondary Schools

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INTERNATIONAL PERSPECTIVES ON SECONDARY GEOMETRY EDUCATION: AN INTRODUCTION

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This book is one of the outcomes of Topic Study Group 13 at the 13th International Congress on Mathematical Education, which took place in Hamburg, Germany, in the summer of 2016. Our Topic Study Group (TSG-13) concerned the teaching and learning of secondary geometry and the chapters in this volume include revised versions of most of the papers presented at the main meetings of the group. Also included are a handful of the shorter papers associated with TSG-13 in the context of short oral communications. In this brief introduction we orient the reader to these papers by first providing an organizer of the focus of our study group.

The International Congress in Mathematics Education gathers researchers and practitioners in mathematics education and pursues a goal of inclusiveness across all sorts of boundaries. In particular, the boundaries between research and practice are often blurred in ICME and this surely applied to our Topic Study Group 13 in ICME-13. Therefore, to orient the reader to the chapters in the book, it might be useful to describe the territory or field of practice associated with the teaching and learning of secondary geometry.

As we engage in such a description, we might benefit from using the metaphor of map-making as a guiding principle. Borges's short story *On exactitude in science* uncovers the futility of expecting that a map be produced on a scale 1:1. Yet the value of maps as containers of geographic knowledge and as resources for travelers cannot be overemphasized, even if the existence of different kinds of projection techniques reminds us that any map has limitations in what it affords its readers. Different maps afford us different kinds of insight on the territory.

There is a constellation of practices that might be spotted as we look toward the teaching and learning of geometry in secondary schools. At the center of this constellation is the classroom practice of students and teacher transacting geometric meanings. Near that center one can find the practice of textbook writing and materials development for secondary geometry; one can also find the practice of preparing teachers to teach secondary school geometry; and the individual practice of thinking and problem solving that youngsters of secondary school age may engage in even outside of school. But as we look closer, finer, relevant distinctions can be made.

The practice of teaching and learning geometry in classrooms admits of one set of distinctions regarding the institutional location of those classrooms: American secondary schools locate that practice in a single high school geometry course, while geometry is integrated with other content areas in most other countries, and also occurs outside of compulsory education, in other organized settings such as summer camps. None of our papers inquires specifically on the institutional situatedness of geometry instruction, though Kuzniak's chapter recommends investigating whether there is a place for the study of geometry in all educational systems, and uses a contrast between work observed in Chile and in France as a way into his approach to questioning the nature of geometric work. Other chapters present inquiries that seem to rely on such situatedness. The chapter by Berendonk and Sauerwein, for example, describes geometry experiences with novel content in the context of a summer course for mathematically-inclined students, and the chapter by Herbst, Boileau, and Gürsel examines how the instructional situations that are customary in the US high school geometry course serve to frame a novel geometry task. Steeped into the institutional location of the teaching and learning of geometry in high school in the United States, Senk, Thompson, Chen, and Voogt examine outcomes of geometry courses taught using the Geometry text from the University of Chicago School Mathematics Program. Likewise Hunte's chapter examines curricular variations situated in the context of textbooks of different eras in Trinidad and Tobago.

Specific geometry content at stake in classroom instruction, as well as in teacher development, textbook writing, and thinking and problem solving is discussed implicitly or explicitly in all chapters. Several chapters focus on specific geometric concepts: area of trapezoids (Manizade and Martinovic's chapter), area of triangles (Cheah's chapter), properties of quadrilaterals (Herbst, Boileau, and Gürsel's chapter), polytopes (Berendonk and Sauerwein's chapter), rotations (Battista and Frazee's chapter), and connections to functions (Steketee and Scher's chapter). Specific geometric processes are also present as Hunte's chapter deals with the work of calculating, the chapter by Chinappan, White and Trenholm include descriptions of the work of constructing, Luz and Soldano's paper addresses the work of conjecturing, and Cirillo's paper deals with the work of proving.

The nature of and difficulties in students' thinking, learning, achievement, and problem solving in geometry are under consideration in several chapters. Across these chapters there is attention to spatial thinking and to aspects of deductive reasoning from conjecturing to proving. Maresch's chapter is focused on students' spatial capabilities, Arai's chapter deals with how students answer spatial orientation tasks, and Battista and Frazee provide detailed descriptions of how students reason in the context of rotation tasks. The chapter by Cirillo describes successful and unsuccessful students' thinking and collaboration in proof tasks. Similarly, Webre, Smith, and Cuevas address the time and quality of students' conjecturing in connection with their engagement in discussions. And the chapter by Luz and Soldano demonstrates how computer-based games engage students in conjecturing and falsifying. Many of those processes are involved in the explorations proposed by Villella and his collaborators. Senk and her colleagues map the variability in students' achievement in a geometry test and look for ways to account for it.

The role of tools and resources in geometry instruction, thinking, materials development, and teacher development is also quite apparent. The technological mediation of materials development in geometry is eloquently illustrated by Steketee and Scher in their chapter showing how dynamic geometry provides a different access to the connections between functions and geometry. Technological mediation of students' thinking and learning is present in the chapter by Battista and Frazee who illustrate the use of iDGi in eliciting students reasoning. Also discussing the mediation of students' thinking, Luz and Soldano demonstrate how games can be developed through dynamic geometry, internet communication, and turn-taking. The role of Dynamic Geometry Software in teacher development is discussed in the chapter by Villella and associates, while Webre and her colleagues make comparable points in the case of classroom instruction. Richard, Gagnon, and Fortuny add intelligent tutoring to dynamic geometry. This chapter's focus is on students' blockage during geometric problem solving and how an intelligent tutor can support students' thinking. Along with Orozco's chapter on the role of writing, these last two help the book connect issues of mediation to metacognition.

Instruments for geometry instruction, thinking, materials development, or teacher development need not be technological though. The chapter by Cheah describes the use of the professional development practice called *lesson study* in the design and planning of a lesson on area by a group of teachers. The chapter by Herbst and his colleagues examines how a teacher made use of instructional situations of exploration, construction, and proof, which were available in her class, to frame a novel geometry task on quadrilaterals as it was implemented in a geometry course. The chapter by Chinnappan, White, and Trenholm describes the work of teaching geometry in terms of its use of specialized and pedagogical content knowledge. As regards the development of ways of assessing teacher knowledge Manizade and Martinovic demonstrate how they use student work to elicit teachers' responses that allow them to assess what they know about specific geometric topics. In contrast, Smith uses the MKT-G test (Herbst & Kosko, 2014) to measure the amount of mathematical knowledge for teaching geometry of practicing and preservice teachers across the domains hypothesized by Ball, Thames, and Phelps (2008). Additionally, Smith uses a questionnaire to access self-reported pedagogical practices of her participants. Also, the chapters by Villella and his colleagues from Grupo CEDE describes how teachers' knowledge of geometry can be developed through experiences framed using ideas from the theory of geometric working spaces introduced earlier in Kuzniak's chapter.

As the chapters address those practices, they do so from multiple perspectives that cover the range between practitioner and researcher. The chapters by Berendonk and Sauerwein and by Steketee and Scher illustrate the work of developing curriculum materials for the teaching of geometry. The development of assessments for teachers is showcased in the paper by Manizade and Martinovic, while the development of games for students is showcased in the paper by Luz and Soldano. The chapter by Cheah illustrates the work of engaging teachers in professional development using lesson study, while the chapter by Villella et al. describes activities used in other professional development activities. The chapters by Maresch, by Senk et al., and by Smith are based, at least in part, on the use of tests. The observation of actual classroom interaction is present in a number of papers including, in particular, Chinnappan et al.'s chapter and Herbst et al.'s chapter. We come back in the conclusion to some methodological aspects of the work presented.

The various ways in which we map the practices of teaching and learning geometry in secondary school highlight many connections and distinctions among the chapters in the book. Surely more can be found through reading and with such purpose we invite the reader to dig in. The book represents a collaborative effort among editors in four different countries (Canada, Malaysia, the United Kingdom, and the United States) working alongside 40 authors, affiliated with 25 different institutions from 14 different countries. These authors put together 19 chapters. In such representation of diversity, this book not only represents diverse perspectives on the practice of teaching and learning geometry in secondary schools, but also represents the diversity among the individuals who attended ICME-13. May this diverse offering of ideas inspire the reader to become a contributor to ICME in the future.

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THINKING ABOUT THE TEACHING OF GEOMETRY THROUGH THE LENS OF THE THEORY OF GEOMETRIC WORKING SPACES

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In this communication, I argue that shared theoretical frameworks and specific topics need to be developed in international research in geometry education to move forward. My purpose is supported both by my experience as chair and participant in different international conferences (CERME, ICME), and also by a research program on Geometric Working Spaces and geometric paradigms. I show how this framework allows thinking about the nature of geometric work in various educational contexts.

Keywords: Construction, Discursive dimension, Geometric work, Geometric paradigms, Geometric working space, Instrumental dimension, Proof, Register of representation, Reasoning, Semiotic dimension, Visualization

Introduction

The purpose of this essay is not to give a general and critical overview of research done in the domain of geometry education. First, this type of survey already exists (e.g., the recent and very interesting ICME-13 survey team report, Sinclair, et al., 2016), and secondly, because given the extension of this field, such surveys are generally partial and, sometimes, even biased. Indeed, geometry is taught from kindergarten to university in many countries, and students engage with it in very different ways, eventually depending on their professional orientation (e.g., architects, craft-persons, engineers, mathematics researchers). Geometry is also a main topic in the preparation of primary and secondary school teachers. Rather, what I want to do in this contribution is to formulate some ideas based on my experience as researcher involved both in the CERME geometry working group, which I was lucky to participate in or chair several times, and in the development of an original model designed for the analysis of issues related to the teaching of geometry, but also for comparative studies of this teaching in various countries.

In one of his rare articles on the teaching of geometry, Brousseau (1987) insists on the need of finding a substitute for the “natural” epistemological vigilance one would expect from mathematicians but which is missing on account of the extinction of any mathematical research on elementary geometry: This substitute would enable the field to avoid the uncontrolled *didactification* of geometry that Brousseau finds in teachers’ practices. Brousseau stresses the essential relationship between epistemology and didactics in the teaching of geometry. In my view, this search for a source of vigilance should pass through well-identified research themes, and be based on development of shared theoretical frameworks in geometry education even if they can be diverse to be adapted in a variety of contexts.

During the symposium honoring Artigue in Paris in 2012, Boero (2016) drew the audience's attention to the fact that the role of researchers in mathematics education depends on strong cultural and institutional components that vary from one country to another. In his country, Italy, researchers in the domain have to be active in two opposite directions: In developing innovation and textbooks with an immediate impact on the country's school life, and at the same time, in developing a research field which can be independent of immediate applications. In all countries, in some form, researchers should be involved to influence education in the country in which they live. But at the same time and independently of any political pressure, they should also evaluate and compare existing teaching activities by researching their effects on the actual mathematical development of students faced with such set of tasks. In addition, research must, as far as possible, highlight and explore invariant parameters that may exist in different contexts. Furthermore, well-accepted findings in didactics of geometry should be known and taken into account by researchers to ensure progress in the domain. Even when this is far from easy, the field of research on geometry education would benefit from being structured around theoretical frameworks and specific research themes to stop being always an emergent scientific domain. Supported by the model of Geometric Working Spaces (GWS) and the related notion of geometric paradigms, I develop a possible approach in this direction. Naturally, the GWS model is only used as an example to show the possible interest of theoretical approaches in the domain. Indeed, a diversity of theoretical approaches is needed to address the wide variety of issues in such an extended field as geometry education.

Travel in a Changing Territory Constantly in Reconstruction

The difficulty of developing research and a common theoretical framework in geometry education comes first from its chaotic evolution over the last decades. In the early sixties, the French mathematician Dieudonné became widely known in the education field by his famous cry "Euclid must go!" At the time, he wanted to denounce a mathematical education ossified around notions that he considered outdated and, in particular, what was called the geometry of the triangle. He did not wish to destroy the teaching of geometry but rather to promote a consistent teaching of this domain, based on more recent mathematical research and, particularly, focusing on algebraic structures. According to Dieudonné, students should enter directly into the most powerful mathematics without any long detours through concepts and techniques that he considered obsolete. This questioning of traditional geometry education initiated a series of reforms and counter-reforms. While some of those reforms sought to bring school geometry closer to the geometry of mathematicians, others have been sought to avoid learning difficulties that students had faced. The teaching of geometry has become more and more utilitarian over time, as exemplified and guided by the PISA expectations.

Furthermore, the teaching of geometry is marked by a great variability among curricula across countries, which makes difficult the consistent networking of researchers on specific topics. This variability can be illustrated by the place that geometric transformations have had since the early seventies to the present in the French curriculum[1]. In the 1970s, heavily influenced by the *mathématiques modernes* (i.e., the new Math), geometric transformations such as translations and similarities were used to separate affine and Euclidean properties. Then in the 1980s, transformations were studied in close relation with linear algebra and analytic work in two and

[1] The French curriculum is set by the central government and official instructions are published in the Journal Officiel. Our short summary on the evolution of the teaching of geometric transformations is based on this material.

three dimensions. There was then also important work on how symmetries generate isometric transformations. In the 1990s, the work became more geometric and transformations were limited to the plane and to explore configurations like regular polygons, as transformations were implicitly associated with the dihedral groups of polygons and the group of similarities associated with complex numbers was the culmination of that mathematical journey. In the 2000s, the importance of transformations decreased again with the disappearance of dilations and similarities. As of 2008, translations and symmetries were the only transformations that remained, as even rotations had disappeared. But in 2016, plans were made to reintroduce geometric transformations from the beginning of secondary school.

That erratic evolution is not without consequence on teachers' mathematical culture. Indeed, new teachers face the challenge of having to teach subjects they do not really know well and from which they do not master even elementary techniques. Surprising situations occur when, as in CERME in 2011, researchers from countries where geometric transformations were just re-introduced in elementary school were wondering whether it is possible to teach them to young students. French researchers could only report that it was possible, but that transformations were just removed from their curriculum.

Taking into account the Diversity of the Teaching of Geometry

What geometry is being taught?

Before dealing with this question, we need to ask ourselves if there is a place for geometry, as a discipline clearly identified, in all education systems. Indeed, one of the main issues of the report on geometry done by the Royal British Society (2001) was to foster the reappearance of the term geometry in the British curriculum. Prior to this, the study of geometry had been hidden under the heading "shape, space and measure." Similar disappearance is apparent in the PISA assessment, in which geometry topics are covered up by the designation "space and shape".

These changes of vocabulary are not harmless as they are not only changes in vocabulary; rather, they reveal different choices about the nature of the geometry taught in school. The choices imply either a focus on objects close to reality or on objects already idealized. The decisions on the type of intended geometry relate to different conceptions of its role in the education of students, and also, more generally, on the citizen's position in society. In the French National Assembly, during the middle of the nineteenth century, a strong controversy about the nature of the geometry taught in school pitted the supporters of a geometry oriented towards immediate applications to the world of work against the defenders of a more abstract geometry oriented to the training of reasoning (Houdement & Kuzniak, 1999). During the second half of the twentieth century, a third more formal and modernist approach, based on linear algebra, was briefly, but with great force, added to the previous two (Gispert, 2002). Thus, over the long term and in a single country[2], the nature of geometry taught fluctuated widely and issues and goals have changed dramatically depending on decisions often more ideological and political than scientific.

[2] This conflicting approach on the teaching of geometry is not typically French. In the US, similar tensions albeit among four different conceptions exist too (Gonzalez and Herbst, 2006) based on formal, utilitarian, mathematical or intuitive arguments.

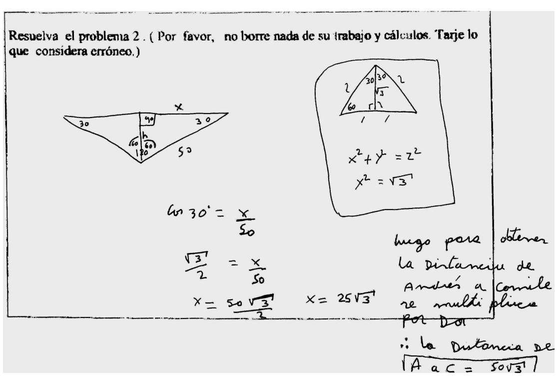
Observations of the choices made nowadays in various countries reveal irreconcilable approaches that seem to resurrect the debate mentioned above.

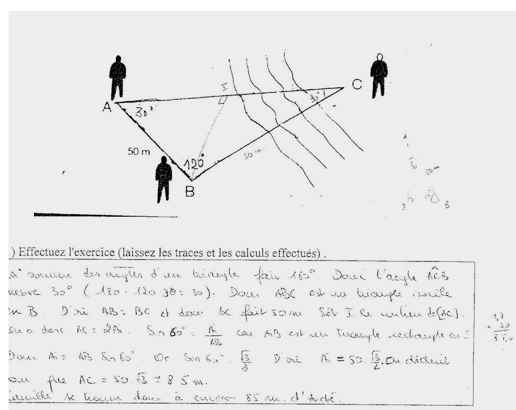
Questions of style

Anybody that has had the opportunity to observe classroom instruction in a country other than his or her own must have noticed differences in style that can hardly be accounted to individual differences. The researchers affiliated with the TIMSS sub-study on teaching practices in six countries noticed such differences in style, and they used the notion of "characteristic pedagogical flow" to account for recurrent and typical styles they observed (Cogan & Schmidt, 1999).

To me, this variety of styles appears when reading Herbst's historical study (2002) on two-column proofs in the USA. This way of writing proofs is similar to nothing existing now in France though it is reminiscent of an old fashioned way used to write solutions of problems in primary school where operations have to be separated from explanations of reasoning. Another case of cultural shock appears too when reading Clanché's and Sarrazy's (2002) observation of a first-grade mathematics lesson in a Kanaka primary school (New Caledonia). This time, the teacher cannot easily assess the degree of understanding of his students for whom customary respect for the elders forbids their expression of doubts and reservations in public and thus they never ask some complementary explanation to the teacher. The analysis of the classroom session allows the authors to claim that the relationship between mathematics teaching and students' everyday life should be analyzed as rupture or obstacle more than as continuity or facilitation.

Let us consider some different styles through an observation made during a comparative study on the teaching of geometry in Chile and France (Guzman et al, 2006). Various exercises were given to high school pre-service teachers in Strasbourg, France and in Valparaiso, Chile. As an illustration, we show two students' work using exactly the same solution method but presenting it in radically different ways. Both are characteristic of what is expected by their teachers.





Chilean student's solution

French student's solution

Figure 1: Comparison of two writing solutions in Chile and France

In Chile, results are given on a coded drawing and the reasoning used is not explicitly given in writing. By contrast, in France, a very long and detailed text is written and no assertion, not even the most trivial, is omitted. This point is clearly apparent in Figure 1 even if Spanish and French texts are not translated.

Observations of Chilean classrooms show that what is written on the blackboard during a session is often similar to the student's written production and only oral justifications are provided, while in France all arguments have to be written (Guzman & Kuzniak, 2006). Knipping (2008) also shows differences in the use of the blackboard and in articulation between the written and the oral in France and Germany. More generally, Knipping (2008) shows that argumentation and proof[3] are not equivalent in both countries; rather they give birth to different ways of developing geometric work in the same Grade.

How can we account for these differences in "style" avoiding, if possible, any hierarchical comparison based on the idea that one approach is fundamentally better than the other? In the following, I will propose a way to explore these differences based on the use of geometric paradigms and the theoretical and methodological model of Geometric Working Spaces (GWS).

Various Geometries and Geometric Work

Three elementary geometries

Houdement and Kuzniak (1999) introduced the notion of geometric paradigms into the field of didactics of geometry to account for the differences in styles in geometry education. To bring out geometric paradigms, three perspectives are used: epistemological, historical, and didactical. The assemblage of those perspectives led to the identification of three paradigms usually named Geometry I (or Natural Geometry), Geometry II (or Natural Axiomatic Geometry), and Geometry III (or Formal Axiomatic Geometry). These paradigms—and this is an original feature of the approach—are not organized in a hierarchy, making one more advanced than another. Rather, their scopes of work are different and the choice of a path for solving a problem depends on the purpose of the problem and the solver's paradigm.

The paradigm called Geometry I is concerned by the world of practice with technology. In this geometry, valid assertions are generated using arguments based upon perception, experiment, and deduction. There is high resemblance between model and reality and any argument is allowed to justify an assertion and to convince the audience. Indeed, dynamic and experimental proofs are acceptable in Geometry I. It appears in line with a conception of mathematics as a toolkit to foster business and economic activities in which geometry provides tools to solve problems in everyday life.

The paradigm called Geometry II, whose archetype is classic Euclidean geometry, is built on a model that approaches reality without being fused with it. Once the axioms are set up, proofs have to be developed within the system of axioms to be valid. The system of axioms may be left incomplete as the axiomatic process is dynamic and has modeling at its core.

[3]In this essay, I mean *proof* more generally than mathematical or formal proof and different ways of arguing or validating are considered.

Both geometries, I and II, have close links to the real world, albeit in varying ways. In particular, they differ with regard to the type of validation, the nature of figure (unique and specific in Geometry I, general and definition-based in Geometry II) and by their work guidelines. To these two Geometries, it is necessary to add Geometry III, which is usually not present in compulsory schooling, but which is the implicit reference of mathematics teachers who are trained in advanced mathematics. In Geometry III, the system of axioms itself is disconnected from reality, but central. The system is complete and unconcerned with any possible applications to the real world. The connection with space is broken and this geometry is more concerned with logical problems (Kuzniak & Rauscher, 2011).

Geometric Working Spaces

The model of GWS[4] was introduced in order to describe and understand the complexity of geometric work in which students and teachers are effectively engaged during class sessions. The abstract space thus conceived refers to a structure organized in a way that allows the analysis of the geometric activity of individuals who are solving geometric problems. In the case of school mathematics, these individuals are generally not experts but students, some experienced and others beginners. The model articulates the epistemological and cognitive aspects of geometric work in two metaphoric planes, the one of epistemological nature, in close relationship with mathematical content of the studied area, and the other of cognitive nature, related to the thinking of individuals solving mathematical tasks. This complex organization is generally summarized using the two diagrams shown in Fig. 1 and Fig. 2. (For details, see Kuzniak & Richard, 2014; Kuzniak, Tanguay, & Elia, 2016):

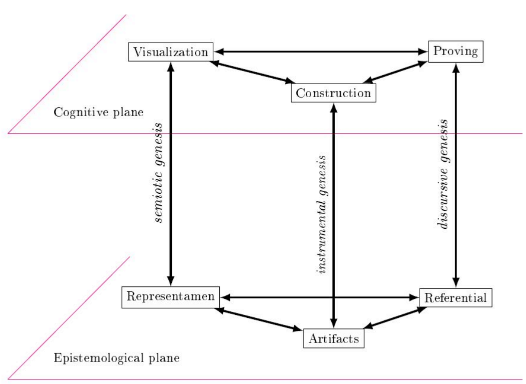


Figure 2: The Geometric Working Space Diagram

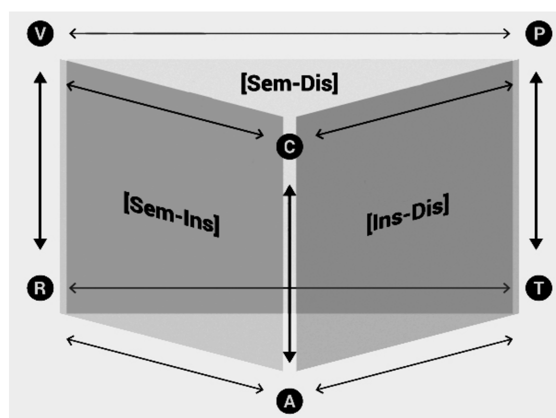


Figure 3: The three vertical planes in the GWS

Three components in interaction are characterized for the purpose of describing the work in its epistemological dimension, organized according to purely mathematical criteria: a set of concrete and tangible objects, the term *representamen* is used to summarize this component; a set of

[4]An extension of this model to the whole of mathematical work has been developed under the name of *Mathematical Working Space* (MWS).

artifacts such as drawing instruments or software; a theoretical system of reference based on definitions, properties and theorems.

The cognitive plane of the GWS model is centered on the subject, considered as a cognitive subject. In close relation to the components of the epistemological level, three cognitive components are introduced as follows: visualization related to deciphering and interpreting signs; construction depending on the used artifacts and the associated techniques; proving conveyed through validation processes, and based on a theoretical frame of reference.

The process of bridging the epistemological plane and the cognitive plane is part of geometric work according our perspective and can be identified through the lens of GWSs as three geneses related to each specific dimension in the model: semiotic, instrumental, and discursive geneses. This set of relationships can be described proceeding from the elements of the first diagram (Figure 2) which, in addition, shows the interactions between the two planes with three different dimensions or geneses: semiotic, instrumental, and discursive. The epistemological and cognitive planes structure the GWS into two levels and help us understand the circulation of knowledge within mathematical work. How then, proceeding from here, can students articulate the epistemological and cognitive levels in order to do the expected geometric work? In order to understand this complex process of interrelationships, the three vertical planes of the diagram are useful and can be identified by the geneses that they implement: [Sem-Ins], [Ins-Dis], and [Sem-Dis] (Figure 3). The precise study and definition of the nature and dynamics of these planes during the solving of mathematical problems remains a central concern for a deeper understanding of the GWS model (Kuzniak, et al., 2016).

A GWS exists only through its users, current or potential. Its constitution depends on the way users combine the cognitive and epistemological planes and their components for solving geometric problems. It also depends on the cognitive abilities of a particular user, expert or beginner in geometry. The make-up of a GWS will vary with the education system (the reference GWS), the school circumstances (the suitable GWS) and the practitioners (personal GWS).

The framework makes it possible to question in a didactic and scientific–non ideological–way the teaching and learning of geometry.

What is the geometry aimed at by education systems? What is the selected paradigm? Does this paradigm get selected or does it emerge from practice in schooling conditions? How do the different paradigms relate to each other? Moreover, the nature and composition of the suitable GWS is to be questioned: What artifacts are used? On which theoretical reference is the implemented geometric work really grounded? Which problems are used as exemplars to lead students in geometric work?

Two Examples Showing the Use of the Framework

In the following, I develop two examples showing the possibilities offered by the framework to deal with the above questions. I refer the interested reader to various papers using the framework and its extensions, and, specially, the *ZDM Mathematics Education* special issue on Mathematical Working Spaces in schooling (Kuzniak, et al., 2016).

An example of a coherent GWS supported by Geometry I

To show what a suitable GWS guided by Geometry I is, I use the findings from a comparative study on the teaching of geometry in France and Chile quoted above (Guzman & Kuzniak, 2006). Education in Chile is divided into elementary school (Básica) till Grade 8 and secondary school (Media) till Grade 12. From 1998 on, the teaching of mathematics has abandoned the focus on abstract ideas which was in place before and turned into a more concrete and empirical approach. As of today, the reference GWS is guided by Geometry I. To illustrate this and point out some differences between France and Chile, let us consider the following exercise taken from a Grade 10 textbook (Mare Nostrum, 2003).

Students starting the chapter on similarity have to solve the following problem, whose solution is given later in the same chapter:

> Alfonso is just coming from a journey in the precordillera where he saw a field with a quadrilateral shape which interested his family. He wants to estimate its area. For that, during his journey, he measured, successively, the four sides of the field and he found them to measure approximately: 300 m, 900 m, 610 m, 440 m. Yet, he does not know how to find the area.
>
> Working with your classmates, could you help Alfonso and determine the area of the field? (Mare Nostrum, 2003, p. 92)

As four dimensions are not sufficient to ensure the uniqueness of the quadrilateral, the exercise is then completed by the following hint:

> We can tell you that, when you were working, Alfonso explained the problem to his friend Rayen and she asked him to take another measure of the field: the length of a diagonal. Alfonso has come back with the datum: 630 m.
>
> Has it been done right? Could we help him now, though we could not do it before? (ibid.)

The proof suggested in the book begins with a classical decomposition of the figure in triangles based on the indications given by the authors. But the more surprising for a French reader is yet to come: The authors ask students to measure the missing height directly on the drawing. This way of doing geometry is strictly forbidden at the comparable level of education in France.

> How can we compute the area now? Well, we determine *the scale of the drawing*, we measure the indicated height and we obtain the area of each triangle (by multiplying each length of a base by half of the corresponding height). (ibid.)

In this example, geometric work is done on a sheet of paper and with the scaling procedures, instruments for drawing and measuring, and a formula for calculating the area of a triangle. In this first GWS, which I call the *measuring GWS*, splitting a drawing of the field into two triangles and measuring altitudes makes it possible to answer the question in a practical way. In that case, geometric work is clearly supported by Geometry I and goes back and forth between the real world and a drawing, which is a schematic depiction of the actual field. Measurement on the drawing affords the missing data. The activity is logically ended by a calculation with approximation, which relates to the possibility of measuring accepted in Geometry I but not Geometry II.

A second GWS, the *calculation GWS*, supported by Geometry II is possible and exists in France where the so-called Heron's formula makes it possible to calculate the area of a triangle knowing the length of its sides without drawing or measurement. The two GWS share a common general strategy: splitting into two triangles. But they do not share the other means of action, the justifications of these actions, and the resulting geometric work.

In the example, the first two modeling spaces do not necessarily organize themselves in a hierarchy where the mathematical model would have preeminence. The GWS supported by Geometry I allows the problem to be satisfactorily solved with a limited theoretical apparatus. The GWS supported on Geometry II avoids drawing and measuring and therefore its accuracy is not limited by the measurement on a reduced scale or the imprecisions of the drawing. The procedure in this GWS allows automation, for example by way of a program on a calculator. The *measuring GWS* favors the use of instruments and therefore their associated geneses, while the *calculation GWS* fosters the use of symbolic signs (semiotic genesis). In both spaces, discursive genesis may be called upon to justify the procedure used but in a different way, which changes the epistemological nature of proof[5].

Intercept theorem current use or incompleteness of the geometric work

To illustrate the interest of the GWS model and develop the question of the completeness of geometric work, we will refer to a classroom session (Nechache, 2014) dedicated to the use of the intercept theorem[6] (in French, *le théorème de Thalès,* or in German *Strahlensatz*) in France at Grade 9 where the Geometry II paradigm is favored by the curriculum. In this session, a restricted use of the mathematical tool, the theorem, leads to a mathematical work that can be often deemed incomplete. Nechache's study (2014) helps to clarify some discrepancies that often arise between the mathematical work produced by the students and the work expected by the teachers. Our analysis is supported by the GWS model, which enables highlighting the dynamic of geometric work through the various planes determined by the model (Figure 3).

In French education, from the 1980s, the use of the intercept theorem has been gradually restricted to two typical Thales' configurations: one named “triangle” and the other “butterfly.”

[5] See note 3.
[6] Also known in English as “basic proportionality theorem;” see https://en.wikipedia.org/wiki/Intercept_theorem

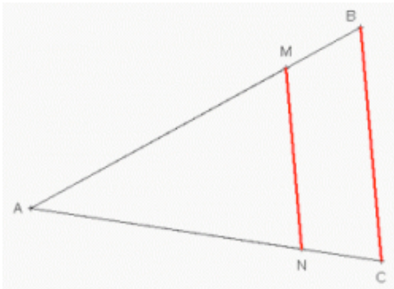


Figure 4. The “triangle” form of the intercept theorem

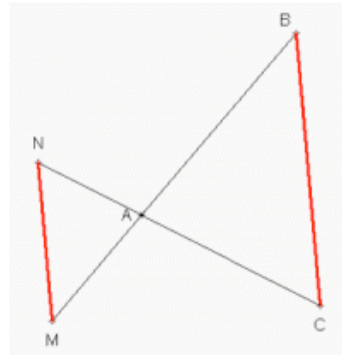


Figure 5. The “butterfly” form of the intercept theorem

During the session observed by Nechache (2014), the teacher asks the students to solve an exercise, taken from the textbook (Brault et al., 2012, p. 311), with nine multiple choice questions having three alternative answers. Two figures corresponding to Thales’ “butterfly” configuration are associated with the statement of the problem.

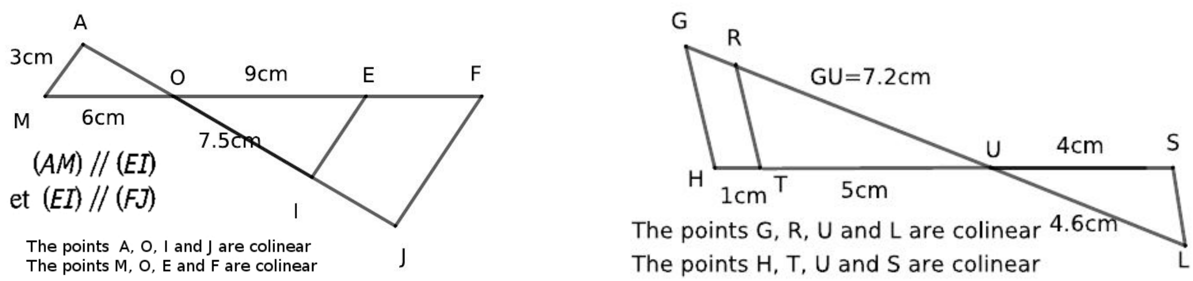


Figure 6a and 6b. The diagrams included with the exercise

The nine tasks can be characterized as simple, requiring a few abilities: determine reduction ratios, check equal ratios, and calculate the lengths of triangle sides. The last four questions relate to the converse and contrapositive form of the intercept theorem by referring this time to the second figure (Figure 6b) to identify the correct parallelism properties. In the textbook, the exercise is designed to train students to identify key figures associated with the intercept theorem and master routinized techniques. The cognitive activity is essentially based on visual and semiotic exploitation of data taken from the diagram: no discursive justification is expected. The mathematical work is fully located in the [Sem-Ins] plane with use of Thales’ diagram as a technological tool for calculation.

In the classroom session observed by Nechache (2014), the teacher first asks the students to investigate the questions for four minutes. Then, he only answers two of the questions he gave and starts with the first question:

> In figure 1, the triangle *AOM* is a reduction of the triangle *IOE* by ratio: 3/9 or 9/6 or 2/3

The question is simple, because it can be answered in a very elementary way by using visual recognition using only the semiotic dimension, as the text specifies that one triangle is a reduction of the other. Different ways to solve it can be used, all of which involve solely the semiotic dimension. The mathematical work is confined to the [Sem-Ins] plane by using the butterfly diagram associated with Thales' theorem as a semiotic tool. The analysis of the entire session allows us to check that students' mathematical work is also confined and closed on the semiotic axis.

The teacher draws the first figure freehand on the blackboard. Before giving the solution to the first question, he urges students to remember methods related to the intercept theorem, which had been studied in an earlier lesson when the theorem was introduced. The solution of the exercise is temporarily postponed in favor of a work exclusively concerned with the theoretical referential in the *suitable* GWS based on Geometry II that the teacher wants to implement. Later, a student reads the question and gives the correct answer. The teacher agrees and asks him to justify the answer. This demand of justification is new and is not part of the initial problem: The student and all classmates remain silent. The teacher reads the question again and addresses the students:

> Teacher: When we tell you that a triangle is a reduction of another one, does this not remind you of any property? No theorem? Well that's a pity, we just saw it 5 minutes ago. So, which theorem has to be applied when we have such a configuration?

Faced with the remarkable silence of these students who, at this level of schooling, only know two theorems (the Pythagorean and intercept theorems), and given that the intercept theorem has just been the subject of an insistent reminder, the teacher comes back again to the figure drawn on the blackboard by commenting on it, then he proceeds to checking each of the conditions required to apply the intercept theorem. He favors the discursive axis in the GWS model by changing the nature of the task: a justification of the result is requested and needs to be based on a theoretical tool. The mathematical work has changed and is now in the [Sem-Dis] plane. The teacher starts by checking the trivial alignment of the points and the fact that straight lines are transversals (*secants* in French).

> Teacher: Are you sure? Do you have what is needed? How are the points supposed to be?
> Students: Aligned.
> Teacher: So, the straight lines must be sec…
> Students: Secants
> Teacher: Which one?
> Student: (ME) and (AI).
> Teacher: (ME) and (AI) are secants in O. We have the five points which intervene.

To move forward toward the solution, the teacher resorts to the Topaze effect that Brousseau (1986) identified when a teacher endeavors to get the expected answer from his student through purely linguistic cues, independent of the target mathematical knowledge. In this instance, the mere utterance of the beginning of the word "secant" with the phoneme "sec" is sufficient to obtain the right answer from the student.

The teacher then guides the student to check the parallelism of the straight lines by using the same effect but with less success because students propose straight lines different from those that are expected by the teacher. These inappropriate answers show that students no longer perceive the goal of the exercise: They persist in carrying out a visual work that is not guided by the theoretical referential. But the teacher remains in his role: He is in charge of developing the theoretical referential and he finishes by applying the theorem to show equal ratios.

The teacher concludes the session by clarifying briefly what he expects from a mathematical work.

> Teacher: The trick is to be able to explain what we have done.

So the teacher has chosen to adapt the task by changing the nature of the geometric work: The results should be justified by using the theoretical referential (the intercept theorem).

In the *suitable* GWS implemented by the teacher, the mathematical work is placed in the [Sem-Dis] plane oriented towards the discursive genesis. The expected validation favors the use of the intercept theorem as a theoretical tool confined in the discursive dimension of the GWS.

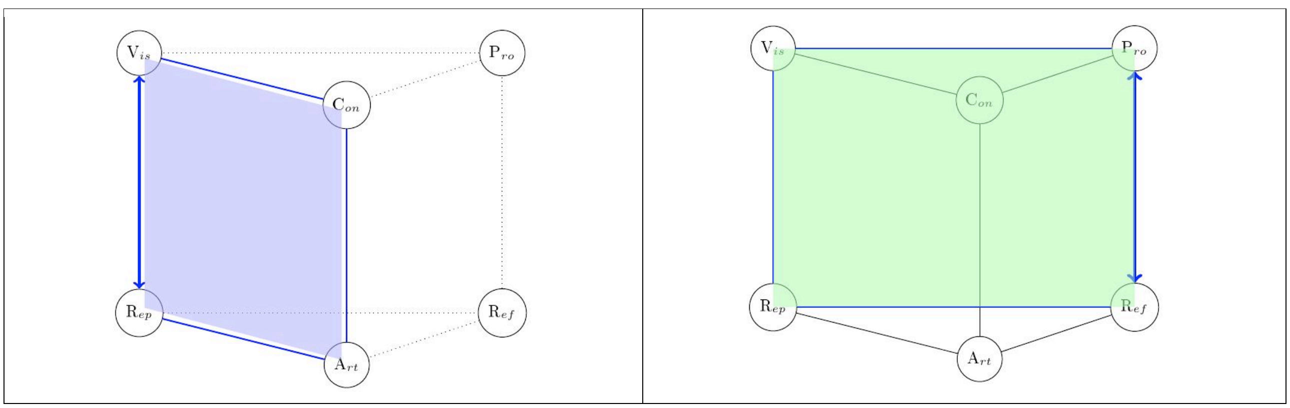


Figure 7. Work done by students vs. work expected by the teacher

The observation of this geometry session shows that students' work is exclusively located in the semiotic dimension favored by the textbook's suitable GWS and not expected in the suitable GWS implemented by the teacher. Hence, a misunderstanding emerges between the work the students do and the work the teacher expects: The misunderstanding relates to the change of validation in what counts as proof. Indeed, no discourse of proof is expected in the textbook, but the teacher does expect proof to be connected to the discourse in the suitable GWS. Both students and teacher carry out their work diligently, but they do not do the same geometric work and this work is incomplete because it is confined to only one or two dimensions instead of all three dimensions of the GWS model.

Understanding and Developing Geometric Work through its Dynamics

The geometric work perspective that I suggest requires coordination between cognitive and epistemological approaches, and the entire work is structured by three complementary dimensions: semiotic, instrumental, and discursive. The research challenge is to identify and understand the dynamics of geometric work by observing, in particular, the role of each of the three previous dimensions, and the interactions among them as suggested by each of the planes used to represent the model (Figure 2 and 3). The successful achievement of this program passes through a better understanding of each dimension of the GWS model.

Geometry is traditionally viewed as work on geometric configurations that are both tangible signs and abstract mathematical objects. Parzysz (1988) has clearly identified this difference under the opposition *drawing vs. figure,* which highlights the strong interactions existing between semiotic and discursive dimensions. In the GWS framework, the semiotic genesis is clearly associated to interpreting and developing a system of signs (semiotic system) and it could be analyzed using the contributions of Duval (2006), who developed very powerful tools (in particular, the notion of registers of semiotic representation) to explore the question. In his view, a real understanding of mathematical objects requires the student to be able to play between different registers, which are the sole tangible and visible representations of the mathematical objects.

Geometry could not exist without drawing tools and study of their different uses makes it possible to identify two types of geometry, which are well described by the Geometry I and Geometry II paradigms. From precise but wrong constructions (like Dürer's pentagon) to exact but imprecise constructions (like Euclid's pentagon), it is possible to see all the epistemic conflicts that distinguish constructions based on approximation from constructions based on purely deductive arguments. This fundamental difference continues to nourish misunderstandings and polemics in the classroom as the "flattened triangle" task shows: Does there exist a triangle with sides 4 cm, 5 cm, and 9 cm? Some students affirm its existence based on a triangle they have constructed with their compass, and others negate its existence by using the triangle inequality and calculation.

The tension between precise and exact constructions has been renewed with the appearance of dynamic geometry software (DGS). As Straesser (2002) suggested, we need to think more about the nature of the geometry embedded in tools, and reconsider the traditional opposition between practical and theoretical aspects of geometry. Software stretches boundaries of graphic precision, and finally, ends by convincing users of the validity of their results. Proof work does not remain simply formal, and forms of argumentation are enriched by experiments, which give new meaning to the classic epistemological distinction between iconic and non-iconic reasoning. The first closely depends on diagram and its construction and relates to the [Sem-Ins] plane and the second tends to be based on a discursive dimension slightly guided by some semiotic aspects [Dis-Sem].

How do the semiotic, instrumental, and discursive geneses relate to each other, and specifically how does the use of new instruments interact with semiotic and discursive geneses in transforming discovery and validation methods? And how can students' geometric work be structured in a rich and powerful way? This is one of the issues that the GWS model seeks to describe through the notion of *complete geometric work* (Kuzniak, Nechache, and Drouhard, 2016) which supposes a genuine relationship between the epistemological and cognitive planes and articulation of a rich diversity between the different geneses and vertical planes of the GWS model. The aim is not only to observe and describe existing activities but also to develop some tasks and implement them in classroom for integrating the three dimensions of the model into a complete understanding of geometric work according to the perspective expected by teachers and that geometric paradigms help to precise.

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EPISTEMOLOGICAL FEATURES OF A CONSTRUCTIONAL APPROACH TO REGULAR 4-POLYTOPES

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The limitations of empirical methods in 4-dimensional geometry demand, but also provide more scope for, alternative ways of understanding, such as analogy. The introduction of students to 4-dimensional objects should thus be considered as an opportunity to enhance students' belief system about mathematics. We will describe and reflect on the characteristics of a constructive approach to regular 4-polytopes and share our experiences with teaching this approach.

Keywords: 4-polytopes, analogy, beliefs, constructional, empiricism, four-dimensional, geometry, hypercube, induction, mental object, platonic solids, reasoning by analogy, subject matter didactics, workshop

A Didactical Challenge: The Pure Empiricist

Consider the following task: Show that the graph of a quadratic function is a curve whose points are at equal distance from a fixed point and a fixed line.

"How to accomplish this task? I know. I open my dynamic geometry software program and let it plot the graph of some quadratic function. Fortunately, there is also a button that creates, by specifying a point and a line, the curve whose points are at equal distance from them. So, I have a second curve on my screen; and, by dragging the specified point and line, I can manage to put that new curve right on top of the old one. Thus, indeed, both curves are exactly the same. QED."

If someone actually solved the above task in the described way, he or she would be convinced that both constructions yield the same curve. Thus, it would be needless to tell them that they don't know the result for sure. In terms of mathematical awareness (Kaenders, Kvasz, & Weiss-Pidstrygach, 2011), the person acquired experimental awareness of the result - no more, no less.

But what is the ontological status of the curves in this solution? Once the curves have been created, they are treated like physical objects that can be dragged around and put on top of each other. In particular, they are compared visually, not mentally. Therefore, this solution serves as a typical example of what has been called naïve empiricism (Schoenfeld, 1985). The pure empiricist discovers his results mostly by induction that is by pattern recognition. If asked for the measure, for instance of an angle, he would go and measure it, for instance with a protractor, and come back with an approximate answer. Several studies (i.e., Balacheff, 1988; Chazan, 1993; Schoenfeld, 1985) have shown that empiricism is a common mathematical behaviour among high school students.

Of course, induction is not something that one should unlearn. The mature mathematician still uses it as a common strategy in his search for new results. De Villiers (2010), for instance, describes geometrical results he found by using empirical strategies and facilitated by employing dynamic geometry software. Pölya (1954) also gathered a whole collection of elementary, but not only geometrical, examples that show the power of inductive reasoning. Leuders and Phillip (2014) highlight inductive reasoning very strongly in order to advocate its dominant role in high school mathematics. De Villiers (2010) holds a similar view, but unlike Leuders and Phillip (2014), he does not abandon the deductive methods from the context of mathematical discovery. Indeed, Pölya's (1954) examples show that inductive reasoning is especially strong when combined with deductive reasoning and also with reasoning by analogy. In order to display and develop these other modes of reasoning in their own right, we looked for a context where inductive reasoning is less effective. An initiation to four-dimensional objects appears to be a good choice in this regard.

A Theoretical Solution

Identifying a cognitive conflict

When introducing regular 4-polytopes on three different occasions to high school students, the students greeted the fourth dimension subject with a curiosity not seen with other mathematical subjects. Thinking of the students, somewhat simplistically, as pure empiricists can lead to the following explanation of this observation: What is the fourth dimension? For the pure empiricist, space has only three dimensions. The fourth dimension, therefore, must be of a different nature than the other three. Typically, the pure empiricist would say that the fourth dimension is the dimension of time. If the pure empiricist was informed the conversation was going to be about a space that has four (identical) spatial dimensions and also about 4-dimensional geometrical objects living within that space, he would, consciously or not, have the following cognitive conflict which raises his interest about the topic:

> A 4-dimensional object, whatever it may be, cannot be treated as a physical object, or can it? I cannot see it and I cannot measure it with straightedge and protractor, or can I? Therefore, my usual (empirical) strategies seem to be quite useless, when it comes to the fourth dimension. But then, how is it possible to determine the properties of a 4-dimensional object?

Resolving the conflict – Analogy takes over

In contrast to the 2- and 3-dimensional setting where every student can effortlessly generate many different object types, the 4-dimensional world seems unoccupied to the beginner. Therefore, conflict starts at the creation of 4-dimensional objects. A plane or solid mathematical object may be the result of an abstraction from some physical reference object. However, 4-dimensional objects cannot be abstracted due to the lack of a reference object. They require construction. Naturally, the beginner does not know how to construct a 4-dimensional object since there seems to be no suitable paradigm at hand. That moment is when the beginner is introduced to the paradigm of the transition from plane to solid objects. Having identified a general construction scheme that turns plane figures into solids, the beginner can try to apply this scheme (at least verbally) to a solid object in order to get an inhabitant of the 4-dimensional world.

A prism, for instance, can be constructed from a plane figure's trace moving in a direction perpendicular to itself. If the plane figure is a square and if the square is moved through a distance equal to the length of its sides, the construction yields a cube (Figure 1). However, if the moving figure is a cube instead of a square, a totally new object is obtained. Since it is a 4-dimensional analogue of the cube, hypercube[1] will be the working definition for this concept. The pure empiricist will object that it is not possible to move a cube perpendicular to itself. This construction, therefore, results not in a physical, but just a linguistic object. It is an object created by means of language. Nevertheless, the linguistic construction in combination with analogy enables the opportunity to identify the properties of this linguistic object. For instance, while a moving square traces a cube, the four edges of the moving square trace four of the square faces of the resulting cube. Together with the starting position and the end position of the moving square those four squares form the boundary of the resulting cube. Analogously, while a moving cube traces a hypercube, the six faces of the moving cube trace six cubical boundaries of the resulting hypercube. Together with the starting position and the end position of the moving cube, those six cubes form the boundary of the resulting hypercube. Thus, apparently, eight cubes bound a hypercube. The given argument, which is typical for our constructive approach to regular 4-polytopes, is an example of a type of reasoning which is in the literature also referred to as operative proof (Wittmann, 2014) or transformational reasoning (Chazan, 1993).

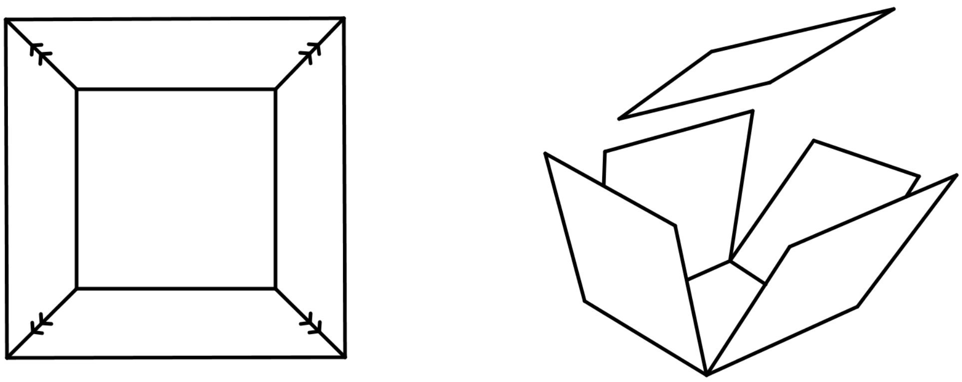

Figure 1: Two different constructions of a cube.

On the one hand, a cube can be created mentally by a moving square; yet on the other hand, it can be created physically out of six congruent squares. Starting with a single square, four more squares are placed around the first one's edges. When folding these four outer squares into the third dimension, the result is an open cube which can be closed by the sixth square (Figure 1). Analogously, it can start with a cube and put six other cubes on the faces of the first cube. Reflection on this alternative construction of a cube results in a new 4-dimensional analogue of the cube. Like the hypercube, it is built from eight cubes. Would it be possible that the new object is actually nothing but the hypercube? Contrary to the situation sketched in the beginning that compared two different ways of generating a parabola, the objects cannot be viewed from outside to determine if they are the same. Instead, the basis of their properties will be the deciding factor. At some point, the connections between the two constructions are realized and are seen in the same picture. This achievement indicates that the hypercube conception has developed. The mere

[1] From now on, we refer by the prefix *hyper* always to the fourth dimension.

linguistic construction has turned into a mental object (Freudenthal, 1991) or figural concept (Fischbein, 1993).

Summarizing the information so far, there is a cognitive conflict about the fourth dimension that stems from the view of the dominant role of empirical methods in plane and solid high school geometry. They are useless in higher-dimensional geometry. The conflict is resolved by displaying the strength of two alternative epistemological tools, analogy and operative proving (Wittmann, 2014). Of course, students might be acquainted with non-inductive methods; but usually, students are accustomed to use these methods to explain results. Here, they need them to find the results. Thus, exploring 4-dimensional objects is epistemologically quite a different activity than exploring plane or solid objects. An accessible and moderate introduction to the fourth dimension might contribute to challenge students' empirical belief systems about mathematics (Schoenfeld, 1985). In view of the long and rich history of the Platonic solids, the Platonic hypersolids are an obvious choice for such an introduction. One way to define these objects is by means of coordinates. There are two challenges when using coordinates to define the Platonic hypersolids. First, this approach demands great familiarity from the learners in working with linear equations. Secondly, neglecting the geometric character of the subject is a risk. Therefore, this study utilizes an approach that establishes the polytopes by means of mental constructions.

The Implementation (Part I): Overcoming Empiricism

Below is a sketch of the beginning of a workshop on the Platonic hypersolids held at the International Mathematical Kangaroo Camp at Werbellinsee, Germany, in August of 2015[2]. The sketch will include reflections on the choices made, observations of the difficulties students encountered, and potential ways to improve the workshop.

Episode 1: Starting predicatively[3]

Edwin Abbott's satirical novella, *Flatland: A romance of many dimensions*, is probably still the most popular early introduction to higher dimensional space. Abbott sketches a society of polygons which live inside a plane. At some point one of the Flatlanders is visited by a three-dimensional being, a sphere. The Flatlander, however, can only see the intersection of the sphere with the plane and thus perceives the sphere as a circle (Figure 2). While the sphere moves upwards and downwards, the Flatlander sees a circle that is growing and shrinking. Thus, the Flatlanders conceive a sphere as a family of circles of different size.

[2] The International Mathematical Kangaroo Camp is an annual event that takes place at the European Youth and Recreation Meeting place (EJB) at the Werbellinsee in Brandenburg, Germany. It is the prize for the best participants of the Kangaroo Competition (grade 9/10) from Austria, Czech Republic, Germany, Hungary, Netherlands, Poland, Slovakia, and Switzerland. Each country sends about 10 students to the camp. The program includes various sports competitions, chess and game evenings, a problem-solving competition, and a trip to Berlin. However, different mathematical workshops, which take place every morning, form the camp's core activity. The workshops usually cover a broad spectrum of topics and try to offer a glimpse into the vast world of elementary mathematics that lies beyond the school mathematics curricula. The focus is more on sharing with the students one's enjoyment in the doing and talking about mathematics than on producing any specific output. The workshop presented here was given to four different groups of 15 students each.

[3] Schwank (1993) distinguishes two cognitive structures of thinking: "Predicative thinking emphasizes the preference for thinking in terms of relations and judgments; functional thinking emphasizes the preference for thinking in terms of courses and modes of actions" (p. 249).

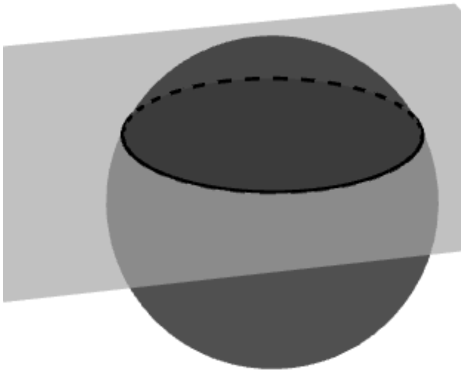

Figure 2: Intersection of a sphere with a plane

Transferring this situation from Flatland to Spaceland leads to the following claim: "As the Sphere, superior to all Flatland forms, combines many Circles in One, so doubtless there is One above us who combines many Spheres in One Supreme Existence, surpassing even the Solids of our Spaceland," (Abbott, 1994, p. 102). Letting our students try to define this One Supreme Existence, alias hypersphere, after recapitulating the definitions of circle and sphere seemed to us a suitable first exercise to become acquainted with analogy:

- What is a circle? Give a definition.
- What is a sphere? Give a definition.
- What is a hypersphere? Guess a definition.

The idea behind the three-part nature of this exercise is to strongly suggest that copy and paste will yield a correct definition of the hypersphere. However, two slightly different answers occurred to the first question (and similarly to the second one): A circle is the set of points…

- …having equal distance to one particular point.
- …satisfying the equation $x^2 + y^2 = r^2$.

Of course, both conditions express the same property of the circle, but while the coordinate-free formulation can be used for the sphere and the hypersphere without alteration, one slightly has to adapt the equation in the Cartesian version. As a result of the exercise, the students created a four-dimensional object as a linguistic object, but they needed to check that its intersection with ordinary space is a sphere. To prevent the students from getting stuck by the lack of basic knowledge about analytical geometry, we decided to go with the coordinate-free definition. Again, the two-part nature of the exercise intended to suggest that copy and paste would also yield a proof of the definition's correctness.

- Prove that the intersection of a sphere and a plane is indeed a circle.
- Prove that the intersection of your four-dimensional object and a space (or hyperplane) is indeed a sphere.

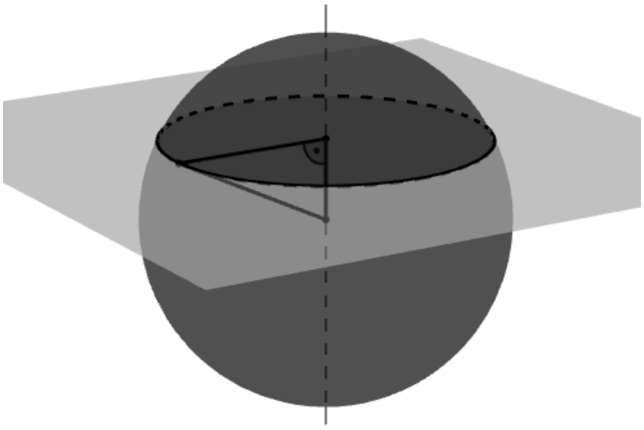

Figure 3: The intersection is a circle.

In the first part of the exercise, one has to find a candidate for the center of the resulting circle: The intersection of the plane and the perpendicular to the plane which goes through the center of the sphere is such a candidate. The Pythagorean theorem then concludes the argument (Figure 3).

Note that we introduced the *hypersphere* by means of a *definition*, not a *construction*. Obviously, the symmetry of all points with respect to the center was crucial in our proof, but in retrospect, this predicative start breaches the strictly constructive approach of the remaining workshop.

Episode 2: Introducing trace constructions

The workshop proceeded with the following question: “You have learned that, if a hypersphere visits us in Spaceland, we will only see an ordinary sphere. How about the other direction? If we encounter a four-dimensional visitor and perceive him as an ordinary sphere, must he necessarily be a hypersphere?”

We intended and hoped for the following answer: The analogous question in a dimension lower has to be denied. The sphere is not the only three-dimensional object that has a circle as a plane intersection. Cylinders (and cones) have circles as plane intersections, too. This is because a cylinder can be generated as the trace of a circle moving perpendicular to itself. Therefore, if we move a sphere perpendicular to itself, this will produce as a trace a four-dimensional object, which consists of spheres, although it is a different object than the hypersphere. Let us call it a *hypercylinder*.

Were the students in a good position to give this answer? Not at all. Due to the predicative start of the workshop, the students naturally looked for a definition, not a construction, of the cylinder, which they could lift to the fourth dimension. For instance: A cylinder is the set of points in space having the same distance to a given line. The students might have come up with the intended answer but only if they have been introduced to trace constructions before. The workshop could have provided them with the following construction of the hypersphere in addition to the definition:

- A circle is the trace of a point rotating in a plane around another point.
- A sphere is the trace of a semicircle rotating in space around its diameter.
- A hypersphere is the trace of hemisphere rotating in four-dimensional space around its equatorial plane.

In any case, having seen the trace construction of the hypercylinder, the students were well-prepared to do the next exercise: Construct a four-dimensional object that, when intersected with a suitable hyperplane, will yield an ordinary cube. Applying the trace construction to a cube instead of a sphere, that is moving a cube perpendicular to itself, will produce such an object. Note that the predicative approach to this exercise would ask for a cube's definition, which could be lifted to the fourth dimension. Finding a suitable definition for the cube, however, appears to be more difficult than the sphere or the cylinder. The constructional approach, on the other hand, produces a suitable object rather easily.

Having solved this exercise, the students then saw Figure 4, which shows the beginning of an infinite sequence of objects. Each object is generated as its predecessor's trace moves perpendicular to itself through a distance equal to the line segment's length, the starting object of the sequence. The sequence's second object is a square, and the third object is a cube. The fourth object, the cube's successor, is called *hypercube*. More generally, the n^{th} object of the sequence is called *n-cube*. Therefore, the sequence's objects are higher-dimensional analogues of the cube.

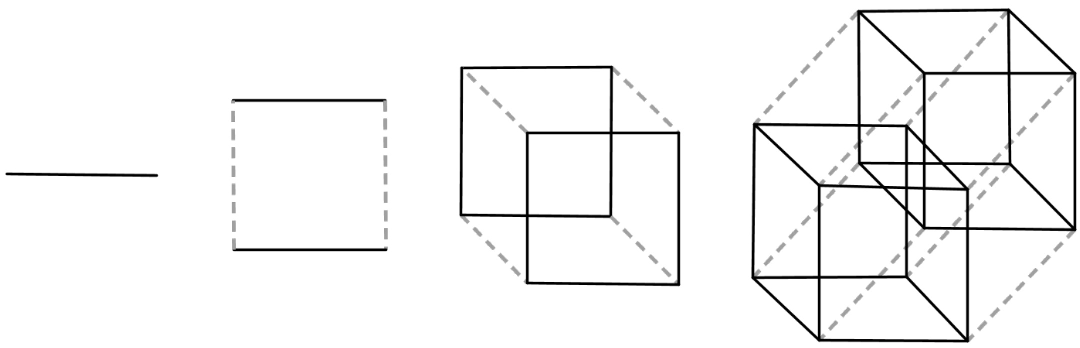

Figure 4: Genesis of the hypercube

How about the other Platonic solids? Do they have higher dimensional analogues, too? Consider the tetrahedron. We are looking for a construction, which yields a plane figure, when applied to a line segment and, which, when applied to the plane figure, gives the tetrahedron. Modifying the previous trace construction does the job: Move the object perpendicular to itself, but shrink it at a suitable pace (to a point) simultaneously. Figure 5 shows the sequence's beginning that belongs to this construction. The second object of the sequence is an equilateral triangle; the fourth object is called *pentachoron*. The n^{th} object of this sequence is called *n-simplex*.

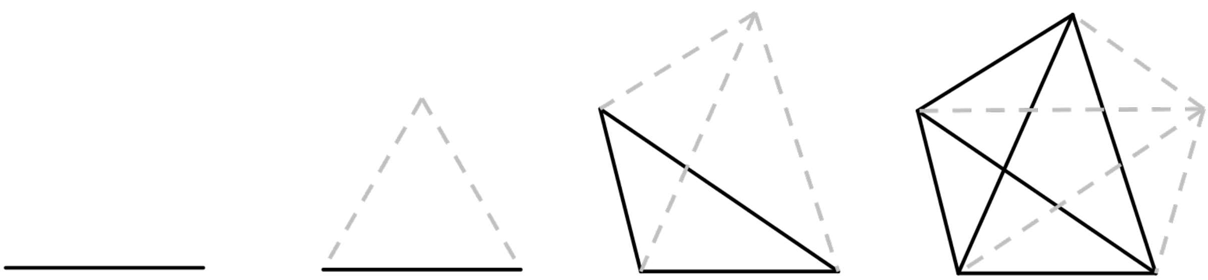

Figure 5: Genesis of the pentachoron

Another modification of the previous trace construction leads to the higher dimensional analogues of the octahedron: Move the object in a direction perpendicular to itself, while shrinking it at the same time, but move it also in the opposite direction, shrinking it simultaneously. Figure 6 shows the beginning of the corresponding sequence of objects. The two-dimensional analogue of the octahedron is a square. The four-dimensional analogue is called *hexadecachoron*. The n-dimensional analogue is called *n-orthoplex*.

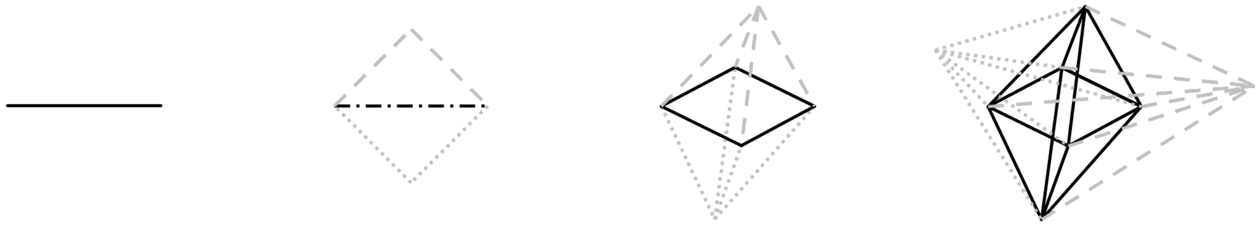

Figure 6: Genesis of the hexadecachoron

At this point, the students should have recognized trace constructions as an effective means to create higher-dimensional objects[4].

Reviewing this introduction: Definitions and constructions

In the following paragraph we integrate the previous introduction into a theoretical framework, which this study refers to as the epistemological scheme (Figure 7).

The predicative approach, depicted on the left-hand side of Figure 7, focused primarily on the definitions of the circle and sphere and the relationships between these definitions. It enables the students to guess and subsequently define higher analogues of the circle in an easy and uniform way. Although this way of action can be seen as dull, mindless, or even misleading, the workshop chose it intentionally to let the students come to play with 4-polytopes. The method of copy and paste can be seen as a door opener to engage the students quickly in their own mathematical activity. It should be mentioned that despite the transition to higher dimensions is performed, it can be doubted that geometric ideas and intuitions have been fostered since they are both not needed. Furthermore, from such a condensed definition it is rather tedious for the students to unravel the definition in order to deduce properties of the given geometric object, and, thereby, to create a mental object eventually.

[4] In (algebraic) topology, these three well-known basic operations on spaces are called cylinder, cone and suspension of a given topological space (Hatcher, 2001).

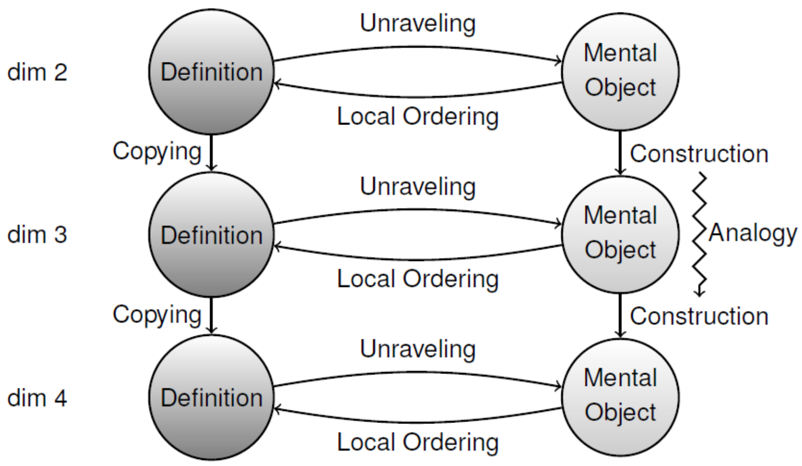

Figure 7: Epistemological Scheme

The second episode focuses on the constructive aspect. This approach can be found on the right-hand side of Figure 7. The idea is that the transition from the plane to the space can serve as a prototype in the construction of higher dimensional analogues of well-known geometric objects. Careful examination of the construction of spatial objects out of planar objects can hint to certain analogy pairs that are crucial for a successful reasoning via analogy. An important point to remember is that readily accessible objects, such as the line, the triangle, and the square are used as building blocks for the construction of new objects. Hence, the geometric notions are more strongly interconnected, and the transition to a higher dimension is perceived as an extension of existing notions. Therefore, the new objects are not produced all alone, but instead they come along with their own individual genesis highlighting certain properties; and thus, constituting much more profound mental objects. These properties can subsequently be ordered locally by the learner in order to learn which properties are defining and can make for a definition eventually.

Episode 3: Beating the empiricist

The students next realize that these trace constructions also provide the opportunity to investigate the resulting objects they created. To this end, the workshop asked students to calculate the number of *k*-faces, i.e. the number of vertices, edges, faces…, of the n-cube, the n-simplex and the n-orthoplex. This section shares the experiences in this exercise.

Consider Table 1. The students spotted easily two distinct number sequences occurring in the table, namely the following:

- The column *Vertices*: 2, 4, 8, 16, …
- The diagonal under the 1's: 2, 4, 6, 8, …

In Table 2, it is just the other way around. The column *Vertices* consists of the even numbers, and the diagonal under the 1's appears to consist of the powers of two. In Table 3, both the column *Vertices* and the diagonal under the 1's apparently show the sequence of the natural numbers, starting with two (2, 3, 4, 5, …).

Dimension	Object	Vertices	Edges	Faces	3-faces	4-faces
1	Segment	2	1	-	-	-
2	Square	4	4	1	-	-
3	Cube	8	12	6	1	-
4	Hypercube	16	32	24	8	1
5	"5-Cube"	?	?	?	?	?

Table 1: Combinatorial Data of *n*-cubes

When students filled in the numbers of the last rows of Table 1, Table 2, and Table 3, to determine the number of k-faces of the 5-cube, the 5-orthoplex and the 5-simplex, some students were only able to determine the number of vertices and the number of 4-cells of each object, while the others found all the numbers.

Dimension	Object	Vertices	Edges	Faces	3-faces	4-faces
1	Segment	2	1	-	-	-
2	Square	4	4	1	-	-
3	Octahedron	6	12	8	1	-
4	Hexadecachoron	8	24	32	16	1
5	"5-Orthoplex"	?	?	?	?	?

Table 2: Combinatorial Data of *n*-orthoplices

Dimension	Object	Vertices	Edges	Faces	3-faces	4-faces
1	Segment	2	1	-	-	-
2	Triangle	3	3	1	-	-
3	Tetrhedron	4	6	4	1	-
4	Pentachoron	5	10	10	5	1
5	"5-Simplex"	?	?	?	?	?

Table 3: Combinatorial Data of *n*-simplices

Obviously, the first group noticed and used the prominent patterns described above. However, their pattern recognition abilities were not strong enough to guess the other numbers. Thus, their inductive approach failed. The second group, on the other hand, stuck to the construction and was thereby able to deduce the numbers of the five-dimensional objects from the numbers of their four-dimensional analogues: Let B_k be the number of k-dimensional faces of the five-dimensional object under consideration and let b_k be the number of k-dimensional faces of its four-dimensional analogue. Then, the trace constructions entail the following recurrence relations:

- 5-Cube: $B_0=2\cdot b_0$, $B_{k+1}=2\cdot b_{k+1}+b_k$ for k from 0 to 3.
- 5-Orthoplex: $B_0=b_0+2$, $B_{k+1}=b_{k+1}+2\cdot b_k$ for k from 0 to 2, $B_4=2\cdot b_3$.
- 5-Simplex: $B_0=b_0+1$, $B_{k+1}=b_{k+1}+b_k$ for k from 0 to 3.

As mentioned previously, the inductive approach should not be discarded. Gathering the combinatorial data and displaying them together properly in a table can be a fruitful activity. The tables may call attention to a phenomenon that would otherwise stay unnoticed. In this case one may, for instance, observe a curious connection between the data of the n-cubes and the data of the n-orthoplices. Apart from the last 1 in each row, the numbers appear in reverse order in each row. This symmetry, which is rather prominently displayed by the tables, can also be discovered by looking at the recurrence relations of the n-cubes and n-orthoplices, but there it might have been overlooked.

The Implementation (Part II): Enriching the Students' Views of Mathematics

The following sections are three different episodes experienced in the different workshops, and reflection is done on each of them individually.[5] These episodes will show that our subject offers good opportunities to challenge some of the typical students' beliefs about mathematics, beyond the empiricism already discussed.

Episode 4: Choosing the wrong candidate

Back to the first exercise: A sphere is the set of all points in space having equal distance to a particular point. What is a hypersphere? Is it the set of points in four-dimensional space having equal distance to a particular *point* or to a particular *line*? The exercise, taking the circle into account, suggests that one should choose the first alternative: since circle and sphere both have a center, the hypersphere should have a center, too. The trichotomy of the exercise, therefore, was important in order to avoid ambiguity. However, at some point, the learner should definitely get the chance to experience this kind of ambiguity, so that he may improve his intuition in choosing the suitable analogue. We decided that lifting Euclid's proof (Heath, 1908) for the fact that there are only five Platonic solids to the next dimension would be a good first exercise which offers this experience.

The construction of the Platonic solids' is uniquely determined by two combinatorial aspects: the type of regular 2-polygon used, and how many of them are adjacent to one vertex. Thus, the question about the number of Platonic solids boils down to the number of vertex configurations with a positive angular defect. For instance, at most five equilateral triangles may fit around a vertex (angular defect: $360° - 5 \cdot 60 = 60°$). By asking the students for a strategy to lift Euclid's argument, it appeared natural to stick to the vertices: "We have to find out how many tetrahedra may fit around a vertex," the students said. However, this strategy failed since "we do not know how to determine the measure of a solid angle." Comparing the cube's second construction (Figure 1) with the corresponding construction of the hypercube suggests an alternative strategy. In the

[5] Episodes 4 (partially) and 6 were observed by both authors, whereas episode 5 was only observed by the first author. However, the reflection is the result of the discussion between both authors.

construction of the cube, three squares met at each vertex of the first square.[6] In the analogous construction of the hypercube, three cubes met at each edge of the first cube. Apparently, there needs to be a consideration of the angular defect at one edge than at one vertex. Indeed, this strategy succeeds if one knows how to determine the dihedral angles of the Platonic solids (Table 4), which is a nice exercise in solid geometry. It can be concluded that only three, four, or five tetrahedra, three cubes, three octahedra, and three dodecahedra may fit around an edge. Thus, there should be at most six (combinatorically) different Platonic hypersolids.

Solid	Dihedral Angle
Tetrahedron	70.53°
Cube	90°
Octahedron	109.47°
Dodecahedron	116.57°
Icosahedron	138.19°

Table 4: Dihedral angles of the Platonic Solids

Episode 5: Struggling with duality

Can all six edge configurations be realized by a Platonic hypersolid? The hypercube (8-cell) realizes the configuration with three cubes around each edge. How about three tetrahedra around each edge? That is easy: Start with one tetrahedron and put another tetrahedron on each face of the first one. Glue the neighboring faces together, and the result is a pentachoron. In a similar way, one can construct the other four Platonic hypersolids (Banchoff, 1990). However, the construction of the polytope with 4 tetrahedra at each edge is much harder to imagine than the construction of the pentachoron since more layers of tetrahedra are needed (Figure 8).

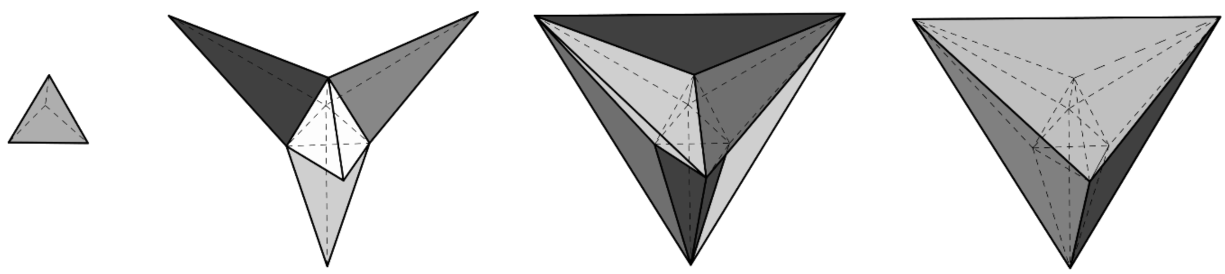

Figure 8: Second genesis of the hexadecachoron

We thought that asking the students to carry out the construction mentally would be an excessive demand, but showing a visualization of the construction process would not be appropriate either,

[6] If Euclid's proof is the *analysis*, this construction can be seen as the corresponding *synthesis.*

since it would appear like a deus ex machina. So, we decided to introduce[7] this polytope in a different way, namely as the *dual of the hypercube* (hexadecachoron). That way, it is also generated naturally, since dualizing is a general method, not just a trick. Moreover, since dualizing interchanges the roles of the vertices and 3-faces and the roles of the edges and faces, the combinatorial properties of the cross polytope can easily be derived from those of the hypercube, by means of a word replacement game:

The hypercube has four vertices at each face and four edges at each vertex.

Thus: The hexadecachoron has four 3-faces at each edge and four faces at each 3-faces.

Although they were able to play this game, the students were suspicious about the resulting insights. They did not trust the method. A potential explanation: The students were required to use duality in the fourth dimension as a tool for gaining new insights. In the third dimension, duality was merely presented as an observable phenomenon to them. The situation might be improved by inserting some additional exercises, like "Dualize the soccer ball," which show duality already in the third dimension as a constructive method to generate new objects and a means to derive their properties.

Episode 6: Seeking for uniformity

"Does Euler's polyhedron formula also hold in dimension four?" asked a student after the hypercube and some other platonic hypersolids had been constructed. Another student (who already calculated $16 - 32 + 24 - 8 = 0$) answered quickly with a definite "No, it is zero!" This short response led to more confusion since many other students calculated 8. It should be noted that the second student adapted Euler's formula to dimension four by taking the eight cubes belonging to a hypercube into account whereas the other students did not feel the urge to adjust the formula and thus obtained 8. After some discussion, the students agreed on the extended formula, but there were still doubts about the result being 0. Shouldn't the correct answer be 2? At that point, the group divided itself into two parts: One group extended the formula to dimension five and announced happily that the result would be 2 again (at least for the 5-cube). The other group checked the formula for triangles and squares, where the result was 0 again. One student summarised the results as follows: Euler's formula yields 2 in odd dimensions and 0 in even dimensions. But there was still an unspoken urge among the students for one unified formula without a case distinction. One student proposed that one could simply add 1, when the result is 2 and subtract 1, when the result is 0. He argued completely on an arithmetic level. Moreover, the student was not able to translate this adjustment geometrically. Another student (rather quick in the construction of hypersolids via analogy) argued that in each dimension the object itself is missing, and thus giving the former reasoning a geometric meaning.

Reviewing the Episodes

The first two episodes present and contrast two different ways to generate 4-polytopes, a predicative and a constructive one. In both approaches analogy is the prominent mode of reasoning. In the third episode we meet a situation where inductive reasoning is possible, but clearly much less effective than reasoning by analogy. The fourth episode broaches ambiguity in mathematics and

[7] Note that we did not consider the trace constructions of the *n*-orthoplices in the workshop in which this episode took place.

demonstrates that *analogizing* is not a mechanical activity. It requires intuition and experience instead of recipes and algorithms. The fifth episode illustrates that symmetry or more precisely *duality* can be used not only for structuring and classification. It is also a useful problem-solving tool when it is used constructively. The final episode deals with the activity of extending mathematical theories and emphasising unification as a motive and driving force of a mathematical investigation.

Taking the episodes together, they offer a broad and rich perspective on the activity of doing mathematics. They address fundamental aspects of mathematics that seem rather neglected in teaching. However, the students might think of these aspects as special features of the fourth dimension or the world of polytopes. They might connect these general phenomena to the mathematical context in which they experienced them. In order to challenge this belief, workshops on other mathematical context on the above aspects and similar aspects are needed.

Finally, it should be noted that this workshop, though it clearly focused on the way mathematics is created, consisted mostly of closed tasks, guided discussions, and guided discoveries. There was not much room for creativity. It could be fruitful and challenging to design a more open version of this workshop without changing its aims and spirit altogether.

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OPPORTUNITIES FOR REASONING AND PROVING IN GEOMETRY IN SECONDARY SCHOOL TEXTBOOKS FROM TRINIDAD AND TOBAGO

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This study examines the quantity and quality of opportunities for reasoning and proving within the geometry content of three secondary school Mathematics textbooks in Trinidad and Tobago. I use an instrument from Otten, Gilbertson, Males, and Clark (2014) to code and analyze the opportunities for students to reflect on or engage in reasoning and proof. My analysis suggests that the three textbooks contain opportunities for students to identify patterns, make conjectures, and construct proofs. At least 30% of the student exercises in two of the textbooks promoted Geometric Calculations with Number and Explanation (GCNE), which provide opportunities for students to develop non-proof arguments or rationales. The findings of this examination can potentially help in guiding curriculum developers, policy makers, and textbook authors with the future design of textbooks, curriculum materials, and other instructional resources that foster the intellectual need of reasoning and proof in students' mathematical experiences.

Keywords: Conjectures, deductive arguments, empirical justifications, geometric arguments, geometric calculations, mathematics textbooks, non-proof arguments, pattern identification, proof, proof construction, reasoning, secondary school geometry, secondary school

Introduction

In Trinidad and Tobago, there have been substantive reform efforts of the mathematics curriculum and policy documents that concern the role of reasoning and proof. However, little is known about how the textbooks from Trinidad and Tobago promote reasoning and proof. According to policy documents, secondary school students should be given opportunities to engage in pattern identification, conjecturing, and formulating proof and non-proof arguments throughout their mathematical experiences (Republic of Trinidad and Tobago, 2009). Additionally, the most recent Caribbean Secondary Education Certificate (CSEC)[1] mathematics syllabus states that students should engage in the practice of constructing reasonable arguments and critiquing the reasoning of others. Underlying these recommendations in the reformers' vision is the assumption that when students engage in reasoning and proving, they have the opportunity to develop a deeper conceptual understanding of mathematical content and appreciate the purpose of reasoning and proof in mathematics. However, despite the policies favoring reasoning and proof, students in Trinidad and Tobago have shown

[1] The Caribbean Examination Council (CXC) administers the CSEC examinations and develops the syllabi for 31 academic and vocational subjects written by students throughout the Caribbean region. A successful completion of the CSEC examinations gives entry into post-secondary institutions in the Caribbean, UK, USA and Canada.

difficulty with items involving reasoning and proof in terminal assessments. The examiners claim that students have difficulty with questions requiring an explanation of why a solution or argument holds or have difficulty constructing proof arguments (CXC Subject Award Committee, 2014).

Several researchers claim that textbooks are an important influence on students' educational experiences in secondary school mathematics (e.g., Moyer, Cai, Wang, & Nie, 2011; Stein, Remillard, & Smith, 2007). Several studies also show that mathematics textbooks have a significant influence on students' opportunities to learn reasoning and proof in secondary school (e.g., Fujita & Jones, 2014; Otten, Gilbertson, Males, & Clark 2014; Stylianides, 2009; Thompson & Senk, 2014). Textbooks influence what students learn, when they learn it, and how well they learn it. On a global perspective, researchers report that efforts to change the content of the secondary school curriculum, in particular textbooks, has been viewed and used as an effective way to influence instructional practices, student learning, and meet the recommendations of curriculum reform (Cai & Cirillo, 2014; Senk & Thompson, 2003). Several studies, including the Third International Mathematics and Science Study (TIMSS), have shown that textbooks continue to play an important role in classrooms around the world (e.g., Fujita & Jones, 2014; Stylianides, 2009; Valverde et al., 2002). Therefore, textbooks have been called a vehicle of change for educational reform (Ball & Cohen, 1996). Mathematics textbooks can play a vital role in students' opportunities to engage in reasoning and proof; and convey the many decisions that teachers make about the construction and execution of mathematical opportunities offered to their students (Stylianides, 2007, 2009). Despite the efforts to make reasoning and proof central to school mathematics in Trinidad and Tobago, there are no existing studies that investigate how secondary school mathematics textbooks promote reasoning and proof. Furthermore, the recent reform recommendations coupled with students' low performance in reasoning and proof items in terminal examinations suggest the need to examine the quantity and quality of opportunities embedded in the secondary school textbooks in Trinidad and Tobago. As a result, my inquiry is driven by the research question: What is the nature of opportunities for reasoning and proof in secondary school textbooks in Trinidad and Tobago?

Theoretical Framework

In this study, I focus on the opportunities for reasoning and proof in geometry sections of the secondary school textbooks used in Trinidad and Tobago. My reason for focusing on Geometry is that traditionally, Geometry has been one of the areas in the CSEC examination[2] wherein students are asked to prove results or engage in pattern identification or conjecturing (CXC Subject Award Committee, 2014). I use the conceptualization of reasoning and proof in Stylianides (2009) to guide my inquiry. By reasoning and proof, I refer to the mathematical activities of (a) pattern identification, (b) conjecturing, (c) providing non-proof arguments, and (d) constructing proofs. Following Stylianides (2009), I refer to pattern identification as the task of identifying a "general mathematical relation that fits a given set of

[2] Candidates for the CSEC examination include in-school and private students seeking full certification for their completion of secondary school in the Caribbean. A full certificate consists of passes in at least five subject areas inclusive of Mathematics and English. All students within the Caribbean must gain full certification in order to pursue higher learning at post-secondary or tertiary institutions.

data" (p. 263). For example, within this mathematical activity, students in Geometry can firstly examine several cases of geometrical objects. Secondly, students can create a data set and then find a general geometrical relation that aptly describes the data set. At the end, students identify a geometrical pattern as the first activity within reasoning and proof.

In the second activity of conjecturing, I refer to the mathematical endeavor of constructing and testing conjectures. Stylianides (2009) defined a conjecture as "a logical hypothesis about a general mathematical relation, which is based on incomplete evidence" (p. 264). The construction of conjectures refers to the actual development of hypotheses about a generalized mathematical relation with some measure of uncertainty about the validity of the hypothesis. The testing of conjectures entails empirical explorations, where a few examples are used to investigate the validity of the hypotheses. In Geometry, students may observe a generalized pattern after exploring several geometrical objects. As a result, students may make a hypothesis (conjecture) describing the generalized observed pattern. At this stage, students may begin to test whether their conjectures hold by testing several sets of geometrical objects.

In the third activity, the development of non-proof arguments pertains to the use of empirical examples and rationales to support one's judgments about the validity of a conjecture. Sentences, diagrams, and examples can be used to construct non-proof arguments. The non-proof arguments could also include and not limited to the use of non-mathematical language, which explains one's reasoning about how and why a conjecture or mathematical claim may be valid. Overall, a non-proof argument is an argument missing some logical deductions in its structure (Stylianides, 2009). A non-proof argument may lack some of the logical deductive arguments that connect the hypotheses to the conclusion.

The final activity is the construction of a proof. A mathematical proof is "a formal way of expressing one's reasoning and justification" (NCTM, 2000, p. 56). Proof as defined by Stylianides (2007) "is a valid argument based on accepted truths for or against a mathematical claim" (p. 195). By an argument, Stylianides referred to a connected sequence of claims. The validity of the argument is determined by accepted canons of mathematical inferences such as modus tollens and modus ponens. The accepted truths that govern the construction of the proof include axioms, theorems, definitions, and modes of reasoning shared by a community such as a group of mathematicians or a classroom of students. The construction of a proof is considered an individual activity framed by the shared understanding of the accepted truths, and criteria for validity defined by a mathematical community. Proof is considered the final product of reasoning activities such as pattern identification and conjecturing (Hanna, 2005; Stylianides, 2009; Thompson, Senk, & Johnson, 2012). During reasoning activities, students make sense of patterns or conjectures, which eventually lead to developing non-proof arguments or proofs that support their sense making. I use the aforementioned descriptions of the activities relative to the conceptualization of reasoning and proof to guide my analysis and descriptions of the Geometry opportunities for reasoning and proof in the secondary school textbooks in Trinidad and Tobago. The main goal of my inquiry is not to compare the textbooks used but to provide descriptions of the characteristics of the textbooks in their offerings of reasoning and proof opportunities.

Methodology

Data Sources

This study involves three contemporary textbooks designed for the preparation of students for CSEC mathematics examination (Table 1). While the schools have the agency to choose their own textbooks, the selections for this study were limited to the recommendations made by the Caribbean Examination Council (CXC) and those offered by the Ministry of Education (MOE) textbook rental program. The MOE governs the centralized education system of Trinidad and Tobago; therefore, all schools receive the same recommended textbooks in the textbook rental program. The first two selections corresponded to the MOE's classification of a traditional textbook *Certificate Mathematics* (CM), and a reform-oriented textbook *Mathematics a Complete Course* (MCC) (Trinidad and Tobago, Ministry of Education, 2005). Secondary Schools have used CM as one of the primary resources for secondary school mathematics in Trinidad and Tobago for the past thirty-three years. MCC replaced CM as the only recommended textbook in the MOE textbook rental system (Trinidad and Tobago, Ministry of Education, 2006, 2007). The final book, *Mathematics for CSEC* (MSCEC), is a supplementary textbook, only recommended by CXC because it reflects the recent changes in the CSEC mathematics syllabus. This textbook is one of the recent ones suggested by CXC for the textbook rental program in Trinidad and Tobago**.**

Table 1: Textbook selections for data analysis

Title	Authors	Year
Certificate Mathematics (CM)	Greer, A. & Layne. C.	1994
Mathematics a Complete Course (MCC)	Toolsie, R.	2009
Mathematics for CSEC (MCSEC)	Chandler, S., Smith, E., Ali, F.W., & Layne, C., & Mothersill, A.[3]	2008

Framework for Coding and Analysis

To code and analyze the various opportunities for reasoning and proof, I utilized the coding instrument developed by Otten, Gilbertson, Males, and Clark (2014), which was based on Stylianides's (2009) conceptualization of reasoning and proof. Otten et al. (2014) used this instrument to analyze six geometry textbooks used in the United States (US). In Trinidad and Tobago, there is no separate geometry course; instead, all mathematics topics are integrated in the secondary school curriculum. Therefore, I examine six common selected geometry topics in three textbooks for instruction in Forms 4 and 5 (US Grades 9 and 10). I find the use of this instrument useful in comparing textbooks in Trinidad and Tobago with other textbooks in the US for which this instrument has been previously

[3] R. Toolsie is a mathematics teacher in Trinidad and Tobago. A. Greer, A Mothersill, C Layne, E. Smith , F. Ali, and S Chandler are mathematics teachers based in the United Kingdom.

used. This comparison could provide material for an interesting discussion about how the nature of reasoning and proof opportunities in an integrated curriculum textbook compares with those offered in a non-integrated or purely geometry textbook.

The coding instrument I adopted, contains two dimensions indicated by the rows and columns (see Figure 1). The first dimension in the columns consists of the units of analysis, namely the textbook expositions and student exercises. The student exercises are further sub-divided to reflect the nature of the expected student activities: (1) activities related to mathematical claims and (2) activities related to mathematical arguments. The former promotes opportunities for students to engage in identifying patterns, making conjectures, and providing non-proof explanations to support claims whereas the latter promotes opportunities for constructing non-proof and proof arguments. The second dimension indicated in the rows, consists of the four components of my analysis. These include: (a) the mathematical statement type, (b) the justification type, and (c) the expected student activity.

	Exposition	**Student Exercises**	
	Properties, Theorems, or Claims	Related to Mathematical Claims	Related to Mathematical Arguments
Mathematical Statement or Situation	General Particular	• General • Particular • General with particular instantiation provided	• General • Particular • General with particular instantiation provided
Justification **(Or environment for exploration)**	• Deductive • Empirical • None	• Deductive • Empirical • Implicit	• Deductive • Empirical • Implicit
Expected Student Activity		• Make a conjecture, refine a statement, or draw a conclusion • Fill in the blanks of a conjecture • Investigate a conjecture of statement • Perform a geometrical calculation with number and explanation (GCNE)	• Construct a proof • Develop a rationale or other non-proof argument • Outline a proof or construct a proof given an outline • Fill in the blanks of an argument or proof • Find a counterexample

Figure 1: Coding Instrument for reasoning and proof opportunities from Otten et al. (2014).

Classifying Types of Mathematical Statements

In their instrument, Otten et al. (2014) classified the types of mathematical statements in the textbook expositions and student exercises. By mathematical statements, I refer to a proposition about a single class or all classes of mathematical objects or situations, that may be either true or false. For example, a statement about all triangles or a single class of triangles such as equilateral triangles. Otten and colleagues used the *necessity principle* (Harel & Tall, 1991) and the field of logic to provide a rationale for distinguishing between types of mathematical statements. The necessity principle highlights the importance of students not only engaging in deductive reasoning but also appreciating the intellectual need for deduction in their mathematical experiences. This principle promotes reasoning and proving as an opportunity for students to understand underlying conceptual relationships, rather than as an arbitrary exercise imposed by an outside authority such as their teacher or the textbook. Otten et al. (2014) posited that deductive reasoning plays a pivotal role in justifying claims about all possible objects or situations under consideration. They captured this role of deductive reasoning by developing a set of codes relating to the mathematical statement or situation of reasoning and proving opportunities. The codes for mathematical statements are *general*, *particular*, and *general with particular instantiation provided*. In Figure 2, I present examples of each code taken from the textbooks I analyzed in this study. I used these codes to classify the quantity and quality of mathematical statements promoting reasoning and proving.

In my analysis, I define *general mathematical statements* as those statements that concern an entire class of mathematical objects or situations without exceptions. Particular statements refer to a statement that concerns a specific mathematical object or situation. A general statement with particular instantiation concerns an entire class of mathematical objects but for which a specific member of the class has been selected for students' use in reasoning (Otten et al., 2014). This type of statement can be considered an exemplar or *a generic example* (Balacheff, 1988) of a class of objects or situation. The main purpose of this type of statement is to elucidate general characteristics of the entire class or situations under consideration. The focus in this case is not on the specific example but its use as a representative of a general class of objects. Therefore, a student can use this exemplar or generic example to help them understand the general characteristics of an entire class of objects.

Within the coding instrument, statement types and justification types are independent dimensions. This separation is due to the fact that general and particular statements can both be justified by empirical or deductive arguments. To highlight this difference, Otten and colleagues used the terms "general" and "particular" to refer only to statements and the terms "deductive" and "empirical" to refer only to justifications. In the same manner, I use these terms as codes for statement and justification types respectively.

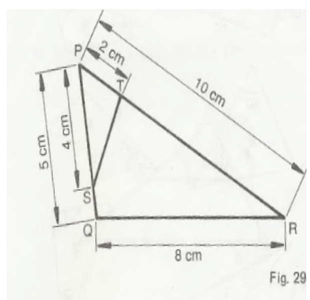
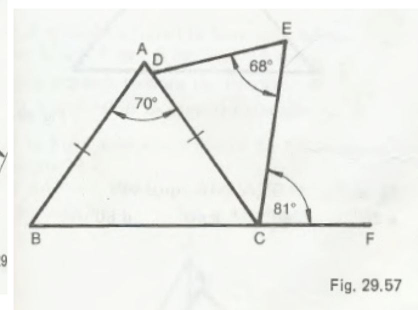
Code	Description	Textbook Exposition Examples	Student Exercise Examples
General	A statement that concerns an entire class of objects or situations.	When two parallel lines are cut by a traversal, the corresponding angles are equal (Greer & Layne, 1994, p. 203).	Prove that all isosceles triangles have congruent base angles (Chandler, Smith, Ali, Layne & Mothersill, 2008, p. 143).
Particular	A statement that concerns a specific mathematical object or situation	In Fig. 29.40 prove that Δs *PTS* and *PQR* are similar and calculate the length of *TS* (Greer & Layne, 1994, p. 215)	In Figure 29.57 below *AB* = *AC* *BCF* is a straight line. ∠*BAC* = 70^0, ∠*CED* = 68^0 and ∠ *ECF* = 81^0. Prove that two of the sides of triangle *CDE* are equal (Greer & Layne, 1994, p. 219).
			
General with particular instantiation provided	A statement that describes an entire class of objects but for which a specific member of the class has been indicated for students' use in reasoning.	NA	Consider an isosceles triangle *PQR* with a perpendicular bisector *OQ*. Prove that the bisector drawn from the apex angle of any isosceles triangle is perpendicular to the base (Toolsie, 2009, p. 457).

Figure 2: Coding of mathematical statements in the textbook expositions

Classifying Justification Types in Textbook Expositions

The codes inherited for the justification types in the textbook expositions are: (a) deductive, (b) empirical, and (c) no justification. Deductive justification refers to a logical argument, which uses definitions, postulates, or previously established results to support or prove a mathematical claim. In an empirical justification, the textbook provides a confirming example to a mathematical claim. Additionally, an empirical justification may consist of a mathematical claim with accompanying diagrams. The sole purpose of the diagrams is for demonstrating examples of cases where the mathematical claim holds. In this case, the narrative text explicitly references the diagrams and highlights the purpose of the examples demonstrated by the diagram. The final code, no justification refers to the case where the textbook does not provide any justification for a given mathematical claim.

Classifying Justification Types in the Student Exercises

In this coding instrument, the following codes were used for the type of justification that a student exercise required. In Figure 3, I present examples of the codes inherited from Otten et al. (2014) for analyzing the justification types in the textbook exercises. In *deductive justifications*, the student exercises explicitly request that students provide a “deductive argument” or a “logical chain” of justifications. This is indicated by the author’s use of the words “prove,” “justify,” or “show” to prompt the requirement for a deductive justification. An *empirical justification* requests that students provide measurements or confirming examples to solve a given task. In the final category, *implicit justification*, the student exercise requests that students engage in reasoning and proving (e.g., “Show…” or “Explain why…”) but does not explicitly specify the nature of the argument to be produced. Otten and colleagues acknowledged that, with their definition of justification types, the majority of student exercises might fall in the implicit category. The inclusion of this code is built on the assumption that students may not necessarily interpret instructions to “prove,” “justify,” or “show” in the same manner that mathematicians or mathematics educators may interpret them. As a result, the code helps capture all of the possible actions students may produce when given these instructions. Furthermore, the inclusion of this code helps distinguish their instrument as one focusing on opportunities for reasoning and proving in textbooks rather than students’ reasoning.

Expected Student Activity

In their coding instrument, Otten and colleagues classified the expected student actions with respect to mathematical claims and constructing mathematical arguments in the student exercises. Using the work of Stylianides (2009), which defined the various activities involved in reasoning and proving, they created the codes shown in row 3 of Figure 1, to ascertain the extent and nature of the opportunities for reasoning and proof offered to students. As a result of a preliminary analysis I conducted, I added a new code to the expected student activity related to mathematical claims. In the following section I introduce the new code.

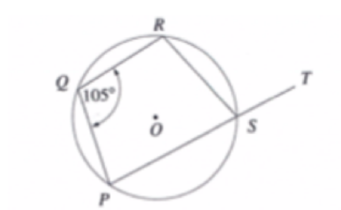
Code	Description	Student Exercise Examples
Deductive	The student exercise explicitly requests a 'deductive argument' or a 'logical chain of justifications'	In triangle ABC, D is the midpoint of BC and E is the mid-point of CA. The lines AD and BE meet at G. Prove that: (a) Triangles ABG and DEG are similar; (b) Triangles AGE and BGD are equal in area (Greer & Layne, 1994, p. 220)
Empirical	The student exercise requests measurements or confirming examples.	Using your pencil and ruler, construct any quadrilateral. Show by measuring with your protractor, that the sum of the interior angles is $360°$ (Toolsie, 2009, p. 468).
Implicit	The student exercise requests that students engage in reasoning and proof (e.g., "Show…" or "Explain why…") but does not explicitly specify the nature of the argument to be produced.	In the cyclic quadrilateral $PQRS$, angle $PQR = 105°$. Evaluate angle RST, giving reasons for your answer (Toolsie, 2009, p.494) 

Figure 3: Coding for justification types in the student exercises

Geometric Calculation with Number and Explanation

The new code I added to the coding instrument is called "geometric calculation with number and explanation" (GCNE). This code is an extension of what previous scholars defined as a "geometric calculation with number" GCN (Ayres & Sweller, 1990; Hsu & Silver, 2014; Küchemann & Hoyles, 2002). A GCN is a mathematical activity involving numerical calculations done on the basis of geometrical concepts, formulas or theorems. In a GCN, the request for an explanation of the steps in one's reasoning is not explicit but is implied as one may use geometrical concepts to obtain the solution. For example, a typical GCN task will request that students calculate the measure of a missing interior angle in a triangle given the measures of two other interior angles, say 30^0 and 50^0 respectively. In this activity, a student is expected to use the interior angle sum theorem for a triangle to calculate the missing angle. The student is not expected to explicitly state how the interior angle sum theorem supports their answer. The reasons supporting their calculations are not mandatory in their solution.

In a GCN, a diagram usually accompanies the given computational task (Hsu & Silver, 2014). The purpose of the diagram is to help students visualize and understand the geometrical situation or object that will guide their reasoning. Based on my preliminary analysis, I define a geometric calculation with number and explanation (GCNE) as a student activity for reasoning and proof, which explicitly requires a geometrical computation and an accompanying reason, or explanation for the resulting calculation. As a result, students are expected to provide a non-proof argument justifying why their result is correct. The main difference between a GCN and a GCNE is that the GCN allows students to reason about a geometric situation using a diagram while performing a computational task, whereas a GCNE goes even further to explicitly afford students the opportunity to provide a justification of the result of their calculation. The justification requests that the student provides a non-proof argument to support their reasoning and computation. Figure 4 shows an example of a GCNE task in the Geometry textbooks I analyzed in this study.

If obtuse $\angle AOB = 96^0$, determine $\angle ACB$, giving a reason for your answer.

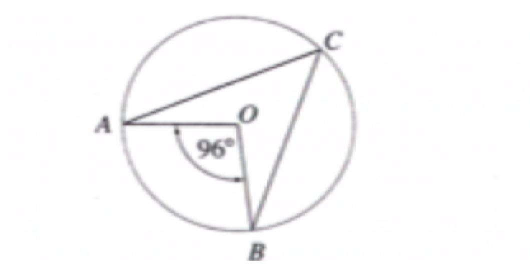


Figure 4: An example of a GCNE task.
Adapted from "Mathematics a Complete Course" by R. Toolsie, p. 492.

As Figure 4 shows, the textbook's author requests that students find the measure of an angle at the circumference standing on the arc *AB*, given the measure of the angle at the center *AOB*, which stands on the same arc *AB*. In addition to calculating the measure of the angle, students are expected to provide a reason for the result of their calculations. Therefore, students will be expected to use geometric theorems about the angle properties of a circle as possible reasons or explanations for the result of their calculation. A possible theorem they may use will be that the measure of the central angle of a circle is twice the measure of the angle at the circumference subtending the same arc. As shown in Figure 4, the given angle *AOB* should be twice the measure of the requested angle *ACB*.

Therefore, students will use this geometric result to help them explain why ACB is equal to half of the measure of AOB (i.e., $96°/2 = 48°$). Thus, the above task illustrates an example of an expected student activity I coded with the new category GCNE.

The expected student activities related to mathematical claims addresses the authors' intent for students to engage in pattern identification, conjecturing, or developing a rationale during reasoning and proving. These activities help students move from inductive reasoning to deductive reasoning as they make generalizations of observed patterns and begin justifying their generalized claims. The aim of these activities is to refine students' abilities in constructing, testing, and critiquing conjectures. The expected student activities related to constructing mathematical arguments help students justify why a mathematical claim holds. These activities help students develop deductive reasoning skills as they write proof and non-proof arguments that explain their reasoning. With regard to the example presented in Figure 4, the expected student activity with respect to the mathematical claim would be to perform a GCNE task. This is indicated by the students' use of the geometrical claim on properties of angles intercepting on the same arc in a circle to calculate the missing angle ACB. The expected student activity related to arguments would be developing a rationale or non-proof argument. This is evident by the phrase "give a reason for your answer." This phrase suggests that students are expected to use the geometrical claims about angles in a circle to explain why they would perform a calculation a certain way to obtain how the requested measure of angle, ACB.

Units of Analysis

In a manner, similar to Otten et al. (2014), I included both textbook expositions and student exercises as my units of analysis. In each of the selected textbooks, I identified and examined all sections dealing with the following six geometry topics: (1) Triangles, (2) Congruent Triangles, (3) Similar Triangles, (4) Pythagoras' Theorem, (5) Quadrilaterals, and (6) Circles. Within each of the topics, I coded the textbook expositions and student exercises for the mathematical statement type, justification type, expected student activity, and type of opportunities about the practice of reasoning and proof. Within the expository sections of each topic, I analyzed sentences or paragraphs of text, which either (1) defined geometrical terms or concepts, (2) explained geometrical properties and accompanying diagrams, (3) demonstrated mathematical claims and properties in worked examples or activities, and (4) justified mathematical claims. In two of the textbooks, I included class activities and investigations about geometry theorems and properties in my analysis since they were part of the authors' justification or explanation of a theorem. I also analyzed and coded exercises that explicitly presented an opportunity for students to engage in reasoning and proof. By such opportunities, I included exercises, which directly asked students to prove a mathematical claim, identify a pattern, investigate or make a conjecture, perform a geometrical calculation with number and explanation (GCNE) or justify a mathematical claim by developing a rationale or providing a non-proof argument. I did not include in my analysis exercises, which did not fall into one of the aforementioned categories of reasoning and proof activity.

Results

Mathematical Statement Types

Table 2 shows the types of mathematical statements that appeared related to reasoning and proof in the textbook expositions and student exercises. Overall, general statements were prevalent in the expositions sections of all three textbooks with over 75% of the statements in each textbook being about a general geometrical object or situation. In contrast, the mathematical statements within the student exercises focused on particular geometrical objects.

Table 2: Mathematical statement types in the textbook expositions and student exercises

Note. CM = Certificate Mathematics; MCC = Mathematics a Complete Course; MCSEC = Mathematics for CSEC

Textbook	Textbook Expositions			Student Exercises			
	No. of Mathematical Statements	No. of General Statements (%)	No. of Particular Statements (%)	No. of Mathematical Statements	No. of General Statements (%)	No. of Particular Statements (%)	No. of General with Particular Instantiation
CM	56	43 (77)	13 (23)	39	10 (26)	21 (54)	8 (20)
MCC	121	96 (79)	25 (21)	185	5 (3)	157 (85)	23 (12)
MCSEC	61	53 (87)	8 (13)	54	2 (4)	40 (74)	12 (22)

Overall, the student exercises offered particular mathematical statements rather than general. Of the general statements, a greater proportion had particular instantiations provided for student's reasoning. For example, in MCSEC, which had the greatest proportion of such statement type (22%), the student exercises required that students prove a mathematical claim by focusing on a selected particular case representative of a general class of objects. Figure 5 presents an example of this case.

As shown in the example given in Figure 5, the student exercise asked students to prove the general result about the type of quadrilateral formed by the alternating vertices of a regular octagon. The question then, further specified by selecting a particular case of a regular octagon with vertices *ABCDEFGH* to prove the result. The author's use of this example demonstrates that students are expected to reason about the general result and construct a proof based on congruency theorems as indicated in the hint.

> 22. Prove that the quadrilateral formed by the alternating vertices of a regular octagon is a square.
>
> Let ABCDEFGH be a regular octagon. Use congruency to prove that ACEG is a square.

Figure 5: An example of a general with particular instantiation exercise.
Adapted from "Mathematics for CSEC" by S. Chandler, E. Smith, F. W. Ali, C. E. Layne, and A. Mothersill (2008), p. 150.

Justification Types in the Textbook Expositions and Student Exercises

In Table 3, I summarize the type of justifications the authors used in the textbook exposition and student exercises. In each textbook, the authors predominantly used empirical arguments to justify the mathematical statements. In Figure 6, I show an example of an empirical justification in one of the textbooks.

Table 3: Justification types in the textbook expositions and student exercises

Textbook	No. of Justification Types (%)						
	Textbook Expositions			Student Exercises			
	No. of Justifications	Deductive (%)	Empirical (%)	No. of Justifications	Deductive (%)	Implicit (%)	*Empirical*
CM	35	12 (21)	23 (41)	39	19 (49)	20 (51)	0 (0)
MCC	96	33 (27)	63 (52)	185	50 (27)	135 (73)	0 (0)
MCSEC	39	18 (30)	21 (34)	54	22 (41)	32 (59)	0 (0)

Note. CM = Certificate Mathematics; MCC = Mathematics a Complete Course; MCSEC = Mathematics for CSEC

> **THEOREM 1:** *The sum of the four interior angles of a quadrilateral is equal to 360^0 (or 4 right angles).*
>
> Take a rule and pencil and draw your own quadrilateral. Now take your protractor and measure each angle. After you have obtained the magnitudes of the four angles sum them. What do you observe?

Figure 6: An empirical justification of a theorem.
Adapted from "Mathematics a Complete Course" by R. Toolsie, 2009, p. 468.

In this example, the author presented a theorem about the sum of interior angles of a quadrilateral. To prove this result, the author suggested that students construct any quadrilateral. When the author stated, "take your protractor and measure each angle," he suggested that students use empirical measurements to obtain the interior angles. The author also suggested that students find the sum of the four interior angles they obtained through measuring. When the author asked, "What do you observe?" he seemed to prompt students to observe that the claim in the given theorem holds for the quadrilateral students constructed. This example demonstrates a case where the author used measurements and student-generated examples to justify a mathematical result. This was the only justification of the given theorem the authors provided in this textbook. The aforementioned example represents a case of an empirical justification.

However, the student exercises required more deductive justifications as shown in Table 3. In all three textbooks, the implicit justifications were the most frequently occurring exercise type. Implicit justification exercises accounted for 51% to 73% of the exercises I analyzed in the textbooks. In MCC, about three-quarters of the exercises required implicit justifications. This means that the student exercises requested that students engage in reasoning and proof (e.g., "Explain with reasons why" or "Give reasons for your statements") but did not explicitly specify the nature of the argument to be produced. The open-endedness of the type of argument expected in the aforementioned phrases indicates that such exercises gave students the agency to choose the type of argument needed for understanding the mathematical claim. Thus, in my analysis, all exercises, which expected students to calculate and explain were considered as exercises that would be justified implicitly. In MSCEC and CM, implicit justifications were expected for 59% and 51% of the exercises respectively. The remaining student exercises in each text expected deductive justifications.

Expected Student Activity Related to Reasoning and Proof

Table 4 presents the number of student exercises providing opportunities for reasoning and proof in each textbook. Overall, I analyzed 519 student exercises combined from the three textbooks. Of all the student exercises within the three textbooks, approximately 54% offered opportunities for reasoning and proof. However, two of the textbooks, CM and MCSEC had less than 40% of their respective exercises offering reasoning and proof-related opportunities. Both of these texts had over 60% of their student exercises having no opportunity for reasoning and proof (see Table 4).

In CM and MCSEC, 35% and 32% of their respective student exercises offered opportunities for reasoning and proof. However, MCC, the textbook with the highest number of student exercises, had highest percentage of opportunities for reasoning and proof among all three textbooks (approximately 77%).

Among the three textbooks, CM offered the most opportunities for construction of proofs, approximately 85% of the student exercises. However, unlike the more recently published textbooks, MCC and MCSEC, CM did not offer opportunities for conjecturing, or developing non-proof arguments. With regard to the expected student activities related to reasoning and proof, the development of non-proof arguments accounted for the type of exercises I labeled as GCNE. As I explained above, due to the nature of the required informal explanation of these exercises, I coded all GCNE exercises as developing a non-proof argument.

Table 4: Types of Reasoning and Proof Exercises

Textbook	No. of Exercises	No. of Reasoning and Proof Exercises (%)	No. of Non-Reasoning and Proof Exercises (%)	Expected Student Activity			
				No. of Pattern Identification Exercises	No. of Make a Conjecture Exercises	No. of GCNE Exercises	No. of Proof Construction Exercises
CM	110	39 (35)	71 (65)	5	0	0	34
MCC	241	185 (77)	56 (23)	5	7	135	38
MCSEC	168	54 (32)	114 (68)	10	4	17	23
TOTAL	519	278 (54)	241 (46)	20	11	152	95

Note. CM = Certificate Mathematics; MCC = Mathematics a Complete Course; MCSEC = Mathematics for CSEC

Geometric Calculation with Number and Explanation

As shown in Figure 7, MCC and MCSEC contained a unique type of exercise requiring the development of non-proof argument. These exercises accounted to approximately 73% and 33% respectively of the geometry exercises I analyzed in these textbooks. CM did not contain any of these exercises. I labeled these exercises as Geometric Calculation with Number and Explanation (GCNE). This new labeling is an extension of a type of geometrical exercise, which scholars previously defined as “geometric calculation with number” GCN (Ayres & Sweller, 1990; Hsu & Silver, 2014; Küchemann & Hoyles, 2002). Overall All GCNE-type exercises offered opportunities for students to engage in or reflect on the practice of developing non-proof arguments, which is one of the processes of reasoning and proof as defined by Stylianides (2009).

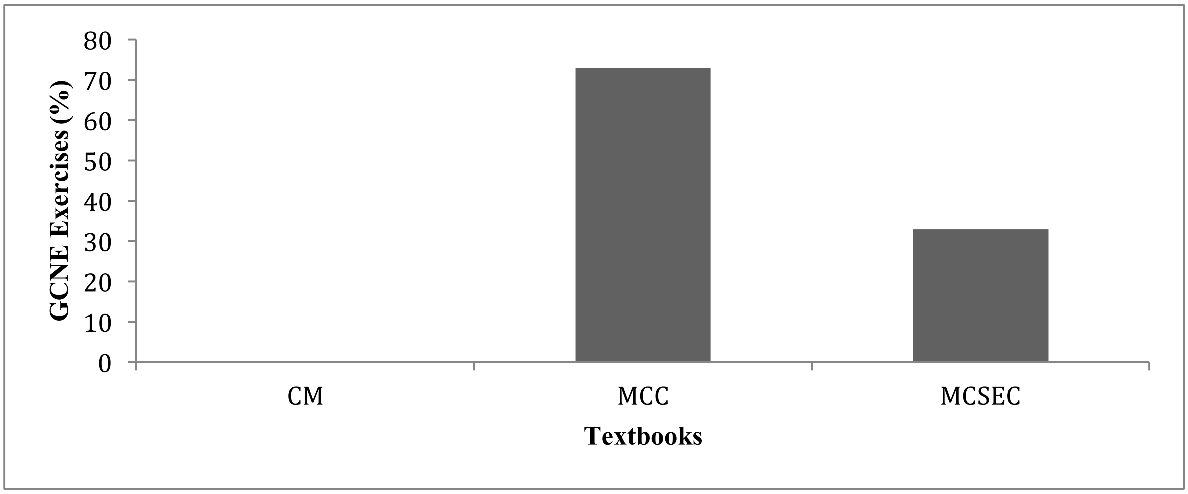


Figure 7: GCNE exercises in textbooks

With regard to the other practices of reasoning and proof, all three textbooks offered opportunities for students to identify patterns and construct proofs. CM predominantly offered the construction of proof arguments in its student exercises. Whereas MCC and MCSEC offered student activities which asked students to identify patterns by empirical investigations with a few geometric objects. In some of these activities, students were motivated to go further and make a conjecture.

Conclusion

The analysis of the three textbooks suggests that there exist opportunities for students to engage in or reflect on the processes of reasoning and proof. However, the type of opportunities varied across the three textbooks. For example, the older textbook, CM specifically offers opportunities for the construction of proof, with limited offerings for conjecturing and writing non-proof arguments. The prevalence of proof construction is important because this aspect of reasoning and proof allows students to use formally introduced theorems and concepts to construct logical deductive arguments that explain why a result may be true (Stylianides, 2009). Proof construction also provides opportunities for students to use their mathematical knowledge to practice deductive reasoning. The emphasis on the construction of proof also aligns with the policy documents in Trinidad and Tobago, which claim “students must be given opportunities to develop logical deductive arguments” (Republic of Trinidad and Tobago, Ministry of Education, 2009, § 2: 1). Therefore, the authors of CM seem to promote the reformers’ vision of increased opportunities for students to engage in the construction of proof arguments.

The more recently published textbooks (i.e., MCC and MCSEC) seem to exemplify more opportunities for all the processes of reasoning and proof. These included the authors’ offering activities that allow students to engage in pattern identification, conjecturing, and developing non-proof arguments. This characteristic is important because it suggests that these textbooks' authors seem to afford the types of opportunities that allow students to engage in all the processes of reasoning and proof. However, this does not necessarily imply that students will gain the type of scaffolding that leads from pattern identification to proof construction. None of the textbooks allowed students to go through the entire process within one exercise. It would be worthwhile for students to engage in, finding patterns, then make and test new conjectures from the patterns observed. This will possibly lead to the revision or validation of these conjectures. The validation process may initially include the developing of non-proof arguments that could develop into the construction of a proof. Several researchers advocate that these activities are important for building the foundations for students' development of writing proofs (e.g., Bieda, 2010; Chazan, 1993; Cirillo & Herbst, 2012; Stylianides, 2009). Thus, the inclusion of all the activities of reasoning and proof within a single student exercise has the potential to help students understand and value the necessity of proof as a culmination of earlier reasoning processes.

The prevalence of GCNE type of exercise in these two textbooks suggests that students may have extensive opportunities to see understand the explanatory role of proof in mathematics. Several researchers argue that the status of proof will be elevated in school mathematics if most and foremost its explanatory role is promoted in curriculum materials (e.g., Bell, 1976; Hanna, 1990; Hersh, 1993). Therefore, the exemplification of explanatory role of proof in the GCNE exercises may be important for helping students understand why a result is valid and promotes insight into the relevance and usefulness of geometrical concepts or theorems when solving problems. However, this depends on how teachers use these exercises during instruction. A future study could investigate students' conceptions of these type of GCNE exercises with regard to developing non-proof arguments. Furthermore, students may not consider these informal explanations as opportunities to further develop a proof.

A major characteristic of the GCNE tasks found in the textbooks was an accompanying diagram, which can initiate mental and physical processes that lead to deductive reasoning about geometrical properties. The inclusion of diagrams in GCNE tasks promotes an important dimension of *cognitive complexity*[4] that requires high-level thought and reasoning of students (Hsu & Silver, 2014; Magone, Cai, Silver, & Wang, 1994). Therefore, the opportunities afforded by solving GCNE tasks have the potential for students to reason with and about relationships between the given and the unknown characteristics in a geometrical diagram. Moreover, the characteristic problem-solving process of GCNE tasks provides students with the opportunity to use algebraic operations with connections to geometric theorems and concept. This latter characteristic seems similar to Geometric Calculation in Algebra (GCA) type exercises found in US Geometry textbooks (Boileau & Herbst, 2015). GCA and GCNE exercises allow students to use multiple-step reasoning in their justification of the steps taken in their algebraic computations derived from creating algebraic expressions for missing components of a geometric diagram. However, my analysis suggested that all GCNE exercises contained the phrase "Give reasons for your answer" thus explicitly requesting that students provide explanations for their algebraic calculations in Geometry, whereas the GCA type questions do not explicitly request students'

[4] Cognitive complexity refers to the features of a mathematical task that promote students' engagement in cognitive process such as making connections among geometrical concepts and mathematical reasoning (Magone et al., 1994).

explanation of their reasoning. There exists the need to examine the possible occurrences of the GCA exercises in the textbooks in Trinidad and Tobago. The aforementioned would provide a useful discussion about comparisons between the type of calculate and explain type of geometry questions in Trinidad and Tobago and US textbooks.

Researchers claim that these aforementioned properties of solving GCNE tasks are characteristics of tasks with highly complex cognitive demand (Henningsen & Stein, 1997; Hsu & Silver, 2014). Therefore, the prevalence of GCNE tasks has the potential for students to develop arguments that could eventually be considered a proof and affords students' opportunity for engagement with highly complex cognitive activity. However, these characteristics lead to the question of whether students realize that these GCNE tasks could foster their development of proof writing skills although they do not formally ask students to do a proof. Additionally, it is worth investigating in future studies, whether teachers see the potential of these GCNE tasks in helping their students' development of reasoning and proof skills. Despite the potential of the GCNE tasks for engaging students in constructing proofs, we are yet to fully understand why students continue to perform poorly on CSEC examination proof items. Therefore, there exists the need for further evaluation of the affordance of GCNE tasks in helping students with constructing proofs.

Although my analysis of the textbooks demonstrates that there exist opportunities which, allow students to engage in pattern identification to conjecturing; there is a need to have more opportunities that guide students even further to constructing proofs. This may allow students to transition from inductive to deductive reasoning. Furthermore, the prevalence of empirical justifications in the textbook demonstrations could possibly indicate to students that the use of a few confirming examples is an acceptable proof of a mathematical claim. Overall the three textbooks to some extent allow students to see the need for explaining why a mathematical statement is true however students should be given a uniform distribution of all four processes of reasoning and proof in geometry. These findings suggest possible guidelines for future evaluations of the quantity and quality of Geometry opportunities for reasoning and proof in secondary school textbooks in Trinidad and Tobago.

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ENACTING FUNCTIONS FROM GEOMETRY TO ALGEBRA

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This paper describes an innovative technology-based approach that enables students to learn function concepts by constructing and manipulating functions in the form of geometric transformations on the plane. Students' direct sensorimotor experiences with variables, function rules, domain and range help them make sense of linear functions, Cartesian graphs, derivatives, multiplication of complex numbers, and Euler's formula. Treating geometric transformations as functions is not a new idea in secondary mathematics, but few curricula take full advantage of the approach to develop students' concept of function. Web Sketchpad, the technology described in this paper, supports a constructionist approach to students' activities of creating, manipulating, and investigating mathematical objects, thus linking their sensorimotor activity to their conceptual understanding. The software provides a simple interface with no menus, based on dragging and on using a small set of tools designed by the activity author. These limited options help create a field of promoted action, encouraging productive student behaviour in accomplishing a specific task.

Keywords: concept image, dynagraph, embodied cognition, enacting, field of promoted action, function, geometric transformation, progressive abstraction, representation, websketch,

Introduction

How does it feel to move like a dependent variable?

Most students would regard this question as nonsense; they view variables as abstract ideas that are unconnected to their sensorimotor systems. Though developing students' understanding of function concepts is a critical goal of secondary mathematics, few students graduate from secondary school with a robust conceptualization of function (Carlson & Oehrtman, 2005). Students have little sense of covariation, and their concept image of function is often at odds with the formal definition (Vinner & Dreyfus, 1989). They graph functions without understanding the link between the behaviour of the variables and the shape of the graph.

Mathematics educators have long stressed the importance of learning by doing, and cognitive scientists have researched ways in which "cognitive structures emerge from the recurrent sensorimotor patterns

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that enable action to be perceptually guided" (Varela et al., 1991, p. 173). Yet curricula often fail to provide students with the sensorimotor grounding for function concepts. The primary visual representation that students encounter is the Cartesian graph, which lacks any explicit representation of variables; the other main representation is the equation, such as $f(x) = 2x - 3$, that lacks any sense of dynamism or opportunity for students to put variables into motion.

Not surprisingly, students' difficulties with functions often begin with the concept of variable, which has so many meanings and serves so many purposes that students have difficulty formulating a coherent sense of the term (Schoenfeld & Arcavi, 1998). Freudenthal (1986, p. 494) argues that mathematical variables "are [an] indispensable link with the physical, social, and mental variables" and observes with approval that "originally 'variable' meant something that really varies" (p. 491). But students seldom experience variables in motion despite evidence suggesting that "if students are allowed to control the movement of an object, for example, or the changing of a variable, their scores and other measures of understanding are much higher than from passive animations or static diagrams alone" (Holton, 2010, p. 5).

If the learning of function begins not with static graphs and equations but rather with variables in motion, with the dance in which independent and dependent variables engage, we argue that students will develop a more detailed and robust concept image of function, and that ideas like the relative rate of change, domain, range, composition, and inverse will be better grounded in their sensorimotor experiences. We believe that with such a concept image as a foundation, students can more easily learn to look at a Cartesian graph and visualize the implicit motion of the variables, mentally seeing x move along the horizontal axis while $f(x)$ moves in synchrony along the vertical axis, and that students can even learn to look at a graph of $f(x) = \sin x$, visualize x in motion, track the rate at which the dependent variable changes, and sketch the graph of the derivative of $\sin x$.

Geometric Functions

Though geometric transformations are functions that have as their variables points in the plane, transformations have seldom been used to introduce function concepts. Coxford and Usiskin's ground-breaking treatment of transformations—first introduced in *Geometry: A Transformation Approach* (1971), and continued in *UCSMP Geometry*—does the converse, introducing transformations as functions, which is not quite the same. Freudenthal (1973) has observed that "[geometry] is one of the best opportunities that exists to learn how to mathematize reality…. [N]umbers are also a realm open to investigation…but discoveries made by one's own eyes and hands are more convincing and surprising" (p. 407). The advent of dynamic mathematics software such as Cabri and Sketchpad enabled students to experience functions by constructing and manipulating geometric objects that depend on each other. As Hazzan and Goldenberg (1997) note, "[the] geometric context may provide enough contrast with algebraic contexts to allow essential aspects of the important ideas [of function] to be distinguished from features of the representation" (p. 287).

One way that researchers and curriculum developers connect geometry to functions is in activities in which students begin with a geometric construction, change one of the construction's elements (commonly by dragging a point), and describe how the dragged point affects other constructed objects or the measurements of those objects. Examples appear in Hazzan and Goldenberg (1997) and Wanko et al. (2012). The independent variable may be the dragged point or a measured value derived from the dragged point. Similarly, the dependent variable may be a constructed point that varies when the first point is dragged or a measured value derived from such a point.

A second way for students to experience function concepts in a geometric context is applying geometric transformations to polygons and other constructed geometric figures (Flores & Yanik, 2016; Hollebrands, 2003, 2007). Many textbooks use a variation of this approach by incorporating tasks in which students transform polygons constructed on a coordinate plane as in Figure 1. In some activities, the independent and dependent variables are pictures or other shapes. In these activities, the independent and dependent variables are not atomic but have structure of their own.

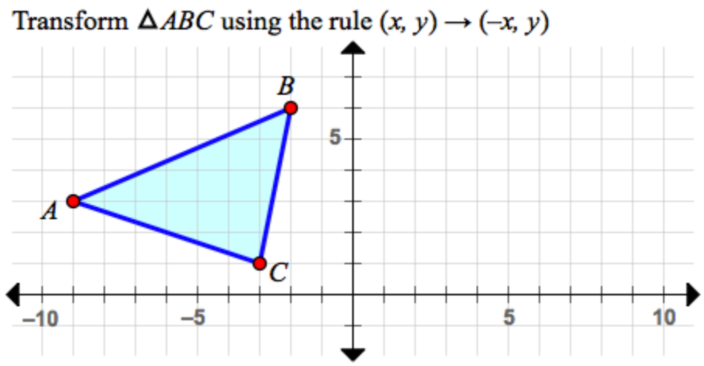


Figure 1: A coordinate-system transformation problem

For the purpose of introducing students to function concepts, both of the above approaches risk creating confusion and misunderstanding due to the presence of extraneous structural elements: Either the function rule is geometrically constructed or the variables themselves have structure. We suspect it is preferable for students to begin with unitary variables and simple, well-defined function rules.

A third way, used here, is based on functions structured similarly to those in *Geometry: A Transformation Approach*. The prototypical function is a similarity transformation (a reflection, rotation, transformation, or glide reflection, possibly composed with a dilation) using geometric points as both independent and dependent variables. The variables are atomic, with no structure of their own, and function rules are limited to the five families listed above with simple parameters (such as a mirror line or a center and angle of rotation) distinguishing one family member from another. We refer to such functions as geometric functions.

Despite a long history of discussion in mathematics education circles about the role transformations should play in the study of geometry, and despite the observations by Freudenthal and others that suggest the potential value of introducing function concepts in this way, the authors are not aware of any published curriculum that uses geometric transformations for this purpose.

Geometric Functions and Dynamic Mathematics Software

Geometric functions are particularly suited for introducing students to function concepts because their two-dimensional nature ($\mathbb{R}^2 \rightarrow \mathbb{R}^2$ transformations in the plane) is well modelled by the two-dimensional input and output interfaces (mouse/finger and screen) that students employ. Similar activities based on one-dimensional dragging using $\mathbb{R} \rightarrow \mathbb{R}$ functions are likely to be less effective: motor actions are less expressive, and visual effects are less compelling in one dimension than in two.

Using dynamic mathematics software, we can leverage this correspondence between the mathematical domain and the computer's affordances to reduce the cognitive distance between the student's concrete sensorimotor system and the abstract mathematical concepts of function. The result is that the Coxford/Usiskin innovation (of treating geometric transformations as functions) is even more persuasive and effective today than when it was introduced in 1971.

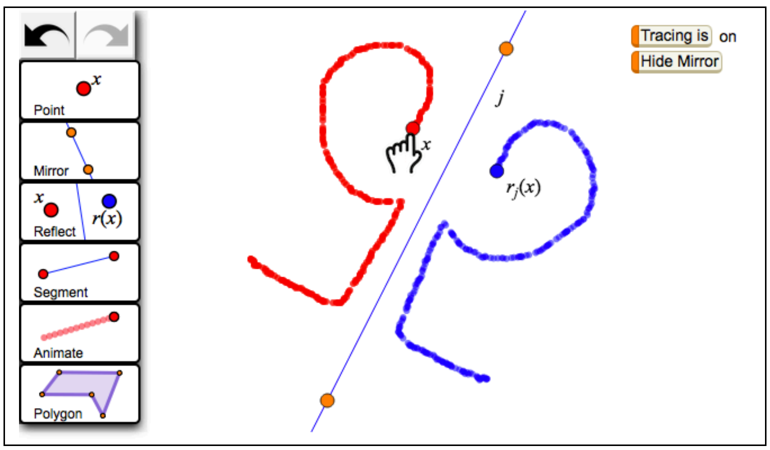


Figure 2: Varying x to make a design and compare rates

When today's student constructs a reflection function as in Figure 2 and drags the independent variable (point x), she can directly observe the motion of the dependent variable $r_j(x)$. (The notation $r_j(x)$ is an abbreviation for "the reflection in mirror j of x".) By comparing the motion of the two variables and observing the traces they leave behind, she might describe the relative rate of change of x and $r_j(x)$ this way: "When I drag along the mirror, $r_j(x)$ moves the same way as x, but when I drag toward or away from the mirror, $r_j(x)$ moves the opposite way from x." Once she verifies that this description is common to all members of the reflection function family, she can identify any other member of this family even if its mirror is hidden, and she can use her understanding of the relative rate of change to locate the hidden mirror.

Innovative Tools in Support of Tasks

Figure 2 shows the work of a student using a Web Sketchpad activity to construct and investigate a reflection function. (This activity, and the other activities illustrated in this chapter, are available online at https://geometricfunctions.org/icme13.) *Web Sketchpad* (WSP) is dynamic mathematics software that runs on all modern browsers that support HTML5 and JavaScript. WSP can open nearly any document created by The Geometer's Sketchpad (Jackiw, 2009), and provides an innovative self-documenting tool interface allowing tools to be customized for each activity.

When a typical student begins the Reflect Family activity in Figure 2, she sees a screen with a Tracing button at the upper right and six tool icons on the left. She uses the first three tools to construct and drag independent variable *x*, to construct a mirror, and to reflect *x* across the mirror to create the dependent variable $r_j(x)$. Dragging *x* while observing $r_j(x)$ allows the student to investigate the relative movement of the two variables. She can turn on tracing, drag once more, observe the covariation that characterizes this geometric function, and answer questions like these: "How can you make *x* and $r_j(x)$ move in the same direction? How can you make them move in opposite directions?"

In this activity students use three different tools to construct the three elements of a function: a tool for the independent variable *x*, a tool for the mirror that corresponds to the function rule for reflection, and a tool for the dependent variable $r_j(x)$. These three tools represent a design choice by the activity developer to emphasize the three elements of a function: the independent variable, the rule, and the dependent variable that results from applying the rule to the independent variable. The combination of the software itself, the carefully crafted tools, and the student task creates a "field of promoted action" (Abrahamson & Trninic, 2015) in which students' actions are gently constrained to help them accomplish the task presented to them.

In later activities students use a single tool for the same purpose: designating or constructing the independent variable, designating or constructing the mirror, and constructing the dependent variable. The transition from three tools to one encourages students to transition from an action understanding toward an object understanding of the reflect function. These are steps in the APOS (action-process-object-schema) sequence that describes students' increasingly sophisticated understanding of functions (Dubinsky & Harel, 1992).

This activity provides students with several additional tools. A student might use the *Segment* tool to construct a restricted domain for the independent variable *x,* to connect *x* to $r_j(x)$, or for some other purpose entirely. Alternatively, she might use the *Polygon* tool to construct a restricted domain, and then use the *Animate* tool to animate *x* around this restricted domain.

The tool interface is innovative, minimizing reliance on language. When the student taps a tool icon, the entire object to be constructed appears on the screen with the tool's given objects highlighted and pre-existing sketch objects backgrounded. This effect provides immediate feedback regarding the entire construction being created; there is no need for the student to be instructed as to what objects to click, in what order, to use the tool successfully. This overview of the entire tool gives the student an opportunity to see what objects the tool will construct and to consider how to integrate these new objects into the existing sketch. A highlighted given object can be attached to an existing sketch object (by dragging the given object onto the sketch object) or located in empty space (by dragging it to the desired location) with no restriction on the order in which given objects are attached. As soon as the last given object is attached or located, the tool's action is complete; the backgrounding of pre-existing objects terminates, and the sketch is again fully interactive.

The tool interface also provides two shortcuts for the users' convenience. Pressing the green check mark above the toolbox instantly completes the tool's action by locating any unmatched given objects in their current locations, and pressing the red ✕ instantly cancels the tool's action. Another shortcut eliminates the need to drag each given object to attach or locate it: At any time during tool use, one given object is glowing to indicate that it can be attached or located by using the finger or mouse to tap an existing object (to attach the given object to the tapped object), to tap in empty space (to locate the given object at the tapped location), or to press and drag (to make the given object jump to the pressed location and follow the drag until finger or mouse is released). A video is here: http://geometricfunctions.org/icme13/using-wsp-tools.html.

The Web Sketchpad tool interface was designed to help activity developers create fields of promoted action. By providing only tools needed for the task at hand (optionally arranged in the order of expected use), there is less need to provide students with prescriptive directions and thus better support for open-ended tasks. And by immediately showing the user detailed visual information about the effect of the chosen tool, there is less need to explain how to use tools with which the user is not already familiar. These innovations enable less prescriptive and more open-ended student tasks, and encourage students' self-reliance and productivity. Students can concentrate on the mathematics of the task rather than following directions from a worksheet or from the teacher.

Design-Based Research

We use a design-based research methodology to iteratively develop, test, and refine the activities described here (The Design-Based Research Collective, 2003; Barab & Squire, 2004; Fishman et al., 2004). Although earlier versions of some of these activities were developed with the support of the Dynamic Number project funded by the National Science Foundation (Steketee & Scher, 2011), development of the current activities began in earnest in late 2014, when customizable tools became available in Web Sketchpad. We first developed 14 activities organized into two units: Introducing Geometric Transformations as Functions (Unit 1) and Connecting Algebra and Geometry Through Functions (Unit 2). Pilot tests occurred with four classes, two in 8^{th} grade while the remaining two in 10^{th} grade, located in inner-city Philadelphia schools. Though designed as an introduction to linear functions, these units appear to be helpful also for students who have already studied linear functions. The pilot tests resulted in substantial changes to the original websketches and student worksheets. They also informed the creation of performance-based assessment instruments both as stand-alone websketches and as pages incorporated into the main activity websketches. We subsequently developed several activities addressing calculus, vectors, and complex functions.

The activities are freely available at https://geometricfunctions.org/icme13 under a Creative Commons CC-BY-NC-SA 4.0 license and can be used with any web browser. Activities from the first two units include online websketches and student worksheets and are available online and as PDF's. We hope to provide detailed teacher support materials soon. Due to ongoing revisions, online activities may differ from the figures and descriptions in this paper.

The remainder of this document describes various activities that emphasize how technology-enabled guided inquiry can enable students to construct and enact mathematical objects and concepts related to function. We also note several instances in which our activities' pilot testing revealed weaknesses in our original instructional design, prompting rethinking and revision of that design.

Enacting Variables and Rate of Change

The act of dragging geometric function variables can help students develop the sense that variables vary. In Figure 2 (above), the student constructs and drags independent variable point *x*, thus enacting the independent variable by moving it directly with her finger or mouse. In Figure 3 (part of the Rotate Family activity), she makes a Hit the Target game. After constructing independent variable *x* and a rotate function to produce dependent variable $R_{C,\theta}(x)$ (again, meaningful function notation: $R_{C,\theta}(x)$ represents the "rotation, about *C* by angle θ, of *x*"), she then uses the Target tool to make a target and create a challenge: drag *x* to make that dependent variable $R_{C,\theta}(x)$ hit the target. Once she hits the target, she generates a new problem by pressing the *New Challenge* button, which changes both the rotation angle θ and the location of the target.

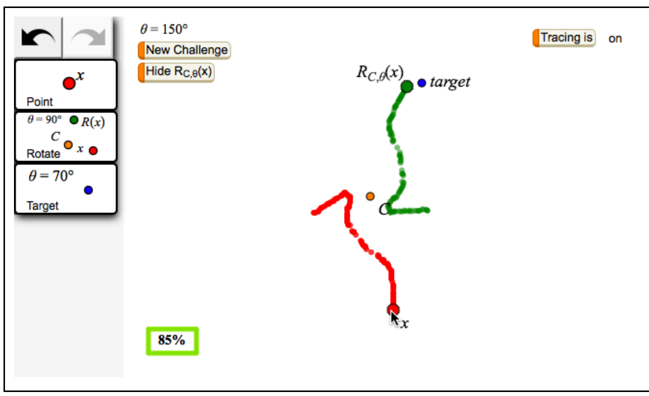


Figure 3: Varying *x* so $R_{C,\theta}(x)$ hits the target

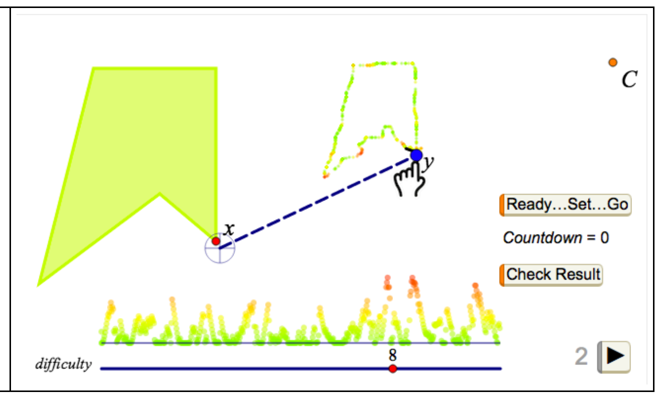


Figure 4: Dragging *y*, trying to co-vary with *x*

When playing this game, students usually begin either by dragging *x* toward the target (as in the top part of the red trace) or by adopting a somewhat random guess-and-refine strategy. As they try to improve their play, students are encouraged to reason backward, using the target location and angle θ to estimate the direction in which to drag *x*.

Figure 4 challenges the student to enact the dependent variable of a dilate function. Her task is to drag *y* according to the function rule, while independent variable *x* follows the polygon border. Even with hints of the dashed segment and cross-hairs showing how close she is and a traced image of *y* that changes from red when she is far away to green when she is close, this is a real challenge. The player must drag *y* both in the correct direction and at the correct speed to match the motion of *x*. In other words, her dragging action must get the rate of change of *y* relative to *x* just right.

In these activities, students' enactment of point variables creates a semantic link between physical movement and mathematical variation. The student drags variables and observes how easy it is to enact an independent variable, free to move within in its domain, and how hard it is to enact a dependent variable, constrained to follow the independent variable based on the function rule.

Enacting Domain and Range

In Figures 2 and 3, the domain of the function is the entire plane, and the student experiences it as the ability to drag x anywhere within the window on the computer screen. This is not in the least remarkable to the student, rendering futile any attempt to introduce the terms *domain* and *range* at this stage. To develop conceptual understanding, students must first have a meaningful reason to restrict a function's domain and observe its corresponding range.

In the Dilate Function activity in Figure 5, the student uses the *Polygon* tool to create a polygon and the *Point* tool to create independent variable x attached to the border of the polygon. She drags x to explore what happens, and how it feels, when x is restricted to this polygonal domain. After using the *Dilate* tool to dilate x about center point C by scale factor s, the student turns tracing on and drags x again to observe the corresponding range traced out by the dependent variable $D_{C,s}(x)$.

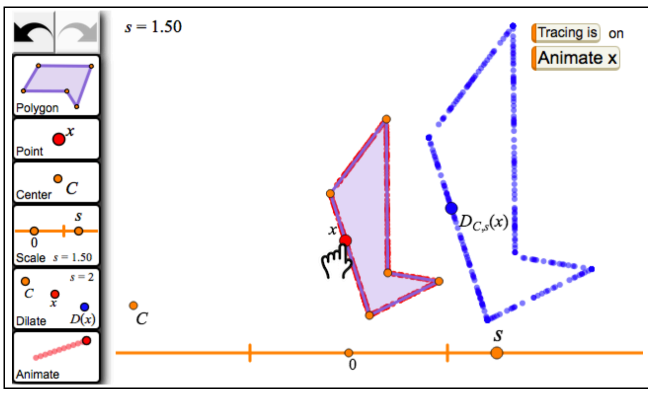
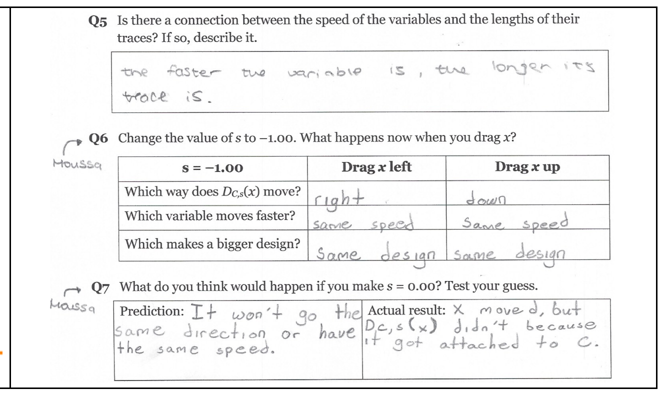


Figure 5: A restricted domain and its range


Q5 Is there a connection between the speed of the variables and the lengths of their traces? If so, describe it.

the faster the variable is, the longer its trace is.

Q6 Change the value of s to −1.00. What happens now when you drag x?

s = −1.00	Drag x left	Drag x up
Which way does $D_{C,s}(x)$ move?	right	down
Which variable moves faster?	same speed	same speed
Which makes a bigger design?	same design	same design

Q7 What do you think would happen if you make s = 0.00? Test your guess.

Prediction: It won't go the same direction or have the same speed.

Actual result: X moved, but $D_{C,s}(x)$ didn't because it got attached to C.

Figure 6: Sample student work (dilate family)

The ability to drag x on its restricted domain while attending to both the path and the relative rate of change of $D_{C,s}(x)$ is an important sensorimotor experience that provides students with grounding for their conceptual understanding of the domain, range, and relative rate of change while also spurring them to consider what it means to apply a function all at once to an entire set of points (a polygon).

By the end of Unit 1 (Introducing Geometric Transformations as Functions), students in the pilot test were using the tools effectively and identifying the roles of the various objects. Most students were already quite comfortable describing function behaviour in terms of the relative rate of change (both speed and direction), as illustrated in Figure 6.

Connecting Geometric Transformations to Algebra

Unit 2 (Connecting Algebra and Geometry Through Functions) explicitly connects the geometric functions of Unit 1 to algebra. It begins by asking students to restrict the domain of these geometric transformations to a number line and to determine which of the *Flatland* (two-dimensional) function families can most easily fit into the *Lineland* (one-dimensional) environment of a number line (Abbott, 1886). Once students determine that the dilate and translate families are particularly suitable because their independent and dependent variables always move in the same (or opposite) direction, they engage in construction activities that connect the geometric behaviour of dilation and translation to the observed numeric values of their variables on the number line.

In Figure 7, a student uses the *Number Line*, *Point*, and *Dilate* tools to create a point restricted to the number line and dilate it about the origin. She measures the coordinates of x and $D_{0,s}(x)$ and drags x to compare the values. When asked to describe what happens when she changes x by 1, she might respond, "When I increase x by 1, $D_{0,s}(x)$ increases by twice as much, which is the same as the scale factor s." By experimenting with different scale factors, the student concludes the coordinates produced by this dilation satisfy $D_{0,s}(x) = x \cdot s$. She then experiments with the translation restricted to the number line and concludes that translation by a vector of directed length v satisfies the equation $T_v(x) = x + v$. Thus, she concludes that dilation on the number line corresponds to multiplication and translation corresponds to addition.

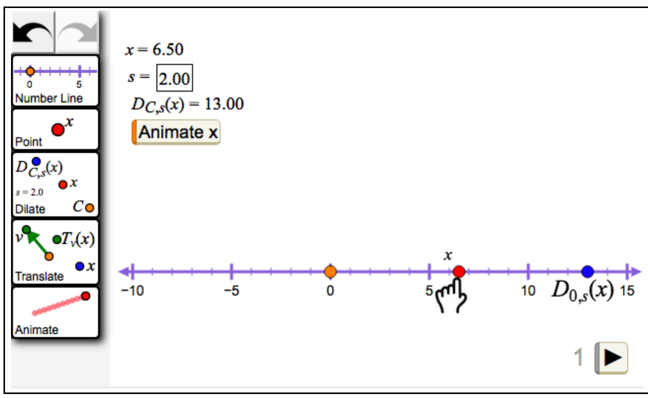


Figure 7: Dilating on the number line

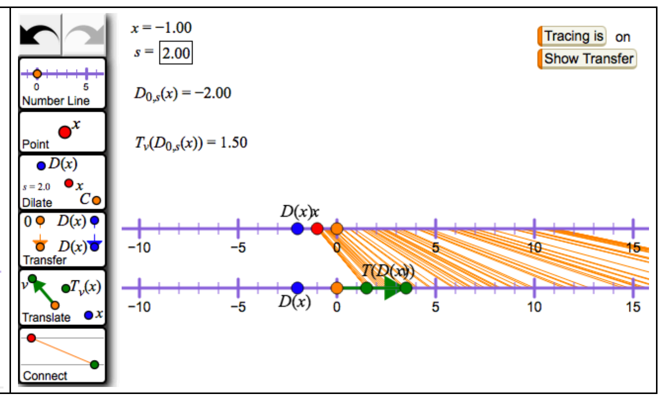


Figure 8: Constructing $T_v(D_{0,s}(x))$ on a dynagraph

Enacting Composition, Dynagraphs, and Cartesian Graphs

Having moved from *Flatland* to *Lineland* and discovered the algebraic meanings of dilation and translation on the number line, students are now ready for a new task: What happens when you dilate x and then translate the dilated image; in other words, how does $T_v(D_{0,s}(x))$ behave? Students' first attempts at this task becomes visually confusing with three variables and a vector stumbling over each other on the same number line. To alleviate the confusion, the next activity incorporates a *Transfer* tool that moves the dependent variable to a different number line, separate from but aligned with the first. In Figure 8, students use this tool to construct a second number line parallel to the original, creating a dynagraph (Goldenberg, Lewis, & O'Keefe, 1992). By varying x and observing the connecting line between the variables, students describe and explain how changing each parameter (scale factor s and vector v) affects the relative rate of change of the variables and their relative locations.

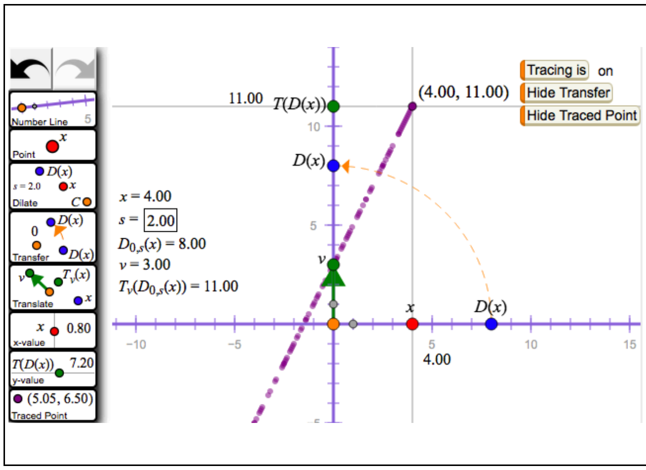


Figure 9: Dilate, rotate by 90°, and translate

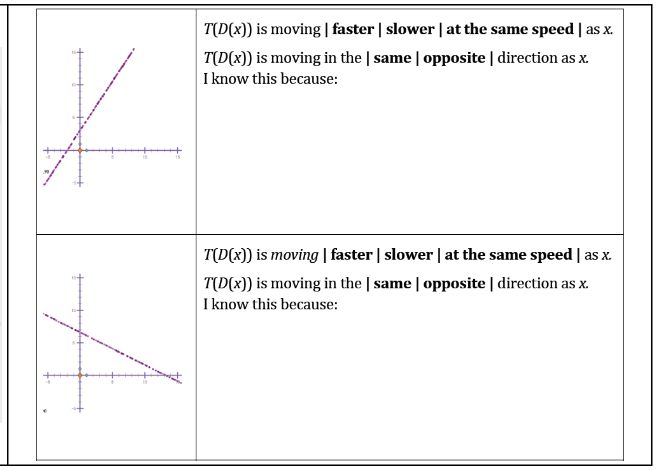


Figure 10: Inferring motion from a graph

In the final activity of Unit 2, students create the Cartesian graph of a linear function using geometric transformations. As Figure 9 illustrates, students start with the same initial tools that they used to create a dynagraph, but this activity's *Transfer* tool rotates a variable by 90°, transferring it to a vertical number line perpendicular to the original, horizontal number line. After using this tool to rotate $D_{0,s}(x)$) to a vertical axis and translating by vector *v,* students use the *x-value* and *y-value* tools to construct lines that keep track of the horizontal location of *x* and the vertical location of $T_v(D_{0,s}(x))$. They then construct a traced point at the intersection of these horizontal and vertical lines and drags *x* to see how the traced point's motion corresponds to the behaviour of the two variables.

After performing the construction, students try different values for the scale factor *s* and the translation vector *v,* and they observe how changing the scale factor affects not only the speed of $T_v(D_{0,s}(x))$ relative to *x* but also the shape of the traced line. For instance, one of our pilot test students looked at the lower traces shown in Figure 10 and explained that this trace indicated that the variables were moving in opposite directions because the value of the dependent variable moved down as the independent variable moved right. She went on to say that $T_v(D_{0,s}(x))$ was decreasing more slowly than *x* was increasing because the traces went down more slowly than they went to the right, and concluded that the scale factor was approximately –½. Such observations suggest that students can use their experiences in geometrically enacting variables and functions to visualize the motion implicit in static Cartesian graphs. (And if this is students' first experience with such functions, they may invent the term *linear function,* and write the formula for linear functions as $y = s \cdot x + v$: dilate *x* by *s* and then translate by *v*.)

Performance-Based Assessment

Our pilot tests have also helped us generate ideas for performance-based assessments. For instance, we created the *Dilate-Family Game* shown in Figure 11 as we discussed assessment issues with one of our pilot-test teachers. The game has multiple levels that require greater precision and provide less diagrammatic scaffolding as a student moves up through the levels. We intentionally did not set a specific number of problems per round, so that a teacher has the flexibility to say, for instance, "To be a dilation apprentice, you must score 8 of 10 at Level 2; to be a dilation master, you must score 7 of 10 at Level 5; and to be a dilation superhero you must score 16 of 20 at Level 9."

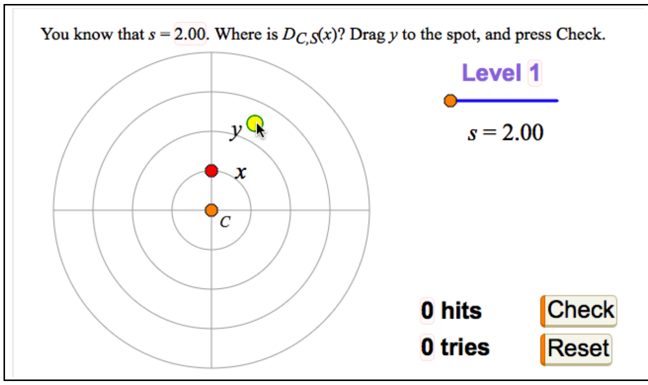


Figure 11: Dilate Family Game

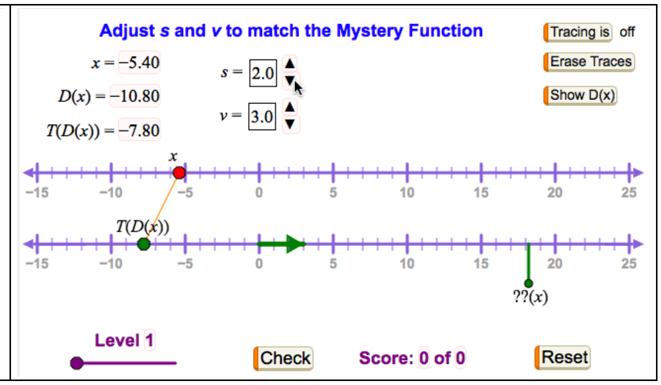


Figure 12: Dynagraph Game

We are not yet satisfied with students' results on this dilation-family assessment. Some students who constructed and investigated *Dilate* functions successfully still had difficulty understanding how the game worked even at Level 1. This activity has already been refined to support students' transition in the game, but we remain concerned about possible gaps in students' visualization of the dilation function. In an upcoming pilot test, we will explore this further by interviewing small groups of students and make additional revisions based on what we learn. Our plan also includes modifying the game to enable direct reporting of students' results to the teacher. (The initial version relies on either visual inspection by the teacher or screen captures submitted by students.)

Figure 12 illustrates the *Dynagraph Game*, a performance-based assessment for the dynagraph activity described above. In this game, independent variable x is always in motion from left to right, and students adjust s and v to control the dynagraph whose dependent variable is $T(D(x))$. There is also a *mystery function* whose moving dependent variable $??(x)$ is shown below the lower axis. The student's challenge is to adjust s and v to match the *mystery function*, so that $T(D(x))$ is always exactly aligned with $??(x)$. Higher levels of the game require greater precision in adjusting s and v.

We conjecture that performance-based assessments such as these can help students solidify their understanding of function concepts while also promoting mathematical fluency, and we are eager to test this conjecture as we continue our effort to refine the activities based on classroom testing.

Enacting the Slope of the Sine Graph

Students are often presented with the definition of *derivative* instead of inventing their own definition based on creating and experiencing the mathematics themselves. In this activity, we present students with five tasks designed to encourage them to connect *slope* to the relative rate of change of variables and to invent their own definition of *derivative*.

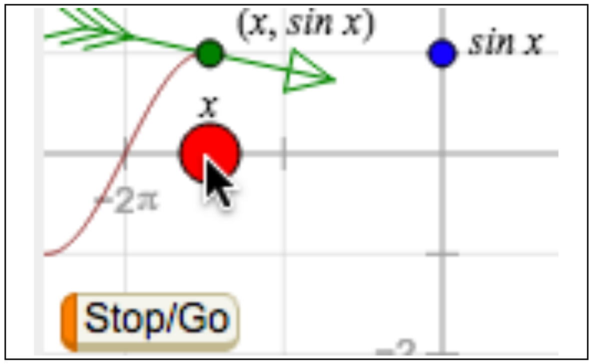


Figure 13: Following the slope

In Figure 13, a student has just begun the first task. She varies *x* while she observes the connection between the green arrow and the behaviour of the dependent variable *sin x*. The student notes that *sin x* has already come to a stop at its maximum value and is about to begin to move down just as the arrow has changed its previous upward direction to horizontal and is now beginning to point down.

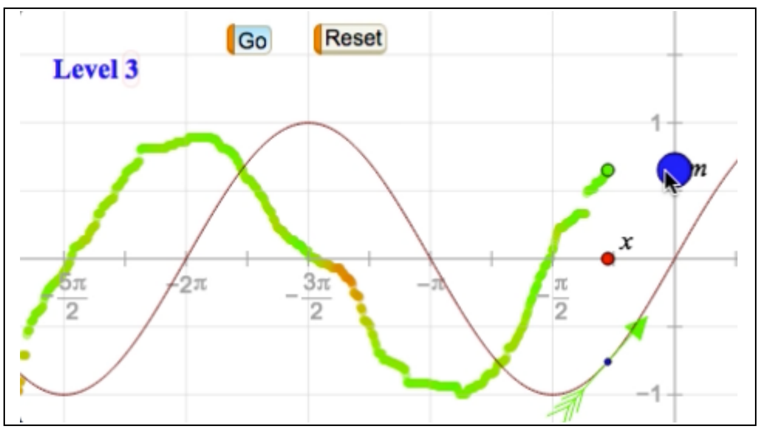


Figure 14: The Slope Game

Figure 14 depicts the second task, the *Slope Game*, in which students control the arrow's slope by dragging point *m* up or down. Their objective is to keep the arrow lined up with the graph. After practicing by dragging *x* and readjusting *m* several times, the student presses *Go.* After a 2-second delay, *x* begins moving along its axis. The student's job is to drag *m* so the arrow stays aligned with the graph. In other words, the goal is to drag *m* so that its value is the derivative of the sine function. As the student drags *m*, the point (*x, m*) is plotted and traced with the colour of the trace ranging from green, when *m* is very close to the function's current rate of change, to yellow to red, when the value of *m* is far from the rate of change. The arrow itself changes colour to match, thus providing the student with immediate feedback as she attends to the relationship between the arrow and the graph. In Figure 14, the student lagged a bit behind adjusting *m* as *x* passed $x = \frac{-3\pi}{2}$, and the slope of the graph became negative. This lag is visible as a reddish-yellow bump in the trace, which is otherwise almost all green.

The gap in the trace shortly after $x = \frac{-\pi}{2}$ indicates that the student again fell slightly behind but caught up by moving *m* so quickly that she left a gap in the trace.

Two pedagogical elements of this activity are particularly worthy of note: its enactivist nature and its incorporation of performance-based assessment into the learning process. While playing the game, the student enacts the derivative of the sine function by dragging m up and down in concert with the rate of change of *sin x* with respect to x. The activity connects the student's physical motion (dragging) to the direction and speed of the plotted point's vertical movement as mediated by the arrow. Though the mediation of the arrow might help the student connect the geometric property of tangency to the function's instantaneous rate of change, it seems more likely that she will attend to the slope of the arrow rather than to the speed of vertical movement of the graphed point.

Our long-term goal for the student is that she directly observe and interpret the motion of the dependent variable, relating her physical actions more closely to the mathematical concept we intend for her to develop. We address that goal in our *Rate of Change Game*, described below and presented in Figure 15. It is preferable for students to begin with the *Slope Game* because the task of attending to the relative orientation of the arrow and the graph, both of which are visually evident, is more concrete and easier for students to master than the task of attending to the speed and direction of the dependent variable. The move from a relatively concrete task to a related task that is more abstract in nature, variously described as *concreteness fading* and *progressive abstraction*, has been found effective in developing students' conceptual understanding (McNeil & Fyfe, 2012; Mitchelmore & White, 2000).

A second important element of these games is that they serve student learning and assessment at the same time. The feedback from the *Slope Game* is immediate. Students see both the colour of the arrow and its relative orientation to the graph, and these behaviours are under their immediate control as they drag *m*. There is no time to dwell on mistakes; as *x* keeps moving, students are encouraged to continue adjusting *m* to keep the arrow tangent to the graph. Nor are mistakes recorded permanently; starting a new game erases the traces from the previous game. Thus, the games provide support for immediate student self-assessment.

As students improve their skills, the teacher can ask students to submit their work: "Please email me a screen capture that shows all green except for at most one relatively short brownish or red area. The higher you set the level, the better, but avoid making it too hard on yourself by skipping levels. Make sure you master Level 1 before moving to Level 2, and so forth." Each game has five levels. As students move to higher levels, they must be more and more accurate in matching the correct slope or rate of change in order to keep their traces green.

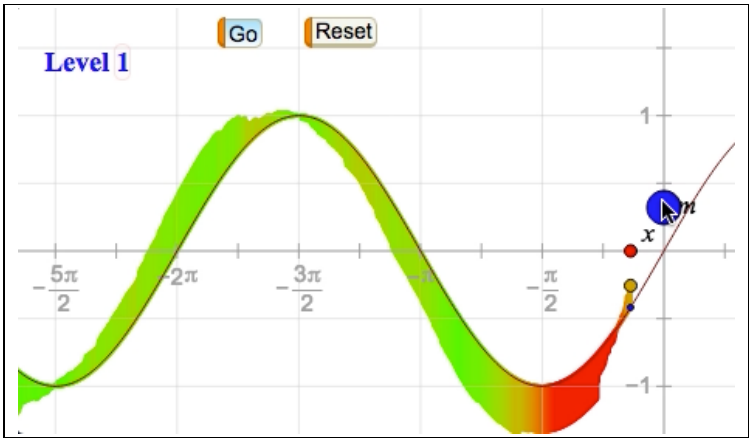


Figure 15: The Rate of Change Game

The *Rate of Change Game* is a performance-based learning task related to the *Slope Game*, but instead of a tangent arrow, it provides a short traced segment, of length proportional to the value of *m,* attached to the moving point. The length of this short segment provides the student with dragging feedback, which allows her to regulate her up-and-down adjustment of *m* while keeping her attention on the moving points. In the meantime, the colour of the point, the segment, and the trace indicate how close the dragged *m* is to the actual rate of change of the dependent variable *sin x.* In Figure 15, as the graph passed the maximum at $x = \frac{-3\pi}{2}$, the student did fairly well at reducing the value of *m* to 0 at the maximum and making it negative thereafter, but as she approached the minimum at $\frac{-\pi}{2}$ she failed to react quickly enough, leaving her value of *m* negative as she passed the minimum. At the moment, she is still recovering, dragging *m* upward towards a positive value that will reflect the current positive rate of change of *sin x.*

We conjecture that this second game will encourage and reward students' direct attention to the rate of change of the function—not just the slope of the graph—and that students who play both games, with a variety of functions, will come to naturally associate the dependent variable's instantaneous rate of change with the slope of the tangent to the graph.

Constructing the Slope and Rate of Change

After completing the initial warm-up task and playing the two games, students are ready to examine the instantaneous rate of change of a function more systematically by means of two more tasks. In both tasks, students begin with an empty screen and use the tools to construct the graph, a secant line, and other objects to approximate the instantaneous rate of change of *sin x* with respect to *x.*

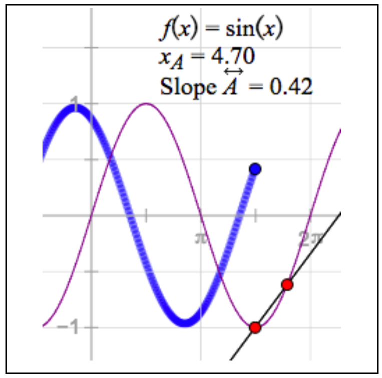


Figure 16: Construct Slope

In the first construction task, *Construct the Slope*, students construct the graph and a secant line, measure and plot the slope of the secant line, and animate the secant line along the graph to track and graph the secant's slope as a function of the position of its defining points (See Figure 16). Based on their *Slope Game* experience and class discussions, students recognize the difference between a secant and a tangent, realizing that the secant will more closely approximate the tangent if the defining points are closer to each other and adjusting the construction accordingly. Students conclude this task by experimenting to find out what happens if they use a button to move one defining point to the other.

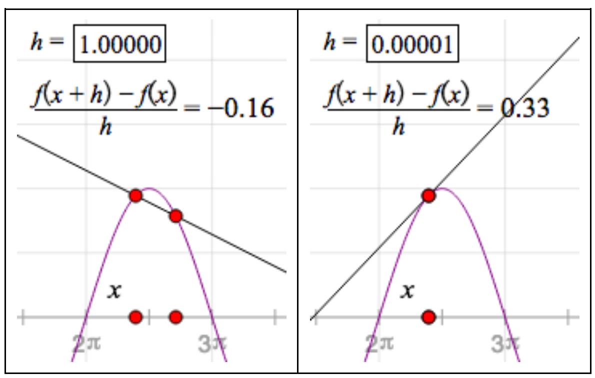


Figure 17: Construct the Rate of Change

The second construction task, *Construct the Rate of Change*, takes a more systematic approach. Like the *Rate of Change Game*, it fades some of the concreteness of the slope construction task. Students create a parameter *h* that they use to precisely control the interval between the x−values at which the function is evaluated. Instead of finding the slope, students calculate the relative rate of change of *sin x* with respect to *x* by calculating the expression $\frac{\sin(x+h)-\sin x}{h}$. Though mathematically equivalent to the slope formula, this calculation is expressed in more abstract language, without any mention of *slope* or *gradient*. By using *h* to control the interval, students can observe the effect of reducing the value of *h* from 1.0 to 0.4 and eventually to 0.00001, as shown in Figure 17.

By using a number of different values of *h,* the first few show two distinct points. Therefore, the student will become aware that even when $h = 0.00001$, the points are still distinct. She is likely to be surprised at the end of the activity when she changes *h* to 0.00000, the line disappears, and the calculation becomes *undefined* instantly.

This surprising action that renders the calculation *undefined* demands explanation and motivates discussion with other individual students and with the entire class. The desired outcome is that students themselves formulate what happened to the calculation and what they can do about it, as a result of making observations such as these:

- As *h* gets smaller, the points get closer and closer together.
- As *h* gets smaller, the line is more closely lined up with the graph.
- As we make *h* smaller, the calculation doesn't change very much.
- When we make *h* tiny, like $h = 0.00001$, we can't even see that there are two points.
- When $h = 0$ the line goes away, because you can't draw a line with only one point.
- Also, when $h = 0$ the calculation is undefined, because you can't divide by zero.
- The calculation gets closer to the real slope the smaller we make *h*—but we can't make it 0.

The pedagogical goal is that that students' experiences and observations lead to a productive class discussion during which students agree on the essential elements of the definition of the derivative. This discussion also presents an opportunity for the teacher to suggest vocabulary useful for naming the phenomena under discussion, including *instantaneous rate of change* and *derivative.*

Enacting Vector Multiplication of Complex Numbers

More than two centuries ago Wessel (1797) and Argand (1874, originally self-published in 1813) independently proposed the two-dimensional complex plane as a geometric way to represent and operate on complex numbers. Complex numbers can be considered either as points in the complex plane or as two-dimensional vectors, and vector addition is essentially identical to complex addition.

However, vector multiplication differs significantly from complex multiplication (described later in this chapter). The former takes two forms: the dot (scalar) product and the cross (vector) product. The dot product is a real number and is readily represented on the real axis of the complex plane, but the cross product is defined as a vector orthogonal to the plane of the vectors being multiplied, thus requiring a third dimension. If the plane containing two vectors **a** and **b** is the *x-y* plane, the cross product $\mathbf{a}\times\mathbf{b}$ lies along the *z*–axis, with magnitude $r_\mathbf{a}\, r_\mathbf{b} \sin(\theta_\mathbf{b} - \theta_\mathbf{a})$ using polar coordinates.

In *Visual Complex Analysis*, Needham (1998) describes a different definition of the cross product $\mathbf{a}\times\mathbf{b}$ that uses only the two dimensions of the complex plane while maintaining several important features of the standard definition. In this redefinition the *z*–axis containing the cross product is rotated into the complex plane to coincide with the imaginary axis, so that $\mathbf{a}\times\mathbf{b}$ retains the magnitude and sign of the standard definition, though it now lies on the imaginary axis, so that its representation in polar coordinates is $\mathbf{a}\times\mathbf{b} = i\, r_\mathbf{a}\, r_\mathbf{b} \sin(\theta_\mathbf{b} - \theta_\mathbf{a})$. The dot product $\mathbf{a}\cdot\mathbf{b}$ is always a real number, so its definition is unchanged: $\mathbf{a}\cdot\mathbf{b} = r_\mathbf{a}\, r_\mathbf{b} \cos(\theta_\mathbf{b} - \theta_\mathbf{a})$ in polar coordinates.

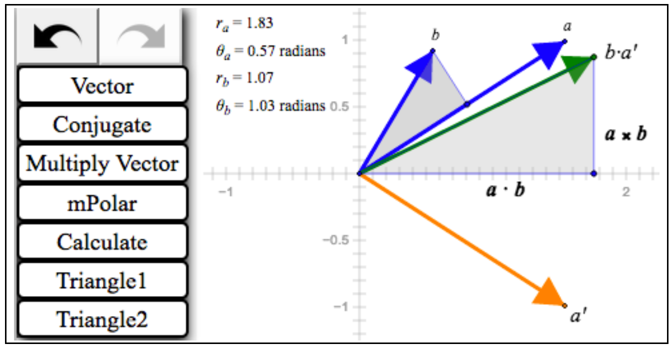


Figure 18: Vector Multiplication

In Figure 18, a student has begun the Vector Multiplication activity by constructing two vectors, *a* and *b*, and projecting *b* onto *a* in the upper triangle. The length of the projection in polar coordinates is $r_b \cos(\theta_b - \theta_a)$. To transform this projection into the dot product on the real axis, she must multiply (dilate) the upper triangle by r_a and rotate it by $-\theta_a$, which is equivalent to complex multiplication by a', the complex conjugate of *a*. To accomplish this task, she multiplies the two vertices of the upper triangle by a' to construct the lower triangle, with hypotenuse $b{\cdot}a'$. As the lower triangle shows, the projection of $b{\cdot}a'$ on the real axis is **a•b**—the dot product—and its projection on the imaginary axis is **a×b**—the cross product. The student can now drag the vectors at will to explore the behaviour of the two vector products she produced.

Enacting Multiplication of Complex Numbers

Though complex numbers can be multiplied algebraically, a geometric method is more elegant and often more useful. In the Complex Multiplication activity, students use the algebraic method to discover the geometric one. They begin with two complex numbers **v** and **w**, both considered as vectors in the complex plane. To multiply them, students represent **w** in Cartesian form ($\mathbf{w} = x_{\mathbf{w}} + iy_{\mathbf{w}}$), write the product $\mathbf{v}{\cdot}\mathbf{w}$ in the form $\mathbf{v}{\cdot}x_{\mathbf{w}} + \mathbf{v}{\cdot}iy_{\mathbf{w}}$, and use transformations of vectors to represent each of the two terms and add them together (Cuoco, 2005, pp. 113–115).

The activity takes place in five parts. The first three parts review some prerequisites: (a) dilation of a vector is equivalent to multiplication by the (real) scale factor, (b) rotation of a vector by 90° is equivalent to multiplication by *i*, and (c) translation of one vector by another is equivalent to adding them. These parts can be omitted if students already have a firm command of the prerequisites.

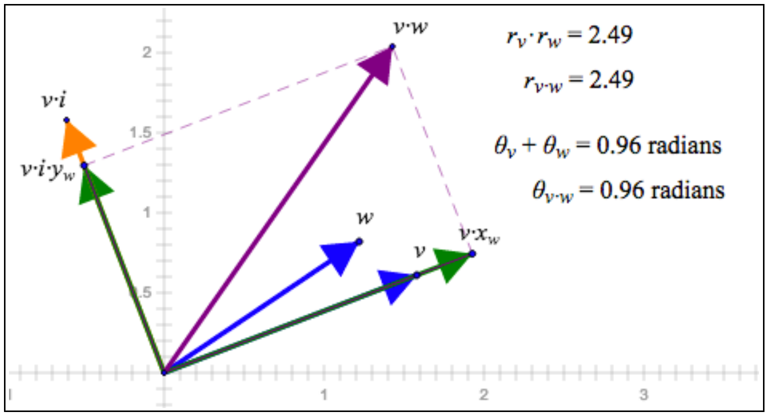


Figure 19: Complex Multiplication

Part four, shown in Figure 19, is the activity's heart. Here a student has rewritten **v·w** as **v**·$x_{\mathbf{w}}$ + **v**·$iy_{\mathbf{w}}$ and used transformations to construct each term of this product. She dilates **v** by the real number $x_{\mathbf{w}}$ to construct **v**·$x_{\mathbf{w}}$, and then rotates **v** by 90° and dilates it by $y_{\mathbf{w}}$ to construct **v**·$iy_{\mathbf{w}}$. The student translates the first result (**v**·x) by the second (**v**·$iy_{\mathbf{w}}$) to add them together, labeling the complex product **v·w**. She measures the polar coordinates of **v, w,** and **v·w,** calculates $r_{\mathbf{v}}{\cdot}r_{\mathbf{w}}$ and $\theta_{\mathbf{v}} + \theta_{\mathbf{w}}$, and makes the remarkable discoveries that $r_{\mathbf{v}\cdot\mathbf{w}} = r_{\mathbf{v}}{\cdot}r_{\mathbf{w}}$ and that $\theta_{\mathbf{v}\cdot\mathbf{w}} = \theta_{\mathbf{v}} + \theta_{\mathbf{w}}$. Expressed in terms of arithmetic operations, to multiply two vectors, you add their angles and multiply their magnitudes. In transformational terms, to find **v·w** you dilate **v** by $r_{\mathbf{w}}$ and rotate by $\theta_{\mathbf{w}}$. As we shall soon see, both formulations are obvious consequences of Euler's formula.

Part Five solidifies and deepens students' understanding as they investigate properties of complex multiplication described in transformational terms by investigating two questions: Is complex multiplication commutative? Do the two transformations *dilation* and *rotation* commute?

This visual approach to complex multiplication encourages students not just to manipulate algebraic symbols but also to visualize the operation geometrically. Importantly, this ability to view complex multiplication as dilation composed with rotation helps provide a window into what is often regarded as the most famous, and most elegant, result in all of mathematics: Euler's Formula.

Enacting Euler's Formula

This activity is based on Euler's extension to complex numbers of his formula for e^x as the limit, as $n \to \infty$, of the quantity $(1 + \frac{x}{n})^n$. The activity begins by having students review the origin of Euler's Formula and then consider how they might use an imaginary value of x by substituting $i\theta$ for x, constructing $(1 + \frac{i\theta}{n})$ on the complex plane, and then repeatedly multiplying this quantity by itself n times (Conway & Guy, 2012).

In Figure 20, a student has constructed angle slider θ, dragged it to an angle of $\frac{\pi}{3}$ radians, and calculated the value of $\frac{\theta}{n}$. (Note that placing the angle slider on the complex plane is a convenience; the value of θ

is real.) The student constructed two vectors to represent 1 on the real axis and $\frac{i\theta}{n}$ on the imaginary axis, added the two vectors, and labelled the vector sum $1+\frac{i\theta}{n}$.

In Figure 21, the student has multiplied four more times by the vector $1+\frac{i\theta}{n}$ in order to construct $(1+\frac{i\theta}{5})^5$. Measuring this point in rectangular form, she finds that its value is $0.57 + 0.96i$. Though this measurement itself does not yet suggest any obvious conjectures, the student may be intrigued to see by how little the vectors increase with each multiplication.

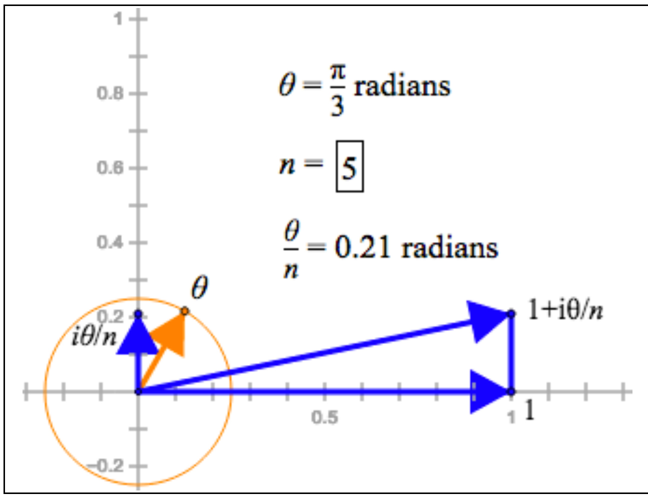


Figure 20: Constructing $(1+\frac{i\theta}{n})$

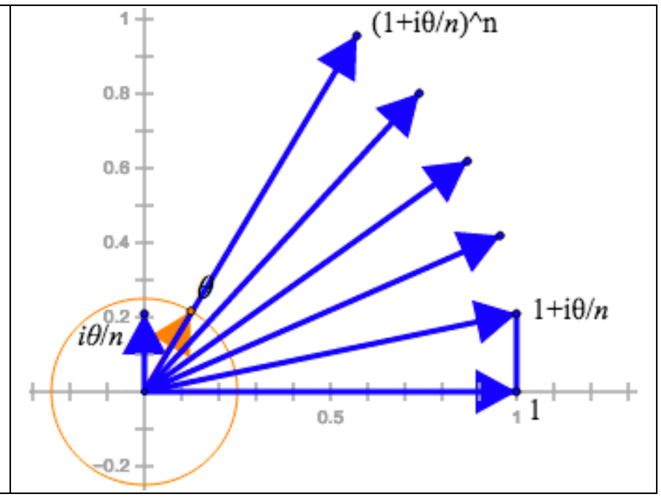


Figure 21: Iterating to construct $(1+\frac{i\theta}{n})^n$

The student changes n to 10, constructing five more multiplications. Finding the terminal vector at $0.53 + 0.91i$, she may begin to suspect that the real part of this value is approaching 0.50. To avoid the labor of continuing to larger and larger values of n, the student goes to the next page of the sketch to use a pre-constructed iteration, allowing her to change n and see the result immediately. She experiments with different values of n to verify that for $n = 100$ and $\theta = \frac{\pi}{3}$, the constructed value of $(1+\frac{i\theta}{100})^{100}$ approximates $\cos\theta + i\sin\theta$ to two decimal places (Figure 22).

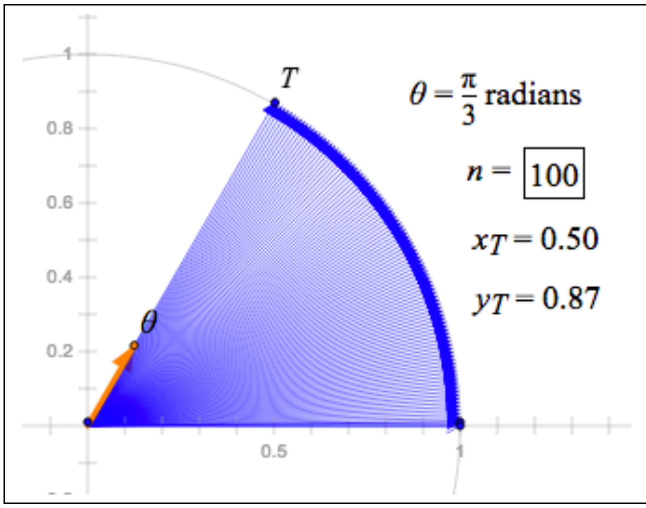


Figure 22: Iterating to construct $(1+\frac{i\theta}{100})^{100}$

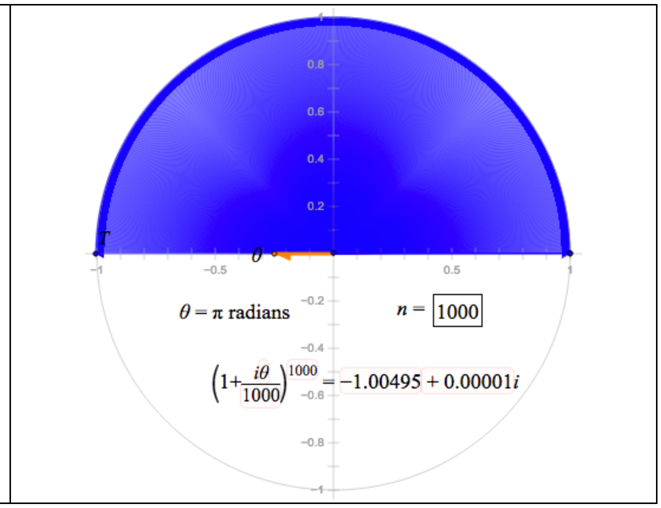


Figure 23: Using n =1000 to find that $e^{i\pi} = -1$

By setting $\theta = \pi$ and using a large value of n in Figure 23, the student concludes that Euler's famous identity $e^{i\pi} = -1$ is true. By changing the θ slider, the realizes that this result for $\theta = \pi$ is only a special case of Euler's formula itself: $e^{i\theta} = \cos\theta + i\sin\theta$.

Thus any complex number expressed in polar coordinates as (r, θ) can be written, and operated upon, as $r{\cdot}e^{i\theta}$. Using this result, the product, $\mathbf{v}{\cdot}\mathbf{w}$ can be expressed as $r_\mathbf{v}e^{i\theta\mathbf{v}} \cdot r_\mathbf{w}e^{i\theta\mathbf{w}}$ and can be easily simplified by applying the laws of exponents: $\mathbf{v}{\cdot}\mathbf{w} = r_\mathbf{v}e^{i\theta\mathbf{v}} \cdot r_\mathbf{w}e^{i\theta\mathbf{w}} = r_\mathbf{v}{\cdot}r_\mathbf{w}{\cdot}e^{i(\theta\mathbf{v}+\theta\mathbf{w})}$. This result confirms both the algebraic multiplication rule to "multiply the moduli and add the arguments" and the transformational multiplication rule to "dilate **v** by by r_w and rotate by $\theta_\mathbf{w}$."

Conclusion

By using web-based dynamic mathematics software and tools tailored to carefully structured tasks, students can enact geometric transformations as functions, creating them, manipulating them, and experimenting with them. Students can perform the mathematics themselves by varying the variables, by describing their relative rate of change, by constructing and using restricted domains, and by composing transformations. In the course of their explorations they can develop a solid understanding of geometric transformations, explore deep connections between geometry and algebra, construct and shed light on the Cartesian graph of a linear function, and make fascinating mathematical discoveries on the complex plane. These results are facilitated by the software's simple interface which, combined with a small number of carefully designed tools, can create a field of promoted action that scaffolds students' work and helps guide them toward meaningful discoveries and understandings.

Pedagogically, the constructive nature of activities such as these has the potential to engage students, to provide opportunities to assess their own work, to encourage meaningful mathematical discussions, and to help students bridge the gap between the concrete, physical world and the profound elegance of abstract mathematical insights.

Early testing suggests that this approach enables students to connect geometry and algebra as they ground function and transformation concepts in sensorimotor experiences, and as they develop their appreciation for the visual beauty of dynamic mathematics. The authors look forward to further refining and extending these activities, and to verifying their effectiveness with a wide variety of students.

[All activities described above are available at https://geometricfunctions.org/icme13/.]

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EXAMINING THE WORK OF TEACHING GEOMETRY AS A SUBJECT-SPECIFIC PHENOMENON

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This paper describes how the notion of instructional situation can serve as a cornerstone for a subject-specific theory of mathematics teaching. The high school geometry course in the U.S. (and some of its instructional situations — constructing a figure, exploring a figure, and doing proofs) is used to identify elements of a subject-specific language of description of the work of teaching. We use these examples to analyze records of a geometry lesson and demonstrate that, if one describes the actions of a teacher using descriptors that are independent of the specific knowledge being transacted, one might miss important elements of the instruction being described. However, if the notion of instructional situations is used to frame how one observes mathematics teaching, then one can not only track how teacher and students transact mathematical meanings but also identify alternative instructional moves that might better support those transactions.

Keywords: Conjecture, construction, contract, description, expectations, doing proofs, exploration, instructional situation, midpoint quadrilateral, norm, tasks of teaching

This paper contributes to the field of mathematics education's theoretical resources for understanding the work of mathematics teaching. Its presence in a volume on secondary school geometry is warranted by our use of examples of secondary school geometry instruction as empirical grounds for our argument that descriptions of the work of teaching mathematics can benefit from subject-specific language if they are going to provide insights into how else that work could be done. As a contribution to a subject-specific theory of mathematics teaching, we show how the notion of *instructional situation* (as instantiated in three instructional situations that have currency in US high school geometry classes -— constructing a figure, exploring a figure, and doing proofs) can serve to construct a first approximation to a subject-specific language of description with which to analyze geometry lessons.

The Teaching of Mathematics as a Subject-Specific Phenomenon

Much research on mathematics thinking and learning pays careful attention to the specifics of the mathematics being learned—using the specificity of schemes or conceptions to describe what students do (e.g., Steffe & Olive, 2010). Yet, when it comes to mathematics teaching, the field of mathematics education is relatively at ease describing the work of teaching without referring to the mathematics at stake—hence the literature sometimes talks of generic (rather than subject-specific)

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tasks of teaching, such as launching tasks, responding to students, orchestrating a discussion (e.g., Stein & Smith, 2011), and sometimes of generic kinds of teaching, such as direct instruction or inquiry-based learning (e.g., Kogan & Laursen, 2014). Clearly, the field can learn from such general ways of describing mathematics teaching and there is abundant literature that provides examples of what can be learned. For example, the video surveys of teaching produced as part of the TIMSS Video Study illustrate that such general ways of coding classroom segments can provide insights about national differences in teaching patterns (Givvin, Hiebert, Jacobs, Hollingsworth, & Gallimore, 2005; Hiebert et al., 2005). But Hill and Grossman (2013) have also recommended the development of ways of describing teaching that attend to the nature of the content being taught; noting that while teachers of different subjects have to "[develop] classroom routines to maximize learning time, [represent] content to a range of learners, [and establish] productive relationships with students [,] how they actually navigate these tasks depends, in large part, on the specific content they are teaching" (p. 374). In this paper, we explore the possibility of describing mathematics teaching in a way that is subject-specific. In this way, the paper can be read as a response to the following question: What might a subject-specific theory of teaching look like and what could descriptions of the work of teaching that draw on it afford mathematics educators? We ground our work in the teaching of secondary school geometry in the United States, in particular, asking what it takes to attend to the specific geometry being taught in this course and how such attention could help us understand the possibilities for improving secondary school geometry instruction.

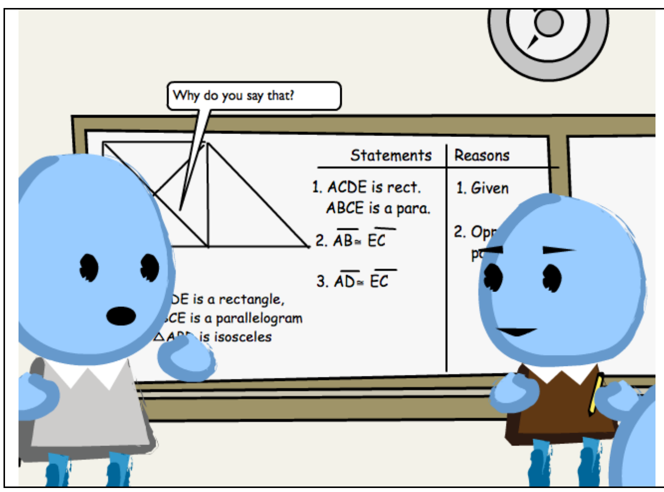

Figure 1a. Pressing for explanation while doing a proof

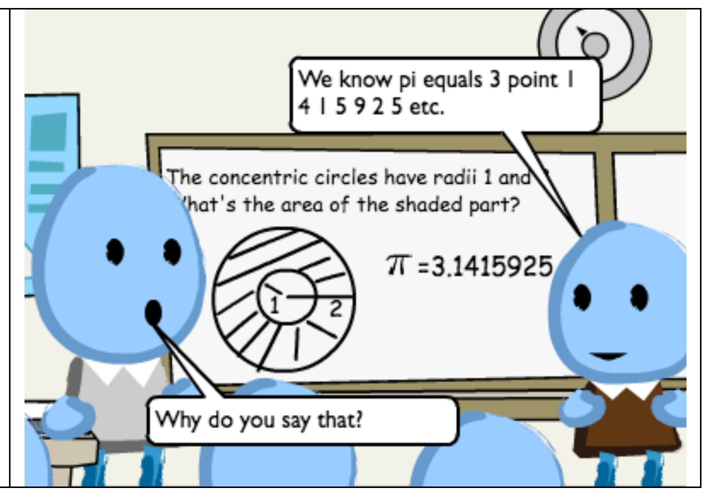

Figure 1b. Pressing for explanation while doing a calculation

By a subject-specific theory of teaching we mean a set of concepts and relationships that include a language of description for classroom instruction and that can help scholars account for how a teacher and their students interact about and work on the specific mathematics at stake. Such a theory should, at the minimum, provide the means to reduce records of actual classroom interaction to accounts that describe the mathematical aspects of the instruction observed; further, such a theory could provide the means to see what actually happened against the background of whatever else could have happened. That is, a theory of mathematics teaching could present the work of teaching as a system of choices that a teacher could make as he or she manages students' mathematical work and learning. If all the work of teaching could be accounted for with generic kinds of teaching (e.g., inquiry-based learning) and generic tasks of teaching (e.g., reviewing homework), that would be tantamount to saying that the work of teaching mathematics is basically the same across mathematics domains, mathematical courses of study, or types of mathematical work, or that mathematics teachers are faced with the same choices for instructional actions regardless of the specific mathematics that they are teaching. We argue that this is not the case: We argue that what appears sensible to do for a teacher depends on mathematical features of the teaching *milieu*[2] (Brousseau, 1997). We elaborate this point below, but Figures 1a and 1b provide a quick initial example. The two images illustrate that the question *why do you say that* might be described generically as a teacher's *press for explanation*; however, the choice to press for an explanation by asking the question "why do you say that?" may be a prompt for different kinds of mathematical work and afford different meaning potential in response to student moves in those different teaching milieux. Those different meaning potentials could be quite consequential for the interaction that ensues, hence entail different cost for the teacher. The request for explanation in Figure 1a addresses a statement the student made in the context of producing a two-column proof (a form of written proofs common in the United States; see Herbst, 2002a), while the request for explanation in Figure 1b addresses a student's statement of an approximation of π when doing a calculation. Our experience in geometry classrooms in the United States suggests that the request for explanation in Figure 1a might be a natural way for the teacher to help a student produce a proof — the question could be interpreted as equivalent to "and what is the reason," which is an expected prompt for what the student would know they have to do. But the request for explanation in Figure 1b might be interpreted as questioning the student's statement of the value of π, which would arguably be a costlier disruption of the work at hand. Our point with this example is that the context in which the teacher *presses for explanation* matters in deciding the meaning (the potential payoff, the potential cost) of the move; and that some aspects of the mathematics at stake are essential to consider when trying to understand which elements of the context are salient to interpreting the meaning of the choice to make such a move, and hence how probable it would be for a teacher to act in that way.

We argue that the work of teaching geometry is subject specific beyond the obvious specificity of the topics a teacher teaches. The examples shown in Figures 1a and 1b suggest that, to the extent that different types of mathematical practices (e.g., making a statement as part of a proof, stating the value of a constant) can be questioned in classroom interaction, the meaning of a given question can differ, depending on the context in which it is asked, even if its wording is the same. This makes sense from an epistemological perspective: To the extent that propositions and concepts are different types of

[2] Brousseau (1997) defines the *milieu* as the system counterpart to the learner in a learning task; the milieu is the recipient of the learner's actions and a source of feedback to the learner. In saying *teaching milieu*, we are using *milieu* analogously and in reference to the teacher's work. The teaching milieu would therefore be the system counterpart to the teacher that contains the teacher's actions and provides feedback to the teacher. Crucially, this teaching milieu contains the students' actions, which, inasmuch as they concern mathematical work, are subject-specific. Margolinas' (1995) studies of the work of the teacher have given a basis for this use of milieu in describing the teacher's role.

mathematical entities, they are amenable to different kinds of justification. But the specificity we allude to goes beyond the topical and the epistemological; it concerns the work of instruction. Our contention is that the meaning potential of the actions of a teacher, when he or she is managing students' engagement with specific mathematical ideas at stake in a given course of studies, is specific to those work contexts in which those ideas are being handled. Our use of the term *meaning potential* is inspired by Halliday's (1978) social semiotics and considers action as semiotic: Actions, inasmuch as they are behaviors in context (including speech and writing, gesture, body position, etc.), are tokens of meaning, and the meaning potential of such behaviors is what those tokens can mean in that context. Our claim that teaching is subject-specific therefore suggests that the meaning of a teacher's action depends on the subject of studies, specifically, as this subject is represented in the students' mathematical work, which the teacher manages through those behaviors. We unpack this statement below and illustrate it with discussion of data from a U.S. secondary school geometry lesson.

An Example: Drawing Diagrams to Enable Student Work

The actions of a teacher could be described with a specificity that addresses how those actions shape the mathematical nature of the work students are expected to do. Geometry teachers often draw diagrams on the board or on worksheets when posing problems for their students. Such work might be described generically as *providing a representation* and perhaps a bit less generically as *drawing a diagram*; but such descriptions are still generic in the sense that neither the drawing action nor the eventual diagram would then be described in relation to the mathematics being transacted. Two things could be meant by the expectation that the description of the action relate to the mathematics being transacted. On the one hand, the object of knowledge to be acquired or assessed could feature in that description: If the diagram was of a rectangle and its diagonals (as in Figures 2a and 2b), one could say *the teacher draws a rectangle and its diagonals*, which is clearly more specific than *the teacher provides a representation*, and relates to the knowledge at stake, for example, if the goal is for students to learn the property that diagonals in a rectangle are congruent. Note that such description benefits from mathematically specific language of the same kind that is used to name the concepts taught in a given course of studies (*rectangle*, *diagonal*). On the other hand, the description could use even more specific language, language that relates to the task at hand, by noting how the characteristics of the drawing achieved might be resources for the task that students will do, hence elements of the milieu. For example, the description could note that the teacher uses different stroke weights that make two overlapping triangles visible in the rectangle and that the teacher labels some points but not others, as shown in Figure 2a below (see Dimmel & Herbst, 2015, for an analysis of semiotic resources available to describe diagrams). Note that a drawing such as Figure 2a features the use of semiotic resources such as line weight and labels whose meaning potential includes stressing that there are two (or three, but unlikely four) triangles of interest, which would be a useful resource if the students were given the task to prove that the diagonals of a rectangle are congruent.

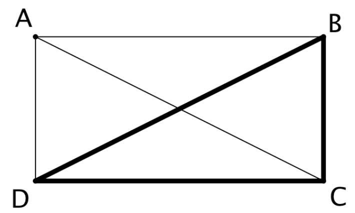

Figure 2a. A diagram of a rectangle and its diagonals, with stroke weights

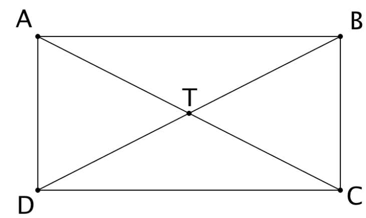

Figure 2b. A diagram of a rectangle and its diagonals, with their point of intersection labeled

The example attempts to support the claim that a description of how the teacher provides the representation should include how the actions of the teacher shape the task that students will do. This could be done by reporting how task resources are made available, as exemplified above: The semiotic resources in the diagram afford a different representation in Figure 2a than in Figure 2b, which is another choice available to the teacher for providing a representation. The same could be said about how the goal of the task is devolved to students: They could be asked to prove that diagonals of a rectangle are congruent or to determine which triangle (*ACD* or *BDC*) has the smaller perimeter, among many other statements; the students could also be given that *ABCD* is a rectangle and asked to prove that $\overline{AC} \cong \overline{BD}$. Additionally, the operations that students have to do, those that they may do, and/or those that they may not do in engaging with the task may or may not be addressed by the teacher, before or during students' engagement with the task (Doyle, 1988). For example, Figure 1a shows how a teacher communicates the need to provide a reason after a statement. Thus, a description of the work of teaching could be subject-specific not only inasmuch as it names the mathematical knowledge at stake but also inasmuch as it helps identify the elements of the mathematical work — that is, the specifics of the task students will do —that provide evidence of the student's understanding of the knowledge at stake. If the knowledge at stake is the proposition that diagonals of a rectangle are congruent, the description of how the teacher engages students in work that installs that proposition as the stake of classroom work may, or may fail to, give us an idea of how students encounter that knowledge. We elaborate on this point below and generalize the notion that a subject-specific theory of teaching would provide the means to describe teaching actions in a way that accounts for their potential impact on the specific mathematical work at hand and/or the knowledge at stake.

Classroom Norms and the Description of Teaching

The notions of *didactical contract* (Brousseau, 1997) and *instructional situation* (Herbst, 2006) are building blocks of a theory that supports the argument that the work of teaching geometry is subject specific, beyond the obvious fact that the object of studies is a domain of mathematics. Brousseau's (1997) notion of didactical contract alludes to a set of relationships among a teacher, their students, and the content being studied that regulate in general and implicitly what it means for the teacher to teach and for the students to study that content: We refer to those implicit regulations as instructional norms. Note that by *norm* we mean an expectation that teachers have of their own work and of the students' work in the context of an instructional exchange, though norms are neither ineluctable nor necessarily explicit. This last point is of particular importance when we think of norms as useful for the observation and description of actual teaching and we come back to it after describing a couple of norms of doing proofs in high school geometry. These norms can vary in their specificity, with some being akin to usual social norms (e.g., that the teacher is expected to respond to students' work; see Wood, Cobb, & Yackel, 1991, p. 599), some more specific to a course of mathematical studies (e.g., what counts as a different solution in a class; see Yackel & Cobb, 1996), and some even more specific to particular types of work that students are asked to do in a given mathematics course (e.g., that students are expected to gather only some information from the diagram when they are doing a proof; Herbst, Chen, González, & Weiss, 2009). Some norms of the didactical contract attest to subject specificity by characterizing the work of doing mathematics in classrooms. For example, in mathematics classes, it is sensible for the teacher to ask a student to justify their responses (e.g., *a rectangle*) to some questions (e.g., *what quadrilateral is formed by the intersection of the angle bisectors of a parallelogram?*), but not so much to justify their responses (e.g., *a diagonal*) to other questions (e.g., *what's the name of the segment connecting two nonconsecutive vertices in a polygon?*). Or, even if asking for a reason was sensible in the second case, the kind of reason that would be sought would be different: While in the first case, the teacher's question might aim at the student's production of a proof that bisectors of consecutive angles of a parallelogram are perpendicular to each other, in the second case, the request to give the reason for a name might pursue extra information on etymology or history (i.e., what *diagonal* means when one analyzes its root in Greek).

The matter is exacerbated if one contrasts a press for justification made by a mathematics teacher and a press for justification made by a teacher of another subject. The epistemology of the subject of studies matters, indeed, but it matters not only in the sense that justification is different across mathematical objects or between mathematics and other subjects. It matters also in terms of the work that students do: What epistemology, in the sense of what relation to knowledge, do the students have the opportunity to construct by way of their interaction with the subject of studies? Furthermore this epistemology concerns the school subject of studies, not only the domain of mathematical knowledge: Norms, such as that teachers rather than students are the ones that choose and assign problems, that tasks are supposed to contain the resources and tools that students will need to complete the tasks and nothing unnecessary, that problems are supposed to take only a few minutes to complete, or that students are supposed to show their work (e.g., see Schoenfeld, 1988), are examples of regulations rather common in mathematics classrooms and that are not issued from the epistemology of the discipline. They also are rather general, applying to a range of mathematical work in a given course of studies, perhaps across mathematical courses of studies. We refer to these as *contractual norms* (Herbst & Chazan, 2012). But we argue that a more specific type of norms, the norms of instructional situations (Herbst, 2006), which we describe in the next section, is particularly useful when describing how teachers shape the mathematical work of students.

Describing How Teachers Organize and Manage Students' Work

Students learn geometric ideas through working on particular tasks.[3] Insofar as the teacher needs to

[3] The word *task* is used as a general concept here, and the emphasis is on a task as a particular chunk of work (task as a proper subset of work). The task might be to do a problem, to discuss a solution to a problem, or to compare solutions to a problem, but the point is that students' engagement is through the particular work called forth by a task (see Brousseau,

manage specific work that mediates students' learning of specific ideas, the actions a teacher takes to enable such mathematical work use elements of a semiotics of professional work that includes language, gesture, physical position and movements, inscription, and material objects (e.g., furniture) and are permeated by similar specificity. This specificity has to do, as we suggest above, not only with the knowledge at stake, but also with the characteristics of the work that students and teachers are expected to do. Doyle (1988) modeled that work by characterizing academic tasks as composed of a goal or *product* that students are expected to seek, *resources* that students have available to use as they work towards that goal, and the *operations* that they do to achieve that goal.[4] This characterization is compatible with Brousseau's (1997) characterization of the learning situation as one in which the learner acts on, and processes reactions from, a *milieu*. But, if describing how teachers organize and manage this work is what is expected, is it sensible to expect that a theory will exist, thus providing some reusable constructs for the description and explanation of mathematics teaching? Or, must we surrender instead to the need for idiosyncratic descriptions of specific tasks? In the rest of the paper, we argue that the construct of *instructional situation* actually provides a way to mediate this paradox of needing a language of description that goes to such specifics as being able to describe tasks, yet is sufficiently general to provide theoretical support for the description of different tasks. In order to enter this terrain, we start with an actual classroom example.

Some years ago, we worked with a high school geometry teacher in designing and using some novel tasks to teach about the properties of special quadrilaterals[5] (see also González & Herbst, 2013). The unit started immediately after the class had studied parallelograms and their properties. At the beginning of the unit, the teacher, Ms. Keating (a pseudonym), defined an M-Quad[6] as the "quadrilateral that is constructed by connecting the midpoints of the consecutive sides of a [given] quadrilateral." She did not provide a diagram with this definition (which is noteworthy, for reasons that become clear below). Ms. Keating then asked the students, "Why would it say *consecutive sides*?" This question elicited a student's consideration of segments between midpoints that "jump around" the sides of the quadrilateral, which Ms. Keating used to note that those figures would not be desirable for the task at hand. She then showed the statement of the task on the overhead projector — "what quadrilateral would you need to start with in order to get an interesting M-Quad?" — again, without drawing a diagram. Shortly after, Ms. Keating restated the task in a way that suggested a synergy between the statement of the task (which is about starting from a quadrilateral and obtaining an interesting M-Quad) and the definition of M-Quad (which is about connecting the midpoints of a given figure): "So, start drawing some quadrilaterals, find the midpoints, connect them."

How should one interpret Ms. Keating's choice to ask her students about the word *consecutive*, in the

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1997, p. 22).

[4] Doyle also included a fourth component, the accountability of a task, or the relative importance of the task when compared to the other work (e.g., other tasks) that the class might do (Doyle, 1988, p.169). We incorporate this notion of the role the task plays in the class's accountability system in our conception of instructional situation and prefer to describe tasks using the three components of goal, resources, and operations.

[5] By special quadrilaterals we mean parallelograms, rhombi, rectangles, squares, etc.

[6] While the instructional goal was to learn about special quadrilaterals, the work assignment was often stated in ways that kept those quadrilaterals hidden. The definition of M-Quad and questions about M-Quad were mere instruments to organize students' work, not what was at stake in the unit (as, obviously, M-Quad is a made-up concept with no status in the curriculum or in the discipline).

definition? Her question could be described generically as asking a comprehension question, or a bit less generically as questioning students' understanding of the definition of M-Quad, but it makes more sense to see it as an attempt to help her students realize that it is they who will be drawing the M-Quads and that the definition should constrain their drawings. Her comments after discussing the meaning of *consecutive (sides)* suggest that her attention to the definition mitigated the possibility that students could just draw any diagram in response to the task. Other elements of the definition (e.g., *midpoint*) could have been questioned as well, but they were not. This is interesting inasmuch as it limited Ms. Keating's prescription of the operations that students could use: Students might have some liberty in terms of how they would find midpoints. To question students about midpoints might have explicitly brought into the discussion control properties such as the equidistance of a midpoint to the endpoints of a segment; these might have further constrained how students undertook the task of drawing.

It appears that Ms. Keating's choice to ask her students about why the definition of M-Quad contained the word *consecutive* had the potential to constrain how the students engaged in the construction task, while her lack of allusion to the meaning of *midpoint* avoided possibly constraining that work too much. The task was scoped to possibly instantiate a situation of *constructing a figure* (Herbst et al., 2010) with some constraints, yet one where not all steps had been proceduralized. We suggest that Ms. Keating's description of the task and definition of M-Quad might have cued students to this situation because the definition included the word *construct*, because the description of the task included the word *draw*, or because she provided students with tools typically used, in high school geometry, to construct figures. All of this may sound idiosyncratic to that task, but it is remarkable for us because we see the work of the teacher assigning a construction task against the background of, or in contrast to, typical construction tasks in U.S. high school geometry classrooms, in which students usually have a specified procedure to produce a figure identified in advance (Herbst et al., 2010). Indeed, the particulars we brought in to make our observations of Ms. Keating's introduction of the M-Quad task were afforded by our knowledge of the instructional situation of *constructing a figure* and its norms (see Herbst et al., 2010).

With this, we illustrate the more general point that existing instructional situations such as constructing a figure (hereafter, the situation of *construction*) can provide language to describe the work of the teacher in organizing and managing students' work on mathematical tasks (be those novel or familiar) and to anticipate what students' opportunities to learn might be. This supports the value of attending to familiar instructional situations in US high school geometry, when studying the instruction of that course (e.g., Ms. Keating's lesson).

Didactical Contract and Instructional Situations

Building on the works of Brousseau (1997) and Bourdieu (1998), Herbst and Chazan (2012) describe the didactical contract for a course, such as high school geometry in the US, as enabling symbolic exchanges of student work for teacher claims on the content at stake (which they refer to as *instructional exchanges*): Students' engagement in a mathematical task allows the teacher to claim that the students have had the opportunity to learn particular mathematical ideas (i.e., accomplish particular instructional goals). These exchanges sometimes require an explicit negotiation of the didactical contract (i.e., negotiations of what students need to do to undertake the task and how doing that attests to their having learned the content; see Herbst, 2003), while in other cases those exchanges are framed under customary instructional situations, whose norms waive the need for such negotiation (Herbst, 2006). Instructional situations are therefore available frames for organizing classroom mathematical work and its exchange for claims over instructional goals; we define instructional situations, operationally, below, after introducing a couple of examples. Herbst et al. (2010) describe various cases of instructional situations in the U.S. high school geometry course, including those of constructing a figure, doing a proof, and exploring a figure.

Instructional situations call for U.S. teachers of high school geometry (hereafter, geometry teachers) to act in particular ways to manage student work, ways in which other mathematics teachers or teachers of other subjects may not need to act. But, do we need to make such observations? Clearly we could consider those actions as cases of the same work being done in two very different manifestations; hence it would be possible to describe the work of teaching in such abstract terms that the differences across the teaching of different mathematical domains might get elided: For example, one could attach the label *posing a problem* both to the actions of a geometry teacher asking her students to construct a figure and to the actions of an algebra teacher asking his students to explore the behavior of a given function. However, the notion that the teaching of mathematics involves specific knowledge that aides teachers in doing their work in specific instructional situations, knowledge that is either available to individual teachers (e.g., mathematical knowledge for teaching; see Ball, Thames, & Phelps, 2008) or recognized by teachers as being required for specific work (e.g., the norms of a situation; see Herbst, Chen, Weiss, & González, 2009), helps us discourage the use of such abstractions to describe the work of teaching (Herbst & Chazan, 2012; Herbst et al., 2010). In the following section, we compare two different examples.

Exploration and Proof Call for Different Work in Drawing Diagrams

Consider two instructional situations in geometry—exploring a figure and doing a proof—and the different demands they pose regarding the teacher's drawing of diagrams. To explore a figure, it is normative for students to be given an artifact (e.g., a diagram, a physical object) and means of proximal contact with it (e.g., measuring tools) and to be asked to state properties of the figure (Herbst et al., 2010). Herbst et al. (2010) explain that the mathematical work done in the situation of exploring a figure may also include the examination of several diagrams for the purpose of conjecturing their common properties and stating them in conceptual language. To facilitate this work the teacher is expected to create one or more representations of the figure for students to use. Inasmuch as students interact proximally with the representations and use those interactions to make assertions that instantiate target properties, we surmise that, in order to enable students' mathematical work, the teacher would have to carefully create accurate geometric diagrams. This might mean drawing the diagram with precise tools and thin strokes, as well as doing as much as possible to have measurements that are whole numbers or that involve simple, common fractions (because, for example, students are more likely to conjecture that the opposite sides of a rectangle are congruent if two sides measure 6 cm and the other two 4.5 cm than if two sides measure 6.05 cm and the other two 5.95 cm.). These actions on the part of the teacher might be interpreted by an observer as extreme attention to detail, but they might also be interpreted as the teacher doing what they need to do to enable students to use their interactions with the diagram to read an instance of the target property of the figure at stake. If the diagram is very accurate, the students will not only be able to abduct the target property (e.g., that opposite sides of a rectangle are congruent) as a possibility but also to confirm empirically their perception when they interact proximally with the diagram, by measuring or folding.[7] We contend that such attention to detail in creating a diagram for an exploration is an example of how the teaching of geometry is subject specific: The mathematical work that students need to do with the diagram makes subject-specific demands on what the teacher needs to do to set up such work. This is clearer when we consider another instructional situation.

In the situation of doing proofs (see Herbst, et al., 2009) the teacher is expected to provide a diagram as well. But this diagram does not need to be very accurate. The diagram needs to be accurate enough to enable students to visualize the statements they want to include as part of the proof, but not so accurate to support verification by measurement, as the students are not expected to measure the diagram. Yet, unlike in the situation of exploration, in the situation of doing proofs the teacher is expected to do more than draw a diagram, the teacher is also expected to label the points of the diagram that will be used in the proof (Boileau, Dimmel, & Herbst, 2016; Herbst, Kosko, & Dimmel, 2013). Labels help keep students' interactions with the figure distal as well as guide attention to relevant geometric objects (Herbst, 2004). This labeling, however, is not necessarily expected when setting up an exploration of a figure, where students can interact proximally with the diagram.

What Can Be Learned from the Examples of these Instructional Situations?

Clearly, one could say that these examples of the work of teaching (in the situations of exploring a figure and of doing proofs) are just examples of the teacher creating the givens of a problem, and, even more generically, that those cases are just examples of the teacher creating the resources that students will need to complete a task. Yet, such generic descriptions would not allow one to distinguish those actions from theoretically-possible, non-normative alternatives, such as drawing a diagram inaccurately yet still asking students to explore it, or asking a student to prove a proposition about a diagram in which points that are not needed are nonetheless labeled. And, if one's language of description did not allow them to notice such things, one could not compare their relative costs and benefits. For example, when exploring a figure with an inaccurately drawn diagram, students might rely on more than empirical reasoning, yet may fail to come up with any conjecture. Likewise, while they might produce a proof that makes reference to all sorts of unnecessary objects, they might also consider the extent to which those statements are needed. That is, the teacher's actions could be described, generically, as creating the givens of a problem, but they could be executed in different ways, in particular, by complying with or breaching the norms of the instructional situations that these norms sustain. These breaches could impact the mathematical work students eventually engage in — in some cases, those breaches could be interesting to track on, as they might improve the quality of students' opportunities to learn (Cirillo & Herbst, 2011)— suggesting why it would be important for the field to adopt a subject-specific language of description, such as the situation-based language that we propose in this chapter. To be clear, if we adopted a generic language of description and described those two events as cases of the teacher creating resources for an assignment, we would need to accommodate within that description (1) the actions of a teacher who does so complying with the norms of the situation and (2) the actions of a teacher who does so by breaching a norm (e.g., provides a diagram for an exploration

[7] Note, however, that our description of the situation of exploration, in which the teacher and students reify concrete artifacts as mathematical objects, does not entail our personal endorsement of such relationship to geometric knowledge. Our descriptive attention to them owes to the fact that such practices exist in intact teaching.

but the diagram is inaccurate). The work of students in response to such variable ways of providing resources for a task would likely offer variability that we would predict is caused by subject-specific differences that a generic language of description would have otherwise ignored.

The observation above suggests that if a language for the description of mathematics teaching will let us understand the mathematical qualities of instruction, it needs to preserve a sense of how the actions of the teacher relate to the mathematical work that the students do. We contend that the actions of the teacher need to be described in subject-specific ways, and that this could be achieved by using categories of subject specificity derived from the norms of the instructional situations that frame the work students are doing. To practitioners, the norms of instructional situations appear as tacit expectations that go without saying when complied with and that are repaired when breached (Herbst, Nachlieli, & Chazan, 2011). For an observer to use those norms in the observation of teaching it is worth noting that instructional situations relate to actual practice not in the sense that their norms provide criteria of objective correctness, but in the sense that norms provide a point of reference, where the word *norm* functions here in the probabilistic sense: The norm is a central tendency around which most of the actual performances cluster. Thus, rather than reduce observation of teaching to rating the work of the teacher in terms of their mathematical correctness in a general, observer-centered way (as is the case with subject-specific rating instruments, such as the MQI protocol; see Learning Mathematics for Teaching Project, 2011), the use of norms of instructional situations for observation requires the observer to subordinate any sense of judgment to the specific expectations practitioners would have of teaching actions in the instructional situation that might most likely frame the work they have organized.

Towards a Subject-Specific Description of Teaching

We contend that the norms of instructional situations provide subject-specific language to describe teaching in ways that can help one understand the qualities of classroom mathematical work. As noted above, we define instructional situations as frames that organize classroom mathematical work—clusters of expectations (norms) of who has to do what and when—that regulate what kind of work the teacher will accept as evidence that a student has acquired a particular item of knowledge. A mathematics teacher has to relate to classroom mathematics in at least two fundamental ways: As knowledge for students to learn and as work students need to do in order to accomplish and demonstrate that learning. Further, the teacher needs to manage many (instructional) exchanges of one or another form of mathematics: In class work, in homework, and in examinations, students propose solutions to a variety of particular mathematical problems that the teacher needs to evaluate insofar as they represent (i.e., stand for, though they are never equal to) the knowledge at stake. In this sense, instructional situations are sets of similar instructional exchanges —exchanges of similar objects of knowledge for similar kinds of work done. The system of norms that regulate instructional exchanges in a given instructional situation can then be considered a specialization of the didactical contract—instructional situations collect exchanges that are regulated by the same situational norms (which are specialized versions of the norms that make up the didactical contract). For example, while the didactical contract may generally authorize the teacher to assign tasks to students, the exchange of specific items of knowledge requires the teacher to issue specific tasks. It is for that reason that the norms of an instructional situation can help an observer frame a particular instructional exchange. In the situation of doing proofs, the contractual norm that it is the teacher who assigns problems to students is specialized in the form of various norms that describe what problems the teacher may assign.

The Situation of Doing Proofs

The high school geometry course, which students in U.S. high schools take in 9th or 10th grade (when they are 14-16 years old), developed historically as a stable place for the notion of mathematical proof and students' engagement in proving (Herbst, 2002a) through the development of an instructional situation that Herbst and Brach (2006) called *doing proofs*: Throughout the 20th century, students in high school geometry have been expected to learn mathematical proof through engagement in proof exercises. Herbst et al. (2009) have characterized the situation of "doing proofs" by spelling out a set of norms that regulate the exchanges between students' work on a proof task and the teacher's claim that they are learning how to do proofs.

As noted above, the didactical contract, in the majority of classrooms, entitles the teacher to assign tasks to students. In the situation of doing proofs, each of those problems is expected to spur students' work that the teacher can exchange for a claim on students' knowledge of how to do proofs—how to logically connect known definitions and theorems to what is known and what is to be verified (a proposition) about a geometric configuration. Yet not every problem does that job. For example, a question such as "what can you say about the angle bisectors of adjacent angles?" (Herbst, 2002b; 2015) would not do, even though a mathematically-educated person would likely see that question as an interesting opportunity for a proof, because one norm of this situation, the *given-prove norm* (Herbst, Aaron, Dimmel, & Erickson, 2013), is for the teacher to state proof problems by parsing the proposition to be proved into 'given' and 'prove' statements. In fact, the teacher is expected to provide students with all of the givens that they will need, and the exact conclusion they will prove. That said, to our earlier point that norms are not ineluctable, note that teachers could breach this given-prove norm by involving students in proposing the givens needed to prove a given conclusion and/or in proposing the conclusion that they will try to prove on the basis of a particular set of givens (Cirillo, this volume; Cirillo & Herbst, 2012; Herbst, 2015). As Herbst, Aaron, et al. (2013) showed through their analysis of teachers' responses to scenarios that depict the assignment of proof problems that deviate from the given-prove norm in these ways, teachers do notice those departures, which suggests that they expect teachers to comply with this norm.

Another norm of doing proofs is what we have called the *diagrammatic-register* norm — that proof problems are stated using a diagrammatic register (i.e., that the statement of the proposition to be proved refers to the characteristics of a provided diagram). Five sub-norms are part of the diagrammatic-register norm: (DRN1) co-exact properties (Manders, 2008) such as collinearity, incidence, and separation are not stated explicitly as givens, but rather given implicitly through a diagram, while exact properties such as parallelism, perpendicularity, and congruence are stated explicitly; (DRN2) the proof problem is accompanied by a diagram; (DRN3) all points to be used in the proof, and no other points, are labeled in the diagram; (DRN4) the given and prove statement are stated in terms of the objects represented in the diagram as opposed to in terms of the geometric concepts that characterize the classes of objects represented; and (DRN5) the diagram accurately represents the figure addressed in the problem. Herbst, Kosko, and Dimmel (2013) showed that teachers recognize those norms when they have to respond to scenarios of teaching (see also Boileau, Dimmel, & Herbst, 2016; Herbst, Dimmel, & Erickson, 2016). Based on observations of geometry classrooms, Herbst et al. (2009) have conjectured several other norms for doing proofs that help characterize doing proofs as an instructional situation. Using multimedia questionnaires (Herbst & Chazan, 2015), we have been able to gather evidence that those conjectured norms are indeed what teachers expect to happen even if they might also conceive the possibility to teach in different ways. It is clear that norms of instructional situations are subject specific in the sense that they are specific to the work that students will do on account of the learning of specific content: If a teacher posed a question (e.g., what can you say about the bisectors of adjacent angles?) rather than state a proposition decomposed into a *given* and a *prove* statements, it is quite possible that students might draw and measure and that some extra maneuvers would be needed for the teacher to get the students to answer the question by formulating and proving a conjecture. But how does this relate to the observation and description of teaching practice?

We went into this discussion of instructional situations and their norms on account of the more general claim that the observation and description of the work of teaching can benefit from being subject-specific. The question that arises is how can instructional situations and norms be used to observe and describe teaching practice. Assuming that the observer has access to a video record of a lesson, can peruse the textbook that the class was using, and collect images of students' work, the observation would proceed at two levels: At a first level of description, the goal of the observer would be to identify one or more instructional situations that could be framing the work that the teacher and students are doing. This can be done first by identifying the items of content at stake by triangulating information from a variety of sources, including the sections in the textbook being referenced, the nouns being used in the teacher's explanations, the teacher's own identification of what the learning goals are, and the observer's recognition of the mathematical concepts conventionally associated with the various symbols and icons used. Simultaneously, the observer could look for self-contained segments of work on problems, either done by students on their own, or by the teacher guiding the students through examples or exercises. Segments that include the work done from the statement of the problem to the sanctioning of an answer can then be associated with one or more instructional situations from a catalogue of available instructional situations. Clearly, classroom work might or might not be an exact instantiation of an instructional situation, but the observer's hypothesis that one instructional situation is framing the work being done, either for the teacher, or for one or more students, can help the observer produce observation questions that elicit a description of the work of teaching. The hypothesis that a known instructional situation can be playing some role in framing a specific exchange authorizes the observer to use the norms of that situation as specific resources for description. Thus, a self-contained segment of work on a problem is a candidate for inspection at a deeper level, with the assistance of hypotheses that a given instructional situation (e.g., doing proofs) is framing the segment. This means, in particular, that the norms of the situation would be used to craft observational questions within the segment of work. The hypothesis that a given situation frames the segment of instruction is provisional and serves to identify norms to be used in asking those observational questions. Confirmation of the hypothesis is less important as a goal than implementing the specific observation grid derived from the norms of a situation as a means; this is what leads to a subject-specific description of instruction and the work of teaching. In other words, an instructional situation provides a language of description that can function like a local theory: The observer's hypothesis that a given situation is framing the instruction being observed warrants using the norms of that situation to look at such instruction and produce descriptions.

Norms of a given situation, such as the given-prove norm and the diagrammatic register norm of the situation of doing proofs, can serve to pose observation questions like the following. Has the teacher indicated that students are expected to do a proof, for example, by drawing a two-column table or writing a proposition, parsed into givens and a prove statement? Has a diagram been provided? How accurate is that diagram in its representation of the givens? Does the statement make reference to exact properties only? Does the diagram have all, some, or none of its points labeled? In what register (conceptual or diagrammatic) are geometric objects described in the statement of the proposition? Note that these questions not only help the observer notice how the problem is initially stated, but they also suggest what the observer could notice when observing the temporal unfolding of the segment of instruction. For example, it is possible that the problem be assigned initially with some of those qualities but not with others and that, during students' work on the problem, the teacher would revise the problem or make special mention of the features of the problem, as that might alter how students work on it. To the extent that practitioners notice (or repair) breaches of norms like these, one can say that, at least for teachers, the grounds for the distinction we have made are not just different examples of the same abstract category, but actual information in Bateson's (1972) sense, "a difference that makes a difference" (p. 315). Other questions, responding to interactive aspects of the work of teaching, would also be posed likewise, originated by other norms of the situation. In the next section, we discuss how this could be done using, as an example, the Midpoint Quadrilateral task introduced earlier as an example.

Return to the Example: The Midpoint Quadrilateral Task

The midpoint quadrilateral task—what quadrilateral would you need to start from to get an interesting M-Quad (midpoint quadrilateral)? —seems to be a novel task, depending only on the definition, given in the classroom a few moments before posing the task, that a midpoint quadrilateral is a quadrilateral that is constructed by connecting the midpoints of the consecutive sides of a quadrilateral. Doyle (1988) had noted that students resist novel tasks. Herbst (2003) later showed how novel tasks may also create tensions for the teacher. At the same time, those scholars and many others have argued for the value of tasks that engage students in doing authentic mathematical work (Stein, Grover, & Henningsen, 1996). As researchers interested in both improving the quality of the mathematical experiences students have in geometry classes and supporting the complexity of the work that teachers need to do, we consider it important to understand both the opportunities the M-Quad task afforded for students and the challenges that it might present for the teacher and her students. The instructional situations of construction, exploration, and doing proofs (introduced above) help us understand those opportunities and challenges, first of all by helping us ask observational questions of the video records of the lesson.

In an earlier section, we discussed the hypothesis that the M-Quad task could be seen from the perspective of a situation of construction, which is warranted by Ms. Keating's definition of M-Quad. Yet, our use of that lens led us to observe how Ms. Keating's discussion of the task highlighted some (e.g., *consecutive*) but not all (viz., not *midpoint*) of the meanings involved, which appeared to help maintain the task as less procedural than usual construction tasks. We observe that groups of students in the class were indeed given construction tools—each group of 4 students was given paper and pencil as well as tools such as a compass, protractor, ruler, and straightedge. Ms. Keating supported the framing of this task as a construction task when she told students to "start drawing some quadrilaterals, find their midpoints, and connect them." That said, certain norms of this situation were also breached. For example, in addition to using the tools provided, students used the edge of their textbooks to draw line segments, which we expect is what led to them to use non-normative methods for constructing parallel and perpendicular lines, congruent segments, and midpoints (i.e., some students were heard guessing where midpoints would be). Indeed, it was faster for them not to use construction procedures, and faster work was encouraged by the task, as it placed a premium on conjecturing which figure would produce an interesting M-Quad, which we expect could have been interpreted by students as a request that they draw several quadrilaterals and compare the M-Quads they led to. As a resource for developing observation questions, the situation of construction suggests that we ask to what extent students' actual constructions were affected by their prior knowledge of straightedge and compass constructions and to what extent their usage of alternative drawing procedures might have blemished the diagrams they drew. The same questions could be asked of the eventual work of the teacher and students sharing their constructions at the board, which we describe below. This is important because the situation of construction is not the only one that is useful as a frame for observing this lesson.

The description of the lesson can also benefit from seeing it from the perspective of a situation of exploration. In fact, Ms. Keating ushered students into exploration and construction at the same time, by asking them to "start drawing some quadrilaterals, find their midpoints, and connect them. Start making some conjectures." As suggested above, it is typical of the situation of exploration that the teacher will ask students to examine several models, then formulate conjectures based on the trends that they observe. She supported them in formulating a conjecture by suggesting that students argue with each other and make statements like, "I started with this and I got this" and "If I start with this, then I always get this." One of the groups came up with two conjectures they stated following deductive rules such as "if 2 sides of the outer quadrilateral are equal, then 2 sides of the M-Quad are equal" (probably referring to two pairs of opposite sides). One of the students wondered if this would be a "great theory."

While it is fair to frame the launch of the task as well as the conjecturing that ensued after students had their quadrilaterals and midpoint quadrilaterals drawn as a situation of exploring a figure, it is equally noteworthy that framing that portion of the lesson in this way allows us to see that several of the norms of the situation of exploration were also breached. For one, Ms. Keating did not provide a diagram, which would be expected of the teacher in the situation of exploring a figure (Herbst et al., 2010). Consequently, the quality of the initial diagrams varied. Therefore, whether students were able to create interesting M-Quads and formulate conjectures depended on the quality of their drawings and/or the tools they used to check whether the midpoint quadrilaterals had some perceived properties. In that sense, the M-Quad task breached a norm of usual situations of exploration—it did not ensure the students' access to diagrams from which the conjectures they were to make could be lifted using empirical means. This was apparent in the interactions students had when looking at the shapes to decide whether they were interesting enough. For example, some groups had individuals who conjectured that the M-Quad is always a parallelogram,[8] but those groups also contained individuals who did not believe the M-Quads were parallelograms because they did not look like parallelograms. In this sense, the task clearly breached expectations of the usual situations of exploration, in which the characteristics of the diagram would be expected to support the students' conjectures, both perceptually and empirically.

The observations above were enabled by what we know about the instructional situations of construction and of exploration, and support understanding the opportunities to learn afforded by the M-Quad task. The task installed some essential uncertainty as to what students could claim was "an interesting M-Quad." While the task provided some means for empirical control of the uncertainty (because construction tools were given), it also discouraged very careful use of tools, as mentioned earlier, because students likely expected that the teacher wanted them to use time efficiently to construct and explore several figures in order to come up with one that produced an interesting M-Quad. If they could activate other means of knowing about the M-Quads (given what they knew about the quadrilaterals with which they started), then that might accelerate their work. Clearly, that was the reason why the task had been designed in that way—to inspect to what extent it would engage students in generative interactions with diagrams that might result in the production of reasoned conjectures (Herbst, 2004). But, was there any reason why students might choose to undertake the task by reasoning their way through from the properties of the quadrilaterals that they started with to the properties of their midpoint quadrilaterals? As they had also been socialized into the situation of doing proofs, one might expect they could use what they knew about doing proofs, even if metaphorically (Herbst & Balacheff, 2009), to help them solve the M-Quad problem.

Therefore, a third way of examining the students' work is to use the instructional situation of doing proofs to look at the M-Quad task. Could the norms of *doing proofs* provide resources for the teacher and students to interact around the task? As was the case with the situations of construction and exploration, several norms of the situation of doing proofs had been breached by the teacher: Ms. Keating did not provide a diagram, nor did she provide given and prove statements. At the time that the task had been stated, no special parallelogram (square, kite, rectangle) had been defined in the class; if students knew them it was because they recalled them from earlier courses. But they did know all the properties that would be put together to define the special parallelograms, so they could use properties to describe both the original quadrilateral and their M-Quads, and to flesh out what they might mean by "interesting." At the same time, by suggesting that students make statements like, "I started with this and I got this" and "If I start with this, then I always get this," Ms. Keating brought the task closer to the realm of proof.

It is noteworthy that, when we framed the situation as one of exploration, these same actions took on different meaning — we interpreted them as a request for students to formulate conjectures, rather than as potential cues that students could engage in the reasoning typical of the situation of doing proofs. We see this as noteworthy as it evidences the type of insights that might be gained by considering that a given instructional exchange could be looked at using different instructional situations as lenses (particularly when the assigned task is novel and the situation cued by the task is therefore less clear).

[8] This is, of course, true, and known in mathematics as Varignon's Theorem (see Coxeter & Greitzer, 1967, p. 51; also http://mathworld.wolfram.com/VarignonsTheorem.html).

The possibility that the teacher's request for if-then statements may have had some students frame the situation as one of doing proofs is supported by the work and discussions that developed when students started to work in their groups. As mentioned above, one of the groups discussed two conjectures that they stated following deductive rules: "If 2 sides of the outer quadrilateral are equal, then two sides of the M-Quad are equal" (probably referring to two pairs of opposite sides). In another group, where some students had conjectured that the M-Quad was always a parallelogram, another student, who had originally objected that in some cases the M-Quad was not a parallelogram, then reasoned her way out of discounting squares and rhombi, saying that those also had properties of parallelograms. Reasoning about the commonalities of figures in terms of properties they had was an affordance that could be traced back to the situation of doing proofs and how definitions are used to support statements about figures.

When the students shared their small group discussions with the class, the need to negotiate what situation they were in became more apparent. For example, when two students went up to the board, they started writing down the group's conjecture in an "if…, then…" format but Ms. Keating intervened: "You don't have to write it all out, I really just want to see your picture." In response to the teacher's comment, one of the students erased the writing, and started drawing a picture as directed, but the other student continued completing the sentence and then drew the picture that went along with the conjecture then written on the board. From our perspective, as the situation unfolded, it distanced itself more and more from one of doing proofs. For instance, points were hardly ever labeled and properties such as parallelism were not explicitly stated. The class ended putting forward the conjecture that the M-Quad is always a parallelogram, though its proof would only be developed several days after, as planned.

Returning to the Problem of Describing the Work of Teaching

Our argument is that a subject-specific account of the work of teaching provides better leverage than generic accounts for understanding how teachers create opportunities to learn and how they manage tensions that appear in that context. The M-Quad lesson could have been described generically: The teacher defined a concept, then introduced to her students a novel problem about that concept, giving them resources to engage with the problem in a hands-on way and organizing them in groups to interact with each other. She also let the students know that the lesson would conclude with a whole class discussion of what each group found, so asked them to write their conclusions on a piece of paper which could be shared. The lesson proceeded as requested by the teacher. Students worked individually and spoke openly with group members when they thought some of their findings were worth sharing in the whole class discussion. The students were not boisterous, yet they were clearly engaged. After about fifteen minutes, the teacher reminded the students to write down what they had observed and how they came to their conclusion. Among the conclusions shared was the statement of a theorem, which summarizes the properties of the concept that had been introduced at the beginning of the lesson. While this generic description is factually true, its lack of attention to subject-specific elements of instruction eludes both the ways in which the given task created conditions for learning and how it created challenges for teaching. This would not be improved if we merely spelled out the concept defined at the beginning (i.e., midpoint quadrilateral) and the theorem conjectured at the end (i.e., Varignon's theorem).

We contend that our subject-specific descriptions of the segment of instruction (framing it as situation of construction, then exploration, then doing proofs), shared in the prior sections, permits us to see how the task could in fact promote learning. It might seem unrealistic to expect that the task as posed would lead to a complete proof of Varignon's theorem. In fact, as mentioned above, the design of the unit was such that the proof would actually be done a few days later. The task had been designed so that it could create three important dispositions that seemed foundational for appreciating the role of proof in coming to know. One of them is the disposition to think of figures in terms of properties, which was supported by the request to get an "interesting" M-Quad. Varignon's theorem, even as an unproven conjecture (which was the case by the end of this lesson) is quite a surprising general result that encourages a bit of skepticism toward organizing quadrilaterals taxonomically. The second one was the disposition to interact with diagrams in a generative way (Herbst, 2004), adding to the diagrams as one goes about reasoning with them, a disposition that would eventually come to fruition a few days later, when a diagonal for the original quadrilateral would be drawn in order to facilitate proving that two opposite sides of an M-Quad are parallel. The third one is the disposition to rectify perception with reasoning, which was encouraged by incorporating the expectation to make interesting conjectures (such as that the M-Quad is always a parallelogram) into an activity whose diagrams purposefully lacked accuracy.

These opportunities to learn were created by making use of existing instructional situations, which brought with them affordances as well as constraints. At each moment when the norms of a situation (of construction, exploration, or doing proof) were breached, there was the possibility that the decision to accept or repair these breaches placed tensions on the teacher, notably around what kind of diagram is needed and who needs to produce it. Observation practices based on attending to the instructional situations that are customary in the U.S. high school geometry class supported our capacity to attend to the events (e.g., the instructional decisions) that might help explain how the creation of that opportunity to learn took place.

Conclusion

The prior sections illustrate the elements of an argument for the claim that the work of teaching geometry is subject-specific and that certain insights into that work can therefore only be afforded by subject-specific language of description. The criteria used to detect differences, whether these are summative measures of achievement and success or analyses of the qualities of the mathematical work, matters in deciding whether these are "difference[s] that make a difference." (Bateson, 1972, p. 315). Additionally, some of the subject-specific differences that the notion of instructional situation permits us to detect are nested in general approaches to teaching (e.g., problem based instruction, direct instruction) that contribute by themselves to making or not making a difference. Having said that, when one views the work of teaching as involving transactions of student work on tasks for claims by the teacher on their mathematical knowledge, some broad tasks of teaching emerge (e.g., creating work assignments, interpreting the students' work) that are intrinsically connected to the subject-specific work that students do. The way in which a specific teacher carries out these tasks of teaching could be idiosyncratic (e.g., he or she might always be careless in the assignments he or she provides), but as mathematics educators, we would not expect to describe the majority of professionals' actions as idiosyncratic. We could, however expect that the qualities of how teachers engage in generic tasks of teaching such as providing a diagram would vary depending on the instructional situations used to frame the work. Furthermore, we would, in general, expect that teachers' recognition of the norms of the instructional situation that frames the work and their knowledge of the mathematics needed to enact such instructional situations would help account for part of the variation in the ways teachers enact these tasks of teaching.

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DIFFERENCES IN SELF-REPORTED INSTRUCTIONAL STRATEGIES USING A DYNAMIC GEOMETRY APPROACH THAT IMPACT STUDENTS' CONJECTURING

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This study inspected the relationships between self-reported implementation of instructional strategies using a dynamic geometry approach and the students' engagement in making, testing, and proving conjectures. Data collected includes a self-reported questionnaire given to all of the project's participating high school geometry teachers, collecting both quantitative and qualitative data. The results of the linear model, with proving conjectures as a response variable, indicate that students spent less time proving or disproving their conjectures when working alone regardless of whether they were in a regular or advanced level geometry class. Time spent making conjectures and testing conjectures were positively and significantly correlated with the frequency of teachers' implementation of class discussions. Furthermore, giving instruction that prompted group work had a significant and positive correlation with students proving conjectures in Regular[2] geometry classes.

Keywords: Dynamic geometry, instructional methods, making conjectures, proofs, testing conjectures

Introduction

Geometry is a high school graduation requirement in the United States. It is important that students possess the ability to reason geometrically and spatially in and outside the classroom. The issue of learning and teaching geometry continues to be a major problem nationally, as U.S. students' geometry achievement level is low, at most 50% of geometry students were able to complete an item that involved proofs (Battista, 2007). To investigate this issue, we conducted a four-year research study, Dynamic Geometry (DG) in Classrooms, funded by a National Science Foundation grant. This project developed a curriculum that uses the Dynamic Geometry software The Geometer's Sketchpad (GSP) to engage students in developing mathematical ideas through experimentation observation and formulation testing and proving of conjectures in the geometry classroom. This project assessed student learning in 64 classrooms randomly assigned to experimental (DG) and control groups (no

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[2] Regular geometry class in this context means not advanced level geometry class

technology). The teachers of both groups were required to complete a DG Teacher Implementation Questionnaire (DGTQ) multiple times throughout the year. This questionnaire asked teachers to report on instructional strategies and the frequency of students' time spent making, testing, and proving conjectures. For this current study, we only analyzed the data from the treatment group due to the large effect size of the Dynamic Geometry curriculum on the Regular class level students' achievement on the standardized state Geometry test. The Regular level DG students scored almost 8% higher on the state standardized test than the Regular control group students. This chapter reports on the following research question: What is the relationship between the teachers' instructional strategies and the time students spend making conjectures, testing conjectures, and proving conjectures?

Literature Review

Dynamic geometry

In this study, the project team randomly assigned teachers into two groups, the Dynamic Geometry (DG) group and the control group. The DG group taught their geometry course using GSP software. Educational software, such as GSP, can assist in developing students' understanding of mathematical concepts and increase their reasoning skills (CBMS, 2001). Students' ability to take advantage of dynamic features such as dragging, measuring, and observing what changes and what stays the same, leads to understanding of "the universality of theorems in a way that goes far beyond typical paper and pencil explorations," (CBMS, 2001, p. 132). After several years of research into the use of technology in the classroom, it has become apparent that beyond solely the technology, teachers are an essential element in overseeing the complexity of the learning situations (Laborde et. al., 2006). Vincent (2005) found that the DG's motivating context and the dynamic visualization fostered conjecturing and intense argumentation; the teacher's intervention was an important feature of the students' augmentations, prompting the students to provide explanations for their statements and check their reasoning's validity. Herbst and Brach (2006) argue that classroom tasks that demand high levels of cognitive activity from the students require teachers to ensure the learner's engagement.

Teacher self-reports of implementation of instructional practices

In this study, teachers were asked to describe the ways they had implemented instructional strategies to address student explorations of geometric concepts, the facilitation of conjecturing, and the approaches to geometric proof. Although teacher self-reports are frequently employed when researching the implementation of instructional strategies, a question often surfaces: How accurate are self-reported data collected through surveys? Cook and Campbell (1979) raise three threats to the validity of self-reports: (a) subjects tend to report what the experimenters expect to see; (b) the reports may reflect the subjects' own abilities, or opinions; (c) the subjects inaccurately recall past behaviors. Some researchers have argued that self-report data is of questionable validity, while others (e.g., Chan, 2009) point to studies of self-reported psychological constructs, which have obtained construct validity. According to Koziol and Burns (1986), teachers' self-reported data are accurate and definitive when the reports are regularly repeated, are retrospective up to six weeks, and concentrated on well-defined instructional practices or activities.. Reddy, Dudek, Fabiano, and Peters (2015) report internal consistency and reliability between measures of teacher self-reports of different general instructional strategies and behavioral management strategies used in the classroom when compared to classroom observations.

Framework

This study uses an adapted version of Van Hiele's Model of Geometry Learning for the foundation of its theoretical framework. Van Hiele's five Geometry learning phases are 1) Inquiry/Information, 2) Directed Orientation, 3) Explication, 4) Free Orientation, and 5) Integration (Crowley, 1987). We modified Van Hiele's framework to align better with classroom instruction using dynamic geometry software and curriculum. Our model has five stages which do not directly correspond to Van Hiele's phases yet maintains the model's essence: Stage 1) Geometry teacher introduces an open-ended problem with proof as an objective and then chooses an instructional strategy that facilitates students' reasoning and problem-solving skills. This stage is similar to Van Hiele's learning Phase One of Inquiry and gathering information for exploration. Stage 2) During this instructional method, the student is prompted to utilize the dynamic geometry technology and investigate the present problem's situation to generate a conjecture. This stage involves both of Van Hiele's phases of directed orientation and explication where students are given an activity of guided questions to explore. Stage 3) Students are prompted to state or make a conjecture. Stage 4) Students are encouraged to test their conjecture. And Stage 5) Students are directed to prove or disprove that conjecture. The last three stages combine the remaining two Van Hiele's learning phases of free orientation and integration since students may need to retrace steps between the three conjecture tasks. As an example of this study's modified Van Hiele's framework,

The researchers observed students progressing through these five stages during a classroom observation where the teacher facilitated an investigation on the sum of the interior angles of polygons. The first stage took place at the beginning of the class, where the teacher introduced the interior angles of a polygon investigation and explained the directions of the activity on the corresponding worksheet. After explaining all the instructions for the activity, the teacher informed the class that they could work in groups of two or three on this activity. The worksheet prompted students by asking them to find the sum of the interior angles of a quadrilateral, then a pentagon, and record their answers in a table. Stage two occurred when students were prompted if they could predict the sum of the interior angles for a hexagon, and then construct a hexagon, find the sum of its interior angles, and verify if their prediction was correct. The third stage prompted students to make a conjecture or devise a formula for an n-sided polygon. Then, the fourth stage prompted students to test their conjecture or formula. The DG software made it quick and easy for students to check to see if their formula was satisfied for as many polygons of size n as they chose. Finally, the fifth stage asked students to prove or disprove their conjecture.

Purpose of study

The study's goal was to compare the teacher's self-reported instructional strategies along with the approximate percentage of class time students spent making, testing, and proving their conjectures. Because this study was only one part of the larger four-year Dynamic Geometry Research Project, the broader context from the overall project may be illuminating. The teacher's choice of instructional strategy was a variable that was not controlled for in the Hierarchical Linear Modeling done for the study. This model showed that students' geometry achievement scores in the classes taught by the DG teachers (the experimental group) were significantly higher than the achievement of students whose teachers were in the control group, with a large effect size for the students in the Regular Geometry classes. Therefore, this study analyzes the differences in the Dynamic Geometry teacher's choice of instructional strategy for the Regular level geometry classrooms versus the honors (PreAP) geometry classrooms. Again, this study focuses on answering the following question: What is the variance in the dynamic geometry teachers' self-reported implementation questionnaire of instructional strategies promoting students making conjectures, testing conjectures, and proving or disproving their conjectures?

Significance of study

The overall research project's study confirmed the hypothesis that the use of DG technology to engage students in constructing mathematical ideas through experimentation, exploration, observation, making/testing conjecturing, and proof results in better geometry learning for urban high school students. This study analyzes only the questionnaires to determine whether there exists a relationship between the teacher's choice of instructional strategy and time that students spent on making conjectures, testing their own conjectures, and proving their conjectures. Many high school students, particularly those in Regular level geometry class, are not accustomed to doing mathematical proofs, as it is a time-consuming process, especially when seeing it and learning it for the first time. The goal is to find which instructional strategies are ideal to use and help promote students' developing and proving their own conjectures.

Methodology

Population and sampling

The study took place in the Southwestern United States and involved a State university in partnership with three school districts from an urban area. The target population was that of practicing geometry teachers; the sample included geometry teachers in those districts and who volunteered to participate in the research project. There were two different levels of geometry courses in this study, Pre-Advanced Placement (PreAP) and Regular level. The PreAP level is an advanced course that primarily consists of 9th-grade students, and the Regular level course mainly consists of 10th-grade students. The research study followed a mixed method, randomized cluster design, with the teacher or the teacher's classroom of students as the unit of randomization. The project team members randomly assigned the 64 high school geometry teachers into two equally sized groups: the experimental treatment group (the DG group) and the control group (commonly referred to as the 'business as usual' or non-DG group).

This chapter focuses on the teachers who were assigned to implement the DG curriculum into their geometry classrooms and to self-report their implementation of this curriculum over a full school year, both fall and spring semesters.

Instrumentation

The DG Teacher Self-Report Implementation Questionnaire (DGTQ) contained six multiple-choice (quantitative) items and ten open-response (qualitative) items. The objective of the DGTQ was to measure the teachers' fidelity to DG approach. This study focused on the DGTQ two of the quantitative questions, the first that asked teachers how often they used the following instructional strategies: class discussions, individual work, group work, teacher demonstrations, student demonstrations, and teacher-student interaction. The researchers also analyzed the DGTQ quantitative questions that prompted teachers to approximate the percentage of class time that students spent making conjectures, testing their conjectures, and proving their conjectures. Jiang (2015), the project's principal investigator, published the results on the reliability and validity of this self-reported implementation of the DG curriculum. He analyzed the data using each time point of the study, 5-6 week intervals, and found that the level of fidelity in teaching with the DG approach, 29% of teachers had a high level of fidelity, 61% of the teachers were in the mid-range, and the remaining 10% of the teachers were categorized in low fidelity range, (Jiang, 2015).

The DGTQ included an instructional method question that asked, "When reflecting on your teaching, how often did you use the following formats during the past 5-6 weeks: class discussion, individual work, small group work, teacher demonstration, student interaction with you (as the teacher), and student demonstration?" The response items were coded using a Likert scale shown in Table 1 below. The research team made the decision to use this coding scheme where zero represented the expected response in a classroom so that negative numbers represent the teachers who are doing less than expected and positive values represent a higher level of implementation than expected. The next item on the questionnaire asked the participating DG teachers, 'What percent of your students did the following (form conjectures, test conjectures, prove or disprove their conjectures) during the past 5-6 weeks?' These responses were coded as follows:

Instructional Strategies Response Scale		Percentage of Class Time that Students did Conjecture Tasks	
Response Choices	Codes	Response Choices	Codes
I have not used this	-2	None	0
Rarely	-1	1 – 25%	12.5%
Every few sessions	0	26 – 50%	37.5%
Most class sessions	1	51 – 75%	62.5%
Nearly all class sessions	2	76 – 100%	87.5%

Table 1*: Coding of Questionnaire's Response Choices*

Results

This study's data collection began with the original 64 questionnaire responses from teachers who participated in the DG project over this two-year period, but preliminary data analysis revealed six teacher's classroom data points as outliers after utilizing the Cook's distance outlier test. Four of these six classrooms were an outlier on the one of the conjecture tasks. The remaining two classrooms were outliers two or more of the instructional methods. Project Year 2 represented the first year of project's data collection and implementation of DG curriculum. Thus Year 3 accounts for the second year of project's data collection. Table 2 below describes the grouping of the remaining teacher data points.

Project Year	Number of PreAP classrooms	Number of Regular classrooms	Total Number of Classrooms
Year 2	10	14	24
Year 3	15	19	34
Total	25	33	58

Table 2*: DG (Treatment) Geometry Teachers separated by class level*

After removing outliers, the researchers explored the potential relationships between the six different instructional methods and the three different conjecture activities by calculating the Pearson r correlation coefficient among the 18 different interactions on aggregate data, followed by class level and then the year of the project. Class discussion was the only instructional strategy with a statistically significant correlation to the conjectures tasks when analyzing all class levels together as a whole. This method of discourse was positively correlated with both making conjectures ($r = 0.36$) and testing conjectures ($r = 0.38$).

There was a significant correlation between the frequency of teacher-student interaction and students' involving in testing conjectures ($r = 0.28$). However, when controlling for the level of geometry class, there was a statistically significant correlation between teachers having students work more individually and less time spent on proving/disproving conjectures ($r = -0.41$). Furthermore, class discussion correlated with making conjectures ($r = 0.27$) and testing conjectures ($r = 0.30$) when controlling for the level of the geometry class. Teacher demonstrations and testing conjectures had a statistically significant correlation of ($r = 0.28$) when controlling for both class level and project year.

The Regular level geometry classes revealed 14 out of 18 positive associations between instructional methods and conjecture activities when controlling for the project year. There was a statistically significant positive correlation between Regular teachers' practice of class discussion and students testing their conjectures ($r = 0.41$). Additionally, there was a positive association between students proving their conjectures with teachers of Regular geometry classes who spent class time allowing students to work in groups ($r = 0.38$) and student demonstrations ($r = 0.37$). However, there was a negative correlation between the frequency with which teachers assigned students to work individually more often and students spending less time on proving or disproving their conjectures (r = -0.42).

	Make Conjectures			Test Conjectures			Prove Conjectures		
	All[a]	Year 2	Year 3	All[a]	Year 2	Year 3	All[a]	Year 2	Year 3
	N=33	N=14	N=19	N=33	N=14	N=19	N=33	N=14	N=19
Class Discussion	0.25	-0.13	0.43	0.41*	-0.01	0.58**	-0.01	-0.01	0.02
Individual Work	0.05	0.01	0.07	0.07	-0.22	0.20	-0.42*	-0.35	-0.46*
Group Work	-0.03	-0.27	0.11	0.24	0.20	0.26	0.38*	0.05	0.58**
Teacher Demo	0.17	0.165	0.18	0.23	-0.13	0.35	0.07	-0.50	0.33
Student Demo	0.23	0.16	0.26	0.21	0.21	0.21	0.37*	0.05	0.52*
1-On-1 w/Teacher	-0.14	-0.39	0.09	0.09	-0.17	0.31	0.17	0.11	0.26

[a] Controlling for project years

**Correlation is significant at the 0.01 level (2-tailed)

* Correlation is significant at the 0.05 level (2-tailed)

Table 3: *Regular Level Geometry Class Correlations of Instructional Methods & Conjecture Task*

Furthermore, when taking the project year into account, more statistically significant correlations are revealed as seen in Table 3 above. Even though the project year was not statistically significant on its own in the aggregated data set, it did have an interaction effect on the Regular level geometry class. In Table 3, the Regular classes in Year 2 reported 10 out of 18 negative correlations between methods and conjecture tasks. Then, in Year 3 of the project, the Regular level classes dramatically increased the percentage of time that students spent on making, testing, and proving/disproving their conjectures which in turn revealed 17 out of 18 positive interactions with three of the correlations being statistically significant. Figures 1 and 2 below show how these negative correlations in Year 2 become positive in Year 3 as teachers gradually became more familiar with the new dynamic geometry curriculum and technology.

	Make Conjectures			Test Conjectures			Prove Conjectures		
	All[a]	Year 2	Year 3	All[a]	Year 2	Year 3	All[a]	Year 2	Year 3
	N=25	N=10	N=15	N=25	N=10	N=15	N=25	N=10	N=15
Class Discussion	0.06	0.49	-0.25	-0.11	0.10	-0.24	-0.22	-0.01	-0.34
	-0.50*	-0.27	-0.66**	-0.52*	-0.22	-0.70**	-0.47*	-0.63	-0.38
Group Work	-0.14	-0.32	0.01	-0.03	0.01	-0.06	-0.03	0.16	-0.17
Teacher Demo	-0.37	0.15	-0.59*	-0.19	0.08	-0.29	-0.63**	-0.26	-0.77**
Student Demo	-0.01	0.19	-0.12	0.12	0.22	0.08	-0.12	0.46	-0.36
1-On-1 W/Teacher	0.08	0.63	-0.61*	0.04	0.46	-0.44	-0.27	0.28	-0.87**

[a] Controlling for project years
**Correlation is significant at the 0.01 level (2-tailed)
* Correlation is significant at the 0.05 level (2-tailed)

Table 4*: PreAP Level Geometry Class Correlations of Instructional Methods & Conjecture Task*

For the PreAP classes, the relationship between each of the six different instructional strategies and three conjecture tasks were predominately negative correlated with one another on 14 of the 18 interactions when controlling for the project year. For example, the method of assigning individual work was consistently negatively correlated with all three tasks: making conjectures (r = -0.50), testing conjectures (r = -0.52), and proving conjectures (r = -0.47). Figures 1 and 2 below illustrate this repeated negative relationship between the time students spent completing the three various conjecture activities and the frequency their classes were assigned to do individual work.

When controlling for project years, the time students spent proving conjectures was strongly and negatively correlated with the frequency with which teachers employed teacher demonstrations (r = -0.63). This relationship is plausible since students cannot gain experience doing proofs themselves if they are only watching the teacher demonstrates proofs. Additionally, PreAP students interacting one-on-one with their teacher in Year 3 and proving conjectures had a statistically significant correlation of (r = -0.87) as shown above in the bottom right of Figure 2. In Year 3, teachers who reported using this instructional method the most, also had students spend less time on proofs. All the data points in the graph of PreAP use of interacting one-one one with teacher are above 0.50 indicating that this was a popular instructional strategy. In general, there was a decrease in the use of teacher demonstrations in both PreAP and Regular geometry classrooms, and an increase in the instructional methods that involved the more student participation. For example, notice in Figure 2 above that the Year 3 data points are further to the left on the teacher demonstrations and interacts one-one with teacher for the PreAP classrooms; But the Year 3 data points are further to the right on the method of individual work which requires more student involvement.

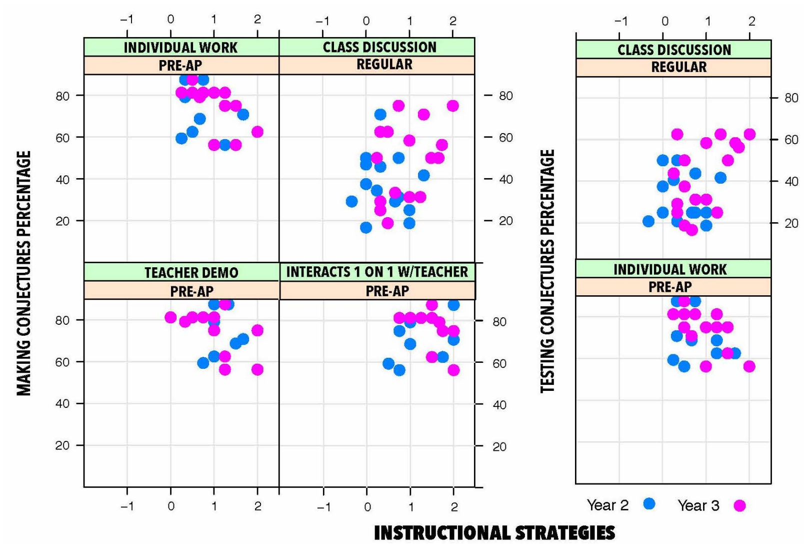

Figure 1: Correlation Scatterplots for Making & Testing Conjectures

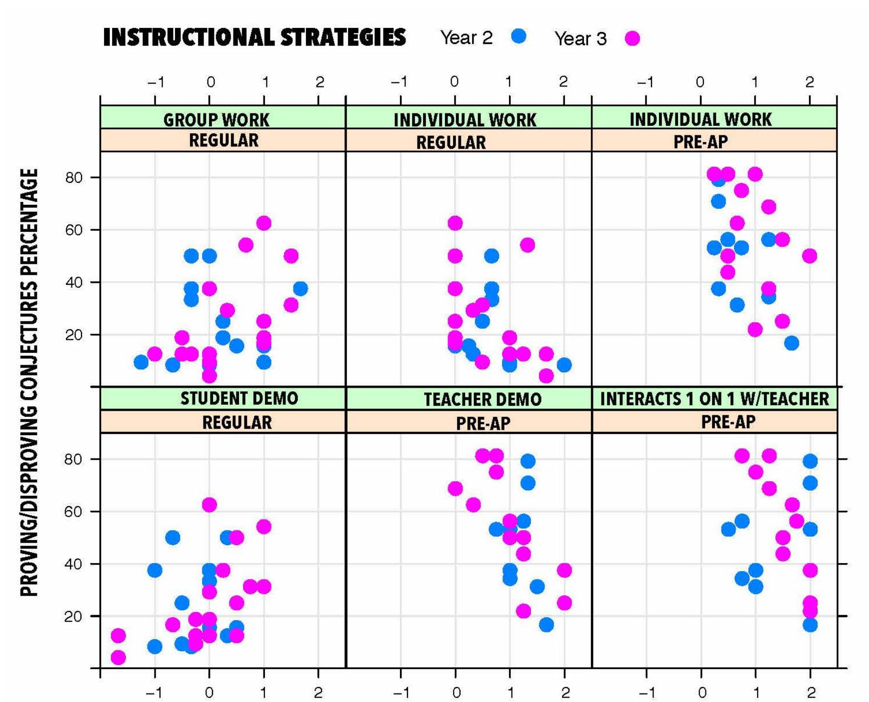

Figure 2: Correlation Scatterplots for Proving/Disproving Conjectures

Next, ANOVA results were examined to explore the difference in means across the PreAP and Regular level classes. There was a significant effect of the independent variable, the class level, on the following dependent variables: individual work [$F(1,56) = 4.20, p = .045$], class discussion [$F\ (1,56) = 4.00, p = .050$], making conjectures [$F(1,56) = 51.48$), $p = .000$], testing conjectures $F\ (1,56) = 81.87, p = .000$], and proving conjectures $F\ (1,56) = 40.15, p = .000$]. There was not a significant effect on the remaining variables: group work [$F\ (1,56) = 0.43, p = .517$], teacher demo [$F\ (1,56) = 2.93, p = .092$], student demo [$F\ (1,56) = 0.83, p = .366$], and 1-on-1 interaction with teacher [$F\ (1,56) = 3.11, p = .083$]. In other words, the PreAP teachers employed the instruction methods of class discussions and assigned individual work significantly more than Regular teachers. The more frequent use of these two methods by PreAP classrooms aligns with the classroom observation data collected by the project's researchers. As hypothesized and observed in the classrooms, the Regular geometry students spent statistically significant less time on making, testing, and proving their conjectures than the PreAP students. This result agrees with the Regular geometry teachers' statements on qualitative portion of the implementation questionnaire where several teachers reported the administration discouraging class time spent on proofs and more time on Algebra topics that would be on upcoming the state standardized end of course exam.

Model 1: Predictors of Making Conjectures

Parameter	B	Std. Error	t	Sig.	95% CI
Constant: β_0	83.76	6.41	12.13	.000	[69.90, 97.62]
Class Level					
Regular β_1	-40.21	6.86	-5.86	.000	[-53.98, -26.45]
PreAP	0	.	.	.	
Project Year					
Year 2	-7.39	4.07	-1.82	.075	[-15.56, 0.78]
Year 3	0	.	.	.	
Instructional Methods					
Class Discussion	5.74	3.08	1.86	.068	[-0.44, 11.92]
Individual Work(IW) β_2	-13.98	5.55	-2.52	.015	[-25.11, -2.85]
Interactions					
Regular * Individual Work (IW) β_3	14.94	7.06	2.12	.039	[0.77, 29.11]
R^2	0.59				

$$y = 83.76 - 40.21\ Regular - 13.98\ IW + 14.94\ (Regular * IW)$$

Table 5*: Regression Model 1 – Predictors of Making Conjectures*

The ANOVA analysis with making conjectures as a dependent variable revealed that geometry class level $[F(1,52) = 34.14, p = .000]$, and the interaction between regular geometry class level with individual work $[F\ (1,52) = 4.47, p = .039]$ were the only significant independent variables. Then, the following independent variables were not significant: year $[F\ (1,52) = 3.29, p = .075]$, class discussion $[F\ (1,52) = 3.48, p = .068]$, and individual work $[F\ (1,52) = 3.30, p = .075]$. This model's results (see Table 5) indicated that these three predictors explained 58.6% of the variance with an adjusted R-squared of 0.546. The researchers then used linear regression to determine which would be the best predictors of students making, testing, and proving their conjectures at the $\alpha = 0.05$ level.

For the linear model with making conjectures as the response variables, statistically significant model intercept coefficient of $\beta_0 = 83.8$ represents the predicted percentage of time that the Regular class level students spend making conjectures. Then, the next coefficient, $\beta_1 = -40.21$, represents the predicted additional time that PreAP students spend on making their own conjectures. There, this model predicts that PreAP Geometry students are predicted to spend 83.8% of class time to making their own geometric conjectures versus the Regular students who spend about 43.6% of their class time on forming conjectures. Furthermore, students in the Regular level Geometry class only spent 44.5% of class time making conjectures when assigned individual work.

The ANOVA results for testing conjectures as a dependent variable revealed that geometry class level [$F(1,52) = 56.18$, $p = .000$], individual work [$F(1,52) = 5.14$, $p = 0.028$], the interaction of level with individual work [$F(1,52) = 6.60$, $p = 0.013$] and the interaction of the project year with class discussion [$F(1,52) = 5.48$, $p = 0.023$] were all significant predictors. Class discussion [$F(1,52) = 3.32$, $p = 0.074$] was not significant. This model's results (see Table 6) indicated that these five predictors explained 70.0% of the variance with an adjusted R-squared of 0.671.

The testing conjectures linear model included the same predictors as making conjectures. However, this model's predictors included more significant coefficients: $\beta_0 = 78.0$, regular level had $\beta_1 = -44.9$, class discussion obtained a $\boldsymbol{\beta_2} = 8.9$, individual work produced a $\boldsymbol{\beta_3} = -15.0$, the interaction of the project year with class discussion revealed a $\beta_4 = -7.9$, and the interaction of individual work with Regular class level was $\boldsymbol{\beta_5} = 15.7$. This model predicts that PreAP students will spend 78.0% of class time on testing their own geometric conjectures, but it decreases to 63.0% if this task is assigned as individual work. Furthermore, the PreAP students utilizing class discussion spent 79% of class time on testing conjectures during Year 2, but it increases to 87.0% during Year 3. Students in Regular classrooms spent about 33.0% on testing conjectures. Additionally, Regular level students spend 33.9% of class time on testing conjectures during Year 2, and this increases to 41.9% in Year 3.

Model 2: Predictors of Testing Conjectures					
Parameter	B	Std. Error	t	Sig.	95% CI
Constant β_0	78.05	5.41	14.42	.000	[67.19, 88.91]
Class Level					
Regular β_1	-44.96	6.00	-7.50	.000	[-57.00, -32.92]
PreAP	0	.	.	.	.
Instructional Methods					
Class Discussion (*CD*) β_2	8.94	2.55	3.50	.001	[3.82, 14.07]
Individual Work (*IW*) β_3	-15.0	4.89	-3.07	.003	[-24.81, -5.20]
Interactions					
Class Discussion*Year 2 (*Y2*) β_4	-7.99	3.42	-2.34	.023	[-14.85, -1.14]
Individual Work*Regular β_5	15.73	6.12	2.57	.013	[3.44, 28.02]
R^2	0.70				

$$y = 78.05 - 44.96\, Regular + 8.94\, CD - 15.0\, IW - 7.99\,(CD * Y2) + 15.73\,(IW * Regular)$$

Table 6: *Regression Model 2 – Predictors of Testing Conjectures*

The ANOVA results for proving or disproving conjectures as a dependent variable revealed that geometry class level [$F(1,52) = 53.62$, $p = .000$], individual work [$F(1,52) = 8.91$, $p = 0.004$], teacher demo [$F(1,52) = 7.23$, $p = 0.01$], and the interaction of level with the teacher's demonstration [$F(1,52) = 11.43$, $p = 0.001$] were the significant independent variables. Project year [$F(1,52) = 0.96$, $p = 0.333$] was not a significant predictor. This model's results (see Table 7) indicated that these predictors explained 61.6% of the variance with an adjusted R-squared of 0.579.

The regression model with the response variable as proving conjectures similarly revealed that the predictor of individual work as an instructional strategy was negatively associated with the percentage of time that students were engaged in proving/disproving their conjectures. This model had a statistically significant intercept coefficient of $\beta_0 = 86.37$, $\beta_1 = -54.66$ (Regular class level), $\beta_2 = -10.92$ (individual work), $\beta_3 = -20.43$ (teacher demo), and $\beta_4 = 22.50$ (teacher demo*Regular). It predicts that PreAP students will spend about 86.8% of class time on the task of proving/disproving conjectures, 75.5% of time on this task when assigned as individual work, and 65.9% of time on this task when teacher demonstration was employed. The Regular classrooms spend about 31.7% of class time on proving/disproving tasks, 20.8% of time on this task when assigned as individual work, and 33.8% on this task when facilitated by a teacher's demonstration.

Model 3: Predictors of Proving or Disproving Conjectures

Parameter	B	Std. Error	t	Sig.	95% CI
Constant β_0	86.37	7.10	12.16	.000	[72.12, 100.62]
Class Level					
Regular β_1	-54.66	7.47	-7.32	.000	[-69.64, -39.68]
PreAP	0	.	.	.	
Project Year					
Year 2	-3.93	4.03	-0.98	0.333	[-12.01, 4.14]
Year 3	0	.	.	.	
Instructional Method					
Individual Work (*IW*) β_2	-10.92	3.66	-2.99	.004	[-18.26, -3.58]
Teacher Demo (*TD*) β_3	-20.43	5.98	-3.42	.001	[-32.43, -8.44]
Interactions					
Teacher Demo*Regular β_4	22.50	6.65	3.38	.001	[9.14, 35.85]
R^2	0.62				

$$y = 86.37 - 54.66Regular - 10.92IW - 20.43TD + 22.5(TD * Regular)$$

Table 7*: Regression Model 3 – Predictors of Proving/Disproving Conjectures*

Discussion

This study's objective was to investigate which instructional strategies are helpful and optimal to further students' developing and proving their own conjectures. The instructional method of individual work was consistently a statistically significant predictor in all three models, as well as a significant predictor when it interacted with class level for both making and testing conjectures. What is particularly interesting about this interaction is when PreAP students are assigned individual work, their time spent making or testing conjectures decrease on average 14.5%. Conversely, when teachers assigned individual work to Regular students who are participating in making or testing conjecture tasks, their time spent on these tasks increases by 0.85%. However, both PreAP and Regular class level students revealed a statistically significant decrease of 11% of class time spent proving/disproving conjectures when assigned individual work.

The Regular Geometry students had a statistically significant increase of 13.6% of time spent making conjectures (p = .028) and 12% increase of time spent testing conjectures (p = .025) between Year 2 and Year 3 of the project. The PreAP teachers' marginal increase of students' time spent on all three conjecture-related activities was not significant. For the explanation of these increases, the project's researchers used the qualitative data collected from the teacher's feedback reports gathered at the monthly professional development sessions over the school year as well as a sample of the teachers' interviews. Obara (2016) found that teachers frequently struggled with learning how to utilize the software and often experienced technical difficulties with the computer labs. Teachers also reported, "[Students] even had a hard time figuring out what the term conjecturing means and how to use the DG tools to come up with conjectures" (Obara, 2016, p. 81).

Both PreAP and Regular teachers reported a marginal increase in time spent proving conjectures, but it was not statistically significant. The PreAP teachers reported an average of 53.7% (SD = 19.34) and the Regular teachers reported an average of 24.5% (SD = 15.61) of students' class time spent on proving conjectures over both years of project's implementation. Again, looking at the project's qualitative data for an explanation on the lack of time dedicated towards proofs, the researchers noted that many reasons mentioned the state's standardized exams (i.e. end of course exam, or E.O.C.). For example, teachers commented that the Regular (lower-level) students already struggle with making connections thus only tested their conjectures since proofs are not on the E.O.C. Additionally, teachers reported being told by their principals that the E.O.C. only tests students on Algebra and not on Geometry. Therefore, there was avowedly no need to cover proofs, and it was avowedly better to use this time to prepare students for the E.O.C. than on proofs. In an interview, one of the teachers commented that her post-secondary institution secondary mathematics methods course did not cover proofs. She also said that she did not have the knowledge or experience to dedicate more time to proofs. Furthermore, this state's high school mathematics certification test to become a teacher does not require proofs.

These teachers' comments from the qualitative data help account for this study's quantitative findings of teachers reporting that the Regular Geometry students spent at least 17% more time on making and testing conjectures than on proving conjectures. The PreAP teachers similarly reported spending at least 19% more on making and testing conjectures than on proofs. Drawbacks of the DG technology also support the significant difference of time spent making and testing, in relation to proving tasks. For example, De Villiers (2006) reported that DG software is largely empirical and best at helping students make and test conjectures but doesn't provide any features, tools, or links to help students prove those conjectures.

This study's findings of a substantial drop in class time between making and testing, on one hand, and proving conjectures, on the other hand, support Herbst and Brach's (2006) statements about how proof tasks require high levels of cognitive activity. Furthermore, they explain how this time-consuming process of developing and proving or disproving a geometric conjecture requires an increased level of critical thinking and problem-solving. For many American high school Geometry students, this is their first encounter with the challenging cognitive proof process. Therefore, if a Geometry teacher assigns these conjecture tasks to their students as individual work, then a majority of students will experience difficulty with making/testing their conjectures and are even less likely to reach the proof stage regardless of their class level.

Although one group of teachers used DG software and the other did not, the one using technology had only been doing so for one year. The project intended for the DG teachers to utilize teaching strategies that incorporated technology, which differ from their prior teaching methods. However, observation and self-report data suggest that both groups were operating under similar didactical contracts defined by Brousseau (1997), ones traditionally embedded in American schools where teachers take significant responsibility for presenting content and where students mostly listen unless specifically prompted to reply or ask questions. The control group's teachers did not differ in as much as our data could infer from this typical didactical contract. Even though the DG teachers presented lessons with the intent to have students take on larger responsibilities for making ideas public and so forth, their students were not always aware of this change and seemed to be operating under didactical contracts that had been operational in classes they had in earlier years. There were frequent comments by teachers in self-reports as well as observations during visits where they directly stated frustration with students. One example were the students waiting for explicit instructions and step by step procedures, implying the students were not yet operating under a different set of expectations. As the year continued, this seemed to change somewhat but not very dramatically. Therefore, we would say there were beginnings of changes in the didactical contract between the two groups of teachers that spread farther apart throughout the year, but many of the students' previous years' expectations of classroom norms were resistant to change. Nevertheless, DG teachers struggled not to fall back into more typical responsibilities for content presentation themselves as a result.

For this current study, we analyzed the data only from the treatment group due to the positive effect on the DG Regular class level students' achievement on the standardized state Geometry test. The researchers wanted to explore what teaching methods were implemented that contributed to the control group students' achievement gains.

Even though the results did not have an instructional practice that positively and significantly predicted students being able to make and test their conjectures, the statistically significant negative predictors revealed which methods were related to lack of success. Future research should focus on generating lesson plans and materials that provide a better link between students making and testing conjectures and proving them.

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CREATING PROFILES OF GEOMETRY TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE

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In this paper the researchers propose a content-specific, short, interactive, online instrument as a way to measure and describe secondary mathematics teachers' pedagogical content knowledge (PCK) related to the area of a trapezoid. The specific components of the PCK are defined with respect to the mathematical content and the process of deriving measures of this construct is described. In this study, 39 inservice teachers were prompted to analyze and report on students' thinking based on interactive samples of students' work provided. Teachers were also asked to propose ways to address students' difficulties and provide suggestions to extend student learning. Their responses were used to develop and modify rubrics for measuring each of the components of PCK and create visual representations of teacher profiles reflecting different levels of teachers' development of PCK. This paper is a result of a mixed methods study where the topic of teaching and learning of geometry at the secondary level is addressed.

Keywords: Area, classroom observations, development of rubrics, in-service teachers, mathematical knowledge for teaching, measures of teacher's knowledge, teacher PCK profile, trapezoid, van Hiele levels, visual representation of teacher's PCK.

Measuring Teachers' Mathematical Knowledge

Among the distinct and often opposing ideas of what content knowledge for teaching mathematics is and how to measure it, it appears that "a unifying theme is the view that teachers' mathematical knowledge is complex, and it has distinctive features that deserve research attention" (Zazkis & Zazkis, 2011, p. 250). Concerns about the adequacy of teacher knowledge of mathematics span across K-12 levels. In their research of elementary preservice teachers' knowledge of mathematics, Goulding, Rowland, and Barber (2002) started by finding out what the teacher brings to the class, including his or her attitudes and beliefs. The authors emphasized that they "cannot subscribe to a commonplace view that good [subject matter knowledge] in mathematics is somehow a barrier to teaching the subject to younger pupils and low achievers" (p. 691). As mathematicians and mathematics educators, we consider that knowing mathematics is a precursor for knowing how to teach it, and embark on a discussion about what constitutes mathematics knowledge for teaching.

There are different conceptual frameworks for describing mathematical knowledge needed for teaching (Carpenter, Fennema, Peterson, & Carey, 1988; Hill, Ball & Schilling, 2008; Kaiser, Blömeke, Busse, Döhrmann, & König, 2014; Manizade & Mason, 2011; Shulman, 1987; Silverman & Thompson, 2008; Tirosh, 2000). Researchers generally agree that pedagogical content knowledge (PCK), as originally introduced by Shulman, connects knowledge of mathematical content and pedagogy but do not agree on its components (Depaepe, Verschaffel, & Kelchtermans, 2013). In this study, we integrated cognitive and situative perspectives on

PCK, considering subject matter knowledge as a prerequisite to PCK. After examining the research literature on mathematical knowledge for teaching and related constructs of teacher knowledge, we created a working definition of PCK and identified four key components of PCK: 1) knowledge of connections among big mathematical ideas; 2) knowledge of learning theories describing students' developmental capabilities; 3) knowledge of students' common challenges and subject-specific difficulties; and 4) knowledge of useful representations and appropriate instructional techniques for teaching the content. This definition changed as a result of data analysis during the study, as discussed in the Data Collection and Analysis section of this paper.

Mathematics education researchers have developed several methods and instruments for measuring mathematical knowledge needed for teaching and related constructs (e.g., Hill et al., 2008; MSU, 2006). A critical review of several PCK instruments is provided in detail in Manizade and Mason (2011). Since it was not possible to develop an instrument to measure different components of PCK for every school mathematics idea, Manizade and Mason (2011) proposed developing short, online, interactive, student response-based instruments that targeted commonly taught content topics.

Positioning PCK Within Teacher Actions

Herbst and Chazan (2011, 2003) hypothesized that teachers' practical rationality shapes their actions in an instructional situation; this practical rationality consists of: 1) portrayal of views of teacher-practitioners about most noticeable people, actions, objects, and instances; 2) notions of what is fair and reasonable and what is unacceptable or unconventional; and 3) values and principles practitioners rely on to rationalize their actions or inactions in professional situations.

In their study, Herbst and Chazan (2003) specifically considered instructional situations which were noted in secondary school geometry classes. They then created animations suitable for teacher professional discussions and attended to what teachers discussed about those animations (Herbst, Nachlieli, & Chazan, 2011). Similar to Herbst and Chazan's work, our research team approached PCK by describing possible student responses in instructional situations that included a geometry task, finding a formula for area of trapezoid, and the elements of the curriculum within which the task was completed. The teachers-participants were then asked to pedagogically react on the student work and elaborate on their actions.

Teachers' Understanding of Geometry

In our deliberations about how to determine levels of teachers' understanding of geometry, this research team utilized the literature related to applications of van Hiele's theory in studies with preservice (i.e., teachers-in-training) and in-service (active teaching professionals) teachers. Van Hiele's theory "suggests that all students progress through a five-level sequence in a particular order and that if one level is not mastered before instruction proceeds to the next level, a student may perform only algorithmically on the higher level" (Mayberry, 1983, p. 58). According to Schoenfeld (1986), the important take-away from this theory is that there exist relatively stable stages in learning geometry and that "empirical grounding is necessary for apprehending and then manipulating abstract geometrical objects" (p. 261). However, these goals are rarely achieved in schools. Teachers as well as students may have inadequate understanding of geometry. Contrary to this scenario, geometry "is a fascinating mathematical microcosm…when it is taught properly, students have the opportunity to do real mathematics in precisely the same way that research mathematicians do" (p. 262).

For example, the study by Gutiérrez, Jaime, and Fortuny (1991) with primary school preservice teachers showed that most participants were at the van Hiele level I (recognition) and van Hiele level II (analysis), but none were at the van Hiele level IV (deduction) or reasoning stage. In Knight's (2006) study, where participants included both elementary and secondary preservice teachers, it was found that elementary school teachers were below van Hiele level III (informal deduction) while secondary school teachers were below van Hiele level IV (deduction).

Mayberry (1983) implemented the van Hiele levels of geometric thought in an instrument designed to study undergraduate preservice teachers and that consisted of a series of tasks ordered to typify geometric thought at the basic and I-IV levels. Her participants were all elementary education majors enrolled in a required science course. Although her results seemingly confirmed van Hiele's theory, Mayberry concluded that further investigation of the hierarchal nature of van Hiele's levels was needed because her study was limited by a small sample size. Erdogan and Durmus (2009) also conducted a study with future elementary school teachers in Turkey and established that the participants' van Hiele levels of geometric thought were low. Also, after their van Hiele-based instructional intervention was proven effective, the authors recommended that preservice teachers should receive instruction based on these levels. Graeber (1999) suggested that preservice teachers' knowledge of students' understanding of mathematics is necessary to make instructional decisions. Pusey (2003) concurred with Graeber's (1999) notion that teachers-in-training need to go through the same kinds of experiences as learners of mathematics to appreciate the benefit of such contexts for their students.

Guided by the notion that practicing teachers and their students usually have similar misconceptions, Swafford, Jones, and Thornton (1997) designed an intervention for middle school (Grades 6-8) in-service teachers. The intervention consisted of a geometry content course based on a problem-solving model and a research seminar, which introduced the van Hiele levels of geometric thought. The authors confirmed that increasing teachers' knowledge about a subject matter and the way students learn it improves the teachers' ability to increase students' mathematical understanding. Regarding the applicability of the van Hiele theory to adult learners, the study suggested that adult learners can progress to higher van Hiele levels rapidly if given proper instruction. However, van Hiele tests have low reliability for adults who have been away from learning geometry for years, and whose performance is sensitive to knowledge recall.

Developing a PCK Instrument

In this study, the researchers developed an instrument to measure and describe geometry teachers' PCK related to the area of a trapezoid. Most of the secondary school teachers were comfortable with this concept, and an assumption was made that they were likely competent to engage in a pedagogical analysis of samples of students' work. In addition, the goal was to develop an instrument that would not discriminate against different teaching styles. This longitudinal study took place over three years during a state-wide, completely online professional development program for secondary mathematics teachers. In the first year, 39 teachers from 12 school divisions across the state volunteered to participate in the study. The study design followed a concurrent mixed-methods approach, in which quantitative and qualitative phases of data collection intermingled to modify the instrument and to develop rubrics as well as profiles of the teachers' PCK. While the work continued with additional cohorts of teachers, this paper presents results based on the data collected from the first 39 teachers; some of whom were later observed in their classrooms. Quantitative results helped to select a subset of participants as representative cases of different levels of content knowledge from 1 to 4, determined by the participants' trapezoid questionnaire results and supported by their results on the van Hiele test (Usiskin, 1982). The observation sample comprised of seven participants from this subset who taught geometry during the school year.

Data Collection and Analysis

Existing standardized measures such as the van Hiele test (Usiskin, 1982) or the Instructional Quality Assessment (Junker et al., 2006), did not focus on the geometry content ideas targeted in this study. However, they were used to gather additional information about teachers' knowledge and backgrounds and to find correlations between the data collected through the newly developed and existing instruments. The instruments that were implemented in this study, their sequencing, and details of data collection methods as well as the materials are shown in Figure 1. Both qualitative (e.g., Trapezoid Questionnaire, Teacher reflections) and quantitative (e.g., van Hiele pre-test, PCK Trapezoid instrument) data for this study were collected in 2014-15. The validity and reliability of these instruments have been established and reported in the literature (Manizade & Mason, 2011; Manizade & Martinovic, 2016; Mayberry, 1983; Usiskin, 1982).

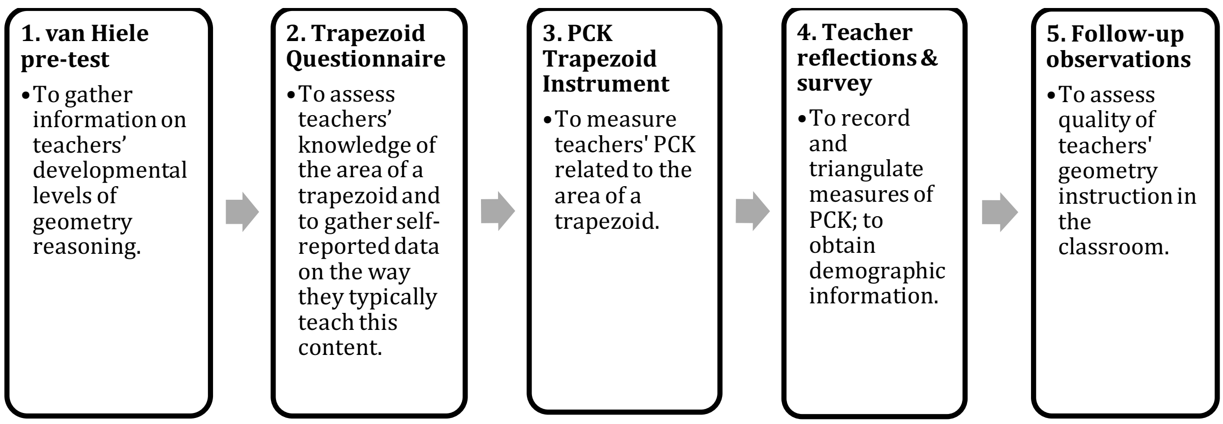


Figure 1. Five main steps in data collection for this study

Teachers' levels of geometric thinking

To summarize the van Hiele pre-test results, our research team followed Usiskin's (University of Chicago, 1982) method of identifying the van Hiele levels. The weighted scores were assigned in the following fashion: 1 point for items 1–5 (Level 1), 2 points for items 6–10 (Level 2), 4 points for items 11–15 (Level 3), 8 points for items 16–20 (Level 4), and 16 points for items 21–25 (Level 5). To calculate the basic score for each level, a strict criterion of 4 out of 5 correct answers (i.e., modified van Hiele levels) was used, given that the participants were practicing secondary school teachers. The results of the van Hiele test are shown in Figure 2. Since the existing van Hiele test measures the teacher's level of geometric development related to a limited number of topics in geometry, as part of the PCK instrument, the researchers included questions to measure the teachers' geometry knowledge of the area of a trapezoid.

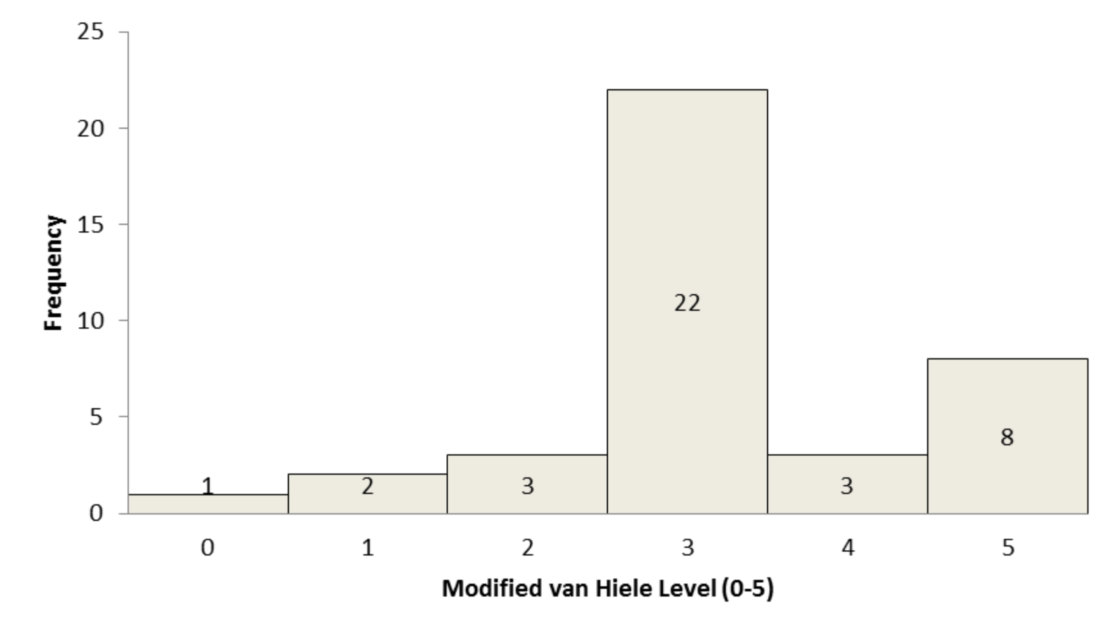


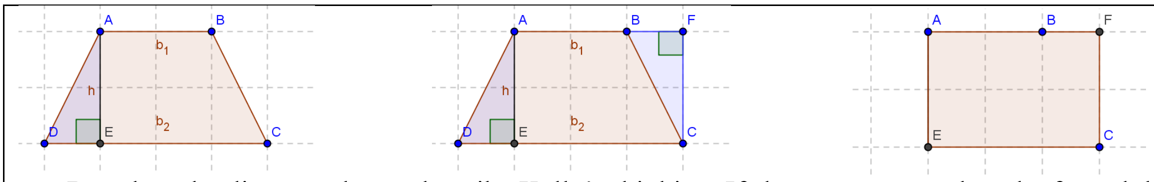
Figure 2. Frequency of teachers' modified van Hiele levels on the scale 0–5 (N = 39)

Teachers' Pedagogical Content Knowledge (PCK) Related to the Area of a Trapezoid

When developing the PCK Trapezoid instrument, the use of multiple choice responses was reduced because of their known deficiencies (e.g., failure to fully capture the complexities of teachers' knowledge and reasoning skills; see Hill, Sleep, Lewis, & Ball, 2007), and participants were encouraged to elaborate and provide detailed reflections of their responses. The original PCK instrument (Manizade & Mason, 2011) was developed using the Delphi methodology (Brown, 1968), and its questions were adapted to accommodate for the mathematical content of this study. The final version of the instrument included six exemplars (one of which is presented in Figure 3). Similar to Herbst and Chazan (2015), who used storyboards and animations of nondescript cartoon characters to explore professional knowledge variables - a cross between a survey and a media enhanced interview, we used an instrument that can be considered a multimedia online questionnaire or virtual manipulative (Manizade & Martinovic, 2016) intended to canvass professional knowledge.

ITEM A: Kelly's Approach

When presented with the task of *developing a formula for the area of any trapezoid* in her high school geometry class, Kelly developed the diagrams as a strategy for deriving the formula for the area of a trapezoid described by the sketches below. She sketched the height $\overline{AE}$ in the trapezoid and constructed a right triangle *AED*. Then she moved this triangle to the opposite side of the trapezoid, constructing a rectangle *AFCE*. Then she calculated the area of rectangle *AFCE*.



a. Based on the diagram above, describe Kelly's thinking. If she were to complete the formal derivation of the area formula in her diagrams, would her method work for any trapezoid? Why, or why not?
b. If Kelly's approach presents a mathematical limitation, what kind of thinking might lead her to the limitation presented in this item?
c. If Kelly's approach presents misconception or misunderstanding, how might she have developed the misconception(s)?
d. What further question(s) might you ask Kelly to understand her thinking?
e. What instructional strategies and/or tasks would you use during the next instructional period to address Kelly's misconception(s) (if any presented)? Why?
f. If applicable, how would you use technology or manipulatives to address Kelly's misconception or misunderstanding?
g. How would you extend this problem to help Kelly further develop her understanding of the area of a trapezoid?

Figure 3. One of the six PCK Trapezoid Instrument exemplars
(adapted from Manizade & Mason, 2011)

Six exemplars with the follow-up questions outline students' strategies for finding the area of a trapezoid. Three of these strategies are generalizable, and three are not generalizable. Figure 3 shows an example of a non-generalizable strategy of "turning" a trapezoid into a rectangle that only applies when the trapezoid is isosceles. In a non-generalizable case, the proposed student's strategy is only applicable for special cases of trapezoids, including but not limited to isosceles or right trapezoids. Generalizable strategies are those that would result in the general formula for the area of trapezoid.

The last item of the PCK Trapezoid instrument consisted of questions designed to gather teachers' ratings (on a 4-point scale, from "1 = not at all" to "4 = very much") of each student's strategies in terms of their mathematical appropriateness, clarity, sophistication, and limitations.

The quantitative data were analyzed using descriptive and inferential statistics to determine characteristic values and differentiate between teachers' levels of geometric development. Other data were coded using an open coding system and analysed for emerging themes related to teachers' PCK, according to the aforementioned theoretical framework.

Based on the teachers' responses to the instrument, the following dimensions of PCK related to the area of trapezoid emerged (see Figure 4): 1) Geometric content knowledge; 2) Knowledge of student challenges and understandings; 3) The ability to ask appropriate diagnostic questions; 4) Pedagogical knowledge of appropriate instructional strategies, and proper use of manipulatives and technology; and 5) Knowledge of geometric extensions designed to deepen students' understanding of the problem.

Development of Rubrics

The Grounded Theory (Charmaz, 2014) approach was used to develop rubrics intended to evaluate and discriminate between the five levels of teachers' PCK in each of the aforementioned five dimensions. The initial versions of the rubrics were created using the literature and the team's professional experiences. The initial coding led us to find new ideas and strategies for further data collection. Next, the qualitative data and the teachers' responses for the instrument described in Table 1 were coded to look for additional emerging themes.

The new themes were identified and included in the corresponding PCK subcomponents of the developed rubrics. These modified rubrics were then checked against the qualitative data collected through the PCK Trapezoid instrument to look for any additional categories and themes. This inspection pulled the researchers into an interactive space where they critically inspected and challenged their preconceived ideas. They conducted coding with gerunds, and grasped directions for exploration and comparison of data. Such methodology asked for an iterative engagement in a cycle of data collection and analysis. The rubrics were then modified three to four times and refined to differentiate between levels of teacher competencies through a reflexive process of linking rubrics to the collected sets of raw data from 39 teachers (related to steps 1–3 in Table 1). Details of the methodological steps for this study are available in Martinovic and Manizade (2017).

Due to the space limitations, only one of the PCK Trapezoid rubrics at levels 4, 3, 2, 1, and 0 is shown (see Table 1), with 4 indicating mastery of knowledge and 0 indicating lack of knowledge. For this dimension of pedagogical content knowledge, 14 sub-components (i.e., A-K) were identified. Based on their presence or absence in data, the teacher's level of knowledge of student challenges and understandings was identified. Details of each of the sub-components are presented in Table 1.

Upon the completion of the analysis of the PCK Trapezoid instrument-related data, teacher profiles were developed. The individual teachers' profiles were presented by diagrams along the five axes (see Figure 4).

Table 1:

Rubric for Evaluating Teacher's Knowledge of Student Challenges and Conceptions.

Level	Characteristics
4	*Teacher is able to identify A and (B or C) and (D or E) and F:* **A.** A student's limited conception of a trapezoid (e.g., isosceles, right), **B.** A student's limited strategy/method (e.g., using only decomposition; composition is basic; strategy that may not always work—decomposing trapezoid into a rectangle and two triangles, transformation may not always work, while enclosing and subtracting excess will always work) **OR** **C.** A special case potentially resulting in a limited or wrong formula. **D.** A student's developmental level in geometry using the van Hiele theory of a trapezoid concept **OR** **E.** A student's developmental level in geometry using the van Hiele theory with respect to area concept (0-not understanding area; 1-basic understanding of adding units; 2-if the shapes match then their areas are equal; 3-if you re-arrange them they will still be the same; 4-using transformational geometry or simple Euclidian proof to claim equal areas). **F.** A student potentially developing these challenges due to the limited experiences

	with different types of trapezoids or tools used or lack of motivation.
3	*Teacher is able to identify A and (B or C) and F:* **A.** A student's conception of a trapezoid as being limited (e.g., to isosceles trapezoid, to right trapezoid). **B.** A student's limited strategy (e.g., using only decomposition; composition is basic; strategy that may not always work—decomposing a trapezoid into a rectangle and two triangles, transformation may not always work, while enclosing and subtracting excess will always work) **OR** **C.** Special case potentially resulting in a limited or wrong formula. **F.** A student potentially developing these challenges due to the limited experiences with different types of trapezoids or tools used or lack of motivation.
2	*Teacher is able to identify A and F:* **A.** A student's conception of a trapezoid as being limited. However teacher does not specify how is it limited, nor proposes any counter-examples in their explanation. **F.** A student potentially developing these challenges due to the limited experiences with different types of trapezoids or tools used or lack of motivation.
1	*Teacher's response covers G and (H or I):* **G.** Teacher recognizes that there is a misconception (if any) in student thinking but does not provide sufficient explanation of the actual misconception or his/her explanation is mathematically incorrect. **H.** The main focus is on the formula, algebra, and counting the area units **OR** **I.** The mathematical terminology is incorrect/poor.
0	*Teacher's response is classified as J or K or L or M or N:* **J.** Did not understand the question **OR** **K.** Did not provide an answer **OR** **L.** Claims that correct approach is wrong (when it is correct) and correct (when it is not) **OR** **M.** The explanation presents a mathematical error **OR** **N.** Does not address geometrical aspect, but focuses only on algebra.

Classroom Observations

To triangulate findings based on the described instruments with information from the real mathematics classroom, observations of each of the seven participating teachers teaching geometry took place twice during the 2015-16 school year following completion of all other data collection. The focus of the observations was on the teachers' instructional quality and the kinds of choices they make in the geometry classroom setting. The seven teachers were observed because at the time when the class observations were scheduled, they were the only teachers who taught geometry. A set of rubrics from the Instructional Quality Assessment (IQA; Junker et al., 2006) instrument served as an indicator of instructional quality focusing on four major aspects to promote students' learning: (1) Accountable talk in the classroom that includes rubrics for the participation rate, teacher's linking ideas, students' linking ideas; (2) Accountability to knowledge and rigorous thinking, including rubrics on asking for knowledge and providing knowledge; (3) Academic rigor of the lesson, including rubrics on the potential of the task (rigor of the text), implementation of the task (active use of knowledge: analyzing and interpreting the text during the whole-group discussion), student discussion following task (active use of knowledge during the small group or individual tasks); and (4) Clear expectations, and the students' self-management of learning, including rubrics on clarity and detail of expectations, academic rigor in the teacher's expectations, access to expectations (Junker et al., 2006). These rubrics used a 4-point scale, with 1 being poor and 4 being excellent. Table 2 presents the summary of observation results for all seven teachers whose geometry classes were each visited twice. The numbers in the table present levels of accountability to knowledge and rigorous thinking, as well as academic rigor of the lesson, according to the IQA rubrics.

Table 2: Sample of the Observation Results for the Seven Teachers

Accountable Talk Dimensions in Classroom Talk	J. Visit 1	J. Visit 2	T. Visit 1	T. Visit 2	C. Visit 1	C. Visit 2	S. Visit 1	S. Visit 2	A. Visit 1	A. Visit 2	P. Visit 1	P. Visit 2	D. Visit 1	D. Visit 2
Accountability to Knowledge and Rigorous Thinking														
Rubric 4: Asking for Knowledge	2	4	4	4	2	2	2	2	4	4	4	4	4	3
Rubric 5: Providing Knowledge	2	4	4	4	2	2	2	2	4	4	4	4	2	2
Academic Rigor of the Lesson														
Rubric 1: Potential of the Task (rigor of the text)	2	4	4	4	3	3	4	4	4	4	3	3	2	2
Rubric 2: Implementation of the Task	2	4	4	4	2	2	3	2	4	4	3	3	2	2
Rubric 3: Student Discussion Following Task	2	4	4	4	3	3	2	4	4	4	3	3	2	2

Note. First letters of participants' pseudonyms are listed.

The following sub-sections focus on three of the observed teachers - John (J, in Table 2), Susan (S, in Table 2), and Anna (A, in Table 2). They were chosen because they exhibit very different cases of the PCK that was targeted.

John. John had four years of experience teaching geometry at the high school level with a high level (4) of geometric knowledge as measured by the PCK instrument. During the first observation, he taught a lesson on the circumference and area of the circle. His second observed lesson was on the midpoint formula. The average scores during the observations were 3 out of 4 for John's accountability to knowledge and rigorous thinking and academic rigor of the lesson. John understood the mathematics that he taught and could solve the problems he presented to the students. During the lesson when teaching the area of the circle, John presented the formula to the students and expected them to memorize it and use it. When the students asked questions, he referred them to the formula sheet. When the students challenged John to explain how the formulas make sense or why the formulas worked, he was unwilling or unable to provide an explanation. On the other hand, John engaged the students in his second lesson by explaining the proof of the midpoint formula even though the focus and derivation were procedural in nature. In the follow-up interviews after the lessons, when asked about his perspective about providing extensions to deepen students' learning or answering questions that emerged in a discussion, John indicated that he did recognize these opportunities for learning but did not have the time to plan for them. He explained, "I wish when we start [with a school year] someone would hand us a curriculum that is well developed and already has all of this built in, instead of me doing it a piece at a time over many years. I would be happy with even a curriculum that is 80% done, and then would adjust to better fit my teaching but it would be still helpful." When asked about using applicable instructional strategies and technologies, John responded, "If someone handed me a package with interactive applications I would use it, but I do not have time to do it myself. I am not paid to do it…" In the case of John, although his geometric knowledge was high, his personal characteristics, which include his attitudes and beliefs, affected the quality of his teaching. These elements were not measured by the PCK instrument directly but could be inferred from the interviews and observations. John's PCK profile in Figure 4 shows that the scores in four out of five categories are between 2 and 3, which is supported by data gathered during the observations and interviews.

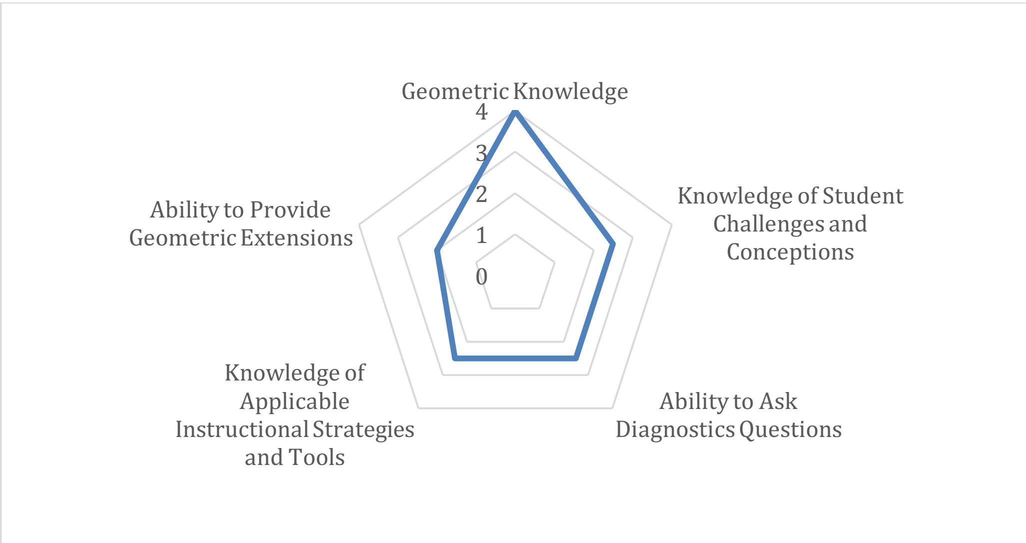


Figure 4. John's PCK profile in five PCK dimensions

Susan. Susan had taught high school geometry for six years. Her level of geometric knowledge as measured by the PCK instrument was 1 out of 4. During the first observation, Susan presented an application problem where students were given three points on a grid and asked to find a location for a fire station which was equidistant to the given points. She liked this problem, which she learned at a recent professional development workshop. Her students generated six mathematically valid approaches for solving this problem, including one approach that was based on non-Euclidian taxicab geometry. Susan was only able to recognize the validity of two of the six approaches. When faced with unfamiliar approaches, Susan acknowledged them by saying, "That sounds nice." She did not make an effort to understand the student's solutions or compare them to the solutions presented by others. Her second observed lesson was on similar solids and their properties, and Susan presented the work as a worksheet where students had to answer a series of questions related to properties of similar solids. The activity was very procedural, and the students were told that they could generalize their findings in the next lesson. Susan had used activities with great mathematical potential; however, she did not recognize the opportunities presented by the students during the whole class discussion. She also posed open-ended questions but was not able to address student answers mathematically. Her average score for academic rigor was three across both observations. Her average score for accountability to knowledge and rigorous thinking was two. Figure 5 shows Susan's PCK profile created using the PCK Trapezoid instrument (see an exemplar from this instrument shown in Figure 3), to compare to the observational data in Table 2.

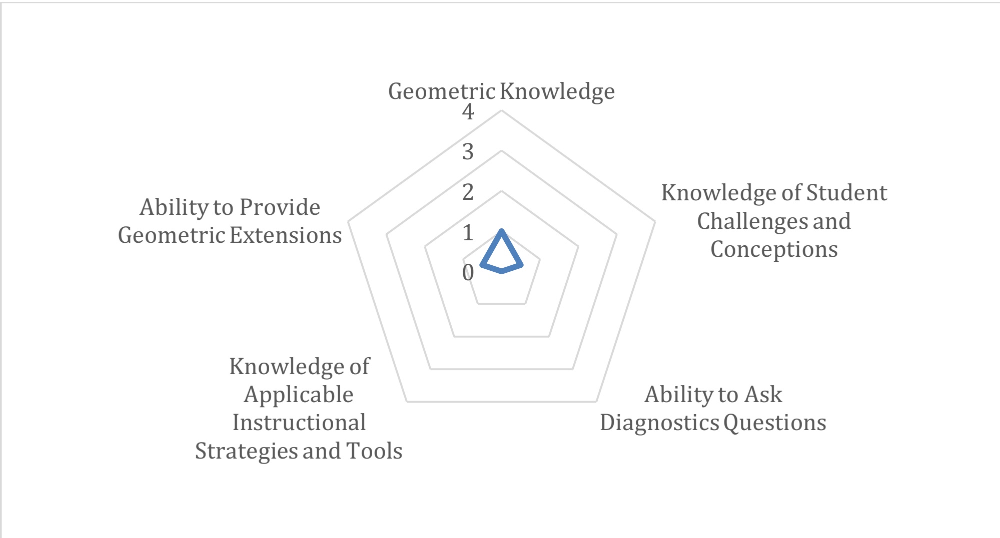


Figure 5. Susan's PCK profile in five PCK dimensions.

Anna. Anna, a novice teacher, was teaching her first year of High School Geometry at a middle school (Grade 8). Her geometric knowledge was rated at level 3, based on the PCK Trapezoid instrument, which can be seen in Figure 6. Anna chose to invite the observer for the class where she taught the area of the trapezoid. During the first lesson, Anna taught the area of the trapezoid lesson. She had previously taught the students the areas of triangles, rectangles, and parallelograms, and Anna expected the students to derive the area of trapezoid based on their previous knowledge on areas of geometric figures. In the lesson, she presented the whole class with one generalizable outline of the proof. Then, Anna asked the students to come up with their own approaches in small groups. The lesson included an in-depth conceptual discussion of the mathematical content where Anna used technology and manipulatives to discuss the proofs presented by the students and challenged them to understand the other students' methods. The second lesson focused on the properties of similar two-dimensional geometric shapes. Anna presented this lesson as a small group activity where the students were asked to create a quilt. Each group needed to select an image of a square for the quilt and scale it to the real quilt's size. Anna's observed scores were 4 in every category for both lessons.

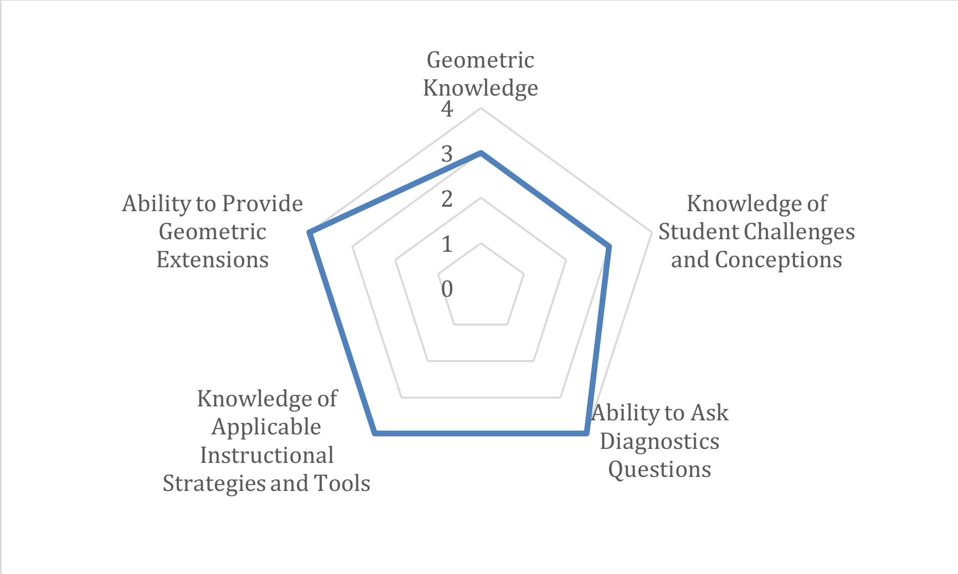


Figure 6. Anna's PCK profile in five PCK dimensions.

Based on the profiles developed in this study, it was found that expertise (measured by the PCK and IQA instruments) did not correlate with length of teaching experience. Some novice teachers performed significantly better in two to three measures when compared to more experienced teachers. It was also noted that if a teacher was lacking geometric knowledge, then he/she was not able to use his/her strengths in other areas in order to synthesize student ideas and summarize the lesson objectives as seen in the example of Susan. This observation confirmed that geometric content knowledge is a prerequisite for the development of other types of teacher knowledge.

Teachers' personal characteristics, including attitudes and beliefs, affected both their PCK profiles and lesson observation results. The PCK instrument was not designed to measure teachers' individual characteristics, which may impact their scores and their teaching quality. An individual teacher's profile with a high level of geometric knowledge and lower levels of knowledge in other areas could indicate a lack of investment of time and effort into pre-active (planning, assessment, and other activities done outside of the classroom in preparation for it) teacher behaviors. For example, John's attitude about his professional responsibilities was a demotivating influence, deterring him from offering appropriate intervention to extend student learning in geometry, using multiple instructional strategies and tools, and asking questions to promote student discussion. Anna's attitude, in contrast, reflected in the time and effort she put towards lesson preparation, addressing her own gaps in subject-matter knowledge, focusing on multiple approaches for solving the problem, and intentionally extending the problem. She also asked diagnostic questions and understood student challenges and conceptions. These differences in attitude might have been reflected in other aspects of their teaching.

Discussion

The purpose of developing teachers' profiles of the PCK was to gain an insight into their strengths and limitations in order to design differentiated professional development experiences that are best suited for a particular teacher or group of teachers. The researchers' intent was not to use these profiles for teacher evaluations.

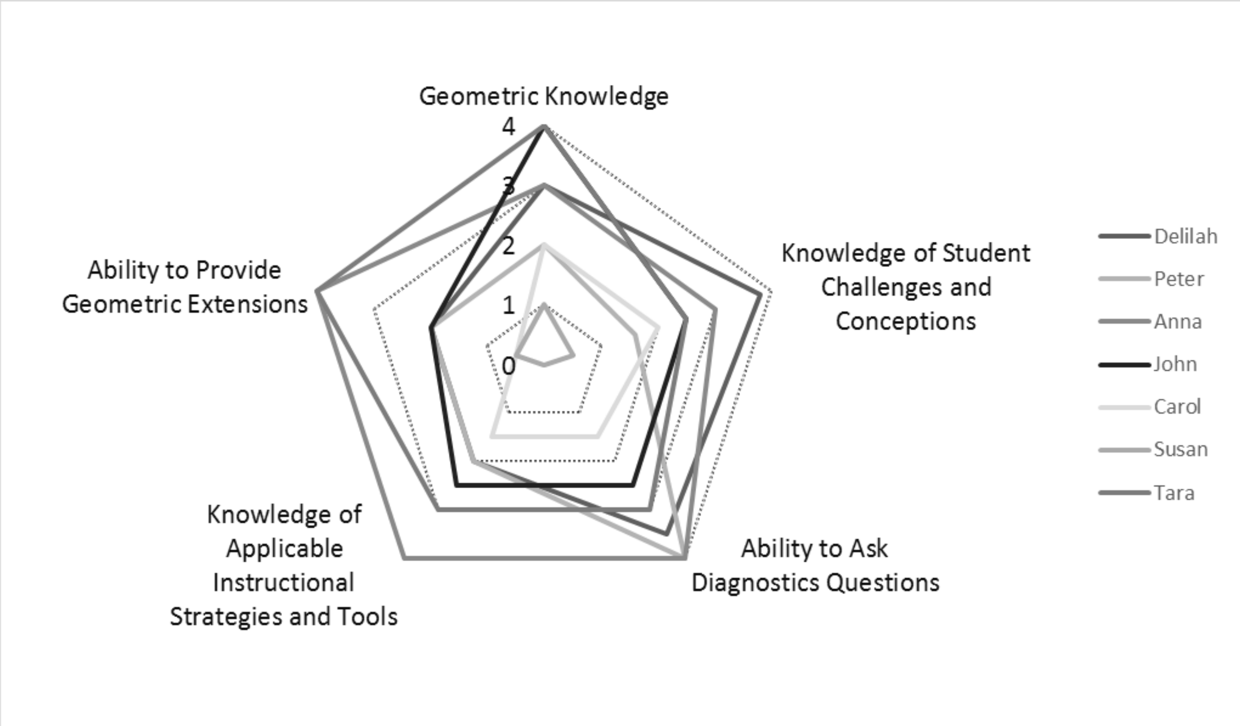


Figure 7. Representation of the seven teachers' PCK mapped on the five dimensions (levels from 0 to 4).

Additional Questions and Limitations

The teachers' profiles of their PCK in geometry raised the following questions for further discussion: 1) What is the importance of years of experience when considering teachers' PCK? 2) In what ways do attitudes and motivations present themselves in teachers' profiles? 3) In what ways, if any, is geometric knowledge a predictor of the other components of PCK? 4) What are the implications of the study when planning and delivering professional development for geometry teachers?

The limitations of the study include: 1) the small sample size affecting generalizability of the quantitative aspects of the study, 2) the sample of teachers chosen for the observations was a convenience sample, 3) the researchers' perspective as social constructivists that might have affected the study design, and 4) known limitations associated with the research method.

Implications

This study presents an approach that can be expanded into other areas of mathematics content. Profiles can serve as predictors of quality of instruction in teachers' classrooms. As a follow-up from this study, the next task would be to create a theory of geometry teacher development based on the rubrics that were created to differentiate between teachers' PCK. The intention is to use additional data related to the area of trapezoid, including the lesson plans, classroom observations, PCK results, van Hiele test results, proofs, interviews, videos, and more teachers, in order to articulate this new framework in future work.

Rather than spending millions of dollars to create long, multiple-choice tests, the research team proposes selecting a small number of carefully chosen commonly taught mathematics domains and developing instruments that will identify a teacher's developmental level in those areas. The

PCK instruments could be used in combination with classroom observations or classroom video analysis (if observations are not possible), along with other types of data such as lesson plans, mathematical proofs/reasoning, etc., to supplement information of teachers' PCK of mathematics. Data presentation and grouping could be done by using methods presented in this paper. Particular teacher's needs could be identified through the profile and the professional development programs could be designed to better address the needs of the individual teachers.

An emergent question is whether the timing of the ongoing long-term PD makes a difference in impacting teachers' PCK. In other words, how is the impact of professional development that takes place immediately after entering the teaching field different from the impact of professional development later in the teaching career? More research is needed to address the aforementioned ideas.

In this paper, a new instrument was presented to measure mathematics teachers' PCK related to the area of a trapezoid. Further, a definition of specific components of PCK, a description of the process of developing evaluation rubrics, and the creation of a visual representation of teachers' PCK using radar diagrams was discussed. The results from this study show the possibility that the development of the instrument, rubrics, and the teacher profiles can be implemented to other topics in Geometry and other branches of mathematics.

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SYMBIOSIS BETWEEN SUBJECT MATTER AND PEDAGOGICAL KNOWLEDGE IN GEOMETRY

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Teacher knowledge that supports effective mathematics teaching has come under scrutiny alongside associated theoretical developments in the education field. Amongst these developments, the Mathematics Knowledge for Teaching (MKT) framework by Ball et al., (2008) has been one of the most influential. While MKT has been useful in helping us identify the knowledge strands teachers need for effective practice, the interplay among MKT's knowledge strands during the course of teaching has received less attention. In this study, we address this issue by exploring interaction between Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK) in the domain of secondary geometry. We provide results of a preliminary study of SMK and PCK in the context of a teacher teaching students how to construct and bisect an acute angle with the aid of compass and ruler only. Our analysis suggests future research needs to consider a) the particular characteristics of the discipline of geometry and b) the developmental knowledge trajectories of teachers of geometry in order to better understand how teachers' SMK influences and influenced by PCK.

Keywords: Conjecturing in geometry, connectedness of knowledge, constructions in geometry, knowledge representation, pedagogical content knowledge, subject matter knowledge, teaching geometry.

Introduction

The instruction's quality students receive in their mathematics classroom is an important, emerging theme in the current debates about enhancing students' learning outcomes. A teacher's knowledge base has a profound effect on the design and delivery of instruction. Thus, it is a productive exercise to develop a nuanced understanding of the knowledge that helps teachers make the content of mathematics more accessible to learners. Mathematics teachers need knowledge that enables them to construct powerful representations to help students visualize concepts, generate explanations that student can relate to and analyze students' responses. This body of knowledge requires a deep understanding not only of content but also of the pedagogy that is built around that content. However, the relationships between, and the changing character of, those two strands of knowledge is a matter of contention among researchers, particularly in relation to actual teaching practice.

Teacher knowledge and teaching mathematics

The knowledge a teacher brings to the teaching-learning context is fundamental to the quality of student learning as it underpins the decisions they make during the course of their teaching (Borko &

Putnam, 1996; Fennema & Franke, 1992; Thwaites, Jared & Rowland, 2011). Mathematics teachers have also identified "teaching for understanding" as an important area of their professional learning (Beswick, 2014). But what knowledge underpins teaching for understanding and student performance?

Research interest in the knowledge that teachers bring to support student learning has gained momentum through recent empirical studies that suggest teachers' mathematics content knowledge contributes significantly to student achievement (Bobis, Higgins, Cavanagh & Roche, 2012). In broad terms, mathematics content knowledge refers to knowledge of concepts, principles, procedures, and conventions of mathematics. Pedagogical content knowledge involves teachers' understanding of students' mathematical thinking (including conceptions and misconceptions) and representing mathematics content knowledge in a learner-friendly manner.

In his seminal work on analyzing teacher knowledge, Shulman (1987) developed the notion of *Pedagogical Content Knowledge*. This pioneering work led Ball and her associates to zero in on the nature of content knowledge and its relationship to teaching mathematics (Ball, Hill & Bass, 2005; Ball, Thames & Phelps, 2008). The outcome of this work was the conceptualization of teachers' knowledge in terms of the influential framework represented in Figure 1.

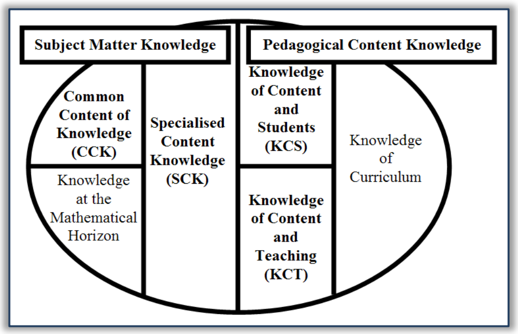

Figure 1: *Mathematics Knowledge for Teaching*
(From Ball et al., 2008, p. 403; © 2008, SAGE publications, used with permission)

Within MKT, there are two main categories of knowledge: Subject-Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). The Subject Matter Knowledge component is further decomposed into Common Content Knowledge (CCK), Specialized Content Knowledge (SCK) and Knowledge at the Mathematics Horizon.

According to Ball et al. (2008), Common Content Knowledge or CCK refers to the body of knowledge that mathematically educated adults are expected to possess. CCK provides individuals with an ability to apply their knowledge to solve mathematical problems. In contrast, Specialized Content Knowledge is considered as "mathematical knowledge *beyond* that expected of any well-educated adult but not yet requiring knowledge of students or knowledge of teaching" (p. 402). Both strands of knowledge are about the content of mathematics, but SCK examines the mathematical demands unique to teaching. SCK is inherently mathematical in nature, is unique to the everyday tasks of teaching, and it demands unique mathematical understanding and reasoning. SCK is topic-specific and includes knowledge about alternative ways to think about a concept, identifying mathematics present in instruction and looking for patterns in students' errors. As CCK and SCK were developed in elementary school contexts, the differentiability between CCK and SCK particularly in secondary mathematics has come under question in recent times. This reason prompts our focus on SMK in the secondary mathematics context.

For a lesson to be effective, however, SMK has to be translated such that learners could develop an understanding of the content of mathematics that underpins that lesson. This translation of SMK while teacher attempt to enact the lesson calls for use of their Pedagogical Content Knowledge (PCK). PCK is concerned with teachers' understanding of how students will learn the content, anticipating students' difficulties with the content (e.g. knowledge of misconceptions) and how to teach that content. Other examples of teachers' Pedagogical Content Knowledge include how to sequence learning experiences, how to present difficult concepts, as well as what tasks to use in teaching. The latter decisions are, in turn, informed by the knowledge of students' strengths and weaknesses. Pedagogical Content Knowledge (PCK) is also further decomposed into Knowledge of Content and Teaching (KCT) and Knowledge of Content and Students (KCS). In our attempts to better understand teacher knowledge needed for supporting the learning of high school mathematics, the framework proposed by Ball and colleagues presents a powerful means to understand the nature of teacher knowledge that anchors students' mathematical thinking and leads to deeper engagement with the content of mathematics.

Ball et al.'s (2008) conceptualization of MKT led researchers to develop tasks to measure the various knowledge components. However, most of this effort has been invested in measuring MKT in the context of primary mathematics. Ball (personal communication, 2015) has suggested the need to analyze the character of MKT in the context of secondary mathematics. We have been working in this area by focusing on the SMK and PCK of prospective secondary and primary mathematics teachers (Butterfield & Chinnappan, 2010; Chinnappan & Forrester, 2014; Chinnappan & White, 2015). SMK and PCK are important strands for two reasons. Firstly, SCK (a component of SMK) has been shown to correlate with high levels of student learning, particularly at the primary levels (Ball & Hill, 2008). Secondly, Hill, Rowan and Ball, (2005) showed that SCK tends to be underdeveloped in most teachers.

We regard MKT as a model for understanding and describing the different strands of teacher knowledge critical to understanding effective practice. While identifying SMK and PCK is significant to extend the field, questions remain about their relationship. Specifically, how does this relationship impact on and play out during the course of teaching mathematics? Knowledge, by its very nature, is interconnected, developmental, and dynamic, but the investigation of this interconnectedness between SMK and PCK, and their growth has not featured prominently in the field. We argue that such an investigation, particularly in the context of in situ teaching, is needed. The results will throw light on and extend current understandings of the relationship between the two key strands of MKT. Indeed, as Ball et al. (2008) suggest, these domains of teacher knowledge are left unexplored and "need refinement and revision" (p. 403).

Moreover, two issues emerge from the work of Ball and colleagues' work: Firstly, while dimensions of MKT have been conceptualized for practice, empirical support for these dimensions have been gathered via test items that refer to tasks involved in teaching. For example, Herbst and colleagues have been actively pursuing CCK, SCK, KCS, and KCT in secondary school geometry (Herbst & Kosko, 2014). Their work has been valuable in generating geometry problems and analyses of teaching scenarios to measure MKT in geometry (see also Smith, this volume). Although these tasks are rooted in and have been informed by the work of teaching geometry, they do not inform us about the changing nature and rationale for the use of these knowledge components *during lesson delivery*. Lesson delivery occurs in a fluid environment and temporality is an important element affecting knowledge use and change. Despite teachers' best efforts at planning, the unfolding events during a lesson are unpredictable. In such a dynamic teaching and learning context, teachers can be expected to adapt their actions and modify their instruction to respond to emerging challenges. The questions are: How do teachers access and exploit their SMK and PCK during lesson delivery, and how does this knowledge contrast with what was measured outside their lesson delivery? Answers to these questions are important to validate MKT, which is conceptualized as a practice-based model of mathematical knowledge used in teaching. Our contentions are that a) there is a relationship between SMK and PCK and b) this relationship should also be examined via events that occur during real-time instruction.

Through a series of investigations, Chinnappan (1998) and Lawson and Chinnappan (2000) showed that, at least within geometry, high school students' conceptual understanding and procedural fluency can be built on a knowledge base that is structured and that teaching ought to find strategies for supporting such structuring of geometric knowledge. This research stream led them to question the nature of teachers' knowledge buttressing students' well connected and usable knowledge. Attempting to answer this question, Chinnappan and Lawson (2005) developed four schemas for categorizing teachers' knowledge about squares. Results of this study showed that even experienced teachers of geometry tend to have limited knowledge about translating geometric content to more learner-friendly representations. Our proposed study results at the end of this analysis is expected to bring insight about why strong content knowledge may remain dormant in the teaching-learning context and how to assist teachers mobilize that knowledge.

Indeed, in highlighting the critical link between content and pedagogical knowledge, Sullivan (2011) directed attention to the importance of ongoing research into experiences that assist teachers in building knowledge of mathematics and how to teach mathematics. Also in recent years, in the area of geometry, Herbst and colleagues have been making inroads into understanding this knowledge. In the next section, we attempt to analyze the SMK-PCK connection in general and apply that analysis to the domain of geometry.

In summary, there is consensus that knowledge of mathematics teachers is an important research area if we are to tackle the question of the quality of teaching. In this regard, the framework of *Mathematical Knowledge for Teaching* has been an important development in identifying two knowledge dimensions: Subject Matter Knowledge and Pedagogical Content Knowledge. However, the relationship among these strands is not clear particularly in the context of in situ teaching of high school geometry.

Relations between SMK and PCK

According to Ball et al. (2008, p. 400), the following routine tasks of teaching mathematics place demands on teachers' SCK:

- recognize what is involved in using a particular representation
- link representations to underlying ideas and to other representations
- select representations for particular purposes
- modify tasks to be either easier or harder
- evaluate the plausibility of students' claims
- give or evaluate mathematical explanations
- choose and develop useable definitions
- use mathematical notation and language and critiquing its use
- ask productive mathematical questions

While there is agreement that SCK undergirds the above tasks, what constitutes SCK in implementing these tasks is less clear (Carreño, Rojas, Montes, & Flores, 2013). Definitions of SCK allude to SCK as content knowledge that is put to use by teachers in performing the above tasks. We suggest that a useful strategy in identifying SCK, and thus SMK, is to capture and analyze the *representations* teachers use to perform the above tasks (Mitchel et al, 2014). We now turn to discussing our interpretation of the role and importance of the *representation* construct.

Representations of the content of mathematics seem central to inform teachers about developing, implementing, and evaluating tasks that teacher use with her students. Tessellations, for instance, is an interesting concept in primary and high school geometry. This concept could be represented as a definition—for instance, it could be defined by saying that a tessellation is a shape which is repeated over and over again covering a plane without any gaps or overlaps (R1). A second representation could utilize tiles on a bathroom floor to demonstrate that shapes such as squares tessellate (R2). Likewise, mosaics from buildings such as churches or mosques could be used to portray tessellation and properties of shapes that tessellate (R3). While R2 assists students in visualizing R1, there are properties unique to shapes in R2 that play a critical role in 'covering' a plane or flat surface without gaps. One such property is that the sum of angles at the corner where the shapes meet in a tessellation is 360^0. For example, squares tessellate because the corner at which four squares meet comprises of four equal angles of 90^0 each. R3 reveals this property. It can be argued that R3 is geometrically more dense and sophisticated than R1 and R2. Thus, we have three representations of tessellations a teacher could utilize in order to a) elicit a question from students, b) explain a definition of the term tessellation, or c) evaluate an explanation provided by students about tessellation. We argue that the above three representations require deep and well-connected content knowledge of tessellation, and that-teachers have to acquire knowledge of tessellation in ways that would allow them to construct the above representations. We regard that knowledge as an example of SMK in this context.

Representations, we contend, can also be used as tools to access PCK and demonstrate interactivity between PCK and SMK. Let us consider two sub-strands that encompass PCK as identified by Ball et al. (2008): Knowledge of Content and Students (KCS) and Knowledge of Content and Teaching (KCT). KCS is expected to assist teachers to anticipate what students are likely to think, predict what students will find interesting and motivating when choosing an example. This strand of knowledge also helps teachers anticipate what a student will find difficult and easy when completing a task, interpret students' emerging and incomplete ideas, and recognize and articulate misconceptions students carry about particular mathematics content. KCT, on the other hand, allows teachers to sequence mathematical content, select examples to take students deeper into mathematical content, and create appropriate representations to illustrate the content.

As an example of a representation where SMK-PCK relations can be observed, we return to the three representations of tessellation provided above. During the course of teaching about tessellations, a teacher could have used combinations of R1, R2, and R3. Why would a teacher use R1 only, or use R2 followed by R1? A teacher using R1 only may have knowledge of his or her students that suggests they can grasp abstract ideas easily (KCS). In contrast, a teacher adopting R2 followed by R1 might do so based on the understanding that contextualizing an abstract concept before defining it is a better sequence for supporting the learning of his or her students (KCT). Thus, while representations provide windows into SMK, the actions and reasons for using a particular representation are sources of data about KCS and KCT.

SMK of geometry

In discussions about SMK in geometry, we are concerned with knowledge of geometry used in tasks of teaching geometry. This knowledge base includes basic geometric concepts, explanations about the key attributes of these concepts and connections between them. Further, different ways of representing these concepts and how they may be contextualized in human activities, and applications of geometric concepts in the solution of routine and non-routine problems are also part of teachers' repertoire of SMK. One example of SMK is the concept of symmetry in 2-D objects. In teaching this concept, teachers could invoke a range of knowledge fragments including an informal definition of symmetry, a formal definition of symmetry, conditions needing satisfaction in order for an object to be judged as symmetric, symmetry of a number of 2-D shapes, reasons as to why some objects have a symmetric property while other do not, extensions of symmetry to coordinate geometry and algebra, relationship between symmetry and tessellations, symmetry in arts, and so on. Throughout these instances, there is a common knowledge strand about symmetry relevant to teaching its multiple meanings and associations. In their analysis of teacher knowledge for teaching, Chinnappan and Lawson (2005) provided evidence of SMK and PCK that teachers have built around the concept of *square*. Results showed that early career teachers tended to build strong content knowledge, but that this knowledge was not in a form that would assist students. These results imply their content knowledge of 2-D geometry was not sufficiently specialized. There was also evidence that experienced teachers' PCK and SMK was not developed as expected. Our assumption is that the greater the range and depth of such knowledge, the greater the teachers' ability to flexibly extend this knowledge to their PCK.

PCK of geometry

Pedagogical Content Knowledge of geometry includes components such as understanding the central geometric topics as generally taught to students at a particular grade levels and knowing the core concepts, processes, and skills to be conveyed to students in geometry. Additionally, this knowledge strand involves knowing what aspects of geometry are most difficult for students to learn, and representations (e.g. analogies, metaphors, exemplars, demonstrations, simulations, and manipulations) that are most effective in communicating the appropriate understandings or attitudes of a geometry topic to students of particular backgrounds. Finally, knowing related misconceptions that are likely to get in the way of student learning forms part of PCK of teachers. For example, teachers' knowledge about how to teach the concept of tessellations and an understanding of why students experience difficulty with problems that demand an understanding as to why some 2-D shapes tessellate while others do not. The latter constitute KCS and KCT – subcomponents of PCK.

Interactivity between SMK and PCK for geometry

While the question of studying SMK and PCK is important for primary mathematics, the issue assumes greater significance for teaching high school mathematics, as the demand for this knowledge are expected to be higher. This is so because in secondary mathematics curriculum, teachers need to assist students examine properties of 2-D and 3-D shapes when they undergo transformations such as translation and rotation, and analyzing the transformations in a coordinate system. Moreover, even though greater emphasis is placed on concept development in the areas of geometry and measurement, Australian school students have been underperforming in this key area of the national mathematics curriculum (Thomson, Hillman, Wernert, Schmid, Buckley, & Munen, 2012). We suggest that one strategy for addressing the problem of underperformance is to examine relational understandings (1978) that students develop or fail to develop with geometry concepts. Relational understandings are constructed on the basis of connections among items of geometric information and organization of that information, the latter constituting structure of geometric knowledge.

Chinnappan (2008) demonstrated within the domain of geometry, high school students' understandings could be supported by knowledge that is structured so that it is accessible for future use. And teaching ought to find strategies for supporting the development of organized geometric knowledge. This stream of research led to a study of teacher knowledge for geometry in which Chinnappan and Lawson (2005) made the distinction between geometric knowledge and geometric content knowledge for teaching. Their work was deemed to have significance for future inquiries of teacher knowledge, practice, and student learning (Lawson & Chinnappan, 2015).

In the above review, we attempted to theorize and generate empirical evidence of teacher knowledge for teaching geometry. It emerges that future research needs to consider a) the particular characteristics of the discipline of geometry, b) the developmental trajectories of teachers' SMK and PCK and c) how these interrelationships are played out during the course of teaching. In summary, what is the overall premise of our discussion? In its totality, we contend that knowledge, by its very nature, is organized into strands that are, in turn, interconnected. The challenge for researchers is to unpack the interconnectedness between strands of SMK and PCK. In order to elucidate the relations between SMK and PCK, we suggest that the construct of *representations* could be employed as a useful analytic lens to generate and analyze data about and interactions between strands of MKT.

Representation

Studies in the field of cognitive science suggest that information is processed and stored in long-term memory. The processing of incoming information involves assimilation of new information with existing information, and reorganization of that information into meaningful entities called schemas. Organized knowledge schemas or entities stand for, reflect, or symbolize a reality. When schemas are activated for later use, humans convey that reality externally via models such as texts and real-life contexts (Lesh, Post & Behr, 1987). In this way, representations have a dual character: internal and external. Mayer (1975) suggested that knowledge presented in the form of representations is better understood and accessed by students. Our earlier example about tessellation is a case in point. The construct of representation has proven to be effective in analyzing teacher knowledge and tasks teachers select to implement their lessons. For example, Mitchell, Charalambous, and Hill (2014) commented that the "ability to teach with representations is critical to teaching well" (p. 43), and that MKT knowledge components can be examined via this construct.

For the purpose of analyzing teacher knowledge, we focus on external representations of that knowledge. *Representations,* as used in the present analysis, refers to vehicles teachers use to model, exemplify, or investigate a concept. *Representational fluency* refers to the ability to move within and between representations. Ball et al (2000) refer to the notion of representations in their discussion about tasks of teaching and associated SCK demands. This includes knowledge of what a particular representation is able to illustrate and explain. In their analysis of teacher knowledge, Ball et al., (2008), argued that "teachers must hold unpacked mathematical knowledge because teaching involves making features of particular content visible to and learnable by students" (p. 400). We suggest that representations provide a powerful window into not only the unpacked mathematical knowledge but also teachers' PCK. A teacher could represent a concept in geometry in the following modes: iconic (pictorial), symbolic, verbal, graphical, as well as real-world examples. Teachers who have developed representations that are wide, rich, and deep can be expected to support more complex understandings.

The study of geometry involves reasoning with diagrams, generating new information from understanding relations between the diagrams' parts and invoking relevant axioms. For example, the concept of *angle of inclination* can be given a diagrammatic and verbal representation. The diagram itself could contain symbols for denoting angle and measure of the angle in degrees (symbolic representation). Further, the concept could be given in meaningful context (real world representation) where the teacher poses a question asking students to use angles of inclination to predict how long it will take for the Leaning Tower of Pisa to fall over.

Emerging Questions

Our review of research suggests that future studies need to explicate the relationship between SMK and PCK as it is activated and mobilized by teachers before and during geometry lessons in order to better understand and support the dimensions of MKT. What do we mean by *relationship* between SMK and PCK? We interpret relationship in terms of translation of knowledge from one to the other representation during the course of teaching. By *teaching*, we mean engagement with students in real-time for the purpose of gaining new knowledge and understandings. We concur that data generated about teachers' SMK and PCK in contexts outside regular lessons are important and indeed necessary. However, the use of that knowledge during lesson delivery may necessitate modification or alteration of that knowledge in subtle ways. Equally, we suggest that the researcher is able to operationalize translation of knowledge in terms of representations. Teachers' representations could be used as an important analytical lens to gain access into both their SMK and PCK and the marshalling of the two bodies of knowledge in teaching. The above line of reasoning leads us to propose that future research should aim to respond to the following three questions:

1. What are the representations of geometry concepts generated by teachers in teaching contexts that provide access to their SMK and PCK?
2. What is the nature of the interaction between SMK and PCK from 1?
3. How does teaching experience impact on the above interaction?

MKT involved in construction and conjecture – Evidence from Preliminary Research

We are pursuing the above questions in a long-term study that examines the access and use of SMK and PCK in different areas of geometry. In this preliminary study, our aim was to generate data that is relevant to a modified version of Research Question 1: What are representations used by teachers to support students to conjecture in geometry?

Participants: The study was conducted in two junior high schools in Australia with teachers (n=3) of Grade 10 (15-year-olds) students (n = 25 per classroom). The students had completed topics in Euclidean geometry during the previous three years of their high school mathematics. In this report, we provide data from one of three schools.

Tasks and procedure: The teacher prepared and taught a lesson involving conjecturing with constructions. The lesson was video-taped, and the teacher was interviewed before and after the lesson. The videos were individually examined by the three researchers without any input from the teacher. This was followed by a group discussion.

We use the teacher's actions in the course of their teaching to make conjectures about their implicated SMK and PCK. In so doing, we adopt a functional view of SMK and PCK as knowledge enabling teachers to carry out tasks during the course of their teaching. In order to make SMK and PCK visible from an identical context, all teachers were provided with a Geometry Construction Task (GCT, Figure 2). The first lesson goal was a) to assist students to bisect angle *AZB* with the aid of a compass and ruler only, b) conjecture why angles *BZC* and *AZC* are equal and c) prove their conjectures.

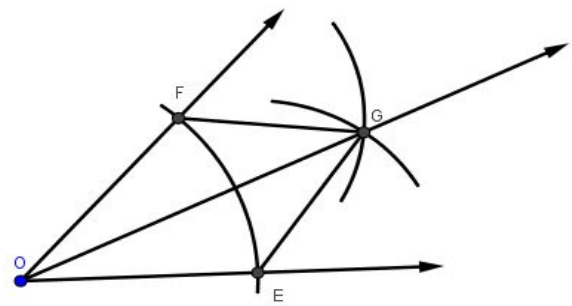

Figure 2: Geometry Construction Task

The lesson's second goal was to scaffold students to transfer the knowledge gained from the GCT to solve other construction problems. A problem of Transfer GCT (TGCT) is: *Construct an angle that is 30^0 in size by using a ruler and compass only*. The solution of TGCT involves students having and using knowledge to construct a 60^0 angle and then bisecting that angle. Construction of a 60^0 angle without the aid of protractors and other tools for measuring angles can be achieved by drawing an equilateral triangle and bisecting one of the three angles of the triangle. We consider that the solution of TGCT requires transferring knowledge and skills the students have on bisecting a given acute angle (covered in the lesson) in a new context with the new, additional knowledge about the equilateral triangles' properties.

SMK involved in implementing GCT

Our analysis of GCT produced the following concepts that we suggest constitute SMK:

Arc, bisect, ray, intersect, parallelogram, radius, centre of a circle, labelling the constructed figure with appropriate symbols (e.g., notations for marking/labelling angles and showing two sides are equal), and representations of equality of angles.

PCK involved in implementing GCT

In the context of our GCT, we conceptualize KCS as involving but not limited to teacher's comments that support students to make correct use of the compass and ruler to construct and bisect the resulting angle. Teachers could ask questions that help students to reflect and justify what they are doing during the construction process.

The KCT sub-strand is likely to address comments about how to represent and sequence the learning experiences that assist students in completing the construction and then extending their understanding to other construction problems such as Transfer CGT.

Data and analysis

As argued before, our aim was to generate data about SMK and PCK by analyzing a) representation of geometry concepts, b) actions, and c) rationale for using the relevant representations during the course of teaching. Table 1 shows a list of actions from Mary (pseudonym), the teacher from our participating school. These actions were observed during Mary's explanation to assist students in solving Transfer CGT. She accompanies her explanation by constructing an angle and bisecting the angle. Mary's actions below reflect a combination of using representations, raising questions and providing assistance to students to complete the task. The three investigators independently coded Mary's actions as reflecting SMK, PCK (KCS and KCT), though no KCT was detected in this excerpt. Following the coding, we met and resolved our differences.

Table 1: Excerpt of Mary's explanation for Transfer CGT

Line	**Mary's Comments**	**Knowledge Used**	
		SMK	**PCK**
1	*Bisecting into half by drawing a line.*	Δ	-
2	*So everyone got Question 1.*	-	$\square_{KCS}$
3	*How can use that knowledge to answer question 2 (Transfer CGT)?*	-	$\square_{KCS}$
4	*Start with a line at the bottom (drawing).*	-	$\square_{KCS}$
5	*Did the same with the other line?*	-	$\square_{KCS}$
6	*What do you have to do?*	-	$\square_{KCS}$
7	*What does the 'cross' represent?*	Δ	$\square_{KCS}$
8	*There is 180 degrees (180^0) in it.*	Δ	
9	*How can we use that triangle to find 30 degree angle?*		$\square_{KCS}$
10	*Put in a triangle and then the same length.*	Δ	
11	*What do we know about equilateral triangle?*	Δ	$\square_{KCS}$
12	*What can you do to both that bisect the 60 degree angle?*	Δ	$\square_{KCS}$
13	*Cut the sixty degree angle into half so each one is 30 degrees.*	Δ	-
14	*Does that make sense to everybody?*	-	$\square_{KCS}$

Note: Δ represents actions related to SMK; $\square_{KCS}$ represents actions related to PCK-KCS; $\square_{KCT}$ represents actions related to PCK-KCT, though none was detected in this excerpt

The series of comments from Mary shows instances where strands of PCK and SMK are activated independently and where the two work in tandem. Initially (Lines 2-3), Mary focused on reminding students about how they went about creating and bisecting an acute angle. Comments on Lines (4-10) are directed at supporting students to activate their knowledge of properties of equilateral triangles and using that knowledge to construct such a triangle. She invites the students to guess the size of each of the angles in the equilateral triangle (Lines 11-12) and proceeds to show how ideas about bisecting an angle could be utilized in constructing an angle that is half the measure of an angle in an equilateral triangle (Line 13). In Line 14, the teacher attempted to draw the attention of all students. As may be seen in the third column of Table 1, with the exception of one instance, the teacher's activation of PCK was reliant on their SMK about properties of geometric figures that included angles, triangles, measurement of angles, arc, ray, concept of bisection, radius and circle.

SMK and PCK for TGCT

Figure 3 shows a student's response when Mary asked them to construct an angle that is exactly 30^0 in size by using a ruler and a compass only.

We can examine Mary's knowledge from the perspective of a) why a teacher would pose such a problem and b) how she would make judgements about the students' response and explore future learning opportunities as suggested by Sullivan (2011). Let us consider the first perspective. By limiting the students to using a ruler and a compass, the teacher would like students to access knowledge of properties of equilateral triangles and the conceptual basis for bisecting angles. The latter involved drawing arcs, one segment that originates from a vertex, then using the cut-off points on the segment to draw another set of arcs, and finally joining the vertex to the point at which the arcs cross each other. Here, one notes evidence of multiple facets of teacher's SMK. If we approach the analysis from the second perspective, she could be expected to arrive at the conclusion that this student had used the knowledge that all sides of an equilateral triangle are equal in length and bisecting an angle of 60^0 will yield the desired outcome (30^0). Again, there is evidence of SMK (Table 1) that is relevant to, and played out during, the course construction.

But what are potential actions of the teachers that could constitute PCK? We are currently generating data to answer this question. We anticipate strands of PCK in this context would emerge from the kind of questions, models, and other scaffolds the teacher could provide in assisting struggling students and extending the knowledge base of successful students such as the student whose work is shown in Figure 3.

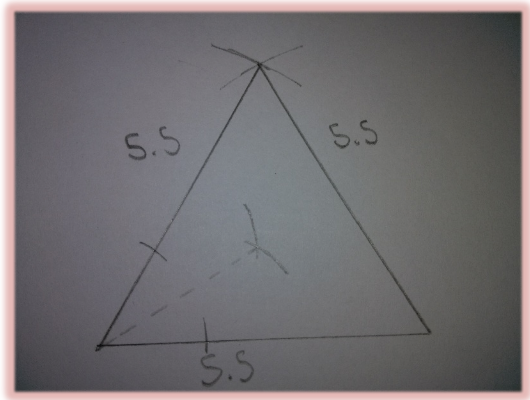

Figure 3: Sample student drawing

Discussion and Conclusion

Teachers and teaching are critical factors that affect students' engagement with and achievement in mathematics. According to the National Council of Teachers of Mathematics (2000) "effective teaching requires knowing and understanding mathematics, students as learners, and pedagogical strategies" (p. 17). In the current era of globalization and information, teachers' knowledge for teaching mathematics is becoming more complex and dynamic. Unpacking this knowledge to support effective learning has been the aim of a number of studies (Beswick, 2014; Sullivan, 2011). Since the conceptualization of PCK by Shulman (1987), the field has been active in developing other constructs to capture content and pedagogy relevant to mathematics. The question of the relative nature and roles of content and pedagogy in teaching mathematics is an issue of major concern to mathematics teachers and educators.

This is a preliminary study where we attempted to gather, code and represent data relevant in untangling relationship between the content and pedagogical knowledge in relation to teaching geometry *in situ.* In identifying, tracking, mapping and interpreting teachers' knowledge in the course of their teaching, we encountered three major challenges. Firstly, the coding of teacher talk as evidence of accessing SMK or PCK was not straightforward. Secondly, as one might expect with geometry, teacher's explanations were almost always accompanied by working with or constructing diagrams. A significant part of teacher knowledge and interactions between the strands of that knowledge occurs during these diagram-intensive activities. Thus, we have to develop a data analysis procedure to capture knowledge transactions in a complex and fluid context.

Thirdly, interpreting the geometry construction tasks within the framework of representations proved to be more difficult than representations of concepts of symmetry and tessellations. Mitchell et al. (2014) alluded to the constraints and affordances in representational use and that each representation has its own conventions. The notion of *conventions of representations* could provide a useful vehicle to better depict teacher's knowledge of SMK and PCK in the contexts of teaching geometric constructions. Our long-term aim is to fine-tune these methodological issues and interpret the data in terms of the representations construct.

While our results are preliminary, we view them as a prelude to a journey to address two important problems: a) develop the notion of *knowledge connectedness* (Lawson & Chinnappan, 2015) that is relevant to the teaching of geometry, which will b) ultimately help improve the quality of geometric knowledge that high school geometry teachers need in order to lift the achievement and participation of young Australians.

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MINDING THE GAP: A COMPARISON BETWEEN PRE-SERVICE AND PRACTICING HIGH SCHOOL TEACHERS' GEOMETRY TEACHING KNOWLEDGE

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This study compares the Geometry Teaching Knowledge of pre-service teachers with that of current high school geometry teachers. Data was collected using items from the Mathematical Knowledge for Teaching Geometry (MKT-G) assessment described by Herbst and Kosko (2014), and a post-assessment survey. The study focuses on the differences found in responses to items belonging to four domains: Common Content Knowledge-Geometry (CCK-G), Specialized Content Knowledge-Geometry (SCK-G), Knowledge of Content and Students-Geometry (KCS-G), and Knowledge of Content and Teaching-Geometry (KCT-G). Data was analyzed using t-tests for independent groups. Practicing high school geometry teachers outperformed the pre-service teachers on the MKT-G assessment in all four domains. Awareness of geometry instructional techniques and methods used in the current high school geometry classrooms was investigated as well. Practicing high school geometry teachers reported using and learning different instructional techniques and methods in their classrooms and professional development when compared to pre-service teachers' techniques and methods used or learned in their education and mathematics courses.

Keywords: Future teachers, geometry teaching knowledge geometry teaching methods, geometry teaching techniques, high school geometry, Mathematical Knowledge for Teaching-Geometry (MKT-G), practicing geometry teachers, pre-service teachers, professional development, teacher education, teacher knowledge.

Introduction

Geometry is a field in mathematics that every student in the United States is required to study in order to fulfill high school graduation requirements. According to the Center for Public Education (2013), all states require that students have two or more mathematics credits of Algebra 1 or higher to graduate. Geometry is listed as the course to immediately follow Algebra 1. The Common Core State Standards Initiative (2010) stresses that geometry is a vital course when preparing students to enter a science, technology, mathematics, or engineering field. According to the National Center for Education Statistics (2012), American students' performance is consistently behind other countries involved in the PISA assessment organized by the Organization for Economic Cooperation and Development (OECD) in two content areas in mathematics: Geometry and Measurement. In 2007, U.S. 8th grade students average score in geometry on the Trends in International Mathematics Science Study (TIMSS) was 20 points lower than the TIMSS scale average, while these students scored at or above the TIMSS scale average on all other content domains (Aud et al., 2010). The literature shows that three possible reasons for poor performance in geometry and measurement are: not enough exposure and emphasis in K-12 curriculum implemented by the teacher, challenges associated with the teaching of geometry and measurement in the classroom, and limited knowledge of the teachers (Steele, 2013).

Teachers that have completed a bachelor's degree in mathematics and a traditional teacher preparation program are considered qualified teaching candidates. According to No Child Left Behind (2002), a highly qualified teacher holds a bachelor's degree in mathematics and has passed a state academic subject test. Teachers with a secondary teaching degree are expected to be able to successfully teach all courses of mathematics study taught in high school, including geometry. According to the topics addressed in teacher certification exams, a pre-service teacher should be prepared to teach geometry when entering the secondary classroom; however, Mitchell and Barth (1999) point out that individuals can pass state certification tests without having to pass all the domains assessed on the test. If a pre-service teacher does not pass the Geometry and Measurement section of the exam, they could still pass the exam. But that pre-service teacher might not have enough content knowledge in Geometry to be a successful Geometry teacher. There is a need to make sure all teachers teaching in secondary schools have enough knowledge of Geometry. Even though teachers follow a traditional teacher preparation program, they may not be prepared to teach the mathematics required of them when they leave the university and enter the secondary classroom.

Geometry Teaching Knowledge: Background

Deborah Ball and her colleagues developed the concept of Mathematical Knowledge for Teaching, also known as MKT. Using Shulman's major categories of teacher knowledge, they developed a theoretical framework for content knowledge for teaching mathematics. Throughout their research, they began to see that "pedagogical content knowledge begins to look as though it includes almost everything a teacher might know in teaching a particular topic" (Ball, Thames, & Phelps, 2008, p. 394). Ball began to focus on how, throughout history, the prevailing assumption that the mathematical knowledge a teacher requires consists of the mathematics that will be covered in the course they are teaching along with some additional study of mathematics at the college level. Deborah Ball and her colleagues decided to develop Shulman's model in the field of mathematics. The primary data used for the analysis was a National Science Foundation funded longitudinal study that documented an entire year of mathematics teaching in a third-grade public school classroom. Many studies have investigated the MKT domains.

The Teacher Education Development Study in Mathematics (TEDS-M) identifies two components to teachers' mathematical knowledge: mathematical content knowledge (MCK) and mathematical pedagogical content knowledge (MPCK) (Tatto et al., 2012). This study developed a framework to measure pre-service teachers' MCK and MPCK in different domains. The domains for MCK included number, geometry, algebra, and data, and in tasks that required knowing, applying, and reasoning. The domains for MPCK included mathematics curricular knowledge, knowledge of planning, and knowledge of enacting mathematics (Tatto et al., 2012). This study found that future teachers in America showed strength in number items but weakness in geometry and algebra items.

The German project COACTIV conducted a study of the connections between content knowledge and pedagogical content knowledge in secondary mathematics among secondary teachers (Krauss et al., 2008). They found that content knowledge and pedagogical content knowledge were distinct factors and highly correlated in the entire sample of teachers; however, teachers considered mathematical experts held knowledge that combined the content knowledge and the pedagogical content knowledge, while those that were not experts kept the factors separate. They concluded that pedagogical content knowledge may be supported by higher levels of content knowledge in ways that lower levels of content knowledge may not (Krauss et al., 2008).

Deborah Ball's model has been cited over 1800 times since it was published. Many studies have been conducted to try to solidify this model, and other studies have focused on specific domains of mathematical knowledge for teaching. For example, Hill, Ball, and Schilling (2008) focused on the domain called knowledge of content and students. They point out that there has been little research in conceptualizing, developing, and measuring teachers' knowledge in each of the domains (Ball et al., 2008). Even though there have been many studies referring to Deborah Ball's MKT model, there is very little research on teachers MKT at the secondary level. Primarily, research has been conducted on teachers MKT of elementary algebra and number sense topics, but very few studies in elementary geometry. Another study of teachers' knowledge of Algebra points out that [while] "the University of Michigan's work marks considerable progress in defining and assessing teachers' mathematical knowledge for elementary and, more recently, middle-grades teaching, there is little systematic evidence about whether, or how different types of mathematical knowledge matter for effective teaching of algebra in grades 6-12" (McCrory, et al., 2012, p. 584).

In describing an MKT test designed to measure the knowledge needed to teach high school geometry, Herbst and Kosko (2014) pointed out that there had been little research into Ball's MKT model for high school specific subjects. At the time of the study reported here, there had not been any quantitative research on MKT-G of pre-service teachers, let alone a comparison between pre-service teachers and in-service teachers MKT of geometry. The literature calls for more research in pre-service and in-service teachers' MKT-G along with an investigation as to where these teachers gain this knowledge. Herbst and Kosko (2014) point out that there is more work to be done to refine the domains of Ball's MKT model with respect to Geometry and by doing so this "could inform the development of coursework in mathematics or mathematics education for future teachers" (Herbst & Kosko, 2014, p. 33).

Theoretical Framework

The theoretical framework used in this study follows the Domains of Mathematical Knowledge for Teaching-Geometry used by Herbst and Kosko (2014) to develop the MKT-G assessment. This assessment was founded on the framework by Deborah Ball and associates (2008). The original framework consisted of Common Content Knowledge, Specialized Content Knowledge, Knowledge of Content and Students, Knowledge of Content and Teaching, Knowledge of Content and Curriculum, and Horizon Content Knowledge. Herbst and Kosko's Mathematical Knowledge for Teaching-Geometry (MKT-G) assessment focuses on four of the six domains: Common Content Knowledge, Specialized Content Knowledge, Knowledge of Content and Students, and Knowledge of Content and Teaching.

Common Content Knowledge-Geometry (CCK-G) is defined as the geometry knowledge and skill also used in settings other than teaching. In particular, CCK-G is the mathematical knowledge needed to simply calculate the solution or correctly solve geometric problems such as those that students do. Specialized Content Knowledge-Geometry (SCK-G) is geometry knowledge and skill unique to teaching, not necessarily used in any other field. For example, the knowledge needed to see what a student's mistake was when solving a geometry problem incorrectly. Knowledge of Content and Students-Geometry (KCS-G) is knowledge that combines knowledge about students and knowing about geometry. KCS-G is the knowledge teachers need to predict how students may react to a new geometry topic, or what misconceptions and confusion students may have going into a geometry lesson. Knowledge of Content and Teaching-Geometry (KCT-G) is a domain that combines knowing about teaching and knowing about geometry. KCT-G primarily focuses on the planning of the teacher, the sequencing of geometry topics so that students are successful, or what geometry examples the teacher decides to show the students.

Purpose of Study

The purpose of this study was to compare what I call the *Geometry Teaching Knowledge* (GTK) of pre-service and practicing high school teachers; GTK includes MKT-G and awareness of geometric techniques and methods used in the geometry classroom. This study examined the differences in knowledge among different groups of teachers and where this knowledge is developed.

This study focused on the knowledge of high school pre-service teachers at a four-year university in the State of Texas (in the United States) and that of practicing high school geometry teachers from multiple school districts in north and central Texas.

Research Questions and Design

The research questions for this study are:

1. What do high school pre-service teachers and high school geometry teachers know about *Geometry Teaching Knowledge*? *Geometry Teaching Knowledge* consists of the following: Mathematical Knowledge for Teaching- Geometry (MKT-G) and awareness of geometry techniques and methods used in the high school geometry classroom.
2. How do pre-service and current high school teachers' *Geometry Teaching Knowledge* compare?
3. Where is awareness of geometry techniques and methods used in the classroom developed?

Sample

The study was conducted at a central Texas university and at school districts throughout the state of Texas. The sample was composed of 53 pre-service high school mathematics teachers at the university and 36 practicing high school geometry teachers in multiple school districts in north and central Texas. The pre-service teachers were chosen based off their completion of their coursework in the program.

The pre-service teachers were in their Junior or Senior years of their degree program and had completed the required geometry content course. The geometry content course taught at this central Texas university is called *Modern Geometry*. This course focuses on Euclidian Geometry and historical aspects of Geometry. This course is a mathematics content course that is required of the secondary pre-service teachers, but there is little pedagogical content covered. Pre-service teachers at this point in their degree plan have at least taken two education courses: *Curriculum and Technology* and *Adolescent Growth and Development*. By choosing pre-service teachers at this point in their degree, there is a guarantee that the pre-service teachers have completed the majority of their required coursework for their specific graduation plan, and are about to enter their student teaching experiences.

The high school geometry teachers were current teachers in multiple school districts in central Texas. Their degrees were obtained from a variety of different universities, and their teaching experience ranged from one to twenty years of experience teaching geometry. Only high school teachers who were currently teaching or had taught geometry within the previous two years were selected to participate in the study.

The pre-service teachers were a convenience sample; however, this sample arguably represents the knowledge base of pre-service teachers about to enter their student teaching experiences. The university uses The Mathematics Education for Teachers II Report (2010), which gives requirements and suggestions for teacher preparation programs in the United States. These requirements are based off the Common Core Standards (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010).

Instrumentation

To investigate pre-service and practicing high school teachers' *Geometry Teaching Knowledge*, data was gathered by means of an online Mathematical Knowledge for Teaching-Geometry (MKT-G) assessment developed by Herbst and Kosko (2014) and a post-assessment survey. The MKT-G assessment consists of multiple choice questions administered through the online platform Lesson*Sketch*. The post-assessment survey consists of demographic questions and questions regarding the experiences of the pre-service and high school teachers with different methods of instruction. The following is a sample item from the post-assessment survey asked to both pre-service and high school teachers.

Read the following techniques and consider which ones you would use in your own Geometry Classroom. You are given a total of 10 points to distribute among 5 techniques however you would like based on what you would think would be best for your students (assign a value between 0 and 10 to all items), with the number of points assigned to the topic reflecting the importance of these techniques in your classroom. You must use all 10 points. Please make sure the points add up to 10 by including a total count at the end.

a. Investigations (Example: Discovery lessons) ________

b. The use of a compass and protractor to construct figures ________

c. Computer Software (Geometer's Sketchpad, GeoGebra, etc.) ________

d. Manipulatives/Models ________

e. Other: (please describe) ____

Total: _______

Figure 1. Example Methods/Technique Problem

Pre-service teachers and practicing high school teachers were asked different questions regarding their awareness of instructional techniques and methods. Pre-service teachers were asked: what types of instructional techniques or methods have they seen in their geometry courses, what types of instructional techniques or methods have they seen in their education courses, and what types of instructional techniques or methods would they use in their ideal classroom. An ideal classroom was described as one for which they would have an unlimited budget and unlimited resources. Due to the selection of pre-service teachers, most of the participants had not been in a current high school geometry classroom as an observer or an instructor, which is why the first two questions addressed what they had seen as students in their geometry course and education courses. Practicing high school teachers were asked what types of instructional techniques or methods do they use in their current geometry classes, what types of instructional techniques or methods have they seen in their professional development, and what types of instructional techniques or methods would they use in their ideal classroom. All participants took the online Mathematical Knowledge for Teaching-Geometry assessment and all but one high school teacher completed the post-assessment survey.

Data Analysis

MKT-G Assessment Results

The MKT-G assessment was given to pre-service teachers and practicing high school Geometry teachers to assess their Mathematical Knowledge for Teaching Geometry. The assessment includes items that address four of the domains of mathematical knowledge for teaching; Common Content Knowledge-Geometry (CCK-G), Specialized Content Knowledge-Geometry (SCK-G), Knowledge of Content and Students-Geometry (KCS-G), and Knowledge of Content and Teaching-Geometry (KCT-G). Because I was interested in comparing scores for each domain, I scored the responses by looking at how many items of each domain participants responded correctly.[1] Because there were different numbers of questions addressing each domain, I calculated the proportion of correct responses for each domain. All 87 participants were combined to form the following descriptive statistics of the proportion correct over each of the domains and the total score. A lower score indicates lower knowledge of a domain and the higher score indicates higher knowledge of a domain. The results are presented in Table 1. When comparing the means of each of the domains, all the participants preformed the best in the Common Content Knowledge-Geometry domain, and performed the worst in the Knowledge of Content and Teaching-Geometry.

Table 1. Descriptive Statistics of Percentage Correct by MKT-G Domain and Total Score.

Domain	Mean	Standard Deviation	N
CCK-G	64.80%	21.94	87
SCK-G	60.00%	14.49	87
KCS-G	39.24%	19.87	87
KCT-G	36.95%	23.52	87
Total	53.67%	13.58	87

In order to better understand the differences between pre-service teachers and high school geometry teachers, a comparison using the raw test scores in each domain was performed. The box plots in Figure 2 show the difference between the two groups in each of the four domains and the total raw scores.

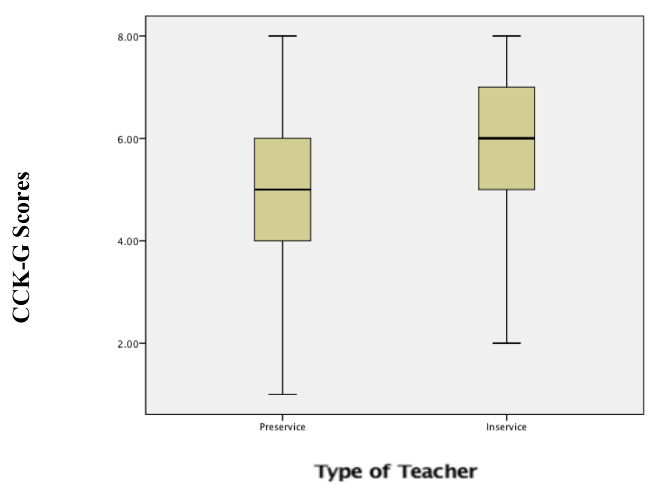

Figure 2a. Boxplot comparing CCK-G Scores of Pre-service and In-service teachers

[1] Because the samples were small, the scores could not be scaled using the Rasch model; hence this analysis does not consider the difficulty level of each of the questions.

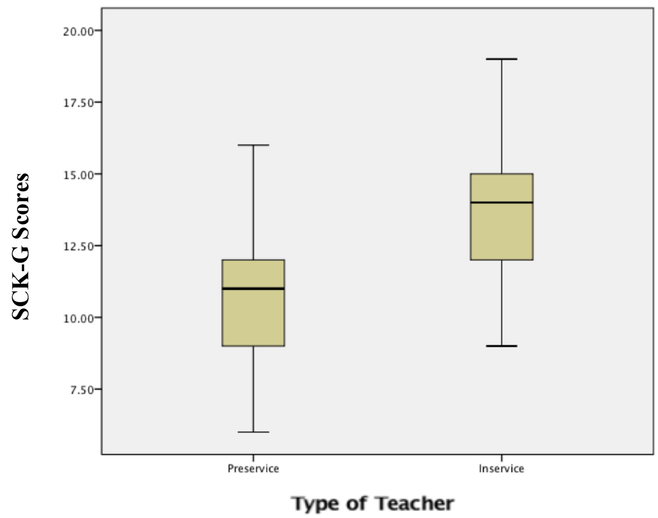

Figure 2b. Boxplot comparing SCK-G Scores of Pre-service and In-service teachers

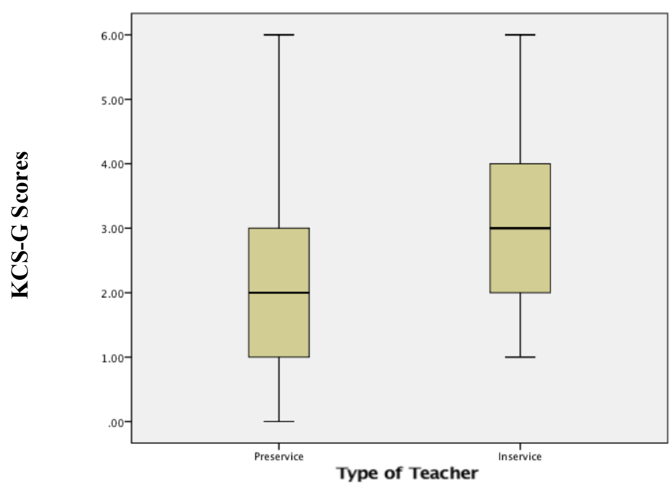

Figure 2c. Boxplot comparing KCS-G Scores of Pre-service and In-service teachers

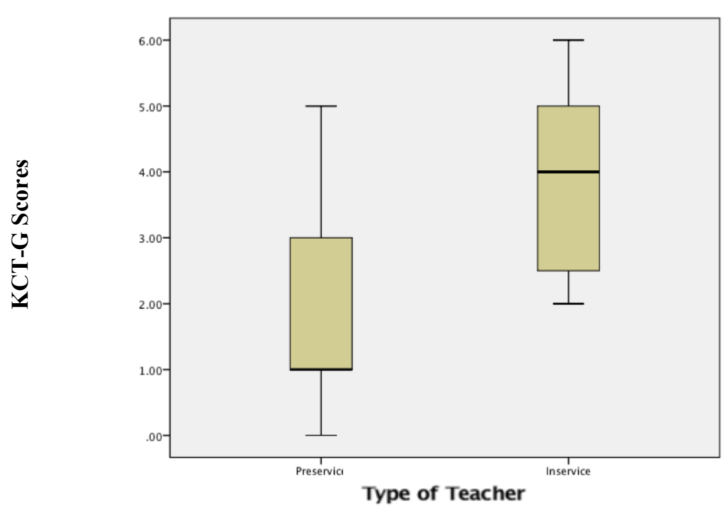

Figure 2d. Boxplot comparing KCT-G Scores of Pre-service and In-service teachers

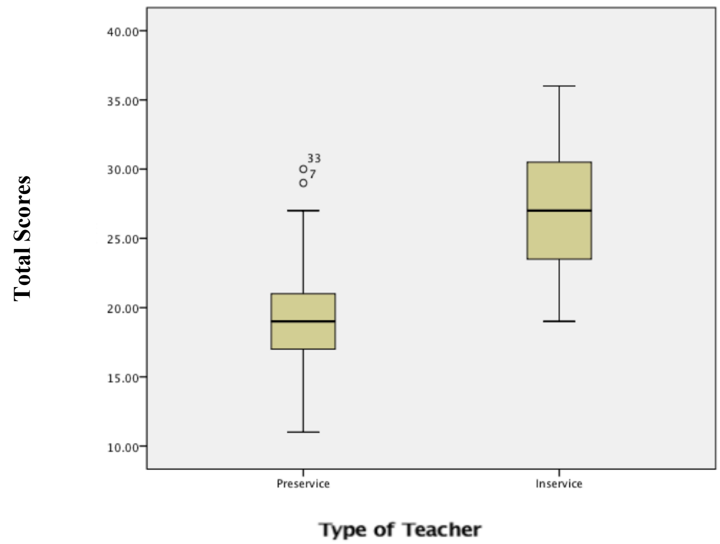

Figure 2e. Boxplot comparing Total Scores of Pre-service and In-service teachers

A *t*-test for independent groups was performed in each of the domains as well as with the total scores. The descriptive statistics for each domain and Cohen's *d* are presented in Table 2. A *t*-test for independent groups was performed in each of the domains as well as with the total scores. Pre-Service teachers had lower CCK-G scores on the MKT-G assessment than current high school Geometry teachers, $t(76.61) = -3.642$, $p < .001$, $d = -.832$. Cohen's effect size ($d = -.832$) suggests a moderate practical significance. Pre-service teachers had lower SCK-G scores on the MKT-G assessment than current high school Geometry teachers, $t(71.899) = -5.882$, $p < .001$, $d = -1.3873$, which suggests a large practical significance. Pre-Service teachers had lower KCS-G scores on the MKT-G assessment than did those that were current high school Geometry teachers, $t\ (72.16) = -3.285$, $p = .002$, $d = -.773$. Cohen's effect size ($d = -.773$) suggests a moderate to large practical significance. Pre-service teachers had lower KCT-G scores on the MKT-G assessment than current high school Geometry teachers, $t\ (80.76) = -6.516$, $p < .001$. Cohen's effect size ($d = -1.45$) suggests a large practical significance. Pre-service teachers had lower total scores on the MKT-G assessment than current high school Geometry teachers, $t\ (70.13) = -7.542$, $p < .001$. Cohen's effect size ($d = -1.80$) suggests a large practical significance.

Table 2. Means, Standard Deviation, and Cohen's *d* by MKT-G Domain and Total Score of Pre-Service and High School Teachers.

Domain	Pre-Service		High School Teachers		Cohen's *d*
	Mean	Standard Deviation	Mean	Standard Deviation	
CCK-G	58.09%	20.74	78.30%	20.25	-.832
SCK-G	53.43%	11.85	69.31%	12.77	-1.387
KCS-G	33.61%	18.26	47.22%	19.56	-.773
KCT-G	25.77%	20.41	52.78%	18.01	-1.45
Total	46.40%	9.99	63.96%	11.16	-1.80

Based on the *t*-tests performed, pre-service teachers had lower scores in all domains and in total scores. There is also large practical significance to all the comparisons.

Correlations between the domain scores are presented in Table 3, and suggest a moderate relationship between the different variables. These correlations were examined to make sure the results from this study were similar to the correlations reported by Herbst and Kosko (2014). These results show similar trends, which suggests that the four domains are interrelated, to a degree.

Table 3. Correlations between MKT-G Domains.

	CCK-G	SCK-G	KCS-G	KCT-G
CCK-G	-			
SCK-G	.343**	-		
KCS-G	.391**	.389**	-	
KCT-G	.361**	.456**	.304**	-

** $p < .01$

I calculated the correlations between each of the domains, total score, and the participants' years of teaching mathematics and years of teaching Geometry. The correlation between the number of years teaching mathematics and Common Content Knowledge-Geometry (CCK-G) and Knowledge of Content and Students-Geometry (KCS-G) were statistically significant, but weak. The correlation between Specialized Content Knowledge-Geometry (SCK-G), Knowledge of Content and Teaching-Geometry (KCT-G), and total score were statistically significant and moderate. The correlation between the number of years teaching Geometry and KCS-G was statistically significant, but weak. The correlation between CCK-G, SCK-G, KCT-G, and total score were statistically significant and moderate.

Table 4. Correlations between Years Experience and Scores.

	Years Teaching Math	Years Teaching Geometry
CCK-G	.239**	.323**
SCK-G	.361**	.352**
KCS-G	.265*	.286**
KCT-G	.448**	.397**
Total	.465**	.471**

*$p < .05$, **$p < .01$

Post-Assessment Survey Results

As part of the Post-Assessment Survey, participants were asked questions regarding their experiences with different Instructional Techniques and Methods that are frequently used in the geometry classroom. Pre-service teachers and current high school teachers were asked different questions regarding their knowledge. Pre-service teachers were asked what types of instructional techniques or methods have they seen in their geometry courses, what types of instructional techniques or methods have they seen in their education courses, and what types of instructional techniques or methods would they use in their ideal classroom. An ideal classroom was described as a situation in which they would have an unlimited budget and unlimited resources. High school teachers were asked what types of instructional techniques or methods they used in their current geometry classes, what types of instructional techniques or methods they had seen in their professional development, and what types of instructional techniques or methods they would use in their ideal classroom. Figure 3 shows the pre-service teacher survey results, specifically the distribution of experience with what types of instructional techniques or methods they had seen in their geometry courses, what types of instructional techniques or methods they had seen in their education courses, and what types of instructional techniques or methods they would use in their ideal classroom.

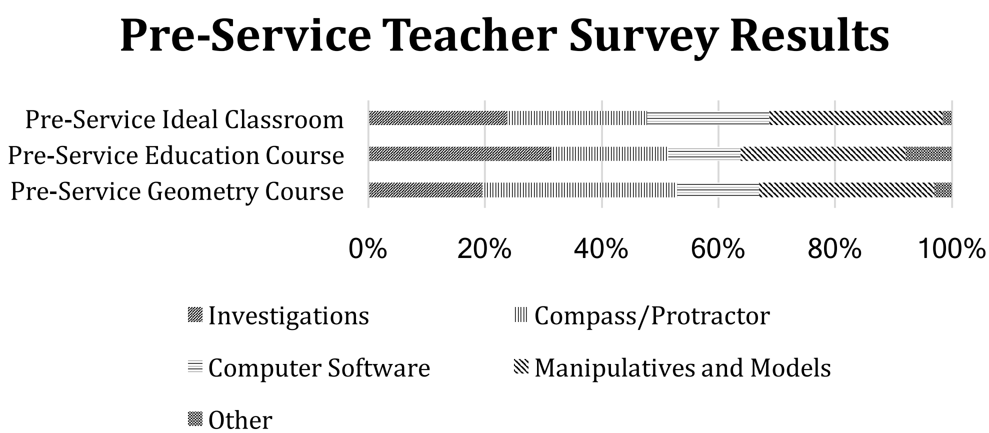

Figure 3. Pre-Service Teacher Survey Results

For pre-service teachers' geometry courses, participants reported experiencing compass and protractor activities (33.3%) and manipulatives and models (30.1%) the most, and computer software (14.1%) the least. In their education courses, pre-service teachers reported seeing investigations (31.2%) the most and computer software (12.3%) the least. Pre-service teachers would use manipulatives and models (29.7%) the most and computer software (21%) the least in their ideal classrooms.

Figure 4 shows the practicing high school teachers' survey results, specifically what types of instructional techniques or methods they used in their current geometry classes, what types of instructional techniques or methods they had seen in their professional development, and what types of instructional techniques or methods they would use in their ideal classroom.

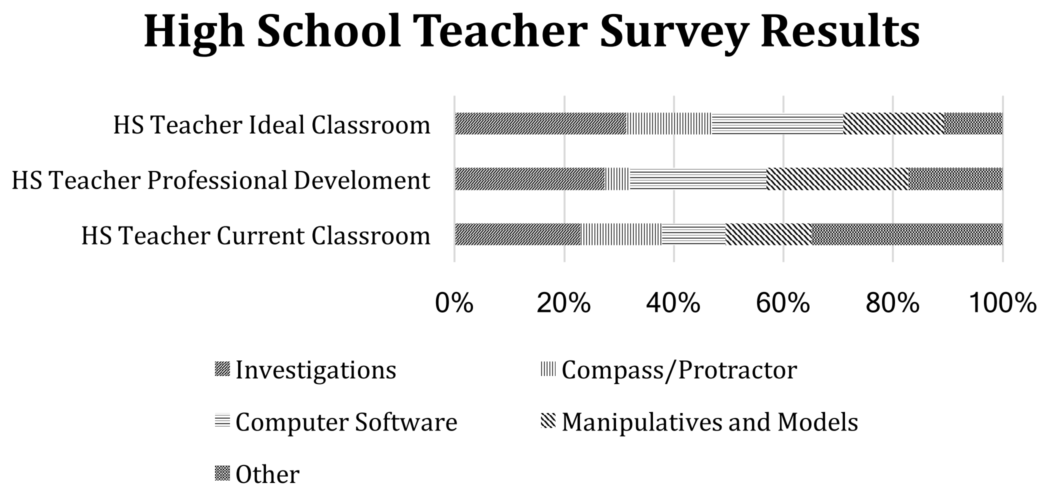

Figure 4. Practicing High School Teacher Survey Results

Practicing high school geometry teachers reported the use of *other* (35%) as most common in their classrooms. Other was defined as Lecture by 80% of the participants. They reported that computer software (11.6%) was used the least in their current geometry classes. High school teachers reported seeing investigations (27.3%) the most and compass and protractor activities (4.7%) the least in their professional development. When teachers were asked about their ideal classroom, high school teachers would use investigations (31.3%) the most and compass and protractor activities (15.7%) the least.

Pre-service teachers were asked which instructional techniques and methods they had used or seen in their geometry and education courses and practicing teachers were asked which instructional techniques and methods they had used or seen in their professional development. Attention was given to this comparison to investigate the methods taught at the university for pre-service teachers and the methods taught in the professional development opportunities given to practicing teachers. A chi-square test for independence was performed to examine the association between pre-service teachers' experience in their geometry and education courses to the practicing high school teachers' professional development. This test was found to be significant, $\chi^2(4, N = 86) = 123.84, p < .01$. This suggests that the pre-service teachers' distribution of what they see in their geometry and education courses and what high school teachers have seen in their professional development are not independent. In Figure 5, the strip diagrams show the distribution among the instructional techniques and methods of the pre-service teacher's Geometry Courses and what they would use in their ideal classroom.

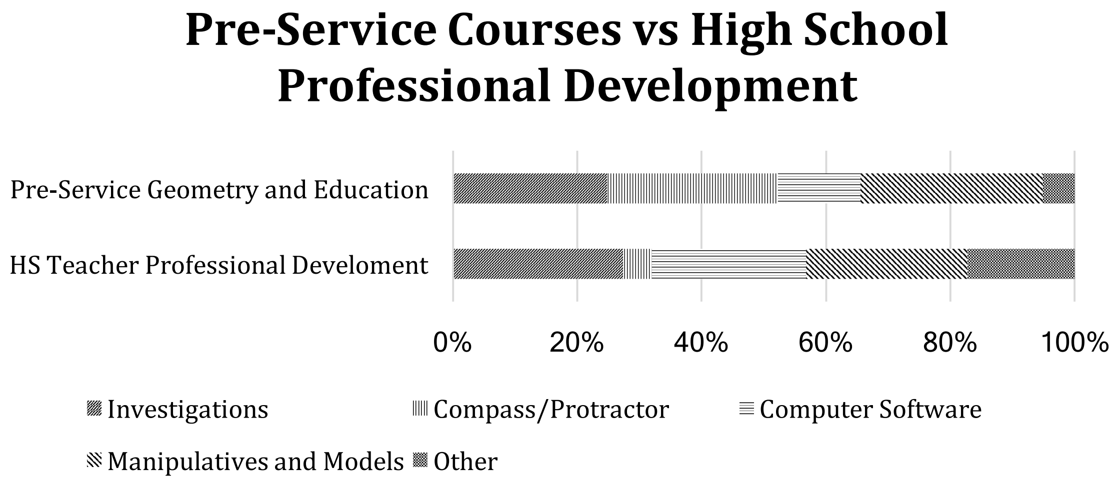

Figure 5. Pre-Service Courses vs. High School Professional Development

Pre-Service teachers have seen more compass and protractor activities (27.5% of the time), and more manipulatives and models (29.3%) in their geometry and education courses when compared to high school teacher's professional development (4.7% and 25.9% respectively). High school teachers reported more investigations (27.3%), computer software (24.8%), and other (17.2%) in their professional development than pre-service teachers have seen in their geometry and education courses (24.7%, 13.3%, and 5.1% respectively). The responses for *other* in professional development included teaching strategies, classroom management, project based instruction, and direct teach/lecture, and the responses for *other* in their geometry and education courses included lesson plans, PowerPoints, projects, and lecture.

Pre-service teachers were asked which instructional techniques and methods they had used or seen used in their geometry and education courses, and practicing teachers were asked which instructional techniques and methods they used in their current classroom. This comparison was chosen because pre-service teachers would expect to see the instructional techniques and methods used in current high school classrooms during their courses at the university. A chi-square test of independence was performed to examine the relation between pre-service teachers' experience in their geometry and education courses to the current high school teachers' geometry classes. This test was found to be significant, $\chi^2(4, N = 86) = 196.19, p < .01$. This suggests that what pre-service teachers see in their geometry and education courses, and what high school teachers are using in their current geometry classes are not independent. In Figure 6, the strip diagrams show the distribution among the instructional techniques and methods of the pre-service teacher's current geometry courses and what they would use in their ideal classroom.

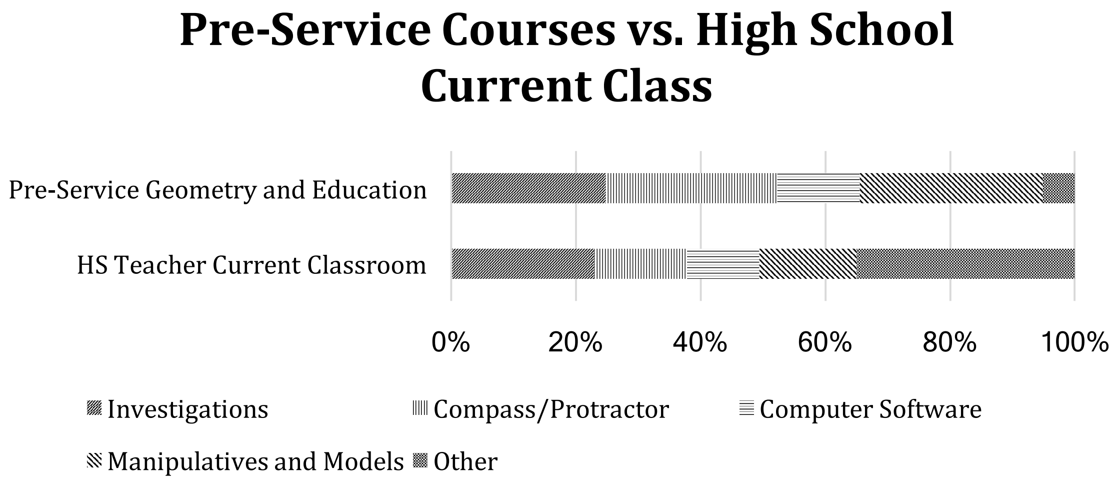

Figure 6. Pre-Service Courses vs. High School Current Class

Pre-service teachers reported more experience with compass and protractor activities (27.5%), and manipulatives and models (29.3%) than high school teachers reported using in their current classrooms (14.9% and 15.6% respectively). High school teachers reported more time spent on other (35%) than pre-service teachers claim in their geometry and education courses (5.1%). Lecture and Direct instruction is what 49% of the high school teachers described as *other*. Pre-service and high school teachers distributed points similarly to the investigations (24.7% and 22.9% respectively) and computer software (13.3% and 11.6% respectively).

Both groups were asked how they would spend time if they had an ideal classroom. An ideal classroom would consist of having unlimited resources and time. A chi-square test of independence was performed to examine the relation between Pre-Service teachers' ideal classroom and current high school teachers' ideal classroom. This test was found to be significant, $\chi^2(4, N = 86) = 59.93, p < .01$. This shows that what high school teachers think would be best for their ideal classroom and what the pre-service teachers think would be best for their ideal classroom are not independent. In Figure 7, the strip diagrams show the distribution among the instructional techniques and methods of the pre-service and high school teachers' ideal classrooms.

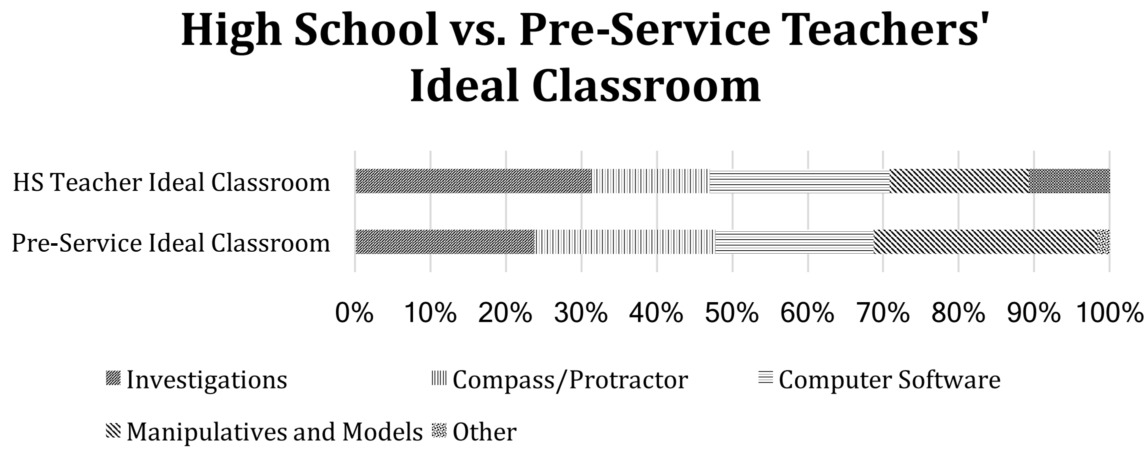

Figure 7. High School vs. Pre-Service Teachers' Ideal Classroom

Pre-service teachers thought that more compass and protractor activities (24% of the time) and manipulatives and models (29.7% of the time) were important to their ideal classes when compared to the high school teachers (15.7% and 18.4% respectively). The high school teachers thought more investigations (31.2%) and computer software (23.9%) would be important to their ideal classrooms, as well as a larger portion dedicated to other (10.7%) when compared to pre-service teachers' distribution of classroom time (23.8%, 20.9%, and 1.6% respectively). Lecture and Direct teach is what 49% of the high school teachers described as other.

Discussion

It could have been expected that the pre-service teachers would not do as well on the MKT-G as the practicing high school teachers because the high school teachers have been actively working with students and refining their geometry knowledge through practice, but this study sheds light on how the groups of teachers compare with one another. The primary domains where pre-service and high school teachers had the largest difference were Specialized Content Knowledge-Geometry (SCK-G) and Knowledge of Content and Teaching-Geometry (KCT-G). Specialized Content Knowledge-Geometry is "mathematical knowledge and skill unique to teaching" (Ball et al., 2008, p. 400). SCK-G is the knowledge of mathematics that is not necessarily used in any other field. Knowledge of Content and Teaching-Geometry is the category that "combines knowing about teaching and knowing about mathematics" (Ball et al., 2008, p. 401). KCT-G primarily focuses on the planning of the teacher, the sequencing of topics so that students are the most successful, or what examples the teacher decides to show the students. These results are not surprising when SCK-G is knowledge of geometry that would not be used in any other activity besides teaching high school geometry and KCT-G would require the pre-service teachers to have some idea of how to present material to students. The pre-service teachers were stronger in Common Content Knowledge-Geometry and Knowledge of Content and Students-Geometry, though they still score lower than the practicing teacher. Common Content Knowledge-Geometry is what they would get from their geometry courses at the university and the Knowledge of

Content and Students-Geometry could come from them interacting with students through tutoring or remembering being a student themselves.

There were statistical differences between pre-service teachers and high school teachers in the knowledge of the different instructional techniques. This was unexpected, but this is a problem that needs to be addressed. One can understand teachers not being able to teach their ideal geometry class because of budgetary restrictions and time, and it seems that professional development would introduce current teachers to other instructional techniques that they might not be using in their current classroom, but the techniques presented in professional development would seem to transfer over to the teacher's ideal geometry class. It seems strange that pre-service teachers are being taught geometry and are in education courses, but their methods of teaching their ideal geometry class do not relate. Where are these pre-service teachers getting these ideas? It seems that there would be differences between the pre-service ideal classroom and the high school teachers' classroom because the pre-service teachers do not have as much classroom experience, and current high school teachers are drawing from their experiences being a geometry teacher. This also could relate to the MKT-G results showing that pre-service teachers have a lower score on the Knowledge of Content and Teaching-Geometry. One surprising result from these comparisons is the difference between the pre-service geometry and education courses and the professional development opportunities for high school teachers. It would seem that both of these types of teacher education would correspond in some way, but statistically they are different. The comparison between the pre-service teachers' geometry and education courses and the current high school geometry classroom is also interesting. If pre-service teachers are not being introduced to what the current high school teachers do in the geometry classroom, is this setting them up for failure?

Significance of the Study

This study sheds light on the *Geometry Teaching Knowledge* that high school pre-service and high school geometry in-service teachers. This study helps fill in the gap in research regarding Mathematical Knowledge for Teaching Geometry and awareness of geometric techniques and methods used in the geometry classroom that pre-service and high school geometry teachers possess and use. The instruments used to address these questions could be used in other pre-service mathematics teacher training programs and in professional development of high school teachers to address any gaps that may exist in their knowledge of geometry and of teaching geometry. This may impact future student performance in Geometry and Measurement since the three main reasons for a lag in performance are weak attention in K-12 curriculum, challenges associated with implementation of geometry and measurement in the classroom, and limited knowledge of the teacher (Steele, 2013).

Limitations of the Study

This study focused on a group of pre-service teachers from a single university in central Texas. The structure of this university's pre-service teacher training program could be different than other universities in Texas and in other states or countries. This study also focuses on currently practicing high school mathematics teachers in Texas. The knowledge level of geometry may be different depending on the state in which the teachers work. The professional development opportunities given to high school teachers varies depending on the district. In general, teachers are given a couple of days of professional development one week prior to the start of the school year and a day of professional development after the Christmas break. While some of the results may be extended beyond the scope of this university and state, any generalizing must be done cautiously.

The MKT-G assessment results were analyzed using the number correct in each of the domains and the total. Difficulty of each individual question was not considered because the sample was too small to estimate item difficulty parameters.

I developed the survey given to all the participants. The intention for the survey was to gather information about the knowledge of instructional methods and strategies of the participants. There is no guarantee that the survey accurately gathered all the knowledge of the participants.

Future Research

This study brought up issues of the differences in *Geometry Teaching Knowledge* between pre-service and currently practicing high school teachers. Pre-service teachers were weaker in all domains, but primarily in Specialized Content Knowledge-Geometry (SCK-G) and Knowledge of Content and Teaching-Geometry (KCT-G). There is a need for future research that focuses on these domains, specifically to target what can be done to increase scores in these domains for pre-service and high school teachers.

This study has shown there are differences in pre-service and high school teachers' experiences with instructional techniques and methods. Further research is needed to investigate the different instructional techniques and methods used in pre-service courses and professional development courses. These two forms of teacher education courses would correspond, and that knowledge would be transferred to the teachers' ideal geometry class. There is also a need for more research into ways they can implement what they learn in their teacher education courses into their current or future classroom.

Further research is needed to elaborate on the origin of *Geometry Teaching Knowledge* in pre-service and practicing high school teachers. If we can pinpoint where the majority of this knowledge is obtained, then we can make sure pre-service teachers have those experiences in their training programs to better prepare them for entering the high school classroom.

While this study is focused on *Geometry Teaching Knowledge*, there is a need to extend this type of research into other secondary mathematics courses (e.g., Algebra 2, Pre-Calculus, and Calculus), and even into post-secondary education. These results provide some insight into how this could be extended to other subjects, but specialized assessments will need to be developed.

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DESIGNING INSTRUCTION IN GEOMETRY: USING LESSON STUDY TO IMPROVE CLASSROOM TEACHING

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The Malaysian Educational Blueprint of 2013 advocated a need to improve mathematics instruction with regards to students' construction and application of mathematical ideas when solving real-world problems. This paper presents a school-based effort to design classroom instruction in geometry that encourages students to mathematize and use mathematical processes towards this purpose. The study used a methodology based on design research and Lesson Study. Qualitative data were collected as the study progressed and were interpretively analyzed. The findings of the study indicate that the teachers were receptive of the approach and made useful contributions in the design of the instruction. The effectiveness of the instruction was gauged by the teachers' active participation in the research cycle as well as the students' thoughtful engagement in solving the tasks and the ability to arrive at solutions through mathematical thinking. The teachers in the study were able to identify three specific key pedagogical points that enabled student learning: a) Using the area of triangle formula to help students make connections to previous knowledge; b) Sequencing the tasks to facilitate the students' progression in learning; c) Realizing the need to further expand and enhance discourse so as to allow more student-student and teacher-student interaction.

Keywords: Area of triangles, design research, lesson study, parallel lines, problem solving, procedures and concepts

Background

The Malaysian Educational Blueprint (MEB) (Ministry of Education, 2013) was documented as a national strategy towards improving education in Malaysia from 2013-2025. Much reference in the MEB was given to the country's performance in two international assessments: (a) The Program in International Student Assessment (PISA); and (b) The Trends in Mathematics and Science Study (TIMSS). Using these two assessments as benchmarks, it was inferred that there has been a decline in students' performance in science and mathematics over the years. Due to these concerns, the authors of the MEB highlighted the need to achieve several aspirations by providing students with a holistic education, emphasizing the necessity to instill in students a love for inquiry and lifelong learning. The authors of the MEB further advocate the inculcation of skills such as critical thinking, reasoning, creative thinking, innovation and the enhancement of students' ability to apply knowledge and think critically outside familiar academic context. These recommendations may seem timely and appropriate for most of the disciplines in the curriculum. Ostensibly, the recommendations may seem superfluous for the mathematics teacher as mathematical processes and problem solving have long been suggested in the curriculum prior to the launching of the MEB. However, the implementation and use of mathematical processes and problem solving in the classroom have continued to pose challenges for the teacher. Malaysian teachers often cite the lack of time, the compact curriculum and examinations as the main constraints that discourage them from including mathematical processes and activities that involve mathematizing in the classroom (Cheah, 2012). These constraints would certainly influence teachers to rely on the more traditional practices in the classroom. There is therefore a continuing need to assist teachers to review and apply pedagogical practices towards realizing the aspirations of the MEB.

The Study

This paper documents a school-based effort to design classroom instruction in geometry which uses a student-centered approach to encourage students to mathematize and use mathematical processes. In the study, a teaching sequence in geometry was designed for the purpose of "developing, testing, implementing and diffusing innovative practices to move the socially constructed forms of teaching and learning … to(wards) excellence" (Kelly, Baek, Lesh, & Banaan-Ritland, 2008; p.3). The aim of the study was to investigate the usefulness of the approach. The three main research questions were:

1) How did the teachers respond in designing and using the classroom tasks?
2) How did the students respond to the tasks?
3) How can the teacher and student responses be used to inform teacher practitioners towards improving classroom instruction and the learning of geometry?

Theoretical Framework

Investigating the usefulness and design of classroom instruction would necessarily involve examining ways to carefully and purposefully design tasks and the subsequent implementation of the tasks in the classroom to gauge their effectiveness and ways of improving the tasks. This involves two main components: (a) A quality assurance component to manage the process of designing, implementing and evaluating for purposes of improvement; (b) A didactical component that examines the quality of teaching and learning mathematics.

The design and implementation of instruction naturally involves teachers who play major roles in the cognitive and formative dimensions of teaching (Mesa, Gomez, & Cheah, 2013). Because of the integral role of teachers in the instructional process, it is imperative that the ideas that are used in the design and implementation of classroom tasks take into account the teachers' views. This applies to classroom-based studies too, where the constant collaboration of teachers and researchers leads to and enriches the learning process of the research team and enhances the synergy among the team members. It is with this purpose in mind that elements of design research and Lesson Study (Zawojewski, Chamberlin, Hjalmarson, & Lewis, 2008; Doig, Groves, & Fujii, 2011; Baba, 2007) were chosen to be included in the design of the study.

The use of Lesson Study as a professional developmental approach is not new. Widely used in Japan, Lesson Study has often been cited as a powerful approach to empower teachers towards better classroom practice (Stigler & Hiebert, 1999). The main characteristic of Lesson Study is the collaborative study of research lessons by teachers and consists of three main phases: (a) Planning the lesson; (b) Observing the implementation of the planned lesson; (c) Reflecting on the lesson to find ways to improve the lesson. While these three main phases may look simple and superficial, Lesson Study has been used to study more deeply various aspects of the lesson including exploring and examining the instructional materials, the role of the lesson tasks, ways to effectively present mathematical tasks as well as mathematical discourse in the classroom (Doig, Groves, & Fujii, 2011). By including the collaborative elements of Lesson Study in this research, teachers become active members of the study team and contribute significantly throughout the different stages of the study as opposed to more traditional design methods where teachers often take more passive roles.

Ensuring a good quality assurance process alone, however, does not guarantee quality didactics. In a sense, the Lesson Study cycle is simply a generic approach to manage lesson improvement, one which can be used in any discipline. It is therefore necessary also to give due consideration and attention to the didactical component that could then serve as a benchmark by which the elements that contribute to, or hinder, the teaching and learning of mathematics can be gauged. Mathematical tasks need to be designed, or selected, carefully so as to engage students in meaningful learning. The design and selection of tasks in this study are guided by the following principles:

1. Children mathematize by organizing and using mathematical means through spontaneous activities (Freudenthal, 1973).

2. Solving the tasks requires that the students use some form of mathematical concept, formula, or method (Brousseau, 1997).

3. The tasks focus on a specific mathematical idea that can be built on and used to solve a related task of higher difficulty.

Since this study relates to the teaching and learning of geometry, the classroom tasks must provide the *geometrical working space* (GWS) (Kuzniak & Richard, 2014; Kuzniak, 2015) for the students to construct the necessary mathematical ideas and concepts and use them to solve problems. GWS, as proposed by Kuzniak and Richard (2014), exist in two planes: (a) The cognitive plane; (b) The epistemological plane. The cognitive plane consists of three activity components: *visualization, construction,* and *proof*. The epistemological plane consists of three kinds of corresponding content components: *representation*, *artefacts*, and *referential*. The cognitive plane describes the kinds of geometric activities that are derived from the corresponding mathematical objects in the epistemological plane through processes referred to as *genesis*. Thus *visualization* is derived from *representation* through *figural genesis*, *construction* from *artefacts* through *instrumental genesis*, and *proofs* from *referential* through *discursive genesis*. For a further discussion on GWS please refer to

Kuzniak and Richard (2014). This study focused on the students' capacity to conceptualize and apply a specific geometrical idea. Therefore the students' work covers mainly the visualization-representation components of their respective GWS.

Methodology

The methodology in this study, which was implemented in a naturalistic classroom setting, involved a research cycle consisting of three phases: (a) The research cycle involved collaborative planning and design of a teaching sequence; (b) Teaching and observations of the research lesson; (c) Reflecting on the lesson and the teaching sequence in order to improve the design of the classroom instruction (Figure 1).

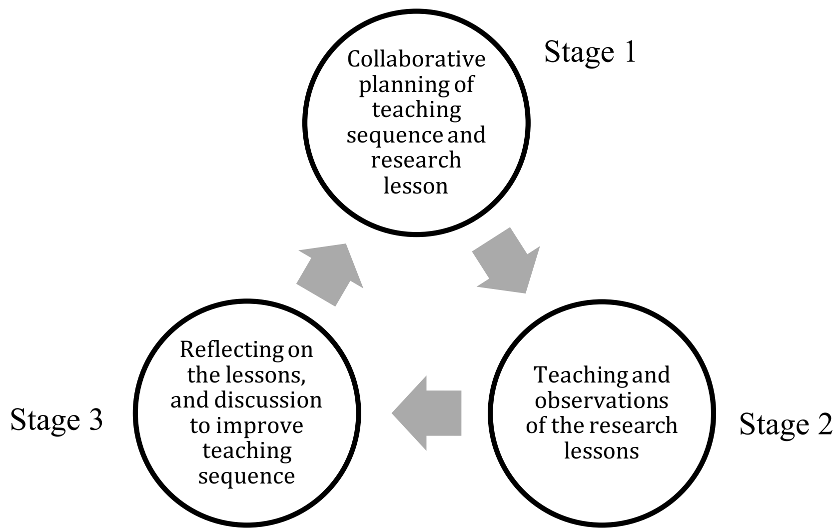


Figure 1: The stages of the study cycle

The study was carried out in a fully residential co-educational secondary school. The research group consisted of four teachers (two males and two females) and the researcher. The teachers have varying teaching experiences ranging from five to thirty years.

Qualitative data for this study were collected from written artefacts, interviews with the teachers and students, still photos and video recordings. The written artefacts include the lesson plans that were drafted by the teachers, students' work, and student responses about the classroom environment which were collected through a post-lesson survey. Informal interviews were also conducted with the teachers and students. The lesson and the post lesson discussions were video recorded. The findings were then triangulated from the data, which were interpretatively analyzed.

Findings

Stage 1 of the Study Cycle (Planning)

This stage covered the initial planning of the teaching sequence and the research lesson. The topic chosen was on geometry; specifically, the area of triangles. This topic is popular amongst teachers and emphasized in the curriculum. During the planning stage, the research team discussed the design and sequencing of tasks to help students develop the idea that the area of triangles between parallel lines with the same base is constant. The students were also required to apply this concept in a variety of problem solving situations. The research team conducted the aforementioned discussions in four two-hour meetings over a two-month period. As a result of the discussions five main tasks were chosen to be used in the lesson. An important consideration during the design stage was to select tasks that would fit into the actual classroom settings. Tasks were chosen and designed so that they would take up minimal classroom time without sacrificing time for students to construct the main mathematical ideas and without the teacher directly telling the answers. The tasks would be able to intentionally foster the creation of a milieu, which could promote students' construction of their own ideas through meaningful student-student and student-teacher interactions (Brousseau, 1997). During the planning stage the team members agreed that the tasks in the lesson would involve the use of dynamic geometry software (DGS) because the dynamic nature of the software, through the click-and-drag feature, hide/show, and measure buttons, allows for a more flexible in-depth discussion. Furthermore, the use of DGS affords more flexibility for teachers to manage the instructional time in the classroom.

The tasks are listed here in sequential order in which they were to appear during the lesson. During the discussion, however, the main anchor tasks, Tasks 4 and 5, were discussed first. As the team members discussed the solutions to Tasks 4 and 5, key mathematical ideas essential for solving the tasks emerged which led to the subsequent design of the other tasks. Tasks 1, 2, and 3 were designed in order to facilitate the students' progression in constructing the geometrical ideas and use them to solve Tasks 4 and 5.

Task 1 (shown in Figure 2) was designed by the team member who taught the lesson. The task, on inferring that the area of any triangle constructed on a common base is dependent on its height, was designed as an enabler to lead the students to Task 2.

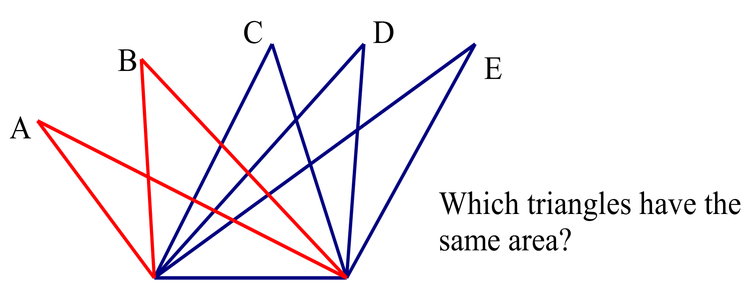


Figure 2: Area of triangles with a common base (Task 1)

Task 2 (shown in Figure 3) was aimed at guiding the students to construct and verify the idea that the area of any triangle between parallel lines is a constant. The measure tools, the click-and-drag feature and the hide/show buttons in the software in the DGS were used to allow the students to arrive independently at the conclusion through investigation. Point A can also be merged or unmerged to the hidden line parallel to $\overline{BC}$. Clicking and dragging point A shows how the area changes as point A moves. The students are asked to infer how the area changes as point A moves.

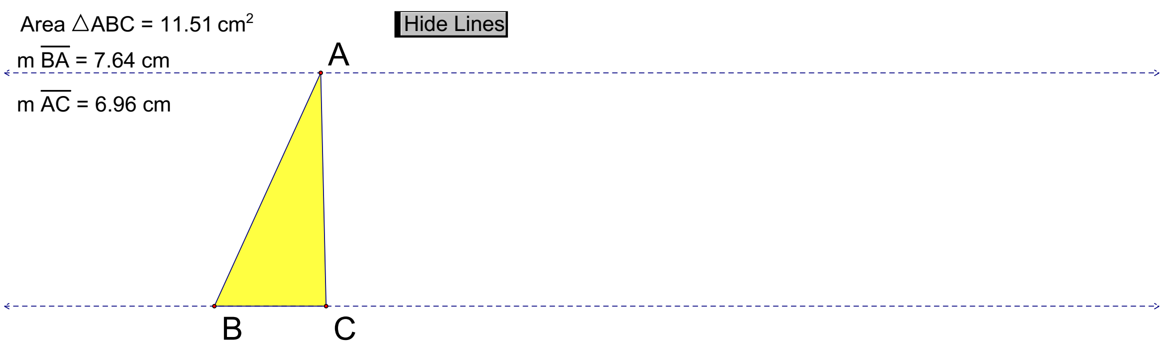


Figure 3: Change in area as point A moves (Task 2)

Task 3 (shown in Figure 4) shows an application of the idea that the area of the triangle with a common base between parallel lines remains constant. The students were required to use the idea to construct a quadrilateral from the pentagon without changing the area.

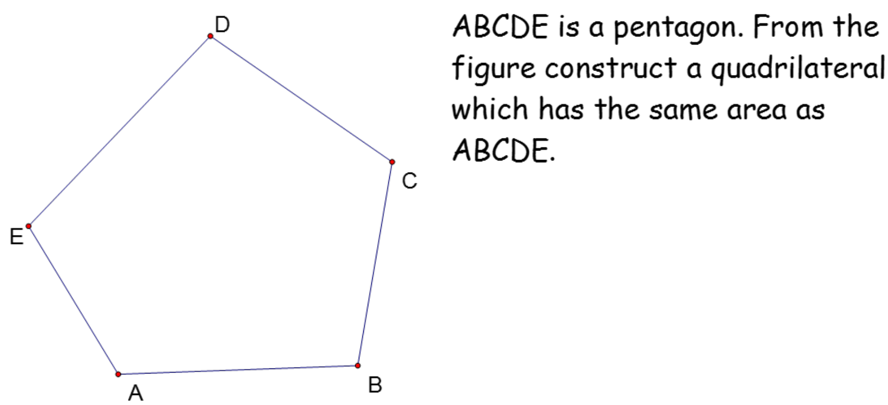


Figure 4: Application of the area concept (Task 3)

Task 4 (shown in Figure 5) shows an application in a real-life situation. Both Tasks 3 and 4 were adapted from the TIMMS video study (TIMSS video, n.d.). Some conditions were intentionally left out in Task 4 so that the conditions could be used as points for classroom discussion. The teacher could initiate this discourse by asking whether it would be fair if any straight boundary is drawn and what conditions need to be considered to ensure fairness.

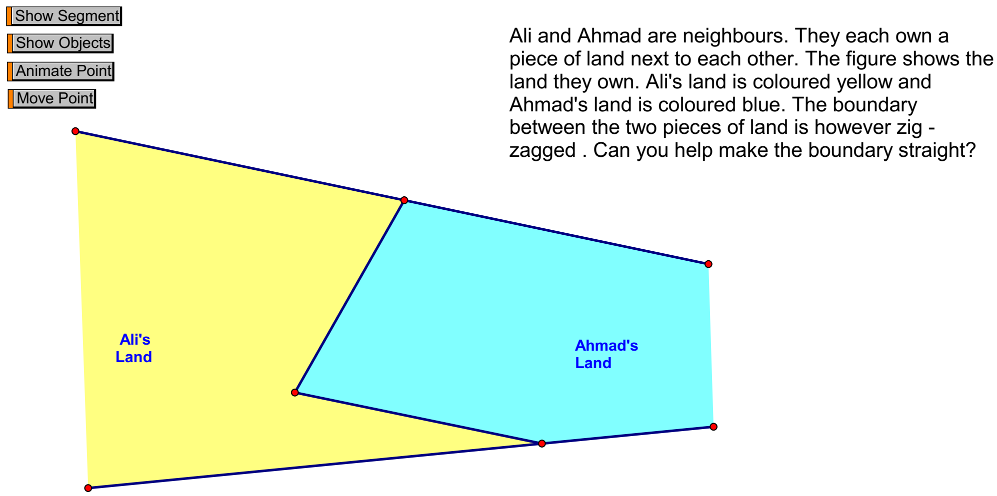


Figure 5: Problem solving task based on real-life situation (Task 4)

Task 5 (shown in Figure 6) is a problem solving task adapted from the Poh Leung Kuk Primary World Mathematics Contest 2002 (c.f. http://www.poleungkuk.org.hk/en/joint-schools-districts-world-competition/primary-mathematics-world-contest.html). One key point in the discussion during the planning stage was that Figure 6 should be drawn so that the location of the point G should distinctly show that it is not the midpoint of $\overline{CD}$. Otherwise, the students would assume that G is the midpoint of $\overline{DC}$, which would lower the complexity of the task.

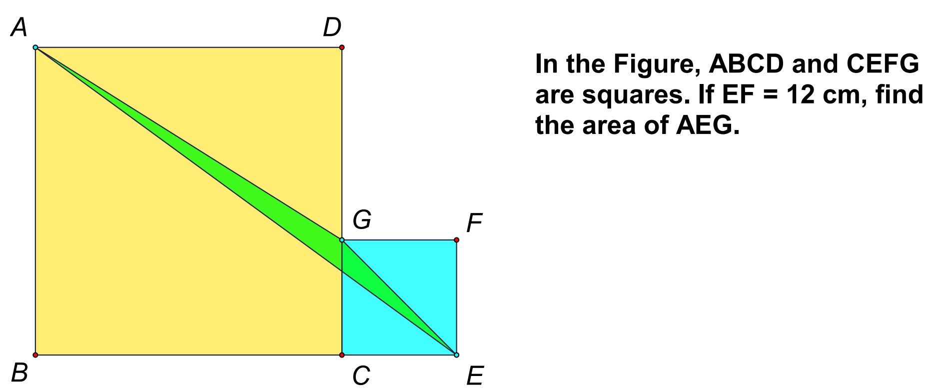


Figure 6: Problem solving task (Task 5)

Stage 2 of the Study Cycle (Teaching and Observation of the Research Lesson)

One teacher from the research team taught the lesson to a Grade 10 class of nine students while the other team members observed the lesson. All the tasks designed in Stage 1 were included in the lesson. However, the teacher who taught the lesson made some modifications to the teaching sequence. He began the lesson by introducing the problem in Task 4. He reasoned that it would help set the tone of a problem solving environment (5 minutes). This was quickly followed by Task 1 and Task 2 (20 minutes) before reverting back to Task 4 to allow time for the students to complete the problem in Task 4 through group work (20 minutes). For all the tasks the students were provided with squared paper.

Task 3 and Task 5 were then given to the students to solve (20 minutes). The final 5 minutes was used for discussion and to wrap up the lesson.

Student responses. The teacher who taught the lesson as well as the teacher observers noted that Task 4 and Task 5 were challenging for the students. The students were observed to be engaged while working on the tasks. This observation was further corroborated by the remarks of three of the students after the lesson. They voiced their wish to have more thinking tasks during lessons. For Task 4, the students' solutions were all similar to the one shown in Figure 7. It was observed that in order to apply the idea that the area of triangles between parallel lines is constant, the students used the procedure of drawing parallel lines on the figures to solve the problem. This procedure was not taught by the teacher during the lesson. The students later clarified that they had learnt drawing parallel lines before. The proceduralizing of the concept by drawing parallel lines was observed to be a key moment that helped the students visualize the locations of the base and the vertex of the triangle and thus identify the triangles with the same area. Task 5 appeared more challenging than Task 4 as the students were observed to initially struggle when solving the problem. All the students, except one, arrived at similar geometrical solutions (see Figure 8a). Just as the students did in Task 4, once they correctly identified and drew the parallel lines in the diagram they were able to identify the triangles with the same area and subsequently found a solution to the problem.

One exceptional case was observed where a student used mathematical calculation to arrive at the solution (see Figure 8b). The student wrote the following solution:

Area $\Delta ADG = \frac{1}{2}(40)(28) = 560\ unit^2$

Area $\Delta GEF = \frac{1}{2}(12)(12) = 72\ unit^2$

Area $\Delta ABE = \frac{1}{2}(52)(40) = 1040\ unit^2$

Area quadrilateral *ABCD* = (40)(40) = 1600 $unit^2$

Area quadrilateral *CEFG* = (12)(12) = 144 $unit^2$

Area of shaded region = (144 + 1600) – (1040 + 72 + 560)

= 1744 – 1672

= 72 $unit^2$

In the solution the student drew the diagram on squared paper. From his drawing, he assumed that the length of $\overline{BC}$ was 40 units although it was not given in the task and proceeded to calculate the area of triangles *ADG*, *GEF*, and *ABE* and the area of the quadrilaterals *ABCD* and *CEFG*. The area of *AEG* (the shaded region) was calculated as the difference of the sum of the two quadrilaterals and the three triangles. When asked later, the student could not explain why he assumed the length of $\overline{BC}$ to be 40 units although he acknowledged that it is possible that the length of $\overline{BC}$ was not necessarily 40 units. It is inferred that he made this assumption from his drawing on graph paper.

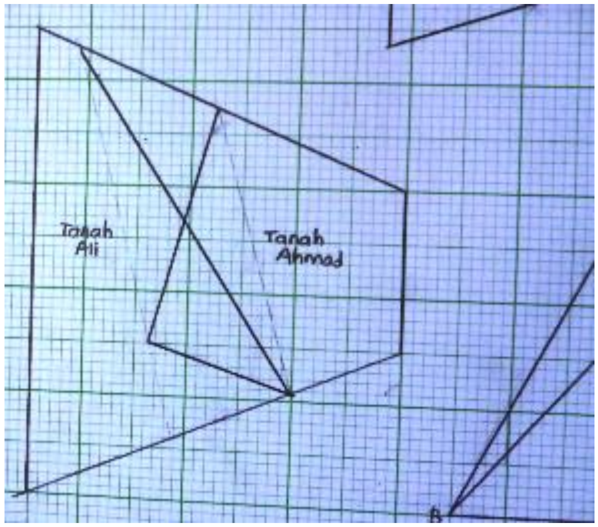
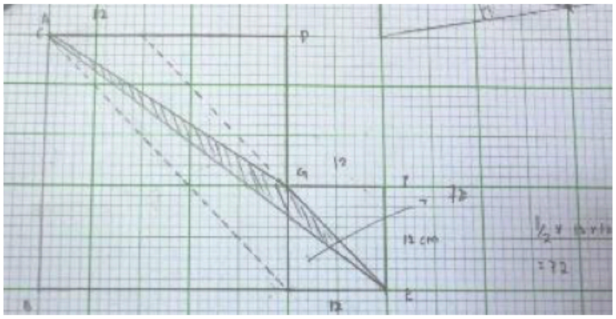


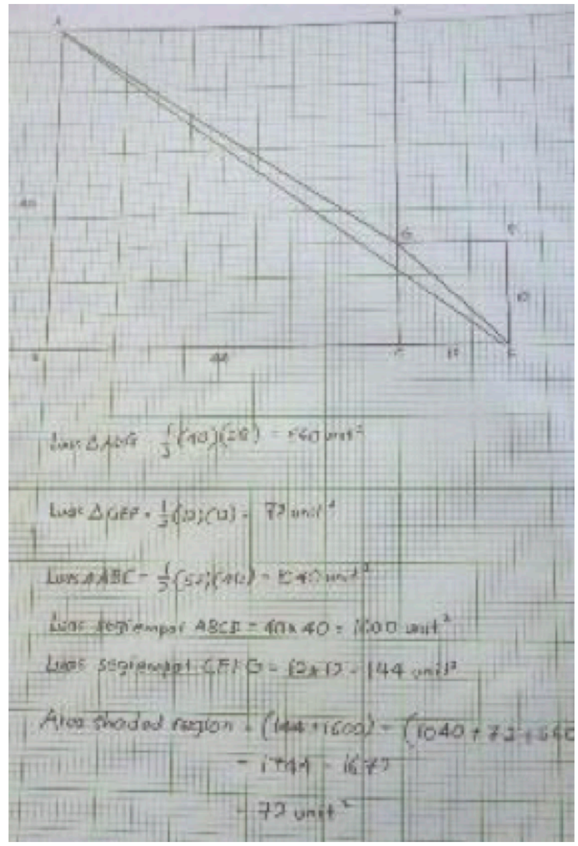
Figure 7: Students' solution for Task 4

At the end of the lesson, the students were asked to complete a 16-item 4-point Likert-scale survey, which had been designed to investigate the students' perception of the classroom environment related to classroom interactions and student learning. The post lesson student survey consisted of 16 items that were scored on a four-point Likert scale (with 1 indicating total disagreement with the statement and 4 total agreement). Items 1 to 3 describe the student's interaction with their peers, 4 to 6 describe whether the teacher, the students or their peers were asking questions, 7 to 9 describe the students interaction with their peers, item 10 describes whether the student felt the teacher was fair, 11 to 13 describe whether the student found the mathematics learnt was useful and interesting, and 14 to 16 describe whether the student liked the lesson (see Table 1).

Stage 3 of the Study Cycle (Reflection and Discussion of Lesson)



a) Geometrical solution



b) Solution using calculation

Figure 8: Students' solution for Task 5

Stage 3 of the study cycle was a post-lesson discussion, which was held immediately after the lesson. The discussion lasted for an hour. The teacher who taught the lesson gave his reflection on the lesson first, followed by each of the other members of the team. All the team members agreed that the tasks posed were challenging and suitable as it was observed that the students were actively engaged in solving the tasks. The team also concurred with the observation that the students initially had difficulty when solving the problem in Task 4 until the teacher suggested the use of the formula for the area of the triangle (Area = $\frac{1}{2}$ x base x height). The teacher who taught the lesson further suggested that it would be better to start with Task 1 and 2 instead of Task 4 for lower achievers in future lessons. Another suggestion by the research team was to allot more time for discussion within groups, and with the entire class. Allowing sufficient time would enable the students to articulate and communicate their ideas. This observation was affirmed by the results of the survey conducted at the end of the lesson (see Table 1). Items 3, 4 and 6 show lower mean scores compared to the other items, indicating that the students felt that there were few opportunities for them to interact during the lesson.

		Mean	S.D
1.	I shared with my classmates what I knew in the lesson	3.30	0.5
2.	I got help from my classmates	3.40	0.7
3.	I helped students who have trouble understanding the lesson	2.63	1.06
4.	The teacher asked questions	3.50	0.5
5.	I asked the teacher some questions	3.00	1.10
6.	I asked my classmates some questions	3.00	0.80
7.	My classmates talked with me about how to do the activities and problems	3.13	0.35
8.	I showed and explained how I solved a problem to my classmates	3.10	1.00
9.	I learned from my classmates in the lesson	3.00	0.76
10.	The teacher was fair to me and my classmates	3.75	0.46
11.	The Math I learned in the lesson can be used at home/the supermarket/store/everywhere.	3.63	0.52
12.	I learned new and interesting things about Math in the lesson	3.80	0.50
13.	What I learned is useful at places outside school.	3.25	0.46
14.	I like the activities in the lesson	3.88	0.35
15.	I understood the lesson.	3.88	0.35
16.	The lesson was fun.	3.88	0.35

Table 1: Student post-lesson survey questions and mean responses

Discussion

The aim of this study was to explore the feasibility of employing a design research/Lesson Study approach to enable students to think mathematically and solve tasks in geometry. The active participation of the teachers in this study showed that the design research cycle was effective in empowering the teachers as well as in developing the teachers' professional knowledge, particularly in specific learning situations in the classroom. Through the discussions, teaching, lesson observation, and the subsequent reflection, the teachers' practitioner knowledge about teaching and learning geometry was further enhanced. All the teachers in the team provided useful inputs in the process of designing the tasks, in teaching the lesson as well as providing constructive feedback to improve the lesson. This study showed that, through the research cycle, the team members were able to identify three specific key pedagogical points that enabled student learning: a) Using the area of triangle formula to help students make connections from previous knowledge; b) Sequencing the tasks to facilitate the students' progression in learning and, c) Realizing the need to expand and enhance discourse through student-student and teacher-student interaction.

The students' solutions showed that they were able to apply the idea that the area of triangles between parallel lines with the same base length remains constant to solve Task 4 and Task 5. Their ability to solve the tasks was facilitated by two key moments in the lesson. The first was the teacher's prompting that led the students to conceptualize their new idea by examining the area of triangle formula. This led the students to conceptualize that the area of triangles between parallel lines with a common base is constant. The second was the procedure of drawing parallel lines onto the figures in the tasks. By drawing parallel lines the students were able to identify the triangles with the same area. This proceduralization of the constant area concept which the students had constructed earlier helped the students to extend their understanding of the concept and solve Tasks 4 and 5. One possible explanation for this is that the students' flexibility and expertise to solve the tasks increased as they make more connections between the procedure and the theorem of constant area of triangles between parallel lines. As Baroody, Feil, and Johnson (2007) argue, making more links between procedures and concepts can lead to a deeper conceptual understanding of mathematical objects which in turn could assist students in problem solving. This is because procedures are not disconnected but rather are linked and intertwined with concepts (Baroody, Feil, & Johnson, 2007; Gray & Tall, 2001; Star, 2005; Tall, 2015). This raises the issue of the importance and necessity of intentionally including appropriate procedures while designing tasks for instruction so as to enrich and deepen the students' understanding of mathematical concepts.

The design and sequencing of the tasks also played an important role in facilitating student learning. The main aim in the sequencing of the tasks was to assist the students to progressively mathematize new geometrical ideas, which they did. They first conceptualized that if the height of triangles is fixed then the area of the triangle with the same base is constant. This led them to conceptualize that the area of triangles between parallel lines is constant and, subsequently, to build on this concept to elicit the procedure of drawing parallel lines and apply the procedure to solve more complex tasks. One pertinent issue in the sequencing of the tasks was raised by the teacher who taught the lesson, whether it would be more appropriate to introduce Task 4 first which would set the tone of the lesson at a higher cognitive demand. The other alternative would be to begin with Task 1 and introduce the other tasks progressively before introducing the main anchor problems of the lesson in Tasks 4 and 5. While this may make the anchor tasks easier to solve, as the students would already know which geometrical idea to apply, it would also make the problem less challenging and take some fun away from problem solving.

While most students gave a geometrical solution to Task 5, one student however gave a solution using only mathematical calculation (Figure 8b). This showed that students at this level were capable of offering different approaches to the solution. An emergent issue here is that this scenario provides teachers with an opportunity to make connections between the geometrical solution and the solution using calculation. In the solution provided by the student using calculation, he assumed that $AB = AC =$ 40 units. Using this special case, he was not able to make any algebraic generalization. This raises the question of whether teachers should extend student learning at this point to create discourse to help students further extend their understanding. Would the method used in the solution still apply if AB is equal to a length other than 40 units? This could lead to more problem posing with possibilities of linking geometric and algebraic solutions and a further blending of mathematical knowledge structures leading to even more mathematizing possibilities.

The teachers and students in the study noticed that there was a lack of opportunities for student-teacher discourse during the lesson. More opportunities could be further incorporated into the instruction so as to encourage a richer discourse in the classroom. Examining the lesson tasks also led to the conclusion that the tasks were able to elicit student work that was centered on the drawing and visualization of geometrical figures and diagrams. Seen from the perspective of the GWS framework proposed by Kuzniak and Richard (2014), the GWS of the students in this study covered mainly the *visualization* activity and *representation* content component in the framework. The students' work was centered on the use of figures and diagrams. Very little working space was covered in the *construction-artefact* and *proof-referential* components in the GWS framework. This indicates that the GWS of the students can be appropriately expanded to include tasks that involve geometrical activities of construction and proofs.

Conclusion

The Malaysian curriculum advocates and emphasizes learning mathematics through fostering mathematical thinking and problem solving. To actualize this vision, it is necessary to carefully design classroom tasks that enable students to mathematize and to progressively learn mathematics by conceptualizing and organizing mathematical structures and subsequently extending and applying them to solve problems (Freudenthal, 1971; Skemp, 1993).

The design research approach used in this study involved a collaborative effort by the research team consisting of teachers and the researcher in designing the instruction, teaching and observing the lesson and reflecting and discussing the lesson. In particular, the design research and lesson study approach was able to facilitate and empower the teacher towards enriching the teachers' practitioner knowledge. In this study attention and focus were also given to the didactical aspects of learning geometry. By considering and examining these didactical aspects, the teaching and learning of mathematics and in this case, the study of geometry, could be examined and improved. The episodes of student problem solving provided some insights into the distinct ways they used to solve problems.

The feedback from the teachers and students also indicated the effectiveness of the lesson in geometry that was able to foster thinking and problem solving among the students. It is significant that, through the design research/Lesson Study approach, the teachers were able collectively to identify areas of instruction that can be continually improved to encourage students to mathematize. Mathematical discourse, which was given minimal emphasis in the lesson, was identified as one aspect that can be given more emphasis in future research cycles. Through cycles of continual improvement, teacher knowledge in both mathematical content as well as pedagogical content can thus be expanded towards crafting instruction that fosters thinking, discourse and problem solving a reality in the classroom.

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A PROFESSIONAL DEVELOPMENT EXPERIENCE IN GEOMETRY FOR HIGH SCHOOL TEACHERS: INTRODUCING TEACHERS TO GEOMETRY WORKSPACES

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Abstract

This article deals with material designed within the framework of a teacher development project coordinated by the authors and aimed at providing support for in-service secondary school teachers who use or consider using Dynamic Geometry Software (DGS) to develop instructional sequences in mathematics classrooms. The project contains worksheets prepared as text support by authors and included in the Teacher's Guide (Fioriti et al., 2014a, 2014b, 2014c), which was provided to students in the context of a teacher development course. A premise of the course is that problem solving is a vehicle for students to learn mathematics meaningfully. To enable this kind of learning, the classroom should be organized as a learning community, and technology should be incorporated as a tool for expanding mathematical knowledge.

Keywords: Dynamic Geometry Software, Geometry, Secondary Level, Teacher Education, Teaching

Introduction

The importance for teachers to study the tasks that secondary school students will be asked to do, cannot be overemphasized. In this contribution, we present a work plan with a sequence of tasks to be carried out in the professional development of in-service, secondary mathematics teachers. The course introduces the teachers to a theoretical framework, includes tasks teachers should carry out during course sessions, and the didactic analysis of such tasks. The course provides opportunities for teachers to reflect on secondary school students' behavior while learning mathematics and when interacting with other students, teachers, and tasks. The course also provides opportunities for teachers to analyze the characteristics of tasks and activities to ensure that they can enable mathematical thinking by students.

With those aims in mind, we presented teachers with different problems relating to real-life and mathematical contexts. The underlying assumption was that the analysis and resolution of these problems in the teacher education classroom may usher the participants into a geometric working space (Kuzniak & Richard, 2014). This working space consists of interactions among:

1. A real space, as support material, with tangible and concrete objects;

2. A set of artifacts such as drawing instruments or construction software (GeoGebra, in this case);

3. A theoretical reference system based on definitions and properties (here, geometric space and area of 2D figures organized in such a way that teachers can ponder on how secondary students using technology to solve problems might be engaged in creating and validating their knowledge on geometry).

Theoretical Framework

Kuzniak and Richard (2014) point out that the teaching that favors the development of students' mathematical work at school requires a certain organization that the teacher is responsible to generate. Thus, in their professional education, it is important to provide teachers with the following:

- Opportunities for those who teach to be involved in formulating conceptual networks or mental schemes whereby teachers can ratify their beliefs and conceptions, and which can be used in class to allow students to produce their own schemes,
- Support materials with related content to encourage teachers to search for mathematical connections throughout the curriculum design,
- Teacher guides that allow teachers to write comments on the software they select and use in the classroom.

We designed a professional development course based on our own experience teaching with this framework (CEDE, 2015) and included work material in the Teacher´s Guide to be used as support for school texts in secondary school teaching (Fioriti et al., 2014a, 2014b, 2014c). The following principles were used to guide the design of the training course:

-The classroom is regarded as a community for the study of mathematics,

- Problems take place in mathematical contexts or occur in extra mathematical context as a learning engine,

- Conjectures and proofs are constitutive tasks of mathematical activity,

- Construction of models of a situation to be studied is the key in mathematics as it entails abstraction that reduces problems of complex nature to their essential characteristics. Students should identify a set of variables, relate them accordingly and transform those relations using any theoretical-mathematical system to produce new knowledge on the problems under analysis.

These guidelines form the framework of the teachers' professional development and how the sequence of activities and their management have been designed. Students decide how to solve the problems, search for the most relevant relationships between variables, and discuss the strategies used with other classmates. The teacher plays the role of a coordinator who chooses problems, encourages student-student as well as student-teacher interactions, and finally organizes students' ideas into a collective production. A teacher, as a real professional, believes that knowledge is produced as a result of the interaction between the problem and the student's peers (Fioriti, 2017).

The problems and activities proposed for didactic analysis are meant for teachers to debate how to manage the class in order to encourage students to try and produce different solutions, then discuss them, all the while dealing with the conceptual networks that involve the passage from arithmetic to algebra, the use of deductive reasoning as a way of justifying in geometry, and the use of different but equivalent representation systems as some of the activities that students beginning secondary school should do. At the same time, these problems and activities aim to encourage teachers to focus on ways of organizing class interaction and think about the validity, accuracy, clarity, and generalizations of students' mathematical statements.

The incorporation of computers into society has brought about such a cultural change that the way in which we see the world and live in it has changed. In the same way, the incorporation of computers in the classroom requires a cultural change in the way we study and acquire knowledge. This change affects mathematical knowledge in how it is studied as well as the organization and management of classroom instruction. Consequently, the teacher should have the skills to deal with this change (Bifano & Villella, 2012). The inclusion of technology in teaching is inevitable; it provides the opportunity to rethink activities and problems that make knowledge comprehensible, and it makes us aware of the powerful tools that we have at hand.

Given this scenario, the incorporation of technology in different ways (to do mathematics, to expand mathematical culture and, consequently, to expand knowledge) should be analyzed as part of the specialized training teachers acquire during their professional development.

A Management Model for Geometry Instruction

The proposal we have described includes topics of Geometry, which are characterized as the branch of mathematics that according to Villella (2008):

- can be *seen*. Geometric figures can be drawn or constructed using the properties that characterize them. This involves being aware of the difference between a diagram and a figure (Laborde,1998), a situation that requires a didactic examination (Charles-Pézard, Butlen, & Masselot, 2012),
- allows for *play*. The development of concepts at the core of the content networks to be studied through the manipulation of concrete objects gives learning an active, playful quality,
- *best connects to reality*. The 3-dimensional and 2-dimensional models it analyzes can be seen in material objects,

- *applies algebra concepts*. The same language and symbols in algebra are used to name and characterize geometric content,

- *helps to reason*. Its axiomatic structure develops thinking and helps generate the use of deductive reasoning in students (González & Herbst, 2006).

The proposed activities center around the connection between geometry and real life situations, which allows working with models and mathematical problems that require the use of geometric properties to justify the solution found. These activities enable teachers to think about the properties of geometric objects that are studied in secondary school. In this reflective process, a model (a mathematical representation for a non-mathematical object) is built, with theoretical developments whose properties become meaningful in terms of how they relate to the situation that originated them, and properties are studied and geometric objects are characterized according to reasoning and procedures of geometry itself.

The development of geometric concepts is presented in activities with the generic name *study* (Chevallard, 2009). We chose this way of identifying them as we believe the classroom will have the same qualities as a learning community when they are solved. This community is made up of a group of students coordinated by a teacher whose main task is to search for a solution to the problem given. In order to do this, the known data is used together with properties studied before or appearing for the first time, which makes the corpus of the answer discussed in groups. This classroom organization, as well as the use of the study content made in it, creates a particular environment that brings about different kinds of methods, qualities of the models used, and justifications of the steps followed as showed in this example:

A candy factory wants to design a pyramid-shaped wrapper for its products. An employee designed a paper like this one:

Answer:

a- How can you fold the paper in order to obtain a pyramid with a square as base?

b- Is your proposal the only possible one? Why?

Figure 1: An example of a problem

When content is set in this way, problem solvers need to apply the necessary conceptual networks to highlight the underlying geometric property in the problem and explain the resulting family of figures (Ferragina & Lupinacci, 2012). Therefore, the classroom becomes a place where debate, argumentation, and the use of properties to explain decision making are more relevant than using a figure as proof, which is common in secondary school classrooms. In addition, ideas about what steps students need to take and what elements they need to use flow freely. It generates communicative competence in the mathematics classroom since students have to justify elements chosen and steps taken (Villella & Ammann, 2012).

Technology as a Tool for Teaching Geometry: Incorporating DGS

The most basic concepts of geometry taught at school can be described as the combination of their properties with the use of relevant and irrelevant attributes (Vinner, 1982) that characterize them. In this identification or construction of a geometric concept, we can distinguish at least four elements:

1. The image of the concept: It refers to the concept as it appears in the mind of the subject who is studying it. It includes everything related to the concept that comes to mind, everything evoked when the word that names it is heard or when a picture or representation is seen.
2. The definition of the concept: It refers to the verbal form with which a certain notion is expressed (when it exists; it does not always include everything the learner knows about the geometric object in question). This definition is not necessarily mathematical.
3. A group of mental or physical operations, such as certain logical operations, that make a comparison with the mental picture easier.
4. Technology: in a broad sense, it refers to a socio cultural product that is useful as a physical and symbolic tool to relate to and understand the world around us.

The construction of the image of a geometric concept results from a mix of visual and analytical processes that are realized in two directions. On the one hand, there is the interpretation and comprehension of visual models. On the other hand, there is the ability to translate symbolic information into a visual image by using certain technology. The interpretation of the image is the product of visual processes where the irrelevant attributes of the visual component are obtained first and act as a distraction between our internal constructions and what is perceived by the senses (Villella, 2008). We believe that from its own conception, there is a certain technology in geometry that contributes to the definition of the geometric concept.

What aspects are to be considered when the translation to a visual image is made through DGS? In the same way that writing has restructured consciousness and the human mind has generated cognitive operations that had not been developed before it, new technologies transform subjectivity, capacity, and practices. (Evans & Levinson,2009; Rogoff & Lave, 1984; Smolensky & Legendre 2006)

Some teachers believe that with the incorporation of Information and Communication Technology (ICT) at school, there is a risk of limiting teaching. In this specific case, the risk exists if the teaching is limited to what can be seen on the screen: the geometric pictures, the graphic representations of functions, the result of calculations, and so on. In traditional mathematics instruction, where many teachers were and still are trained, it is common to focus on techniques, which usually appear before the problems that make them meaningful or needed. Mathematical software and calculators are tools that solve algorithms effortlessly and in the case of graphs and figures, DGS allows for some properties to be seen. Thus, it is necessary to modify classroom work and start solving problems that will enable students develop three cognitive processes of geometric activity:

1. Visualization, related to the representation of space and support material;

2. Construction, determined by the instruments used (GeoGebra) and geometric configurations;

3. Discursive, aimed at producing arguments and proofs (Kuzniak & Richard, 2014).

To overcome these processes, our teacher development course first provides meaningful concepts and then assigns work on the mathematical techniques.

Just as in oral language it was impossible to manage concepts associated with geometric figures, in written language it is impossible to think of dynamic geometry objects. In a teacher development classroom, this makes a good starting point for a discussion:

- A *technology* for dynamic geometry constitutes a new system of representation of geometric objects when using new ostensive objects: computerized pictures. These pictures differ from the ones made on paper precisely because of their dynamic nature. They can be moved and deformed on the screen while keeping the geometric properties that have been assigned by the construction procedure;
- *A production means* that uses a device (the computer) as a fundamental requirement for its use;
- *A particular language* that integrates not only the language of geometry but its articulation with computer language,
- *A semiotic tool* with particular characteristics that combines different models, particularly the geometry model in the software embedded in the computer language.

Using DGS allows for a new means of producing knowledge, with a specific language that must be known. Learning processes built in this way are encouraged through the design of teaching processes. Listed below are some of the goals students are expected to achieve:

1) *Interpret* the problem posed.
2) *Understand* the given information and establish relationships with the commands in the program.
3) *Formulate and test conjectures* about the concepts being taught.
4) *Design* strategies to confirm or refute conjectures.
5) *Summarize* information given.
6) *Communicate* the result of findings while trying to define what they managed to build.

Teachers' Professional Development

In this section, two activities are provided to exemplify what was described. The first one serves as an example of a model construction, and the second one exemplifies the study of the geometric object from the discipline itself. These activities will be used to describe the management of classroom work as well as the meaning that content is given through the use of technology.

We propose a collaborative task where it is important to consider what a teacher needs to know to develop a successful teaching process in which students gain more understanding about the nature of mathematical knowledge. With the analysis and resolution of this kind of teaching situation developed in the project, teachers are given the opportunity to discuss the different variables they should deal with in order to give students the possibility of reasoning, arguing, making conjectures, refuting, and modeling in order to provide meaning to the mathematical knowledge students are learning. In furtherance of this aim, we selected mathematical content in the specific context in which it would be used. Then, we analyzed the processes involved in teaching it, and made conjectures about how learning would be achieved. The whole procedure makes this mathematical content specialized and limited to teaching professionals. It is included in a sample about mathematics teachers' specialized knowledge MTSK (Muñoz-Catalán, 2015).

Applying mathematics to situations originating outside of mathematics

The following paragraph sets out a situation described as fiction from reality:

A farmer wants to install a water tank to provide water to the main house, the housekeepers´ house and a work shed. The tank should be as close to the main house as possible. However, due to the leafy trees surrounding the house which cannot be moved, the tank can only be installed 500 meters from the house. The idea is to place the tank at the same distance from the housekeepers´ house and the work shed. Where should the tank be erected?

In this example, the first decision to take leading to the solution of the problem above is to construct on the screen representations that model the two conditions set out in the problem:

a) The distance from the tank to the main house must be 500m,

b) The tank must be placed at the same distance from the housekeepers´ house and the work shed.

For that purpose, scales must be used and the points representing each element must be named (Figure2):

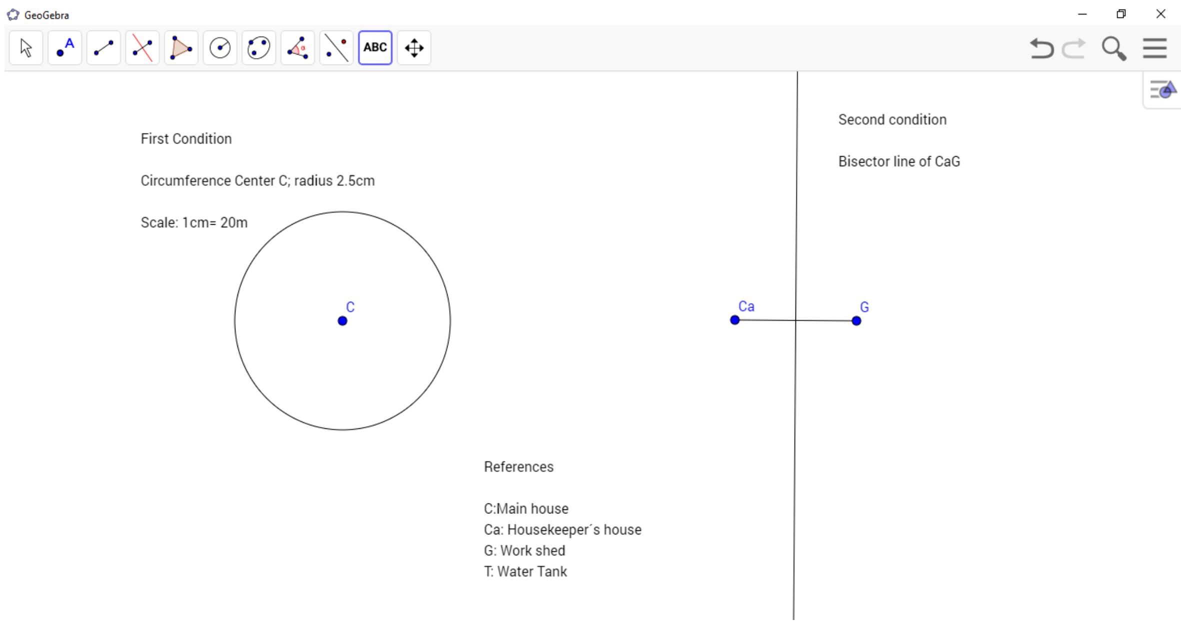


Figure 2: Possible answer attempt

Now, it is possible to establish the construction steps to be followed, what software tools are available, and what hidden conditions are being taken for granted by understanding the logic of the software.

Some teachers' (Tn) responses when they worked on this task:

T1: We need to draw a circumference. The problem says the tank must be placed at the same distance from the housekeepers´ house and the work shed. But, where do we draw the center of the circumference?

T2: Anywhere. The only important information is the radius' length.

T3: But we need to see them on the screen. So…point it near the center of the screen, please.

T1: Ok. Can you remember me the radius' length?

T2: I think it´s 500 m.

T1: So, We will need to use a scale.

T3: 1cm = 20m.Do you agree?

T1: Yes.

T2: Yes, it can be a good one.

T3: Use the command that shows circumferences to draw it…

T1: We need to use the second condition too!

T2: Uhh… You´re right. I´d forgotten it.

T3: Draw this figure near the other one. We´ll be able to compare the two figures all at once.

T1: It´s necessary but we need to use them at the same time in order to find the answer

T3: Uhm…let me see.…

All these activities need to be justified: the circumference has a given center and radius, the segment can have any length, however, the required line can only be its bisector, although the axes system cannot be visualized, the software assumes its orthogonal reference system, etc. The cognitive problem to be solved requires that both conditions be fulfilled simultaneously. The original screen (Figure 2) must be changed, so both conditions can lead to a model upon which conclusions can be drawn. The screen may show several pictures to be analyzed in terms of the dynamic nature of some points or figures. For example, if segment C_aG is moved, a possible figure of analysis is:

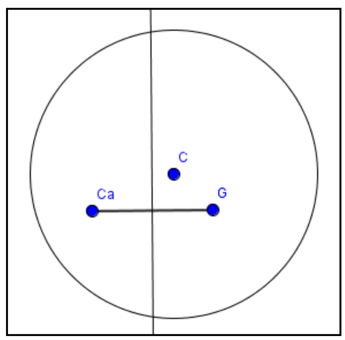


Figure 3: Dynamic study of the figure (case 1)

Some teachers' responses:

T1: This is a good answer (showing Figure 3).

T2: Um…it´s an answer, but not the answer!

T3: What do you mean?

T2: If I move *G* to the right, C_aG changes its length, then the bisectors line change too

T1: - Yes… and if we move C_a we obtain another line, so…

T3: Move them all around the screen, and let me see what happens...

T3: There are many answers…

T1: But…what happens if we choose a bisectors line by *C*?

T2. It´s a particular case.

T3: No, I think it´s the best answer, isn´t it?

Discussions, debates, and arguments based on certain properties arise by analyzing some possible answers to these questions: Does the result reflect the target model? What if the moving figure is another one and the screen obtained is the one below?

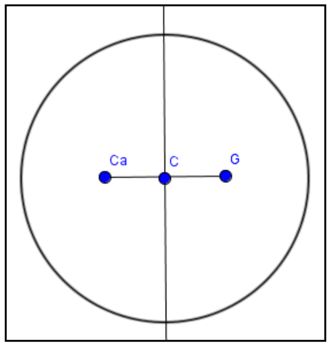


Figure 4: Dynamic study of the figure (case 2)

Answers may vary depending on the problem solver’s perception. The dynamic nature of the point moving throughout the screen and the presence of other many infinite figures may change the answer. However, the logical reasoning leading to such an answer is still valid and so are the conclusions: The circumference of center *C* and radius 5cmrepresents the geometric locus of the points modeling the first condition of the problem, and the C_aG segment bisector is the geometric locus of the points modeling the second condition.

The figure of analysis becomes a knowledge object. This picture is no longer enough to solve the problem since the screen becomes the justification. Thus, the answer can only be found in the properties defining the geometric properties.

Some teachers ‘responses:

T1: There are two conditions and two geometric properties: The circumference and the C_aG segment bisector.

T2: But, we need to use both of them to find the answer.

T3: If this is true, draw the only figure that uses both geometric properties.…

T1: Umm..Another problem. There are two points of intersection that satisfy both geometric properties.

T2: We need to study which of them is the appropriate one.

T3: I think both of them.

T1: Why?

T3: I can see it in the screen.

T2: No, it´s not enough.

T1: We need to justify… properties!, properties!…

The study of the teachers´ answers led to the construction of a model fulfilling both conditions. Such model being the one showing the intersection of both geometric loci requires another decision to be made: Which of the intersection points P_1 or P_2 will be considered point *T* (tank location)? Is it necessary to make this decision? Is it required by the formulation of the situation that gave rise to this study?

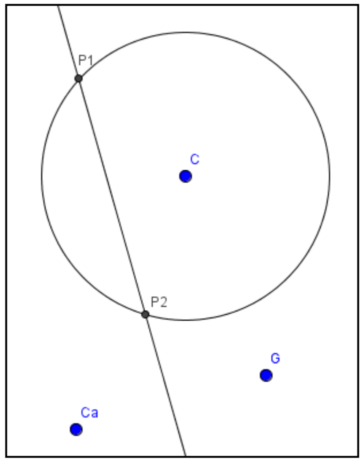


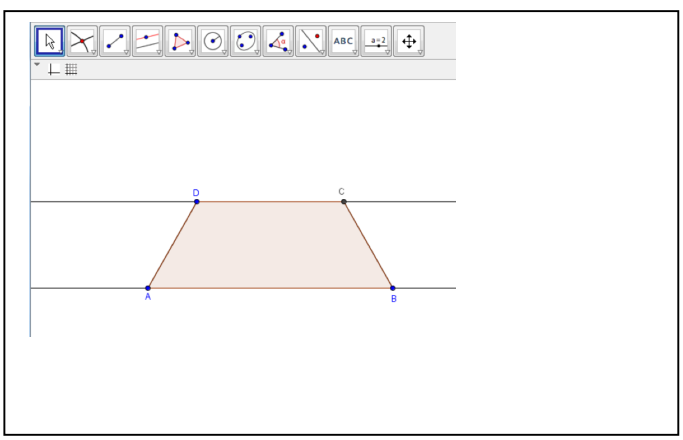
Figure 5: Dynamic study of the figure (case 3)

With these new questions, other questions arise which support the didactical analysis and make up the specific knowledge teachers must acquire as part of their training.

A situation within mathematics: Study of the geometric figure

In order to study a property of the isosceles trapezoid, we introduced the following activity:

In the GeoGebra screen below there is a trapezoid, which, by construction, is isosceles.



a) Determine the ratio between the areas of triangles *DAB* and *ACB*. Justify your answer.

b) If the trapezoid *ABCD* were not isosceles, would the answer to the question above still be valid? Explain why.

The first decision to make when solving the problem is to reproduce the figure on the screen, so both triangles can be seen (Figure6):

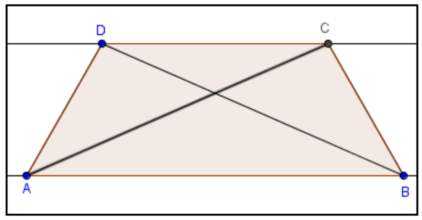


Figure 6: Reproduction of the trapezoid with the triangles drawn

Some teachers' responses:

T1: We need to reproduce the screen figure. This is an isosceles trapezoid, so we need the length of segment *DA* to be the same length of segment *CB*.

T2: Use circumferences!!

T1: Perhaps another tool is available. Let´s explore the tool bar.

T2: Yes...

T1: This is an isosceles trapezoid (showing Figure6 without the triangles).

T2: The problem says: "triangle *DAB* and *ACD*." Draw them, please.

T1: Here they are (showing Figure6).

T2: We need to determine the ratio between their areas. We need to calculate each one. So, base multiplies height and then we divide …

T1: Yes…but we don´t know the measurement and, if we move the baselines of the trapezium they will change. So, it´s not easy…

T2: Let me think…

The above dialogue leads to establishing the steps that must be followed in the construction and their pertinent justification: base lines are parallel; sides *DA* and *CB* have the same measurement. The cognitive problem lies in the lengths of *DA* and *CB* and in the area of *ABD* and *ABC*: They that are not measured directly and are visually considered equal. The solution entails designing a task that involves conceptual networks already studied: triangle height, bases, and similarities. In this case, both triangles have a common side (*AB*), and they both have the same height. The ratio between the areas is 1 as the areas are equal. Once question (b) is answered, the screen shows different pictures to be analyzed in terms of the dynamic nature of some of its vertices. For example, if vertex *A* is moved, possible figures of analysis (height is marked in dotted lines) are shown below (Figure 7):

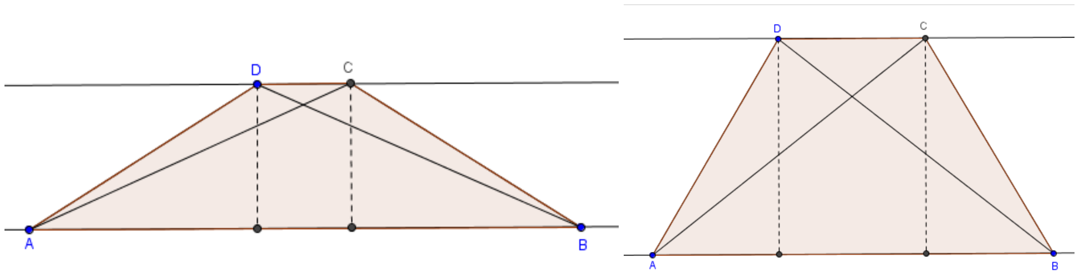


Figure 7: Dynamic modification of the figure

Some teachers' responses:

T1: The areas are equal. The ratio between them is one.

C (Coach): Why?

T2: We used properties!

C: Which ones?…

C: It´s OK. But, what happens if you move vertex *A*?

T2: Nothing!! The trapezoid is always isosceles.

C: Move it.

T2:I see it on the screen!

T1: Stop! You are changing the baseline length.

T2: But not the height.

C: So…

T2: Nothing happens.

C: The ratio between the areas doesn´t change, does it?

T1: Let me think, please.

C: OK.

T1: And if I move vertex *C*…

T2: It´ll be the same.

C: Try.

T1: It´s not the same.

T2: I agree…I need more properties!!

The questions arising from the analysis of these figures are: Does the ratio between the areas change because the shape also changes? If the moving point is a different one, could another figure be obtained? Once more, the presence of many other infinite figures may change the answer. However, the logical reasoning leading to such an answer is still valid and so is the conclusion: The triangles have the same area. The figure to be analyzed is not enough to solve the problem since the screen becomes the justification; thus the answer can only be found in the properties defining the area and the triangle similarities.

If we compare the areas of triangles *AOD* and *COB*, with *O* the point of intersection of the diagonals, can we reach the same conclusion? Upon exploring the figures obtained when *O* is located in different places on the screen, the shape of the figure changes but the ratio of the areas is the same. *AOB* is part of the two triangles compared in the original problem, by subtracting it from the new triangles to be compared, "the same area" is subtracted; thus such areas are equal. Now, we may wonder: What properties are brought into play if we compare triangles with similar areas but with different bases and height?

When considering the problem, teachers may raise doubts about the mathematical knowledge they think they possess after analyzing the ratio between areas not measured directly and studying the ratio reaction after obtaining different figures of analysis. Furthermore, the need to use properties that go beyond what is seen on the screen challenges the knowledge teachers have on geometric structure, and sets in motion a more active way of solving mathematical problems.

An analysis of specialized content knowledge for teaching (Muñoz-Catalán, 2015) prompts the assumptions teachers make about the way geometry is taught and learned. In our case, we add the use of DGS, which besides adding dynamism to answers leads to a series of assumptions regarding the geometric object of study that need to be confirmed. The problem described above is meant to analyze the specific knowledge about geometry each teacher has, considering what each teacher knows about Geometry, and the specialized content knowledge for teaching (SCK; Ball & Bass, 2009) each teacher has acquired. This allows teachers to interrelate content, to weigh student reasoning and mathematical solutions, and to recognize the validity of the arguments that may arise.

Conclusions

In our proposal, the mathematical content to be learned includes problems from two different work contexts: the modeling of a real situation that requires the construction, study and analysis of a model so that the conclusions drawn may be applied to solving the situation from which it originated, and the study of figures within mathematics where the use of properties and the construction using valid reasoning lead to the targeted solution.

The presentation of these two different types of problems in the teacher training classroom is relevant for teachers as it allows them to study the underlying structure of geometric working spaces: An epistemological level, linked to mathematical content, and a cognitive level, linked to visualization, construction, and proof. In order to articulate these two levels and obtain sound mathematical work, we propose discussing with teachers the development of figural genesis, relating space and figures (epistemological level) with visualization (cognitive level), instrumental genesis, relating artifacts (DGS, paper and pencil, etc.) from the epistemological level with construction (cognitive level), and discursive genesis, relating the reference framework (epistemological level) with proof (cognitive level; see Kuzniak & Richard,2014).

For teachers, this articulation into two levels includes a wide range of teaching situations that lead to the development of a mathematical work space inside the classroom and the use of a learning community. Our interest in the use of DGS lies in its capacity to support the discussion with teachers about the acquisition and construction of geometric knowledge in the secondary school classroom.

In addition to the specific knowledge of teachers, the work proposed supports reflection about the mathematical performance of secondary school students at the moment of studying and how they solve specific geometric situations. Some of the points discussed with teachers include how students design models, use metaphors to communicate findings, and organize explanations and reports to communicate discoveries and verifications. Other times, the points discussed were how students design strategies to find solutions justifying the procedure used, select material, spend time, appreciate both their own and their classmates' performance, accept mistakes, and correct the models used. It is important to analyze how students transfer the knowledge acquired to other learning contexts analyzing the wrong ideas acquired from the physical representation of objects, realize the double status of geometric objects, since the drawing of an object is sometimes considered the object itself, and the need of a description characterizing the object with the purpose of removing any ambiguity related to its representation.

The management of instruction—the design, performance, assessment, and generalization of teaching strategies performed by the teacher—leads to a process of negotiating the interests of students and teachers, where teachers act as stewards of a learning environment. The interests of students are based on meaningful content to be developed and on the naturalization of the use of DGS in the world of mathematics instruction. The interests of the teachers are based on the epistemology of the given content. In this negotiation, teachers act as natural mediators between the content and students; while teachers design and pose problems to be solved, students develop strategies to solve such problems where both teachers and students are part of a classroom project.

Regarding mathematics in secondary school, the use of DGS generates several ways to introduce tests as an unavoidable element of conceptual networks that are essential to the learning process. Teachers can suggest situations for graphic and dynamic research for students to analyze the behavior of geometric objects and the relations among them and thus, understand mathematical concepts and procedures, to justify and to do some more formal tests. DGS helps teachers lead a learning process by dealing with contradictions and causing students to learn about the formal demonstration process, explain why a result is mathematically true, communicate mathematical relations and properties used and discover by manipulating dynamic objects develop logical and abstract thinking, systematize by organizing results into a deductive system of axioms and theorems and to discover and construct mathematical knowledge.

In our proposal, the technological tool is used as a means to explore different types of graphic representations interactively. Thus, geometric objects can be constructed out of a variety of primitive objects (points, segments, lines, etc.) in this creative environment thought by the mathematics teacher as a professional.

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DEVELOPMENT OF SPATIAL ABILITY:
RESULTS FROM THE RESEARCH PROJECT GEODIKON

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This chapter discusses results from GeodiKon, a research project that analyzed the spatial ability of 903 students with the aim to find out whether or not training in each factor of spatial ability and its repertoire of strategies to solve spatial tasks would lead to an improvement in an individual's spatial ability. The chapter focuses on the findings regarding the use of the different strategies, the promising strategies for solving spatial tasks, gender-specific results, the results of the Spatial Orientation Test (SOT), and the connection between the individual's sport/leisurely time activities and spatial ability. Finally, the chapter offers suggestions for mathematics and geometry education based on its findings.

Keywords: Factors of spatial ability, gender, mental rotation, spatial ability, spatial ability tests, spatial orientation, spatial relations, strategies, visualization

The Research Question and the Aims of the Project

The research project GeodiKon was funded by the Austrian Ministry for Education and the Salzburg University of Education. The Austrian project team includes members from eight Universities and Universities of Education. The project investigates supports for and development of the factors of spatial ability, and the deliberate training of different strategies for solving spatial tasks. The underlying hypothesis of the project is that training (making aware, categorizing, internalizing) each factor of spatial ability and training in a repertoire of strategies for solving spatial tasks will lead to an improvement of spatial ability.

The major aims of the project are:

To develop **specific learning material** for the training of the four factors of spatial ability: visualization, spatial relations, mental rotation, and spatial orientation (Linn & Petersen, 1985; Maier, 1994; Maresch 2014b; Thurstone, 1950) so as to create balanced and extensive developmental materials for learners

To produce a **user-friendly book** with all the learning material from the project (Maresch et al., 2016; Maresch & Scheiber, 2017) to train teachers and lecturers on how to use the material in classes as well as disseminating the project's results in conference presentations and papers (further details in regards to tasks, materials, papers, and talks: www.geometriedidaktik.at).

To build a contemporary **model of the factors of spatial ability**. Which of the large number of existing psychological models for spatial ability should be taken as the scientific basis for this project? During the factorial phase of spatial ability research (Maresch, 2014b) between 1950 and 1994 many psychometric factor based models of spatial ability were described (e.g., from Thurstone, 1950; French, 1951; Guilford, 1956; Rost, 1977; Lohman, 1979; McGee, 1979; Linn & Petersen, 1985; Lohmann, 1988; Carroll, 1993; Maier, 1994). Maier's (1994) approach was formulated as an aggregation of the models existing at that time. Maier (1994) took Thurstone's (1950) model with the three factors of visualization, spatial relations, and spatial orientation as the basis of his approach. Linn and Petersen’s (1985) model of the three factors of visualization, spatial perception, and mental rotation turned out to be “an outstanding supplement” (Maier, 1994) to the first model. Maier (1994) combined these two models and formulated his approach which finally consisted of the five factors of visualization, spatial perception, spatial relation, mental rotation, and spatial orientation. Detailed analyses of Maier’s approach showed that the four factors of visualization, spatial relation, mental rotation, and spatial orientation had also been formulated in other researchers’ models (Maresch, 2014b). The factor of spatial perception was only included in Linn and Petersen’s (1985) model. The description of this factor according to Linn and Petersen (1985) defines the factor of spatial perception as the ability to identify the horizontal and the vertical. This very specific ability is considered to be an integrative part of the spatial orientation factor of Thurstone (1950). Thus we no longer consider the factor of spatial perception as a discrete factor. So Maier's (1994) approach – but without the factor spatial perception – was taken as the scientific basis for the development of the learning materials and the test battery in the project GeodiKon. The factor-based model of spatial ability for the project GeodiKon contains the four factors (Maresch, 2015):

- Visualization
- Spatial Relation
- Mental Rotation
- Spatial Orientation

Development of a structured model of strategies. One of the challenges with classical spatial ability tests is: "The classical factor-analytical-psychometric research perspective requires implicitly that all tasks on spatial ability can be solved by individuals or subjects using the same solving strategy" (Gruessing, 2002). The assumption that there is a consistent and homogeneous strategy for finding solutions of tasks had to be abandoned because of inter-individual varying solving strategies and intra-individual change of strategies (Souvignier, 2000). Because of the diverse strategies used by the individuals, there are highly reciprocal effects and dependencies between the diverse factors of spatial ability (Maier 1994). Such findings indicate that in some cases, intended solution strategies are hardly used at all (Maier 1994). To quote Lohmann (1979): “One of the major problems is that tests are solved in different ways by different subjects. Subjects change their solution strategies with practice or when item difficulty increases" (p. 174). Because of such findings, the analysis of factors became of decreasing importance. Souvignier (2000) pointedly stated that the interpretation of factors was based solely on the description of test requirements with great emphasis on the factors, and that therefore their corresponding definitions represent only an abstract list of test procedures in the respective analyses.

Emphasis of spatial ability research is now increasingly placed on the identification and description of the solution strategies used. It is asserted that conventional alternative solution strategies […] should be regarded with due attention (Maier, 1994), or that especially the strategies used should be the focus of interest (Gruessing, 2002), and it is also stated that the flexible use of strategies or the use of one adequate strategy – depending on the task – forms an important aspect in gaining optimal test results (Glueck, 2005).

The analysis of current studies on strategies showed that four pairs of solution strategies (Figure 1) could be identified. The four pairs of strategies, formulated and explained below, are not claimed to be a complete set. The majority of publications, however, acknowledge these four pairs of strategies or parts thereof as the relevant strategies. Examples of spatial ability solution strategies are found in publications. Key features strategies move-object, and move-self strategies are featured in Barrat's (1953) work. Just and Carpenter (1985) found mental rotation around the global coordinate system, mental rotation around a user coordinate system, comparing the characteristics of objects with another, and change of perspective strategies. Duenser (2005) wrote about moving oneself or moving the object, concentration on details or the whole, and reflection and visualization. And Schultz (1991) documented mental rotation, perspective-change, and analytic strategies. In addition to the four pairs of strategies which are described below in Figure 1, there are further terms frequently formulated: avoidance strategies, complementary strategies, mixed strategies, verbal-analytical strategies, and logical consequential thinking (Maier, 1994; Gruessing, 2002; Souvignier, 2000). After close analysis, these strategies can be regarded as parts of one of the pairs of strategies.

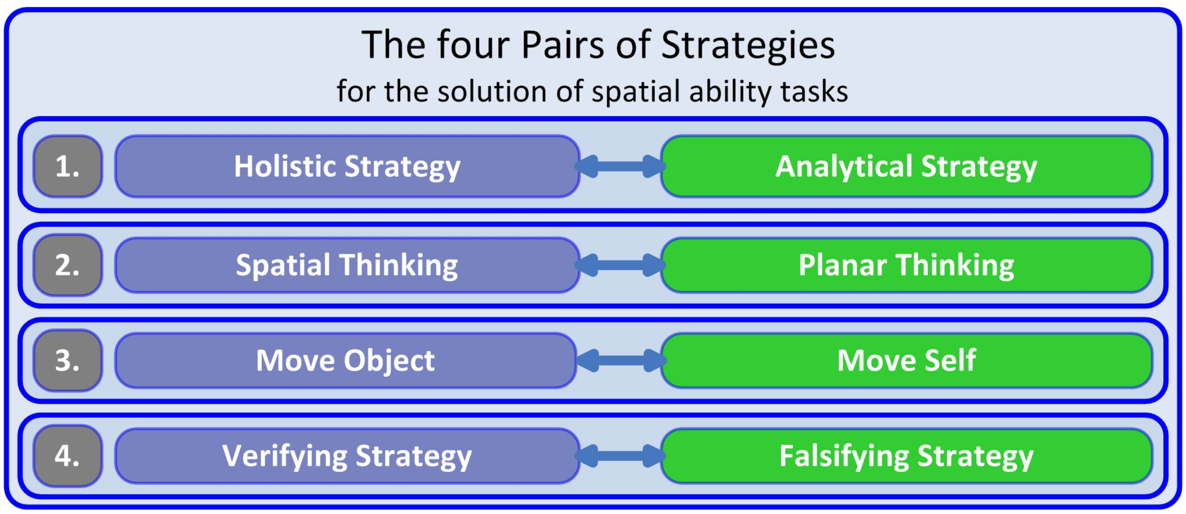

Figure 1: The model of the four pairs of strategies
for the solution of spatial ability tasks (Maresch, 2014a)

The individual pairs of spatial ability solution strategies form dialectical pairs. In tests, geometrical objects are generally comprehended either holistically or analytically. Individuals either construct a mental spatial model of the objects depicted (spatial strategy) or they just see a planar image of the object (planar thinking). When solving spatial ability tasks, individuals often position themselves outside the scene. Conversely, some individuals – particularly in tasks of spatial orientation – put themselves into the proposed setting and mentally move around the objects. Individuals, in general, prefer verifying and falsifying solutions in solving the given tasks. If there are several acceptable solutions, they either try to find the right solution straight away or exclude false solutions one by one until only one solution is left as the correct one.

The four pairs of strategies are not independent of one another. Numerous studies in the literature identify crosslinks between the diverse eight strategies mentioned. Individuals using the holistic approach tend to think spatially (Kaufmann, 2008). Females tend more frequently to use analytical solution processes, whereas males prefer to use holistic processes (Glueck, 2005). The strategies individuals use for solving spatial ability tasks depend on intrapersonal preference, size of the individual strategy repertoire, type of task, level of difficulty and complexity of the task, and individual experience in solving similar and related tasks (Souvignier, 2000; Gruessing, 2002; Kaufmann, 2008).

With tasks of high complexity, strategies are used to reduce task difficulty. With challenging tasks, complementary and avoiding strategies are used, requiring a less challenging spatial-visual cognitive demand and thereby enabling a more successful handling of the task (Maier, 1994, p. 69). Complementary and avoiding strategies can be the following: logical thinking, verbal-analytical strategies, the use of several strategies in solving a task, change of strategies within parts of the task, concentrating on parts instead of the whole setting, or also the reduction from three to two dimensions. Several strategies are often used within one task. Therefore, it seems to be of particular importance that students have a wide range of strategies in order to be able to choose the optimal strategy suiting the situation. Lohmann (1988) states that individuals use all the strategies at their disposal in spatial ability tasks. Glueck and Vitouch (2008) found that the range of strategies and the flexibility in adapting them to the requirements of the task is more relevant than basic cognitive processes. The phenomenon of strategy changes within a task occurs more often in complex than in simple tasks.

Thinking about one or more changes of strategy within a task on the one hand requires the individuals to have command of a broad spectrum of strategies, but it also compels the test person to adopt meta cognitive processes. The choice of the best possible strategy to solve a task in a specific situation requires reflection, calculation and decision-making at a higher level. (cf. Kaufmann, 2008). For these reasons, identifying a model of strategies is important to this study's findings.

The Tests and Questions

In the pre-tests and the post-tests, we used four spatial ability tests (Three Dimensional Cube Test (3DW; Gittler, 1984), Differential Aptitude Test (DAT; Bennett, Seashore, & Wesman, 1973), Mental Rotation Test (MRT; Peters, Laeng, Latham, Jackson, Zaiyouna, & Richardson, 1995) and Spatial Orientation Test (SOT; Hegarty & Waller, 2004). We asked additional questions such as which strategies students used to solve spatial tasks, age, gender, computer usage, leisure activities, school marks in Mathematics, German, and English, and learning style. The allocated time for the pre-tests was 85 minutes and for the post-tests 77 minutes.

We wanted to know which strategies individuals used to solve the tasks on the four spatial ability tests. So after each of the four tests the students once again got one of the tasks, which was arbitrarily chosen. When the students solved the task, they were asked to observe themselves accurately with which spatial strategy they solved the task. Then, students answered questions concerning the different strategies they used from the model of the four pairs of strategies – each in an eight-part scale (Figure 2). The 13-year-old students appeared to have no problems self-reporting with which strategy they solved the different tasks.

Looked at the object in its entirety (whole approach – holistic strategy):		**Looked at parts of the object (part approach – analytic strategy):**
You looked at the whole object. You did not concentrate on parts of the object only. You visualised the whole object and found the solution right away.		You concentrated on parts of the object only. You did not have to use the whole object for the solving process.
1.) holistic	□ □ □ □ □ □ □ □	**analytic**
Spatial thinking:		**Planar thinking:**
You created a mental, three-dimensional model of the object and solved the task by working on this mental model.		You saw a planar (two-dimensional) image and solved the task by working with this planar image.
2.) spatial	□ □ □ □ □ □ □ □	**planar**
Move self:		**Move object:**
You placed yourself inside the setting and moved around mentally and changed your perspective.		You positioned yourself mentally as an observer outside the setting and moved (rotated, translated, …) the individual objects.
3.) move self	□ □ □ □ □ □ □ □	**move object**
Falsifying strategy:		**Verifying strategy:**
You identified all the incorrect solutions first and excluded them step by step.		You had the correct answer in mind and worked on it directly.
4.) falsifying	□ □ □ □ □ □ □ □	**verifying**

Figure 2: The questions for students concerning the four pairs of solving strategies for spatial tasks

To support better understanding and traceability of the results of the project, the four spatial ability tests we used are explained as follows. Each of the test addresses specific factors of spatial ability. The Three-Dimensional Cube Test (3DW) addresses visualization factor; the Differential Aptitude Test (DAT) the visualization and spatial relations factors. The Mental Rotation Test (MRT) focuses on the mental rotation factor, and finally, the Spatial Orientation Test (SOT) addresses the spatial orientation factor. These classifications had been specified in the best possible way. They do not raise the claim to be fully selective and accurate. Being fully selective and accurate is not the main point because the analysis will not go into detail of the varying improvements of the four factors. In the following sections is a fuller explanation of the selected tests.

Three-Dimensional Cube Test (3DW)

This test investigates whether any one of the six cubes A, B, C, D, E or F is exactly the same as the given cube X or whether the right answer is G (no cube matches; German: kein Würfel richtig). If individuals did not know the solution, they had to choose H (I do not know the answer; German: ich weiß nicht). Each pattern at the side faces of the cube occurs only at one side face. Thus, each side face has a different pattern. The test author, G. Gittler, (1984) provided a special version of the 3DW-test for this project with 13 tasks (Figure 3). The first one is a hidden warm-up task and is not being counted. The test lasts for 15 minutes. You can find an example of the test online at Gittler & Glueck (1998).

Differential Aptitude Test (DAT)

The tasks of this test, created by Bennet, Seashore, and Wesman (1973), consist of handling folding nets with shades and patterns. The templates can be folded to three dimensional objects. Each task shows one folding template and four three dimensional objects. Individuals have to choose which of these three-dimensional objects A, B, C, or D can be made by folding the template provided (Figure 4). The test consists of 15 tasks and lasts for 8 minutes. For each task, exactly one answer is correct. You can find an example of the test online at https://www.researchgate.net/figure/268982370_fig2_Figure-2-Differential-Aptitude-Test-Space-Relations-DATSR-example-problem-Bennett.

Mental Rotation Test (MRT)

In the test created by Peters et al. (1995), an object is presented on the left. The individuals have to determine which two of the four sample stimuli A, B, C, and D on the right are rotated versions of the target stimulus (Peters et al., 1995). A task is solved correctly if both correct answers are marked (Figure 5). Only then the individual gets one point. The test consists of 24 tasks and lasts for 6 minutes. You can find an example of the test online at Titze, Heil, and Jansen (2008)

Spatial Orientation Test (SOT)

Hegarty and Waller's (2004) test is on one's ability to imagine different perspectives or orientations in space. In each task, one can see a picture of an array of objects. For each task, there is what could be called an *arrow circle* along with a question about the direction between some of the objects. For each task, one needs to imagine oneself standing next to one object in the array (which is placed in the center of the circle) and facing another object, placed at the top of the circle. The task is to draw an arrow from the center object showing the direction to a third object from this facing orientation (Kozhevnikov & Hegarty, 2004). In this test, no points are awarded for each answer; instead, in each task, the deviation angle from the correct answer is measured. The angle is measured without regard for orientation, so therefore, all the deviation angles are in the range between 0° and 180° **(**Figure 6**)**. The score on the SOT for each individual is the arithmetic mean of deviation angles. The SOT consists of 12 tasks and lasts for 8 minutes. You can find download the test at http://spatiallearning.org/resource-info/Spatial_Ability_Tests/PTSOT.pdf

Description of the Study

The project was carried out in a pre-test/post-test-design. During the project's first phase, from January until September 2013, the project team compiled learning material for 12 weeks of lessons in geometry and mathematics. In Austria most students have both subjects: geometry and mathematics. The learning material contains specific spatial ability tasks to train students in the four factors of spatial ability and the different strategies for solving spatial tasks. The structured model of the four pairs of strategies for the solution of spatial tasks was developed and the tests and questionnaires were set up. Pre-tests were given in September and October of 2013. Immediately after the pre-tests, the twelve-week long learning phase began for the treatment groups. Post-tests took place in all the school classrooms in January and February of 2014. From March until November 2014, the research team digitized, prepared, and analysed the collected data, and compiled the user-friendly book with all the special learning material (Maresch et al., 2016) as described earlier. The team trained teachers and lecturers on how to use the material in classes, and disseminated results of the project in conference presentations and papers.

The participants of this study came from 46 classes from the Austrian provinces of Salzburg, Styria, and Lower Austria, totalling 903 students in ages ranging between 12 and 14 years old from various types of secondary schools: Hauptschule (HS), Neue Mittelschule (NMS), Bundesrealgymnasium (BRG), and Bundesgymnasium (BG). A digital newsletter served as the invitation to participate in the study, which was sent out to 2,260 teachers (606 at BG/BRG and 1,654 at HS/NMS). This newsletter periodically addresses geometry teachers in the German speaking area (mainly Austria). Originally, the project was designed for 10 classes. Because of the great interest (96 teachers and their classes), we accepted 46 classes to take part in the project. The project focused on selecting its participants from the three provinces' residents and aimed for a balanced distribution of individuals across sex, age, school type, and rural and city schools. Province coordinators supervised all the pre-tests and posts-test, working with the same time schedule for the test. Two coordinators oversaw the 12 project classes of Styria; one coordinator oversaw the 12 project classes in Salzburg, and two coordinators oversaw the 22 project classes in Lower Austria. We had 39 classes, were students worked with the specific learning material and got information about strategies for solving geometry tasks, and we had 9 control classes, were students had no additional material or information about strategies. They had "just" their usual lessons.

All the teachers in project classes participated in training sessions where they learned to work with the learning materials (Figure 3) and provide information about the different strategies to solve spatial tasks to the students. The sessions were organised to make sure that all the classes would work in (nearly) the same way during the 12 weeks of the treatment.

Students' usual schedule for "Geometrisches Zeichnen" (Descriptive Geometry for Lower Secondary Schools) allocates 1 hour a week for this class. For half of each treatment lesson the project classes worked with the special learning materials. During the second half, of each lesson, the teachers worked with their classes on materials unrelated to the project. In the treatment part of the lessons, students had to solve about four to six tasks in the given time (25 minutes). The learning material's tasks were set to train students on all of the four factors of spatial ability in a well-balanced way. Every week, students had to solve one to two tasks for every factor.

Before the treatment period, all students took the pre-tests. Students then took the post-test after the treatment. After the post-tests, all the data were aggregated and differences in the performances of the students were analysed. According to the classification of spatial training studies by Newcombe et al. (2002), GeodiKon was set as a general training study as well as a long duration study because it lasted for at least a semester.

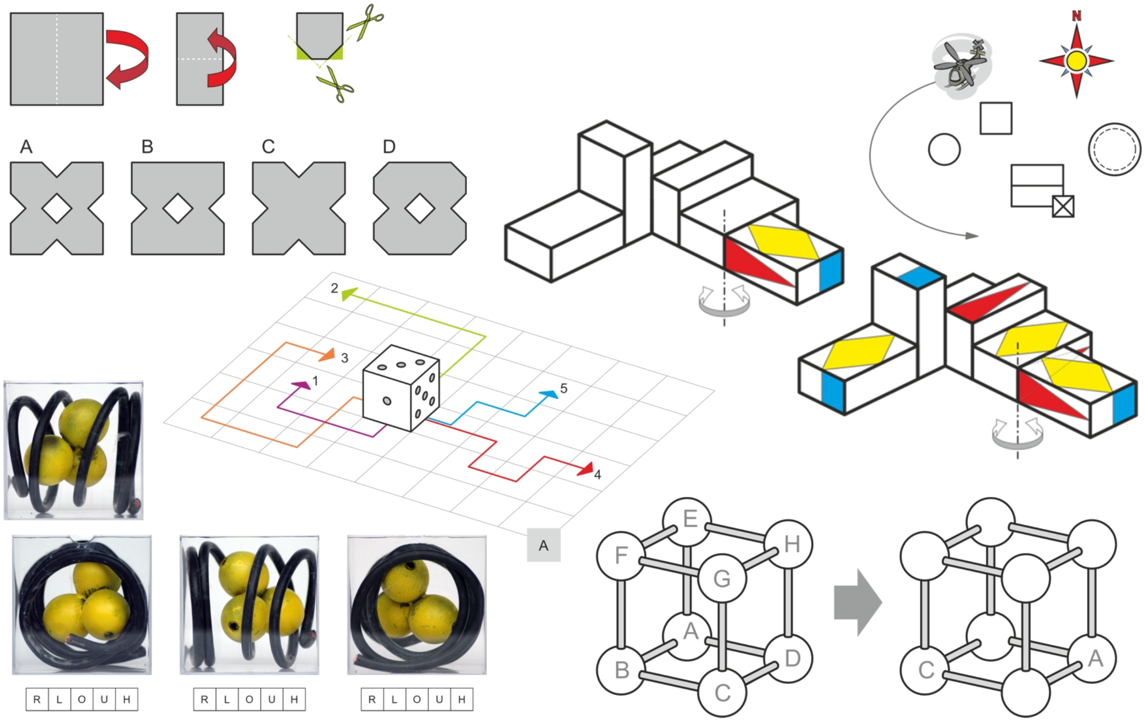

Figure 3: Some images of the learning material

Results

This section describes gender-specific results, findings regarding the use of different strategies for solving spatial tasks, promising strategies for solving spatial tasks, results of the SOT, and connections between sport/leisure time activities and spatial ability.

Gender Differences

The analysis of the project data of the groups who worked with the learning material showed clearly that female and male students have different basic strengths regarding the factors of spatial ability.

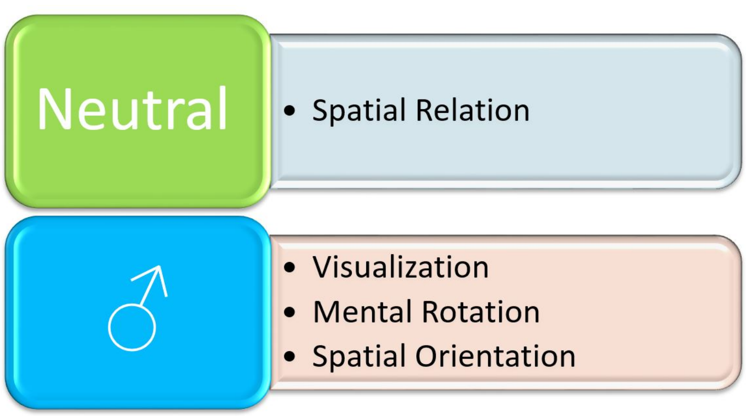

Figure 4: Different basic strength of female and male students in regard to spatial ability

The pre-test results show that male students have greater basic strengths in the factors visualization, mental rotation and spatial orientation. The factor spatial relation is gender neutral (Figure 4).

The difference between the pre-test and post-test results show that female and male students have different growth potential regarding the factors of spatial ability. Female students have a greater growth potential in the three factors of visualization, spatial relations and mental rotation. Male students have a greater growth potential in the factor of spatial orientation (Figure 5).

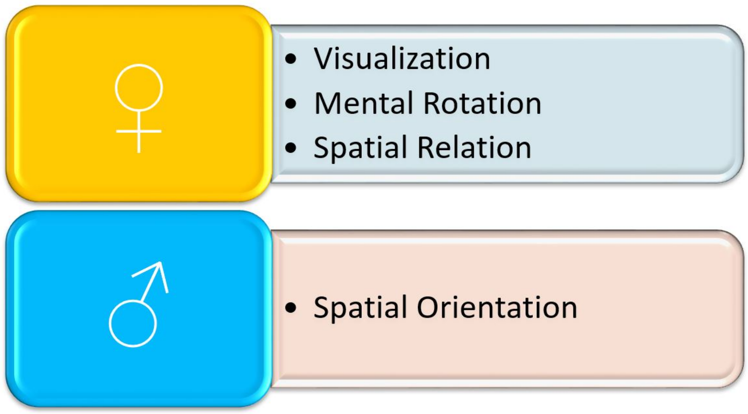

Figure 5: Different growth potential of female and male students in regards to spatial ability

Change of Strategies from the Pre-Tests to the Post-Tests

The focus of these analyses was to determine how students changed their strategies from the pre-tests to the post-tests.

A highly significant change in strategies used in the 3DW-Test was evidenced ($F_{4;\,694} = 12.026$; $p < 0.001$). In the post-tests, the students used the holistic strategy and the move-object strategy much more. Also, there was a highly significant change in strategies students used on the DAT ($F_{4;\,682} = 13.491$; $p < 0.001$). We can see that the individuals more often used the holistic strategy and the move-object strategy in the post-tests. As in both tests above, we found in the MRT a highly significant change in strategies ($F_{4;\,706} = 11.497$; $p < 0.001$). Here, the students changed from the move self strategy in the pre-tests to the move object strategy in the post-tests. Finally, in the SOT, we found a highly significant change in strategies ($F_{4;\,673} = 3.518$; $p = 0.007$). Individuals more often used the analytic strategy and the planar strategy in the post-tests (Svecnik, 2014).

Do promising strategies for solving spatial tasks exist?

To investigate the influence of the types of strategies used by students in solving the test questions, we used regression models where gender, school type, school level and all the items of the strategy questions were included. We found that analytical strategy and spatial strategy were used in the 3DW-Test and in the DAT. In contrast, we found that other strategies seemed to be more promising (holistic strategy, spatial strategy, and move object strategy).

Results of the Spatial Orientation Test (SOT)

In the SOT, we see that the performance of the 12-year old and 14-year old students is lower (average error angle of 59.04°) than the performance of 17 years old students (average error angle of 30°) (Duenser, 2005).

We analyzed the hypothesis that the absolute angular error increases with the angular deviation of one's imagined heading (perspective) from the orientation of the array (Figure 6). Figure 6 shows that the absolute angular error increases with the angular deviation of one's imagined heading from the original orientation of the array. This result confirms that of Kozhevnikov and Hegarty's (2001).

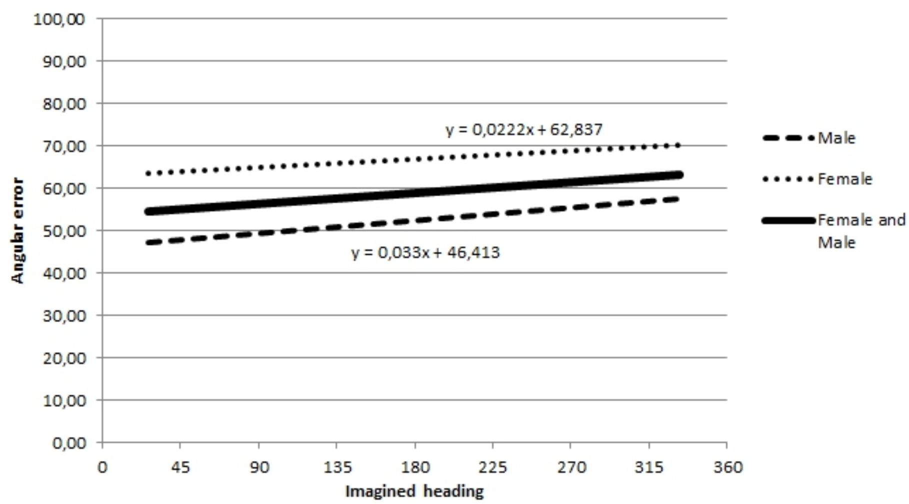

Figure 6: Absolute angular error increase with the angular deviation of one's imagined heading (horizontal axis) from the orientation of the array (vertical axis)

Because of the challenge in analyzing the SOT's data and the wish to provide meaningful feedback on its results, we developed a new method to analyze the SOT (Maresch, 2016). The new method is called the "differentiated presentation and feedback method" (DIAM). This method's core is the fact that students solve the SOT in two different steps. Step one is to locate the solution angle in the correct quadrant/semicircle, and step two is to place the best possible solution angle. DIAM provides two kinds of results. The first result is information if individuals draw their solution in the correct quadrant or had a left/right error or had a front/back error or both errors. Its second result is if the individual drew the solution in the correct quadrant and gave the information about the error angle. Thus, DIAM provides enough information for researchers to make a more detailed analysis of the SOT's results, and it offers a differentiated and therefore helpful feedback for individuals (Maresch, 2016).

Leisure time activities and spatial ability

During the pre-tests and post-tests, we asked the students about leisure time activities. All students got a list of 25 sport activities (soccer, tennis, swim, dance, …) and other leisure time activities (handcraft work, pottery, sewing, …). The question asked if the students participated in any of the activities of the given list. If the answer was "yes," then the question asked how often she/he participated in the activity. The data analyses provide a clear indication of significant gender difference (Table 1). If boys participated in technical drawing, or model making or/and construction toys (like Lego or Geomag) they had significant higher spatial abilities then other boys. Girls had significantly higher spatial abilities than other same-aged girls if they worked with construction toys (like Lego or Geomag) and puzzles.

Table 1: Sport and leisure time activities which showed significant results in regard to spatial abilities

	3DW-Pretest	3DW-Posttest	MRT-Pretest	MRT-Posttest	DAT-Pretest	DAT-Posttest	SOT-Pretest	SOT-Posttest
Male Students								
Technical Drawing	*	*	*	*	*	*	*	
Model Making			*	*		*		
Construction Toys	*	*		*	*	*		*
Female Students								
Construction Toys		*			*	*		
Puzzles	*	*		*	*	*		

Discussion and Prospects

It is remarkable that even during the very short treatment phase of 12 weeks students in all four groups (test group and control group) showed highly significant and substantial increase of performance in all four spatial ability tests. Many factors might be responsible for this trend: learning effects due to test repetition, maturation process effects, development process effects, treatment effects, and combinations of these effects. The highly significant and substantial increase of performance could be a verification of Thurstone's (1955) research. He had argued that children between 5 and 14 years of age show a very high potential for the development of their spatial ability (Figure 7). This project and Thurstone's (1955) work imply we should put in more effort to train, support, and encourage spatial ability in school from the very beginning (age of 5 or 6 years) up to 14 years.

It can be noted that those groups who have spatial treatment performed much better than the control group on each of the four spatial ability tests used in the project. In two tests (3DW-Test and MRT), the students in the spatial treatment had a significantly higher performance than the students of the control group.

It should be noted that the four spatial ability tests that were used in the project are "classical" paper-pencil-tests. These tests are apt to show the students' abilities in the four "classical" factors of spatial ability. Other spatial abilities (e.g. dynamic spatial ability, small scale/large scale spatial ability, and working memory), that have been identified in the past 20 years were not in the project's focus. This leads to follow up questions such as: Which kind of spatial abilities do we train in school? Is it mainly the "classical" spatial abilities, or also the "new" spatial abilities as mentioned above? Should we include more training of "new" spatial abilities? Further projects will pay attention to these questions.

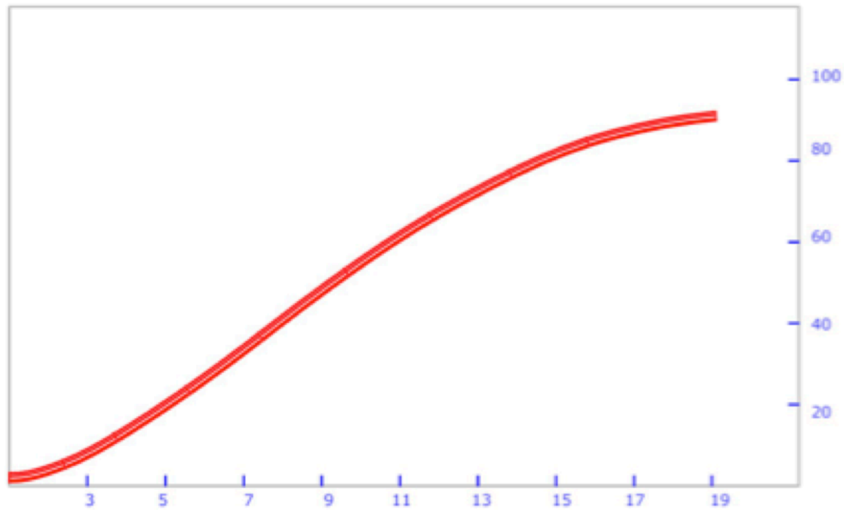

Figure 7: Development of spatial ability. The vertical axis shows the percentage of the development and the horizontal axis shows the age of individuals (see also Thurstone, 1955)

The gender differences in the project showed that female students had a significant treatment effect in the 3DW-Test. It is remarkable that in all three treatment groups the increase of performance in the 3DW-Test is much higher for girls than for boys, and that it is exactly the other way round in the control group. Here the male students have a higher improvement than the female students. The MRT was the only speeded-power test in the test battery of the project. Male students worked with more tasks, and they also had more items correctly solved than female students. In the SOT, male students had a better performance in the pre-tests and in the post-tests. The gender sensitive analyses point out that male and female individuals have different basic strengths in regards to spatial ability and different growth potential in regards to the factors of spatial ability.

Individuals use a large variety of different strategies for solving spatial ability tasks and can combine them in many different ways. This finding suggests that students should be familiar with a large repertoire of different solution strategies for spatial ability tasks and be able to use them in many different ways and combinations. Students must develop a kind of meta-knowledge to be able to handle this wide repertoire consciously. Students very often change their strategies between the pre-test to the post-test for the same tasks. This is an indication that with growing routine individuals may get to work with tasks in a different way. Individuals use more new and efficient strategies only when they have sufficient routine in a topic. This leads to the following didactical guiding idea: Teachers should discuss special and selected topics long enough that students can develop a sufficient routine in these fields. Only then students will get to know new and efficient solution strategies even in school and learn how to use them in a meaningful way.

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MIDDLE SCHOOL STUDENTS' USE OF PROPERTY KNOWLEDGE AND SPATIAL VISUALIZATION IN REASONING ABOUT 2D ROTATIONS

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In recent years, there has been increased attention on teaching transformational geometry. There is also increased recognition of the importance of spatial reasoning in mathematics and science. As a way of integrating research on these interconnected topics, we investigated middle school students' developing understanding of geometric rotations in the plane as they were working in a special dynamic geometry environment.

Keywords: geometry, imagery, properties, reasoning, rotations, spatial, transformations, visualization

Investigating students' understanding of 2D rotations—an important topic in transformational geometry—provides a fertile environment for integrating two of the major strands in research in geometric reasoning: analyzing students' use of spatial visualization in geometry and analyzing students' understanding of properties of geometric objects. On the one hand, almost all geometric reasoning, sense making, and problem solving are intimately connected to spatial reasoning. Even more, the National Research Council claims that, "Underpinning success in mathematics and science is the capacity to think spatially" (NRC, 2006, p. 6), a statement backed by research (Newcombe, 2010; Wai et al, 2009). On the other hand, an essential element of geometric reasoning is the use of a property-based conceptual system to analyze shapes (Battista, 2007). This system uses concepts such as angle measure, length measure, congruence, parallelism, and isometries to describe spatial relationships and movement. As part of our research and development of a special computer dynamic geometry environment—*Individualized Dynamic Geometry Instruction* (iDGi)—one topic we are investigating is the relationship between middle school students' knowledge and use of properties of rotations and their visualization of rotations. This research is not only important for extending and refining research on students' understanding of geometric transformations, but it is also relevant to the important general question of how spatial visualization and analytic-measurement-based property knowledge interact in geometric reasoning (Clements & Battista, 2001).

Theoretical Frameworks

Subscribing to a psychological constructivist theory of mathematics learning, we posit that students construct new mathematical understandings out of their current relevant mental structures. Consistent with this view of learning, maximally effective teaching is based on detailed knowledge of students'

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current mathematical ideas and ways of reasoning. An abundance of research has shown that mathematics instruction that is guided by knowledge of student thinking and supports students' personal sense making produces powerful mathematical thinking, conceptions, and problem-solving skills in students (Hiebert, 1999).

Differentiating Geometric Properties

An essential component in developing conceptual understanding of geometric objects is to understand the properties of those objects (Battista, 2007; van Hiele, 1986; Gorgorió, 1998). A critical issue in this development is specifying which properties of isometries are most important for students to learn. Based on previous research (Battista, 2007), we contend that, initially, the properties most critical to students' learning about geometric objects are properties that express prototypical, defining characteristics of those objects, which we call "*prototypical defining properties*." As an example, the prototypical defining properties of parallelograms are: *opposite sides congruent and parallel*. These are the properties that express mathematically the most visually salient spatial characteristics that students use in identifying parallelograms. Of course, there are other, less visually salient properties of parallelograms. For instance, in parallelograms, opposite angles are congruent, and all pairs of adjacent angles are supplementary. Certainly, the property that all pairs of adjacent angles are supplementary could be used to define parallelograms, as could the property "the diagonals bisect each other." However, prototypical defining properties are the properties that students derive from visual examples of parallelograms, and ones that students use to determine if a shape is a parallelogram through *visually-based*, conceptual analysis.

Similarly, and analogous to the properties of shapes described in Battista (2007), we take the prototypical defining properties of an isometry to be properties that express mathematically the visually salient spatial relationships among the preimage, image, and determiners of the motion defined by the isometry—that is, its parameters (e.g., see Coxford, 1973). The parameters for rotations are the position of the turn center and the amount of rotation, both of which have to be specified by the students in the iDGi tasks we discuss [see Table 1]. Thus, we agree with Hollebrands (2003) that understanding transformations requires understanding their parameters as well as the effects of parameter changes on the transformations.

We assert that although isometries are distance- and angle-measure-preserving, and one-to-one mappings of the plane onto itself, these characteristics are not prototypical defining properties. This aspect of our perspective contrasts with Hollebrands' (2003) who focused on how well high school students understand transformations as 1-1, onto functions of the plane. Although a function perspective is valuable for older, more experienced students, beginning instruction for middle school or beginning secondary students seems more appropriately focused on prototypical defining properties and transforming single figures instead of the whole plane. Indeed, mathematicians Wallace and West (1992) argued that isometries provide a mathematically precise way to reformulate Euclid's "common notion" idea of shape congruence by superposition, which involves transformations of *specific objects* in the plane.

Prototypical Defining Properties of Rotations
P1. Rotations are determined by a turn center and an amount of turn specified as a signed amount of degrees.
P2. Preimage and image polygons have corresponding points (preimage and image point pairs).
P3. The angle between the turn center and any pair of corresponding points equals the rotation angle.
P4. Pairs of corresponding points are the same distance from the turn center.

Table 1. Prototypical defining properties of rotations

Previous research: Can middle school students learn isometries?

Previous research on middle school students' ability to learn transformations yielded inconclusive results. While some research found middle school students have difficulty mentally performing transformations (Kidder, 1976) and as few as 50% of 10-11 year olds are able to master transformations (Shah, 1969), more recent studies show that students are able to make sense of transformation properties and parameters (Olson, 1987; Edwards, 1991; Panorkou et al., 2014). Our iDGi results reaffirm that middle school students can develop substantial understanding of the properties of isometries, which may be especially true in dynamic geometry environments (Battista, Frazee, & Winer, 2017). In fact, Dixon (1997) and Johnson-Gentile et al. (1990) reported that students learning about isometries in a computer environment outperformed students using a paper and pencil approach.

Components of spatial reasoning

Many cognitive psychologists (e.g. Hegarty, 2010) have discussed two types of spatial reasoning: (a) mental imagery/simulation and (b) spatial analytic thinking. For instance, on the Vandenberg Mental 3D Rotation Test, many students use a mental imagery strategy of either imagining objects rotating or imagining themselves moving around the objects. Many students also use spatial analytic strategies including counting the number of cubes in the different arms and decomposing cube configurations into parts easier to rotate mentally (Hegarty, 2010). In Table 2, we hypothesized adaptations to these strategy definitions to describe students' reasoning about rotating polygons ±90° or 180° in the plane. The hypothesized strategies were constructed to be consistent with our observations of student work in iDGi rotation modules.

Mental imagery strategies
1.1. I imagined the polygon turning in my mind.
1.2. I looked at the turn center and imagined the polygon turning about it in my mind.
1.3. I visualized the preimage and image, each connected by line segments to the turn center.
1.4. I visualized a vertical-horizontal "L" connected to the turn center and a polygon vertex turning in my mind.
Spatial analytic strategies
2.1. I noted the directions of corresponding sides of the polygons and decided if that was the correct angle measure.
2.2. I looked at the two polygons to decide what the angle of rotation was. Then I counted the number of units up/down/right/left between the turn center and corresponding vertices.
2.3. I visualized a vertical-horizontal "L" connected to the turn center and polygon, and counted units in each leg of the L.
2.4. I visualized rotating a polygon side, then counted how long its preimage was to know how long the image is.
2.5. I found images of the two perpendicular triangle sides one at a time. I knew that one side must make a right angle with the other side, so I could tell by visualizing where the side images should be located. I counted units to know how long to make the images of each side.

Table 2: Adaptation of Hegarty's (2010) strategies for polygon rotations tasks

Previous Research on Properties and Visualization

In their extension of the van Hiele levels to 3D shapes, Gutiérrez and colleagues (1992) integrated descriptions of students' property knowledge and spatial visualization. At Level 1, students compare solids globally with no attention given to properties such as angle size, side length, or parallelism. Students cannot visualize solids, their positions, or motions if they cannot see them; they manipulate solids using guess-and-check strategies. At Level 2, students move to visual analysis of solids' components and properties and are able to visualize simple movements. At Level 3, students compare solids by mathematically analyzing their components; they can visualize movements involving positions that are not visible, and in reasoning about movements, students match corresponding parts of images and preimages. At Level 4, students mathematically analyze and formally deduce properties of solids; visualization is strong and linked to property knowledge.

However, extending the theoretical integration of property knowledge and visualization is a difficult task because visualization may be connected to property knowledge in complex ways (Battista, 2007). On the one hand, some students who are not high visualizers develop analytic (property-based) strategies to help them compensate for a lack of pure visualization skills (Battista, 1990; Hegarty, 2010). On the other hand, some students possess very high visualization skills well before they develop property-based reasoning. Indeed, some high visualizers can mentally imagine movements of solids so well that, for many problems, they have no need to analytically examine the solids' components (Battista, 1990). The present study continues and deepens these extension efforts.

Methods

In the context of creating and field-testing a learning-progression-based, dynamic geometry environment and curriculum for elementary and middle school (ages 9-14 years), we conducted one-on-one teaching experiments with 8 middle school students on iDGi's isometry modules (2-3, 1-hour sessions). Because the target audience was middle school students, the iDGi isometry modules' goals were for students (a) to begin understanding the prototypical defining properties of the three basic isometries, and (b) to help develop their spatial visualization ability in 2D geometry. To promote these goals in iDGi, for each type of isometry, students first made predictions for problem answers, then checked their predictions using motion animations. To make both visual and analytic strategies accessible to students, rotation problems were presented on a square grid, the snap-to-grid feature was activated, and parameters were restricted: rotation turn centers were at grid points and rotation angles were limited to ±90°, 180°. The iDGi modules presented a variety of problem types in which students had to choose or create the correct parameters for a given isometry. In the iDGi rotations module, students first explored rotations of single points then rotations of right triangles. In the first right triangle task, students had to choose the amount of turn for a given preimage, image, and turn center; in the second, they had to create the rotation image of a right triangle given the turn center and amount of turn. Then, and the focus of this chapter, students were given two additional types of iDGi rotation tasks. The first type required students to find the amount of turn to rotate the preimage triangle onto the image and to determine which of several given points on the grid was the turn center (Figure 1). In the second type, students had to determine the amount of turn and the turn center to rotate a preimage onto its image, but they were not shown possible turn centers, which made finding the turn center much more difficult.

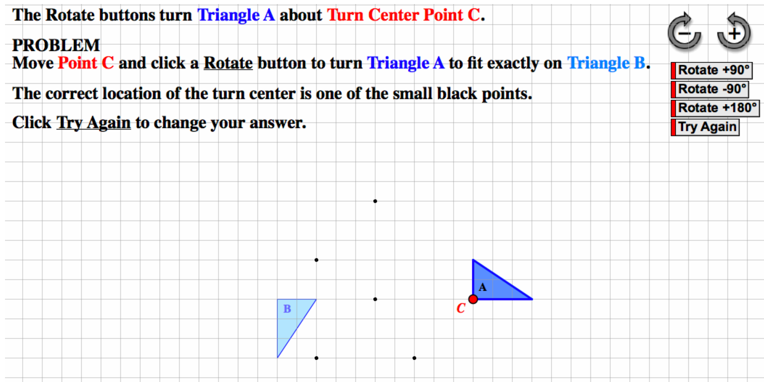


Figure 1. iDGi Rotation Task

In the iDGi environment, students made predictions for locations of rotation images and turn centers, and amounts of turn—which required them to come to know and utilize the prototypical defining properties of rotations—then, when students specified an angle of rotation and turn center, the computer performed the associated motion. By focusing on motion and properties in this linked way, the iDGi environment helped students transition from a strictly motion conception of rotations to a more abstract, property-based mathematical conceptualization of rotations (Clements & Battista, 2001).

To collect data, we had students work on rotation modules individually while sitting with an iDGi researcher who asked them to think aloud while working. Often, we asked questions: What are you thinking? Why did you do that? All work was video and audio recorded, both with a screen capture program and an external camera focused on the screen (to record student screen-related gestures).

Comparison of Case Study Students

To illustrate the nature of students' reasoning, we compare the work of three students: MR, a 7th grader, and two 8th graders, PG and YJ. Each student developed a spatial reasoning strategy with both visualization and analytic components: MR's strategy was predominantly analytic, PG's strategy favored visualization, and YJ integrated analytic and visualization. All three students experienced success with their strategy in solving some rotation problems. However, MR and PG experienced difficulties when solving complex problems due to the lack of coordination between visualization and analytic reasoning as well as to visualization errors. Though YJ experienced some difficulty with visualization, she combined her visual and analytic reasoning to accurately complete most of the problems.

Student MR

First, we examine problems where MR chose the turn amount and one of five possible turn centers when given preimage triangle *A* and image triangle *B* (Figure 2; only Point *C* is labeled in the actual iDGi module; points *A, B, C, D,* and *E* are possible turn centers).

MR: That one doesn't form a right angle [traces path *RAS*], that doesn't form a right angle either [traces *RBS*]. That might form a right angle [traces *XCY*]. Yeah, that might form a right angle. Oh, wait…this one I think…that does not form a right angle [traces *XDY*].

I: When you say it doesn't form a right angle…what were you talking about?

MR: … If you connect the two similar points like this [*X*] and that [*Y*], they have to make either a 180° or 90° angle, and they do neither… you can see this one [*D*] is like way like out there.… [Motioning *X* to *W* to *C*] 1 to 6 that way. So I think it's this one [*C*] because, this might seem silly, but there is like one space distance between this point [motioning *X* to *W*] and there is one space distance between that point [motioning *Y* to *Z*, then to *C*], so they're off by the same degree.… If that's [the rotation] going that way, that's counterclockwise, which is positive.

In this problem, MR used Properties 1-3. In other problems, she also used Property 4: "they [corresponding points] have to be the same distance away [from the turn center]." She used the properties to develop an analytic strategy for testing possible turn centers, which, like in the next example, she successfully applied in a number of problems.

MR: [Figure 3] This one [turn center *C*] … This is 1-2-3-4-5 and this is 1-2-3-4-5. And this one [*XCY*] is a right angle because they [*X* and *Y*] are both the same degree off [referring to the 1 unit horizontal distance between *X* and the point marked 5 and the 1 vertical unit between *Y* and the point marked 5].

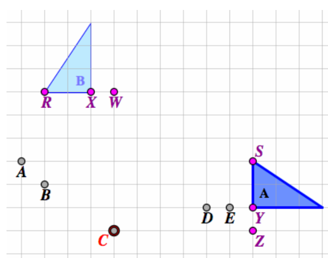


Figure 2

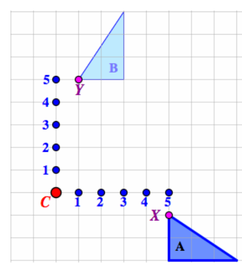
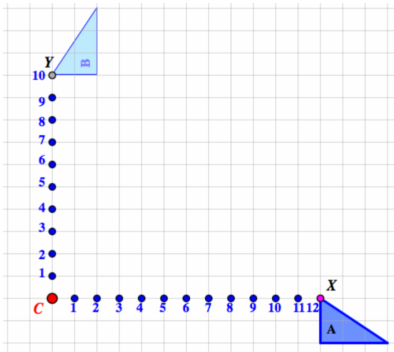
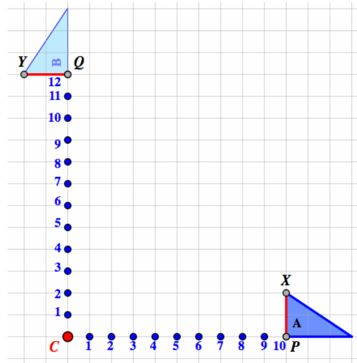


Figure 3

As these two examples illustrate, MR had a well developed property-based, counting strategy for locating the correct turn center when a small set of possible turn centers was provided. As MR moved to solving problems in which no possible turn centers were shown, she adapted her counting strategy to include a "one and one" strategy for adjusting "failed" turn center counts.

MR: [Figure 4] I'm guessing this is another 90° problem so if I match these two [*X* and *Y*]. This is 1-2-3-4-5-6-7-8-9-10-11-12, so 1-2-3-4-5-6-7-8-9-10…. So if I move it [turn center *C*] down 1 and across 1. Cause if I want to move it [*C*] across 1 to reduce this [distance from *C* to triangle *A*], I have to move this [*C*] down 1 so that it doesn't match up with this [triangle *A*] but not that [triangle *B*]. Because when it does that [not 'match up'], it forms an angle like this [gesturing off-screen], which doesn't work, cause that's definitely not going to be 90°. So I'm going to move it down 1 and [across 1]… again [to *C*; Figure 5], so 2 again 2 again [indicating segments $\overline{XP}$, $\overline{YQ}$; Figure 5]. So that's 1-2-3-4-5-6-7-8-9-10-11-12 and that's 1-2-3-4-5-6-7-8-9-10 [Figure 5]. Ok, so up 1, across 1 [Figure 6]; so that should work [which she verifies by clicking on the appropriate angle rotation button].





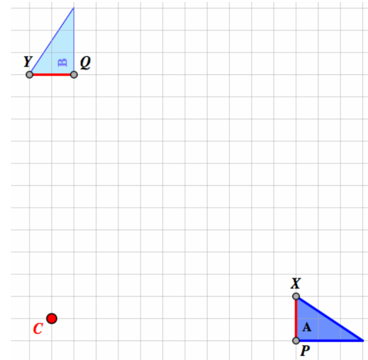
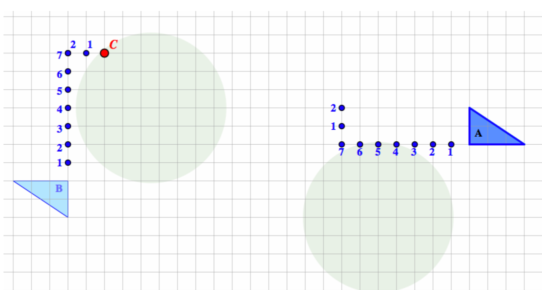


Figure 4 Figure 5 Figure 6

Note that MR did not recognize that her first move (right 1, down 1) after her first count (Figure 4) was correct. Perhaps at first she was trying to match corresponding vertices via vertical and horizontal segments. Her final move matched midpoints of corresponding sides XP and YQ.

However, despite her successes, MR's reasoning often seemed hampered by visualization difficulties on more spatially demanding problems. For instance, she sometimes failed to recognize correct rotation angles, as shown below.

> MR: This is going to be 90° so it's going to be here or there [indicates circular regions in Figure 7]. ... You know I think I'm going to actually put it [turn center *C*] up here [in the upper left circular region in Figure 7]. So that's 1-2-3-4-5-6-7 and 2 across [counts up from Triangle *B* and left from *C*; Figure 7]. So that has to be 1-2-3-4-5-6-7 and 2 up [counts left and up from Triangle *A*; Figure 7]. Which doesn't work.... [Moves turn center *C* as in Figure 8] So then this is 1-2-3-4-5-6-7-8 across and 1-2-3-4-5-6-7 up [counting from Triangle *B*]....Ok, so then this is 1-2-3-4-5-6-7-8 ac---[counting from Triangle *A*]. Wait 8 across and 7 up [from Triangle *B*], so this would be 7 across and 8 up [from Triangle *A*—moves cursor along segments indicated in Figure 8], right?



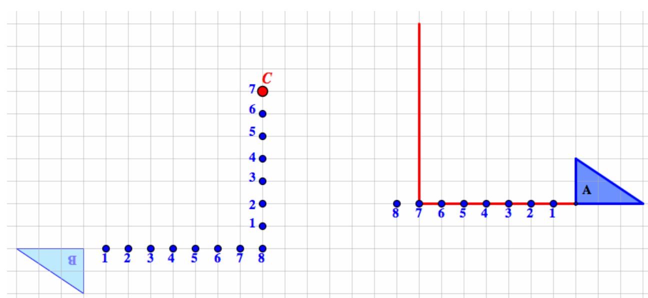


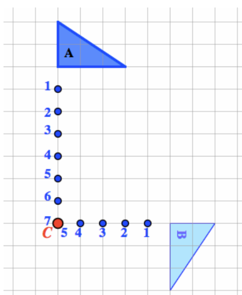
Figure 7 Figure 8

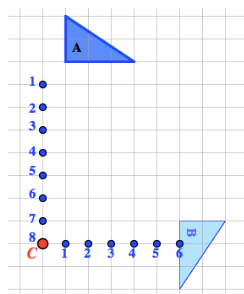
In this problem, MR did not use her "one and one" strategy as she did in Figure 4. Instead, she understood that for 90° rotations, the across moves from the preimage triangle to the turn center turned into up/down moves for the image triangle. She repeatedly tried to use this up-down/across strategy, failing to recognize that this was not a 90° rotation until later when her interviewer asked her about the rotation angle (she also sometimes confused positive and negative 90° rotations).

On other problems, MR seemed to get disoriented in her "one and one" strategy, again, possibly because of spatial disorientation.

> MR: [Figure 9] So this is 1-2-3-4-5-6-7 down, and 1-2-3-4-5 across [Figure 9]. So when you move it that way [left], you also have to move it down. So 1-2-3-4-5-6-7-8, 1-2-3-4-5-6 [Figure 10], [moves the turn center 2 left, Figure 11] so that's 3 across now and you have to move it [turn center] 1 up [Figure 12]. [Sighs and moves the turn center to the location in Figure 13]. So that's 2 and then 2 [segments indicated in Figure 13].

Note that initially MR moved in a way that increased both distances (Figures 9 & 10). However, something in what she observed caused MR to stop following her one-and-one adjustment strategy (Figure 13). Moreover, in her reasoning about the possible turn center location in Figure 13, MR made a spatial error. That is, if she was trying to visualize the -90° rotation of the configuration "up from C then right 2," then the configuration's correct image would be "right then *down* 2" as shown in Figure 14, not "right then up" as she motioned in Figure 13.





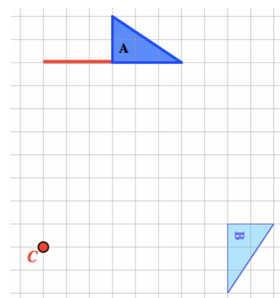
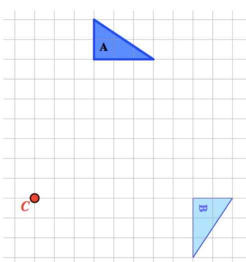
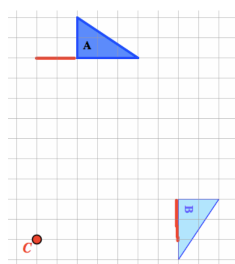


Figure 9 Figure 10 Figure 11





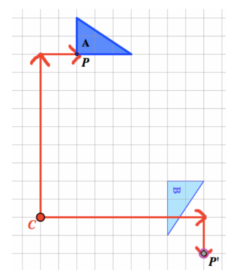


Figure 12 Figure 13 Figure 14

> MR: So these form a right angle [corresponding points in Figure 15], but they don't match up [not equidistant from *C*]. This is like 2-4-6-7 and this 2-4-5 [Figure 15]. If you want this to become 7 [horizontal distance between *C* and Triangle *B*], or no, you want them both to become 6, and then 1 across. So, [Figure 16] that's 1-2-3-4-5-6, then 3-6, yeah, and that forms a right angle and this would be this way, which is negative. [Enters correct rotation.]

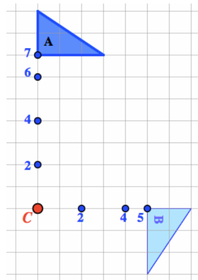


Figure 15

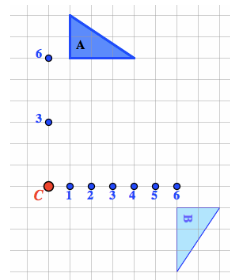


Figure 16

When MR decided to restart her thinking, she returned to a strategy of starting at the intersection of vertical and horizontal lines that contain corresponding points. By moving in a way that increased the smaller distance and decreased the larger distance to corresponding points, MR successfully solved the problem.

MR always attempted to make the up-down/right-left distance from the turn center to corresponding points on Triangles *A* and *B* equal by using her "one and one" strategy to adjust the turn center location, moving 1 unit up-down and 1 unit right-left. Because the interviewer thought that this two-step process was too difficult for MR to fully understand, he asked MR about moving just one space at a time.

> I: Suppose you just move it 1 at a time. Would that help? If you just move the point like instead of this way and this way [up 1, left 1 from C in Figure 17], just 1 unit at a time.

> MR: [Moves turn center up 1 from *C* in Figure 17 to the location of *C* in Figure 18] But then what happens is this has a distance up of 3 [segment above Triangle *A* in Figure 18], but this has a distance across of 2 [segment left of Triangle *B* in Figure 18]. So then it definitely won't work if I do that. So I need to like move it both ways.

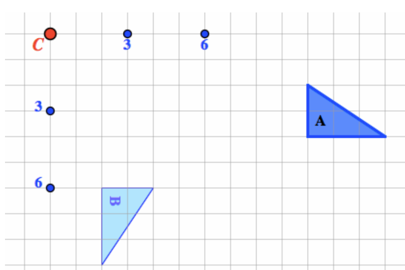


Figure 17

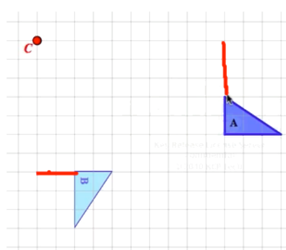


Figure 18

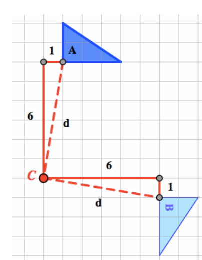


Figure 19

In her last explanation, MR did not count to corresponding points. She again made a mistake in visualizing rotations of her up/down-right/left movements.

In summary, MR developed a property-based analytic strategy to test whether corresponding points were the same distance from possible turn centers. She did not use the hypotenuse of the right triangle to find the straight-line distance between the turn center and corresponding points, but instead used the lengths of the horizontal and vertical legs of the right triangle (Figure 19). In 8 of 12 problems, MR used her strategy to find the correct turn center before she checked her answer with the iDGi rotation command. But, seemingly due to the complexity and resulting cognitive load of the visualizing and counting she did with these right triangle L's, and especially when her turn center predictions were incorrect, she sometimes made spatial errors as in Figures 13 and 18.

Student PG

In contrast with MR, PG developed a predominantly visual strategy. In problems for which PG was given a preimage, an image, and turn center, he was able to reliably visualize the amount of turn. For instance, in Figure 20, PG immediately stated the answer should be +90°.

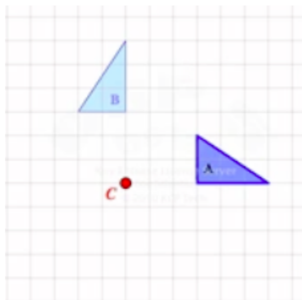


Figure 20

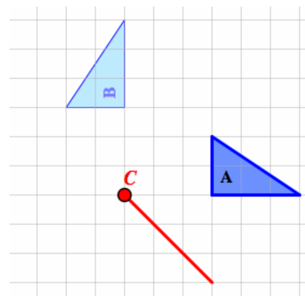


Figure 21

I: How did you know?

PG: Because negative 90° would be this way [moves cursor left-to-right as indicated by the line segment in Figure 21].

I: How did you know it was 90° in the first place?

PG: [pause] I'm not quite sure.

I: That's ok! So you can tell by looking?

PG: Yeah.

PG's responses to the interviewer's questions, along with the fact that he immediately and correctly found turn centers and determined the amount of turn for many problems given the preimage and image, support our contention about the visual nature of his reasoning (see also Figure 25).

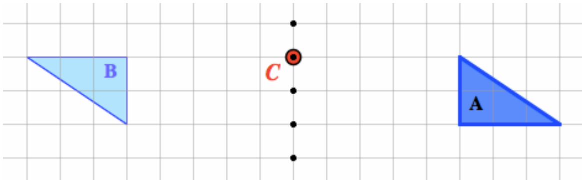
However, PG's visual strategy was not supplemented by a sophisticated understanding of the properties of rotations. Rather than identifying corresponding points as stated in Property 2, PG focused on corresponding *parts* of the preimage and image. For instance, as shown in the next two examples, PG often spoke of the angle of rotation *between the two triangles as whole shapes, not between corresponding points on the triangles*.

I: [After PG chose turn center *C* in Figure 22] So before you click anything, can you explain to me how you are getting this?

PG: If I put it here [turn center *C* in Figure 22], there is an equal amount of distance between this *side* of *B* [vertical side] and [puts cursor on the vertical side of *A*]…

I: I think all those points are equidistant, so how do you know which one of those equidistant points to choose?

PG: I don't [moving the *C* from gray dot to gray dot]…



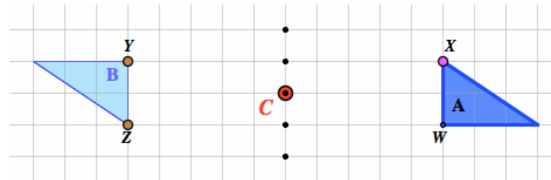


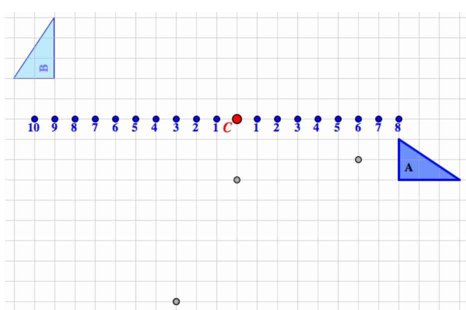
Figure 22 Figure 23

I: So what made you move [from *C* in Figure 22 to *C* in Figure 23]? ...

PG: Because then it's [C in Figure 22] on this, it's on the top part of A but the bottom part of B. So I decided to do the middle [gestures to the middles of the vertical legs of the right triangles and places *C* as shown in Figure 23]….

PG's lack of attention to corresponding points continued as he added an analytic counting component to his strategy for problems that were harder for him to visualize. But he often seemed to count to determine the distance between the turn center and whole triangles, not between the turn center and corresponding points. For instance, as indicated in Figure 24, PG counted from turn center *C* to near the triangles. But then, as he also often did, PG switched from an analytic strategy to a visual strategy.

PG: [After counting] I'll just give this [turn center *C* in Figure 24] one a try. Actually, I think I'll give this one [turn center *C* in Figure 25] a try.



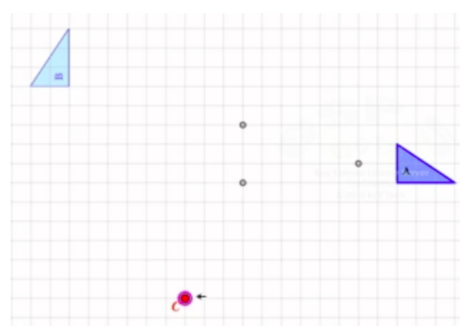


Figure 24 Figure 25

As PG explains in the next example, he used a counting strategy to locate a point in the middle of the two triangles, but he used visualization to approximate where the turn center was located.

I: How about I drive [control the mouse] and you tell me what to do?

PG: OK. Um, first count the distance of squares between both of them. ... Start from here [points to *X*; Figure 26] and go down to here [points to *Y*]....

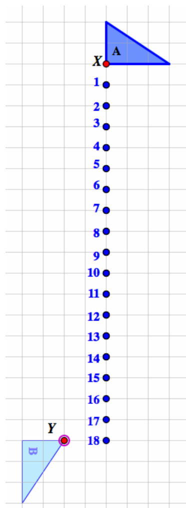
I: [Counts 18 as in Figure 26]…and then over, [moves left 1] 19?

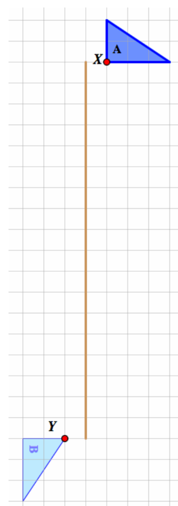
PG: No just like…

I: Here? [motions along segment indicated in Figure 27]

PG: It's [the turn center] somewhere on this line [Figure 27], the middle line between *A* and…. [Thinks a while] So then there is 18 so move it [turn center] to the 9th line…the ... center *C* to the 9th line.

I: 1-2-3-4-5-6-7-8-9 [as indicated in Figure 28].





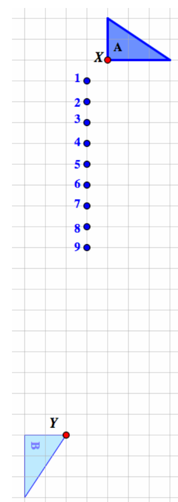


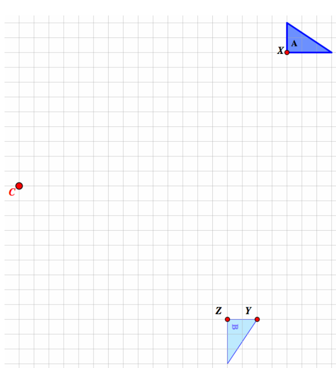
Figure 26 Figure 27 Figure 28

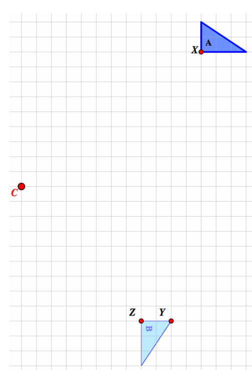
PG: Yeah, now rotate...no, actually, and then you've gotta move it back [points at screen toward the left] so it can make a big rotation....

I: Left? OK [moves point *C* to location in Figure 29].

PG: No, that's too much.

I: How would I know? [Interviewer moves *C* right a little] You're going to have to help me.





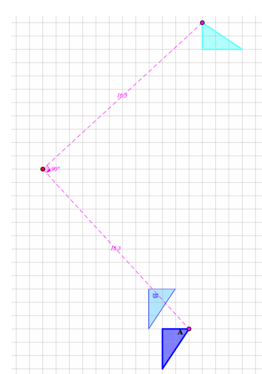


Figure 29 Figure 30 Figure 31

PG: I’m trying to draw an imaginary line from [triangle] *B*…to both angles. It’s hitting *B* right here [points to *Z* in Figure 29] and A right here [points to *X* and makes a 90° angle shape with points *X*, *C*, and *Z*]. That looks like it might be it [Figure 30].

I: So I should do?

PG: Negative 90. [checks and sees result Figure 31]

I: So how could we adjust?

PG: If you move the turn center point one square over…

I: This way? [right]

PG: No, no, no, left.

I: Left? Ok. What will that do?

PG: This triangle [*A*] will come over here [one unit left] then if you move it 3 squares up, then it will—I’m pretty sure it will match this [*B*; checks answer and sees it is still incorrect].

PG used a similar kind of visualization supplemented with analytic-counting reasoning on other problems of this type. However, PG never developed an effective analysis-dominated strategy like MR, figuring out only 1 of 8 problems before he checked his answer with the iDGi rotation command. PG used some analytic reasoning, but visualization always dominated. The last example also illustrates the effectiveness of PG's visualization to approximate the location of the turn center when no turn-center options were given. Like MR, PG evidenced some understanding of Properties 1-4. However, unlike MR's explicit and completely correct statements about the properties, PG's knowledge seemed embedded in his visual strategies or focused on corresponding triangle parts, not points. Finally, PG, like MR, never figured out a reliable method for adjusting the placement of the turn center after seeing where the chosen turn center placed the image triangle. Neither student saw any patterns that determined how the image moved for specific moves of the turn center. Given the complexity that existed for turn-center movements (see Table 3[2]), it is no wonder MR and PG could not detect them. For example, to interpret the cell in the first column second row, suppose we rotate a point P +90° about a given turn center C to get P'. Now suppose we move C to the left 1 unit and rotate P +90° about the new position of the turn center, getting point P''. Then P'' is up 1 unit and 1 unit to the left of point P'. We believe that without appropriate instructional support, it is unlikely that students at this age level would be able to sort out the complex patterns depicted in Table 3 and implement this knowledge in a reliable analytic strategy. Thus, to be successful on these tasks, students had to use visualization to guide their analytic strategies.

[2] One way to prove these movements is to think carefully about how a vertical/horizontal L-shape connected to the preimage moves when the turn center moves. Another way is to use coordinates and matrix concepts in transformation geometry. For example, to compare the image of a point rotated about the origin to the image of the point when rotated about (0, 1), we first translate the plane down 1 unit, do the rotation about the origin, then translate the plane up 1 unit. For us, this is where Hollebrands’ focus on transforming the whole plane can become practically useful for students.

Move Turn Center Left 1	Move Turn Center Right 1	Move Turn Center Up 1	Move Turn Center Down 1
Moves 90° Image Left 1, Up 1	Moves 90° Image Right 1, Down 1	Moves 90° Image Up 1, Right 1	Moves 90° Image Down 1, Left 1
Moves -90° Image Left 1, Down 1	Moves -90° Image Right 1, Up 1	Moves -90° Image Up 1, Left 1	Moves -90° Image Down 1, Right 1
Moves 180° Image Left 2	Moves 180° Image Right 2	Moves 180° Image Up 2	Moves 180° Image Down 2

Table 3. Movements of image in relation to movements of turn center

Student YJ

Student YJ integrated visual and analytic strategies more than MR and PG. When solving problems for which turn center options were shown, YJ often successfully employed a purely visual strategy making use of Property 3 for one pair of corresponding points.

YJ: [As shown in Figure 32, places turn center *C*, then moves the cursor in *L*'s from corresponding points to *C*] Rotate, and it would go that way [motions clockwise as indicated, chooses -90°].

On a later problem (Figure 33), YJ first visually estimated a turn center and an amount of turn but then used an analytic strategy, employing Properties 1-4, to test her estimates.

YJ: [Moves turn center *C* to the location indicated in Figure 33] So this takes 8 [motions from *C* as indicated in Figure 33; no counting aloud] and then 1 [motions down as indicated in Figure 34]. [Moves cursor as indicated in Figure 35; no counting aloud] And it's [the image triangle] not there. Yeah it's not there.

I: What do you mean it's not there?

YJ: I just counted 4 units and 4 units and that's 8 [indicates how she counted in Figure 33] And then 4 units and 4 units [indicates how she counts in Figure 35] and not there… Maybe it's this one [*C* in Figure 36], and it's a rotate 90 [motions as indicated Figure 36]. Well—I believe so [checks and sees answer is correct]. [Originally] I kind of thought it would be...like right here [*C* in Figure 37], and would rotate 180.

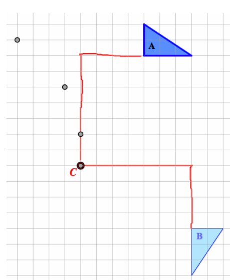


Figure 32

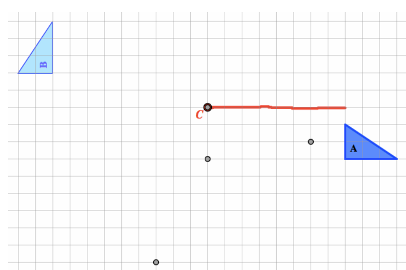


Figure 33

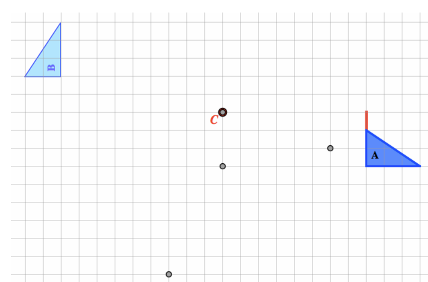


Figure 34

So, in this example, YJ's application of an analytic strategy, implicitly based on Properties 1-4, helped her see that her initial angle estimate was incorrect. She quickly switched to a visual strategy that led her to the correct answer. In the next example, she uses all four rotation properties.

YJ: [Figure 38] So it obviously has to be 2 here or 2 here [as indicated in Figure 38], I think. Ah [moves *C* to location in Figure 39]. So 1-2 [counts as in Figure 39], 5 [motions up 5; Figure 39]. 2-5 [motions Figure 40]. So that would be—go this way. Oh, not really.

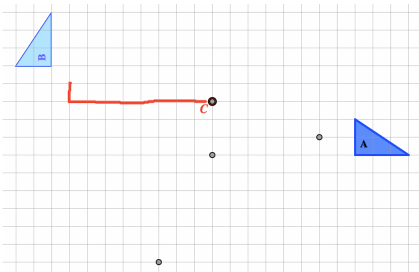


Figure 35

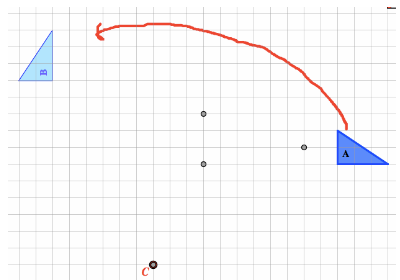


Figure 36

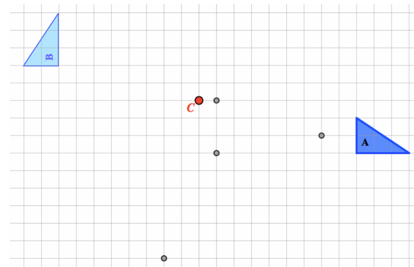


Figure 37

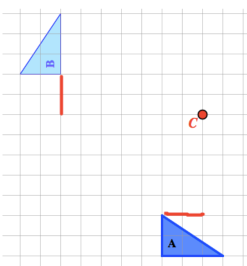


Figure 38

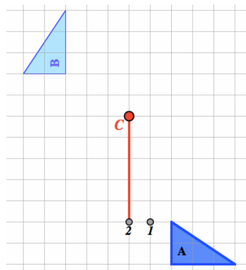


Figure 39

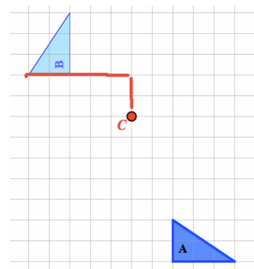


Figure 40

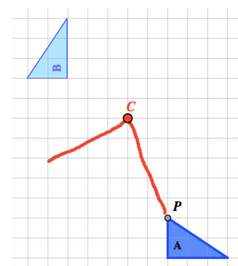


Figure 41

I: Not really what?

YJ: Like, um, this point [*P* in Figure 41], I don't think it would make a right angle because then it would have to be like somewhere here [motions cursor in 90° angle as indicated Figure 41]. So it would be right there or something [motions to area where *C* is in Figure 42].

I: To actually make the 90° rotation?

YJ: Yeah. 1-2-3-4-5-6-7 [Figure 43]. And 7 [Figure 44]. So I think this is the right one, it's clockwi—counterclockwise, [checks] yay!

In summary, YJ coordinated spatial and analytic reasoning in a way that enabled her to make adjustments when her initial predictions were incorrect. However, similar to MR, the analytic strategy that YJ used in Figure 39 and 40 erred in visualizing the wrong angle. Nevertheless, YJ overcame this error by accurately visualizing the approximate location of the turn center, enabling her to use her analytic strategy to locate the correct turn center.

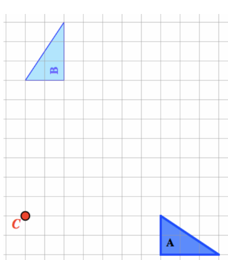


Figure 42

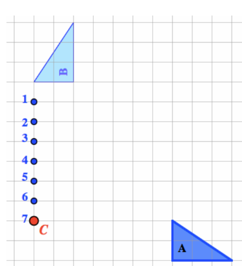


Figure 43

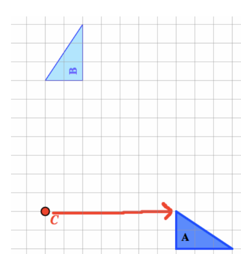


Figure 44

Conclusion

Our research focuses on the important general question of how spatial visualization, analytic-measurement-based strategies, and property knowledge interact in students' geometric reasoning. Much of the cognitive psychology research in spatial visualization has investigated the spatial-analytic relationship by analyzing individuals' performance on assessments of spatial ability such as the Vandenberg Mental 3D Rotation Test. Only Hegarty and colleagues (e.g., Hegarty, 2010; Stieff, Hegarty, & Dixon, 2010) seem to be descriptively investigating the nature of spatial strategies, doing so for tasks in science and engineering. What we do not have enough of in mathematics education are detailed descriptive studies that explicitly and deeply investigate the nature of spatial analytic reasoning in geometric contexts. The present study, along with that of Ramful, Ho, & Lowrie (2015), are first steps in this direction. They describe in detail the specific visual and analytic strategies, and property knowledge, that students use in one particular geometric context and the difficulties that students face in implementing these strategies. In particular, the present study found that each student used knowledge of all four prototypical-defining properties of rotations either explicitly expressed in analytic strategies or implicitly embedded in visual strategies. But this study also showed how these analytic strategies often failed because of students' difficulties with spatial visualization. Such descriptions are critical to genuinely understanding the role of spatial visualization in geometric reasoning.

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EXPLORING MODELS OF SECONDARY GEOMETRY ACHIEVEMENT

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Thompson and Senk (2014) described variations in 12 secondary school teachers using the same geometry textbook in enacting the curriculum. In this paper, the researchers investigated factors that might account for the achievement of the 544 students enrolled in the 25 geometry classes these teachers taught. Multilevel regression analyses showed that the students' prior achievement, teachers' reports on their use of questions applying the mathematics studied, and students' opportunity to learn the content of the posttest have significant positive effects on the geometry posttest achievement. The percent of lessons taught, writing emphasis, and frequency of use of activities with concrete materials had negative effects on the posttest achievement. The researchers' final model accounted for about 95% of the variance. School size or type, instructional time, teacher's certification and experience, and other aspects of curriculum enactment were not significant. Other factors and more reliable ways to measure and combine those factors in determining curriculum enactment may lead to developing more precise models of students' achievement.

Keywords: Curriculum enactment, geometry achievement, instructional practices, multilevel analysis, opportunity to learn, Rasch analysis, reading mathematics, regression analysis, textbook questions, use of concrete materials in geometry

Introduction and Research Questions

Over the years, various models of school learning have been proposed to explain variations in students' achievement in school subjects. For instance, Carroll (1963) and Bloom (1976) postulated variables such as aptitude, opportunity to learn, and quality of instruction to account for variations in school learning. In a 25-year retrospective and prospective view on effects of his 1963 model, Carroll (1989) noted that virtually all the variables in his proposed model had been substantiated by research, but many studies had neglected "the basic issue of how the content of instruction is to be organized and presented" (p. 29).

In recent decades, researchers (e.g. Li & Lappan, 2014; Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002) have begun to look more closely at issues related to the mathematics curriculum, instruction, and their effects on learning. Remillard and Heck (2014) proposed a conceptual model where both the instructional materials used and the curriculum enacted by the teacher influence students' learning. As an example of that model in use, Thompson and Senk (2014) document how 12 teachers from different schools implemented lessons on congruence from the same geometry textbook. In particular, they report considerable variation in the number of lessons taught or skipped as well as variation in instructional approaches, including the use of reading and writing mathematics and the use of technology.

Due to the hierarchical nature of schooling, namely that students are taught in classrooms, which are within schools, scholars have also begun using multilevel modeling to analyze school and classroom effectiveness variables (Hill & Rowe, 1996). For instance, researchers working on the COSMIC project in Missouri (Chávez, Tarr, Grouws, & Soria, 2015; Grouws et al., 2013; Tarr, Grouws, Chávez, & Soria, 2013) have engaged in a large-scale investigation about achievement when students study from curriculum-specific textbooks (Algebra I, Geometry, Algebra II) or a textbook series that addresses the content in a more integrated manner. As part of their study, they investigated various factors that might influence achievement, including both student and classroom instructional variables. As in previous studies (Bloom, 1976; Carroll, 1963; De Jong, Westerhof, & Kruiter, 2004), they found prior student knowledge to be a main predictor of student achievement. They also found gender and ethnicity to be important predictors of performance although the predictive level depended on the test-type (standardized test or curriculum-specific test). However, results were mixed relative to classroom instructional factors. In two studies, increases in Opportunity to Learn [OTL], or the level of curriculum implementation defined as the percent of lessons taught, resulted in increases in student performance (Grouws et al., 2013; Tarr et al., 2013). However, in a third study, OTL was not a statistically significant predictor of achievement (Chávez et al., 2015). In addition, teacher experience mattered as students of teachers with three or more years of experience performed better on assessments than students taught by less experienced teachers. In all three COSMIC studies, teachers greatly varied in how they used curriculum materials, but curriculum fidelity was not a significant predictor of mathematics achievement.

In this chapter, we follow up on Thompson and Senk (2014) by investigating the extent to which variations in classroom enactment predict students' geometry achievement. Based on our review of related literature, we hypothesized that students' achievement on a posttest would be predicted by student factors, school factors, teacher factors, and curriculum enactment factors. Specifically, we investigate the question: Which characteristics of students, schools, teachers, and classroom enactment by geometry teachers contribute to students' end-of-course achievement?

Design and Methods

The data set used to explore models of students' achievement is a subset of data collected by the University of Chicago School Mathematics Project [UCSMP] during the 2007-08 school year as part of a curriculum evaluation study.[1] Founded in 1983, UCSMP aimed to upgrade and update mathematics education in elementary and secondary schools throughout the United States (Usiskin, 2003). The instructional materials emphasize reading, problem-solving, everyday applications, and the use of calculators, computers, and other technologies. Unnecessary repetition of concepts studied in earlier courses was eliminated, so that by the end of high school, the diligent average student could learn mathematics once reserved only for honors students. Since its inception, UCSMP has been the largest university-based mathematics curriculum project in the United States. In 2017, estimates indicate that UCSMP materials were being used by about 4.5 million elementary and secondary students in schools in every state in the United States.[2] The UCSMP *Geometry* textbook (Benson et al., 2007) is the fourth in a sequence of seven textbooks developed for students in Grades 6 – 12. In this section, we describe the textbook, the sample and instruments, and procedures which were used for this investigation.

UCSMP *Geometry* Textbook

The main goal of UCSMP *Geometry* is to provide students with a clear understanding of two-dimensional and three-dimensional figures and the relationships among them (see http://ucsmp.uchicago.edu/secondary/curriculum/geometry/). Transformations are used to introduce

[1] While the data is 10 years old as of the publication of this book, there is no reason to believe the phenomena they document has changed substantially.

[2] Data retrieved from http://ucsmp.uchicago.edu/about/overview/ on February 14, 2017

general definitions of congruence, similarity, and symmetry that enable students to connect the abstract notions of geometry with figures on a page and the real world. Transformations also provide an opportunity to integrate geometry with concepts in algebra that students have previously learned and provide practice with function notation and composites of functions. Special lessons are devoted to aspects of geometry in art, architecture, sports, and music; activities using concrete materials or geometry drawing software appear throughout the textbook. By starting from the assumed properties of points, lines, and angles, as well as selected definitions, UCSMP *Geometry* aims to develop a coherent mathematical system in which students learn to make deductions from definitions and then write direct and indirect proofs in various formats.

During the evaluation study of UCSMP *Geometry* (Third Edition, Field-Trial Version), at the beginning of the school year, teachers received a Table of Contents and the first four chapters with the rest of the textbook provided in groups of 2–4 chapters. The version used in the Field Trial contained 114 lessons organized into 14 chapters as denoted in Table 1.

Table 1
Chapter Titles for UCSMP Geometry (Third Edition, Field-Trial Version)

Ch	Title	Ch	Title
1	Points and Lines	8	Lengths and Areas
2	The Language and Logic of Geometry	9	Three-Dimensional Figures
3	Angles and Lines	10	Formulas for Volume
4	Transformations and Congruence	11	Indirect Proofs and Coordinate Proofs
5	Proofs Using Congruence	12	Similarity
6	Polygons and Symmetry	13	Consequences of Similarity
7	Congruent Triangles	14	Further Work with Circles

Each lesson ends with four types of questions: *Covering the Ideas*, *Applying the Mathematics*, *Review*, and *Exploration*. The *Covering* questions in UCSMP focus on the basic ideas of the lesson. The *Applying* questions extend the concepts to new types of problems or require students to relate concepts to each other. *Review* questions provide an opportunity for students to develop mastery of the mathematics by continuing to work on new mathematics ideas throughout the chapter and into subsequent chapters. *Exploration* questions provide an extension for interested teachers and students. The curriculum developers recommend that teachers assign all of the *Covering*, *Applying*, and *Review* questions in each lesson. Samples of these question types are shown in Figure 1.

Several textbook activities and examples from UCSMP *Geometry* (Third Edition, Field-Trial Version) are described in Thompson and Senk (2014). Examples from an earlier edition of the textbook appear in Hirschhorn, Thompson, Usiskin, & Senk (1995), which includes examples that illustrate how concepts are addressed from a multi-dimensional approach to understanding that focuses on skills, properties, uses, and representations. Additional interactive demos are available at http://ucsmp.uchicago.edu/secondary/curriculum/geometry/demos/.

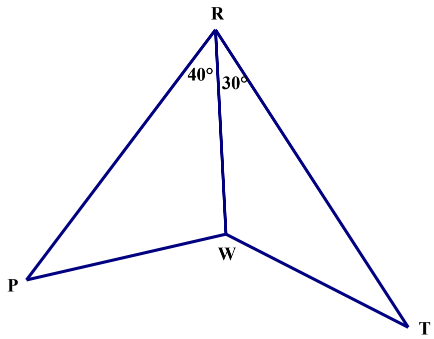
<table>
<tr><td>Covering the Ideas</td><td>How many symmetry lines does each type of triangle have?
a. equilateral b. isosceles c. Scalene</td></tr>
<tr><td>Applying the Mathematics</td><td>In nonconvex quadrilateral RPWT, PW = RW = WT = 18 in. m∠PRW = 40° and m∠WRT = 30°. Determine m∠PWT.

</td></tr>
<tr><td>Review (from previous lesson)</td><td>If F and G are figures and $r_m(F) = G$, then $r_m(G)$ = _____.</td></tr>
</table>

Figure 1. Sample Covering Applying, and Review questions from Lesson 6-2 on isosceles triangles (From Benson et al. (2006/2007), pp. 344-346. © 2006 by the University of Chicago School Mathematics Project. Reprinted with permission.)

Sample

The sample was drawn from eight public and four private schools in nine states from the Midwest and South of the United States of America (USA). Size of the schools ranged from 300 to 2,200 pupils. Time allotted for mathematics instruction ranged from 215 to 300 minutes per week.[3]

One teacher in each school taught from the UCSMP *Geometry* textbook (Benson et al., 2007) with each teacher teaching one, two, or three classes of geometry for a total of 544 students in 25 classes. One teacher taught advanced Grade 8 students in a middle school, and one teacher taught students in Grades 8-10 in a K-12 school. The other ten teachers taught in high schools with most students in Grades 9 or 10. The class sizes, determined by the number of students who completed all instruments, ranged from 6 to 31 students.

Instruments

In this paper, students' achievement is reported on two multiple-choice instruments: (a) a 35-item *Geometry Readiness Pretest* on geometry and algebra which were considered prerequisite knowledge for the course, and (b) a 35-item *Geometry Posttest* assessing the intended content of the course. Thirteen items were common to both the pretest and posttest. These common items test mathematics concepts that are considered part of the U.S. Common Core State Standards for Grades 6–8 (Council of Chief State School Officers, 2011), including determining angle measures, lengths or areas of triangles, quadrilaterals, and circles, and using vocabulary about lines and angles. Rasch model equating with the 13 common items was conducted using BILOG-MG software (du Toit, 2003) to obtain item difficulties as well as estimates of students' pretest and posttest knowledge on the same logit scale. The Rasch logit scale is a *z*-score with mean of 0 and standard deviation of 1.

[3] At School L, *Geometry* was taught on a 4 × 4 block schedule during the Spring semester only, and students had 490 minutes of instruction per week. For purposes of comparison with schools at which *Geometry* was taught during the entire year, we divided the weekly instructional time at School L by 2.

Both pretest and posttest had similar test quality. The test reliability (Cronbach's alpha) was .80 for both tests; the 95% confidence intervals for pretest and posttest were 0.78-0.82. The posttest (Rasch test difficulty = .375) was more difficult than the pretest (Rasch test difficulty = -0.50). Because the two tests have different difficulty levels, statistical equating is needed to compare students' performance on pretest and posttest.

Stems of seven sample items from the posttest illustrating a selection of geometry concepts from the posttest and their item difficulties are shown in Table 2. An increase in item difficulty shows that the item is more difficult. Thus, the easiest item shown in Table 2 is an item common to the pretest and posttest about the image of a vertex of a triangle after a translation. The most difficult item appeared only on the posttest. It concerns the effects on the volume when tripling the dimensions of a toy truck.

Data about teachers' backgrounds, their use of the UCSMP *Geometry* textbook, and their instructional practices come from five additional sources:

- A *Beginning-of-the Year Questionnaire* about the teachers' backgrounds;
- *Chapter Evaluation Forms* that teachers completed at the end of each chapter taught, indicating which lessons had been taught, which questions had been assigned, and the instructional practices specific to that chapter that the teachers had used;
- An *Opportunity-to-Learn Form* for each posttest item, on which the teachers reported if they had taught or reviewed the mathematics needed for their students to answer that item;
- An *End-of-Year Questionnaire* about instructional practices, including questions about the teacher's emphasis on reading and writing mathematics and students' engagement in mathematical activities using concrete materials;
- A Structured Interview with each teacher after observing his or her geometry classes.

Data about school enrollment, school type, teachers' certification and experience, as well as additional sample questions and activities appear in Thompson and Senk (2014). The complete set of instruments for the evaluation study is described in Thompson and Senk (in preparation).

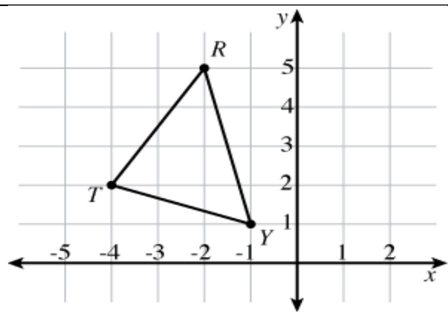
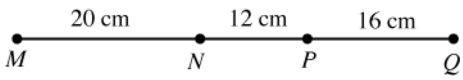
Procedures

Rasch models produce an estimate of knowledge of geometry, called a theta estimate, for each student. These theta estimates have a distribution with mean of 0 and standard deviation of 1. These estimates on the same scale allow pretest and posttest performance to be compared directly. Because negative Rasch theta estimates are sometimes difficult to understand, each theta estimate was converted to a T-score with mean of 50 and standard deviation of 10 (i.e., T-score = 50 + Rasch theta * 10). Descriptive statistics for measures of geometry achievement by school, gender, and grade level were then calculated.

Teachers reported the overall lesson coverage and instructional strategies rather than by individual geometry class. Therefore, the data about curriculum enactment by teacher were aggregated when exploring the models of geometry achievement. We ran a series of multilevel analyses using SAS 9.4 (SAS Institute, 2013) with the PROC GLIMMIX procedure to examine effects of various factors on students' achievement. The dependent variable for all regressions was the posttest T-score. Level 1 predictors were variables about each student (n = 544). Level 2 predictors were variables about the schools, individual teachers, or features about the teachers' reported enactment of the geometry curriculum (n = 12). Because each school had only one teacher in this study, the teachers' relevant variables were included as school level (Level 2) predictors. The 21 variables used as predictors are given in Table 3.

Table 2

Stems of Sample Items from the Posttest and Rasch Item Difficulties[4]

Pretest Item number	Posttest Item Number	Rasch Item Difficulty	Item Stem
4	2	-1.287	Triangle *TRY* is translated 3 units to the right and 4 units up. What will be the coordinates of the image of point *Y*? 
na	3	-0.206	*M*, *N*, *P*, and *Q* are collinear, as shown below. What is the distance between the midpoint of $\overline{MN}$ and the midpoint of $\overline{PQ}$? 
6	10	0.342	In a quadrilateral, each of two angles has a measure of 115°. If the measure of a third angle is 70°, what is the measure of the remaining angle?
na	27	1.636	The midpoints of the sides of Δ*ABC* are connected, forming Δ*XYZ*. Which is NOT always true? (Choices were statements about similarity/congruence, sides, angles, or area.)
35	19	2.111	Due to a chemical spill, the authorities had to evacuate all people within 5 km of the spill. To the nearest square kilometer, how much area had to be evacuated?
na	13	2.446	Which picture shows a counterexample to the statement *If a figure is a parallelogram, then it has a diagonal that bisects two of its angles*? (Choices were pictures)
na	34	3.490	Two toy dump trucks are similar. The dimensions of one truck are 3 times the dimensions of the other. If the smaller truck can carry 2 cubic inches of dirt, how much can the larger truck carry?

[4] From: Geometry Readiness Test and/or Geometry Test developed by the University of Chicago School Mathematics Project (UCSMP). Posttest items 2, 27, 19, 13, and 34 were developed by UCSMP personnel, © 2006/2007 and reprinted with permission of UCSMP. Item 3 on the posttest was a released item from the National Assessment of Educational Progress and used in accordance with its policies, U.S. Department of Education, Institute of Education Sciences, National Center for Education Statistics, National Assessment of Educational Progress (NAEP), 1990 Mathematics Assessment (Grade 12). Item 10 was a released item from the TIMSS 1999 Assessment (Grade 8) and used in accordance with its policies. © 2001 International Association for the Evaluation of Educational Achievement (IEA). Publisher: TIMSS & PIRLS International Study Center, Lynch School of Education, Boston College, Chestnut Hill, MA and International Association for Evaluation of Educational Achievement (IEA), IEA Secretariat, Amsterdam, the Netherlands.

Table 3

Independent Variables Used as Predictors in Multilevel Analyses

Level: Category	Predictor Variables
1: Student	Gender (0 for female, 1 for male)
	Grade (7 – 12)
	Pretest score (Rasch T-score)
2: School	Type (public or private)
	School Enrollment (rounded to the nearest hundred)
	Instructional time (mins/week)
2: Teacher	Secondary certified? (no = 0, yes = 1)
	Number of years teaching mathematics
	Number of years teaching UCSMP *Geometry*
2: Curriculum enactment	Percent of lessons taught from *Geometry* textbook
	Percent of *Covering* questions assigned from lessons taught
	Percent of *Applying* questions assigned from lessons taught
	Percent of *Review* questions assigned from lessons taught
	Posttest Opportunity-to-Learn (OTL) as a percent
	Reading emphasis (index is sum of values for 3 separate questions)
	Writing emphasis (index is sum of values for 3 separate questions)
	Percent of class time reported spent on whole class instruction
	Percent of class time reported spent introducing new content
	Time expected for students to spend on homework (in intervals)
	Percent of class time reported spent reviewing homework
	Reported frequency of use of activities with concrete materials

Results

First, we present descriptive statistics for scores on the pretest and posttest with the factors we hypothesized that may affect students' achievement. Second, we present the models we built for predicting posttest scores. Finally, to illustrate how the specific significant factors found in our final model may affect achievement, we describe specific characteristics and actions of four of the 12 teachers in our sample.

Descriptive Statistics

Table 4 presents the mean percent correct for pretest and posttest by school before equating as well as the Rasch theta estimates and T-scores after equating. Because the two tests have different difficulty levels, the mean percent correct for pretest and posttest before equating are not appropriate for comparison purposes. In contrast, the Rasch theta estimates and T-scores for pretest and posttest are placed on the same scale, so they can be used for comparisons.

Table 4 also shows the change in T-score from pretest to posttest, denoted as Δ T-score, as well as the output from paired *t*-tests of the statistical significance of those changes for each school. As seen in Table 4, all schools showed significant increases in T-scores for the posttest compared to T-scores for the pretest. The average T-scores were 47.46 and 55.60 for pretest and posttest, respectively, an increase of 8.14 or almost one standard deviation of T-score ($p < .001$).

Table 4

Mean Geometry Scores by School, as Percent, Rasch theta, and T-score, and Output of Paired t-test

School	*n*	Pretest			Posttest			Paired *t* test	
		Percent correct	Rasch theta	T-Score	Percent Correct	Rasch theta	T-score	Δ T-score	*t*(*df*)
09	19	51.88	-0.34	46.63	57.29	0.75	57.47	10.84	8.69 (18)***
25	67	52.11	-0.35	46.49	54.29	0.59	55.95	9.46	14.97(66)***
26	61	35.69	-1.09	39.11	35.27	-0.31	46.87	7.76	9.90(60)***
27	79	55.33	-0.19	48.13	62.03	0.99	59.89	11.76	20.85(78)***
28	50	47.94	-0.53	44.71	51.43	0.47	54.74	10.03	13.88(49)***
29	37	55.83	-0.18	48.19	51.58	0.47	54.74	6.55	8.44(36)***
30	47	60.61	0.06	50.61	51.85	0.46	54.64	4.03	5.47(46)***
31	51	77.37	0.86	58.63	69.52	1.35	63.46	4.83	5.88(50)***
32	56	50.61	-0.41	45.88	46.53	0.24	52.39	6.51	7.99(55)***
33	12	55.24	-0.22	47.82	56.43	0.72	57.23	9.41	6.35(11)***
34	11	63.38	0.19	51.86	65.46	1.12	61.24	9.38	5.91(10)***
35	54	53.02	-0.31	46.87	51.53	0.48	54.77	7.90	11.36(53)***
Total	544	53.93	-0.25	47.46	53.36	0.56	55.60	8.14	32.45(543)***

Note: *** indicates the *p*-value for a paired *t*-test is less than .001.

Students in School 31, a public suburban school where the geometry students were gifted 8th Graders, had the highest performance on the pretest and posttest, but their increase was the second lowest, perhaps reflecting a ceiling effect for these students. Students in School 27, a public school in a small town, increased their T-score by more than one standard deviation, the highest increase of any school. Their pretest scores were slightly higher than average, but their posttest performance was significantly higher than average. Other schools with gains in T-score of more than one standard deviation were School 9, a private suburban religious school, and School 28, a public rural school. Students in School 26, a private religious urban high school for boys, had the lowest performance on the pretest and posttest, and their increase was about three-fourths of a standard deviation. The pretest performance of students in School 30, another suburban public school, was higher than the average, but their posttest performance was lower than average. They also showed the lowest gain from the beginning to the end of the year.

Table 5 shows descriptive statistics of the means and standard deviations for the pretest and posttest by grade level and gender. The mean scores of all grades and both genders showed an increase from pretest to posttest of around 8 points except for Grade 8 (an increase of 5.65). However, Grade 8 students had the highest average T-scores for pretest and posttest compared to other grades. Apparently, as grade level increases, the T-score decreases. The performance of male and female students on pretest and posttest was similar.

Table 5

Mean and Standard Deviation of Geometry Scores by Gender and Grade Level

		Pretest T-score		Posttest T-score		Δ T-score	
	n	Mean	SD	Mean	SD	Mean	SD
Grade							
8	58	57.87	6.55	63.51	6.63	5.65	6.25
9	144	50.54	5.72	59.00	6.30	8.46	6.05
10	302	44.50	6.60	52.96	7.10	8.46	5.73
11	40	43.69	4.80	51.91	4.95	8.22	4.66
Gender							
Male	290	47.30	8.33	55.65	8.47	8.35	6.14
Female	254	47.65	6.86	55.55	6.75	7.90	5.50

Note: There was only one student in grade 7 and one in grade 12, so they were grouped with grade 8 and grade 11, respectively.

Table 6 reports data on selected aspects of teachers' curriculum enactment. The number of lessons taught by each teacher is given as a percent of the 114 lessons in the textbook. Percentage of homework questions assigned is based on the number of questions in the lessons taught, which varied by teacher. The emphasis given to reading and writing is quantified as an index created based on teachers' reported responses to three questions about the frequency of their practices related to reading/writing in geometry class. The maximum value of each index is 10. The questions about reading and writing have been reported in Thompson & Senk (2014). The value for the third instructional strategy in Table 6, Use of Concrete Materials, was based on a single item which asked the teachers to state the frequency of opportunities for students to engage in activities using concrete materials with *almost never* = 1, *sometimes* = 2, *often* = 3, and *almost all* = 4. Posttest OTL is the percent of questions on the posttest for which the teacher indicated that he or she had taught or reviewed the material needed for the student to answer the item.

Teacher G (School 31) taught the highest percentage of lessons in the textbook (92%). Teachers A (School 25), B (School 26), F (School 30), and K (School 35) taught the least with each reporting having taught less than 60% of the textbook's lessons. Teacher E (School 29) assigned almost all questions in the lessons that he taught. In contrast, Teacher F (School 30) assigned only 24% of the questions in the lessons he taught. Large variations were also observed in the percent of questions assigned from each of the *Covering*, *Applying*, and *Review* sections in the lessons and in the three instructional practices noted in Table 6. All but two teachers reported that they had taught or reviewed the material needed by their students to answer at least 80% of the posttest questions. The exceptions were Teachers B and K, two of the four teachers who taught the least number of lessons.

Table 6. *Teachers' Textbook Use, Reported OTL on the Posttest, and Instructional Practices*

School/ Teacher	Percent Lessons Taught	Percent of Questions Assigned Based on Lessons Taught				Instructional Practices			Post-test OTL
	n = 114	Covering the Ideas	Applying the Math	Review	Total	Reading	Writing	Use of Conc Matl	
09/I	62	96	91	85	91	10	7	2	80
25/A	53	53	32	19	37	4	7	1	91
26/B	54	59	41	18	42	6	3	2	66
27/C	68	79	82	54	73	7	5	1	80
28/D	66	37	89	78	65	8	10	2	91
29/E	85	100	100	99	100	8	9	2	94
30/F	57	31	24	12	24	3	8	3	97
31/G	92	85	84	86	85	4	8	1	100
32/H	73	87	62	21	61	4	4	3	80
33/J	74	94	79	78	85	8	9	3	94
34/L	78	89	77	19	67	8	7	2	100
35/K	55	80	79	62	75	6	7	3	77

Models of Posttest Achievement

A series of two-level regression analyses with different sets of predictors were conducted to explore the best-fit model of posttest achievement. Using the Rasch T-scores on the posttest as the dependent variable, we first ran a two-level regression analysis without any predictor (referred to as an unconditional means model, also known as a one-way ANOVA with random effects) as a baseline model to obtain between-school and within-school variances. Variances between-school and within-school can be used to calculate the interclass correlation (ICC) and proportion of the dependent variance explained by the predictors in the following models. The ICC in this study was 0.30, a moderate to large level, supporting use of multilevel regression analysis. We then added the three Level 1 variables as predictors (see Table 3). Pretest T-score and grade were significant at $p < 0.001$ and gender at $p < 0.05$. This model with three variables related to students accounted for approximately 73% of the school-level variance.

We kept all three Level 1 variables and further added different sets of Level 2 variables (i.e., school characteristics, teacher characteristics, and curriculum enactment; see Table 3) in the models to explore which other variables affect students' performance on the geometry posttest. No school characteristic or teacher characteristic variables were found to be significant. However, in each model, student-level factors continued to be strong predictors of end-of-year achievement. Several aspects of curriculum enactment were also found to be significant. Table 7 shows all factors in our final model together with their regression coefficients.

Table 7
Unstandardized Coefficients and Significance for a Multilevel Linear Regression Model for Posttest T-Score

Effect	Solutions for Fixed Effects				
	Estimate	Standard Error	*df*	*t*	*p*
Intercept	27.26	7.41	7	3.68	0.008
Pretest (T-scores)	0.60	0.04	528	16.49	<0.001
Gender	0.88	0.45	528	1.94	0.053
Grade	-1.45	0.39	528	-3.70	<0.001
Posttest OTL	0.40	0.12	528	3.43	<0.001
Percent of Lessons Taught	-0.30	0.07	528	-4.26	<0.001
Percent of *Applying* Questions Assigned	0.17	0.04	528	4.60	<0.001
Writing Emphasis	-1.42	0.49	528	-2.93	0.004
Use of Activities with Concrete Materials	-1.14	0.47	528	-2.43	0.015

All predictors except for gender significantly influenced posttest performance at the $p < 0.05$ level. The proportion of the between-school variance of the dependent variable explained by these predictors was 95%. Our final model shows that prior knowledge is the strongest positive predictor of future achievement. For every increase of one point in T-score on the pretest, the posttest T-score increased by approximately 0.6 points after controlling for other variables. Posttest OTL and percent of *Applying the Mathematics* questions assigned also contributed to increased posttest scores, whereas increases in the grade level, percent of lessons taught, emphasis on writing mathematics, and use of activities with concrete materials resulted in lower total posttest scores.

A Closer Look at Four Cases

In order to examine more closely how the statistically significant factors identified in our multilevel models affect achievement, we identified several teachers whose students started the school year with comparable scores on the *Geometry Readiness Test*, but whose posttest scores are quite different.

Teacher B (School 26) and Teacher D (School 28) were identified because their students had the two lowest mean scores on the pretests. However, by the year's end, the scores of students in School 28 had increased considerably more than those of students in School 26. Specifically, as shown in Table 4, at the end of the school year, the students in School 26 still had the lowest mean score on the *Geometry Posttest*, and their T-score had increased by less than the average gain (Δ T-score = 7.76 vs. 8.14). In contrast, at the school year's end, T-scores of students in School 28 had improved by 10.03 points, which is more than the average gain. Teachers C (School 27) and F (School 30) were also identified as potentially interesting because their students started the school year at or above the sample average. However, at the end of the school year, the gains made by their students differed dramatically. During the year, the improvement in T-scores of students in School

30 was less than those in any other school (Δ T-score = 4.03). These students scored below average on the posttest, in fact, lower than the students of Teacher D. In contrast, students of Teacher C in School 27 showed the largest gain in geometry achievement (Δ T-score = 11.76). By examining practices of teachers whose students' scores improved more than their colleagues who taught students with similar scores in the *Geometry Readiness Test*, we had hoped to uncover factors beyond those we had examined quantitatively

Teachers C and F had more instructional time (55 and 60 min/day, respectively) than either Teacher B (48 min/day) or Teacher D (45 min/day). Teacher B had more experience teaching mathematics (25 years) than either Teachers C (1 year), D (3 years), or F (4 years). Teacher B was certified to teach mathematics only in Grades K-9, whereas the others were certified to teach in middle and high school. But neither teachers' backgrounds nor school characteristics were significant predictors in our final model.

Our final model indicates that on average, for every one percent increase in the *Applying the Mathematics* questions assigned, the posttest T-scores increased by about 0.17 points. As shown in Table 6, Teachers C and D assigned more than 80% of the *Applying the Mathematics* questions in the lessons they taught. In contrast, Teachers B and F assigned less than half of the *Applying the Mathematics* questions. Thus, Teachers C and D tended to assign tasks encouraging higher cognitive demand more frequently than Teachers B and F. The percent of *Review* questions assigned was not significant. This may be due to the fact that *Review* questions might have been similar to either *Covering the Ideas* (basic knowledge) or *Applying the Mathematics* (higher cognitive demand), and the percent of *Review* assigned does not indicate which type of review the teacher provided. However, Teachers C and D adhered more closely to the recommendations of the curriculum developers about assigning questions for homework than Teachers B or F.

Posttest OTL also has a significant positive effect on posttest scores, with each increase of one percentage point of OTL resulting in an increase of about 0.4 on posttest T-score. The posttest OTL reported by Teacher B (66%) was the lowest among the 12 teachers in our sample.

Grade level had a negative impact on posttest performance. Each increase of one grade resulted in a decrease of about 1.45 in the posttest T-score. In School B, all students were in grade 10. In the other schools, the geometry students were in mixed grades: School C: grades 9 – 12, School D: grades 9 – 11, and School F: grades 10 – 12. So, how grade levels related to posttest scores in these schools is not evident without disaggregating the data.

Percent of lessons taught has a small ($\beta = 0.3$) but significant ($p < .001$) negative effect on performance. This result means that, on average, for each increase of 1% in lessons taught, the posttest T-score decreases by 0.3. However, classes of these four teachers did not follow this general pattern. Students of Teacher D (66%) did better than those of Teacher B (54%), and students of Teacher C (68%) did better than those of Teacher F (57%). Thus, the use of this predictor seems to lead to inconsistent results. This could be due to some interaction between percent of lessons taught and the number or type of questions assigned that the model was not able to capture.

Summary and Conclusions

In this research, we investigated factors that contribute to achievement at the end of a course in secondary school geometry in the USA. Using multilevel regression analysis, it was found that students' prerequisite knowledge had a significant positive effect on posttest achievement, a result consistent with research reported by Carroll (1963), Bloom (1976), and De Jong, Westerhof, and Kruiter (2004). Gender was not significant. Grade level had a negative effect on posttest achievement while none of the school variables (type, enrollment, or instructional time) or teacher variables (certification or teaching experience) were significant. Of the 12 factors related to curriculum enactment, five had statistically significant effects on posttest achievement. Percent of *Applying the Mathematics* questions assigned and Posttest OTL had positive effects on posttest T-scores, whereas percent of lessons taught, writing emphasis, and use of activities with concrete materials each had negative effects. In all, the seven significant predictors (two student factors and five curriculum enactment factors) account for about 95% of the variance when posttest T-score is the dependent variable.

These results have practical as well as statistical significance. The finding about prerequisite knowledge underscores the importance of building a strong foundation in geometry concepts in lower grades in order to maximize success in secondary school. Curriculum enactment factors, unlike student and school characteristics, are variables within the control of the geometry teacher. The significance of the percent of *Applying the Mathematics* questions assigned illustrates the importance of regularly assigning multi-step tasks or tasks that require students to apply their knowledge in new settings. The use of cognitively demanding tasks, especially in ways that encourage multiple solution strategies, multiple representations, and explanations, has been shown to result in learning gains by Stein and Lane (1996). Senk, Thompson, and Wernet (2014) found that posttest OTL was a positive predictor of achievement on functions in an advanced algebra course. In this study, posttest OTL was a strong and consistent predictor of posttest achievement because teachers were answering questions about very specific test items and linking them to what they have taught.

The negative effect of grade level on achievement likely reflects a practice in the USA in which students of high ability are often encouraged to study geometry at earlier grades than students of average or low ability. Our finding of negative effects of percent of lessons taught is puzzling. As noted earlier, researchers in Missouri reported that the percent of textbook lessons taught had significant positive effects on achievement in two studies (Grouws et al., 2013; Tarr et al., 2013), but was not significant in a third (Chávez et al., 2015). We found that percent of lessons taught had a negative effect. Clearly researchers should continue to study this variable and how it is related to other opportunity-to-learn variables. As Burstein et al. (1995) reported, teachers tend to answer questions about whether they had taught the mathematics needed to answer a specific item more reliably than whether they had taught more general topics (e.g., congruence or linear functions). This suggests that Posttest OTL is a more reliable measure of learning opportunities than lesson coverage, and that percent of lessons taught is not as meaningful as a predictor. As Thompson and Senk (2017) have advocated, teachers' reported posttest opportunity-to-learn measure is an important variable in considering the content validity of an achievement assessment, and as shown here, is especially important when building predictive models.

In retrospect, the negative effects of writing emphasis and engagement with concrete materials may be related to how these variables were measured. Each score was determined by only a few questions about the teacher's frequency of use of a particular instructional practice. Hence, they may not be sufficiently sensitive in reflecting the constructs that they were intended to measure. For instance, on the *End-of-Year Questionnaire*, we did not ask what concrete materials were used (e.g., geometric solids or patty paper) or how the students used the materials. Future research should investigate how to measure such constructs reliably and how to weight such variables in analyses. Perhaps in future research, factor analyses could be administered using a larger number of questionnaire items that utilize Likert scales to help identify key constructs for building more precise models of students' achievement.

The statistical power of the models resulting from our regression models is limited because only 12 teachers were studied. We visited each of the 12 teachers for two days as we would not have been able to visit 100 or 1,000 teachers. Using electronic surveys, it is now possible to scale up data collection for some variables that were found to be significant, such as Posttest OTL, percent of lessons studied, and percent of questions assigned. Additional work is needed to determine how researchers can measure other aspects of classroom enactment such as expectations in order to model achievement for a large school district, state, or country. Researchers working on the COACTIV Project in Germany (Kunter, et al., 2013a, 2013b) and the COSMIC Project in the USA (Chávez et al., 2015; Grouws et al., 2013; Tarr et al., 2013) have also worked on building models of secondary students' mathematics achievement. More sharing of research methods would be helpful, particularly those engaging in such investigations at scale.

Some of the variance not accounted for by the regression models in this study may be due to other factors directly related to the students. For instance, the time students devote to homework, their use of technology, or their persistence when studying geometry. Some student self-reported data on these issues were aggregated at the class level, and originally, the researchers had hoped to include such factors for further analyses. However, because of the type of permission that was granted by the Institutional Review Board, we were not able to link this data to individual student's test scores and use this data in the predictive models.

As other researchers (e.g., Hill, Rowan, & Ball, 2005; Kunter et al., 2013a, 2013b) have found, teachers' knowledge may also be a factor in students' achievement. In particular, it is not clear how the mathematical background of Teacher B, who was not certified to teach high school mathematics, affected her ability to enact the geometry curriculum or set high expectations for her students. However, we do not have any direct measures of teachers' knowledge, so we were not able to investigate this issue in the present study but recognize the need for researchers to examine teachers' knowledge as a factor in future studies. Researchers and developers may also need to consider what professional development is needed to help teachers implement geometry curriculum materials to promote the instructional practices that we found resulted in higher achievement.

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ENGAGING STUDENTS WITH NON-ROUTINE GEOMETRY PROOF TASKS

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Students who earned high marks during the proof semester of a geometry course were interviewed to understand what high-achieving students actually took away from the treatment of proof in geometry. The findings suggest that students had turned proving into a rote task, whereby they expected to mark a diagram and prove two triangles congruent.

Keywords: Conjecturing, diagrams, doing proofs, drawing conclusions, figures, focus group interviews, proof, student thinking, tasks, theorems, triangle congruence, two-column proof

Introduction

Although there have been ongoing calls to improve the treatment of reasoning and proof in school mathematics, success in teaching proof has remained elusive. For example, in the introduction to their chapter on the teaching and learning of proof, Harel and Sowder (2007) noted: "Overall, the performance of students at the secondary and undergraduate level is weak….it is clear that the status quo needs and has needed improvement" (p. 806). There is evidence that this need for improvement exists in many parts of the world (Hershkowitz et al., 2002; Reiss, Heinze, Renkl, & Gross 2008). This study's focus is on proof in the context of high school geometry. The research question for this study is as follows: How do students who earned high marks (i.e., earned As and Bs) in a high school geometry course respond when asked to engage in non-routine geometry proof tasks? This work falls under a larger study aimed at understanding the challenges of teaching proof in the high school geometry course (see Cirillo, 2014; Cirillo, McCall, Murtha, & Walters, 2017, for more detail).

Theoretical Perspective

Past research has suggested that most students do not enter high school geometry prepared to learn proof. For example, at the end of the school year, after a full course in geometry, Senk (1985) found that only 30% of U.S. students reached a 75% mastery level. This low percentage might be explained by the statistic that more than 70% of students begin the course at van Hiele Levels 1 or 2, and only those students who enter high school geometry at Level 3 (or higher) have a good chance of becoming competent with proof by the end of the course (Shaughnessy & Burger, 1985). Nearly two decades after Senk (1985) published her work, McCrone and Martin (2004) modified some assessment items from Senk (1985) and TIMSS (IAEEA, 1995), only to find results similar to those of Senk (1985), Healy and Hoyles (1998), and Chazan (1993). That is, geometry students continue to have great difficulty constructing original, deductive proofs.

Unfortunately, geometry teachers do not fare much better when it comes to feeling confident in their ability to teach proof. Researchers have found that teachers view the teaching of proof in geometry to be a difficult endeavor (Knuth, 2002). In fact, Farrell (1987) indicated that the high school geometry course is a feared teaching assignment for beginning teachers. Cirillo (2011) conducted a case study on secondary teacher, Matt, who claimed that one cannot teach someone to write a proof. Matt believed that when students look at proof problems, they either see how to do them or not; he also said, "seeing it is nothing that I can teach you" (Cirillo, 2011, p. 246). While conducting classroom observations, Cirillo also observed two different teachers telling their students that a “shallow end” to teaching proof did not exist. Rather, teachers simply needed to throw students into the “deep end” of a metaphorical proof pool (Cai & Cirillo, 2014). Clements (2003) cited impoverished curriculum materials as one potential explanation for these kinds of findings.

Proof in U.S. Geometry Textbooks

Analyses of U.S. textbooks verify that a compartmentalization of proof in the high school geometry course still exists (see Thompson, 2014). Yet, even within the six most popular U.S. geometry textbooks analyzed by Otten, Gilbertson, Males, and Clark (2014), a 30% sample from each textbook only yielded 5% of textbook exercises that asked students to construct a proof on their own. In addition, the majority of expository mathematical statements were general, while the student proof exercises tended to involve particular statements. This means that students rarely had the opportunity to prove actual theorems. Instead, they received the "Given" and the "Prove" statements as well as a diagram to go with them. Even in instances where a student exercise did involve a general statement, more often than not, the textbook then provided a particular diagram labeled for students to use.

Sears & Chávez (2014) reported on the interaction between students’ opportunities to engage in proof through two geometry textbooks and its influence on enacted lessons. They found that even though the geometry textbooks had proof tasks of higher level cognitive demand, there was no guarantee that those tasks would be assigned, or that the levels of cognitive demand would be maintained from the written to the enacted curriculum. The three teachers in the study all admitted that they tended to pose lower-level tasks to students because they had not had much experience with proof before the geometry course. For example, one teacher described the proofs taught as "very basic, very obvious proofs" consisting of no more than 10 steps that were "never anything that's complicated" (Sears & Chávez, 2014, p. 776). Overall, these results indicate that current textbooks and classroom experiences may not provide students with many opportunities to appreciate the generality of proof or develop proving competencies. After observing this situation themselves, Cirillo and Herbst (2012) suggested a set of alternative problems that could allow students to play a greater role in proving by, for example, having students make reasoned conjectures, using conjectures to set up a proof, and evaluating mathematical proofs by looking for errors or determining what was proved.

"Doing Proofs" in Secondary Geometry

Over the past three decades, several researchers have provided classroom accounts of what proving in geometry looks like. For example, Schoenfeld (1988) claimed that in most tenth-grade geometry classes there is a strict protocol, wherein one lists what is given and what is to be proved; one then draws a *T*, which divides the space below the problem statement into two columns, labeled "Statements" and "Reasons." These statements are numbered with one statement per line - the right-hand column contains justifications which are numbered to correspond to statements; and the last entry in the statements column is the result to be proved. Schoenfeld also observed that, particularly when proof is being introduced, a great deal of time is spent on the form over the content of proofs.

In her study, *Teachers' Thinking about Students' Thinking in Geometry: The Effects of New Teaching Tools,* Lampert (1993) outlined what doing a proof in high school geometry typically entails. According to Lampert, students are first asked to memorize definitions and learn the labeling conventions before they can progress to the reasoning process. They are also taught how to generate a geometric argument in the two-column form where the theorem to be proved is written as an 'if-then' statement. After students write down the "givens" and determine what it is that they are to prove, they write the lists of statements and reasons to make up the body of the proof. In this context, there is never any doubt that what needs to be proved can be proved, and because teachers rarely ask students to write a proof on a test that they have not seen before, students are not expected to do much in the way of reasoning.

More recently, Herbst and colleagues (Herbst & Brach, 2006; Herbst et al., 2009) described a traditional sequence of what doing proofs looks like in modern-day geometry classrooms. For example, Herbst et al. (2009) described instances of student engagement with proof in various geometry courses in a high school. Through this work, they unearthed a system of norms that appear to regulate the activity of "doing proofs" in geometry class. The authors contended that a collection of actions related to filling in the two-column form are regulated by norms that express how labor is divided between teacher and students and how time is organized as far as sequence and duration of events. For example, the first 5 of 25 norms reported by Herbst et al. (2009) are listed below:

> Producing a proof, consists of (1) writing a sequence of steps (each of which consists of a "statement" and "reason"), where (2) the first statement is the assertion of one or more "given" properties of a geometric figure, (3) each other statement asserts a fact about a specific figure using a diagrammatic register and (4) the last step is the assertion of a property identified earlier as the "prove"; during which (5) each of those asserted statements are tracked on a diagram by way of standard marks. (pp. 254-255)

The authors argued that despite the superficially different episodes in which doing proofs were observed, there were deep similarities among those events. This model of the instructional situation of doing proofs as a system of norms is helpful to those who wish to investigate what it might mean to create a different place for proof in geometry classrooms (Herbst et al., 2009). The authors concluded that in the classrooms that they observed, the students' main responsibilities continue to be the production of statements and reasons in sequence. Students were rarely, if ever, responsible for fashioning an appropriate diagram or making connections to concepts that have not been activated by the problem or the diagram. The absence of these types of tasks may add to students' difficulties with proof.

Sub-Goals of Proof

Many researchers have generated ideas and findings about what makes the teaching and learning of proof in geometry a challenging task (Cirillo, 2014; Cirillo et al., 2017; Gal & Linchevski, 2010; Laborde, 2005; Smith, 1940). These findings support the work of decomposing the practice of proving so that the teaching of proof can be built in progressive steps towards a larger goal. Cirillo et al. (2017), for example, identified several sub-goals of proof in geometry. Here, four of those sub-goals are discussed with respect to the research literature.

Coordinating Geometric Modalities. The mathematics register draws on a range of modalities. What is important to this paper is the idea of working with diagrams. Although working with diagrams is central to geometric thinking (Sinclair, Pimm, & Skelin, 2012), doing so has proved to be a challenge for students (Laborde, 2005; Smith, 1940). Textbooks tend to define a term, perform a construction, or prove a theorem using the simplest possible figure and then expect students to apply what they have learned to more complex figures (Smith, 1940). For example, a figure such as a right triangle can be made complicated by turning it so that it rests on its hypotenuse rather than being oriented on one of its legs as students might expect to see it. Although Smith made these claims over 75 years ago, they remain true today. More recently, Gal and Linchevski (2010) identified several difficulties in geometry from the perspective of visual perception. These difficulties include: identifying a right angle, using the perpendicular symbol, naming angles, and naming polygons. For example, students might label a rectangle according to its verbal representation (reading letters from left to right) rather than using the convention that we name polygons in a clockwise direction. Finally, Laborde (2005) wrote about the diagram's hidden role in students' construction of meaning in geometry. Relevant to this paper, she highlighted the ways in which some information used in proofs is actually taken from diagrams such as the notion of betweenness of points. As another example, the intersection of two lines is often taken for granted from the diagram. Yet notions related to parallelism and perpendicularity cannot be directly assumed (Laborde, 2005).

Conjecturing. Stating the importance of conjectures, Lampert (1992) wrote: "Conjecturing about...relationships is at the heart of mathematical practice" (p. 308). Similarly, related to the importance of determining statements to prove, Meserve and Sobel (1962) wrote:

> Many people think of geometry in terms of proofs, without stopping to consider the source of the statements that are to be proved....Insight can be developed most effectively by making such conjectures very freely and then testing them in reference to the postulates and previously proved theorems. (p. 230)

If we are to engage students in meaningful mathematics, then we must allow them to discover and conjecture (Cirillo, 2009). This practice can start early, where students of all ages are capable of engaging in conjecturing.

Drawing Conclusions. The drawing conclusions sub-goal is about the ability to draw valid conclusions based on the information provided. One makes a deduction through the use of definitions, postulates, and previously proved theorems, or by discerning that something valid is true from a diagram (Cirillo et al., 2017). However, it is not uncommon for students to erroneously assume things about diagrams such as equality of angles from the appearance of a figure and their lack of understanding about how to draw valid conclusions (Smith, 1940). This is complicated by the notion discussed above related to how some textbook tasks require that students use information from a diagram even though teachers typically warn against it and may not be explicit about when it is okay or not.

Understanding Theorems. One important aspect of understanding theorems is choosing the hypothesis and the conclusion from a verbal statement. In Smith's (1940) study, half of the students assessed did not have an understanding of the if-then relationship that would allow them to correctly write the hypothesis and conclusion in terms of a figure. In her study published about

45 years later, Senk (1985) similarly found that only 32 percent of students assessed were successful in proving a theorem about congruent diagonals in a rectangle. To prove the theorem, students needed to identify the "Given" and the "Prove" statements from the theorem stated as: "The diagonals of a rectangle are congruent" (Senk, 1985, p. 451). Smith had also noted that students are likely to have trouble discerning a difference between a conditional statement and its converse such as those below, because the diagram for both will have a pair of sides and a pair of angles marked congruent:

- If two sides of a triangle are equal, the angles opposite those sides are equal.
- If two angles of a triangle are equal, the sides opposite those angles are equal.

Additionally, there is much to understand about theorems beyond identifying hypotheses and conclusions, such as understanding that a theorem is not a theorem until it has been proved and that theorems are only sometimes biconditionals (see Cirillo et al., 2017, for more on this sub-goal).

Methods

This study was part of a larger three-year project aimed at understanding the challenges of teaching proof in high school geometry. The data for this paper was collected during the baseline data collection year in the second term of a year-long high school geometry course, after the students had completed a semester-long study of proof in geometry. These students came from two different teachers' classes in an all-boys private school where conventional geometry textbooks were used, and the norms documented by Herbst and colleagues (2009) were frequently observed in the classroom lessons. In the first semester of the geometry course, students studied logic, geometric objects, triangle congruence proofs, and quadrilateral proofs.

Participants

The data set includes interviews of 15 students from Mr. Mack's and Mr. Walden's classes. The students attended a private boys school in the mid-Atlantic region of the U.S.A. Students were interviewed during a free period in groups of 3, 3, 3, 4, and 2 based on when they were available to meet with the interviewer (the author). Prior to conducting these interviews, the author had already observed two non-consecutive weeks of one section of Mr. Mack's and Mr. Walden's geometry classes. The observations were conducted while the teachers were introducing proof, in this case, using triangle congruence conditions, and again when they were working with students on quadrilateral proofs.

Interview Protocol

Students worked on tasks that appeared in or were inspired by Cirillo and Herbst's (2012) article: *Moving Toward More Authentic Proof Practices in Geometry.* Each student received a packet with the assigned tasks to complete. The goal of those tasks was to expand the role of the student in ways that differ from how they might engage with typical geometry textbook tasks (e.g., as described in Otten et al., 2014) or classroom tasks (as described by Herbst et al., 2009 and Herbst et al., 2013a) where (a) the "Given" and the "Prove" statements are provided to the students, and (b) students are expected to write two-column proofs.

Students who earned high marks (grades of A or B) in the first semester of the geometry course were interviewed in focus groups. The rationale for interviewing students with high marks was to understand what high-performing students were taking away from the course with respect to proof. The rationale for using focus groups was that students would engage in the tasks together, in groups, and the researcher would be able to capture students' thinking as they worked through the task aloud. The researcher interjected with questions when students seemed to be straying from the task's goal in order to maximize the time spent with the students during the interview. The focus group interviews were video-recorded. In addition, each student's written work was collected at the end of the interview. Interviews lasted about 40 minutes each.

Data and Analysis

Using Transana (Fassnacht & Woods, 2005), software that allows qualitative researchers to transcribe and analyze video or audio data, collection reports were developed. In particular, each interview was segmented by tasks attempted so that the researcher could conduct an item analysis, which looked across how each group approached each individual task. The researcher analyzed one task at a time, going back and forth between the student work, the video, and the transcript, looking for patterns across how the groups approached each task. Four tasks that were completed by all five groups were analyzed.

Findings and Discussion

In the sections that follow, the findings from the analysis of the student work and interview videos are presented. Descriptions that illuminate how the students thought about and solved each task are provided.

Task One: The Conjecturing Task

In the Conjecturing Task (see Figure 1), students were provided with a conjecture (the diagonals of a rectangle are congruent) and a diagram of a rectangle. They were then asked to write the "Given" and the "Prove" statements that could be used to prove the conjecture. Despite the fact that students were not asked to write a proof for this task, all five groups of students started to work on a proof at some point in the discussion of the task to prove that the two triangles were congruent. Most students began by calling out statements that they thought they should write in a proof before discussing what it was that they were trying to prove. The following transcript excerpt typified the discussions across the groups:

Mark: So, *AB* and *CD* are congruent.

Larry: Yeah, it's a rectangle.

Jamal: The diagonals are congruent, so *AC* and *BD* are congruent.

Mark: So, *AD* and *BC* are congruent….

Mark: And then we can have the triangles *DBC* and *DBA*, so all angles are congruent as well, so if all the angles are congruent you can break out the triangles *ABC* and *A*, or yeah, you can get *ABC* and *ADC*.

Larry: Oh, I see what you're doing now….

In most cases, students began working on the task without discussing what a conjecture was and by calling out things that they believed to be true. Figure 2 contains every "Given" and every "Prove" statement written on students' sheets. For each group, any unique statement appears only once. None of the groups were able to correctly solve the task, and three of the groups assumed the conclusion (i.e., that the diagonals of the rectangle were congruent).

Suppose you conjectured that the diagonals of a rectangle are congruent and your teacher started you off with the diagram on the right.

Write the "Given" and the "Prove" statements that you would need to use to prove your conjecture.

GIVEN: PROVE:

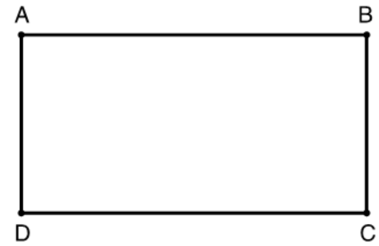


Figure 1: The Conjecturing Task: Students are asked to write the Given and Prove statements (Reproduced from Cirillo & Herbst, 2012, p. 17; used with permission under a CC license)

After calling out statements that they believed to be true, eventually, two groups did ask the interviewer what a conjecture was. Another group asked what they were trying to prove. Compared to the other groups who never discussed this explicitly before calling out statements, Group 5 asked the question pretty quickly, where the conversation went as follows:

John: So, the diagonals are congruent. So *AC* would be congruent to *BD*.

Lin: That means angles – angle *DAC* and angle *BCA* are congruent? What do we need to prove? (to the interviewer)

Interviewer: That's what I'm asking you, actually, to figure out what you could assume as given and what you would want – what you're wanting to prove, based on that conjecture.

Lin: Okay, *ABCD* is a square you would need to be given, or a, not a square, a rectangle. Um…

John: Yeah so the only given we have is uh, *AC* is congruent to *BD*.

Here, this group ultimately reversed the "Given" and "Prove" statements as shown in Figure 2.

Group	**Statements Written on the "Given" Line**	**Statements Written on the "Prove" Line**
1	$\overline{DB} \cong \overline{AC}$	$\Delta ABD \cong \Delta CDB$
2	*ABCD* is a rectangle Diagonals $\overline{AB} \cong/\| \overline{DC}$ $\overline{AB} \cong \overline{DC},\ \overline{AB} \parallel \overline{DC}$ $\overline{AD} \cong \overline{BC},\ \overline{AD} \parallel \overline{BC}$	$\overline{AC} \cong \overline{DB}$
3	$\overline{AD} \cong \overline{BC}, \overline{AB} \cong \overline{DC}$ $\overline{AE} \cong \overline{BC},\ \overline{AD} \cong \overline{DC}$	$\overline{AC} \cong \overline{DB}$
4	$\overline{AB} \cong \overline{DC}$, *ABDC* is a rectangle, All $\angle$'s are congruent $\overline{AD} \cong \overline{BC}$	$\overline{AC} \cong \overline{BD}$ $\overline{AC} \cong \overline{DC}$
5	$\overline{AC} \cong \overline{BD}$	*ABCD* is a rectangle
Correct Answer	*ABCD* is a rectangle	$\overline{AC} \cong \overline{DB}$

Figure 2: All "Given" and "Prove" statements written on students' papers in each group

Task Two: The Diagramming Task

In the Diagramming Task, students were provided with the "Given" and the "Prove" statements but were asked to draw a diagram that could be used to prove that two segments drawn within a parallelogram were congruent (see Figure 3). Three of the five groups of students had at least one student who incorrectly drew parallelogram *PQRS* as parallelogram *PQSR* (see Figure 4). Most of the students also had trouble drawing $\overline{ST}$ and $\overline{QV}$ wanting instead to draw the diagonals. Students commented that this setup was unusual: "We never did one like this before" and "We're trying to prove something about the diagonal." One student said he drew the diagonals in the parallelogram because it was "just a habit." Below is an example of a typical group discussion:

Ben:	I think *PQ* and *SR* are diagonals. I mean like, we -.
John:	They're not diagonals, but they're parallel. (Pause) You mean *T* and *V* are diagonals?
Ben:	Like, I, no, yeah. I think it's like this it's um like [draws *PSRQ*].
	PSRQ and *P* and *Q* are diagonals and *R* and *S* are diagonals.
Jeff:	Where would you put this midpoint? Just in the middle of the thing? But it says -
Ben:	Um, do like, when it says parallelogram *PQRS*, does it, is there any specific order that it has to be in, that the points have to be in?
John:	Um, did you make those diagonals?
Ben:	I think they are.
Interviewer:	Why do you think that they're diagonals?
Ben:	Um, I don't know. I just, I don't know. Cause like, I've never seen a problem where –
John:	Yeah, I've never seen like a variable in the middle of a line before.
Ben:	They're usually drawn –
John:	And I don't know how to, like, it's asking can we prove that *ST* is congruent to *QV*. So I mean does it, parallel lines by looking at them, but I need a way to prove that.

So here, the students seemed quite thrown off by the fact that the line segments drawn in the parallelogram are not the diagonals. It seemed that they were even trying to reorder the parallelogram's vertices, so they could somehow force the diagonals to be the line segments in the figure.

Draw a diagram that could be used to prove the following:

Given: Parallelogram PQRS where T is the midpoint of $\overline{PQ}$ *and V is the midpoint of* $\overline{SR}$.

Prove: $\overline{ST} \cong \overline{QV}$

Figure 3: The Diagramming Task: Students are asked to draw a diagram for the proof (Reproduced from Cirillo & Herbst, 2012, p. 17; used with permission under a CC license)

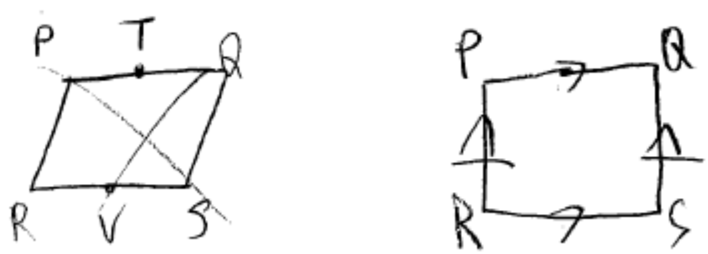


Figure 4: Parallelogram *PQRS* drawn as *PQSR* by two different students

Students from two different groups very quickly moved from drawing a diagram to drawing a "T" to write their two-column proofs (see, for example, Figure 5). In one case, the students attempted this after they quickly drew an accurate diagram, saying, "Wow! *ST* and *QV* – it's a cool problem." In another case, however, the students seemed to believe that writing a proof was their main goal. After being unsure of what to do about the diagram, they decided to try to write a proof, saying "Make a chart" with another following, "Yeah, let's make a chart." When asked why they said that, they explained that "chart" meant a two-column proof "because this is how we did proofs." A similar discussion occurred with another group as shown below:

Mark: You make a chart.

Jamal: Yeah, we should make a chart.

Interviewer: Okay, what's that you just said?

Mark: Oh yeah, I said that you should, uh, since we're trying to prove that *ST* is congruent to *QV* you're going to want to make a chart, at least this is how we did, uh proofs, so..

Interviewer: You mean a two column…[overlapping talk]

Mark: Two-column chart, yeah, like that. The statements and reasons….

So even though the task did not ask students to write a proof, Mark explained that this is how they did proof in their class, by making a "chart." The fifth group began discussing a plan for writing a proof, but it was unclear whether or not they realized that they completed the task after drawing the correct diagram.

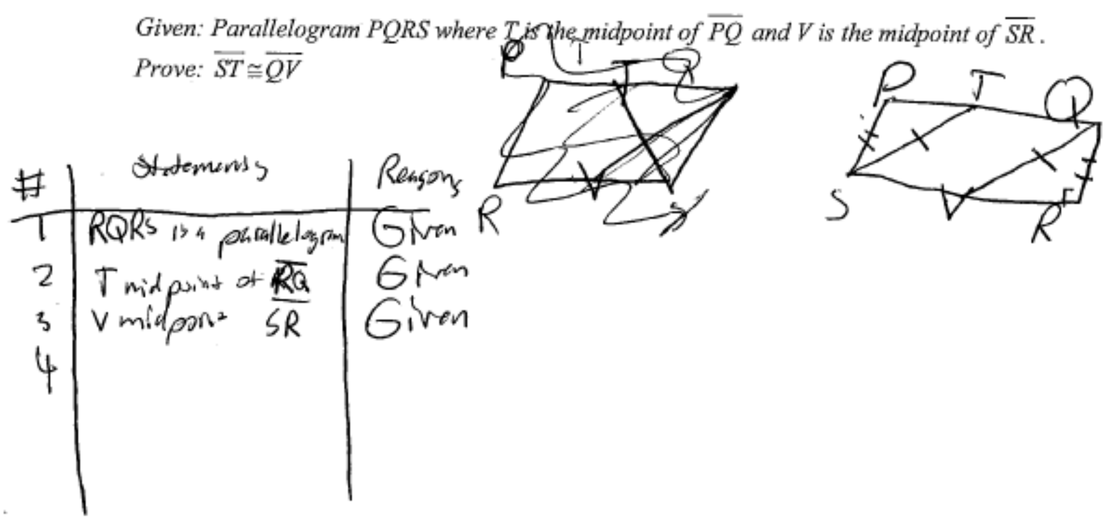


Figure 5: A student starts to write a proof after another group member suggests doing so

Task Three: The Drawing a Conclusion Task

In the Drawing Conclusions Task (see Figure 6), students were asked to draw a conclusion when provided with a particular "Given" condition and a diagram. In this case, students were not asked to write a proof of anything in particular, but rather to use the "Given" statement and the diagram to draw a valid conclusion. As they began this task, students in each group typically started by marking their diagrams (see Figure 7). Most noted that two triangles were formed and started making hash marks. Each group eventually drew a valid conclusion, but all groups were distracted by the diagram and put forth invalid assertions. For example, students from three of the five groups asserted that the angle bisector at *W* formed two right angles. Students from three groups also asserted that *W* was the midpoint of $\overline{XZ}$. Two groups debated these ideas and suggested alternate diagrams that would serve as counter-examples to these claims (see, e.g., Figure 8). All five groups thought that it was important to note that $\overline{YW}$ was congruent to $\overline{YW}$ by the reflexive postulate. When asked whether or not $\overline{YX}$ was congruent to $\overline{YX}$, students said no.

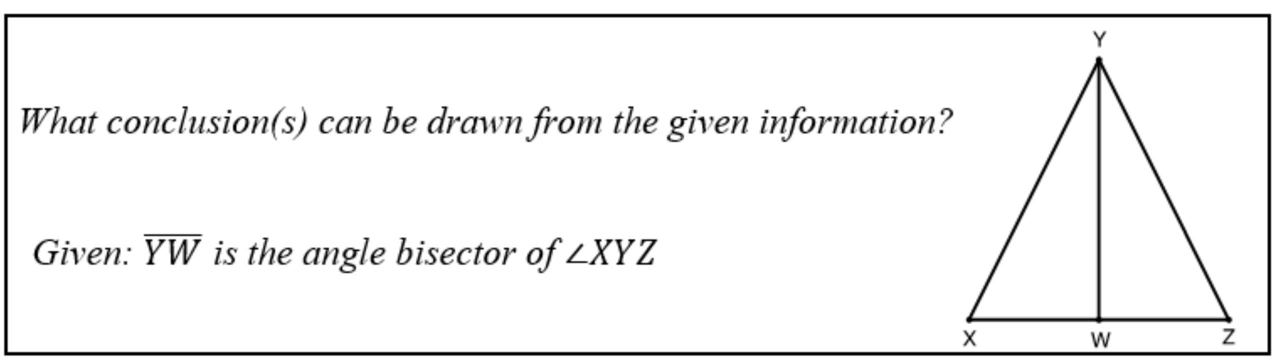


Figure 6: The Drawing Conclusions Task

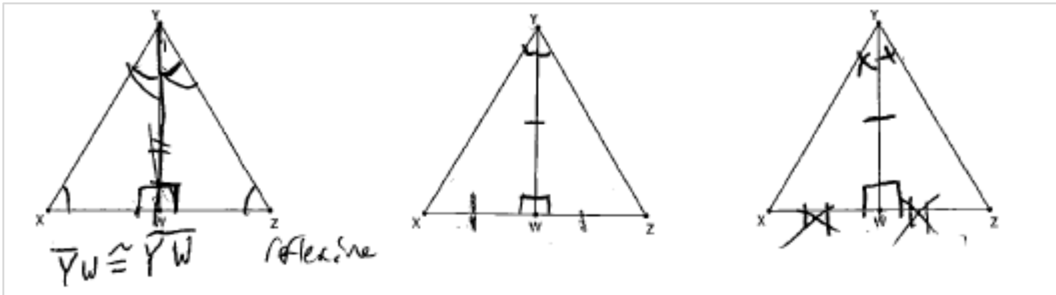


Figure 7: Student work samples where students marked up their diagrams

When the interviewer asked students “if you can go by the picture or just go by what is given,” one student said, "You have to go by what you're given." A second student said, "You have to go by what you're given, but you can also assess from the picture."

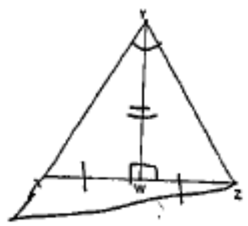


Figure 8: Student diagram for Task 3

Task Four: The Determining a Theorem Task

In the Determining a Theorem Task, students had the opportunity to analyze a completed proof. More specifically, students were provided with a proof of the Base Angles Theorem and asked to determine what theorem was proved (see Figure 9).

Because most groups seemed to have trouble getting started on this question, the interviewer typically said something like this to each group: “So, sometimes you’re given a theorem, and you’re asked to prove it. So this time I gave you the proof. What are you proving?” After still seeming confused by the question, the interviewer reminded students that theorems were typically statements written in the “If…, then…” form.

Students struggled quite a bit with this task. Initially, they struggled to try to understand what the task was asking them to do. Then, they were unsure about how to do it. Many of the groups started by saying that what we were proving was either that the triangles were congruent or that angles *A* and *B* were congruent. Below is a typical discussion of this task:

Liam: Wait, what? I’m confused. What was proved in the first place?

Kyle: Everything.

Liam: Is it just trying to prove all the angles or is it trying to say angle, uh, triangle *C*, *CBD* is congruent to angle, or uh triangle *CAD*?

Interviewer: That's actually the question I'm asking you. What's proved in the theorem? What's the theorem?

Kyle: Angle *A* is congruent to angle *B*, okay.

Jeremy: No, gotta write a theorem.

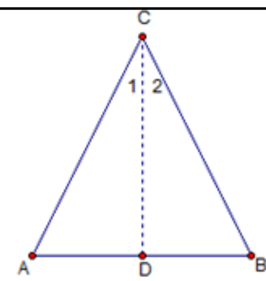
Liam: I think it's triangle *CAD* is congruent to *CBD*. I think that's what it's asking.

Kyle: If you look at the end, angle *A* is congruent to angle *B*. So, that's what was trying to be proved.

Jeremy: But what theorem is that?

Determine the theorem that was proved in the given proof.

Write the theorem that was proved by the proof below.



Statements	*Reasons*
1. $\overline{CA} \cong \overline{CB}$	1. Given.
2. Let $\overline{CD}$ be the bisector of vertex $\angle ACB$, *D* being the point at which the bisector intersects $\overline{AB}$.	2. Every angle has one and only one bisector.
3. $\angle 1 \cong \angle 2$	3. A bisector of an angle divides the angle into two congruent angles.
4. $\overline{CD} \cong \overline{CD}$	4. Reflexive property of congruence.
5. $\Delta ACD \cong \Delta BCD$	5. Side-Angle-Side ≅ Side-Angle-Side
6. $\angle A \cong \angle B$	6. Corresponding parts of congruent triangles are congruent.

(Adapted from Keenan & Dressler, 1990, p. 172)

Figure 9: The Determining a Theorem Task: Base Angles Theorem
(Reproduced from Cirillo & Herbst, 2012, p. 23; used with permission under a CC license)

Most of the groups ultimately got to a point where they were on the right track for stating the theorem, but then it took quite a while for them to articulate their thinking. For example, students would say things like, "If you have the two that are congruent in a triangle, then the opposite angles are congruent" or "If two sides of a triangle are congruent, then the corresponding angles are congruent" before either stating the Base Angles Theorem correctly or never getting there at all. One group wrote the theorem symbolically, first writing the converse of the Base Angles Theorem, and then the Base Angles Theorem, using notation that their teacher allowed them to use in their proofs (see Figure 10). Perhaps because they had so much trouble getting from the diagram to the verbal statement, students commented that maybe they should not have been allowed to write the theorem this way since they could not actually say what it meant. For example, when the interviewer commented about them having trouble putting it into words, one student remarked: "Yeah, cuz when we proved it, Mr. Mack just told us that it was alright if, in the reasons, if we just drew the picture." When asked if they thought this was a good idea, that same student responded, "I mean I liked it, but I guess that this just kind of proves that we know how to draw it, but we don't actually know the theorem."

(a) (b)

Figure 10: Student representations of the Base Angles Theorem (a) and its converse (b)

Students' Reactions to the Alternative Proof Tasks

After solving the tasks, students were asked to comment on the work that they did with the interviewer. They seemed to recognize that the tasks were different from the ones that they typically worked on in class, for example, noting, "They're different because we usually just have to write a proof." Students were generally positive about the tasks even though they were clearly challenged by them. They noted benefits of doing tasks such as the ones described here, saying, for example: "I think they make you think more about what you're actually proving…maybe think about what you're trying to prove and that helps to think about how you get there and how you prove it." Other students commented that they liked drawing the diagrams themselves: "I think having them draw the picture kind of gives you a better understanding of it…[since] I'm a visual learner." Some students seemed to prefer the "normal stuff" with one student commenting:

> Yeah actually I think we should like stick with the normal stuff we do for homework like proving the regular stuff, but then um, I guess also, once we've learned how to prove the regular stuff, we can have some fun with it I guess, because, so just like kind of change it up a bit, and try new things with it.

Responding to this comment, another student said that the tasks presented to them in the session were more challenging, and he guessed that "they help you learn proof better."

Summary

Students who earned high marks during the proof semester of the geometry course were interviewed to understand what they had taken away from the treatment of proof in geometry. It was observed that students struggled with similar things as in past studies. During the Conjecturing Task, students struggled greatly with determining what the conjecture's hypothesis and its conclusion were, exchanging the two in most cases. In the Diagramming Task, all students struggled to draw the diagram; some even struggled to properly draw and label parallelogram *PQRS*. The Drawing Conclusions Task elicited multiple assertions that should not have been claimed. These assertions resulted from what the diagram looked like rather than what students were told was "Given." Finally, in the Determining a Theorem Task, students thought that what was being proved was particular to the diagram, and they struggled to generalize the theorem. Even when they finally moved close to doing so, they tended to begin with the converse of the Bases Angles Theorem, seemingly not realizing that the Base Angles Theorem and its converse are different propositions.

Across the evidence, one can conclude that the students were accustomed to engaging in particular types of tasks where they were asked to write a two-column proof that somehow involved congruent triangles. The findings suggest that students had turned proving into a rote task, whereby they would identify two triangles in the diagram provided, mark the diagram, and then brainstorm as many conclusions as possible based on some of the written text in the task and the diagrams themselves. Students were challenged to complete tasks that did not follow their prototype of what "doing proofs" looks like. For example, Herbst and colleagues (2013a & b) documented normative classroom practices such as: students *are* typically provided with "Given" and "Prove" statements and they *are not* typically asked to sketch diagrams that could be used to write their proofs. The results of this study are reminiscent of the "bad results" of "good teaching" demonstrated by Schoenfeld (1988). More work is needed to understand how we can teach students to better understand the reasoning behind the proving. Future studies should also incorporate more typical types of proof problems to see how students think through those in contrast to atypical tasks. Finally, technologies such as smart pens could be used to better coordinate the discussions with the student work.

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ASPECTS OF SPATIAL THINKING IN PROBLEM SOLVING: FOCUSING ON VIEWPOINTS IN CONSTRUCTING INTERNAL REPRESENTATIONS

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What difficulties do seventh grade students have in constructing internal representations and in their mathematizing processes while considering external representations from various viewpoints? Students received a photograph and were asked to mark where on a map they think the photograph was taken. The results reveal seven types of places where students mark a point and six specific perspective cues they use. Different kinds of difficulty students had in each category are found by examining the relational terms, such as in front of*, or* right side*, used by the students. The study suggests that a possible cause of difficulty in constructing internal representations is a lack of connection between the objects in terms of their position and direction from several perspectives. Finally our data indicates that crating positional relation with information of real world is significant ability in mathematizing process.*

Keywords: internal representation, mathematizing process, spatial thinking, viewpoints

Introduction

Various situations occur in daily life where spatial thinking serves a purpose. Such examples include working with virtual reality like 3D maps on web sites and reading an instruction manual for assembling furniture. Due to the development of information and communication technology, more types of 3D representations like automobile's navigation systems are more prevalent than ever before. This increase indicates the importance of spatial thinking. According to the comprehensive report, "*Learning to Think Spatially*" published by the National Research Council Committee on Geography (2006), spatial thinking is a powerful tool, and it is fundamental to problem solving in a variety of contexts in living space, physical space, and intellectual space. In addition to recognition from educational researchers, spatial thinking has been getting attention in school curricula in Japan (Murakoshi, 2012). For example, map reading in geography, understanding solar trajectories in science, and reasoning geometrically in mathematics require students to think spatially. Compared to other subjects, mathematics plays a specific role in fostering students' ability to transform real-world phenomena into mathematical-world problem then solving problems in the mathematical-world.

In the Japanese geometry curriculum, learning goals related to spatial thinking are mainly related to sketching diagrams that include nets and projection views. There has been much research and ideas for practice in this area (e.g., Yamamoto, 2013) However, the majority of such research and ideas for practice deal with abstract objects such as prisms and pyramids. Moreover, results of the national achievement test in Japan report the difficulties students have with mathematizing real world problems (National Institute for Educational Policy Research[NIER], 2014).

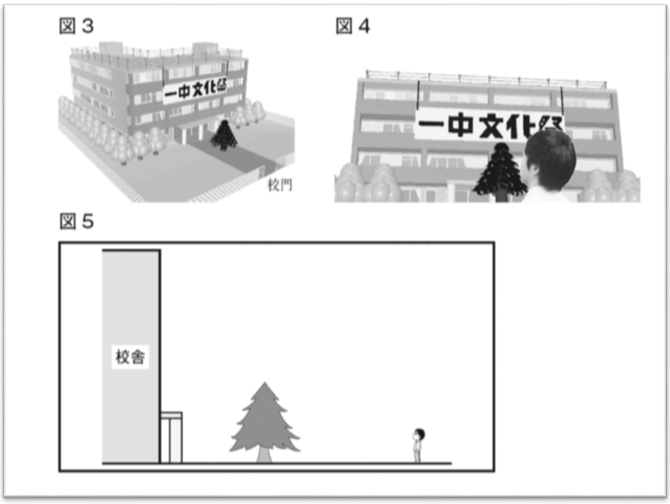

Figure 1: A test item in the Japanese National Assessment of Academic Ability
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Figure 1 provides an example of a real world problem. The question is: “there is a cultural festival. A hanging sign needs to be installed on our school building. Decide the lowest position possible for the display so that it is not eclipsed by the tree when someone looks at it from the sidewalk, and explain how to find the position of the sign using words or figures.” (ibid., p.98[1]) The A 61.3% of students answered this item correctly but this percentage is lower than achievement on other problems formulated with abstract objects. Therefore, NIER raised the issue that secondary school students have difficulties to simplify phenomena in order to interpret the results mathematically (ibid., p.102). These mathematical processes are very difficult for students to do in Japan. Therefore, research is needed to understand how students think spatially in real world situations and what difficulties they encounter in their mathematization processes.

In order to examine the role that spatial thinking about real world objects plays in students’ ability to mathematize those real world objects, this study explores students’ spatial thinking process while they solve problems with planar representations including photographs and maps. A photograph is an “in-between” representation of the actual object and its geometric diagram while a map represents the space with some information from real world. Bishop (1986) considers both photographs and maps as promising avenues in mathematizing space.

Theoretical Background

Research has shown some spatial abilities are present at birth but are slowly realized over years of development (Sarama & Clements, 2009). From a psychological perspective, according to Krutetskii, 1969), Thurston clarified the structure of human intelligence using factor analysis and showed that the primary mental abilities include a spatial factor. Thurston’s notion of primary mental abilities offers a provocative idea that if there is an appropriate combination of primary

[1] Author’s translation from the original in Japanese.

abilities which constitute mathematical ability, it is possible that mathematical ability could be developed by suitably stimulating those primary abilities besides teaching mathematics (Bishop, 2008). Therefore, spatial ability could be developed through stimulating spatial factor in mathematics education.

From a review of studies on factor analysis regarding spatial abilities, McGee (1979) distinguished two spatial factors, spatial visualization and spatial orientation, . Mathematics education also fosters them as competencies. Spatial visualization is the comprehension and performance of imagined movement of objects in 2D and 3D space; spatial orientation is the understanding and operation on the relationship between the objects' positions in space with respect to one's own position (Clements & Battista, 1992). For this paper's focus, spatial thinking is the intellectual exercise of mental operations to create mental spatial images that is supported by intuitive ideas in problem solving situations related to the real or abstract spatial world (Hazama, 2004). From this standpoint, spatial thinking is the activity supported by the competences of spatial visualization and spatial orientation.

The results presented in this paper focus on how students change their viewpoints, which is one of the important intellectual activities related to both spatial visualization and spatial orientation. Saeki (1987) mentioned that changing viewpoints contributes to the reconstruction of internal representations to solve a problem. Also considering an image as a coherent, integrated representation of a scene or object from a particular viewpoint (Eliot, 1987), we believe that looking at viewpoints offers the key to understanding how students create internal representation.

In cognitive psychology, perspective-taking has been discussed since Piaget's "Three Mountain Task." Voluminous literature on the development of perspective-taking provides evidence to support modifying Piaget's theory that young children are spatially egocentric until the age of nine or ten years. Recently, Watanabe and Takamatsu (2014) pointed out that there are processes used to solve a perspective-taking task, one of them being the imagination of body movement from another vantage point in 3D space. Therefore this study takes two types of viewpoint which are considering the part of the object the viewer sees from his or her position (Level 1) and considering the relationship the observer sees among objects as indicated by the cues he or she takes from viewing the objects while solving problems (Level 2) (Flavell, 1974).

With viewpoints thus defined, it is important to refine how spatial descriptions are formed. Spatial descriptions contain statements that locate objects from a reference frame, which includes an origin, a coordinate system, a point of view, terms of reference, and reference objects (Taylor & Tversky, 1996). In order to describe how students construct internal representation, the study focuses on the reference frame. The study's goal is to identify the difficulties students have in solving real world problems by analyzing the terms they use to relate the location of a landmark to a certain origin (i.e. the viewer's position).

Methodology

The participant sample included 60 seventh graders (33 males and 27 females) in a public school in July 2015. They had not learned how to create nets, map reading, or the topic of similarity. Each student received a questionnaire, which had two components. The first component asked the students if they had seen the objects in a photograph (Figure 2). The second component included two tasks: Task X and Task Y. These tasks were designed based on representational correspondence methods (Liben, 1997). Figure 2 shows that the given tasks required students to make a connection between two external representations for one particular place. The closed oval line with arrows at the center of the Figure 2 represents making a connection in the process of solving the tasks.

In Task X, representation 1 is a photograph, that is a 2D representation, of an elevation view. Representation 2 is a map, that is a 2D representation, representing a view from the top. Students were asked to place a point on the map (Figure 4) to indicate from where the photograph (Figure 3) had been taken and describe the reason for their choice. Both representations show three landmarks: TOKYO SKYTREE℠, East Tower, and a river.

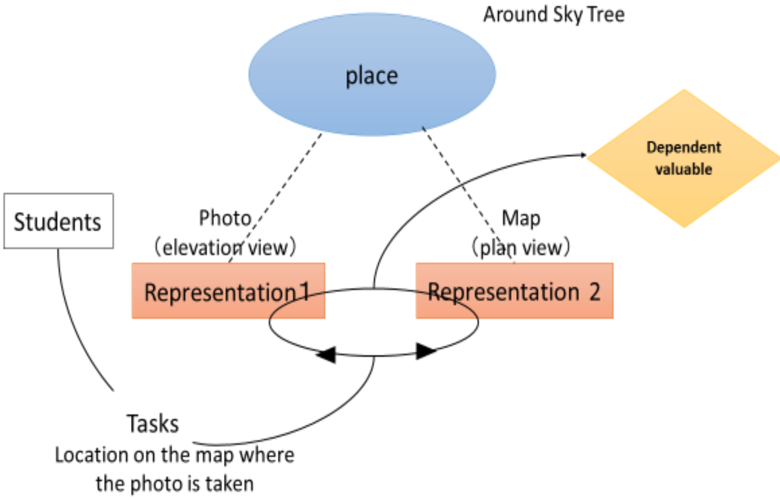

Figure 2: Representational Correspondence Methods
(Task X)

The relational terms *back* and *front* are shown on the photograph. Task X's purpose was to discover what kinds of difficulties seventh graders have in constructing internal representations through focusing on their viewpoints at level 1 and level 2.

In Task Y, Representation 1 included two photographs: One photograph had been taken from an airplane with information about the height and the distance between landmarks; the second photograph gives the appearance of the heights of the two landmarks looking the same from the front (Figure 5 and 6). Representation 2 was a map, a 2D representation with view from the top. The two landmarks, TOKYO SKYTREE℠ and Mt. Fuji, are well known in Japan. So, every student could have some images of them easily. Task Y asked students to estimate the location in which the photograph was taken (Figure 5) and put a point on the map (Figure 6) or explain it in words. Then, they needed to describe the reason for their location choice with figures and sentences. This task's purpose was to clarify how seventh graders mathematize the given problem and what difficulties exist in their mathematization processes when students analyze the external representations.

Figure 3. The photograph in Task X

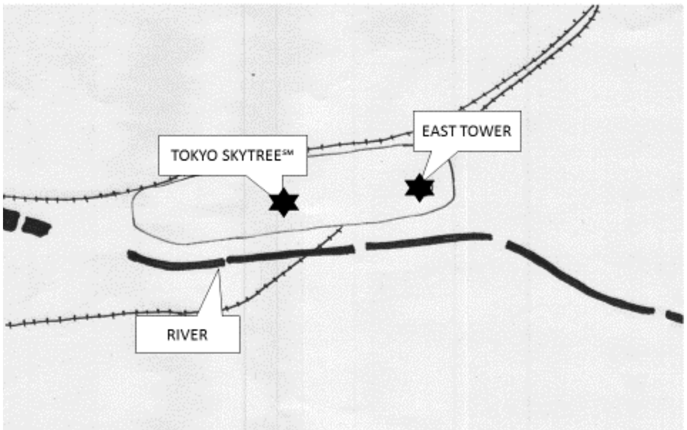

Figure 4. The map in Task X

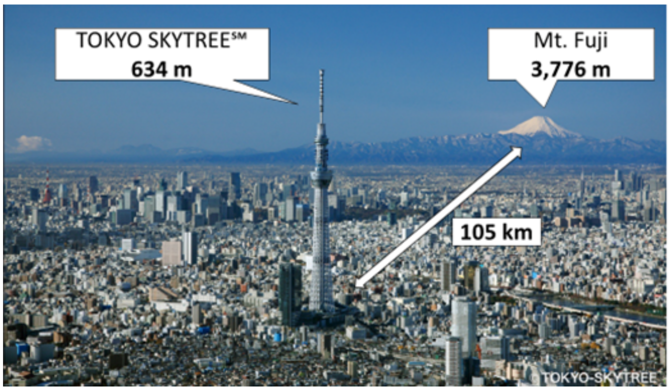

Figure 5. The photograph with information in Task Y
©TOKYO-SKYTREE, used with permission

Figure 6. The photograph in Task Y
Courtesy of Shiroi City Hall, used with permission

The analysis of the two tasks is as follows. In the case of Task X, the cues students described are grouped, and the points students marked are positioned accordingly. Then, specific cues are categorized. The next step is identifying the relationships between the positions and cues using correspondence analysis in order to find strong relations between them. Following the correspondence analysis, the groups are compared based on their descriptions. Finally, our attention shifts to focus on the reference frame expressed in spatial terms. In the case of Task Y, the stages are set based on students' description. Then, cues are selected to solve the problem in each stage. The final part of the analysis of Task Y is examining the relationship between the selected cues in Task Y and Groups A~F in Task X.

Results And Discussion

All students have had experience seeing TOKYO SKYTREE℠ on TV (93%), magazines (60%), from the window (95%), from a distance (55%), from nearby (53%), from the inside of TOKYO SKYTREE℠ (35%). All students have seen it in some ways it. Their familiarity with TOKYO SKYTREE℠ differs only slightly.

Task X: In this task, there are seven groups of points marked by students, Group A (n = 5), Group B (n = 7), Group C (n = 26), Group D (n = 7), Group E (n = 6), Group F (n = 5), and Group G (n = 4), in the answers (Figure 8). Also identified in the task are six perspective cues: positional relation, distance, direction of stream, curved point, drawing lines, and photograph information (Table 1). Through correspondence analysis based on the data (Table 2), there are three strong relationships between the answers and perspective cues: Groups A & E and Curved Point & Direction of Stream, Group B and Drawing Lines, Groups C & F and Positional Relation (Figure 9). For example in the case of strong relation between Group A and curved Point, the student in Group A describes that "There are three conditions, on the river (bridge), TOKYO SKYTREE℠ should be back and East Tower should be front, the river curved to the right".

Based on the strong relation mentioned above, some groups are compared with the focus on the relational terms, which means terms relating the location of landmark. Comparing Group A and Group E, we observe that students in group A wrote "The river curves *towards* East Tower", "The river curves to the *right*", in contrast students in group E wrote "The river curves to the *side*". Thus, Group A is different from Group E in that using specific terms related to direction. Comparing Group B, C, and Group F, we observed that students in group B drew straight lines connecting buildings and certain point on the river. Thus Group B exploited a mathematical way of drawing lines. On the other hand, 22 students out of 26 in Group C described the river as "The river is curved" and "The photo must have been taken from the bridge." Group C shows lack of connection between direction of river and position of buildings. Two students out of five in group F described the river's existence, "The river *is there."* Group C and F have strong relationship with positional relation yet they only focus on two buildings such as "East Tower is *right."* The students in group G wrote some words relating to their experience instead of relational terms.

Keeping these conditions of spatial thinking in each group in mind, the study shifts to look at the difficulties in constructing internal representation. Table 3 shows the viewpoints in each group. Building, River, and Curve in columns are landmarks students use as viewpoints, showing what they see on the photograph and the map. Positional Relation (Buildings), Positional Relation (River and Buildings), Direction of River, and Direction of Curve are selected as viewpoints, showing how students see or use viewpoints on the photograph and the map. To explain the process, here is an examination of Group C. The viewpoints students in Group C are buildings and the river in the photograph and the map except the curve. When they construct internal representation, the students use these viewpoints and make relationships among them. Some of these relationships are the positional relation of the buildings, right and left, and the front and back from the position on the river, but students do not include the river's direction. These results indicated that students have difficulty in paying attention to the relationships among objects even if they have the information about them. In short, level 2 viewpoints are not sufficient to construct an internal representation under the condition of isolated information.

View Point (perspective cues)	Concrete examples
Positional relation	East Tower is in front of TOKYO SKYTREE℠. TOKYO SKYTREE℠is to the left of East Tower.
Distance	It looks close.
Direction of stream	The river goes to TOKYO SKYTREE℠.
Curved point	The river is curved to the right.
Drawing line	Drawing the line connecting landmarks on the paper.
Photograph information	It might be taken on a bridge

Table 1: Perspective cues

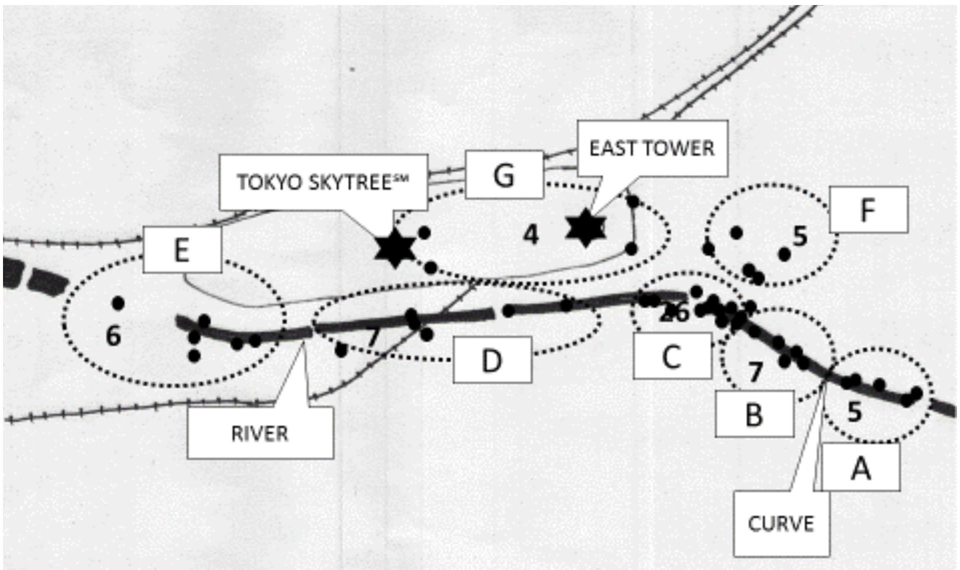

Figure 8. Students' answers (points) in Task Y

	positional relation	distance	direction of stream	curved point	drawing line	photograph informations
A(n=5)	80.0 (4)*	20.0 (1)	40.0 (2)	80.0 (4)	20.0 (1)	20.0 (1)
B(n=7)	71.4 (5)	0.0 (0)	14.3 (1)	14.3 (1)	42.9 (3)	0.0 (0)
C(n=26)	73.1 (19)	7.7 (2)	3.8 (1)	23.1 (6)	15.4 (4)	11.5 (3)
D(n=7)	14.3 (1)	14.3 (1)	0.0 (0)	0.0 (0)	0.0 (0)	14.3 (1)
E(n=6)	33.3 (2)	16.7 (1)	33.3 (2)	16.7 (1)	16.7 (1)	0.0 (0)
F(n=5)	60.0 (3)	20.0 (1)	20.0 (1)	0.0 (0)	20.0 (1)	0.0 (0)
G(n=4)	50.0 (2)	0.0 (0)	0.0 (0)	0.0 (0)	0.0 (0)	0.0 (0)

*The figure in parentheses is the number of the students.

Table 2: Ratio of cues in each group

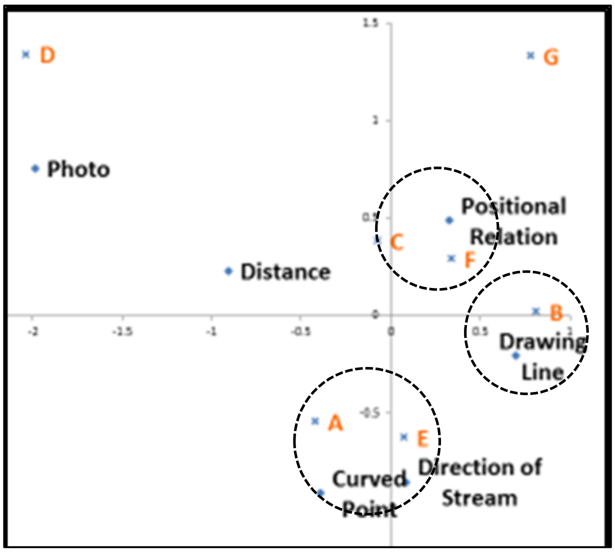

Figure 9: Strong Relations

	Photo			Map			Construction of internal representation			
	Buildings	River	Curve	Buildings	River	Curve	Positional Relation（Buildings）	Positional Relation（River & Buildings）	Direction of River	Direction of Curve
A	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
B	✓	✓		✓	✓		✓	✓	✓	
C	✓	✓		✓	✓		✓	✓		
D	✓	✓		✓	✓					
E	✓	✓	✓	✓	✓	✓				
F	✓	✓		✓	✓		✓			
G	✓									

Table 3: The viewpoints in each group

Task Y: This task does not require to find the place the photograph was taken exactly because seventh graders have not learned homothetic ratios. Task Y's purpose is to understand how students construct internal representation in the process of mathematizing through analyzing their descriptions. In order to solve Task Y, students needed to draw figures from the side like Figure 10. Mathematizing process involves making a transformation from the photograph information to the mathematical figures in this task.

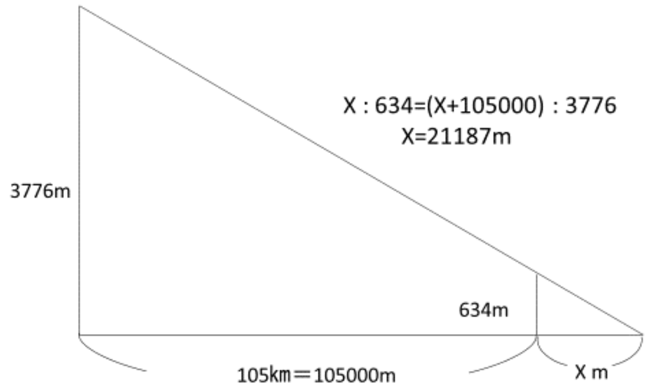

Figure 10: Solution of Task Y

Figure 11 shows the position of answers on the map. The places students mark are classified in five groups: (1) mark near TOKYO SKYTREE℠ (41%), (2) mark far from TOKYO SKYTREE℠ (27%), (3) mark vaguely or write "around here" (17%), (4) use words in the answers (12%), (5) wrong answer (3%). Table 4 shows ten perspective cues found in the description. The students drawing the line or pictures were divided into three types according to from where they look at, landmarks are standing on a line from the front of TOKYO SKYTREE℠ (Straight line (front), Figure (front)), from the sky (Straight line (above), from the side (Figure (side)). They have other cues such as Size, Photograph information, Height of camera. The average of number of cues in each group are that Near (2.0), Far (2.2), Vague (1.2), Words (1.2), Wrong answer (0). It is clear that lack of cues make a decision vaguely.

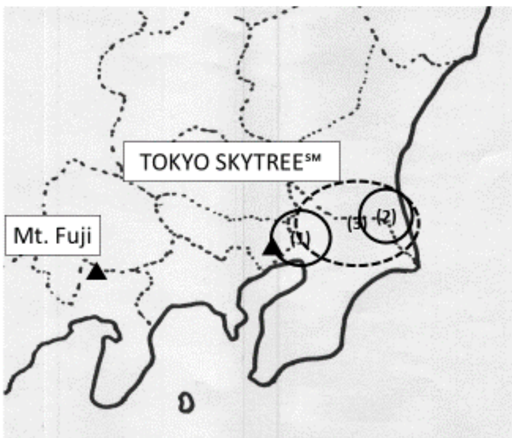

Figure 11: Students' Answer

Ten cues	Straight line (front)	Straight line (above)	Size	Height of camera	Figure (front)	Figure (side)	Line of sight	data	Calculation	Photo inform-action
Near (n=25)	9	11	6	5	2	2	2	4	4	4
Far (n=16)	3	4	3	5	0	7	3	3	6	1
Vague (n=10)	3	3	2	1	0	0	0	1	2	0
Words* (n=7)	1	0	0	2	1	2	1	1	0	1
Wrong**(n=2)	0	0	0	0	0	0	0	0	0	0

*Students describe the location by words. **The answers are not on the straight line.

Table 4: Ten cues in Task Y

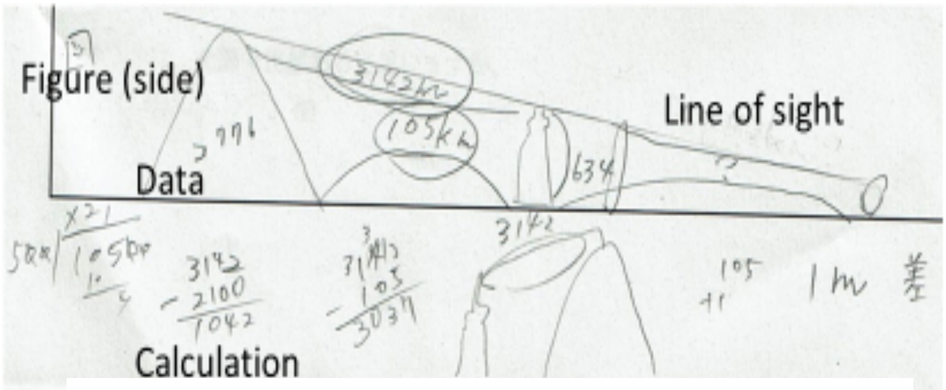

Figure 12: Example of student's description

As previously mentioned, knowledge of homothetic ratio is needed to solve Task Y (Figure 10). Before reaching this stage, students must construct internal and external representations according to the following steps: Step 1 is to recognize that the objects stand on a straight line and estimate the position of the camera should be to the right side of TOKYO SKYTREE℠ and close to it. Step 2 is to think that the height of the camera should be on the line of sight connecting the top of Mt. Fuji and the top of TOKYO SKYTREE℠. Step 3 is to construct internal representations and external representations like figures from the side. Step 4 is to estimate the height of Mt. Fuji as six times as TOKYO SKYTREE℠ in order to draw a figure like Figure 10. In these steps drawing the line of sight is the key point in the mathematical process.

First of all, we would like to describe what kinds of difficulties seventh graders have in these procedures, from Step 1 to Step 4. After that, connecting with the results of Task X, it is shown that the difficulties in each group in Task X are related to the difficulties in the mathematization process in Task Y. Here is Table 5, which shows that 87% of seventh graders pass Step 1, however, in Step 2, there is only 22% of seventh graders paid attention to the height of camera with the line of sight. The implication is that realizing the line of sight is the most difficult in the key point of the mathematical process. In Step 3, it clearly appears that drawing a figure from the side is difficult, but the students who understand the positional relations between buildings and river could construct internal and external representation between Mt. Fuji and TOKYO SKYTREE℠ from above and from the side (see Figure 12). Ten out of thirteen students who described the line of sight belong to Group A, B, and C in Task X. To find the reason why students had difficulties in drawing figures, the focus shifts to the students who tried some cues. The students belonging to Group E had difficulties in drawing figure from the side (Figure 13). They might have been bound to the photograph taken from the front. A student in Group A could build an internal representation among landmarks judging from the description, “the angle of camera is a little bit oblique,” however she did not try to draw a figure included a line of sight (Figure 14). Her case indicated that expressing external representations is difficult even if she has an internal representation.

In summary, although it is important to draw the figure with the line of sight from a side in the mathematization process, the results of this analysis indicate some obstacles to the next step. The students in Groups D, E, F, and G who could not use the viewpoint of level 2 could not mathematize Task Y. Furthermore, even if the students have the internal representations using the viewpoint of level 2, they have the difficulty to express external representations. Additionally, persistence of the picture may have led to create obstacles in the mathematization process.

	A (n=5)	B (n=7)	C (n=26)	D (n=7)	E (n=6)	F (n=5)	G (n=4)	Total (n=60)
Step 1	4	7	24	4	5	5	3	52
Step 2	2	3	5	1	0	1	1	13
Step 3	2	1	4	2	1	0	1	11
Step 4	0	0	0	0	0	0	0	0

Table 5: Number of students in each step

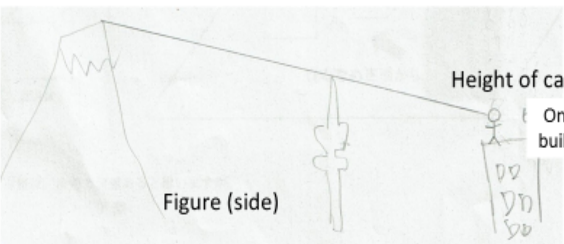

Figure12 Example of the height of camera (Group C)

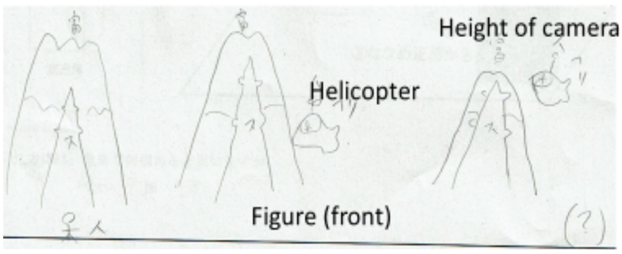

Figure 13 Example of the height of camera (Group E)

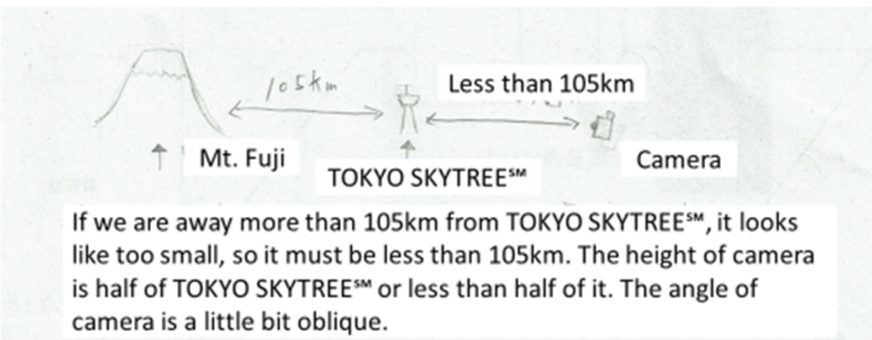

Fig.14 Example of awareness of line of sight (Group A)

Conclusion

These results lead to the conclusion that there are different types of difficulties. In the case of Task X, the difficulties include the lack of information from the photograph (Group B), making a connection between the direction of the river and the position of buildings (Group C), making a connection among three objects (Groups D, E, F), and few specific cues (Group G). Besides considering the reference frame in the case of Groups D, E and F, there are other difficulties. These difficulties include the lack of relation back and front (Group D), the lack of distance to the buildings (Group E), and the lack of position on the river (Group F). In the actual problem solving situation, the difficulties are to find specific cues, to decide a standing point, and to make a connection among objects relating to their position and direction in the process of structuring the internal representation. Considering these difficulties in each group, it is significant to foster not only the viewpoints of relational position but also utilizing the information about the objects. In the case of Task Y, the difficulties are being aware of line of sight, constructing internal representation that is a figure from the side to include the line of sight, and drawing external representations. However, some of the students in Groups A, B, and C in Task X could recognize the line of sight and draw the figure from a side, enhancing the viewpoint of level 2, which is how objects are seen using cues in real world, the implication is that it is critically important to mathematize real world problems. To foster spatial thinking in mathematics education, two types of viewpoints of level 1 and level 2 need development. In relation to solving real world problems, level 2 viewpoints with utilizing information of real world and expressing internal representation in mathematical way such as drawing line of sight are the key ability in spatial thinking.

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PLAYING WITH GEOMETRY: AN EDUCATIONAL INQUIRY GAME ACTIVITY

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In this study, we present a new approach to teaching based on the Logic of Inquiry (Hintikka, 1998)*, which develops students' investigative and reasoning skills and may promote a deeper understanding of the meaning and the validity of mathematical theorems. Starting from a game played in a Dynamic Geometry Environment (DGE) and guided by a questionnaire, students discover and become aware of the universal validity of the geometric property on which the game is based. In this paper, we present two game-activities. The first is an activity in which students play the game against a schoolmate and use a worksheet questionnaire to reflect on their findings. The second is an online game-activity in which the students play the game against the computer and reflect their findings in an online questionnaire. Using the theory of didactical situations (Brousseau, 1997) we describe and analyse the work, diagrams, dialogue, and question responses, showing the importance of the strategic thinking activated by the game-activity for students' mathematical inquiry and reasoning development.*

Keywords: cyclic quadrilateral, DGE, discernment, falsifier, game-activity, investigation, logic of inquiry, logic of not, online activity, parallelogram, semantic games, verifier.

Introduction

At the start of high school, the teaching and learning of geometry requires students to explore geometric properties and encounter the associated theorems and proofs. Activities such as exploring properties and constructing proofs are not procedural or algorithmic by nature, requiring students to develop their own solutions using their conceptual understanding and strategic thinking. More precisely, in order to solve inquiry activities, students often need to behave as detectives: they have to observe facts, link them through cause-effect relationships, and formulate probable explanations of what they noticed. Aspects of inquiry strategies are not frequently discussed in standard classroom teaching; they are often left to the students' personal learning. Teachers often skip the inquiry phase and present mathematics as an already systematized discipline.

The goal of this paper is to present the design of teaching activities meant to develop an inquiry approach to the learning of mathematics. These activities, which we call *game-activities*, are inspired by the studies developed by the Finnish philosopher and logician Jaako Hintikka (1998) in the field of pure mathematics. Differently from classical logic, the logic Hintikka created, called the *logic of* *inquiry,* is not only a logic of justification but also a logic of discovery. Within this logic, the basic rules of inferences are described through *semantic games,* which are two-player games between a Verifier and a Falsifier who argue on the truth of a statement.

Our game-activities adapt Hintikka's logical constructs for educational purposes. Through the activities, students inquire about the geometric situations inside Dynamic Geometry Environments (DGE) and discover new geometric theorems within a game-theoretical approach developed on the use of existential and universal quantifiers. The focus of learning shifts from knowledge to higher-order and deeper understanding, which include some of the following strategic aspects: exploring new situations, making conjectures from empirical evidence, investigating conjectures, and reasoning about their validity. All these aspects are framed and described within Brousseau's (1997) theory of a-didactical situations. Our research focuses on the ways in which such games can promote students' strategic thinking and on how students' learning can benefit from it.

Theoretical Framework

As underlined by Hintikka (1999), the central idea of the Logic of Inquiry consists in assuming the scientific inquiry and the knowledge acquisition as question-answer processes. The eminent logician described it using an extract from "Silver Blaze," a Sherlock Holmes episode:

> "The background is this: the famous racing-horse Silver Blaze has been stolen from the stables in the middle of the night, and in the morning its trainer, the stablemaster, is found dead out in the heath, killed by a mighty blow. All sorts of suspects crop up, but everybody is very much in the dark as to what really happened during the fateful night until the good inspector asks Holmes:
>
> "Is there any point to which you would wish to draw my attention?"
>
> "To the curious incident of the dog in the night-time."
>
> "The dog did nothing in the night-time."
>
> "That was the curious incident," remarked Sherlock Holmes.
>
> Even Dr. Watson can see that Holmes is in effect asking three questions. Was there a watchdog in the stables when the horse disappeared? Yes, we have been told that there was. Did the dog bark when the horse was stolen? No, it did not even wake the stable-boys in the loft. ("That was the curious incident.") Now who is it that a trained watchdog does not bark at in the middle of the night? His owner, the stable-master, of course. Hence it was the stable-master himself who stole the horse… Elementary, my dear Watson."
>
> (Hintikka 1999, p. 31)

Through the dialogue, Sherlock Holmes obtains the answers to three implicit questions, which are the inquiry transposition of the following non-mathematical argument: if there was a watchdog in the stables and the dog did not bark when the horse was stolen then, probably, the thief was the owner, since generally a trained watchdog does not bark only at its owner.

The same interrogative process accomplishes inquiry and justification. This logic of inquiry involves deductive, abductive, and inductive inferences. Abductions are logical operations fundamental in inquiry processes; they allow the subject to introduce new elements for explaining the facts observed. Peirce characterized them as follows:

> "abduction looks at facts and looks for a theory to explain them, but it can only say a "might be", because it has a probabilistic nature. The general form of an abduction is:
>
> - a fact A is observed;
> - if C was true, then A would certainly be true;
> - so, it is reasonable to assume C is true"
>
> (Peirce 1960, p.372)

If we consider the previous Sherlock Holmes's episode, we can notice that an abduction allows Sherlock to discover the murderer. The observation that the dog did not bark at the time when the horse was stolen requires an explanation. The best explanation for this fact is that the thief is the horse's owner. Once the abduction is formulated, it is possible to rewrite Sherlock's reasoning in a deductive way. The abduction marks the transition from an inquiry to a deductive approach.

Hintikka (1999) characterized the Logic of Inquiry with two types of rules/principles that govern it: definitory rules, which tell the subject what is possible to do, and strategic principles, which tell the subject what is more convenient to do. These rules are typical of strategic games, such as the chess game:

> "The definitory rules of chess tell you how chessmen may be moved on the board, what counts as checking and checkmating, etc. The strategic rules (or principles) of chess tell you how to make the moves, in the sense of telling which of the numerous admissible moves in a given situation it is advisable to make."
>
> (Hintikka 1999, p. 2)

Hintikka (1999) modeled the inquiry processes through the so-called *interrogative games*, which are two-player games between an *Inquirer*, who asks questions, and an *Oracle*, also called *Nature*, who answers him. The answers given by the Oracle furnish the Inquirer with the hypotheses from which the conclusion is derived. The strategic principles guide the inquirer in the formulation of the best question to ask.

Using games, Hintikka, also modelled the processes for establishing the truth of a mathematical statement. He defined *semantic games*, which are two-player games between a Falsifier who tries to refute the statement and a Verifier who tries to verify it. For example, consider the formula $\forall x\ \exists y \mid S[x,y]$, it is possible to verify the formula through a semantic game between a Falsifier who controls the variable x and a Verifier who controls the variable y. The Falsifier's aim is to find a value x_0 of x for which there is no value y_0 of y, such that $S[x_0,y_0]$ is true. The Verifier's aim is to find a value y_0 such that $S[x_0,y_0]$ is true, for each x_0 presented by the Falsifier. If the Verifier has a winning strategy that allows him to win for each value x_0 proposed by the Falsifier, then the formula is true. The truth of the statement is defined by Hintikka employing the concept of *strategy* developed by von Neumann and Morgenstern (1945) inside Game Theory:

> "It is a rule that tells a player what to do in any conceivable situation that might come up in the course of a game. Then the entire game can be reduced to the choice of a strategy by each player. These choices determine completely the course of the play and hence determine the payoffs. And these payoffs specify the value of the strategies chosen. Strategic rules hence concern in principle the choice of such complete strategies. "
>
> (Hintikka 1999, p.3)

By designing the inquiry of geometric theorems as a Hintikka's (1999) semantic game we create a learning environment that engages the student in producing winning strategies, not being fully aware with the didactical intentions of the underlying knowledge. This learning environment enables the student to establish a relationship with the knowledge, regardless of the teacher, and creates an a-didactical situation (Brousseau, 1997). The milieu, which is the game's rules, constraints and available resources, allows and directs students' a-didactical actions. The feedback produced by the milieu allows students to check the effectiveness of their strategy and may lead them to accept or reject it. The interactions between the student and the milieu constitutes what Brousseau calls the *situation of action*. Continuing in the game the students pass through what is called the *situation of formulation* that consists in "progressively establishing a language that everybody could understand... makes possible the explanation of actions and models of action." (Brousseau, 1997, p. 12). Situation of validation occurs when spontaneous discussions about the validity of strategies or efficacy take place and include explanations and elements of a proof. Brousseau suggests that while all three situations are expected from students, it is through situations of validations that genuine mathematical activities take place in the classroom. We show that the design of the activities presented in this paper lead to situations of validation.

Methodology

Taking inspiration from Hintikka's (1999) notion of semantic game, we developed game-activities based on a geometric property or theorem. The property is unknown to the students. They are expected to discover it by playing the game and answering a questionnaire. In order to develop a winning strategy, the players should generalize the different winning shapes generated and understand their common properties. The game serves as a guided inquiry, which calls students to integrate empirical work with conceptual work and take an active role in the learning process (Yerushalmy & Chazan, 1992). By playing the game, the students generate a wide range of examples that constitute the example space of the solution (Sinclair, Watson, Zazkis, & Mason, 2011). For developing a winning strategy, the Verifier should discover their common properties. The different modes of the game raise uncertainty, which drives students to test the validity of their conjectures and to reason about them (Buchbinder & Zaslavsky, 2011).

In this paper, we present two games: the first is based on the geometric statement, "If the diagonals of a quadrilateral mutually bisect each other, then the quadrilateral is a parallelogram;" the second is based on the geometric statement, "If all the intersection points of the perpendicular bisectors of a quadrilateral coincide, then the quadrilateral is inscribable in a circle." We tested the first in the form of a game played between two students in the 9^{th} grade of a scientifically oriented high school in Italy and the second in the form of an online game in three 10^{th} grade classes from three different schools in Israel.

The collected data consist of videotapes of the group activities and transcripts of the conversations. In the analysis of the students' dialogues and example spaces, we identify Brousseau's (1997) three a-didactical situations: action, formulation, and validation. These situations focus on the activated strategic thinking in the *transition* from a situation of actions, in which students do not reason the actions and strategies they take; to the situation of formulation in which students are conscious of the strategies they would use; and to the discussion about the validity of the strategy can involve intellectual, semantic and pragmatic reasons (Brousseau, 1997). A pragmatic reason occurs when students declare to test what he/she says by really playing the game, a semantic reason when students validate their claim using the results of the matches, an intellectual reason when students detached from the concrete situation and gives theoretic reason of what they claim. By identifying the three types of situations and reasons in students' dialogues we wish to describe the process of knowledge acquisition in students' inquiry.

A Game between Two Students

Game description

The activity involves two students playing a non-cooperative game in a DGE and then reflecting on it using a worksheet with guiding questions. The object of inquiry is a dynamic diagram (Figure 1) that each player controls through one of its constructed elements. *ABCD* is a quadrilateral whose vertices *A* and *B* are fixed, while *C* and *D* are free to move. The points *E*, *F*, and *G* are respectively the midpoints of diagonals *BD* and *AC* and their intersection point. By moving *C* and *D*, the screen position of these

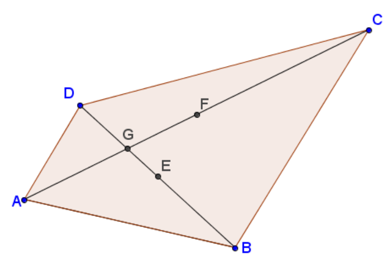


Figure 1: dynamic diagram on which the game is played

points change, but they still conserve their constructed properties.

Player C controls the point *C* and his goal is to make points *G*, *E*, and *F* coincide. Player D controls the point *D*, and his goal is to prevent player C to make the three points coincide. The students do not know either the geometric nature of points *G*, *E*, and *F* nor the property that characterizes the diagonals of a parallelogram. It is expected they will discover it through the game-activity.

The game is played in turns. We ask students to play four matches. Each match has a given number of moves and a given player who makes the first move. In the first match, for example, the player who moves point C is the first to play, and the number of moves is six.

Student D plays the role of Falsifier of the statement "for any position of point *D*, there exists a position of point *C* such that *G*, *E* and *F* coincide." Thus, his or her goal is to find a position of *D* in which student C cannot reach his goal. Student C plays the role of Verifier of the statement, because he or she should show the truth of the statement for any position of *D* proposed by the Falsifier.

Questionnaire description

The questions in the worksheet guide students to investigate the geometric properties of the game and the importance of having the last move. These questions include:

1. What is the geometric nature of points *E, F* and *G*?
2. How do you suggest the player who moves *C* should modify the quadrilateral?
3. Suppose that the given number of moves is odd and that you are the player who controls *C*. If you could choose whether to be first or second, what choice allow you to win the game?
4. Which true statement is it possible to discover through the game? The statements should be of the following types:

   If A then B, $A \rightarrow B$
   A if and only if B, $A \leftrightarrow B$
   If B then A, $B \rightarrow A$

A and B must be replaced with one of the following propositions:

A: The diagonals bisect each other

B:
1) The quadrilateral is a rectangle
2) The quadrilateral is a parallelogram
3) The quadrilateral is a square
4) The quadrilateral is a rhombus
5) The quadrilateral is a trapezoid

The first question intends to draw students' attention to what varies and what is invariant. Its aim is to guide students to discover the geometric nature of the points *E*, *F*, and *G*. These points are robustly constructed as midpoints and intersection of the diagonals; hence, they conserve their nature under both Verifier and Falsifier's moves. The second question focuses the students' attention on the invariant configuration that characterizes the Verifier's moves, namely the parallelogram configuration. The third question aims at triggering a reflection on the fact that the winning strategy of the player who makes the last move in a single match depends on the parity of the number of moves in the game and the identity of the player who plays the first move. Finally, the fourth question intends to create cause-effect links between the geometric invariants discovered through the first three questions, guiding students in the construction of the following *if and only if* statement:

The diagonals of *ABCD* bisect each other if and only if *ABCD* is a parallelogram.

Once the nature of *E*, *F* and *G* and the invariants of the Verifier's moves have been discovered, the semantic game triggered by the game can be reinterpreted in the following equivalent forms:

- For all positions of point *D*, there exists a position of point *C*, such that the midpoints of the diagonals and the diagonals' intersection point coincide.
- For all positions of point *D*, there exists a position of point *C*, such that the diagonals *AD* and *BC* bisect each other.
- For all positions of point *D*, there exists a position of point *C*, such that *ABCD* is a parallelogram.

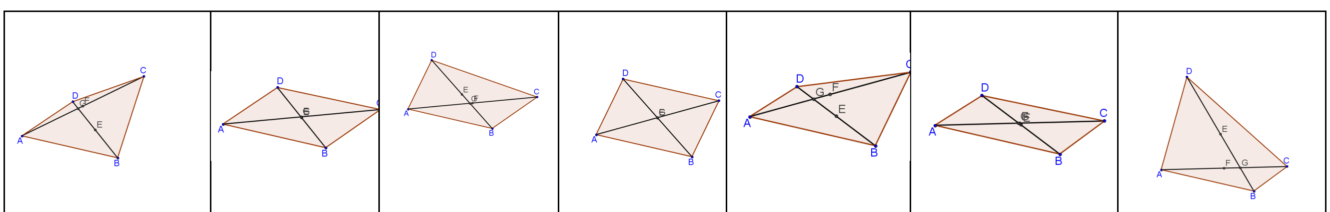
Analysis of the Game-Activity

One of the videotaped student pairs includes Marco, as the Verifier, and Vittoria, as the Falsifier. Vittoria, after making her first move, reflects loudly over it

Vittoria: How can I do? Before points *G*, *E*, and *F* were wider… Then if I tighten this *(making the gesture of moving D toward the centre of the screen)* became wider theoretically…*(Vittoria makes the move)* Done!!![1]

The students are in the situation of action: while playing, Vittoria is describing the effects of the previous moves on the position of the points *E*, *F*, and *G* in order to plan how to act in the next move. She is looking for a winning strategy and to this end she activates her strategic thinking: by reasoning backward, she is selecting the best move to make according the fact observed in the previous moves. Vittoria's reasoning focuses on properties which are not relevant for the game: the possibility to win does not depend on the size (extension) of the diagram.

Figure 2 demonstrates the example space generated in the first match where the number of moves is six, and the starting player is the Verifier. As it is possible to observe in Figure 2-g, the Falsifier won the match because within the last move he reached his goal, since the game ended in a configuration in which the three points do not coincide.



a) Starting configuration	b) Verifier move	c) Falsifier move	d) Verifier move	e) Falsifier move	f) Verifier move	g) Falsifier move

Figure 2: Example space of the first match

[1] English translation from Italian sentence: "Come faccio? Prima erano più larghi no I punti? Quindi se io *stringo (gesto di portare il punto D verso il centro dello schermo)* si allargano teoricamente… *(Vittoria* fa la mossa) Ecco!!"

Analyzing the dialogue, it is noticeable that Vittoria and Marco's attention focus on the number of moves and the rules of the game, rather than the type of diagrams produced:

Vittoria: You have to move *C (looking at Figure 2-a)*.

Marco: Only *C*? *(making Figure 2-b)*.

Vittoria: Yes.

Marco: Go! I caught you!

Vittoria: We did 2 moves *(making Figure 2-c)*.

Marco: Write it!

Vittoria: We did move three *(making Figure 2-d)*.

Marco: Yes

Vittoria: Four. I did move four *(making Figure 2-e)*.

Marco: Five *(making Figure 2-f)*.

Vittoria: And now? *(Making Figure 2-g)* "Player X makes the first move and the moves are 6" *(reading the task)*. We did case A, because you started, we made six moves, and I won because I didn't make them coincide.

Except for Vittoria's first sentence, while playing, the students do not discuss the winning shapes or the strategies they use implicitly in their moves. The students are opponents and do not want to reveal their strategies for not advantaging each other. The example space shows us the diagrams implicitly explored within this match. Just at the end, Marco claims: "At the end, I won if I created a parallelogram." With this claim, Marco is shifting into the situation of formulation to respond to the need of communicating the action accomplished in his moves. His words demonstrate that he discovered the advantage of making the moves guided by the parallelogram configuration instead of the screen position of the points *E*, *F*, and *G*. Marco develops a geometric strategy, namely reasons for moving point C in a given direction based on observed geometric property or configuration The evidence for its use is given by the time spent to make the move and the way he moves the point *C* in the DGE. Marco drags C in the position in which the for vertex of the parallelogram is supposed to be in few second. This way of moving would not be possible without noticing that the parallelogram configuration causes the coincidence of the three points.

After playing, the students proceed to the questionnaire, moving from the situation of action to the situation of formulation. The following dialogue reports the discussion that was triggered by the third question, in which they are required to understand whether it is better to play first or second when the number of moves is odd.

Marco: First, first, first! Don't even think about it! First!

Vittoria: I'd go second!

Marco: First.

Vittoria: No, take a look: here you went first, and then you lost, here… *(looking at the matches' results)*

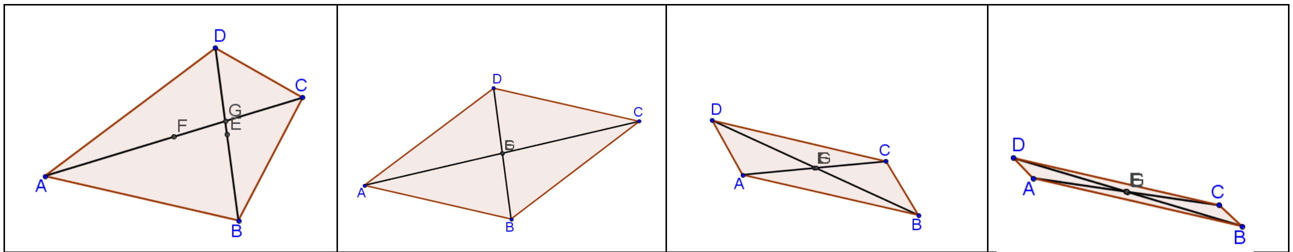
Marco: No, that *(referring to the matches' results)* doesn't count, I am a bad player!

Vittoria:	Yes, you are right, here the Falsifier goes first and 5 is odd.
Marco:	That doesn't count! If I had played bad, I would have lost.
Vittoria:	When the number of moves was odd, the first player has always won.
Marco:	No, because here *(pointing the third match)* I would have won as well.
Vittoria:	Indeed, here there were 4 moves.
Marco:	If I go first, I have the possibility to put them in parallel, create the parallelogram. Anyway, if you are the last to play, you can ruin it, so I lose. I can win only if I am lucky and I go first.

The students are in the formulation/validation phases. The dialogue's first exchange shows that Vittoria's sentences refer to what has happened in the game while Marco's sentences are formulated according to what could have happened in the game. Vittoria uses the results of the four matches as pure truth (an Oracle); from them she formulates conjectures and checks Marco's conjectures. Using Brousseau (1997, p. 17) terminology, Vittoria's reason is a *semantic reason* derived from the game experience. Since the matches' results do not coincide with perfect players' results, this way of reasoning leads Vittoria to false conclusions. Marco, instead, does not activate just a semantic control but also an intellectual one, as demonstrated by his last sentence: "If I go first, I have the possibility to put them in parallel, create the parallelogram. Anyway, if you are the last to play, you can ruin it, so I lose. I can win only if I am lucky and I go first." His intellectual control allows him to look at the matches' results critically, and considers what would have happened if they were perfect players. Using Brousseau's (ibid.) terminology Marco's reason is an *intellectual reason.*

Marco tries to explain his point of view by employing the result of the third match in which he lost even if he could have won. In this way, he can explain to Vittoria that the matches could end in a different way, and her semantic way of reasoning based on the matches' results is fallible. Marco is trying to establish a dialogue, between his intellectual reason and Vittoria's semantic reason. His desire to make Vittoria understand causes Marco to improve the logical structure of his argument as demonstrated by his last sentence.

In this moment of the dialogue, the students are in the validation phase. However, since they did not develop a shared strategy in the transition from the situation of action to the situation of formulation, they have some difficulties understanding each other's point of views. Figure 3 displays the example space generated while students are trying to answer question two: "How do you suggest to modify the quadrilateral [to the player who moves *C*]?"



a) Vittoria's move | b) Vittoria's move | c) Marco's move | d) Marco's move

Figure 3: Example space generated during the discussion over the game

The example space includes quadrilaterals that grow thinner and thinner. The students move to the exploration of degenerate quadrilaterals in order to check if Marco's winning strategy is always true. The search for a counterexample triggers the transition from the a-didactical situation of formulation to that of validation. The following is the dialogue between them, while constructing the example space shown in Figure 3.

Vittoria:	In the first move, you always tighten the extension [of ABCD], right?
Marco:	I could also widen it! The important thing is that it is a parallelogram!
Vittoria:	If you widen it, you win. Look! *(Making Figure 3-a,b)*
Marco:	Even though you make it smaller, I do it! *(Making Figure 3-c).* As you lessen.
Vittoria:	But it is more convenient widen it.
Marco:	Yes because it is easier! But for how small it is… *(Making 3-d)*
Vittoria:	Marco now you are moving player D, not C!
Marco:	If it is larger, it is easier to find, but you can find it even if it is smaller. You must always keep in mind that we are humans, we are not machines!

In this extract, students are rethinking the a-didactic situation of action and are repeating the strategies that shift them to the a-didactic situation of formulation. Vittoria's strategy relies on visual/empirical properties of the diagrams, "to tighten the extension; If you widen it, you win" Marco's strategy relies on the geometrical properties of the diagrams, "to make a parallelogram." In order to validate this strategy, Marco uses pragmatic reasons, "Even though you make it smaller, I do it!", proving counterexamples to Vittoria' claims, namely diagrams that shows he can win even if the extension is not widen The type of logic that guides Marco's claims is the 'logic of not' (Arzarello & Sabena, 2011), since he provides counterexample to Vittoria's strategy and at the same time tries to convince the schoolmate that there is not a counterexample that can falsify his strategy; in fact, Marco is showing Vittoria that even in the worst conditions, the parallelogram's strategy is not fallible while the strategy proposed by Vittoria is fallible.

The Activity as an Online Game: Students vs. Computer

Game description

The online activity includes a game played by one or two students against the computer in a DGE. The game and the questionnaire operate in an online assessment system (Luz & Yerushalmy, 2015) and are followed by an online questionnaire that guides the students in their reflection on the game. The system provides an immediate automatic feedback on each move and displays counters of winning vs. losing moves. The system stores the submitted diagrams and answers, which provides the means for the student or the teacher to later review the course of games for feedback or class discussion purposes.

The game is based on the theorem: A convex quadrilateral is cyclic if and only if the four perpendicular bisectors to the sides are concurrent. We used the dynamic construction shown in Figure 4.

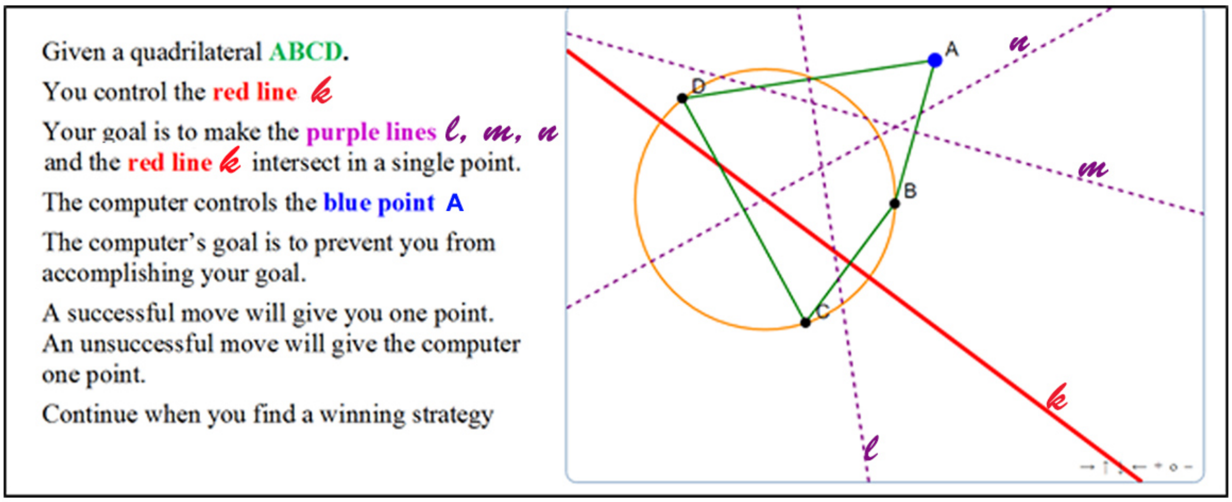


Figure 4: The online game

The basic elements of the construction are the point *C* and the lines *n* and *k*. Points *B* and *D* are the reflections of the point *C* across the lines *n* and *k*. This construction also includes the circle that passes through the points *B*, *C*, and *D*, the point *A*, and the quadrilateral *ABCD*.

The students start the game as a Verifier, who controls the line *k*. Their goal is to drag the line to a location where the four perpendicular bisectors are concurrent. In this game, the computer plays the Falsifier's role and controls the point *A*. As such, the computer chooses a random position on the board for the point *A*. There is a winning solution for the Verifier as long as *ABCD* is a convex quadrilateral or *ABCD* is a degenerated quadrilateral in the form of a triangle. Later, the players switch their roles. As Verifier, the computer automatically moves the line *k* to the locations of the concurrent perpendicular bisectors. The students, who now play the Falsifier's role, are challenged to find a location of *A* that will prevent the computer from winning. Such a location exists in creating a non-convex quadrilateral or a non-polygon shape.

The case of Itay and Harel

Itay and Harel play together against the computer. Harel controls the mouse. They start playing as Verifiers.

Harel: Wait; first let's see what they *(dashed lines ℓ,m,n) are*. They are perpendicular…. The thick line (*k*) is perpendicular to *BC*.

Itay: So, we need to make *AB* and *BC* the same line, like this *(Figure 5-a)*.

Harel: No *(drags line k back and forth. He ignores Itay's strategy)*. We need to have a way…

Itay:	To make *AB* and *BC* on the same line *(Harel intuitively finds the right position and stops Figure 5Figure 5-b).* Here, now they intersect.
	(Harel submits the diagram and they receive a winning feedback and a new diagram)
Itay:	We need a strategy, now it's a new shape.
Harel:	But, you can only move the thick line (*k*)? Well, the strategy is very simple. You can only move the thick line (*k*), so just move it until you see it all meet.
	(Intuitively drags to the intersection Figure 5-c) Here. You see.
Itay:	You need to make *AB* and *BC* the same size. That is a strategy.
Harel	Yes, but this is a different strategy

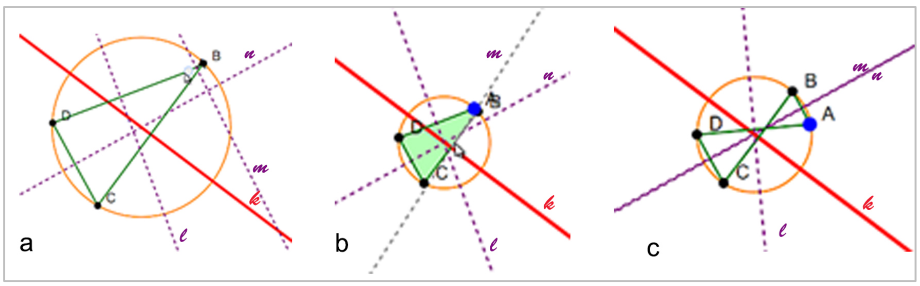


Figure 5: Diagrams generated by Itay and Harel while playing the online game

Harel and Itay are in the situation of actions. Their first step involves understanding how the objects in the game work. They start with identifying the invariants of the diagram. Before they played one move, they notice that the lines are perpendicular to the polygon sides. The pair does not cooperate; Itay suggests an intuitive action, and Harel performs a different, intuitive action. They suggest intuitive actions, check and reject them if they don't see that they work. Their strategies are visual, pragmatic based, strategies ("move until they meet").

After playing these matches, Itay and Harel start answering the questionnaire:

Itay:	The dashed lines (*ℓ,m,n*) are perpendicular to the sides.
Harel:	So does the thick (*k*). It is perpendicular to *BC*. What happens when they meet?*(drags the thick line (k) and generates Figure 6-a)*
Itay:	We already said, it's *AB* = *BC*. *(Re-examines the figure).* No, it's *AD* = *BC* it's an isosceles trapezoid.
Harel:	Why?
Itay:	Is it isosceles? We can't be sure it is isosceles.
	(Harel drags point A along the circle. The lines keep intersecting in a single point, but it is no longer an isosceles trapezoid, Figure 6-b).
	It's a quadrilateral inscribed in a circle.
Harel:	Why?
Itay:	*(talks slowly, as if he thinks while talking)*…They are all perpendicular bisectors, which means this equal to this *(points to the bisected chords)*…Ah… there is something with

perpendicular bisectors… because there is a theorem that a perpendicular bisector to a chord always passes through the center of the circle.

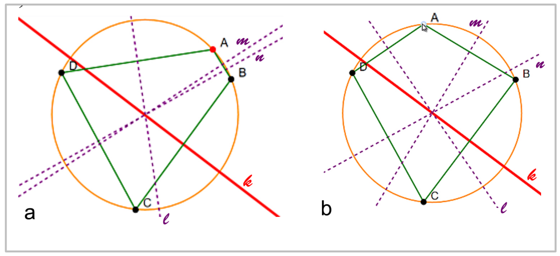


Figure 6: Harel and Itay's diagrams when answering the questionnaire

Answering the questionnaire guides Harel and Itay to the formulation situation. They gathered some information and they cooperate to understand it. They establish a common language and use geometrical terms (e.g. isosceles trapezoid, perpendicular bisector). They suggest strategies ($AB = BC$, $AD = BC$, and isosceles trapezoid) based on empirical results. They seek possible explanations of actions. Harel, who suggested a pragmatic reason, now stresses for an intellectual reason by asking "why." The questions posed by Harel motivate Itay into rejecting the insufficient explanations. By posing them, Harel encourages his partner to come up with a better explanation. Finally, Itay provides an intellectual reason based on his mathematical knowledge and validates that a single intersection of all bisectors yields an inscribed quadrilateral.

Continuing with the questionnaire, Harel and Itay try to validate their conjecture that a parallelogram cannot be inscribed in a circle.

Harel: You see, if it was a parallelogram then it would just not be possible... *(drags A to generate a parallelogram)*

Itay: I get it, but what is the theorem behind it?

Harel: Look. I guess you can say that it will not intersect.

Itay: Yes, but why will they not intersect?

Harel: Because it won't. If you do it, it just won't intersect.

Itay: Why?

Harel: Because they are not at the same place, and they are in the same size *(points to the parallel perpendicular bisectors of the parallelogram opposite sides)*. Look, you can say that if it's a rectangle or a square…

Itay: But how do you explain?

Harel: Look, if it's not a square or a rectangle, the lines are the same size, so unless they are in the same exact position… if there isn't a 90 degree angle between… I don't know…

Itay: No, no, no! Think about geometry, not just logic.

Itay: *(Talks slowly)* Let's say you have something with four sides, and these are chords, then these angles *(points to A and B)* are equal, because they lay on the same chord, but it can't be the same if it's a parallelogram that is not a rectangle or a square.

Harel and Itay are in validation phases. Itay starts with a conviction, but Harel challenges his statement, seeking for an intellectual explanation. They reject some explanations. They search their previous geometrical knowledge about quadrilaterals and perpendicular bisectors, finally coming up with the explanation.

The case of Hila and Gaya

When playing as Falsifiers, the students take the investigator's role, and the computer functions as an Oracle, one who knows everything. Students have no previous knowledge on how to approach the task. There is no information about the diagram's properties or about its construction. The students must discover the construction in order to come up with a winning strategy. The following dialogue demonstrates the investigation of Hila and Gaya while playing as Falsifier. As Verifiers, Gaya and Hila concluded the statement: "when the dashed lines intersect in a single point the quadrilateral ABCD is inscribed in the circle". They were not able to validate their statement, since they were not aware of the dashed lines property as perpendicular bisectors.

Hila: It should be parallel, because then they will not intersect…

Gaya: Now, it is parallel ($l \parallel k$), maybe he *(the computer)* will not make it. Let's try *(Figure 7-a)*.

Hila: He did it.

Gaya: No, he didn't. One line is missing... *(she drags A and finds out k and m coincide (Figure 7-b)*. Maybe not this one (k) should be parallel, but the other one...

Hila and Gaya are in the situation of actions. They make the line parallel. They base their selected action on a rational reason: "We want to prevent the intersection of the lines, parallel lines do not intersect, hence drag the lines to parallel positions." The game feedback shows that the lines do intersect, yet there is a need to clarify the situation since the feedback shows only three lines. Hila and Gaya accept the solution silently, after checking that two of the lines coincide. They start looking for a new action, which shows that they rejected their initial strategy. Their actions show that they are in the phase of dialectic of action.

Gaya: Should we make a specific shape in the circle? Maybe we can place the red point (*A*) on one of the other points. (She drags point *A* and places it on *B*, and submits the diagram. She then drags *A* onto *C* and *A* onto *D*, but the computer successes at each of these moves.)

Gaya selects a set of actions, intuitive this time. She tests the different actions and rejects them as she fails to win.

Hila: Let's think. How does the thick line (k) move? *(She switches roles).* Whenever this line (k) moves, another line (ℓ) moves with it.

Gaya: And the other two (m,n) already intersect.

Hila: So maybe we should make the other two not intersect. *(They try to drag A to make ℓ and m parallel, and fail)*

Gaya: Moving the line (k), another line (ℓ) moves. Moving the point (*A*), two lines move *(ℓ and m)*.

Hila: Let’s say this is a worst case scenario because the lines *(ℓ and m)* don’t meet *(Figure 7-c).*

Gaya: One line is static.

Hila: Yes. This one(*n*). Therefore, we need to move the other one.

Gaya: We need this (*m*) will not intersect this (*n*).

(They spend almost five minutes trying to make m and n parallel and fail, and discuss other strategies.)

Gaya and Hila focus their efforts on finding the variants and invariants of the diagram. They use similar words to describe situations (moving, static, intersect) and progressively establishing a shared language, making possible the explanation of actions and modes of actions. They shift to the dialectic of formulation.

Hila: Oops. We moved the wrong lines! Which line moves with the thick line(*k*)?

Gaya: This (*m*). Therefore, we need to make sure about the other one. The left one (*ℓ*).

(They drag A to generate figure 6d with ℓ parallel to n, computer fails! They move to fill the online questionnaire)

Gaya and Hila shift to a dialectic of validation. They produce intellectual reasons, validate them, and find a winning strategy. However, they do not transition to the mathematical language. Since they did not identify the perpendicular bisectors property, they did not conclude the mathematical theorem on which the game is based. At this moment the teacher steps in and draws their attention to the perpendicular bisectors property. With this additional knowledge Gaya and Hila can repharse their statement to: when the perpendicular bisectors of a quadrilateral meet in a single point then the quadrilateral is inscribed in a circle. Examining previous knowledge about perpendicular bisectors they can justify their statement.

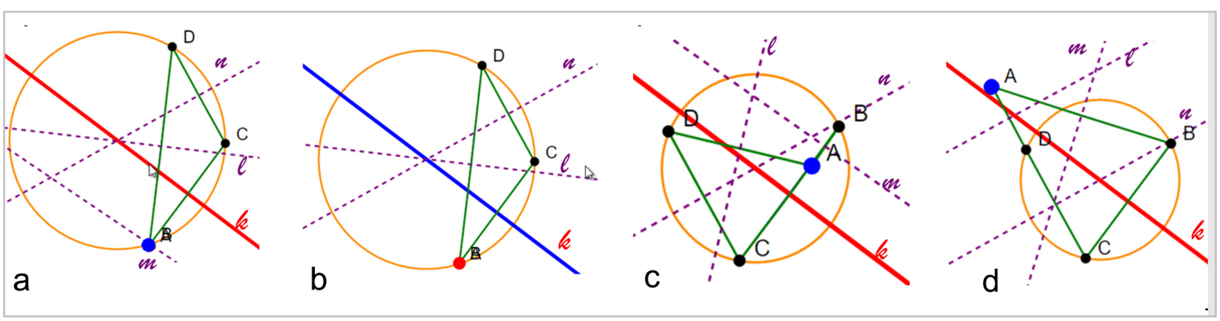


Figure 7: Drawings from Gaya and Hila game

Discussion

Bowden and Marton (1998, p. 7) define *discernment* saying that “To discern an aspect is to differentiate among the various aspects and focus on the one most relevant to the situation.” From our analysis of the students’ games, we see that even though each student has reached a different level of discernment, all students have shown some progress in discernment. In the first game, Vittoria discerns the aspect of the parity of the number of moves, and Marco discerns the winning shape of a parallelogram. In the second game, Harel discerns the perpendicular lines, and Itay discerns the winning shape as a quadrilateral inscribed in a circle. Gaya and Hila both discern the variants and invariants of the diagram's constructions. The desire to discover the winning strategy of the game prompt students in the discernment of the aspect of the game.

When the Verifier discerns the geometric invariants produced by his or her moves, he or she can use it as a winning strategy and, by playing the game and discussing it with his or her classmate, he or she can validate the strategy in different ways, using pragmatic, semantic or intellectual reason. By experiencing different geometric interpretations of the game, the students can comprehend the universal validity of the property. For example, Marco discerns the universal aspect of the game, when he says, "If I go first, I have (always) the possibility to … create the parallelogram."

Playing the role of the Falsifier students can investigate non-prototypical situations. Students are naturally engaged in the search for a "counterexample of the game," namely a configuration in which the Verifier cannot reach his aim. When students are in the validation phase, this attitude can trigger a pragmatic way of validation guided by the *logic of not* (Arzarello & Sabena, 2011). The students validate the strategy by showing and discussing the non-existence of counterexamples (see the case of Marco). In order to validate the strategy, the students produce large and varied example spaces, which include not only standard examples, but also extreme and degenerate examples that are not frequently demonstrated in mathematics teaching. The search for a winning strategy widens the boundaries of the exploration of geometric properties.

Working in pairs motivates reasoning. The game encourages students to explain their different points of view and helps them to improve their arguing abilities. By posing an incorrect conclusion, Vittoria motivates Marco to provide a more comprehensive explanation of his strategy. By posing why-questions, Harel motivates Itay to come up with a geometrical proof. On the way to reasoning, we are able to see the three types of reasoning (Brousseau, 1997): pragmatic (Marco: "Even though you make it smaller, I do it!"); Semantic (Vittoria: "When the number of moves is odd, the first player has always won"); and intellectual (Itay: "Because perpendicular bisectors to a chord always passes through the centre of the circle").

When students answer the questions, in the worksheet or in the online questionnaire, they shift from playing the game in order to defeat the opponent, to a "reflective game" (Soldano & Arzarello, 2016), where the students play the game in order to investigate and answer the questions. The game, along with the students' knowledge, takes the role of Oracle, or the milieu. The guiding questions are not sufficient to all students. Vittoria, for example, did not discern the geometric aspects of the game, despite Marco's explanations. Hila and Gaya partially interpreted the game using mathematical theory (parallel lines), but did not discern all invariants of the game (perpendicular bisectors). Their validation remained in the context of the variants they were able to discern. A teacher-guided class discussion, where students share and discuss their strategies, can highly benefit from the game-activity. The teacher can use the language developed by the students to present and clarify the approach with game. The teacher's guidance can help to close the gap and complete missing knowledge.

The use of a game as a mathematical inquiry requires careful design. The invariant properties of the dynamic diagrams that are given or hidden from the players require adjustments based on the students' level of knowledge and their inquiry experience. Different design aspects can shift students from mathematical inquiry to pure game playing such as the limitation on the number of moves in the game between two students. In the online version, the computer is taken as an Oracle or the milieu. It is important that the computer's feedback be accurate, though a small amount of inaccuracy could be neglected. The accuracy level depends on the instruments used as there is a difference between dragging in tablets and mouse dragging. Understanding how different students approach inquiry can assist teachers in guiding their students through the curve of learning to inquire. Being able to retrieve students' submissions in the game enables a visual way in which a teacher reconstructs with students the course of the game and point out possible obstacles in the inquiry process. The display of the example space generated in the game can be used as a visual aid to class discussion. We find that the challenge of taking interesting teaching activities and design them as a game can open up many opportunities for further research.

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THE USE OF WRITING AS A METACOGNITIVE TOOL IN GEOMETRY LEARNING

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This work reports on a teaching intervention that explored the use of writing as a metacognitive tool in high school geometry problem solving. Specifically, this qualitative research study investigated how explicit writing directives can help students understand, organize, and monitor the steps involved in the different phases of activities for geometry problem solving in the third year of secondary school. Possible gains of the intervention are assessed by comparing the performance of students who participated of the intervention with that of students who did not.

Keywords: geometry, learning, metacognition, metacognitive tool, writing

Background and Research Problem

Secondary school students often experience systematic difficulties during problem solving. One difficulty is interpreting the problem statement from the provided information. For example, Figure 1 shows an incorrect interpretation of the information. While the drawing fulfills the condition to divide the trapezoid into four parts, the edited figure fails to satisfy any of the conditions stated in the problem.

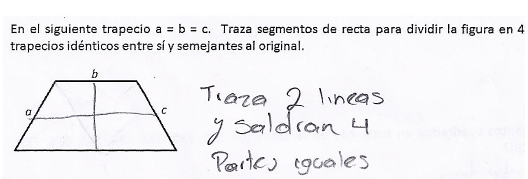
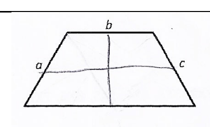


Figure 1a. Student's original answer

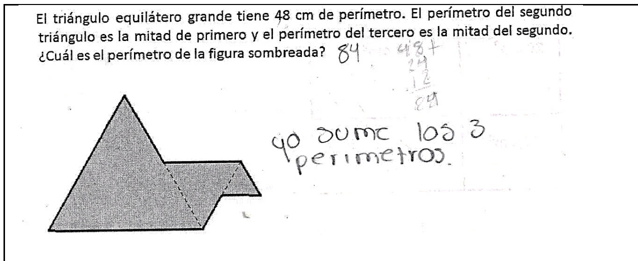
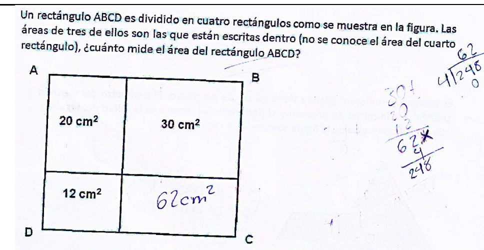
In the trapezoid below, a = b = c. *Draw straight-line segments to divide the figure into 4 trapezoids which are identical to each other and similar to the original.*



Draw 2 lines
and you will obtain 4
equal parts

Figure 1b. Translation of student's original answer

Another difficulty observed is that the student only solves part of the problem by using only some of the information, and fails to utilize the information required by the problem, as shown on the left side of Figure 2. Another difficulty arises from an unclear presentation of student operations and answers, which complicates the matter of understanding their reasoning when attempting to find the solution (as shown in Figure 2b).

 “I added the 3 perimeters”	
Translation: The perimeter of the biggest equilateral triangle is 48 cm. The perimeter of the second triangle is half of the first and the perimeter of the third is half of the second. What is the perimeter of the shaded figure?	Translation: A rectangle *ABCD* is divided into four rectangles as shown in the figure. The areas of three rectangles are written inside them (the area of the fourth rectangle is unknown). Find the area of the rectangle *ABCD*.
Figure 2a. Worksheet with student's answer	Figure 2b. Worksheet with student's answer

This study’s research questions, which emerged from the examination of the previously mentioned difficulties, are as follows: How should a cycle of activities for working through problem solving be designed? How can writing help students understand the information provided in the problem, reflect on their work, clarify their ideas, and organize their thoughts? We designed an intervention to study the answers to these questions. The study’s objective was to carry out a cycle of activities with students in the last year of basic schooling (9th Grade) to facilitate the problem-solving process and to develop metacognitive skills by combining writing with the solving of geometry problems.

The purpose of this project was to improve student’s problem-solving skills through reflective activities directed by open-ended questions. The students received explicit writing directives to guide them through the process of expressing their understanding of geometry problems, thereby helping them to organize, monitor, and justify the steps for their solutions.

Theoretical Framework

In a meta-analysis of research literature to understand the role of metacognition in scientific education, Veenman et al. (2006) examined the differences of how each author described the concept and the current lack of congruence between the components of metacognition and their relationship. Veenman et al. (2006) further found that there is a useful distinction between metacognitive knowledge and metacognitive abilities.

According to Flavell (1979), metacognitive *knowledge* refers to one’s declarative knowledge about the interplay between the individual, the task, and the strategy characteristics. Veenman (2012) also theorized that both metacognitive experiences and metacognitive knowledge originate from a monitoring process. However, metacognitive knowledge is retrieved from memory whereas metacognitive experiences concern the on-line feelings, judgments, estimates, and thoughts that individuals become aware of during a task performance.

Veenman (2012) details how metacognitive skills are refined primarily through four types of learning processes: text reading, problem solving, discovery and writing learning. Veenman states that in the field of exact science teaching, reading, problem solving, inquiry, and writing activities are always connected. Orientation, goal setting, planning, monitoring and evaluation are essential for all learning processes in science education. Although it clarifies that the reflection is not always mentioned in the investigations, perhaps because it appears after ending the tasks.

Metacognitive skills are mechanisms that take place inside the head and remain concealed (Veenman 2006) as a consequence cannot be directly evaluated, but have to be deduced from their behavioral results (Veenman, 2007). The way to assess metacognitive skills is through two methods: online and offline (Veenman, 2005). Online methods are evaluations during the completion of the task, such as: observation, thinking aloud, recording in a computer of the learning process. The off-line methods are questionnaires or interviews that can be applied before or after the execution of the tasks, which suffer from the same problems of validity as the evaluation of metacognitive knowledge.

Veenman (2012) describes metacognitive abilities as those that enable regulation of cognitive processes. These include the capacity for oversight, orientation, direction, and control of proper behavior in learning and problem-solving. Metacognitive abilities are learning activities per se and are critical for determining the results of learning. Veenman (2012) makes a distinction among the activities that he considers representative of metacognitive abilities, dividing them into three categories as shown in Table 1.

Table 1. Metacognitive Abilities

Learning Activities		
At the beginning of task execution	In the process of task execution	After task execution
− Reading	− Following a plan	− Performance assessment
− Analysis of the tasks	− Changing the plan	− Recapitulating
− Activation of prior knowledge	− Follow up	− Reflecting on the learning process
− Setting goals	− Control	
− Planning	− Note taking	
	− Time and resource management	

Veenman (2011) suggests that metacognition might adopt the perspective of a self-instructional model for the regulation of task execution. This process can be activated as a program acquired through a list of self-instructions that are applied each time the student is faced with performing activities. For Veenman (2011) it is important to recognize that both cognitive processes and metacognitive self-instructions that are involved in the execution of instructions are part of the same cognitive system. Cognitive activities are always necessary for the execution of any process related to a task at the object level, while metacognitive activity represents the directive as a function of meta-level for the regulation of cognitive activity.

In order to explain more clearly the situation of cognitive and metacognitive activities involved in a task, Veenman (2012) likened cognitive activities to soldiers and metacognitive self-instructions to the general. He explained that a general cannot win a war without soldiers, but a large unorganized army will not be successful either. Metacognitive instructions always manage cognitive processes, and without the instructions overseeing the processes, accomplishing the proposed task is more challenging. Many school subjects require metacognitive skills, but according to Veenman (2012), they are honed mainly through four kinds of activities: reading texts, problem-solving, discovery learning, and writing.

Skillful reading and writing has a great impact on problem-solving activities (Hyde & Hyde, 1991). Hyde (2006) emphasized the importance of students in basic education to be involved in mathematical problem-solving. More explicitly, students need to try to describe and represent mathematical concepts, questions, assumptions, and solutions. In this way, students can identify and clarify previous knowledge in the problem-solving processes, which can better prepare students to organize, monitor, and reflect on their work, strengthening their thought processes. The philosophy is that language, mathematics, and thought that uses both cognitive and metacognitive dimensions are better together as a braiding model (Hyde, 2006).

Hyde (2006) is guided by the principles of cognitive psychology and uses the term braiding to indicate that language, thought and mathematics can be intertwined into a single entity, making it possible to make connections between these three important processes result is stronger, more durable and more powerful than if you work individually. With the term braiding it suggests that the three components are inseparable from mutual and necessary support. It states as much stronger the connections between the related ideas are, deeper and richer is the understanding of the concept.

Hyde (2006) emphasizes that the context of braiding benefits children to imagine, visualize and connect mathematics with context. He states that this Model has been used effectively in the instruction of a class with small groups and with teacher support. The questions are effective in order to discuss the problem in small groups as well as strategies of representation in oral language, in this way students begin to internalize these questions to use them for themselves during subsequent tasks.

Methodology

The intervention design was based on the list of self-instructions suggested by Veenman for regulating tasks and on Hyde's Braiding Model (2006) described in the theoretical framework. Hyde (2006) designed the braiding method directly for teachers in the classroom where the teachers could elicit which parts of the model to employ that would be appropriate for the topic and situation, thus using only those items that were necessary in guiding the students through the problem-solving process.

This research seeks to explore implementation of a less detailed procedure for teaching metacognition skills, one that students may apply by themselves without needing total support from the teacher. Students are provided with very simple directives that are nonetheless useful for them to find the problem's solution. Therefore, our intervention aimed at supporting students with solving geometry problems. To accomplish this aim, we established a five-phase plan that focuses on the use of representations and writing as metacognitive tools (see Figure 3).

We guided students with simple prompts, given in the form of questions to guide them through each phase that leads up to the solution. Veenman (2012) originally proposed this list of self-instructions for regulating the problem-solving process. In response to that prompting, students gradually incorporated writing as a support tool during the activities. They were encouraged to use this tool repeatedly on their worksheets. Even though the students may have considered writing to be merely a means of communication, it provided them all the support necessary to control and regulate the process of problem-solving.

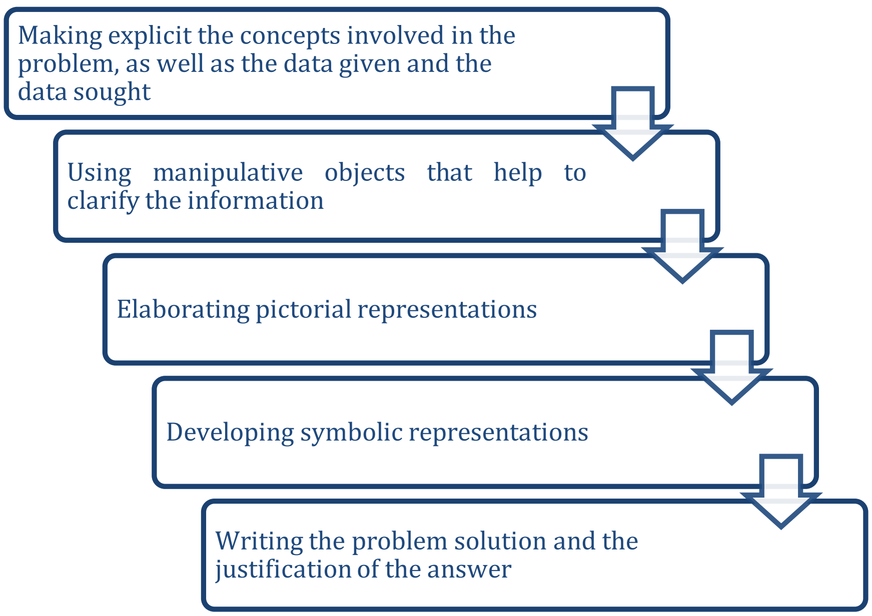


Figure 3. Phases of the problem-solving cycle of activities

The intervention's five phases of the problem-solving cycle were based on the strategies identified by Hyde et al. (1991) which focuses on student representations. However, unlike Hyde, we placed special emphasis on writing, highlighting it as a means of recording, and exteriorizing, and communicating one's thoughts to others. We also noted the function of writing as a metacognitive tool; as an instrument for amplifying and exploring one's own knowledge.

The designated prompts or self-instructions as suggested by Veenman (2012) in this intervention first focused on writing down the information given in the problem. Writing down the information clarifies what we know and understand from the problem. The writing next focused on what was still vague such as the parts that required further clarification and was followed by the writing of what needs to be determined and where that will lead.

The entire problem-solving process used the writing of representations, which helped traverse the path towards attain the solution. The worksheet then captured the student's comprehension and initial reflections about the given information, what was asked and the process to be followed in order to solve the problem. Finally, the students needed to write their justification of the results they obtained and demonstrate why they consider it as the correct solution. Following this path in writing-based problem-solving, students were able to monitor, encode, and establish processes in a reflective manner, which strengthened their learning. The five phases activate a metacognitive process of self-regulation.

Prompts in the form of simple questions or self-instructions guided the students through the learning activities to develop their metacognitive abilities during the problem-solving process. Each question focused on a learning activity as described in Table 2. The analysis of the answers followed the scope suggested by Veenman (2012).

Table 2. Link between Self-Instructions and Veenman's Metacognitive Abilities

Self-instructions for applying writing as a metacognitive tool in problem resolution		TASK	Learning activities representative of Veenman's metacognitive abilities
1	What information am I given in the problem?	START	Reading
2	What do I need to find?		Analysis of the task
3	What knowledge do I have about the topic?		Activation of prior knowledge
4	How am I going to solve it?		Planning
5	What steps will I follow?	DURING	Follow or change the plan
6	Do you thing, the notes you take inside the drawings could help you to solve the problem?		Note taking
7	How do I justify the answer I found?	AFTER	Performance assessment
8	Is this the only way of arriving at the answer?		Recapitulate
9	What other forms can apply?		Reflection on the process

Description and selection of the problems

Problems for the intervention were chosen so as to have particular characteristics. The main characteristic was that the solution did not merely require sentences to be translated into mathematical equations, but rather the answer required a process of inquiry, not merely the application of routine procedures. We also consider it necessary to work on problems that give the students the opportunity to increase their knowledge, develop their skills and abilities and also allow indications of the functioning of the guiding questions when they are answered in writing.

I chose problems that could be solved in multiple ways, as suggested by the Ministry of Education of Jalisco (Mexico). The questions chosen had served as practice questions to prepare students for the Primary and Secondary School State Mathematics Olympics (whose acronym in Spanish is OEMEPS). These problems required students to reason creatively, justify, and explain their solutions. In addition to having the above features, these problems were in Spanish and compatible with the Mexican mathematics curriculum. I selected twelve secondary school level geometry problems from the 2010, 2011, 2012, and 2013 OEMEPS (Secretaría de Educación Pública, 2013a, 2013b) for participants to work through during the teaching intervention.

Characteristics and implementation of the intervention

This research was primarily qualitative in nature and used the line method -does not mean by internet- for assessing metacognitive abilities (Veenman, 2005, 2012) where the written compilation of the students' entire problem-solving process was examined. All notes made by the students on the worksheets were used to facilitate our analysis of the students' written expressions. In addition, through the use of these notes we were able to consider the influence of the context in the development of the solution to each problem. Another source of information used in the analysis was the record of observations logged by the researcher in the work sessions.

Ten 9th graders students served as the participants in this intervention. Was proposed to the Daytime Secondary School principal, to accept the intervention, a problem-solving workshop, where students were encouraged to participate and prepare for their tertiary school admission examinations. This study focuses on the interpretation of writing according to Henning, Gravett and van Resburg (2002), as part of the procedures that can be used to think clearly and build a knowledge, in and of itself, writing is a thought in action. This is in the interest of using writing as a metacognitive tool in problem-solving. Hoping that the intervention generates a clear and orderly thought during the whole process of the activities.

For the teaching intervention, students worked in the classroom during their mathematics class time (45 minutes). There were 20 work sessions, conducted three times per week (Monday, Wednesday and Friday). The first three sessions were dedicated to construct a glossary of fundamental geometry concepts that the students should have acquired by the third year of secondary school: point, segment, line, triangle, and quadrilateral, among others. The researcher guided the participants to describe the basic concepts and properties of geometric figures based on their prior knowledge. These sessions' main purpose was to activate the students' prior knowledge and to help the participants gain some confidence in their work.

The fourth and fifth sessions involved students solving problems taken from the sixth grade OEMEPS (Primary and Secondary School State Mathematics Olympics); worksheets were provided containing the prompts in the task. It was inspected each worksheet after the students finished solving the problem. It was agreed that there were several paths leading to the solution and each option was discussed. The only condition was that the questions in the prompts had to be followed. For this purpose, a poster with all the prompts was put up on the board for the next session. These questions would guide the students through solving the 12 problems. During the remaining 15 sessions, students worked on solving the problems individually and at their own pace for the duration of the session.

Results

All students got the correct answers for all the problems, most of them after reviewing failed attempts. Some students directly applied the initial directions by writing what information was given and what information they needed to find in a complete, clear, and orderly fashion. Figure 4 shows a student's correct answer, who gave a detailed reconstruction of his or her train of thought by writing a detailed description of each relationship used and operation performed. This student also tried to write an orderly narrative sequence, providing clear visual description of mathematical expressions, and was one of the few students to use punctuation marks.

The student began his or her writing with a correct description and interpretation of the information given in the problem. Then, he or she provided some useful representations to exteriorize the information, which clearly indicates the sum of the interior angles of each polygon and the measure of each of the angles. These assertions imply the activation of prior knowledge. Figure 4 also shows that the student's knowledge and assertions combines with symbolic writing (the sum of the interior angles of an equilateral triangle is 180°, and each of them measures 60°).

The student then planned the next steps to solve the task, indicating the procedure: "Mark triangle *RNO* as an isosceles triangle since two of its sides are the same... and I don't know the measure of [angle] *RNO*." This narrative demonstrates a correct identification of the information. He or she then followed the proposed plan and showed step-by-step the results with accompanying explanation why each operation was performed, as can be inferred from the last comment "... and then I divide the answer by two and get 39°, (the triangle is isosceles)," thus justifying the operation of dividing by two and confirming that the answer is correct (Figure 4).

Figure 5 shows the worksheet of a student who solved another problem following the prompts and numbering the steps to taken. The student first worked with the starting prompts upon identifying the characteristics of the equilateral triangle and the square, which were represented in the drawings and then wrote down the measures of the corresponding angles.

The student`s worksheet shows how the numbered steps were followed in the solution of the problem, specifying first how the triangle and square were joined to produce the main figure of the problem and then indicating that the measure of angle *ACE* must be found. In the third step, the student wrote, “I know how to measure the angles,” and that the shapes were “a square and an equilateral triangle,” confirming the characteristics of each.

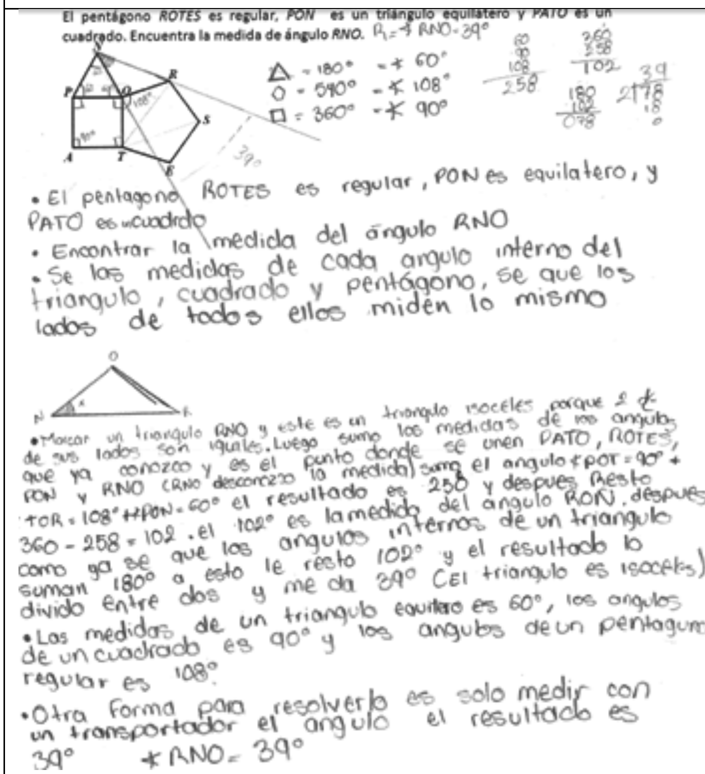
Problem. The pentagon *ROTES* is regular, *PON* is an equilateral triangle and *PATO* is a square. Find the angle measure of *RNO*.		
	Translation: • The pentagon *ROTES* is regular, *PON* is an equilateral triangle and *PATO* is a square. • Find the measure of angle *RNO*. • I know the measurements of every internal angle of the triangle, square and pentagon, all the sides are the same length.	Note taking Reading and analysis of the task Activation of prior knowledge
	• Mark triangle *RNO*, which is an isosceles triangle because two of its sides are equal. Then I add the measures of the angles that I know meet at that point *PATO*, *ROTES*, *PON*, and *RNO* (*RNO* I do not know the measure). I added the angle *POT* = 90° *TOR* = 180° *PON* = 60° the result is 258° and after I subtract 360 -258 = 102. 102° is the *RON* angle measure, then as I already know that the interior angles of a triangle add 180°, I subtract 102° from it and then the result it’s divided by two to get 39° (the triangle is isosceles). • The measure of the angles in an equilateral triangle is 60°, the angles of a square are 90° each and the angles of a regular pentagon are 108°. • Another way to solve the problem is to simply measure the angle with a protractor and get the result 39°∠*RNO* = 39°.	Note taking Following the plan Performance assessment Recapitulation and reflection on performance

Figure 4. Sample worksheet with student’s answer

In the fifth step, the student stated, “I know the measures of the angles of a square... I know each internal angle of an equilateral triangle measures 60°,” then further continued the narrative with “I will measure each angle in the figure.” From that moment, even though all the information written down was correct, the student decided to change plans and indicated that the above description “[didn’t] count”. Upon changing plans, the student used the representations of the figures separately, constructing each of the three triangles by joining the square and the equilateral triangle together. Then, the student used these representations to measure each angle and arrive at the answer. The student then described the answer, starting with the representation of the original figure together with the measures of each angle, followed by the answer, and lastly included an explanation of the path to the answer.

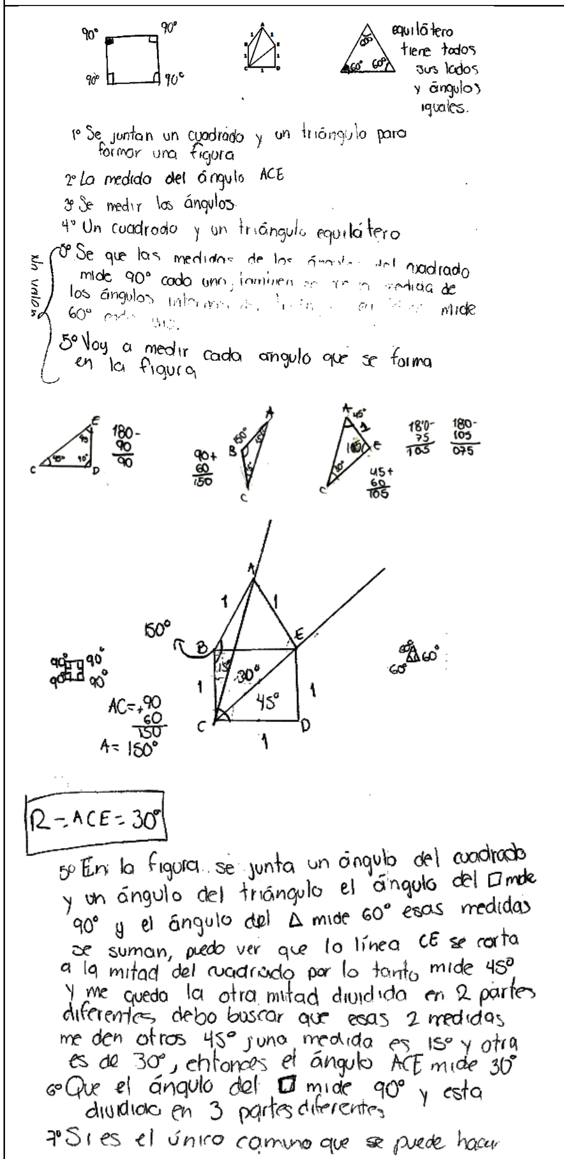
Problem: A square and an equilateral triangle are joined to form a figure shown: what is the measure of angle *ACE*?		
	Translation: Equilateral has all equal sides and equal angles. 1° A square and triangle are joined to form the figure. 2° The measure of angle *ACE*. 3° I know how to measure the angles. 4° A square and an equilateral triangle. Does not count (5° I know that the measures of the angles of a square are 90° and that each internal angle of an equilateral triangle measures 60°. 5° I will measure each of the angles in the figure). 5° One of the angles of the square and one from the triangle are joined in the figure, the angle of the square is 90° and the angle of the triangle is 60° so we add those measures, I can see that line *CE* cuts the square in half, so that angle is 45° and then I must divide the other half in 2 separate parts, which must add to 45° and one of them is 15°, so the other must be 30°, therefore angle *ACE* measures 30°. 6° That the angle of the square measures 90° and is divided into 3 parts. 7° Whether it is the only path to the solution.	Reading and analysis of the task Activation of prior knowledge Planning Following the plan Changing the plan Note taking Performance assessment Recapitulation and reflection of performance

Figure 5. Sample worksheet with student’s answer

The most important observation from the worksheet is how even though the initial assertions were correct, the student decided to change course and modified the work plan. This change led to establish a relation between the isosceles triangles in figures *ABC* and *CED*. The student then founded the measures of all other angles, in particular the measure of angle *ACE*, which is the problem's solution by using the measures of some of the angles given and the characteristics of the square and the equilateral triangle.

The participant then narrated the steps taken to obtain the answer. Most notable is the statement that "The angle of the square (in the figure) measures 90° and is divided into three different parts," a description which confirms the student narrative and provides certainty in the answer (Figure 5).

In the case of both Figure 4 and 5, we note the development of students' metacognitive abilities during the intervention, reached gradually using the questions described in Table 2. The students acquired orientation and planning abilities during each problem's resolution, which we can see when they noted the steps taken in their problem-solving, described the procedure, provided reasons, and justified their entire process.

Due to the favorable results obtained from the students participating in the intervention, the investigation expanded to include other students in the third year of secondary school (9th Grade). Something to note is that the application of this worksheet was not intended to show the contrast between using and not using suggested prompts to facilitate the problem-solving process. To adjust for the space constraints as well as to limit distractions or communication between students, we used four different worksheets.

This expanded group included 50 students: 10 students who had participated in the intervention and 40 other students who had not. Non-participants of the problem-solving workshop (NP-PSW) only received the instructions to solve the problem and write down the procedure they used to obtain the answer, and justify their answer within one class period (50 minutes). The difference between the two groups was that non-participants had no prior knowledge of the prompts.

Table 3 presents the results obtained in the application of these worksheets. The second column shows the number of student participants of the problem-solving workshop who obtained correct and incorrect answers while the third column shows the corresponding results of the students who did not participate in the workshop. We observe in Table 3 that only 12 students obtained the correct answer, 10 belong to the workshop participants' group and only two to the non-participants' group. The 38 students who were not able to solve the problems were all in the group of non-participants.

A closer look at the answers shows that 76% were incorrect answers, and they belong to the group of the students who were non-participants at the workshop. Of the 24% who obtained correct answers, 4% were students who did not attend the workshop, while the remaining 20% were participant students.

Table 3. Results of the Participants and Non-Participants of the Workshop

	Participant of Problem-Solving Workshop (P-PSW)		Non-Participant of Problem-Solving Workshop (NP-PSW)	
	Correct answers	Incorrect answers	Correct answers	Incorrect answers
Problem 1	2	0	2	10
Problem 2	2	0	0	8
Problem 3	3	0	0	10
Problem 4	3	0	0	10

The type of answers we obtained suggest that the use of writing through questions produces favorable results. Due to space constraints, this paper only shows the analysis of four types of answers to the problem shown in Figure 5, given by 12 of the non-participating students (Problem 1 in Table 3). Four of the students who did not participate in the problem-solving workshop used the protractor immediately to measure the angle, using no prior knowledge. Given what we could examine from the answers, they did not have a clear idea on how to use the protractor to measure the angles (Figure 6).

In the first answer shown in Figure 6, the student asserted, "I first took the protractor and placed it correctly to find angle *ACE*" and wrote at the end of the question "$R = 33°$". Another student stated, "Well I took the protractor and placed it over angle *C*, measured the angle and got 150° as my answer" (second answer). Both narratives show that the students only considered the simplest procedure, which is to measure the angles by using the protractor, although some had issues using the protractor.

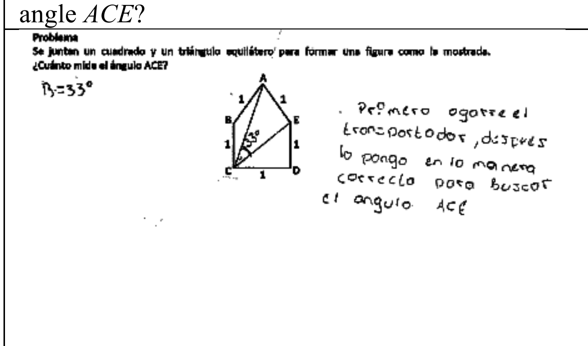
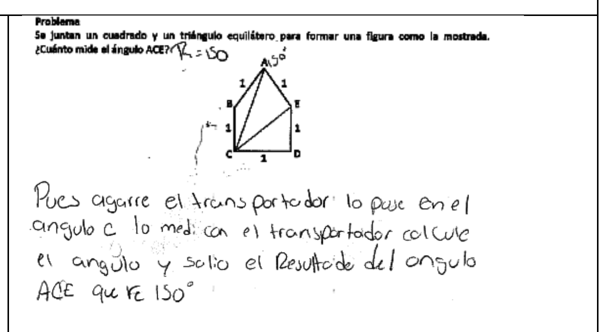
<table>
<tr><td colspan="2">Problem: A square and an equilateral triangle are joined to form a figure as shown: what is the measure of angle ACE?</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>I use the protactor in order to measure the angle ACE</td><td>I grab the protractor and I put it in the angle C, then I measured the angle and I got the result for angle ACE, that is 150 °</td></tr>
</table>

Figure 6. Answer to Problem by four students from the NP-PSW group

The first student correctly placed the protractor and then properly measured the angle, although the response was three degrees greater than the correct measure. The second student, from the researcher's point of view, placed the protractor correctly but read off the incorrect value of 150° from the protractor. This error was made because protractors, which students have been using at a very basic level, have the measures of angles in both directions (from left to right and from right to left). When the concept of angle measurement is unclear, the students misread the measure on the protractor.

Two other students who did not participate in the problem-solving workshop mentioned using trigonometry to obtain the measure of the angle even though the second student started by using the protractor to measure the angle. Another student reached an incorrect answer of 70°, without leaving any trace of his or her reasoning, then stated that trigonometric functions would be appropriate for this problem but had no idea how to use them (See figure on the right in Figure 7).

Another student (see the left-hand side in Figure 7) used trigonometric functions, starting with a description written down on the left hand side, related to the area of one of the triangles, which would be correct if it referred to triangle *EDC* "$\frac{1x\ 1}{2} = 0.5$", although it is unclear why they obtained the area. Other operations were then performed, and trigonometric functions were used with incorrect data because the hypotenuse of *EDC* equals $\sqrt{2}$ and not 2 as written by the student. The hypotenuse of triangle *ACE*, which is not a right triangle but was considered to be one by the student, would have a measure of $\sqrt{3}$ if it were right angled, not 3 as the student stated.

The student explained, "First, I get the area of everything, so I can know the value of the sides of the triangle, then I use trigonometry to get the angle. I used the cosine because it gives me an angle with the values I am asked for." This assertion confirms that the student assumed that both triangles were right-angled, in the student's view, even though two of them were not, as angle *AEC* measures 105°, a value that appears to be obtained from the information given in the problem, joining a square and an equilateral triangle.

Next, we examine the answers of two other students (A and B) who did not participate in the workshop, who used the formula for the area of a triangle to solve the problem, as shown in Figure 8. It remains unclear how the area could help the students find the measure of angle *ACE* and how they could even find the triangle's height in order to use the area formula.

They both used the same procedure but with different data. Neither of them arrived at the correct answer nor provided much description or justification for any of their work. As the second student stated, "I used the area," highlighting it with an arrow. They did not explain their responses. Although the problem asks, "What is the measure of angle *ACE*?" both their answers were for the area of a triangle, not the measure of an angle. We believe neither of the students attempted to justify nor analyze their answer. If they tried to justify their results, then they might have realized what the problem really asked.

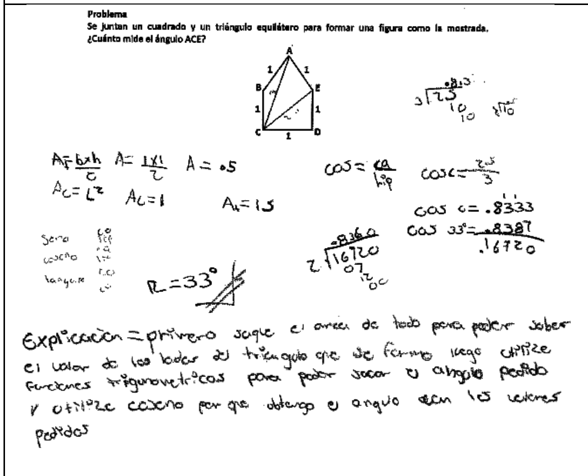
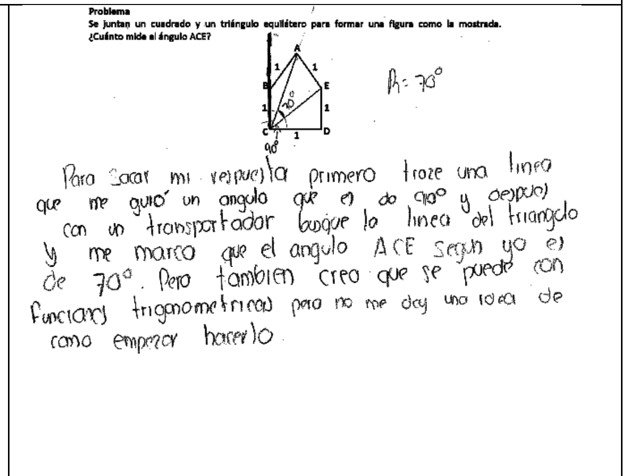
<table>
<tr><td colspan="2">Problem: A square and an equilateral triangle are joined to form a figure as shown: what is the measure of angle ACE?</td></tr>
<tr><td></td><td></td></tr>
<tr><td>Explanation: First, I obtained the whole area in order to know the value of the sides of the triangle that is formed. Then, I used trigonometric functions to get the angle requested and I used cosine because I get the angle with the requested values</td><td>To have my answer, first I draw a line that guided me an angle that is 90°, then with a protractor I searched the line of the triangle and it showed me that the angle ACE is 70°. But, I also believe that it is possible to do it with trigonometrical functions but I do not have an idea about who to start solving it</td></tr>
</table>

Figure 7. Answers to Problem from two students NP-PSW

Both answers reflect poor reading comprehension, which in turn failed to activate necessary prior knowledge for solving the problem. They also did not attempt to analyze the task in order to decide if they were providing the information requested in the worksheet, what is the measure of angle *ACE*?

The last four students who did not participate in the workshop solved problem 1 using the representations given in the problem, but the representation confused them, and this led to an erroneous interpretation of some information, just like the student who had used trigonometric functions (Figure 7). The above contradicts the situation of the representations given in the problem.

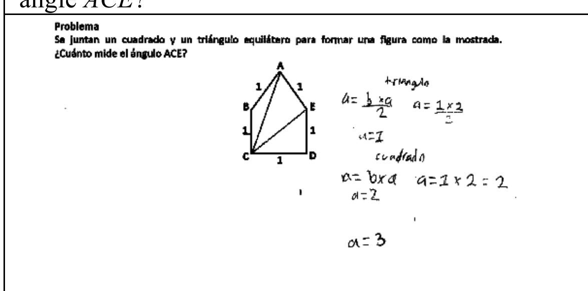
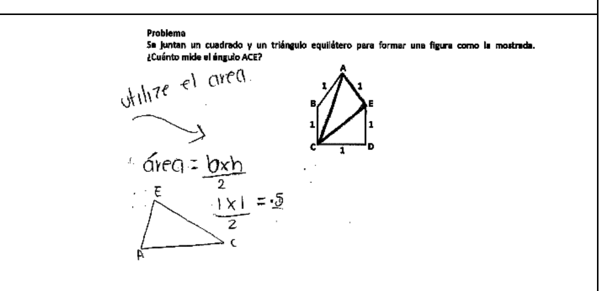
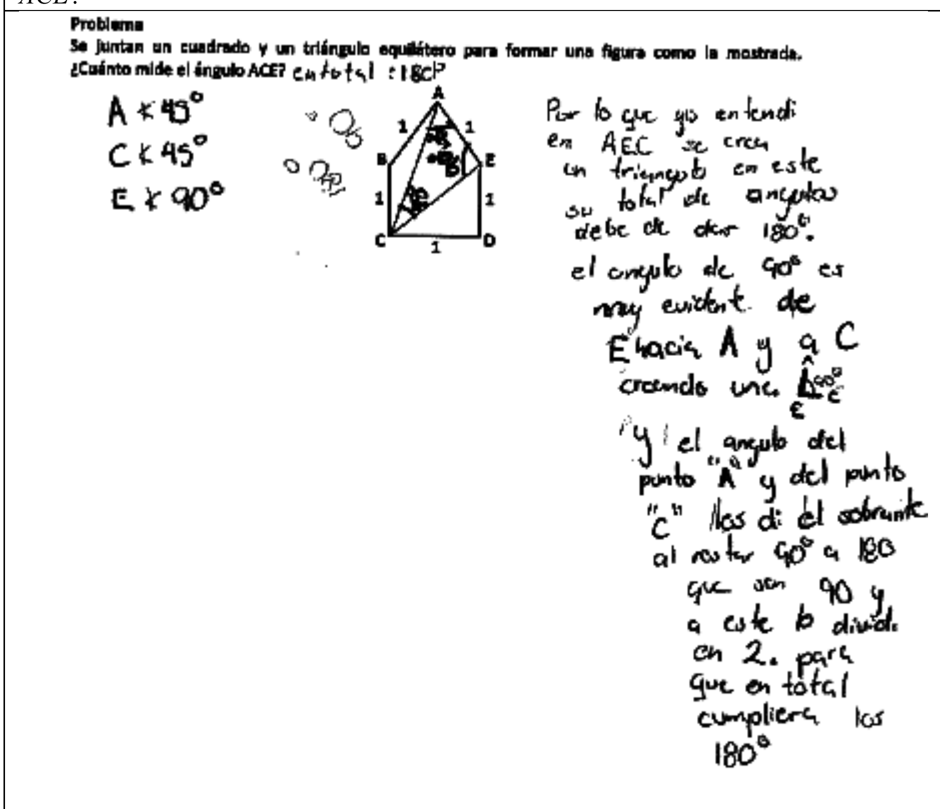
<table>
<tr><td colspan="2">Problem: A square and an equilateral triangle are joined to form a figure as shown: what is the measure of angle ACE?</td></tr>
<tr><td></td><td></td></tr>
<tr><td>Triangle $A = \frac{b\ x\ h}{2}$</td><td>Use the area. Area = $\frac{b\ x\ h}{2}$</td></tr>
</table>

Figure 8. Worksheet of two student NP-PSW, student A left and B right

For instance, the answer given by the student in Figure 9 is in relation to the measure of the angles and stated, "it is evident that from *E* to *A* and *C* creates a ∠ 90° and I added the remaining measure to the angle from point A and C... so 90° divided by 2...", therefore obtaining the other two 45° angles. The student failed to notice the importance of the given information "a square and equilateral triangle are joined," which does not give an angle of 90° at point E. In addition, only the sides of the pentagon are equal, and therefore, triangle *ACE* is not isosceles, which would be necessary to establish that the other two angles measure 45°; this condition is only satisfied by triangle *ABC* and triangle *CED*.

We have observed how students found obstacles at different points in the problem-solving activities. Some had issues during the reading and analysis phases of the task. As seen in Figure 8, some had trouble understanding what the problem asked them to do. Both students obtained the area of different triangles, which was not what the problem asked them to find. Other students were unable to activate prior knowledge, which is seen in Figures 6 and 7, such as measuring angles, using the protractor, though the measures of the angles could be deduced from what they knew about squares and equilateral triangles. The lack of prior knowledge prevented students from reaching the answer. The remaining students managed to establish a plan, but the imprecision of their notes or the lack of required prior knowledge led them to incorrect answers (Figure 9).

Problem: A square and an equilateral triangle are joined to form a figure as shown: what is the measure of angle *ACE*?



From what I understood, in AEC a triangle is created, its total from all the angles must be 180°. The 90° angle is very evident from E to A and to C creating an angle the 90°(represented by the symbol of angle <).
And the angle from point "A" and point "C", I gave them the remaining by subtracting 90° to 180. which are 90 and then I divided it into 2, so that it was fulfilling 180°.

Figure 9. Answer to problem taken from one student in the NP-PSW group

The use of the problem with non-participants allowed us to realize that the prompts used with the participants had indeed been like a plan of action, which guided the student through the steps of the problem-solving process. The use of the problem also allowed us to realize that the participants carefully reviewed the steps that they had followed when writing down the justifications to their answers. Not only did they check whether they found the right answer, but they also discerned whether the steps were successful in solving the problem. The act of writing under the guidance of the prompts given at the start of the intervention helped them understand the solution process.

From our point of view, identifying what is given what is looked for in the problem formulation, and beginning to work explicitly writing these elements, makes a big difference for the students. Write the data given provides an initial orientation, which remains on sight, and functions as a control elemental that helps to correct mistakes and take into account relevant relationships and conditions.

Conclusions

With this intervention we realize, firstly that the self-instructions are in themselves a plan of action, which guide the student step by step during the whole process of problem- solving and secondly, that when they wrote the justification of their responses carefully reviewed the steps that followed. That is, they not only analyzed if they achieved to the correct answer, but recognized that steps were successful in the resolution, that is, writing helped them to understand the solution process, guided by the questions given at the beginning of the experiment.

In this study, I have shown that the decision to coordinate different elements of mathematical thinking through prompts was associated with stronger performance and increased sophistication of students' problem-solving behaviors. The intervention relied on purposefully selected problems that provided opportunities to develop concepts while also allowing the students to be free to pursue other paths to the answer based on their prior knowledge.

The chosen problems met the intended conditions and together with the use of prompts, served to reinforce certain habits among the students that participated of the intervention workshop, such as having steps to follow in a certain order, as well as gaining the confidence to communicate their thoughts through the worksheets thus expanding their points of view. Additionally, the worksheets provided evidence in the analysis of two participants' answers that by writing, all ten students in the workshop activated their prior knowledge, organized ideas, established a plan to follow, supervised the entire process, evaluated, and used feedback about their answer, implying a metacognitive process.

As Schoenfeld (1985) described, the metacognitive process is exteriorized when students reflect on the thoughts they had while performing a mathematical task. Therefore, we may assert that at the problem-solving workshop, metacognition occurred when students, while following the prompts as self-instructions for solving the geometry problems:

- Reflected about how to proceed in the problem and on the processes that were generated in the solution.
- Developed the justifications that backed their problem-solving procedure in each problem.
- Evaluated their results and reflected upon whether there are other ways of finding the correct answer.

We therefore consider that the objective of our intervention was achieved, which was to facilitate the problem-solving process and develop metacognitive abilities, combining writing with solving geometry problems. On the other hand, the students' disposition and confidence in their own knowledge increased throughout the problem-solving workshop.

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CONNECTEDNESS OF PROBLEMS AND IMPASSE RESOLUTION IN THE SOLVING PROCESS IN GEOMETRY: A MAJOR EDUCATIONAL CHALLENGE

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Our contribution shows the anticipated effect of what we call connected problems *in developing the competencies of students and their acquisition of mathematical knowledge. Whilst our theoretical approach focuses on didactic and cognitive interactions, we give special attention to a model to reason about learners' conceptions, and the ideas of mathematical working space and zone of proximal development, in order to explore how connected problems can help to resolve moments of impasse of a student when solving a proof problem in geometry. In particular, we discuss how the notion of interaction moves our theoretical framework closer to the methodological challenges raised in the QED-Tutrix research project jointly being realized in didactics of mathematics and computer engineering.*

Keywords. CHSM variables and HPDIC graphs, conception and mathematical working space, complexity of connectedness and decision-making, devolution and learning, didactic and cognitive interactions, geometric thinking, impasse and connected problems, intelligent tutorial system QED-Tutrix, problem solving.

Introduction

We begin with a brief story. Once upon a time in their mathematics class, two 14/15-year olds were attempting to solve a geometry proof problem using an intelligent tutorial system. The problem involved a comparison of the area of two triangles with that of a parallelogram and a demonstration of the selected conjecture. After reading the statement and constructing, or moving, elements of the figure in the dynamic geometry module (Fig. 1), the students quickly agreed that the areas were equal. They began to write their first sentences using the interface of the tutorial system and, from the outset, they were delighted to see Prof. Turing, a virtual tutor, telling them with a smile (emoticon) that their first

[1] This research has been made possible by a grant from the Conseil de Recherches en Sciences Humaines (CRSH 435-2015-0763), Gouvernement du Canada.

answer was correct. As good students, they were aware that they could sometimes become blocked in their work. Thankfully, through the messages, Prof. Turing always managed to restart their solution process. It must be said that whilst not claiming to be a substitute for a human teacher, this virtual tutor had access to a memory of 69,000 possible solutions and could quickly target the solution envisaged by the students. In its personal support facility, Prof. Turing also recognized any persistent difficulties students had, and when appropriate, could suggest that the student re-contact their teacher.

It was then that something happened that we did not expect. Upon the students reaching an impasse during the next stage of their solution, and the teacher having seen the appropriateness of the messages that students had been receiving from Prof. Turing, we thought that the teacher's intervention would have placed greater emphasis on the meaning of the messages in the context of the problem. Instead, after a brief analysis of the situation, the teacher asked the pupils to solve a new problem, explaining: "Looking at [the statement of the problem on paper], it makes me think of this [pointing to another problem on the sheet]. If you can solve that, you will see what you are currently missing." The students, accustomed to this type of intervention in their usual classes, began to solve on paper the new problem. Then one of them said to the other: "look I know it... look, that's why it works!" The solution to the original problem at the interface was thereby restarted. This prompted us to wonder whether, like the teacher, we could give Prof. Turing a set of problems to generate help messages of a new kind.

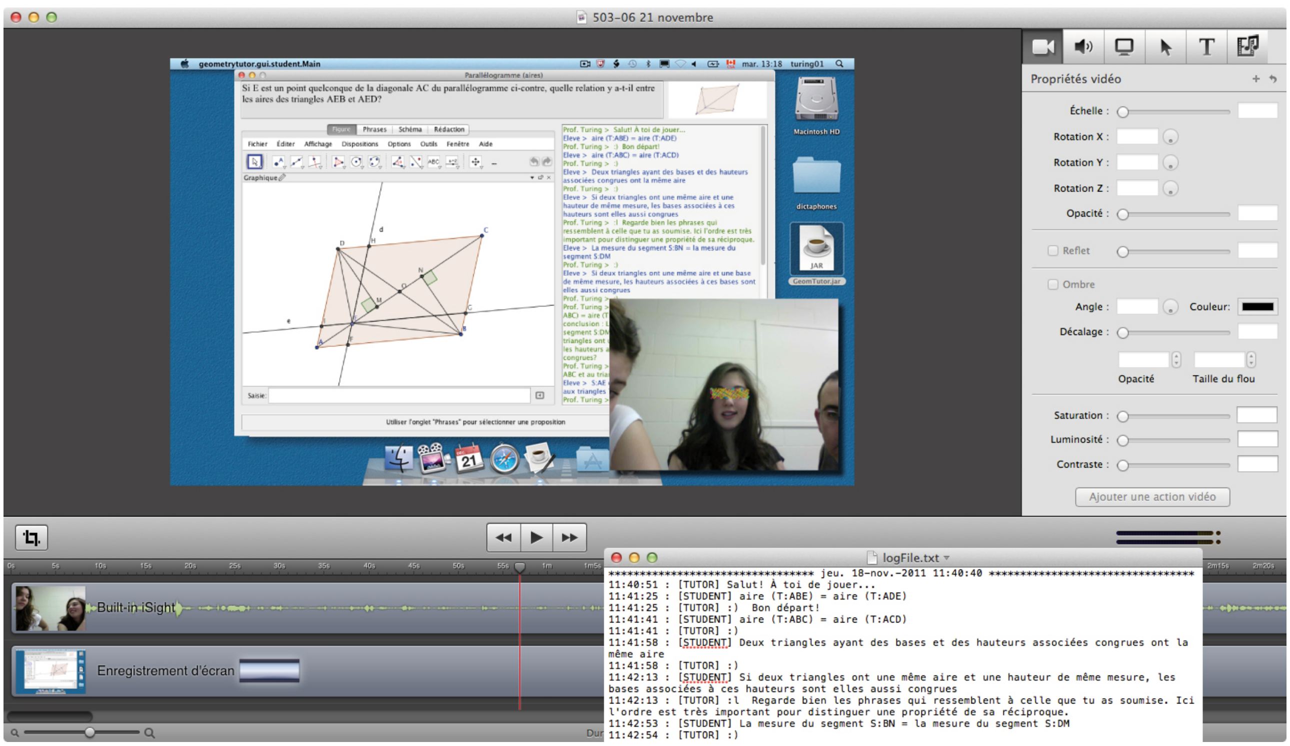


Figure 1. An analysis of the interactions between students and GGBT system, which inspired the implementation of QEDX.

This brief story shows how the first version of our system GeoGebraTUTOR (GGBT) (created to study, amongst other things, real teacher interventions) worked and what is the basic idea that inspired the implementation of our second version, QED-Tutrix (QEDX), emphasizing problem solving as a fundamental mathematical competence (see *Research context* section). These systems, and the passage of GGBT to QEDX, are described and analyzed by Tessier-Baillargeon (2016), from the perspective of the didactics of mathematics, and Leduc (2016), from computer engineering.[2] In the following, we situate the context of the research around the problem solving before introducing our theoretical framework centred on the notion of interactions. We first introduce key concepts and axes of references (in italics). We then propose two approaches that show how the connected problems can intervene to resolve moments of impasse and we conclude briefly with some expected results.

Research Context: Problem Solving at the Heart of the Teaching and Learning of Mathematics

According to the Theory of Didactical Situations (TDS; Brousseau, 1997, p. 31):

> We know that the only way to 'do' mathematics is to investigate and solve certain specific problems and, on this occasion, to raise new questions. The teacher must therefore arrange not the communication of knowledge, but the *devolution* of a good problem. If this devolution takes place, the students enter into the game and if they win learning occurs.
> But what if a student refuses or avoids the problem or doesn't solve it? The teacher then has the social obligation to help her and sometimes has to justify herself for having given a question that is too difficult.

In the spirit of the TDS, we illustrate the challenges of a research project based on three key ideas: The need to find and solve specific problems in the learning of mathematics in secondary school, the help that constitutes the *devolution* of "right problems" (see § 4 in the next section) for the development of *competencies* and the *geometric thinking* of the student, and the voluntary, but surprising, action of the teacher who chooses to pose a new problem to jumpstart an initial solving process that has been halted (Richard, Gagnon & Fortuny, 2015). The original solving process focuses on a *root problem* and a new problem put forward, such as a message returned by a problem-management system, is called a *connected problem* (Richard, Gagnon & Fortuny, 2013).

Our research proposes two questions as overall aims 1) in the management of connected problems, which conditions allow for the restarting of a halted solving process with a student? 2) What information brings us root problems and connected problems, posed by a tutor, in the teaching and learning of mathematics? There are theoretical and methodological issues at the origin of these questions, but before addressing those we first turn to why the interactive management of problems is so important. In learning, if the right problem is characteristic of mathematical work, it is also a component of the construction of mathematical concepts in the course of *cognitive interactions* with the milieu, complementary to *didactical interactions* with the tutor. This joins with the notions of the *mathematical working space* (Kuzniak & Richard, 2014) and *conception* as knowledge that is actually built by the student (Balacheff & Margolinas, 2005). The concepts of *conception* and *working space* offer two insights into the same subject-milieu system (Fig. 2). We return to this in our theoretical framework section below.

[2] For a comparative study and a complete update on related tutorial systems, see Tessier-Baillargeon, Leduc, Richard, & Gagnon (2017).

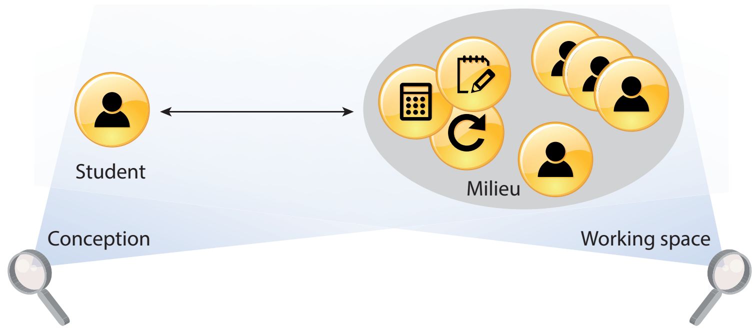


Figure 2: Two insights into the subject-milieu system: from the point of view of conception and that of the workspaces, the first looking at the pupil in the foreground and the second, the milieu.

In terms of teaching, when a problem choice occurs through of a moment of impasse, or the success of the root problem, we create a learning scheme tailored to the student's competencies. This view pushes, in an innovative way, the boundaries of traditional teaching, which involves posing problems in series without regard to the proximity of problems already solved or the knowledge acquired during the learning process. If we reflect on the mutual commitment between the student and the teacher with regard to mathematical knowledge, the management of connected problems respects the specificity of the *didactical contract* and offers a response to the paradox of devolution.

Theoretical Framework: An Approach Centered on Didactic and Cognitive Interactions

The general framework follows five conceptual reference axes that have been published in journals of the social sciences (Richard, Fortuny, Gagnon, et al., 2011) and computational mathematics (Richard, Gagnon, & Fortuny, 2013). These axes are *epistemological* [in reference to the dialectical proofs and refutations of Lakatos (1984), the heuristics for problem solving of Polya (2007) and the breaking points in the mathematical discovery of Mason (2005)], *semiotics* [the theory of the functions of language of Duval (1995), the functional-structural approach of Richard and Sierpinska (2004) and the register of dynamic figures of Coutat, Laborde, and Richard (2016)], *situational* [the theory of didactical situations of Brousseau (1997) and the model to reason on learners' conceptions of Balacheff and Margolinas model (2005)], *instrumental* [the theory of the instrumentation of Rabardel (1995), the geometric working space of Kuzniak (2006) and the instrumented reasoning of Richard, Oller, and Meavilla (2016)] and *decisional* [the didactic paradoxes of Brousseau (2004) and the theory of decision-making of Schoenfeld (2011)].

In the TDS (Brousseau, 1997), the milieu appears as the system antagonist to the student. Given the fact that the milieu is a vehicle for knowledge, the latter can only be revealed when the student questions it. It is therefore not an opposite response, but rather a partner in the creation of meaning. The first system that interests us is therefore the *subject-milieu system* (Margolinas, 2004, pp.13-14):

> Brousseau goes on to consider the subject-milieu interaction as the smallest unit of cognitive interaction. An equilibrium state of this interaction defines a state of knowledge, where the subject-milieu imbalance is producing new knowl*edge (search for a new balance)*.

This contribution of the TDS is well documented in the literature. We highlight the first two results (see below) when the observables in our project are then grouped according to didactical and a-didactical intentions.

§ **1**. If the TDS determines all knowledge by specific situations, the model to reason on learners' conceptions of Balacheff and Margolinas (2005) — known in literature as the *cK¢ model* (conception, knowing, concept) — places conceptions in the subject-milieu interaction, while initially characterising a conception created by the problems in which it is involved. Specifically, this model characterizes conceptions **C** as a set of defining problems (P) for which they provide tools (R) by relying on representation systems (L) and a control structure (Σ) that allows for judgments and decision-making. The result is a strong relationship between a moment of impasse and the arrival of a connected problem. The a-didactical observables are modelled by problems (P), operators (R), languages (L), and controls (Σ) of the conceptions.

§ **2**. A didactical intention cannot simply develop mathematical competencies, since it must also seek knowledge recognized by the institution and allow the student to carry out their work as a mathematician. Thus, in the exercise of geometric meaning, it is still necessary that the competencies at stake adhere to a theoretical reference: geometry. With its plans (epistemological and cognitive), genesis (instrumental, discursive, and semiotic) and cognitive math competencies (reasoning, communication, and discovery), the model of Mathematical Working Space (MWS) allows for the design and organization of the environmental thought process and enables the work of individuals solving mathematical problems (Fig. 3; from Kuzniak and Richard, 2014). In geometry, when the focus is on the learning process of students in a didactic situation, the epistemological plan can also be seen as an epistemological milieu and the cognitive plan, as an epistemic subject (Coutat & Richard, 2011; Coutat, Laborde, & Richard, 2016). It follows that the specific interactions within the geometric approach are part and parcel of the working space, and a characterization of these interactions, from a set of tasks (problems to solve chosen by the teacher), reveals issues with the mathematical competencies of the subject during their geometric work. The didactical interactions are manifested in the choice of problems to solve and their meaning can be interpreted from the components of the working space. The links between the didactical interactions and cognitive interactions are possible because the MWS incorporates both subject-milieu interactions and the intention to amend the system with new problems. Moreover, the model of the MWS joins in particular the model to reason on learners' conceptions with the notion of *fibration*. The set of defining problems (P) belongs to the epistemological plane (pose a problem/problem at issue) or to the cognitive plane (solve a problem/solving at hand), the operators (R), languages (L) and controls (Σ) of the conceptions can be associated respectively with the fibrations of the type: semiotic, material, and notional tools; semiotic, material, and discursive-graphic representations; semiotic, material, and discursive-graphic controls (Richard, Marcén, & Meavilla, 2016; Kuzniak, Richard, & Michael-Chrysanthou, in press).

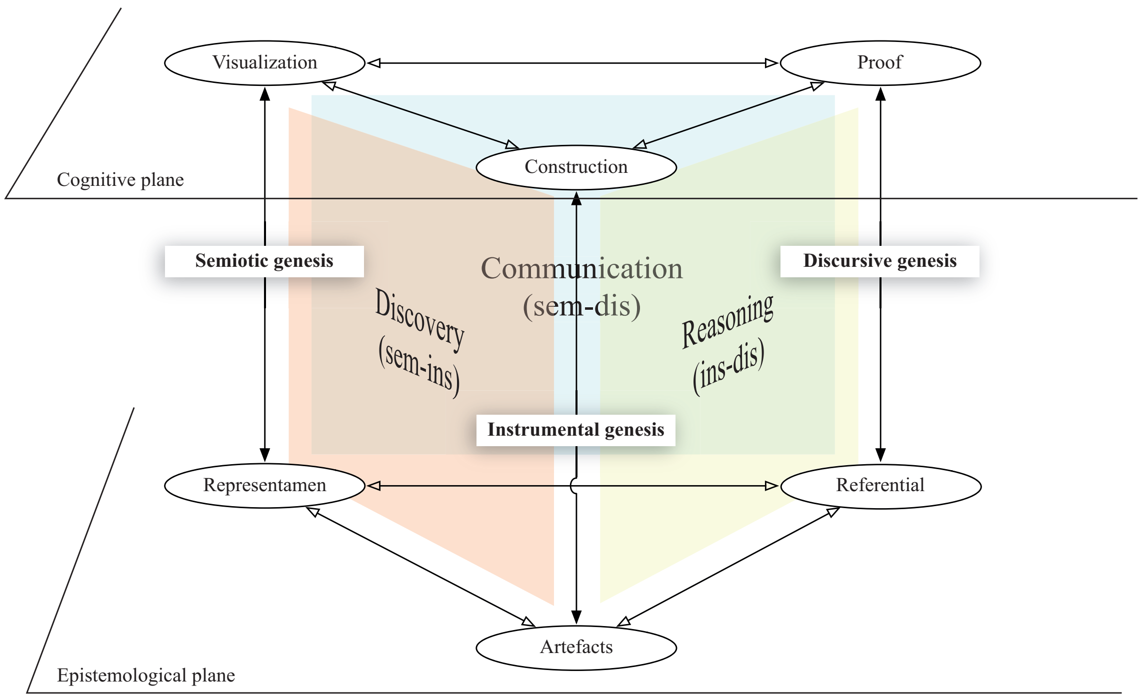


Figure 3: The vertical planes in the MWS join the three math competencies of the educational programme of the Quebec school (MÉLS, 2016), from primary to secondary, as training vectors (Coutat, Laborde, & Richard, 2016).

§ **3**. As a central concept in the work of Vygotsky (2013), the Zone of Proximal Development (ZPD) represents the distance between what a child can learn if they are alone and what they can learn if they receive the assistance of a competent person. Since the ZPD represents primarily what the learner is not able to do without help, it appears that the level of potential development is greater when the learner is accompanied by a human teacher or an expert system. With regard to the theory of Vygotsky, if the arrival of a connected problem adapted to a moment of impasse already contributes to the normal development of the student, the reconciliation between *impasse → connected problem* has considerable potential to facilitate and accelerate learning[3]. In other words, the *impasse → connected problem* consequence allows for focus on a possible evaluation of the zone of proximal development for the purposes of facilitation, based on both current and potential gains. In some ways, the idea of a zone of proximal development is similar to the notion of conception within the cK¢ model in the sense that the knowledge acquired by the learner is focused locally and demonstrated in terms of validity and efficiency in the context of the root problem.

§ **4**. In light of our approach, the *right problem* is a concept whose choice and intervention are placed in didactical and cognitive interactions. In everyday language, the adjective *right* means that the problem has met or has the useful qualities we expect. The utility area that interests us here is based on our research questions, i.e. a problem is right if it allows the exercise of a new conception, which means

[3] For example, if we know that a student cannot solve a problem because he or she does not confront the hypotheses, then we ask him or her a new problem in which the main issue is the discovery of incompatibility between hypotheses.

that there will be learning after a first root problem, or if it is used to restart a blocked solving process to facilitate and accelerate learning. This is a relative definition that assumes some knowledge of the issues of mathematical work by posed problem solving — whether in the process of discovery and exploration, justification and reasoning and presentation and communication (see the mathematical cognitive competencies in vertical plans as in Fig. 3).

The notion of connected problems is innovative in the didactic literature but we have included, specifically, Iranzo and Fortuny's (2009) structure of learning routes where the transition from one problem to the other at a time of impasse or interaction responds with a tutorial system. This results in a tree structure that reconfigures at each point of impasse (Fig. 4). A learning route can be seen as a tree branch and the configurations for root problems create a problem *forest*.

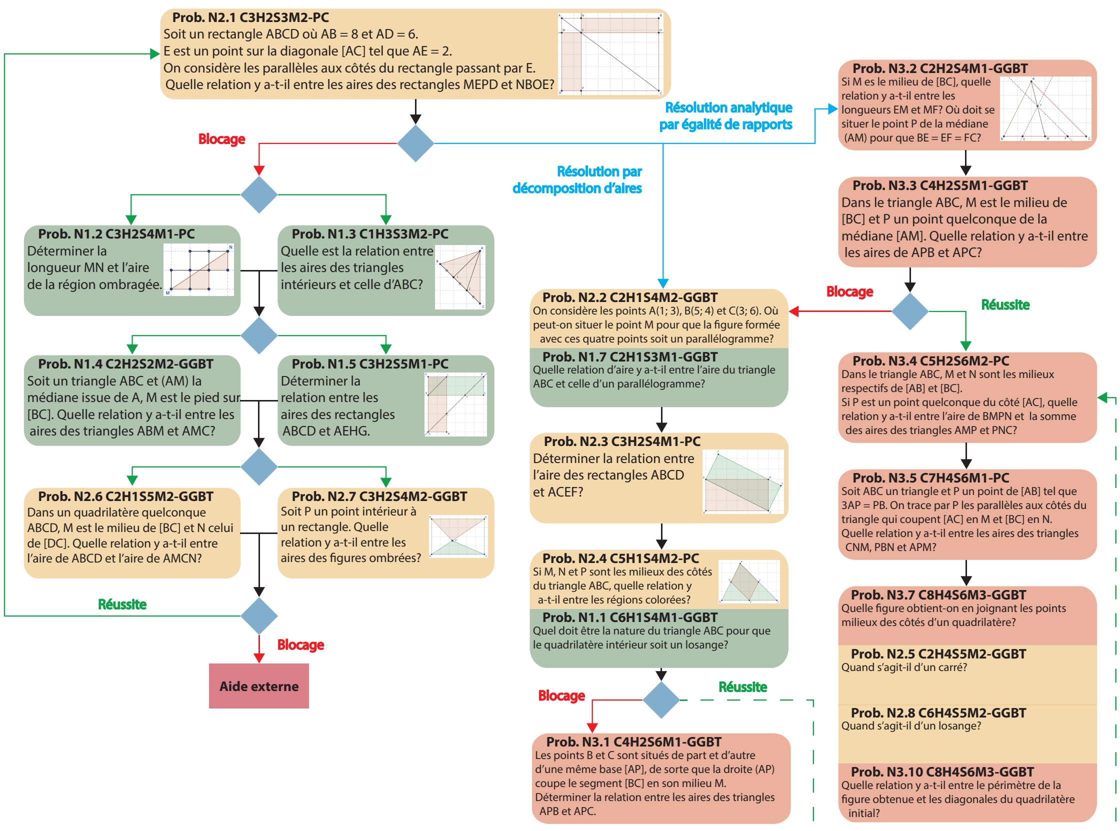


Figure 4: Tree diagram of connected problems for the same root problem (stated at the top).

Choice of Problems: Complexity of Connectedness and Decision-Making

The question of connectivity arises in characterising each problem in a number of variables and comparing the values of variables. Two connected problems are similar when variable values are shared. In this paper, we propose two possible avenues to translate mathematical problems into computable variables, therefore allowing us to easily assess the similarity of two problems.

C-H-S-M variables

In the first approach, we suppose that a problem can be uniquely defined by answering four questions, related to the statement and possible solutions of the problem. The answer to each question represents a variable. Therefore, the problem can be visualized as a point in a 4-dimensional state, the 4 variables each representing an axis. The questions are the following:

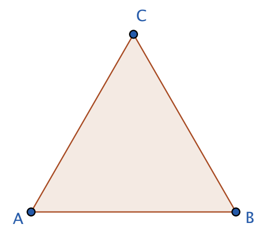
- What is the curriculum content (concepts, processes) that is involved in the solving of the problem and what mathematical competencies are involved (*content* variable)?
- How is a solution viewed in the process of solving (*heuristic* variable)?
- By what means (signs, tools) are ideas expressed, developed and communicated (*semiotic-instrumental* variable)?
- Under what conditions is the treatment of the problem controlled (*metamathematical* variable)?

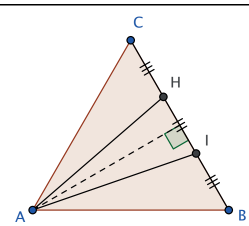
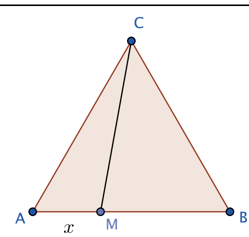
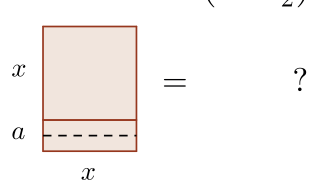
Indicatively, these variables can take the values:

- Content (C): triangle, height, base, length, measure, area, scale, isometric, description, construction, analysis, transformation, etc.
- Heuristic (H): breakdown, compare, equate, customize, limits, singularity, formulas, auxiliary, apprehension, exemplification, generalisation, iteration, etc.
- Semiotic-instrumental (S): interpret, represent, translate, model, accentuate, instrument, exploit, decode, communicate, de-contextualize, coordinate, move, etc.
- Metamathematics (M): identify, describe, conclude, hypothesise, figure, define, demonstrate, speculate, validate, assume, argue, induce, etc.

While sensitivities may vary between regions or from one author to another, the values of the C-H-S-M variables in terms of *content* and *heuristic* values are fairly standard in the didactical tradition. The difficulty of assigning these values to a problem is due mainly to an anticipation of possible solutions, which presupposes knowledge of the solving context such as the status of mathematical cognitive competencies or habits cultivated by didactic contracts. However, the assignment of the two other variables requires a bit more creativity and reflection on the variation of the statements. To illustrate the links between a statement and a possible value for *semiotic-instrumental* (Table 1a) and *metamathematical* (Table 1b) variables, we outline several archetypal attributions. However, the proposed examples do not exhaust the set of possible values, i.e. normally, the same statement may take several values of the same variable, and all statements may be given at least one value per variable.

Table 1a, b. Archetypal attributions for semiotic-instrumental (*bottom*) and metamathematical variables (*top*)

Metamathematical	
Statement	***Example of value type***
In the following problem: “Divide the equilateral triangle *ABC* into three equal triangles from two straight lines passing through point *C*.” What kind of result will we get in from equilateral triangle *ABC*?	**Conclusion** (identify, describe the conclusion)
In the following problem: “Divide the equilateral triangle *ABC* into three equal triangles from two straight lines passing through point *C*.” What do we know before dividing the equilateral triangle *ABC*?	**Hypothesis** (identify, describe the hypotheses)
What geometric object is missing from the figure for it to represent the following problem: “Divide the equilateral triangle *ABC* in three equal triangles from two straight lines passing through point *C*”? We are not asking for the construction, you only need to say what is or what objects are missing. 	**Figure** (identify, describe the figure)

Semiotic-instrumental	
Statement	***Example of value type***
In the situation opposite, what can be said about the areas of the triangles *ACH*, *AHI*, and *AIB*? 	**Interpret** (a drawing)
Draw three triangles of same area that together form an equilateral triangle.	**Represent** (a figure)
If *M* is a point on the base $[\overline{AB}]$ of an equilateral triangle *ABC*, where should *M* be located so that the area of triangle *MBC* is double that of triangle *AMC*? 	**Translate** (from the figural register to the analytical register)
Explain the equation on the right hand side using a geometric drawing: $x^2 + ax = \left(x + \frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2$ 	**Model** (geometrically)

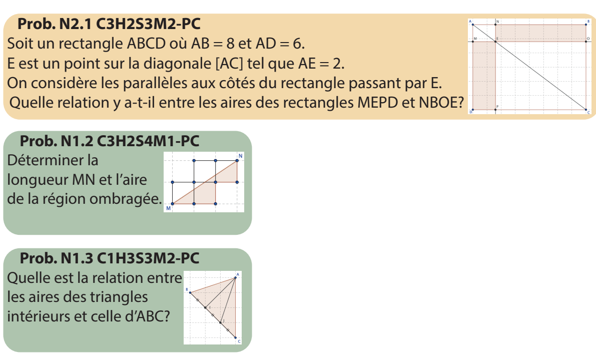
Articulation of the problems

To show how an approximation problem mechanism can be established, we use again the problems shown, N1.2 and N1.3 in Fig. 4, which are all conducted in a pencil-paper environment. We assume here that these problems are intended for 14/15-year olds in Quebec, and the variable *content* resumes the concepts and processes related to geometric figures and the spatial meaning of the educational programme of the Quebec schools (MÉLS, 2016). In this programme, the mathematical content is tiered, so the possible values for the previous problems are equal (below). For economy and to facilitate the indexing of problems, we characterize them using the following numeration:

1	Plane figures > Triangles, quadrilaterals, and convex regular polygons > Segments and remarkable lines: angle bisector, perpendicular bisector, median, altitude
2	Plane figures > Triangles, quadrilaterals, and convex regular polygons > Base, height
3	Plane figures > measurement > Length
4	Plane figures > measurement > area, lateral area, total area
5	Geometric transformations > Dilation of positive ratio
6	Finding unknown measurements > lengths > Segments resulting from an isometry or a similarity
7	Finding unknown measurements > lengths > Missing measurement in a segment of a plane figure
8	Finding unknown measurements > Areas > Area of polygons broken down into triangles and quadrilaterals
9	Analysis of situations using the properties of figures > Description and construction of objects
10	Analysis of situations using the properties of figures > Finding unknown measures > lengths > sides of a triangle (Pythagorean theorem)
11	Analysis of situations using the properties of figures > Finding unknown measures > Lengths > Segments resulting from an isometry, a similarity, a plane figure, or a solid
12	Analysis of situations using the properties of figures > Finding unknown measures > Areas > Figures resulting from a similarity

where the symbol “>” separates the hierarchical levels from the classification of the concepts or processes of the curriculum — for example in 3: “Plane figures” is the class, “measurement” is the subclass, “Length” is the sub-subclass defining the concept at stake.

Again, for economy, we limit the complexity of the values of other variables to those we have listed above, and we keep in reserve all possible hierarchies of these values. Under these conditions, a possible characterisation of the problems N2.1, N1.2 and N1.3 is shown in Fig. 5.



$C_{N2.1}$ = {1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12}
$H_{N2.1}$ = {breakdown, compare, amount, ancillary, apprehension}
$S_{N2.1}$ = {interpret, translate, communicate, coordinate}
$M_{N2.1}$ = {identify, describe, figure, define, conjecture}

$C_{N1.2}$ = {3, 4, 6, 7, 8, 10, 11, 12}
$H_{N1.2}$ = {break down, compare, equate, apprehension}
$S_{N1.2}$ = {interpret, communicate}
$M_{N1.2}$ = {identify, describe, validate}

$C_{N1.3}$ = {1, 2, 3, 4, 6, 7, 8, 9}
$H_{N1.3}$ = {break down, compare, equate}
$S_{N1.3}$ = {interpret, decode, communicate}
$M_{N1.3}$ = {identify, describe, figure, define, conjecture}

Figure 5: possible characterisation of the problems N2.1, N1.2 and N1.3

From Fig. 5, it can be immediately seen that problem N2.1 is essentially richer than N1.2 and N1.3, in the sense that its characterization involves more values for almost every variable. However, this feature does not make N2.1 very different from the others. Indeed, in the transition N2.1 → N1.2 and N1.3, it is noticeable that there are many common values, meaning that problems N1.2 and N1.3 are close to N2.1, as in Fig. 6.:

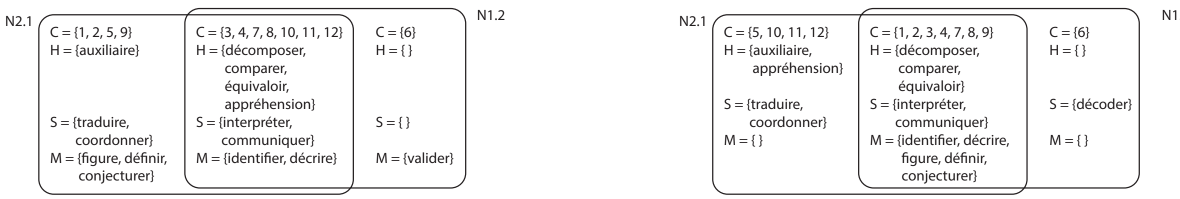


Figure 6: problems N1.2 and N1.3 are close to N2.1

In other words, even if the statements are independent, solving a problem similar to another risks influencing the solving based on common values. Let us look more closely at the relationship between connectivity and moments of impasse to form a decision-making process associated with it.

Impasse and decision

In our theoretical framework, we have associated a moment of impasse with an imbalance within the subject-milieu system, impasse bringing with it a potential opportunity for learning. In principle, overcoming an impasse creates a dynamic transition from one conception to another. Thus, with regard to the model cK¢, considering p_1 the root problem and $\mathbf{C}_1 = (P_1; R_1; L_1; \sum_1)$ the conception of subject-milieu system before solving p_1, when $\mathbf{C}_1$ solves p_1, then p_1 belongs to P_1. This means that learning does not occur. Nevertheless, when $\mathbf{C}_1$ is insufficient, then the solving of p_1 requires learning $\mathbf{C}_1 \rightarrow \mathbf{C}_2$, where $\mathbf{C}_2 = (P_2; R_2; L_2; \sum_2)$ and $p_1 \in P_2$. Following an impasse, the arrival of a connected problem p_2 should also belong to P_2. However, as learning is not yet achieved, p_2 is likely to throw off balance the conceptual consistency of $P_1 \cup \{p_2\}$, for P_1 that rightly excludes p_1. It follows that the set difference $p_2 - p_1$ represents a potential conception imbalance. For two similar problems, the decision process should be established on this difference and correspond, as far as possible, with the cause of the impasse. Therefore, variables of a connected space act as interpretation variables for impasses. In the previous example, the choice between N1.2 and N1.3 depends on where the impasse is situated. If we manage to identify that the student is blocked on H = {apprehension}, which is present in the description of N1.2 but not N1.3, we present the former to unblock the student.

In other terms, we can visualize each problem as a point in a 4-dimensional space, with the variables C-H-S-M as the four dimensions. The identification of the impasse in terms of these variables gives us a direction to look into. Finally, the choice of the next problem is simply to choose the closest problem in that direction.

HPDIC graphs

Another promising avenue to compare problems is to rely on HPDIC graphs (from French *Hypothèses*, *Propriétés*, *Définitions*, *résultats Intermédiaires* and *Conclusion*). These graphs introduced in the researches of Leduc (2016) and Tessier-Baillargeon (2016), display all the possible deductive paths from the hypothesis to the conclusion of the problem.

To demonstrate the utility of the HPDIC graphs, here is an example of a simple geometry problem:

> Given three lines, *AB, BC,* and *CD* all in the same plane, with *AB* perpendicular to *BC* and *BC* perpendicular to *CD,* what can be said about the lines *AB* and *CD*?

A HPDIC graph is composed from the *hypothesis* to the *conclusion*, through the *intermediate results*, each of which is justified by a mathematical *property* or *definition*, in an inferential process (figural and discursive; Richard, 2004a, b). First, we extract the hypothesis and the conclusion:

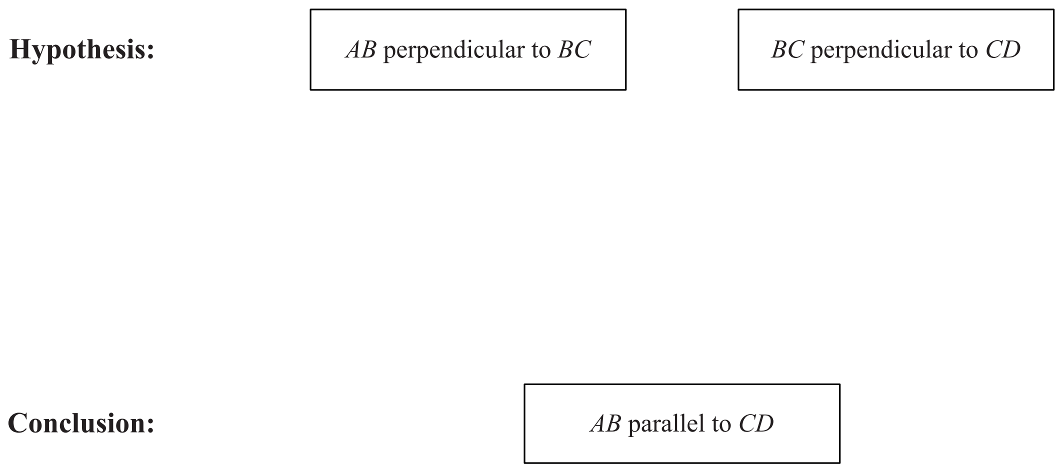


Figure 7. The start of an HPDIC graph

In this trivial problem, the answer is immediately given by the property: "If two lines are perpendicular to a third, they are parallel". By combining the two hypotheses with this property, we obtain the conclusion. This process is called an *inference*. The resulting graph is the following:

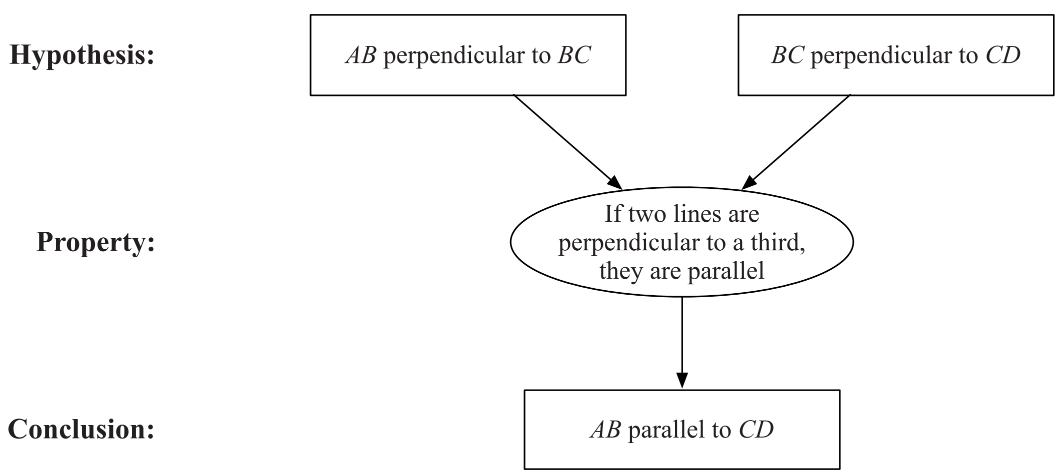


Figure 8. A simple example of an HPDIC graph

By combining such inferences on a more complex problem, a graph can be obtained that represents all the possible proofs for the problem. The meaning of *all* here is conditioned by the properties (which serve as justification for inferences) that are authorized at the level of the class and the habits of the didactical contracts as the tolerance in the inferential shortcuts, the effects of the counter-examples in the proof, etc., that is to say, the logic of the players who reason (Paillé & Mucchielli, 2016). For instance, there are without a doubt other ways to answer this very simple problem by using complex geometry or orthonormal coordinates, but it is not what we expect here.

On a more realistic problem, the interests of such a graph are more visible. Let us study the following problem (Richard & Fortuny, 2007):

Prove that a quadrilateral with three right angles is a rectangle.

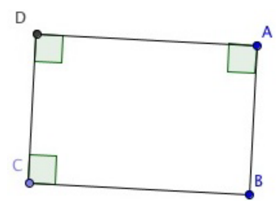


Figure 9. A quadrilateral with three right angles

There are various possibilities to solve this problem: with the sum of the angles in a (convex) quadrilateral, or by combining properties on lines such as the one we used in the previous example to prove that *AB* is perpendicular to *BC* for instance. The full HPDIC graph presents all the possible paths for this problem (Fig. 10).

Such a graph allows for interesting processes. In the *QEDX Tutor*, we are able to identify what path (i.e., what specific proof) the student is working on, and use this knowledge to provide a series of targeted advice to help or unblock. One of our objectives for the future is to exploit these graphs to find a connected problem as a way to help a blocked student. For instance, instead of giving the student some advice, and avoiding the connected problem being, in fact, a more directive sub-problem, if we discover a student well-engaged on a proof for the rectangle problem using properties on parallel / perpendicular lines, but is blocked on the step “‘*AD* perpendicular to *CD* and *AB* perpendicular to *AD*’ + ’two lines perpendicular to a third are parallel’ => ’*AB* and *CD* are parallel’”, then the student could be presented with a slight variation of the first problem (Richard, Oller, & Meavilla, 2016). This example is deliberately very simple and not really applicable in a real situation, but the idea of using the similarity of the graphs to find similar problems seems to us like a promising avenue. Besides, as opposed to the C-H-S-M method, we are already know, with a good deal of certainty, *where* in the graph the student is blocked, and what properties / results are needed to finish the proof.

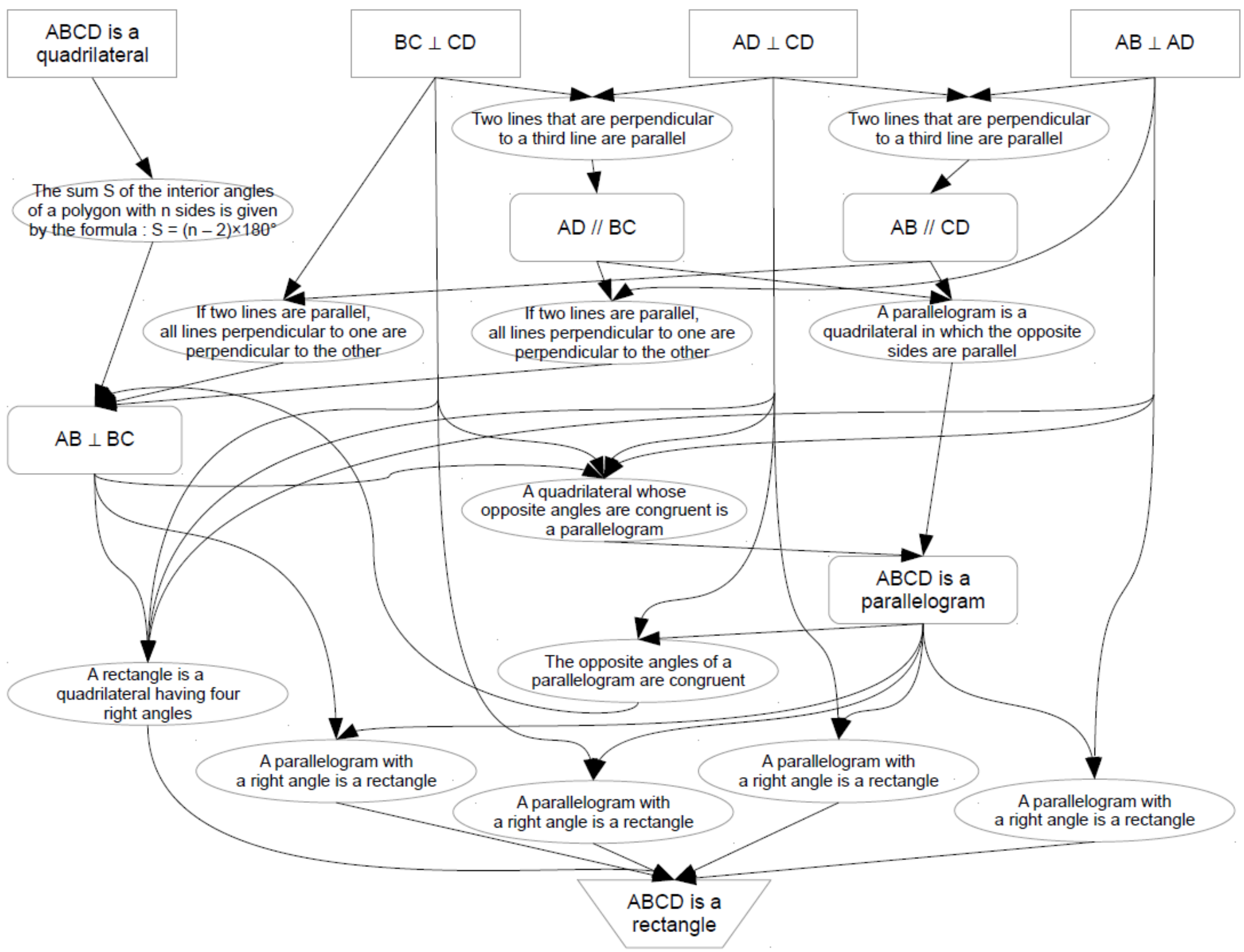


Figure 10: The HPDIC graph of the problem of the rectangle.

Another possible use is more global. After a student has solved a problem, all the information obtained during the solving process (properties used, time spent on each step, number of possible paths explored outside of the final path he presented…) is stored. This allows the proposing of problems that are adapted to the current knowledge of the student. For instance, if a student never uses angle properties for any problem, we can exploit this information to present a problem that exclusively uses angle properties to ensure that the student is acquainted with all the elements seen in class. Ideally, we would be able to utilise a detailed profile of the student. This could allow, first, the program to choose intelligently the problems given to students when they are stuck or have solved the previous one, and second, the teacher to know exactly what are the students' strengths and weaknesses.

This presents difficulties. We currently only have a small number of problems that have been translated into HPDIC graphs. This work has to be done manually in order to respect the customs of the didactical contracts, particularly those that involve working in a natural geometry paradigm (Kuzniak, 2006). In the rectangle problem, the graph is very simple, but for one of our five problems, that is not much more complicated than the rectangle problem, the graph contains hundreds of nodes and more than five million possible paths. This represents a considerable amount of processing time. One of our goals is to be able to generate automatically, or at least mostly automatically, the HPDIC graph of a problem through a better understanding of logic of the *deductive isles*[4] in class.

Working Conclusion: An Important Expected Result

The idea of responding to a student impasse by offering timely opportunities to solve problems is an effective solution to one of the major difficulties of teaching: To avoid giving answers at the same time as questions when the student is experiencing difficulties. In this sense, our project theoretically relieves a paradox of Brousseau (1997), the so-called *paradox of devolution*: Everything that the teacher does to produce in students the behavior that is expected tends to reduce the uncertainty of the student and thereby deprive the last of the conditions necessary for the understanding and learning of the concept in question. If the teacher says or means what is wanted from the student, then this can only be obtained as the execution of an order and not through the exercise of knowledge and judgment. The concept of devolution, as a didactic lever for the teacher and prerequisite for the development of student autonomy, gains strength and reinforces the idea that a connected problem belongs to the working space of a root problem and that the teacher seeks to relinquish that working space so that the student is left in charge of the solution process. The development of independent learning remains the major issue.

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[4] *Deductive isles* is our translation from the French *îlot déductif* that considers the network of mathematical properties and definitions accepted or actually used in a given class, which includes the implicit hypothesis and the inferential shortcuts tolerated in the didactic contract.

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CONCLUSION: PROSPECTS FOR DEVELOPMENTS AND RESEARCH IN SECONDARY GEOMETRY EDUCATION

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The chapters in this book provide a snapshot of where the international community is in regard to its scholarship on the teaching and learning of geometry in secondary schools. The contents of the book also reveal the absence of some themes that readers might have expected to encounter. In this final essay, we elaborate on such themes as a way of suggesting possible next steps in development and research on secondary geometry education.

Inasmuch as the chapters in the book address the practices of thinking, learning, and teaching geometry, they discuss those practices as mediated by a range of tools and signs. Among those tools and signs are traditional ones such as diagrams constructed on paper with straightedge, compass, ruler, protractor, etc. or more contemporary ones such as dynamic geometry software and Internet communication. All this has come along with increased focus on theories of cognition and learning that attend not only to mental activity but also to embodiment, discourse, social expectations, and instrumentation, with concomitant research emphases on visuospatial reasoning, on the use of gestures and diagrams, and on digital artifacts (Sinclair et al, 2016). Some of that progress in the field has been visible in this book and a lot more of it can be expected in the future.

Yet the range of available tools and signs to engage geometric thinking, learning, and teaching is larger than listed above. Traditional instruments for the construction of objects in the mesospace,[1] such as the tools of carpenters and mechanics, and signs of mesospace objects such as photographs or assembly blueprints, and the software used in engineering design, game design, robotics (Moore-Russo & Jones, 2012), and 3D modeling for animation (Jones & Moore-Russo, 2012), provide additional ways of thinking about practices that might make their way into our field. The literature on ethnomathematics has documented the use of geometry at work, for example by carpet layers (Masingila, 1994), carpenters (Milroy, 1991), or tool-and-die makers (Smith, 2005), while the use of historical artifacts, such as instruments to draw parabolas (e.g., Bartolini Bussi, 2010), by secondary school students also provides a context for geometrical exploration.

The popularization of design software and 3D printers, the emergence of engineering programs for high school (e.g., Project Lead the Way; www.pltw.org), the development of a Maker culture (e.g., at Maker Faires, the MIT Hobby Shop, and so on), and the increased emphasis on modeling in mathematics education suggest that some interesting new geometric work could be on our

[1] The mesospace is the space of objects of size commensurate with that of the human body (Berthelot and Salin, 1998). Likewise, Berthelot and Salin (1998) also talk of the macrospace and the microspace. The former can be defined as the space of objects whose size is one or more orders of magnitude larger than the human body, and the latter as the space of objects whose size can be handled by the human hands (see Laborde, 2000).

radar screen. Specifically, real world activities in which it might have seemed expensive or unsafe to engage students in the past may now be done in school or at home at low cost and increased safety. And they may afford opportunities to investigate geometric conceptions used to solve problems at the mesospace scale, to design activities in which those conceptions may be challenged and developed, and to investigate the work a teacher does managing students' work in such activities.

Herbst, Fujita, Halverscheid, and Weiss (2017) argue for the value of activities that engage microspace conceptions of *figure* (such as those addressed in traditional school geometry work) to model geometric work in the mesospace. The same software used to design immersive 3D games involving running and shooting could be used to design immersive 3D games where avatars build or move large objects: Imagine, for example, a virtual carpentry shop where students control avatars who cut wood pieces, then assemble them to make artifacts such as a dog kennel; or, imagine a virtual household moving game environment, where users are challenged to direct avatars to move variously sized and shaped household objects through more or less constrained spaces such as staircases. Tasks could be designed to initially elicit embodied conceptions when students merely control their avatars, then to make such conceptions more explicit, for example using what Brousseau (1997) calls situations of communication. Likewise motion sensors such as those used in the animation industry to capture human movement could be used in designing activities where students can bring their embodied cognition into the screen, for example to combine the use of the body in mesospace problem solving with alternative ways of visualizing such interactions, as in screen displays of such movements from different perspectives. For example, the improvement of bodily form in activities such as lifting weights, running, or yoga could be the apparent purpose in connecting students to computers using motion sensors, eliciting embodied geometric conceptions (e.g., of angle; see Fyhn, 2008) in their interaction with their bodily image on the screen (which might be seen from different perspectives). Again, making those conceptions explicit might require the design of communication tasks thus bringing the geometry of the mesospace into the space of classroom discussions.

The macrospace (or large scale space; see Battista, 2007) of buildings, landscapes, and seascapes also presents opportunities for various forms of geometric thinking aided by new tools and signs. New software and devices could help bring such thinking closer to what secondary school students can do. The chapter by Arai in this volume anticipates some of these possibilities. Devices such as drones, geographic information systems, and virtual reality glasses can be used to either visualize or experience the macrospace. Goodchild (2014), for example, illustrates the potential of spatial technologies for exploring caves, one of the most challenging navigational problems because cave systems are geometrically and topologically complex. Another application of spatial technologies is the MathCityMap project (https://mathcitymap.eu).

Finally, technologies like video recording have made it possible to represent and study transformations of space over time and explore the geometry of movement, as shown for example in Vi Hart's videos (http://vihart.com/). We wonder whether in the near future, perhaps at ICME-14 in China, the contributions to practice in our field might include more frequent uses of these technologies by teachers and their students to engage in geometric problem solving. The use of video could serve, for example to study the geometry of mechanical transformations such as those one makes when one uses exercising equipment (e.g., ellipticals, rowing machines, weight lifting) or to analyze form in dance or martial arts. The emergence of applications that can annotate and draw over images in video may facilitate such study of form and movement. Such possible practices would create new opportunities for scholars to ask questions of student reasoning and teacher decision-making about the nature of the tasks and student work (Richard, Oller & Meavilla, 2016).

All this takes us to an important research connection. When mathematics education researchers started using video recording in their studies of cognition and classrooms, it became possible to conduct studies of the microgenesis of inscriptions such as diagrams or equations (e.g., Chen & Herbst, 2013). Earlier research technologies, such as audio recording or collecting students' written work, might not have allowed researchers to account fully for how students were interacting with figures or in what way a figure had been constructed. Likewise, the development of dynamic geometry software has not only provided tools for students to develop or express their understanding; such software has also brought in, at least potentially, the capacity to record users' work through the keystrokes that might be stored in the scripts that could be made for a construction or more simply through the possibility to record a screen (for an example of using a dynamic geometry software to generate an *image map* of student work with the tool; see Leung & Lee, 2013).

The field of data science has been growing quickly as researchers and businesses have realized the value of click data and Internet footprints. We wonder whether the mathematics education research community can take advantage of related analytic possibilities. Motion sensor data, for example, can be used not only by the computer to render screen representations for the user to see on the screen, but also to analyze the mediating data structures collected to facilitate such visualizations. The tools of data science can be used to make sense of those data structures. Researchers studying embodied geometric cognition may be able to make use of those data structures to distinguish, for example, between different embodied conceptions of geometric ideas.

For every device that supports the creation of computer-mediated experiences with shape and space that has been listed above, there are data structures generated in computers where researchers can find geometric conceptions and their management by people over time. It seems that while our traditional data collection tools (the field note, the survey instrument, the video and audio record) are likely to continue to be useful, we also face the opportunity for exploiting new forms of data collection, new data structures, and new methods for data analysis. While some of that analysis may require us to collaborate with computer scientists or statisticians, there is clearly a role for mathematics educators in identifying the meanings of those data representations. May we see some of that in the years to come.

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