Seeing Algebraic Structure: The Rubik's Cube; Online Appendix 2

Amanda Milewski[1] and Daniel Frohardt[2]

Background Information:
This is the online appendix #2 for the following article:

Milewski, A., & Frohardt, D. (2020). Seeing Algebraic Structure: The Rubik's Cube, *Mathematics Teacher: Learning and Teaching PK-12 MTLT, 113*(5), 397-403. https://pubs.nctm.org/view/journals/mtlt/113/5/article-p397.xml

Online appendix #1 featuring animations of the moves from the article can be found here: http://hdl.handle.net/2027.42/143506

Invariances on the Rubik's Cube

For the novice, the number of possible configurations of the Rubik's Cube feels infinite. No doubt, if one considers the possibility of disassembling and reassembling the cube, there are 519 quintillion possible configurations. However, only 1/12 of those configurations of the cube can actually be reached using only "legal" moves (including any number of combinations of the basic moves described above).

This can be proved by considering three "invariants" of the Rubik's Cube—structural features of the cube that no combination of the basic moves can change. The invariants are (1) the parity of the permutation of the cubies (corners and edges), (2) the net twisting of the corner cubies, and (3) the net flips of edge cubies.

Parity of the positions of the corners and edges: **A key property of permutations is that they are either even or odd.** Every permutation is a product of 2-cycle permutations (*transpositions*). The product is not unique, but its *parity* (even or odd) is unique. For example,

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1 **Amanda Milewski,** amilewsk@umich.edu is an assistant research scientist at the University of Michigan in Ann Arbor. A former high school mathematics teacher and consultant, her primary research interest is on design and implementation of innovative forms of professional development for shaping teachers' instructional practices.
2 **Daniel Frohardt**, dan.frohardt@gmail.com, is currently a professor of mathematics at Wayne State University in Detroit, Michigan. A former department chair, his primary research interest is group theory, and he regularly gives talks on the Rubik's Cube and other topics related to group theory.

the corner permutation produced by the basic move R featured in Figure 1—(*URF*, *URB*, *DRB*, *DRF*)—is an odd permutation because it can be represented as a product of the three transpositions (*URF*, *URB*), (*DRB*, *DRF*), and (*URB*, *DRF*). The key property mentioned above implies that it cannot ever be represented as the product of an even number of transpositions. Note that basic move R also permutes the edge cubies in an odd permutation, namely (UR, RB, DR, RF). A consequence of this is that any combination of basic moves will have the same parity of the edge cubies as it does the corner cubies.

One of the ways that students may notice this invariant is that if all of the corner cubies are in their original locations and at least 10 of the 12 edge cubies are in the right locations, then all of the edge cubies must be in the right locations. Another way of saying this is it is impossible to find a combination of legal moves that leaves everything in place and simply puts only two edge pieces out of place.

Net Flips of Edge Cubies: Similarly, if all of the edge cubies are correctly located and at least 11 of the 12 have the right orientation then they must all have the right orientation. This can be restated by asserting that there will be no combination of legal moves that will allow for a single edge flip.

Extended Exploration of the 2nd invariant: A simple exploration of this invariant is outlined to enable one to explore the structure of the cube. Suppose we gave each of the 12 edge cubies a preferred orientation, for example by putting a white dot on one facelet of each as follows: (1) a white dot on each of the four edge facelets on the top; (2) similarly each of the four edge facelets on the bottom; and (3) the two middle-layer edge facelets on the front and the two on the back. Each basic movement will either leave white dots in exactly the same places they were originally or leave exactly 4 of the white dots in places where they were not.

So, for example, by simply rotating the top face using U, U^{-1}, or U^{2}, all of the white dots are in the exactly the same places they were before. However, if one uses the beckon middle layer movement instead, the white dots in *UF*, *UB*, *DF*, and *DB* will have flipped from the top and bottom facelets to appear on the front and back facelets. In general, any combination of basic movements will result in flipping an even number of edges.

Net Twisting of the Corners: Finally, if at least 7 of the 8 corner cubies are correctly located and have the right orientation, then the 8th corner cubie must have the right orientation as well as the right location. Analogously as the previous two invariants, this means there exists no combination of legal moves that will allow for a single corner twist.

Extended Exploration of the 3rd invariant: A similar exploration of the corners is possible by placing white dots on the top facelets of the corners in the top layer and the bottom facelets of the corners in the bottom layers. Again, every basic movement will either leave the white dots in exactly the same places (as with the rotation of the top face again) or twist the corners clockwise or counterclockwise in such a way that their combined twist cancels out. For example, take a rotation of the right face clockwise. The result will be that *URF* has one counterclockwise turn, *URB* has one clockwise turn, *DRB* has one counterclockwise turn, and DRF has one clockwise turn. In general, any combination of basic movements will result in twisting of an even number of corners in opposite directions, a set of three corners in the same direction, or some combination of the two.