Analysis of Mathematics Curriculum Materials to Ascertain the Potential for Students to Develop Agency and Autonomy

by

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Dedication

To all high school students who can be empowered through their learning of mathematics

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Abstract

Mathematics textbooks are commonly used around the world to teach mathematics during lessons. They provide mathematical tasks and theory to support student learning. Given the centrality of textbooks as a vehicle for mathematics teaching and learning, prior research has examined ways in which texts support students' learning of a wide variety of mathematics knowledge and skills. Less examined, however, has been the potential role of textbooks in supporting development of agency and autonomy relation to mathematics learning. This dissertation examined the treatment of functions in two textbook series to identify ways that each positions students to develop distinct forms of agency and autonomy while solving mathematical tasks.

To study how the two textbook series position students to develop agency and autonomy, I investigated and systematically categorized the types of mathematical tasks and the linguistic structures found in the texts. The mathematical task features were examined from a cognitive perspective drawing on analysis of tasks with different levels of cognitive demand. The linguistic analysis drew on Systemic Functional Linguistics. Data consisted of selected lessons on chapters on the topic of functions.

The findings show that for the topic of functions, both textbook series provide students with opportunities to develop agency and autonomy that align with the instructional orientations each text supports. One textbook series supports a so-called reform-oriented approach to teaching and learning whereas the other supports a traditional-oriented approach. One textbook series also positioned students to develop greater varieties of agency and autonomy than the

other. For example, for the topic of functions, this textbook series provides students with a broader range of tasks than the other textbook series. These include simpler tasks that develop disciplinary agency and more complex and challenging tasks that develop conceptual agency and intellectual autonomy.

The findings contribute to an understanding of different ways textbook series with particular orientations make opportunities available for students to develop forms of agency and autonomy during classroom learning. The findings also contribute to methodology for analyzing textbooks based on the mathematical tasks and other supporting texts for a lesson.

Chapter 1 : Introduction

This dissertation concerns the study of mathematics textbooks to learn ways that mathematical tasks can empower students to develop independent thinking and learning skills. Over the past few decades, mathematics textbook research and use have both evolved from mainly traditional textbook formats to include those of the reform era. Traditional format textbooks tend to adopt a structural approach to learning mathematics. In them, students learn mathematics by solving tasks not necessarily connected to real-life contexts. Reform format textbooks on the other hand promote a functional approach, wherein the learning of mathematics is connected to situations students may encounter in real life (Cai & Ni, 2011). This is meant to develop students' mathematical thinking skills during classroom work in order to use those skills in their lives outside of the classroom (Wijaya et al., 2015). This dissertation is about the study of two reform textbooks of different orientations.

In recent times, there has been increased research on the opportunities reform textbooks offer teaching and learning. Although some of these studies have examined textbook use from the teacher's point of view (e.g. Ball & Cohen, 1996), other researchers have studied mathematics textbooks from the point of view of what opportunities they offer students for learning mathematics. Textbook opportunities come in many forms. Some of these opportunities include encouraging students' reasoning ability (Stylianides, 2009) and learning through word problems (Xin, 2007).

Research into the use of reform and non-reform textbooks for teaching and learning has shown some support for students' development of problem solving skills when learning mathematics with reform curricula (Ni & Cai, 2011). In their longitudinal study of mathematics learning in China and in the US, Ni and Cai (2011) learned that in both countries, students who studied mathematics with reform curricula improved on their ability to solve challenging mathematical tasks while still retaining basic mathematical skills. These challenging mathematical tasks were such that students could not thoughtlessly apply standard algorithms or procedures to solve them. Students had to think in order to solve the tasks. This is an important result because recent research suggests that students need to develop independent and critical mathematical thinking skills. This can be achieved if they are given the chance to "learn by doing" while working on worthwhile mathematical tasks that engage their critical faculties and that allow students room to express their own ideas through methods and solutions. Lester & Cai (2016) in their synthesis of decades of research on problem solving assert that students need more opportunities to rely more on themselves as they work on challenging mathematical tasks. By having opportunities to work on challenging mathematical tasks, students can become empowered to learn mathematical thinking and self-management skills. These skills can serve them well in class and in their daily lives.

I distinguish 'doing mathematics', a process students engage in as they work on challenging mathematical tasks from 'solving exercises', a process in which students work on procedural or routine mathematical tasks. Although mathematical tasks of the exercise variety can help students learn procedures, such tasks may not give students many opportunities to exercise more of their own cognitive abilities. These include deciding on methods or making connections between concepts and procedures when producing solutions. Working on challenging mathematical tasks can also give students opportunities to draw on their own abilities when they need to for instance create mathematical conjectures, justify them, check for correctness and attempt to extend solutions to mathematical tasks (Silver, 2013; Mason et al., 2010; Schoenfeld, 1985). Investigative and real-life mathematical tasks presented in the curriculum can challenge students to learn to think mathematically as they encounter similar situations in real life Martin (2009). This is a major goal of mathematics learning.

The Research Problem

There currently appears not to be enough research done that foregrounds ways to empower students to exercise agency and autonomy while learning mathematics so that they can eventually develop independent and critical mathematical thinking skills to use outside of the classroom. If mathematics educators seek to empower middle and high school mathematics students to become independent, persistent and confident thinkers and problem solvers in school and in daily life, they must discover opportunities to promote these qualities in students during mathematics learning.

Consider two scenarios: in the first one, students in a class work hard at solving mathematical tasks. The tasks are from a textbook. The students are working at their own paces. The tasks are varied, challenging, interesting and require students to engage with and to connect different mathematical objects such as formulas, tables, graphs. Students can also resort to other resources outside the textbook. These resources include manipulatives, computer resources and other physical objects they can use to model and understand mathematics they are learning. The classroom is alive, buzzing with activity, sometimes almost seeming chaotic. Students are allowed to work with one another. At one table, three students are working together. At another, four students have teamed up. At yet a third table, only one student sits, deeply engaged in thought, and occasionally expressing those thoughts in diagrams and writing. The students seem to be enjoying themselves and engaged in the tasks. Even though tasks are challenging, students persist because the learning orientation positions them to explore mathematics, to make connections and to be able to express those mathematical connects well with peers and with the classroom teacher. The classroom teacher moves around from one group of students to another, sometimes asking questions to extend students' thinking, sometimes giving hints to guide students along, and sometimes listening to students' theories and explanations.

In the second scenario, students are also using textbooks. These students however sit by themselves. On occasion, two form a pair. They working individually and they do not seem to be very excited about the tasks they are working on because they are used to this routine. They are all working at the same pace, because the solutions for the tasks are periodically displayed on the whiteboard, after which the teacher may work through one or a few tasks that many students struggled on. Students care about finishing each batch of tasks otherwise they will fall behind when the tasks are assessed as a whole class. Students also care about how many tasks they answer correctly. When they continuously underperform compared with their peers, they feel less able at mathematics.

These two scenarios describe familiar situations for mathematics learning environments, especially at the middle and high school. The first scenario depicts a reform oriented lesson, where students are learning mathematics primarily by engaging with each other and with the tasks. Through this collaborative learning orientation and by working on tasks that engage students' thinking, they can develop mathematics problem-solving and thinking skills. The second scenario depicts a more traditional classroom setup where students work primarily by themselves. In the second setup, students are more focused on getting to the end of the task correctly rather than engaging with it to learn mathematics. Collaborative and individual learning setups each have affordances. It is certainly possible to have students working by themselves on challenging tasks and it is also possible to have students working in groups on routine tasks. However the situation where students collaborate with one another while working on challenging tasks offers opportunities for them to learn by doing and by learning from one another. Recent thinking within mathematics education is that problem solving should be the basis for what students learn in reform classrooms (Hiebert et al., 1996). Through the functional approach to understanding mathematics, where students learn the subject by participating within a community of learners, they can be empowered as learners. Essential for this empowerment is for teachers to know how to utilize available teaching resources. These resources very often include the mathematical tasks students work on during lessons. Research has shown that when students learn through problem solving involving challenging mathematical tasks in a learning context fostering their empowerment, they can develop a positive and persevering attitude toward mathematics (Boaler, 1998). Their understanding of mathematics also becomes more complex and deeper (Higgins, 1997). In the absence of conditions allowing students to assume and develop greater control of their learning while working on challenging tasks, students risk experiencing mathematics as procedure driven, static, fragmented, and even disempowering (Boaler & Selling, 2017).

Given that textbooks serve as a key repository for mathematical tasks, that mathematics education researchers have identified the importance of having students develop independent thinking skills while solving mathematical tasks and that "there is a growing body of research that approaches textbook analysis through the opportunities afforded by task content" (Watson &

Thompson, 2015, p. 144), one might expect to find numerous studies by mathematics education researchers foregrounding agency and autonomy as learning opportunities available to students. Unfortunately this appears not to be the case. Since there are studies on other opportunities textbooks provide (Wijaya et al., 2015), I observed that there seem to be only few studies (e.g. Herbel-Eisenmann, 2007; Clarke & Mesiti, 2013) that even address the need to study opportunities for students to develop agency as they learn mathematics by solving tasks. Therefore considering the importance of mathematics problem solving in the learning of mathematics (Schoenfeld, 1992), the amount of research already focused on problem solving over the past few decades (Silver et al., 2005), and research work aimed at revealing ways that textbooks provide learning opportunities to students, there appears to be a need for more studies that shed more light on the said opportunities for agency and autonomy. These needed studies are important because they will strengthen the knowledge that researchers and practitioners have of ways they can empower students in their learning of mathematics.

Study design

The goal of my dissertation is to meet this need by presenting a study of how features of texts and mathematics tasks in textbooks that promote problem-based learning (hereafter, PBL) position students to develop agency and autonomy when studying mathematics. The idea behind PBL is that students to learn mathematics primarily through solving mathematical tasks. In other words, when solving mathematical tasks, students learn about mathematics theory and also apply that theory in context, thereby learning in the process. To accomplish the dissertation goal, I study two PBL textbooks with different orientations. One textbook, College Preparatory Mathematics (CPM) is designed based on complex instruction (Cohen et al., 1999). In complex instruction, students assume different roles and frequently collaborate as they learn mathematics.

This learning orientation is student oriented in the sense that the teacher acts as a guide rather than the leader. The outcome CPM curriculum authors aim for is for students to be able to use their mathematics knowledge in real life, beyond the duration of a semester or even a year. The CPM curriculum authors stress that it is through active and collaborative participation in doing mathematics that students acquire long term mathematics knowledge. The authors of the CPM curriculum also appear to believe that their PBL curriculum can develop independent thinking and problem solving abilities, "students who are educated using a problem-based learning style are developing another useful skill – the attitude that they do not need someone else to tell them how to tackle a new problem" (CPM, 2018b, p. 2). In their mission statement, they state:

"CPM envisions a world where mathematics is viewed as intriguing and useful, and is appreciated by all; where powerful mathematical thinking is an essential, universal, and desirable trait; and where people are empowered by mathematical problem-solving and reasoning to solve the world's problems." (CPM, 2018)

This mission statement reveals that through its curriculum, CPM aims to develop students' independent mathematical problem solving and thinking skills so students can apply these skills in their lives. One would therefore expect analysis of CPM curriculum materials to reveal opportunities for students to develop these skills during mathematics learning, in order to be able to achieve the outcome of learning envisioned by CPM. Analyzing the features of CPM lesson text and tasks would reveal these opportunities.

The other textbook series is Pearson Integrated High School Mathematics (hereafter, PI). Although PI also supports PBL, PI lessons resemble the traditional, teacher-led orientation. In PI lessons, technology and the teacher play a more leading role in student learning than in CPM lessons. The outcome PI textbook authors aim for is for students to be able to use their mathematics knowledge in their daily lives. In their note to students found at the beginning of each PI textbook, the textbook authors state that “The problem-solving and reasoning habits and problem solving skills you develop in this program will serve you in all your studies and in your daily life.” (Pearson, 2018). From this note, we can infer that PI also aims to empower students with independent mathematical problem solving and thinking skills. PI however has a different approach to leading students to their envisioned outcome. While CPM lessons aim to have students rely on one another during their learning, PI lessons are more individually oriented, designed to prepare each student to succeed in “next-generation assessments” (Pearson, 2018). There is a strong emphasis in PI textbooks on assessment.

To summarize, even though both CPM and PI are instances of PBL curricula, and even though both CPM and PI aim to empower students to use mathematics in and out of class, the two textbook series take different approaches to achieving this outcome. CPM positions students to collaborate with one another and to sometimes struggle to figure out the mathematics being learned. PI differentiates its lesson texts by guiding students through basic material that meets the needs of all students, and offering a range of more challenging tasks at the end of the lesson or as homework for students who can access more difficult tasks.

Studying both textbook series can reveal similarities and differences between them. By analyzing samples of the variety of tasks in each textbook series from cognitive and linguistic perspectives, I aim to make a case for how opportunities they offer to empower student learning can be revealed. The cognitive perspective reveals opportunities for students to develop independent problem solving and thinking skills by analyzing variety among tasks. The idea is that by solving different task types, especially the more challenging ones, students can practice and develop the problem solving and thinking skills to be applied outside the classroom. Among the challenging tasks are those that include investigations of mathematical patterns and those that include real-life contexts that simulate situations that students may encounter in real life. The linguistic perspective examines language choices made in the texts to understand how the textbook authors give information to or demand action from students. Studying how different clauses give information to students can reveal whether students are compelled to directly incorporate the information in their workflow or whether they are at liberty to use the information in creative and individual ways for their work. Likewise, studying how different language choices demand action of students can reveal when students are set up to carry out routine procedures and when they are set up to act in ways that bring out their independent thinking and problem solving skills. The combined perspective reveals how the function of language with respect to the task features can reveal the ways students can be set up to be more independent when learning. The task features refer for instance to what serves as an appropriate method and what would be an acceptable solution for the task. Studying how clauses are used in the statement of the task or in the lesson text in connection with a task variable can reveal ways that students can be set up to experience more opportunities for developing their thinking skills. For example, if a particular clause type grants students opportunities to be creative in how they generate solutions, and if this clause type is used sparingly in the text of a particular textbook series, analysis could reveal that low use of this particular clause type impacts students' opportunities for developing thinking skills.

Research Questions

1) To what extent does a cognitive perspective analysis of mathematical tasks in two PBL textbook series reveal differences and similarities in opportunities for students to develop different forms of agency and autonomy with respect to topics on functions?

2) To what extent does a linguistic perspective analysis of lesson texts in two PBL textbook series reveal differences and similarities in opportunities for students to develop different forms of agency and autonomy with respect to topics on functions?

3) In what ways does the interaction of cognitive and linguistic perspectives in the analysis of mathematical tasks in two PBL textbook series reveal differences and similarities in opportunities for students to develop different forms of agency and autonomy with respect to topics on functions?

Contributions of the study

This dissertation brings attention to questions of agency and autonomy in the context of solving mathematical tasks in textbooks. Problem solving and problem posing have been key areas of research in mathematics education over the last few decades (Lester, 2013; Silver et al., 2005). Problem solving is also of foundational importance to mathematics teaching and learning, as it is emphasized in Principles and Standards for School Mathematics (NCTM, 2000) from prekindergarten through to the final years of high school. During the earlier part of research on problem solving, the focus was more on strategies, methods and heuristics that students can learn in order to become more adept at solving and posing problems. The goal of problem solving has always been to improve students’ mathematical understanding and to equip them with thinking skills that they can use in their everyday lives. With the plethora of research on problem solving and problem posing and its emphasis in the mathematics curriculum, a study that foregrounds the importance of researching questions around opportunities for students to exercise agency and autonomy as they solve mathematical tasks in textbooks could bring renewed attention to research interests that have already appeared in the mathematics education literature over the years. To be fair, earlier research on problem solving has touched upon questions of agency and autonomy. For instance, studying aspects of mathematical problem solving such as metacognition (Schoenfeld, 1992) has implications for student autonomy. The study carried out in this dissertation adds this new perspective to the extant literature on problem solving and problem posing, by placing particular focus on opportunities for students to exercise agency and autonomy that are built into textbook tasks.

Organization of the dissertation

This dissertation has five chapters. Chapter 2 reviews the literature pertaining to textbooks and curriculum materials, to opportunities to learn, to agency and autonomy, problem-solving and frameworks in the literature on analyzing textbooks for opportunities to learn. I also state my research questions in chapter 2. Chapter 3 delves into the methods I adopted for my study, including data selection, conceptual framing, and analytic methodology. In chapter 3 I describe in detail the perspectives I have adopted for the study. Chapter 4 lays out findings from the study, while chapter 5 enters a discussion around the findings, including implications, limitations, future directions and the conclusion.

Chapter 2 : Literature Review

Introduction

It is well established that curriculum materials in the form of mathematics textbooks are a main staple for classroom instruction (Valverde et al., 2002). They provide mathematical tasks which have been and remain a central feature of classroom mathematics learning (Kilpatrick et al., 2001). This is true across many countries (Fan et al., 2013). Serving as a source of mathematical tasks, textbooks have a long association with problem solving, and problem solving is intrinsically linked with mathematics learning, discovery, and research. Contemporary efforts have influenced the thinking around what kinds of mathematical tasks provide opportunities for students to develop the kinds of mathematical thinking skills that are deemed to empower them in life. There is a common understanding among mathematics educators that different types of mathematical tasks in textbooks can give students different opportunities to be empowered in their learning (Wijaya at el., 2015). However, a question that is not well investigated is, do textbooks empower students with opportunities for them to rely more on their own agency and autonomy when solving mathematical tasks, and do we have suitable means for analyzing how well textbooks empower students?

In order to answer these questions, I shall review extant literature for conceptions of agency and autonomy that influence the study in this dissertation. I shall then discuss a brief account of problem solving as an activity of foundational importance in mathematics learning. This account is important because it is through problem solving activities that several opportunities for developing agency and autonomy become available to students. I shall next review literature on how what I will explain as traditional and more recent efforts to reform mathematics teaching and learning relate to problem based learning, to problem solving and to the two textbooks I study in this dissertation. That will set the stage to discuss different kinds of mathematical tasks, the cognitive and linguistic frameworks that have been used to study textbooks along with how and why these frameworks are important to or are drawn upon in this dissertation.

Conceptions of agency and autonomy in mathematics education research

In this section, I review literature pertaining to the conceptions of agency and autonomy I adopt for this dissertation. Agency and autonomy are important to mathematics learning because they are means through which students can be empowered while learning mathematics to gain the independent thinking and problem solving skills they can apply outside of learning situations. If students can learn to develop and rely on their own thinking and problem solving skills while engaging with the mathematics, with each other, with the classroom teacher, they can be empowered to use the same skills in real life.

Notions of agency: Notions of agency that have become more commonly referred to in mathematics education research (c.f. Boaler & Greeno, 2000; Boaler & Selling, 2017; Cobb, Gresalfi, & Hodge, 2009) derive from the work of Pickering, whose field is the sociology of scientific knowledge. In Pickering's view, every physical object in the world has the potential to express agency, in its interactions with other object. Pickering outlines two forms of agency: human and material. Human agency involves the capability to express choice and to act in order to affect other objects. Material agency involves the forces of nature. These are the tangible forces that can be expressed for instance through wind, heat, and cold and that can act upon and influence other objects. It is in his notion of human agency that we find connections with expressing choices in mathematics. According to Pickering (1995), conceptual systems, algebra being an example, “hang together with specific disciplined patterns of human agency, particular routinized ways of connecting marks and symbols with one another” (p.115). It is from this idea that Pickering’s notion of disciplinary agency emerges. For disciplinary agency, Pickering explains that such conceptual systems entail established conceptual practices that function based on the rules of the discipline not on personal predilections of individuals using the system. Those who use the system engage in disciplinary agency, as the conceptual practices for these conceptual systems are standardized.

Cobb et al. (2009) take up Pickering’s ideas and extend them to establish two poles of agency that are applicable to mathematics education research. On one hand, they outlined the idea of conceptual agency, wherein students are at liberty to think about and assert relationships between concepts and to develop approaches to theory and to solving problems. They define conceptual agency as “choosing methods and developing meanings and relations between concepts and principles” (p.45). Conceptual agency is an important idea because it implies a scenario where students are actively engaging concepts, principles and methods to develop understanding. If students are to achieve deep learning, they have to go beyond merely accepting information. They have to understand why methods work, what concepts methods are built upon and how to use methods and concepts when in situations they may not have previously encountered. In order to achieve this, the learner, the agent, must create connections. They must assert links between concepts. They must actively develop meanings. In the case of mathematics learners working on tasks, among other things, this means being able to think mathematically about patterns, make conjectures about them, search for counterexamples and construct proofs.

On the other hand, Cobb et al. (2009) also outline disciplinary agency. Their definition for disciplinary agency is "using established solutions methods" (p. 45). This conceptualization of disciplinary agency is consistent with Pickering's idea of a passive agent executing established routines and it is very much linked with the procedural. This conceptualization of disciplinary agency is useful as a counter pole to conceptual agency. By having both poles, it becomes possible to discuss the extent of agentic experiences available to students during lessons. In mathematics learning involving tasks, disciplinary agency involves actions such as being adept at using established formulas and methods for solving tasks of particular topics. Cobb et al. (2009) suggest that for effective mathematics learning to occur, students must experience some conceptual agency. Without the experience of conceptual agency, they argue, students will lack the mathematical understandings underlying the nature and purpose of the disciplinary tools at their disposal. In their study, Cobb et al. (2009) documented middle school students' obligations in the classroom as learners and doers of mathematics, to determine what kinds and to what extent they could exercise agency during episodes of learning. In this dissertation, I draw on Cobb et al. (2009)'s definitions of conceptual and disciplinary agency in my analysis regarding the cognitive, the linguistic and the combined perspectives for analysis mentioned in the introduction.

Notions of Autonomy: Research conducted by Cobb and colleagues (Cobb et al., 1991; Yackel & Cobb, 1996) provides one foundation upon which to draw on notions of autonomy for studying classroom teaching and learning settings. The foundation they provided is based on elements from constructivist and sociocultural aspects of mathematics teaching and learning in the classroom. The constructivist branch of their view accounts for each individual student's mathematical thinking and sense-making processes that occurs as they engage with mathematical content and work to understand and apply mathematics. Out of the constructivist branch emerged the notion of intellectual autonomy, that is, Piaget's idea that individual learners develop intellectually to be able to act based on their own thinking and convictions (Kamii, 1984). Yackel and Cobb (1996) define intellectual autonomy in terms of how students participate in classroom practices as members of a learning community. They assert that "students who are intellectually autonomous in mathematics are aware of, and draw on, their own intellectual capabilities when making mathematical decisions and judgements as they participate in these practices." This definition is useful because it emphasizes giving students a chance to practice decision making and judgment during lessons. These are thinking skills they will need to apply in life outside of lessons. The practices to which this definition refers are those of inquiry mathematics. This is the classroom orientation they studied. In inquiry mathematics, "students have frequent opportunity to discuss, critique, explain, and when necessary, justify their interpretations and solutions" (Cobb et al., 1991, p.6). This description of inquiry learning that Cobb et al. (1991) agree with Boaler (2002) on how students communicate in mathematics classrooms that support reform-oriented learning. Reform and traditional approaches to teaching and learning mathematics are discussed further below.

In order to clarify the *extent* to which students can express intellectual autonomy, I also draw on the ideas of Littlewood (1999) who introduced the idea of autonomy as being proactive or reactive, wherein proactive autonomy connotes an expansive state or a greater degree of autonomy whereas reactive connotes a more restrictive state of autonomy. As concrete examples, learners may experience proactive autonomy when working on a mathematical task if they have the conceptual agency to set all conditions for solving the task. On the other hand, if some constraints are given to learners before they work on the task, the extent to which students can exercise conceptual agency will be lessened and consequently they will experience reactive autonomy as they may then work within those constraints. Reactive autonomy constitutes a decrease in students' opportunity to experience intellectual autonomy, due to a decrease in conceptual agency. The more opportunity for reactive autonomy, the less intellectual autonomy there is for students to experience. When all that students can experience is disciplinary agency, the extent of reactive autonomy extinguishes intellectual autonomy altogether. Including proactive and reactive qualifications for intellectual autonomy therefore allows for a spectrum of experience available to students, depending on whether or not they can experience conceptual or disciplinary agency.

Cobb and colleagues also recognized the importance of the social dimension of classroom interactions and therefore included it in their notion of autonomy. Not only is intellectual autonomy important. Social autonomy too plays an important role in inquiry mathematics. In social autonomy, groups of students assume responsibility for the processes and outcomes of their work on mathematical tasks. Group refers to more than one student. In this dissertation, I conceptualize social autonomy as involving groups of students working together, largely independent of the teacher and toward a common goal of learning mathematics for understanding. At the other pole, opposite to social autonomy is individual autonomy, which consists of individual students working alone. In individual autonomy, students work independently of the teacher and of classroom peers. At any given time during lessons, students can switch from individual autonomy to social autonomy. They can also engage in whole class sessions led by the teacher. For example in a classroom where students work primarily by themselves, individual autonomy is at work. In instances where students are working independently in small groups, social autonomy is in action. There could be exceptions, such as the instance of pair work described in the second scenario I described in the “problem” section of the introduction, where social autonomy manifests in a classroom in which individual autonomy is more the norm. Conversely, in a classroom where students work primarily with others, there will be instances where students work individually, as was depicted in the first scenario in the introduction.

Students can experience the different forms of agency and autonomy individually or socially. Individuals working alone can experience conceptual agency and proactive or reactive autonomy. They can also experience disciplinary agency. For the collaborative experience of social autonomy, Cobb et al. (1991) explain that with the appropriate social norms in place, students collaborating in small groups can persevere together through challenging tasks. Independent of the teacher, they can support one another’s attempts at making sense of their study of mathematics. They can resolve conflicts involving different ideas to arrive at a consensus solution as a group.

To summarize, a useful overall definition of agency encompassing both conceptual and disciplinary aspects which I adopt in this dissertation is agency refers to opportunities for students to exercise choice with concepts and procedures available to them in order to solve a task. In the same vein, autonomy refers to opportunities for students to take control of the processes and outcomes involved with solving a mathematical task. The type of agency students can experience (conceptual or disciplinary) as a result of the type of task students work on in turn influences how much intellectual autonomy (proactive or reactive) they can experience in the process of solving the task.

Problem solving in mathematics teaching and learning

For more than four decades, the mathematics education research community has extensively researched problem solving in mathematics teaching and learning (Lester, 2013; Silver et al., 2005). Earlier decades of problem solving research drew on theories from cognitive science (Silver, 1987) and in particular on the view that the problem solver was essentially an information processor (Mayer, 1996). Mathematics education researchers in these earlier times examined mathematical tasks for their complexity, for how well students fared at solving them, for what methods students used, and for how well students regulated their problem-solving activity. These dimensions, according to Schoenfeld (1985) have often factored in the problem solver's prior knowledge, ability to draw on heuristics and the problem solver's ability to self-regulate. Mathematics education researchers were also interested in figuring out how teaching and learning through problem solving leads students to develop their mathematical understanding (Lester, 2013).

It is clear from this brief description of prior research that problem solving is a foundational aspect of mathematics teaching and learning. The Principles and Standards for School Mathematics (NCTM, 2000) emphasizes problem solving at each level of schooling from prekindergarten through to the final years of high school. With this much research on problem-solving, where do agency and autonomy fit into the scheme of problem solving research? From reviewing research on problem solving, it appeared that much of the discussion has scarcely foregrounded the notions of agency and autonomy in relation to working on challenging mathematical tasks. Yet with a careful look at some earlier attempts at providing mathematics instruction in history of problem solving, I argue that there are elements of agency and autonomy that surface in these studies, even though these may not have been called so.

For instance, the mathematician George Pólya who galvanized the importance of problem solving in mathematics education in his promotion of heuristics gave a four step approach to guide students when solving challenging problems for which there was no immediate method (Pólya, 1971). In Pólya's approach, students first need to understand the problem they are attempting to solve. To understand the problem, they must ask themselves a number of general questions that can serve as a plan to scaffold students toward a solution. The questions students need to ask themselves concern what the unknowns are, what the data or the givens are and what the conditions are for those data. Once students understand what the problem requires, they can then put together a plan that serves as a method for the problem. I argue that this general approach is a way to assist students to draw on their own conceptual agency and intellectual autonomy. Pólya's approach inspired several other mathematicians and mathematics educators to create even more specific methods for helping students work through difficult problems by themselves (c.f. Mason et al., 1982; Schoenfeld, 1985)

Likewise, the research done on metacognitive behaviors that aid in problem solving can be reframed into a discussion foregrounding opportunities for students to develop self-regulation skills while solving problems working on mathematical tasks (Schoenfeld, 1987; 1992). In relation to problem solving, Schoenfeld explains metacognition in terms of a student's ability to monitor and control their thinking and to make "executive decisions" optimizing the best use of resources available for the task. At least one mathematics education researcher (Crosswhite, 1987) made an explicit link between metacognition and autonomy. In this study, although I do not study metacognition, I point to instances where students are given opportunities to make decisions regarding the best use of resources.

There is also at least one mathematics education researcher (Pea, 1987) who has explicitly associated the idea of agency with some of ways of learning mathematics that later became known as “reform mathematics” (see the section that follows). Pea (1987) promoted the idea of learning mathematics by solving problems in functional and social environments where students can collaborate with one another, engage in dialogue, and use tools that aid their learning. In his view, such a learning environment promotes students’ agency.

So there have been some linkages, even if a few, with agency and autonomy in problem solving related to the teaching and learning of mathematics. In all of this history, mathematics textbooks have been an intimate part of the evolution of problem solving especially because problem solving of one form or another has been an integral part of mathematics teaching and learning for as long as textbooks have existed, going perhaps as far back as the time of Euclid, the Greek mathematician (Fan et al., 2013). The next section addresses two broad orientations of textbooks used for mathematics instruction in the US today, and how each of these orientations is related to problem solving through PBL.

“Traditional” versus “reform” orientations and textbooks

In this section, I present a brief overview of developments in curriculum ideas in the US that align with curriculum orientations in CPM and PI. The discussion is important because it sets the stage for understanding why PBL textbooks promote a functional approach to mathematics learning and why they focus more on student learning.

Two broad categories span modern textbooks. There are those textbooks written in the era prior to the National Council of Teachers of Mathematics (NCTM) reforms of 1989 and 1991, and those that were written after the reform. Pre-NCTM reform textbooks had some variations but for the most part, they were prepared to be used in US classrooms where students experienced what I am terming the “traditional” approach to mathematics teaching (Steak & Easley, 1978). In the traditional approach to mathematics teaching and learning, the teacher leads instruction. Much of the mathematics work especially that which is done from a textbook, involves repetition practice to master procedures. It has elements of students memorizing mathematical facts. Students tend to work individually and in silence. Textbooks written to conform to a traditional approach to mathematics teaching and learning therefore had an ample supply of mathematical tasks largely meant for students to practice memorization of facts and working through procedures.

These traditional textbooks existed prior to and emerged after the era of the “new math” movement. The new math movement promoted the discipline of pure mathematics through the study of abstract mathematics structures based on axiom and proof. The new math movement generally did not succeed in its aim to have all pre-college students learn mathematics from a rigorous basis familiar to what students encounter at the college level. There were a number of reasons why the new math movement did not succeed, which go beyond the scope of this dissertation. One reason that is however to this discussion in relation to textbooks was the lack of support for teachers to use the new math curriculum. As a result of its overall failure, there was a push back and a movement back to preparing students to master mathematics basics. This “back to basics” approach to teaching and learning mathematics “became the hallmark of textbooks and instruction programs” (Stanic & Kilpatrick, 1988, p. 413). This approach too was eventually not successful.

In the years that followed the new math and the back to basic movements and leading up to the NCTM reforms, there was a lot of thinking around what the important elements are for mathematics learning. The scholarly discussion of the time recognized the failure of both the “new math” era and the “back to basics” movement. Each approach failed for different reasons. The “new math” approach failed because it was not accessible to all who learned mathematics, being very abstract and complex. “Back to basics” was not successful because it focused primarily on basic skills, routine practice and procedural learning. An approach to teaching and learning mathematics was required that would be conceptually oriented, that would foreground the student’s role in teaching-learning process, that would emphasize problem solving, discourse among learners, mathematical understanding and sense making. These ideas led to the so-called reform.

In reality, these were not new ideas. This teaching and learning orientation had existed prior to 1989, only that that it was not mainstream prior to the NCTM reform. The NCTM’s efforts led to the Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989) and the Professional Standards for Teaching Mathematics (NCTM, 1991) which provided support for teachers to cope with and use these reform ideas, unlike the new math movement had done. Focusing on the abstract or the procedural structure of mathematics was no longer enough. Students now had to make meaning out of mathematics through a functional approach to learning. In the functional approach, both theory and application are merged, where students can connect procedures with concepts in given contexts. Through this so-called reform effort, the practice of teaching mathematics by having the teacher transmit to students was given less attention in favor of having the responsibility for learning fall more on students. It did not mean that teachers would no longer teach. Rather, it meant that mathematics teaching and learning should give more opportunity for students to do the work of learning.

Important to our discussion of textbooks, this reform effort led to the creation of new reform-based curricula such as Everyday Mathematics and Connected Mathematics (Noyce &

Riordan, 2001) and Core Plus (Hill & Parker, 2006). All three are reform-based textbooks which emphasize learning mathematics by solving problems in context. It is in the same vein that Hiebert et al. (1996) foreground the importance of problem solving to curriculum and reform.

These first NCTM reform efforts (NCTM, 1989; NCTM, 1991) and later additions (NCTM, 2000) align with the PBL teaching and learning approach for mathematics, insofar as they lay emphasis on students learning mathematics by solving problems in context, to make sense of mathematical procedures and concepts. This is in spite of the fact that PBL is a teaching and learning tradition onto its own, outside of mathematics education (Hmelo-Silver, 2004). Yet, according to Cotič & Zuljan (2009), expressing the broader view of using mathematical skills to solve problems that students encounter in life, “If we want our students to be sensitive to problems, to tackle them as challenges and be equipped for tackling and solving them, problem-based learning (PBL) should play an important role in their education” (p. 297). Therefore PBL is a germane approach to meeting the tenets of reform mathematics.

CPM and PI, the two textbooks I study in this dissertation, align with the PBL tradition. They both provide mathematical tasks of various kinds for students to solve. Solving these tasks serves as the main learning mechanism for students. The discussion around traditional and reform oriented textbooks is particularly relevant to CPM and PI because CPM’s orientation leans more toward what has been discussed in this section in relation to so-called reform mathematics through a version of it known as complex instruction (Cohen et al., 1999). PI’s orientation on the other hand resembles that of traditional mathematics teaching and learning. The analysis of tasks in both textbook series should thus reveal some interesting differences along the lines of these two orientations. In the next section, I discuss literature pertaining to opportunities that solving tasks give students for learning mathematics.

Opportunities tasks provide students to exercise agency and autonomy

Several studies (Sherman et al., 2016; Sood & Jitendra, 2007; Son & Senk, 2010; Stalianides, 2009) have shown that tasks in mathematics textbooks can offer students opportunities for learning. These learning opportunities come in different forms, depending on the structure and the content of the textbook (Wijaya et al., 2015). In particular, mathematical tasks in textbooks give students opportunities to experience and learn mathematics based on the features or characteristics of the tasks (Wijaya et al., 2015). So it matters what task content features in mathematics textbooks. In this section, I argue that whether or not task features are open or closed and the task types determined by task features make a difference in revealing what opportunities students have to develop different kinds of agency and autonomy.

The multiple ways of describing "open tasks": Yeo (2017a) whose work I build on in this dissertation made the argument that researchers have used the terms open and open-ended in quite ambiguous ways. In some instances, the same researcher uses the two terms to refer to certain kinds of tasks (c.f. Boaler, 1998). In other instances, the same researcher uses the term open-ended to refer to different kinds of mathematical tasks (Wolf, 1990). There are those researchers who distinguish between "open" and "open-ended" (Orton & Frobisher) while others use only open-ended (Becker & Shimada). There are also researchers who relate the terms open, open-ended and ill-structured (Silver, 1995). So the conceptual terrain for describing openness is varied. Part of the reason for this is because different researchers were referring to different features of tasks in order to classify them as open.

These task features are discussed in the section below on opportunities based on task features. Based on task features, Yeo (2017a) also suggested task types. In this dissertation, for simplicity, I shall use one term, "open" to refer only to task features and not to the overall task.

Instead, I shall resort to the term “degree of openness” when referring to tasks. Those tasks that have a higher degree of openness have more open task features. Those tasks that have a lower degree of openness have fewer task features open.

Opportunities for agency and autonomy based on task features: In an attempt to simplify the situation, Yeo (2017a) define task openness in terms of the open or closed state of five task features. These task features are the goal, method, complexity, answer and extension. Each of these task features can be open or closed. Yeo (2017a)’s work in pointing out the five task features was preceded by Silver (1995), who also suggested that a task can be open in terms of its goal, methods of solution and the potential for the task to be extended. In my methods section, I draw on these task features and descriptions of them, to identify the kinds of opportunities students have to exercise conceptual or disciplinary agency based on what task features are open or closed.

The open or closed state of a task feature helps determine what opportunities for agency and autonomy are available to students. For instance Schukajlow and Krug (2014) studied how prompting students to generate different solutions for real-world tasks impacted students’ interests in mathematics and their sense of autonomy and competence. The real-world tasks that students worked on were of three kinds. The first kind allowed students to explore various methods to arrive at a solution. The second kind required of students to make assumptions or to set conditions for the task. In essence, because the task was stated vaguely, students were at liberty to determine the constraints of the task in order to model a solution. They found that prompting students to seek multiple solutions positively impacted students’ autonomy, experience of competence and interest in mathematics. These results are important because they come from one of the few studies that have studied experiences of student autonomy in terms of opportunities for students to explore open method and open solution.

Opportunities based on task type: The QUASAR project (Silver & Stein, 1996) engaged students and teachers in meaningful mathematics learning experiences that emphasized mathematical reasoning and thinking skills and problem solving. Students in QUASAR classrooms constantly had opportunities to work on challenging tasks that involved multiple representations, multiple solution strategies and in collaborative groups. This was in contrast to the traditional method of teaching mathematics, where students learn by memorizing and then practicing procedures. One result from this project was that a subgroup of QUASAR students from disadvantaged backgrounds did about as well on computational tasks as students nationally, but then did much better than other students from disadvantaged backgrounds when it came to problem solving on challenging tasks. Other studies have reported similar findings at the elementary level of mathematics learning (Cobb et al., 1991; Riordan & Noyce, 2001).

Boaler (1998) also demonstrated in a study now well-known study that the types of tasks students work on in class can influence their engagement. Her study involved two schools – Amber Hill and Phoenix Park. Amber Hill students experienced a traditional approach to mathematics learning. In Amber Hill mathematics lessons, there was a large emphasis on working exercises out of a traditional textbook. The traditional textbook had mostly closed tasks, or those that primarily involve the application of mathematical procedures. Phoenix Park students on the other hand experienced learning mathematics in an open-ended, project-based approach. In this latter approach, students had the liberty to explore mathematical ideas, methods and patterns and to largely determine for themselves how to do so. Boaler (1998) stated that the Phoenix Park students she studied worked on open-ended mathematical tasks such as “The volume of a shape is 216. What can it be?" (p. 49) while Amber Hill students worked on more traditional routine exercises. When she interviewed a sample of students from each school, those from Amber Hill emphasized memorization and procedures in recounting their mathematics learning experiences. Those from Phoenix Park emphasized having choice in deciding how to solve the mathematics tasks assigned to them. They stressed that they were able to decide how to solve the task, that even when some of the task was set up for them, they had the liberty to take things further on the task. In relation to Phoenix Park students' experiences, Boaler (1998) stated that "if students are given open-ended, practical, and investigative work that requires them to make their own decisions, plan their own routes through tasks, choose methods, and apply their mathematical knowledge, the students will benefit in a number of ways" (p. 42). In spite of this statement, and similar to the QUASAR study, Boaler (1998) did not frame Phoenix Park students' mathematics learning experiences in terms of them having (conceptual) agency. However, in a follow-up study which revisited earlier participants in the 1998 study to determine how their school mathematics learning experiences had impacted them later in life, Boaler and Selling (2017) connected the empowering experiences that Phoenix Park students had to the idea of them having agency while doing so.

The findings from the QUASAR project and also from Boaler's study are important to this dissertation because they both demonstrate that the types of tasks that students work on are important to students' learning outcomes. In both cases, students developed their thinking and problem solving skills. They had the opportunity to work on challenging tasks that encouraged conceptual agency and intellectual autonomy. In the case of Phoenix Park, students had opportunities to even experience the proactive extent of intellectual autonomy. In relation to this dissertation, mathematical tasks in PBL textbooks which are reform focused can give students similar learning opportunities to exercise agency and autonomy as they develop their thinking and problem-solving skills.

Frameworks for analyzing textbooks from a cognitive perspective

There is a clear understanding among mathematics education researchers that different types of tasks give students different opportunities for learning. This is important because textbooks typically feature different kinds of mathematical tasks. For this reason, a number of research groups have put forward task classification schemes that attempt to capture the different tasks students encounter in the classroom (e.g. Stein et al., 1996; Kolovou et al., 2000; Yeo 2017a). In this section, I review these literature on the frameworks they provide to analyze tasks in textbooks, in order to address the question "do we have suitable means for analyzing how well textbooks empower students?" I shall review the literature in two categories: those that provide means for analyzing tasks by type and those that provide means for analyzing tasks both by type and by feature.

Task analysis by task type: One influential task classification scheme was put forward by Stein et al. (1996). This scheme classifies tasks based on levels of cognitive demand. Cognitive demand is a reference to the kinds of thinking required for solving tasks (Stein et al., 2000). In very simple terms, simple mathematical tasks that require little or no thinking can be classified as tasks of lower cognitive demand. On the other hand, more complex and challenging mathematical tasks for which students have to think and do more in order to solve the task. Based on the task type, different kinds of thinking are needed. The task types are memorization, procedures without connections, procedures with connections and "doing mathematics" (Stein et al., 1996). These task types are arranged in a hierarchy of levels of cognitive demand. For instance, memorization tasks require a low level of cognitive demand because they require students to recall mathematical facts. Procedures with connections tasks on the other hand require a higher level of cognitive demand because in addition to having students execute procedures, they also make connections to the concepts underlying those procedures. Students therefore have to think beyond the mere application of procedures. These four task categories together make up the Mathematical Task Framework (MTF). It is important to mention that the MTF was not originally created for the purposes of analyzing textbook tasks. Rather, it was created for studying changes in levels of cognitive demand between when classroom tasks are set up by teachers and when they are implemented by students. These classroom tasks could be from a textbook or from other sources. Subsequent researchers have commonly used the MTF as a framework for analyzing mathematical tasks in textbooks and other curriculum materials (c.f. Bieda, 2010; Hsu & Silver, 2014; Jones, 2007; Kotsopoulos et al., 2010; Özgeldi & Esen, 2010; Ubuz et al., 2010). It is also important to mention that Stein et al. (1996) made no explicit connection between the task types and the opportunities they may offer students for agency and autonomy. Additionally, the task categories that the MTF offers are not suitable for analyzing the variety for the tasks that the MTF will classify as "doing mathematics". This is because "doing mathematics" is a catchall category for more challenging or complex tasks. The MTF however distinguishes between procedural tasks that make connections with concepts and those that do not. That categorization is useful for the types of tasks encountered in textbooks. I therefore adopt that aspect of the MTF in this dissertation.

Unlike Stein et al. (1996) Kolovou et al. (2009) adopted a framework for analyzing textbook tasks. This study analyzed mathematical tasks in fourth grade Dutch textbooks to learn whether they offer students opportunities to develop higher-order thinking. According to them, higher-order thinking involves “both an insightful approach to the problem situation and strategic thinking” (p. 36). In their study, Kolovou et al. (2009) also clearly link the notion of higher-order thinking with Stein et al. (2000)’s idea of levels of cognitive demand, where higher-order thinking is associated with a higher level of cognitive demand. As a result of analyzing mathematical tasks in Dutch textbooks, Kolovou et al. (2009) created their task classification system comprising three levels of tasks: straightforward, “gray area” and puzzle-like tasks. Straightforward tasks require only routine application of procedures to generate a solution. Puzzle-like tasks require higher-order thinking, creativity and genuine problem solving. Gray-area tasks are neither straightforward nor puzzling but rather include some procedural aspects and some non-routine problem-solving aspects as well. To clarify their levels further, they created subcategories for gray area and puzzle-like tasks. They subdivided gray area tasks into numbers and operations, patterns and combinatorics. Combinatorics is a field of mathematics concerned with counting. The subcategories for puzzle-like tasks are context and bare number problems, which are symbol based tasks with little to no context. Kolovou et al. (2009) discovered very few opportunities for students to engage with puzzle-like tasks in the Dutch textbooks. None of the six textbooks they analyzed had more than 2.5% of puzzle-like tasks. Overall, puzzle-like tasks constituted 1% of tasks while gray area tasks constituted 8% of the tasks, meaning that problem solving tasks constituted 9% of tasks overall. These results show that in comparison with the total number of tasks there were relatively few opportunities for students to exercise conceptual agency and intellectual autonomy. Conversely, there were many more opportunities for students to exercise disciplinary agency while working on straightforward tasks, which constituted 91% of tasks. Since Kolovou et al. (2009) is one of the few studies that analyzed textbooks based on task types for the learning opportunities the offer, it is important to this study. The task categories in this study are however specific to the textbook types they analyzed. They are not general enough to be useful for the kinds of tasks encountered in the two PBL textbooks used for this study.

Task analysis by task type and task features: Yeo (2017a)'s framework for task types provides four task categories based on five task features. The four task categories are procedural tasks, problem solving tasks, investigative tasks, and real-life tasks. The five task features are goal, method, complexity, answer and extension. Each of these can be open or closed. Collectively, the task features determine the task types. For instance procedural tasks have all five task features closed, whereas investigative tasks have them all open. Yeo (2017a) only presents the framework without testing it on textbooks. This framework provides useful categories for distinguishing between task types. Also the task features allow for the creation of new task types. For this reason, Yeo (2017a) is a framework I adopt in this dissertation to study mathematical tasks in the two PBL textbooks.

Therefore with respect to frameworks for the cognitive perspective, the answer to the question "do we have suitable means for analyzing how well textbooks empower students?" is in the affirmative. However none of the appropriate frameworks reviewed in this section is by itself sufficient. There is the need to pick aspects of the MTF and of Yeo (2017a)'s framework for the analysis of the two PBL textbooks.

Frameworks for analyzing textbooks from a linguistic perspective

In contrast to the studies on frameworks reviewed for the cognitive perspective, a number of active mathematics education researchers employ well developed linguistics based frameworks in their work. As part of addressing the question "do we have suitable means for analyzing how well textbooks empower students?" I shall review the literature on frameworks that analyze mathematics texts in their entirety as well as those that only analyze mathematical tasks. I shall assess each in relation to the study in this dissertation.

Mathematics texts: Dowling (1996) provides a theoretical framework which he calls a language of description, based on language for analyzing sociological aspects of school mathematical texts. This theoretical framework analyzes all the contents of school mathematics texts, including tasks and supporting texts. In his language of description, he makes a distinction between "esoteric domain" and "public domain" mathematical practices where the former refers to mathematical activities done in abstract formalism while the latter refers to mathematical activities done within the context of daily life. Dowling considers how demarcations between higher and lower social class dimensions in society are reflected in the esoteric and public domain distinctions of school mathematics texts. Although Dowling's analysis encompasses the entire lesson text and his work is important in the linguistic analysis of mathematics textbooks, his categories for analysis are only generally related to agency and autonomy and are therefore not suitable for the analysis needed in this dissertation.

Where Dowling's framework appears to be related only at a very general level to issues of agency and autonomy as they may appear in textbooks, Morgan provides an analytic framework useful for analyzing whole texts in ways that more closely align with this dissertation. Morgan (1996) argues on the basis of Systemic Functional Linguistics, or SFL (Halliday & Matthiessen, 2014) that imperatives such as "consider, suppose, define" (p. 6) used in mathematics texts "implicate the reader, who is addressed implicitly by the imperative form, in the responsibility for the construction of the mathematical argument" (p. 6). This view is relevant to the study in this dissertation because it shows that the imperative form used in mathematics texts addresses readers to take action. In mathematics textbooks, imperatives can be used to direct students to execute mathematical procedures or to express their thinking. Equally important, Morgan also stresses the importance of the study of modality in understanding how author, reader and subject matter interrelate. Modality refers to the degree of obligation or probability expressed in a clause. In this dissertation, I analyze modalization and modulation, two aspects of modality. Modalization is concerned with the degree of probability in a clause while modulation is concerned with the degree of obligation in a clause. These interrelations are important to revealing how textbooks through their authors provide opportunities for students to exercise agency and autonomy. According to Morgan (1996), modality can manifest in communication between teacher and students in the expressions of authority and certainty between the two.

Herbel-Eisenmann and Wagner (2007) present a framework for analyzing ways that mathematics textbooks position students in their learning of mathematics. Their framework includes how mathematics textbooks position students in relation to mathematics, in relation to other people, and in relation to their experience of the world. Herbel-Eisenmann and Wagner (2007) operationalize positioning in their study as realized by certain linguistic features of the textbooks they studied. These framework features include a study of imperatives and modality. This dissertation draws on Herbel-Eisenmann and Wagner's (2007) discussion of imperatives and modality to understand how a mathematics textbook positions students to exercise agency and autonomy. In this regard, their work connects to that of Morgan (1996) but is more relevant to this dissertation as Herbel-Eisenmann and Wagner (2007) focused on mathematics textbooks whereas Morgan (1996) focused on different genres of mathematics texts including textbooks.

One study that was particularly relevant to my dissertation was Herbel-Eisenmann (2007). In her study of middle school mathematics reform curriculum materials, Herbel-Eisenmann (2007) examines the “voice” of the text. Through the “voice” of the textbook, the textbook authors establish their role as well as the roles of teacher and of students. The textbook “voice” enables the textbook authors to position students and teachers in relation to the intended learning. In order to study textbook “voice”, Herbel-Eisenmann (2007) also draws on methods from SFL. Also drawing on methods appearing in Morgan (1996), Herbel-Eisenmann focuses in particular on language that realizes interpersonal meaning. Analysis of such language makes apparent the interactions, relationships and roles among individuals through the study of text. To reveal the role of the text’s “voice”, Herbel-Eisenmann (2007) operationalized three aspects of the interpersonal function. These are imperatives, personal pronouns and modality. Of these three, imperatives and modality are relevant to this dissertation. Imperatives are commands that direct students’ actions while solving mathematical tasks. Imperatives that middle school mathematics students commonly encounter in textbooks are “draw”, “explain”, “find”, and “solve”. For her conceptualization of imperatives, Herbel-Eisenmann drew on Rotman (1988)’s ideas of exclusive and inclusive imperatives. According to Rotman (1988), inclusive imperatives are those verbs such as “explain”, “justify”, “predict” for which “the speaker and hearer institute and inhabit a common world or that they share some specific argued conviction about an item in such a world” (p. 9). Therefore, for mathematics learning, inclusive imperatives allow individuals to express and share their mathematical thoughts to one another. For exclusive imperatives, the requirement is that “certain operations meaningful in an already shared world be executed.”(p. 9). Herbel-Eisenmann conceptualized modality in terms of Hodge and Kress

(1993)'s definition of the term, which is "indications of the degree of likelihood, probability, weight or authority the speaker attaches to the utterance" (p. 9).

In this dissertation, I conceptualize imperatives in the same ways that Herbel-Eisenmann does, although her study does not draw strongly on the analysis of imperatives in connection with agency, as I do. In connection with agency, I think of inclusive imperatives as affording students opportunities to exercise conceptual agency while exclusive imperatives give them opportunities to exercise disciplinary agency. My approach to modality is however different. Morgan (1996), Herbel-Eisenmann (2007) and Herbel-Eisenmann and Wagner (2007) all appear to conflate modalization and modulation (Halliday & Matthiessen, 2014), two aspects of modality. In their analyses of modality, they examine verbs such as "must", and "will" that indicate modulation as well as adverbs such as "possibly" that indicate modalization under the overarching category of modality. In this dissertation, I chose to have them separate in order to study how these two aspects of modality impact student positioning and the opportunity that positioning affords for agency and autonomy. One other way in which this dissertation differs from the research carried out in Herbel-Eisenmann (2007), is the manner in which I analyze the overall lesson text. In addition to the methods from Herbel-Eisenmann (2007) that I draw on, I also adopt an approach based on text genres and stages for analyzing texts section by section, and foregrounding the sections in the analysis. Herbel-Eisenmann (2007)'s initial analysis covers the entire lesson text however the subsequent focus is on the interactions happening primarily in mathematical tasks. My analysis, because it is based on different sections of the text, covers both tasks and supporting lesson texts.

Mathematical tasks: in a recent study, Morgan and Sfard (2016) draw on SFL and on Sfard (2008)'s communicational theory (hereafter, CT) to study how changes due to reform mathematics have made their way into high stakes mathematics examinations. The high stakes examinations referred to are the more recent UK GCSE (General Certificate of Secondary Education) examination which became the UK national examination in 1988 for students who have completed the equivalent of the US tenth grade. They also selected examination questions from the older General Certificate of Examination Ordinary Level (GCE O Level) and the Certificate of Education (CSE). These last two were the precursors to the GCSE. The collection of examination problems from these three examinations then became the Evolution of the Discourse of School Mathematics (EDSM) database, spanning the years 1980 to 2011, which was the corpus of data for this study. As a result of the data they studied, this study is concerned primarily with analyzing mathematical tasks with no accompanying lesson texts.

The conceptual framework for this study is complex and extensive. Fundamentally, Morgan and Sfard (2016) promote the notion that "mathematics may be usefully conceptualized as a discourse and that mathematical thinking is a form of communicating" (p. 101). This means that all of mathematics itself – the objects, the notations, the ideas and the relationships can be thought of as a form of discourse. Mathematical thinking, in their view, entails communicating information about objects, notation, ideas and relationships. With this outlook, they study what mathematics examination problems communicate by analyzing discourse elements in them.

There are a number of similarities as well as differences between Morgan and Sfard (2016) and this dissertation study. Both studies investigate questions of agency and autonomy. Both studies also analyze mathematical tasks. For autonomy, Morgan and Sfard (2016) are interested in the decision processes students go through as they first interpret an examination problem, chart out a solution path and construct a response. In this regard, there are similarities with the study in this dissertation that investigates closed and open features of tasks. Open features of tasks such as the task method and solution give students opportunities to develop intellectual autonomy. One difference is that Morgan and Sfard (2016) study agency assigned to mathematical objects such as equations, graphs and tables whereas this dissertation concerns how mathematical tasks can afford students opportunities to develop agency. Another difference is that Morgan and Sfard (2016) only study mathematical tasks whereas this dissertation studies mathematical tasks and supporting lesson texts. A third difference is in the analysis of tasks at different grainsizes. Morgan and Sfard (2016) analyze words, phrases and sentences, and parts of questions and entire problems to get a larger picture of the happenings in a task. In this dissertation, I approach grainsize differently, through the idea of different stages of a lesson text. The scope of the grainsize of analysis in Morgan and Sfard (2016) is limited to a task. The scope of the grainsize of analysis in this dissertation goes beyond tasks to encompass the entire text for a given lesson. Therefore the analytical tools available in Morgan and Sfard (2016) are of limited scope for the required version of textbook analysis in this dissertation. It is however clear from this comparison that a close study of language is useful for studying agency and autonomy in curriculum materials.

Therefore with respect to frameworks for the linguistic perspective, the answer to the question “do we have suitable means for analyzing how well textbooks empower students?” is also in the affirmative. As with the case of the cognitive perspective, none of the linguistic perspective frameworks can be applied in full. This is because the purposes for which those frameworks were created do not exactly match those of the study in this dissertation. In particular, I will adopt the approaches that Herbel-Eisenmann (2007) used with imperatives and a similar approach to what she used with modality for this dissertation. In so doing, I draw on a subset of the analytic methods Herbel-Eisenmann (2007) used. On the other hand, this dissertation takes into account analysis of lesson stages which widens the scope of analysis beyond the clausal level to include the entire lesson text for a given lesson. In order to study entire lesson texts by stages, I will have to also draw other linguistic tools not used by any of the frameworks reviewed in this section.

Summary

In this literature review, I have presented varying notions of agency and autonomy. In particular, I defined conceptual and disciplinary agency. I also defined intellectual, individual and social autonomy. I reviewed research on problem solving in mathematics education as it pertains to working in textbooks. I then discussed reform and traditional approaches to teaching and learning mathematics, and their connection with problem based learning. The review then proceeded to examining the notion of open and closed task features and task types, followed by a review of frameworks for analyzing mathematics textbooks both from a cognitive and a linguistic point of view. In the next chapter, I detail methods for analyzing textbooks from cognitive, linguistic and combined perspectives.

Chapter 3 Methods

Introduction

This dissertation investigates ways that lesson texts and mathematical tasks in College Preparatory Mathematics (CPM) and Pearson Integrated (PI) position students to develop agency and autonomy as they work through texts and tasks. I chose these textbook series in particular because even though both support problem-based learning (PBL), each presents lesson texts and tasks in particular ways. The orientation of CPM tasks and texts align more with so-called reform mathematics while those of PI align more with traditional mathematics textbooks. Analyzing both textbook series can thus reveal differences and similarities in making opportunities to develop agency and autonomy available to students.

To restate, the research questions for this study are (1) To what extent does a cognitive perspective analysis of mathematical tasks in two PBL textbook series reveal differences and similarities in opportunities for students to develop some forms of agency and autonomy with respect to topics on functions? (2) To what extent does a linguistic perspective analysis of lesson texts in two PBL textbook series reveal differences and similarities in opportunities for students to experience some forms of agency and autonomy with respect to topics on functions?, and (3) In what ways does the interaction of cognitive and linguistic perspectives in the analysis of mathematical tasks in two PBL textbook series reveal differences and similarities in opportunities for students to develop some forms of agency and autonomy with respect to topics on functions?

In this chapter, I shall detail three analytical approaches I developed for the analysis of tasks and texts in each textbook series. These analytic approaches reveal how students are positioned by CPM and PI as they work. I call these approaches the cognitive, the linguistic and the combined perspectives. The cognitive perspective analyzes task types on the basis of specific task features. These task features are the task goal, which is what the task asks students to accomplish, the task method, which is how the task asks them to accomplish it, the task solution, which is what the task expects students to produce as a solution and the task extension, which is an opportunity for students to work toward generalization. Each of these task features can be coded as open or closed, based on certain factors which I describe further below in this chapter. A given task type provides students opportunities to develop agency and autonomy depending on which task features are open. The linguistic perspective allows for the analysis of both tasks and texts. This is done through analysis of the clauses in the task. I study declaratives, or clauses that usually give students information, imperatives, which usually demand action of students, and interrogatives, which usually demand information of students. Whether clauses are demanding information or action from students or giving them information, and how this is done in the clause can to reveal opportunities for developing agency and autonomy. The combined perspective associates clause type with task features to reveal ways students can be positioned to develop agency and autonomy. Clause functions together with task purposes can reveal insights into how the task as a whole gives those opportunities of interest in this dissertation.

For this chapter, I shall first explain my interest in studying tasks and texts in PBL textbooks. This will set the stage for subsequently explaining the methods I develop in the chapter. After presenting the rationale for this study, I shall then describe the data I selected for analysis, followed by detailed descriptions and explanations for the three analytic perspectives.

Describing these perspectives is necessary to show how I generate my findings and the discussions based on them.

Rationale for embarking on this study

Across the world, mathematics textbooks guide the teaching and learning of mathematics. As such, the importance and ubiquity of mathematics textbooks in learning mathematics is universally acknowledged (Fan et al., 2013). Mathematics textbooks are important because they contain instructional texts detailing mathematics theory and mathematical tasks with examples meant to guide students during learning. What those mathematical tasks are and how they are presented to students can make a difference in how students develop into independent thinkers and problem solvers. Recent studies (e.g. Lester & Cai, 2016) have emphasized the need for students to learn mathematics by working on worthwhile tasks that drive their interest and provides them with opportunities for independent thinking for developing higher-order thinking skills. While exercising higher order thinking skills as they work on mathematical tasks, students can exercise their agency and autonomy.

As textbooks are a key resource for students' mathematics learning, analyzing textbook tasks and texts for features that can support students' development of agency and autonomy while working on challenging tasks is thus an important undertaking both from the point of view of research and of teaching and learning. In terms of research, developing and forwarding ways to make apparent such opportunities in textbooks can advance what we know and understand in our field about how students can be empowered through their mathematics education. In terms of teaching and learning, it is important for teachers to know that textbook tasks that challenge students offer among other things the opportunity for them to hone their independent thinking skills. As such, the practitioner and research communities may both benefit from more studies that reveal opportunities mathematics learners have to experience agency and autonomy as they work on mathematical tasks.

Data

I collected mathematical tasks and sections of lesson texts from selected lessons in chapters of Core Connections (CC) Integrated I (2013 edition), II (2015 edition), & III (2015 edition) of College Preparatory Mathematics and PI Mathematics I, II & III (all of which are 2014 editions). I used purposive sampling (Miles & Huberman, 1994) to select chapters addressing the subject of functions. Functions are a central topic in mathematics, encountered by students over multiple years. Learners of mathematics see functions in numerous ways across the mathematics curriculum of high school, and this topic has importance in the future study of mathematics at the college level and beyond. As such, if are taught and learned in a way that promotes agency and autonomy then as students encounter functions again in the study of higher mathematics, they may activate and draw on their developed autonomy. Research suggests that students often have difficulty transitioning from the more supportive secondary environment where there are more supportive classrooms and more teacher attention, to the more autonomous learning environment that is represented by college learning activity.

Studying functions across three CPM and three PI textbooks would give me access to a breadth of topics on functions beginning with the introduction of functions found in the first textbook of each series to Trigonometric functions found in the third textbook of each series. It happens that the topics on functions are not all found in one textbook in the series. As such, it is important to analyze opportunities for students found in different topics on functions across all three textbooks in order to get a broad and detailed view of what such opportunities are when students study functions.

In both CPM and PI, each chapter has some text that guides instruction for a given lesson, in addition to mathematical tasks. Tables 1 and 2 below show the textbooks, chapters and lessons from those chapters that I shall analyze. I have chosen to select the first, middle and end lesson for each chapter. Such a purposive sample will allow me to study a wide range of mathematical tasks and associated lesson texts across the chapter. Tables 1 and 2 below show the selected chapters.

Table 3.1: Purposive sample of CPM and PI chapters on functions

Textbook	**Chapter numbers and names**	**Lessons**
CC Integrated I	01 Functions	1.1.1; 1.2.1; 1.3.2
	02 Linear Functions	2.1.1; 2.2.2; 2.3.2
	08 Exponential Functions	8.1.1; 8.1.5; 8.2.3
CC Integrated II	05 Quadratic Functions	5.1.1; 5.2.1; 5.2.6
CC Integrated III	01 Investigations and Functions	1.1.1; 1.1.4; 1.2.3
	09 Trigonometric Functions	9.1.1; 9.1.6; 9.2.3
PI I Vol I	02 An Introduction to Functions	2.1; 2.4; 2.7
	03 Linear Functions	3.1; 3.4; 3.7
	05 Exponential and Radical Functions	5.1; 5.5; 5.9
PI II Vol II	12 Quadratic Functions	12.1; 12.6; 12.12
PI III Vol III	05 Rational Expressions and Functions	5.1; 5.4; 5.7
	08 Trigonometric Functions	8.1; 8.6; 8.11

The Cognitive perspective

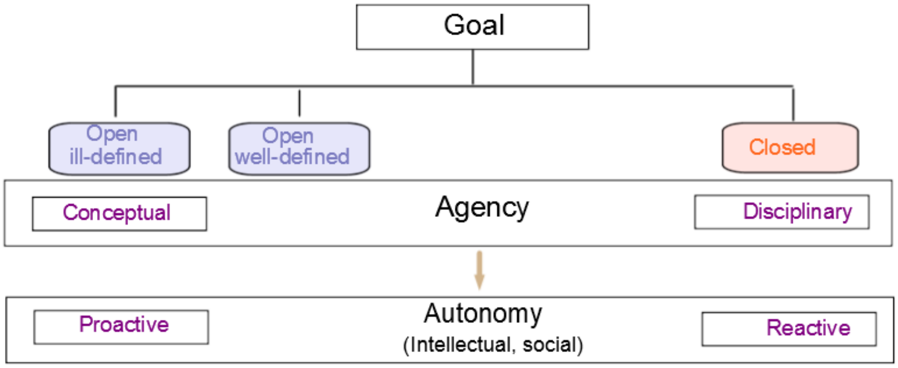
The cognitive perspective involves the idea that different task types require different levels of thinking needed to solve the task. In this section, I present certain task features which I used to classify tasks. Task features are those aspects of the task that influence the work done on the task. Yeo (2017a) developed a task framework based on five task features. These features are helpful ways of understanding the purpose and function of tasks. The features can be closed or open. Tasks with different degrees of openness depending on which task features are open or closed in turn offer different opportunities for students to experience agency and autonomy when solving them. I go into more detail about what it means for a task feature to be closed or open in the section below on task features.

For analysis involving the cognitive perspective, the unit of analysis is a mathematical task. Before outlining different types of mathematical tasks, I shall first describe each task feature in detail. With each task feature, I shall explain how and why I coded example tasks from CPM and PI. I shall also point out what implications the degrees of openness for the given task feature being coded have for opportunities for students to experience agency and autonomy.

Detailed descriptions of the five task features: In this section, I elaborate on what each task feature is and what it means for the feature to be open or closed.

Goal: the task goal is the purpose for which students work on the task. This purpose outlines the direction students are meant to take as they work on the task and what they are meant to produce as a result of working on the task. The goal was coded as one of three different states, depending on the task: closed, open and well-defined or open and ill-defined. I describe each of these states in Figure 3.1 below.

Figure 3.1: Open and closed states for the task goal



The task goal is closed when it is clear, explicit and specific what the task asks students to do, with no room for students to pursue their own goal. This means that students cannot modify the goal or follow a sub goal. When the task goal is closed, students do not have to figure out what the task is about or how to scope their responses, because all of that is clear. Consider the following task from PI:

Figure 3.2: Example of a task where the goal is closed

Write a recursive formula for the arithmetic sequence below. What is the value of the 9th term?

a. 3, 9, 15, 21, …
b. 23, 35, 47, 59, …
c. 7.3, 7.8, 8.3, 8.8, …
d. 97, 88, 79, 70, …

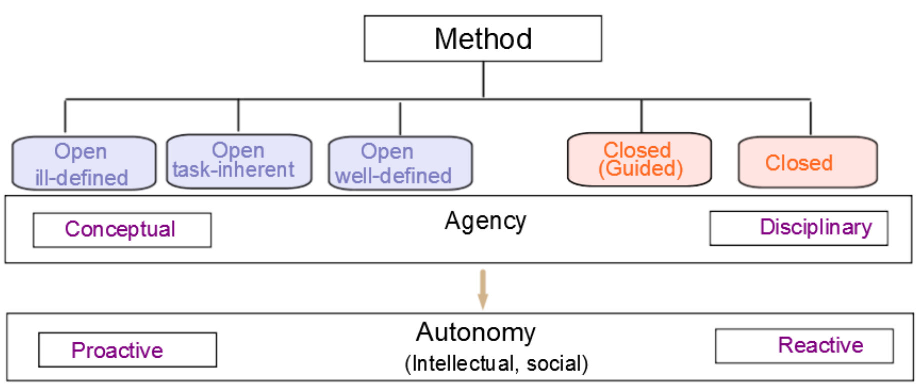
I coded the goal in tasks such as the one above as closed. This is because it is clear that students have to write a recursive formula, and then work out the 9th term. Yeo (2017a) terms the task goal 'closed' when it is clearly stated. In the task above, the goal is clear, explicit and specific. There is little room for students to pursue their own goal as they work on the task. Therefore the task goal is closed. In Figure 3.1, I show that when the goal is closed (in red), students' experience limited conceptual agency. They can exercise reactive instead of proactive autonomy, as they are constrained by the closed goal.

On the other hand, when the task goal is not explicitly stated, or when it is only generally stated, I termed the task goal ‘open’. According to Yeo (2017a) tasks can be open and well-defined or open and ill-defined. When the task is open and well defined, it provides some guidance on the direction to take for the task but still leaves room for students to determine the ultimate direction for the task. Consider the following example from Yeo (2017a), “Powers of 3 are 3^1, 3^2, 3^3, 3^4, …Find as many patterns as possible.” (p. 180). This task directs students to find as many patterns as possible, as a result of investigating the task. Students are not directed to work on one particular pattern or another. So the task goal is open, but well-defined, because although students are directed in general to investigate patterns, they are not told which particular patterns to investigate. Students have room to decide for themselves which patterns they want to investigate. They can thus exercise conceptual agency and reactive intellectual autonomy. It is reactive because the task already set constraints for the goal – to find as many patterns. Students will be working within this constraint.

When the task goal is ill-defined, the goal is vague, leaving it for students to set general and particular directions for the goal of the task. Yeo (2017a) gives the following example task as having an ill-defined goal, “Powers of 3 are 3^1, 3^2, 3^3, 3^4, …Investigate.” (p. 178). For this task, the only direction students get is to investigate the task. They are not told what to investigate, so the goal is vague. Students have to decide what they wish to investigate. They could investigate patterns, they could investigate sums of powers, or they could investigate something else of interest to them. In the ill-defined case, students could very well pick one pattern and investigate it at length, rather than exploring various patterns. Therefore the manner in which the goal is stated has a bearing on whether it can be classified as closed, open and well-defined or open and ill-defined. When the goal is open and ill defined, students have the opportunity to exercise conceptual agency and proactive autonomy, as they can take it upon themselves to determine the goal of the task on which they will work.

Method: The task method involves the steps students follow as they work through a task. These steps could be in the form of procedures laid out in the task. For instance the textbook authors can lay out for students the steps students need to follow in order to generate solutions to the task. The method could also comprise standard or well-known mathematical procedures for solving particular kinds of procedural tasks. For instance the method for factorizing a quadratic equation is a well-known mathematical procedure. The method can also comprise steps that students assemble or come up with by themselves. For example through their use of heuristics, students can come up with a method to solve a given task.

Figure 3.3: Open and closed states for the method task feature



The requirements for the task method can be evident in the task in a number of ways, which can help distinguish the task method as being either closed or open. For one, if the task makes clear which procedures students should draw on to work through the task, then the task method is closed. As an example, consider the following task from PI on Exponential and Radical Functions:

Figure 3.4: Example of a task for which the method is closed

What is the simplified form of each expression?

a. 9^{-2}

b. $-(3.6)^0$

PRENTICE HALL, HIGH SCHOOL MATH 2014 COMMON CORE INTEGRATED MATH 2 WRITE-IN STUDENT EDITION 2-VOLUME + DIGITAL COURSEWARE 1-YEAR LICENSE (REALIZE) GRADE 8/9, 0, ©2014. Reprinted by permission of Pearson Education, Inc., New York, New York.

In the statement of the task above, one would expect students to draw on commonly known mathematics properties for simplifying indices. For part 'a', students may use the property of indices stating that $a^{-b} = \frac{1}{a^b}$ to simplify the expression. For part 'b', students may use the property $a^0 = 1, a \neq 0$. The use of these properties is a typical approach students take to solve these two indices sub-tasks. Therefore, I would code the method for solving this task as closed, as students will likely use the standard procedures rather than their own methods.

As shown in Figure 3.3, another scenario for which I considered the task method to be closed was when the task directed students to use a method already encountered in a previous and related task without asking students to extend the method. This second scenario occurred on a number of occasions with tasks I analyzed in the CPM sample. For those cases, the CPM textbook authors appeared to have designed tasks with learning progression in mind, so that earlier tasks impacted later ones within the same text for a given lesson. I also considered the method to be closed when the task detailed all the directions students need to follow in order to solve the task. In this case, the directions serve as scaffolds to guide students' activity in generating solutions to the task. This case was common among CPM tasks. In Figure 3.3, when the method is closed, as with the goal, students exercise limited conceptual agency and reactive autonomy.

The task can however leave the method open to varying degrees, as shown in Figure 3.3. The task method in the open state gives students opportunities for exercising agency in what methods they choose when working to solve a task. The following task also from the CPM chapter "Investigations and Functions" on combining linear functions gives students enough direction but also leaves room for them to decide on what steps to take to solve the task:

Figure 3.5: Example of a CPM task where the method is open and well-defined

First investigate the graphs for the sum and difference of your two functions. Predict what happens if you add or subtract any two linear functions. Can you think of any exceptions?

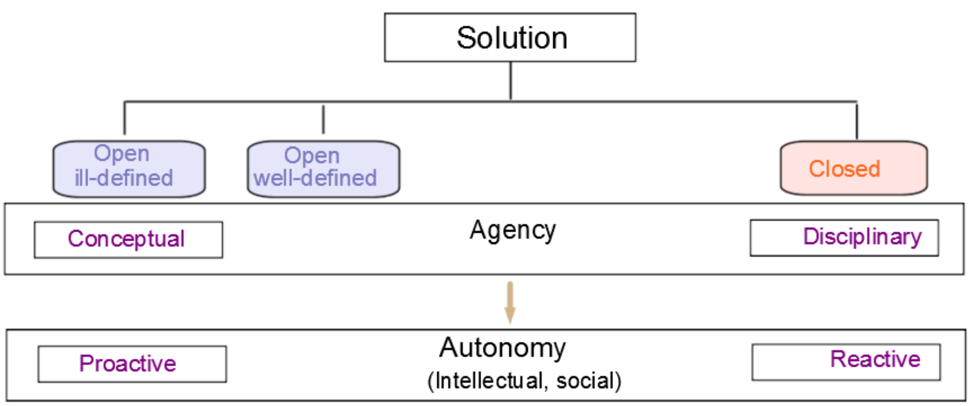
In the task above, students are given two linear functions and asked to investigate what happens during addition and subtraction of the functions. Students are then asked to predict what happens when any two linear functions are added or subtracted. For this latter part of the investigation, they are at liberty to decide which test cases to select for making the prediction. They are also at liberty to decide which cases to investigate as exceptions, if they can come up with any. So for task method, I code tasks such as the one in Figure 3 and others like it as open and well defined. The method is open because students have some (reactive) autonomy in deciding on and executing aspects of the method. This is true when students are drawing on the mathematics they know. This open nature of the task method is well defined because the correctness of the method students come up with can be checked by referencing extant mathematics knowledge.

For some tasks, the method can also be open and ill-defined, as shown in Figure 3.3. This happens when the task gives no specific or general guidelines for proceeding with the task and where it is not immediately obvious how the textbook authors expect students to work on the task. In this case, students can come up with the intermediate steps needed to arrive at the solution. For the ill-defined case, the intermediate steps students come up with can be subjective.

The distinguishing feature between the ill-defined and the well-defined cases of open task method is that in the well-defined case, the ideas from which students compile their method are objectively established in formal mathematics. In the ill-defined case, the ideas students draw from may not relate to established or formal mathematics. Instead, they may create or use their own mathematical ideas as methods. In this case, the teacher or student colleagues may appraise the method subjectively. Students can experience a high degree of conceptual agency and proactive autonomy when the method variable is ill-defined.

Solution: the solution consists of the final product students give for their work on a task. Yeo (2017a) refers to this task feature as the "answer" rather than the "solution". I choose "solution" over "answer" as "solution" implies a task that is solved. Figure 3.6 shows the various states of open and closed that I coded tasks as.

Figure 3.6: Open and closed states for the solution task feature



The task solution can take numerous forms. For the tasks I analyzed, the required solution was to be in the form of a mathematical object such as a number, a function, a graph, a table or roots of a function. Other times, the required solution was a theoretical argument such as a conjecture, justification, a prediction or a proof. I coded the solution variable as closed when the task demanded a specific and unequivocal mathematical object. If for instance the expected solution consists of a single number, a graph or an expression, I coded the solution variable as closed.

Additionally, if the required solution was a set of mathematical objects that was finite and could be predetermined, I coded the solution as closed. As such, I coded the solution as closed. Consider the following task in Figure 3.7 from a PI lesson:

Figure 3.7: Example of a PI task with method closed

"What is the graph of the function $y = x^2 - 6x + 4$?"

This task requires students to produce a graph for the given function. Students could come up with their graphs via a number of different methods. They could for instance create a table of x and y values. They could alternatively use the quadratic formula to come up with roots for the quadratic equation. They could also convert the equation from standard form to vertex form and draw on information from that to draw the graph. They could also use graphing software on a calculator or on a computer. All of these methods can lead students to generate a graph of the function. While there may be slight variations in the graph produced using different methods, the solution will consist of one mathematical object – a graph. As such I code the solution variable as closed in this case.

There are also cases where I coded part of the expected solution as closed and part as open. Consider the following task in Figure 3.8 from a chapter in CPM on linear functions:

Figure 3.8: Example CPM task with solution variable in closed and open states

"The slope of a line can represent many things. In this lesson you concentrated on situations where the rate of change of a line (the slope) represented speed. However, rate of change can represent many other things besides speed, depending on the situation. For each graph below: (i) Explain what real-world quantities the slope and y intercept represent. (ii) Calculate the unit rate for each situation. Be sure to include units in your answer. (iii) In each of the situations, would it make sense to draw a different line with a negative y intercept?"

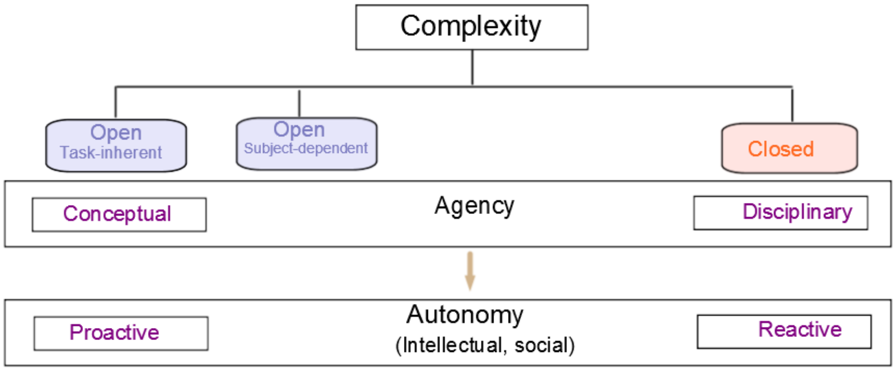
In this task, the student is asked to calculate the unit rate. I would code this part of the solution as closed because it is a specific numerical value. The task also demands mathematical explanations and justifications from the student. I would code explanations and justifications as open. This is because there may be so many different ways that students may express correct explanations.

If it would be possible to distinguish mathematically correct explanations from mathematically incorrect ones, I coded the solution as open and well defined. The solution is well defined in the sense that the explanation or justification is either mathematically correct or mathematically incorrect. I coded the CPM example task above overall solution as open, even though subsections of the solution comprise a closed response. There are also tasks for which the solution is open with no closed components. Consider for instance the task in Figure 3.5. That task requires students' solution to make a generalization about what happens when two linear functions are added or subtracted. Students may present solutions that include different individual ways of expressing their predictions. Some of these expressions may be mathematically accurate and others may not be. I code the solution variable of tasks of this kind as open. For this task, students have the opportunity to exercise conceptual agency and intellectual autonomy in determining the scope of the solution, as the solution space is left open to them to explore, yet the solutions students generate will still be either mathematically correct or mathematically incorrect. I therefore code tasks of this nature as open and well-defined. What constitutes a correct solution is well-defined from a mathematical point of view.

The final category under the solution variable consists of those tasks whose solutions I coded as open and ill-defined. These are tasks for which the required solution is open but which the correctness of a student's solution is subjectively determined by a teacher rather than on the basis of its mathematical correctness. An example of such a task is given in Yeo (2017a), "Choose any mathematics project to do. Submit a report at the end of the year." (p. 181). In such a task, the solution is the actual mathematics project. In this case, the solution to the task, the project itself, is not correct or incorrect in the same sense as when the solution variable is open and well-defined. In this ill-defined case, as the problem-statement stands, students are at liberty to determine what project they choose to work on. It is left to the teacher to decide whether the solution meets criteria for correctness or incorrectness.

Complexity: task complexity refers to the degree to which students are able to access a task in order to produce correct solutions for it. Students can access a task if they understand what the task asks them to accomplish and how to they are to accomplish it. In coding tasks in the two textbooks for complexity, I considered whether or not the task gave students scaffolding in the form of guidance on how to go about the task. In other words, I considered whether the task was closed in terms of giving students all the components they needed to work on methods that led to solutions or whether the task left gaps for students to figure out certain steps so as to solve the task. If gaps were left in the task, students would have opportunities to exercise agency and autonomy in determining the necessary steps to fill in those gaps. Figure 3.9 shows the various categories.

Figure 3.9: Open and closed states for the complexity task feature



According to Yeo (2017a), complexity can be closed, open and subject-dependent or open and task-inherent. In this dissertation, I shall adopt the term “student-dependent” rather than Yeo (2017a)’s term “subject-dependent”. This is because the subject of concern solving tasks in this study is always the student. I shall give examples to illustrate each case. The first task we shall discuss appears in Figure 3.4. In that task, students are asked to simplify two expressions. If students have the required prior knowledge for simplifying expressions, they will apply the needed procedures to simplify the expressions and that will be all that is needed. The next task in Figure 3.10 from the CPM chapter on quadratic equations provides enough scaffolding for students to have all the components they need to solve the task:

Figure 3.10: Example CPM task where the complexity variable is closed

“Consider the equation $x^2 = 2$.

a. When you solve $x^2 = 2$, how many solutions should you get?
b. How many x-intercepts does the graph of $f(x) = x^2 - 2$ have?
c. Solve the equation $x^2 = 2$. Write your solutions both in exact form and as decimal approximations.”

In this task, both the method and expected solution are straightforward. Assuming students know the procedure for solving the equation and for determining the difference of two squares they can proceed with the task without having to explain why they get the number of solutions they get, because the task does not ask for an explanation. For these reasons, I coded complexity for this task as closed.

When I coded the task as open and student dependent, it was because the task provided students with opportunities to exercise conceptual agency and intellectual autonomy in offering their thinking or explanations. For example in Figure 3.11, students are given some of the information they need to solve the task. They are however not told how to use the table to determine the growth and starting value. That step is omitted from the task. Instead, the textbook authors pose the question to students to figure that out by themselves. They did this by asking "How can you use the table to determine the growth and start value?" In so doing, the textbook authors give students an opportunity to exercise their conceptual agency and (reactive) intellectual autonomy in figuring out how to use the table to solve the task.

Figure 3.11: Example CPM task with open and student dependent complexity variable

"The growth of tile Pattern C is represented by the equation $y = 3x + 1$.

a. Copy and fill in the table for Pattern C.

Figure # x	0	1	2	3	4
# of tiles y					

b. By how many tiles is each figure in Pattern C growing? What is the starting value?

c. How can you use the table to determine the growth and starting value?

d. Where do you look in the equation to see the growth and starting value?"

I coded the complexity variable in tasks such as the one in Figure 3.11 as open and subject dependent, the subject being the student, as it will depend on the knowledge and ability of each individual student whether they find the task challenging or whether they will need the teacher to provide scaffolding to close the task. In this case, the scaffolding the teacher (or a peer) can provide will be in the form of explaining to the student just how to use the table to determine the growth and starting value.

The third way I coded tasks for complexity, based on Yeo's (2017a) framework, is as open and task inherent. Tasks that are inherently complex require students to generalize. Figure 3.12 gives an example.

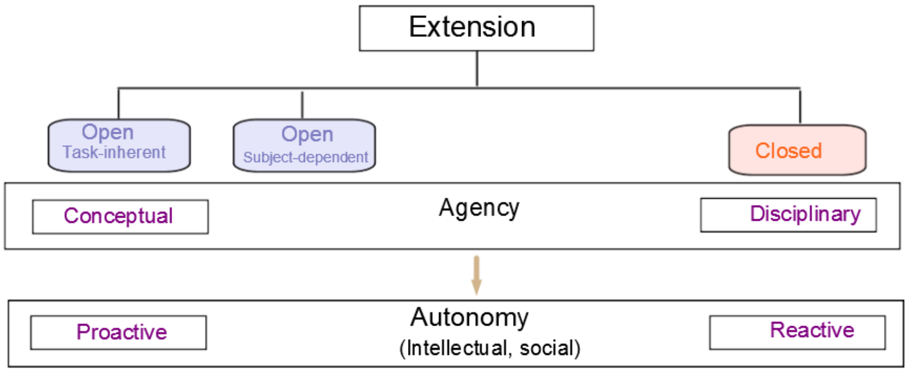
Figure 3.12: Example CPM task with open and task inherent complexity variable

"Now multiply the functions and graph the product $y = f(x) \times g(x)$. How well did your team predict the result? Predict what the resulting graph looks like if you multiply any two linear functions. Can you think of any exceptions?"

For the task in Figure 3.12, students are required to predict what happens when any two linear functions are multiplied. By its nature, the task is complex, because it involves coming up with a general rule. A teacher or a peer can provide some scaffolding for a student who needs it, such as helping pick the linear functions to investigate. However the task is inherently complex, according to Yeo (2017a) because such scaffolding will not close the task. There are other selections of linear functions that may also work. So for a task to be coded as open and task inherent, the task must be complex by itself, so that providing scaffolding will not close the task.

Extension: the extension task variable involves opportunities in the task for students to explore the boundaries of the task as they work toward understanding the conditions around how a mathematical pattern generalizes.

Figure 3.13: Open and closed states for the extension task feature



In the process, students may pose problems that explore alternative situations to the one they are currently working on and test possible counterexamples to the pattern. Given the nature of the task, students may or may not be able to engage in such task extension. Figure 3.13 shows various open and closed states that pertain to the extension task variable. According to Yeo (2017a), task extension is closed if an attempt to extend the task by posing alternative scenarios only leads to a new task. The closed case tends to occur with simple procedural tasks or in tasks where an explicit directive to generalize is absent. As an example of a task for which attempting to extend will lead to a new task, consider the task in Figure 3.10.

In that task, the mathematical object of interest is the equation $x^2 = 2$. Considering any other equation will mean dealing with a different problem. For instance, considering $x^2 = 3$ gives a new mathematical object and as such a new problem. I would thus code tasks of the kind above as closed for task extension.

When the task is open and student-dependent, extending the task can happen if the task itself does not ask students to generalize and if posing alternative scenarios will not create a new task. It entirely depends on the student whether or not to try out alternative scenarios. One type of task that allows for these situations is what Yeo (2017a) terms a “problem solving task”. An example of this task is discussed in the next section.

On the other hand, consider the task in Figure 3.5. For that task, students will have to explore different mathematical objects and scenarios. In order to make a meaningful prediction, students must consider other linear functions different from the pair they start with. They can get linear functions from other students in class or they can come up with some on their own. The task also prompts students to consider counterexamples or exceptions. In order to make that consideration, students will have to pose questions and scenarios that test the boundaries of what happens when any two linear functions are added. So in this task, the statement itself gives explicit direction for students to explore and extend the original conditions they began with. I therefore code tasks of this kind as open and task-inherent, as they involve generalization.

Task types based on task features: In order to classify different types of tasks, I adopted a syncretic approach. By this, I mean that I combined task classifications from the Mathematical Task Framework (Stein et al., 2000) and from Yeo (2017a). I adopted this approach because neither classification system offered enough categories to cover the variety of tasks I encountered during coding. The Mathematical Task Framework (hereafter, MTF) has four task classifications: memorization, procedures without connections, procedures with connections and "doing mathematics". These task categories are useful. The "doing mathematics" category is however overly broad. Within the "doing mathematics" category, there can be several subcategories of tasks. This is precisely where Yeo (2017a) complements the MTF. Yeo (2017a) gives four types of tasks: procedural tasks, problem-solving tasks, investigative tasks and real-life tasks. The latter three categories are tasks of higher cognitive demand that fall under MTF's "doing mathematics" category.

On the other hand, Yeo (2017a) does not distinguish between procedural tasks that make connections to concepts and those that do not. Instead, Yeo (2017a) has just one generic category for procedural tasks. Therefore the syncretic approach allows for more specific categories across tasks of lower and higher cognitive demand. The combined categories comprise two from the MTF (procedures without connections, and procedures with connections), and three from Yeo (2017a) (investigative, problem-solving and real-life tasks). During coding, I encountered almost no tasks that fell under the "memorization" category, so I exclude it from the new combination. I also include new categories of tasks that emerged during coding. The new categories of tasks are conceptual explanations, guided investigative, guided real life and synthesis tasks. These four new categories are all challenging tasks that fall under the “doing mathematics” category of the MTF. Their properties, based on open and closed task features are shown in Figure 3.13.

Figure 3.13 shows the task classification system I used for analyzing and classifying tasks. Tasks classified as “procedures without connections” require students to execute standard procedures in order to generate solutions. As such, they can only exercise disciplinary agency when working on such tasks. Procedures with connections tasks not only involve working with standard mathematical procedures. They also require students to make connections with underlying mathematical concepts. In making these connections, they may have opportunities to exercise intellectual autonomy in how they express their solutions. Conceptual explanations tasks are those where students need not execute or enact an algorithm or a procedure to solve the task. They are primarily asked and required to explain concepts. Guided investigative tasks are those where students set up to carry out an investigative task where the method for the task is scaffolded to provide direction on how to go about the task. A guided real life task also has the method variable scaffolded so as to provide students direction to work on a real-life task. Synthesis tasks require students to summarize or integrate the learning they have acquired up to that point by carrying out an activity that gives students the opportunity to present a solution drawing on earlier learning. Problem-solving tasks are those that cannot be solved only with procedures but that require the use of some problem heuristics. Real-life tasks give students the opportunity to model real life situations with mathematics. For investigative tasks, students need to investigate underlying mathematical patterns and structures. These task types with the states (closed or opened) of their task features are shown in Table 3.14. In this table, I kept the color codes identical to those shown in earlier figures. The color red in Table 3.14 stands for instances where task features are closed, whereas blue stands for instances where task features are open.

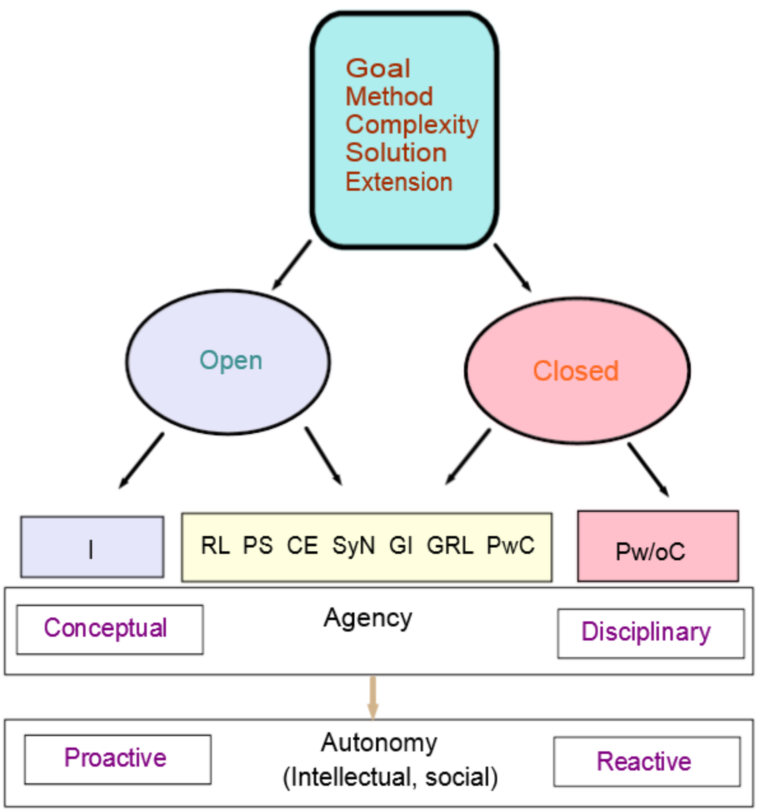
Figure 3.14: Degrees of openness of tasks based on open or closed task features

	Goal	Method	Complexity	Solution	Extension
Procedures without connections (Pw/oC)					
Procedures with connections (PwC)				(Sometimes)	
Guided Real-Life (GRL)					
Guided investigative (GI)					
Synthesis (SyN)					
Conceptual Explanations (CE)					
Problem-Solving (PS)					
Real-Life (RL)	(Sometimes)				
Investigative (I)	(Sometimes)				

This rich variety of tasks appearing in the two textbooks offer a range of opportunities for exercising agency and autonomy as they work on the tasks. It is important to point out that the tasks in Figure 3.14 are in a hierarchy of degrees of openness. In other words, procedures with connections tasks have a higher degree of openness than procedures without connections tasks, because the solution task feature can be open in the former and not in the latter. Likewise, guided investigative tasks have a higher degree of openness than procedures with connections tasks, because the extension task feature can be open in the former and not in the latter. Overall, procedures without connections tasks having the lowest degree of openness whereas investigative tasks having the highest degree. Each of these tasks gives students opportunities to develop different forms and extents of agency and autonomy. For instance, procedures without connections tasks give students opportunity to develop disciplinary agency. Procedures with connections tasks afford students the opportunity to develop disciplinary and conceptual agency as well as the reactive case of intellectual autonomy. This is due to the open nature of the solution variable, which allows students to explain their thinking. It is also due to the open nature of the complexity variable, which means that students will have to make connections to figure out some of the gaps in the method needed to execute procedures. Guided investigative tasks allow students to develop conceptual agency and reactive autonomy. And so on. All the tasks give students different opportunities.

Coding task types based on task features: When coding tasks features to determine task types, I first determined whether the task feature was closed or open.

Figure 3.15: Overview of task types and opportunities for agency and autonomy



The closed case was the straightforward case, because it only had one option. If open, I coded the task feature based on the open categories pertaining to it. Once all the task features were assigned a closed or open code, it was then possible to categorize the task as one of the nine in the task classification system shown in Figure 3.14. Where the task had subsections, I coded those subsections as closed or open in terms of each of the five task features. I then selected the most open state for each task feature across the entire task to decide on task type. As already discussed, each task type offers students opportunities to develop agency and autonomy based on which task features it has open. This is what is captured in Figure 3.15. Figure 3.15 shows the coding process, beginning with task features and ending with task types and the opportunities they offer for agency and autonomy. In figure 3.15, the five task features are shown in the green oval at the top. Two arrows point from the green oval to two more ovals, one with the word "open" and the other with the word "closed". These two arrows represent the coding process for deciding for a given task whether each task feature is open or closed (see Appendix 1). Once the open or closed state of each task feature is determined, the task type can be determined. Thus the arrows from the ovals with the words "open" and "closed" to the rectangular boxes with acronyms such as "I", "Pw/oC" and "GIT" represent the process of determining task types after coding task features. Full names for the task types represented in acronym form can be found on the left hand side of Figure 3.14. The rectangular box beneath the task types shows the different extents of agency that each task type can give students opportunities to develop. The idea is more open task types afford students the opportunity to develop more conceptual agency. Closed task types give students the chance to only develop disciplinary agency. The second rectangular box beneath the one for agency shows extents of autonomy that students can experience based on their experiences of agency. When students have the chance to experience more conceptual agency, on tasks with more open features, they can in turn be more proactive in their experience of intellectual autonomy. Conversely, when they experience less conceptual agency on tasks with fewer open features, they can experience reactive autonomy. To put numbers on the demarcation between proactive and reactive cases of intellectual autonomy, when students have control over four or five task features (i.e. when the task has a high degree of openness), students can experience proactive autonomy. Otherwise they can experience reactive autonomy if the task has one, two or three open features.

I shall now explain how the process shown in Figure 3.15 functions in practice. In order to do so, I shall code all five features of selected tasks in the different task categories of Figure 3.14 to show why I coded the features as such, leading to categorizing a task as a specific kind.

Procedures without connections task: For such a task, we can refer back to Figure 3.2. For the task in Figure 3.2, the goal is two-fold and is stated in the first two sentences. Students are first asked to write recursive formulas for each arithmetic sequence. They are also asked to find the value of the 9^{th} term in the sequence. The goal variable is closed because it is stated clearly in the task what students are meant to accomplish in order to solve the task. The method variable is closed because students are expected to use the recursive formula to generate the 9^{th} term for each sequence. The complexity variable is closed because there are no "gaps" in the task for students to fill. They are given all the information. Provided students know how to work with recursive formulas, they need not interpret the task to discover hidden conditions or to figure out aspects of the method. The solution variable is closed because the solution consists of a single mathematical object, a number representing the 9^{th} term. The extension variable is closed because were students to extend this task by for instance examining a different arithmetic sequence, they would be solving a different task. There is no need to generalize in this task so there is no need to examine special or outlier conditions or other example sequences. Because all task features are closed, the task in Figure 3.2 will be classified as a "procedures without connections" task.

Procedures with connections task: We can refer back to Figure 3.11 for such a task. For this task, the goal is clearly for students to answer questions about Pattern C. This can be seen in the introductory description of the task "The growth of tile Pattern C is represent by the equation $y = 3x + 1$". The goal can also been seen in the sub-questions a, b, c, and d, all of which refer to Pattern C. So the goal is closed, because students are directed to carry out specific actions and to answer specific questions relating to Pattern C. The method variable is also closed, because students are directed to complete the table by substituting values into the equation $y = 3x + 1$. The complexity variable is open and student-dependent in the sense that the task does not give students all the steps they need to solve the task. Instead, the textbook authors chose to pose a question to students concerning how they can use the table to determine the growth and starting value. In order to answer this question, students will have to think about what the values in the table represent and how those values relate to the equation $y = 3x + 1$. In so doing, students are connecting procedures, that is, substituting values into an equation to populate a table, with concepts. The concepts entail understanding what the values in the table represent in relation to the equation and to Pattern C. The solution variable enables students to express their understanding of these relationships in their own words and thoughts. The solution variable also gives students the opportunity to makes connections between two mathematical objects and representations: the table and the equation. So the solution variable is open in the sense that students can respond in all kinds of ways, however the responses are well-defined because they are about specific mathematical objects and patterns. Students' solutions either have correctness or incorrectness as they can be judged based on the wider knowledge of mathematics. The extension variable is closed because the task is about a specific pattern and its representation in equation and table form. Any attempt to pose a new situation will result in a new task.

Guided real-life task: Figure 3.16 shows such a task. The goal is closed because students are positioned to analyze given data in order to advise the city manager on whether each home should be given a dumpster for yard waste. The method variable is also closed because students are also because students are guided to construct a box plot with particular dimensions.

Figure 3.16: Example of a guided real-life task

The city of Waynesboro is trying to decide whether to provide each home a dumpster for yard waste.

To be cost effective, the city needs to collect a sufficient amount of yard waste each week. To measure the effectiveness, the city manager chooses a random sample of 25 homes and provides them with a dumpster. After one week the following amounts of yard waste (in pounds) are collected from the dumpsters at the 25 homes.

8.9	2.9	11.2	15.1	0	2.6	1.7	10.1	36.5
5.1	4.4	6.4	0	22.2	13.5	0	0	21.4
9.7	4.2	17.7	16.3	22.0	8.0	5.6	*Checksum 245.5*	

a. Create a combination boxplot and histogram for the data. Use a bin width of 6 pounds.

b. As you learned in the Math Notes box in Lesson 1.2.2, the standard deviation for a sample is calculated slightly differently than the standard deviation for a population. The population standard deviation is often represented by the symbol $\boldsymbol{\sigma}$, while the sample standard deviation is represented by $\boldsymbol{s}$.

What are the mean and sample standard deviation for the data? Use the sample standard deviation reported by your calculator, which is abbreviated S_x.

c. The city can sell the compost they create from the collection of yard waste, and engineers estimate the program will be profitable if each home averages at least 9 pounds of material. The city manager sees that the mean is nearly 10 pounds and she is ready to order dumpsters for every residence. What advice would you give her? Include a complete analysis of the data set to justify your recommendations, including a description of the center, shape, spread, and outliers.

They are also directed to find the mean and standard deviation, and how to do this with their calculators. The complexity is open and for this task, the complexity relates particularly to the way students will use information in part ‘c’ of the task to formulate a solution to the dumpster problem. The complexity is open and student-dependent. After working out measures of spread (mean, standard deviation) it is up to students to make meaning of the analyzed data in the context of the problem the city of Waynesboro faces, and to subsequently synthesize the rest of the information given in the task in order to come up with a solution in the form of a recommendation. This takes us to the solution variable. The solution is open and ill-defined because it will be left to the teacher or whoever appraises the solution to decide what constitutes a “correct solution” or an “incorrect solution”, based on how students argue their case. Because the dumpster problem is a practical one, students can argue in all kinds of plausible or implausible ways, backed by data or not, to make their points as to what they think should be the best advice for the city manager to follow. The task extension variable is closed because this task is about the dumpster problem in Waynesboro. It is one particular situation. The task does not require students to generalize their advice to cater to any other situation apart from the on in Waynesboro. To conceptualize any other situation will be to work on a different task. Thus based on the closed and open variables for this task, I classify it and others like it as a guided real-life task.

Guided Investigative tasks: For this task type, we refer to Figure 3.17. In this task, the goal is closed because students are instructed to investigate a specific pattern. The second sentence in the task states that students are supposed to work specifically on questions relating to pattern A. The task method is also closed because the manner of investigation is laid out for students in the task. The textbook authors guide students in a step-by-step fashion to consider and sketch Figure 0, to sketch Figure 4, to consider where growth is occurring on the task by how much. The method given to students is for them to figure out growth patterns for small cases so that they can then extrapolate their findings to predict larger cases such as for the 100^{th} growth tile and to generalize for any growth pattern number.

Figure 3.17: Example of a guided investigative task

TILE PATTERN INVESTIGATION

Obtain a Lesson 2.1.1B Resource Page and find Pattern A, shown below. Complete the following tasks for Pattern A, recording your work on the resource page or on your paper as appropriate. (Do not consider Patterns B or D yet.)

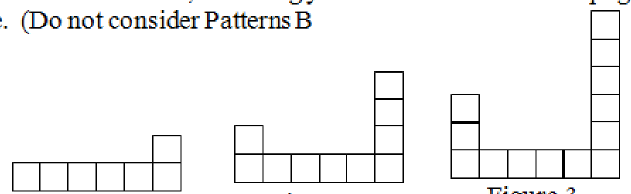


Figure 1 Figure 2 Figure 3

a. What do you notice about Pattern A? After everyone has had a moment on his or her own to examine the figures, discuss what you see with your team.

b. Sketch the next figure in the sequence (Figure 4) for Pattern A on your resource page. Figure 0 is the name of the figure that comes before Figure 1. Sketch Figure 0.

c. By how much is Pattern A growing? Where are the tiles being added with each new figure? Color in the new tiles in each figure with a marker or colored pencil on your resource page.

d. What would Figure 100 look like for Pattern A? Describe it in words. How many tiles would be in the 100th figure? Find as many ways as you can to justify your conclusion. Be prepared to report back to the class with your team's findings and methods.

e. Assume the **starting value** of any tile pattern is the number of tiles in Figure 0. Assume the **growth** is the number of tiles that are added from one figure to the next. What are the growth and starting value for Pattern A?

f. Write an equation that relates the figure number, x, to the number of tiles, y.

For this task, the solution variable is open and well-defined. Based on their investigations regarding Figures 0, 1, 2, 3, 4 and perhaps others, students may be able to discover the pattern of growth, so that they can predict any figure number, including the 100th growth pattern, which the task asks students to predict. If students are successful in discovering the pattern of growth, they would be able to express it in different ways in their solutions, without having to draw it out. The solution variable allows students to demonstrate their success in using the method given, and the connections they made. The extension task variable is also open in this case, because of the elements of prediction and generalization. In order to predict the 100th pattern, students may need to draw other figures beyond the first four. This action will still be within the scope of the work required for this task in the sense that examining Figure 8 will not generate a new task. Students can thus extend the original conditions given in the task (i.e. Figures 1, 2, and 3) in order to arrive at generalization. They may be able to express the generalization algebraically, using an equation that relates the figure number x to the number of tiles. They may also be able to express their understanding of the generalization through words or figures or a combination of all three. So this task gives students opportunities to express the solution and extension aspects of the task by connecting different mathematical objects and representations. The complexity variable is open and subject dependent. This is because it will depend on students' own abilities how complex the investigation into the growth pattern is for them. Some students may be able to discern the pattern after trying out a few cases. Others may need more. The mathematical sophistication of their understanding of the pattern will also vary by students. Some students may be able to express the pattern algebraically, as is required by section 'f' of the task.

Synthesis tasks: For a task of this kind, we shall refer to Figure 3.18. The task in this figure is related to the one in Figure 3.17, so it will provide a connection to the discussion so far. The task in Figure 3.18 has a closed goal. The goal is to outline information needed to predict the 100th figure for a new pattern students were not aware of. This task essentially asking students to summarize what they have learned while working on Patterns A, B, and C to come up with a general method that will work for all patterns, even those they have not come across.

Figure 3.18: Example of a synthesis task

Look back at the growth of Patterns A, B, and C. Imagine that the team next to you created a brand new tile pattern, but they refused to show the pattern to you. What information would you need in order to predict the number of tiles in Figure 100? Explain your reasoning.

For this task, the method variable is closed, because students will base their method on the methods they used to investigate Patterns A, B, and C. What will be interesting is the solution variable, which is open and well-defined. The solution variable will consist of students' ideas regarding what is common with Patterns A, B and C that can generalize across all investigations of like patterns. As to whether or not students can succeed in drawing such links and generalizing across patterns will depend on their own abilities, so the extension variable too is open, but student-dependent. The complexity variable too is open and student-dependent. It depends on students' abilities, on their knowledge and understanding after working on Patterns A, B, and C how complex this synthesis task will be for them.

Conceptual explanations tasks: For this task type, we shall refer to Figure 3.19. In this task, students must first explain minimum and maximum pointes of the vertex of a parabola. They must then compare two quadratic equations to note the differences between them. The two purposes serve as the goal of the task. The goal is thus closed. The method of the task is open and well-defined. When the vertex of a parabola is at a minimum or at a maximum is well established in theory. Students' contribution in answering the question is in *how* they choose to explain that theory. They may choose to explain it by using a diagram and explaining that diagram in words. They may choose to simply state the mathematical fact.

Figure 3.19: Example of a conceptual explanations task

Do you understand?

- **Vocabulary** When is the vertex of a parabola the minimum point? When is it the maximum point?
- **Compare and Contrast** How are the graphs of $y = -\frac{1}{2}x^2$ and $y = -\frac{1}{2}x^2 + 1$ similar? How are they different?

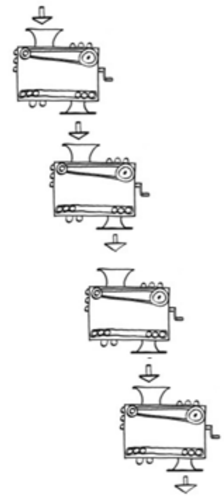
Alternatively, they may explain it with algebra, by working from the standard form of a quadratic to the vertex form or by drawing diagrams of maximum and minimum and explaining those diagrams. They may instantiate standard formulas with numerical examples to show. In short, it is up to students what method they will use to explain this aspect of standard theory on quadratic curves, but the methods they select will be grounded in mathematics theory, hence well defined. Likewise when explaining the difference between $y = -\frac{1}{2}x^2$ and $y = -\frac{1}{2}x^2 - 1$, students will be able to make up their own method to support their explanation. The complexity variable is open and subject dependent as the methods and explanations students give will depend on their individual abilities and knowledge. The solution variable is open and well-defined because just as with the method variable, students' solutions will consist of explanations of mathematics theory. This theory has well defined notions. Explanations of notions such as when the vertex of a parabola is minimum or maximum or what makes two quadratic equations different are mathematically correct or mathematically incorrect. So, students' solutions are well defined. The extension variable in this case is closed, because the task does not require students to make generalizations. Instead, the task focuses on specific explanations about specific mathematical objects. In the case of the parabola, the task does not ask students to then figure out when all curves are minimum and maximum, a situation which may be explained with calculus theory. The task also does not ask students to explain differences between all quadratic equations. So the conceptual explanations are for the particular mathematical objects given in the task.

Problem solving tasks: For the task in Figure 3.20, the goal is closed because the task states that students must discover how four function machines produce particular outputs when arranged in a specific order. With respect to method, students will have to work out how order the four functions $f(x)$, $g(x)$, $h(x)$ and $k(x)$ such that starting with the initial value for the first function, and with the output of one function serving as the input of another, the stacked function machines will produce the required outputs stated in parts ‘a’ and ‘b’ of the task.

Figure 3.20: Example of a problem solving task

FUNCTION MACHINES

Obtain a set of four function machines (Lesson 1.1.1C Resource Pages). Your team’s job is to stack the machines in a particular order so that one machine’s output becomes the next machine’s input. As you work, discuss what you know about the kind of output each function produces to help you arrange the machines in the order that produces each result described below. The four functions are reprinted below.



$f(x) = \sqrt{x}$ $\qquad$ $g(x) = -(x-2)^2$

$h(x) = 2^x - 7$ $\qquad$ $k(x) = -\frac{x}{2} - 1$

a. In what order should you stack the machines so that when 6 is dropped into the first machine, and all four machines have had their effect, the last machine’s output is 11?

b. What order will result in a final output of 131,065 when the initial input is 64?

With respect to method, students will have to work out how order the four functions $f(x)$, $g(x)$, $h(x)$ and $k(x)$ such that starting with the initial value for the first function, and with the output of one function serving as the input of another, the stacked function machines will produce the required outputs stated in parts ‘a’ and ‘b’ of the task. In order to obtain those results, students will have to work out the order of stacking the functions. They can do that by taking into account the properties of each function to determine how to stack them. So the method that students use to get their solution is open and well defined. The complexity of the task is open and student-dependent. This is because the task has “gaps” in the sense that it does not give students all the intermediate steps to solve it. Students may for instance have to work out properties of the four functions in order to know how to stack them in ways that will give them the desired outputs. Their success in this endeavor depends for instance on their knowledge and their mathematical ability. The task solution variable is however closed, because the solution consists of the stacking that produces the desired output. There is no need to explain the process that led to the solution. There is only a need to achieve it. The extension task variable is open but student dependent. This is because as students work out the stacking order, they are at liberty to test theories by stacking the function machines in different orders based on their understanding of the functions to find out whether or not they will attain particular outputs. Students can test two machines or even three machines at a time to support their understanding. So they can extend the task in many different ways that help them understand how to eventually arrive at the solutions demanded in the task.

Real-life tasks: The example I give in this section comes from Yeo (2017a), because both textbook series lacked complex tasks that had such little detail to guide students. According to Yeo (2017a), an example of a real-life task is the following, "Choose any mathematics project to do. Submit a report at the end of the year." (p. 181). This task is open in terms of goal. The goal, method, complexity and solution of the task all depend on the mathematics project students decide on. Because the solution is a specific artifact (a report), the task extension variable is closed.

Investigative tasks: For this kind of task, we will refer to Figure 3.5. The goal for the task in Figure 3.5 is closed because students are instructed first to investigate graphs of given functions and then to make a prediction for what happens when any two linear functions are added. The method variable is open and well-defined because the task instructs students to make their predictions based on addition and subtraction of two linear functions. Students are thus expected to explore what happens when two linear functions are added or subtracted. The task however does not tell students how to carry out the investigation. Students will have to figure that out by themselves. The solution variable is open and well-defined because in order to make predictions about adding or subtracting any two linear functions, students' solutions will likely include notions about addition and subtraction and about linear functions. Because their solutions will be a mathematical prediction, students' solutions can be judged as being correct or incorrect based the generally known mathematics theory concerning arithmetic operations and linear functions. The extension variable is also open because the task specifically asks students both to make a prediction and to consider exceptions. In both cases, students will have to consider different cases from the ones they are originally given. So the task by necessity has to be extended.

The Linguistic Perspective

For analysis via the linguistic perspective, I draw on tools from functional grammar, also known as Systemic Functional Linguistics (SFL). SFL is a linguistic theory of language in social context that enables us to study meaning-making in the use of language. SFL tools for discourse analysis can be used to examine language used to make sense of objects and phenomena experienced in the internal world of the mind or in the external world of observable reality. SFL gives researchers tools to examine language used when people interact and to make sense of the relationships formed during those interactions. Finally, SFL considers sense-making in terms of the use of language to communicate in spoken or written form. In making sense of language use, the unit of analysis is 'text' of some kind (Zolkower & Shreyer, 2007). According to Halliday and Matthiessen (2014), "When people speak or write, they produce **text**; and text is what listeners and readers engage with and interpret; The term 'text' refers to any instance of language" (p. 3). As such, 'text' refers to spoken or written language that can communicate meaning. SFL can be a powerful tool with which to learn about how use of language potentially influences mathematics teaching and learning situations (Morgan, 2006).

For this dissertation, the particular texts I shall analyze are lesson texts found in PBL mathematics textbooks. These lesson texts consist of written information that textbook authors expect readers, in this case students, to consider and act upon. Readers are primarily students and teachers as well as others such as researchers. In the sections that follow, I shall first draw on text categories from SFL theory to describe the manner in which CPM and PI curriculum text are organized to communicate information or interact with students. Presenting how curriculum texts are organized will help give context to the important features of text I focus on for analysis. I shall then present concepts from SFL that help me detail the manner in which textual analysis at different grain sizes can help reveal how textbook authors communicate and interact with students. I shall finally explain how I plan to compare CPM and PI selected lesson texts across all textbooks to reveal ways that texts position students to experience agency and autonomy.

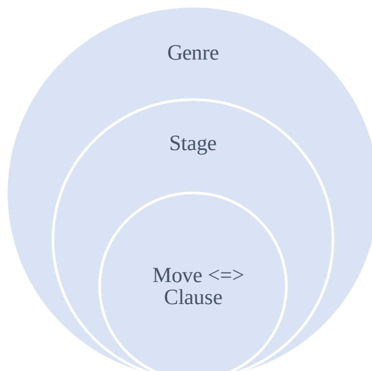
Organization of CPM texts: For data analysis, I shall draw on the notions of curriculum genres (Christie, 1991) to organize the texts found in a given textbook chapter and a given lesson. Curriculum genres can be thought of as a means of categorizing texts in terms of their social purposes. For my dissertation, I am considering each chapter of a given CPM or PI textbook as a curriculum genre. The social purpose for this curriculum genre is learning a particular topic of mathematics. As chapters are composed of lesson sections, the chapter as curriculum genre comprises a collection of lessons as different curriculum stages. Each curriculum stage progresses the learning purpose of the chapter. CPM textbook chapters for instance have a chapter opening section, a chapter lessons section comprising several lessons, and a chapter closure section. So the CPM chapter genre has three stages within it. PI textbook chapters on the other hand have the following stages within a given chapter: "Get Ready", a diagnostic test at the start of the chapter; a chapter overview; chapter lessons; a "Lab"; a chapter review, and finally "putting it all together", which consists of assessments at the end of the chapter.

In the same way that chapter genres have stages, each day's lesson text can also be thought of as a genre onto itself. When the day's lesson text is thought of as a genre, the chapter that comprises all the lesson texts can then be thought of as the macrogenre. The lesson genre also has stages through which the lesson progresses. These stages are the main sections in which the CPM lesson text is organized for work done during class time: orienting text, classwork, Math Notes. PI is organized around the following lesson structure for work done during class time: interactive learning, Guided instruction, lesson check, assess and remediate.

Apart from examining the stages in a given lesson text, communication from textbook authors to readers can also be examined at the level of a clause. A clause is a part of a sentence organized around a verb. When two or more clauses are linked together, they form a clause complex (Thompson, 2013; Halliday & Matthiessen, 2014). Often but not always, a single sentence with two or more clauses is an example of a clause complex. For example, the sentence, "When Dorothea came to my house she had some pudding as she conducted research on the linguistic features of noun phrases" has three clauses "when Dorothea came to my house", "she had some pudding", and "as she conducted research on the linguistic features of noun phrases". Clauses and clause complexes are important in my dissertation because they are the main grammatical structures to which I shall apply SFL analysis.

For this dissertation, clauses and clause complexes encapsulate the language moves that textbook authors make as they communicate information or interact with students. That is, a move consists of a clause or a set of clauses in a sentence. Moves can take the form of giving students information about how to solve a problem, asking students questions to check their understanding or directing students during their work. I am thinking of the textbook authors' moves as being realized through clauses and clause complexes. The diagram below shows the nested nature of curriculum genre, stage and clauses as moves:

Figure 3.21: The nested nature of the categories within a lesson genre



Analyzing lesson texts: The purpose of categorizing a lesson text into stages and moves is to be able to conduct textual analysis at different grainsizes order to draw out meaning in relation to opportunities for students to experience agency and autonomy. Analysis at different grainsizes is similar to an approach taken by Morgan and colleagues (Morgan, 2016; Morgan & Sfard, 2016; Morgan & Tang, 2016). Morgan and Sfard (2016) indicate that in order to analyze examination questions based on certain indicators, they decided to "attach codes to different units of text: individual words, phrases, sentences, sub-questions/tasks or complete questions" (p. 103). In my dissertation, the focus of analysis will be at the grainsize of a clause, which I have selected as my unit of analysis. Analysis at the clause level is meant to support analysis at the stage level. It is at the level of the stages of the lesson text genre that I will make comparisons across lessons.

Language functions in text: SFL analysis involves experiential (sometimes also referred to as ideational), interpersonal and textual functions of text. The experiential function involves the way text communicates ideas and processes going on the world. The interpersonal function involves how text communicates interactions between individuals. The textual function involves the way text is organized. In my analysis of lesson texts, I shall draw primarily on analysis of the interpersonal function. Analyzing the interpersonal function in lesson texts can help make apparent the power relations in the communications and interactions between textbook authors and students (Eggins and Slade, 1997). These power relations involve whether or not the textbook authors grant students opportunities to exercise agency and autonomy in the extent to which they give students choices to take control of learning. With the interpersonal function, I shall draw on the notions of mood to show how textbook authors communicate information or interact with the reader through clause moods. This is one aspect of interpersonal meaning I will investigate with the linguistic perspective. The other aspect is modality, which I shall discuss further on in this section.

Moves and mood functions: For this dissertation, I categorize moves textbook authors make to communicate information and to enact relationships through language. The information communicated is in the form of clauses and clause complexes. Thus the moves that textbook authors make depend on the kinds of clauses and clause complexes they draw on. In order to analyze clauses and clause complexes as moves, I shall analyze the mood of each clause or clause complex. The mood of a clause or clause complex refers to whether it is declarative (which usually functions as a statement, in speech or written text), interrogative (which usually functions as a question) or imperative (which usually functions as a command). As textbook authors communicate information and interact with students, they may do so either by stating information for students to think about or use, by posing questions to students that demand that they provide a response (often in written answer form) or by directing students to act or think in particular ways, especially as students work on mathematical tasks. Table 3.2 below lists the three moods with a few examples.

Table 3.2: Mood and congruent speech functions with examples

Mood function (grammar)	**Code used in textual analysis**	**Congruent Speech function**	**Examples**
Declarative	<d>	Statement	A function is given a name, that can be a letter, such as *f* or *g*.
Imperative	<im>	Command	Examine the input (*x*) and output (*y*) values in the table below.
Interrogative	<i>	Question	Is there a relationship between the input and output values?

I should point out that although clauses in the three moods are associated with congruent speech functions, there can be cases where a clause may be associated with an incongruent mood. Mood and speech functions are said to be congruent when the mood of a clause as determined by grammar matches up with what the clause is used for, as reflected in the speech function of the same clause. For example the clause “Is there a relationship between the input and output values?” from Table 3.2 above has an interrogative mood with question as its congruent speech function. There are however also cases of spoken English where there is a mismatch between the mood of a major clause and the corresponding usual speech function that matches up with the mood. These are instances of incongruent (Eggins & Slade, 1997), or non-congruent (Zolkower & Shreyar, 2007) alignment of mood and speech function. For example, a clause that ordinarily would function as a question, presented in the interrogative mood in the congruent case could function as a command (typically presented in the imperative mood) instead. The following example shows the incongruent case: “Can you indicate in your answer all the various ways that you think the problem can be solved?” In this example, although the clause complex ends with a question mark, and one would think that the student could simply answer with a “yes”, what students are really being directed to is to come up with all the methods they can think of for the mathematical task. In essence, they are being directed or commanded to produce a certain solution, for which only answering the question in the affirmative is not sufficient. Speaking in these kinds of ways where a clause functions in an incongruent mood reflects some common ways that English is spoken in this (Western) culture. Such ways of communicating and interacting that involve incongruent mood-speech function associations can appear in the textbook authors’ communications and interactions with students. In this study, the majority of clauses are congruent with respect to mood and speech function. In the few exceptions where this is not the case I use the speech *function* as the determining factor for assigning a role in regard to agency and autonomy.

For this dissertation, I analyze clauses in the declarative and imperative mood functions to investigate how textbook authors are communicating information to students and/or enacting relationships and how they demand specific kinds of actions of students in ways that may position them to exercise agency and autonomy. Depending on how students are given information and how their actions are directed when solving tasks, students can follow more standard ways of thinking about mathematics theory and executing procedures or be more engaged to offer their own thinking and justifications in pursuit of understanding and making meaning of mathematics they learn. In analyzing clauses to determine textbook authors’ positioning, I shall look for the degree to which they allow students to have choices in how students go about their work, when communicating to or interacting with students. Clauses in the declarative mood typically give students information that could be theory or fact about mathematics. Depending on the context of clauses in the declarative mood, students can use information in them to exercise either conceptual or disciplinary agency. With respect to clauses in the declarative mood, I focus on another feature of clauses that presents interpersonal meaning: modality (Hallidan & Matthiessen, 2014). Modality refers to the degree of obligation or probability in a clause. Modality has two aspects: modalization and modulation. Briefly, modalization refers to the level of probability indicated in a clause. This can be judged by the presence of words in a clause such as “may” or “might” that lessen the authoritative nature of the clause. For example, a clause may read, “You may want to explore the function using a graph or with another method”. In this clause, students are positioned to choose what they intend to consider in order to explore a function. They may thus have a chance to exercise conceptual agency and intellectual autonomy in the process. Modulation on the other hand presents meaning related to obligation in the clause. Words such as “must” or “should” present *high* obligation. A clause that reads “you must check your solution by inserting the roots of the equation into the function” communicates high obligation on the part of students. On the other hand, words such as “can” indicate *low* obligation. This can be shown in a sentence such as “you can use a table to help plot the graph.” So there is low obligation. In clauses with modulation, students are directed by the textbook authors. This restricts students’ autonomy and ability to exercise conceptual agency to varying degrees.

Clauses in the imperative moods typically demand action. For example, the imperative clause “Examine the input (x) and output (y) values in the table below” from Table 3.2 is demanding action. For clauses in the imperative mood, I focus on two kinds: inclusive and exclusive imperatives. Inclusive imperatives can position students to exercise conceptual agency and intellectual autonomy in terms of giving students opportunities to think and to offer justifications for their own ideas. They include the use of verbs such as "explain", "justify", and "predict". Exclusive imperatives on the other hand can position students to exercise disciplinary agency and little opportunity for intellectual autonomy. They involve positioning students to carry out standard mathematical procedures, and include verbs such as "find", "solve", and "calculate".

Beyond the clausal level, the analysis of tasks and texts will be separated by the stages that each text is organized into. The stage level of text is shown in Figure 3.22 below.

Figure 3.22: Linguistic perspective analysis at the level of genre, stage and clause

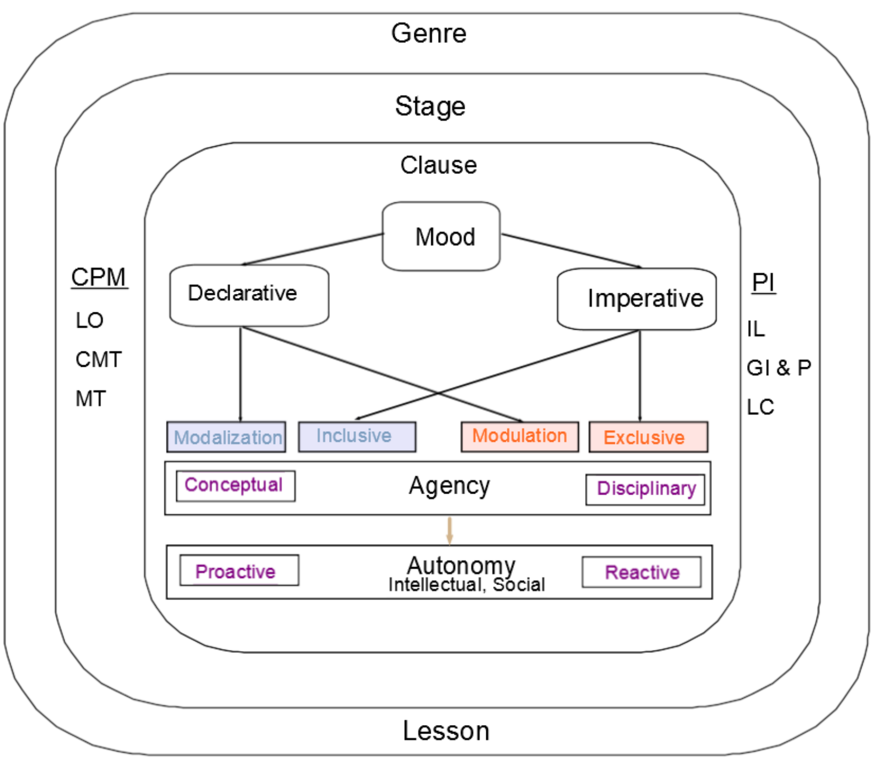


Figure 3.22 encapsulates the analytic methodology of the linguistic perspective. It is an elaboration of Figure 3.21 for each textbook series. The genre level involves the lesson text for a

given lesson. The genre level nests the stage level. These are the divisions of the lesson text. For CPM, lesson texts are divided into three stages: Learning Orientation (LO), Classroom Mathematical Tasks (CMT) and Math Notes (MN). Each of these stages of CPM lesson texts performs a specific function. The LO stage introduces the lesson's goals. The CMT stage consists of mathematical tasks students will work on in class and the MT stage presents mathematics theory for the lesson. Likewise, in PI, the Interactive Learning (IL) stage introduces students to the lesson with a mathematical task. The Guided Instruction and Practice (GI & P) stage of the lesson engages students in solving several more mathematical tasks. The Lesson Check (LC) stage of the lesson tests students' understanding of the lesson's learning with tasks that check students' knowledge of concepts and of procedures learned during the lesson. The stage level nests the clause level in the sense that the text in each stage consists of several clauses. There are two moods for which I analyze clauses in lesson texts. These are the declarative and the imperative moods (described earlier). For the declarative mood, I analyzed instances of modalization and modulation. For the imperative mood, I analyzed instances of exclusive and inclusive imperatives. Together, these can give indications of how language choices position students to develop agency and autonomy.

In order to show how I analyzed lesson texts with the linguistic perspective, I shall use the excerpt of lesson text in Figure 3.23 as an example. In Figure 3.23, the clause "Write a recursive formula for the arithmetic sequence below" is in the imperative mood. The verb "write" in the clause functions as an exclusive imperative, as it directs students to execute a standard mathematical procedure. The standard mathematical procedure in this case is writing a recursive formula for a sequence. The imperative clause "explain" directs students to explain their thinking. "You can find the value of any term of an arithmetic sequence using a recursive formula." is in the declarative mood. The phrase "you can" in this declarative clause indicates an instance of low modulation. In this clause, the textbook authors are positioning students to consider a particular method to help them solve the given tasks.

Figure 3.23: Example of analysis of a task in a PI lesson stage

Guided Instruction and Practice

Got it? Write a recursive formula for the arithmetic sequences below. What is the value of the 9th term?

a. 3, 9, 15, 21, …
b. 23, 35, 47, 59, …
c. 7.3, 7.8, 8.3, 8.8, …
d. 97, 88, 79, 70, …

e. **Reasoning** Is the recursive formula a useful way to find the value of an arithmetic sequence? Explain.

Practice Write a recursive formula for each sequence.

5. 2.3, 2.8, 3.3, 3.8, …
6. 4.6, 4.7, 4.8, 4.9, …

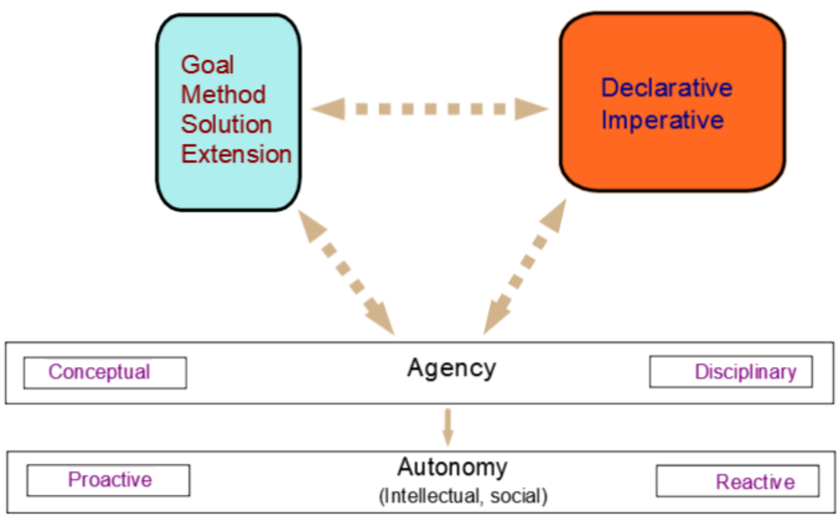
[Note] You can find the value of any term of an arithmetic sequence using a recursive formula. You can also write a sequence using an explicit formula. An explicit formula is a function rule that relates each term of a sequence to the term number.

The Combined Perspective

The cognitive and linguistic perspectives described in the two previous sections detail ways in which analysis of lesson texts can reveal opportunities for students to experience agency and autonomy. For the cognitive perspective, the opportunities stem from the degree of openness of tasks based on open or closed task variables. Task variables such as the solution, when open, offer students the chance to provide justifications for their thinking. In providing justifications, students can exercise their conceptual agency and intellectual autonomy. For the linguistic perspective, the opportunities for exercising agency and autonomy come in the form of linguistic positioning. Linguistic positioning refers to how the textbook authors communicate information and interact with students through the content of the text. The linguistic perspective analysis involved how each textbook series positioned students using clauses in the declarative and in the imperative moods. The declarative clauses examined modalization and modulation while the imperative clauses examined exclusive and inclusive imperatives. I have associated each perspective with an analytic framework (Figures 3.15 and 3.22). With the cognitive perspective, task features determine different categories of tasks. With the linguistic perspective, clause types in lesson texts reveal the means of communication from textbook authors and students. The interaction of cognitive and linguistic perspectives in the combined perspective can reveal how declarative and imperative clauses function through task features such as goal, method, solution and extension variables to position students as they work on tasks.

Figure 3.24 shows a bidirectional arrow connecting task features and clause moods and two other bidirectional arrows, one linking task features with agency and autonomy and another linking clause types with agency and autonomy. The three bidirectional arrows represent relationships. These relationships are manifest in clauses in a task and can inform us about those opportunities for students to develop agency and autonomy. For instance clauses in the declarative mood with modulation position students to consider information pertinent to the task. This information could be relevant to the task goal or the task method. Clauses in the imperative mood direct students to take actions in other to solve tasks. Two typical actions students would take when working on mathematical tasks is to execute methods or to generate solutions. Clauses in the imperative mood can have relevance to what actions related to method and solution that the task demands of students. Depending on what information is given to students and what actions are demanded of them, analysis can reveal opportunities to develop agency and autonomy. For instance, one clause in the imperative mood with an inclusive imperative verb that directs students to explain their thinking can be linked with the solution task variable and shown to support students' development of conceptual agency and intellectual autonomy. Another clause in the declarative mood with modulation that restricts what students can work on in the task may limit students' expression of conceptual agency and intellectual autonomy. This is the essence of analysis for opportunities via this third perspective. As an example, we can consider Problem 3 in Figure 3.23. In this problem, the imperative clause "write a recursive formula for the arithmetic sequences below" can be associated with the method variable of the task. This is because this clause is directing students to carry out actions on mathematical objects, in this case the arithmetic sequences. In order to come up with the recursive formulas, students may have to manipulate these mathematical objects, thus executing a method.

Figure 3.24: Task features, clause types, agency and autonomy



Chapter 4 Findings

Introduction

This study involved the investigation of different mathematical tasks and associated lesson texts found in two PBL textbooks for the opportunities they give students to experience agency and autonomy. I highlight these opportunities through the cognitive and linguistic lenses I adopted for analysis. The analysis of the data that resulted from coding was based on answering the three research questions I posed for this study. The first set of results pertains to the cognitive perspective. They detail opportunities in mathematical tasks CPM and PI. The second set of results pertains to the linguistic perspective. These detail opportunities in both tasks and supporting texts. The third set of results combines both cognitive and linguistic perspectives.

I shall first revisit my research questions and then proceed to stating results specific to each research question. The first research question involves the extent to which a cognitive perspective analysis of mathematical tasks in CPM and PI reveal differences and similarities in opportunities for students to develop agency and autonomy with respect to topics on functions. For the second research question, I investigated the extent to which a linguistic perspective analysis of the same textbook series revealed differences and similarities in opportunities for students to experience agency and autonomy with respect to topics on functions. For the third research question, I probed ways that the interaction of cognitive and linguistic perspectives in the analysis of tasks and texts in CPM and PI revealed differences and similarities in opportunities for students to develop agency and autonomy with respect to topics on functions. I shall now delve into the findings for each section.

Findings from the cognitive perspective

In this section, I first present overall findings comparing the opportunities CPM and PI give students to exercise agency and autonomy as they work on tasks on the topic of functions in each textbook series. I shall next unpack the overall finding to display what those opportunities look like at the task level. At the task level, we can observe distributions of task types in CPM and PI to learn which tasks appear more frequently in students' learning. I shall then present more detailed findings at the level of task features in order to demonstrate how a given task can afford students opportunities to exercise agency and autonomy. I shall also interpret the findings at each level to explain what they mean and why they matter to student learning.

Figure 4.1: Overview of results of analysis from the cognitive perspective

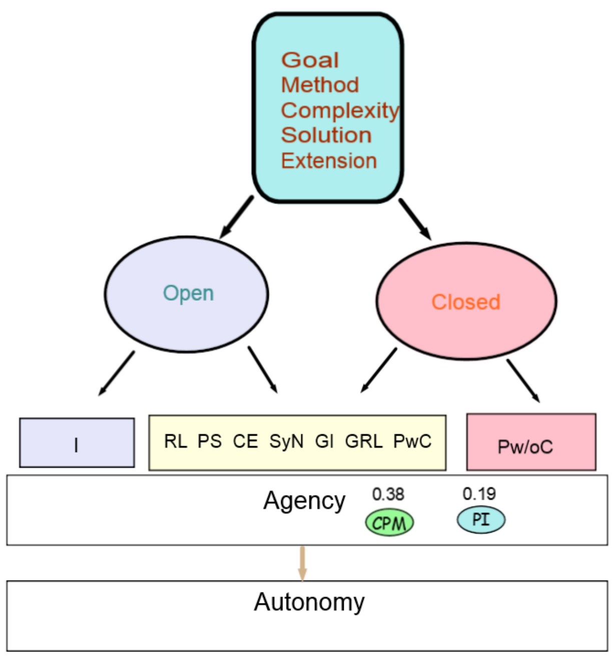


Figure 4.1 shows the overall process of cognitive perspective analysis of mathematical tasks applied to each textbook series. Each task was analyzed based on five task features, which are

task goal, method, complexity, solution and extension. These task features were coded as either open or closed, as shown in Figure 4.1. The aggregate of the coding resulted in classifying the analyzed task as one of nine task types, shown in Figure 4.2. The task types appear as acronyms in Figure 1, however the full names are given in Figure 4.2. For instance, the acronym “PwC” that appears in Figure 1 stands for “procedures with connections”, as shown in Figure 4.2. Figure 4.1 arranges the tasks in order of degrees of openness. The most open tasks are the investigative kind. In acronym form, they are represented on Figure 1 as “I”. For investigative tasks, all features re almost always open. The least open tasks are those classified as procedures without connections. They are represented on Figure 4.1 as “Pw/oC” and they have all task features closed.

Figure 4.2: Degrees of openness of tasks based on open or closed task features

	Goal	Method	Complexity	Solution	Extension
Procedures without connections (Pw/oC)					
Procedures with connections (PwC)				(Sometimes)	
Guided Real-Life (GRL)					
Guided investigative (GI)					
Synthesis (SyN)					
Conceptual Explanations (CE)					
Problem-Solving (PS)					
Real-Life (RL)	(Sometimes)				
Investigative (I)	(Sometimes)				

Figure 4.2 gives details on which task features are open and which are closed for each task type. I represented closed in red and open in blue to conform to the representation of open and closed in Figure 4.1. Based on the task type, students have opportunities to exercise different forms of

agency and autonomy. For example for procedures without connections tasks, which have all task features closed, students can only exercise disciplinary agency. This is because for this category of task, all task features are closed. Students are therefore constrained to using standard methods and representing their answers in standard formats. At the opposite extreme, for investigative tasks which have all task features open, students can exercise extreme cases of conceptual agency and proactive autonomy. In this scenario, students are at liberty to select methods and task constraints in order to determine goal and method of task needed to generate a solution. In the moderate case, for instance with guided investigative and guided real-life tasks, students can experience some conceptual agency and some proactive or reactive autonomy depending on which task feature is open.

Figure 4.1 shows that overall, CPM gave twice the opportunity for students to exercise the forms of agency and autonomy characterized by opportunities for students to draw more on their own thinking. On a scale ranging from 0 to 1, CPM scored 0.38, while PI scored 0.19. In this scale, 0 represents a case where all tasks have all features closed and are all classified as procedures without connections. 1 represents a case where all tasks have all features open and are all classified as investigative tasks. In effect, for each task feature, I assigned 0 if the feature was closed and 1 if the feature was open. The aggregate for the task, depending on the number of open and closed features the task has (as shown in Figure 4.2) gave a weight for the task. For example, guided investigative tasks (see Figure 4.2) 3 features that are open and 2 that are closed. So out of the five features, the score for this category of task is 3 out of 5, which is 0.6. Where a task sometimes had a feature open, such as with procedures with connections tasks, I found the average based on task numbers in either configuration to come up with an average for that task type. This manner of assigning weights to tasks made it possible to take a weighted average of all CPM and PI tasks, in order to arrive at 0.38 for CPM and 0.19 for PI.

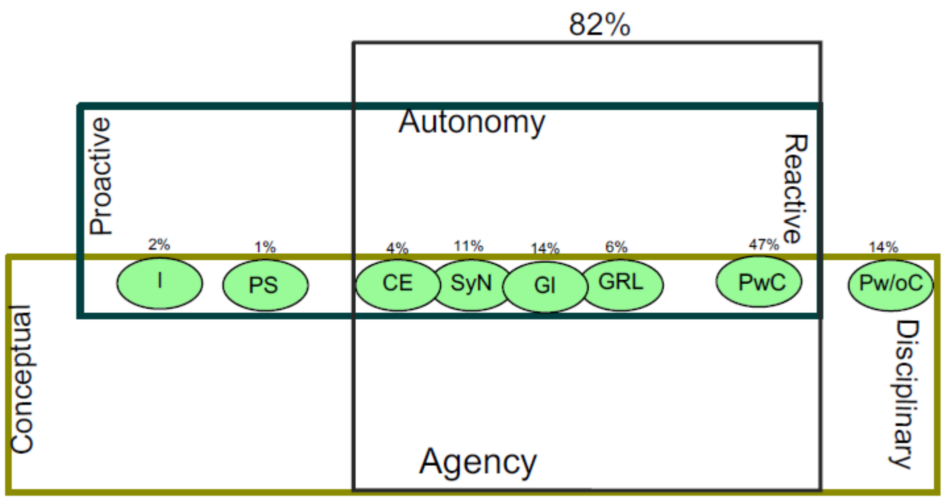
What these two numbers mean, in relation to Figure 4.2, is that the majority of tasks in both CPM and PI are procedural. In this overall regard, CPM is similar to PI. Specifically, both textbook series skew on procedures with connections tasks, with CPM registering more openness and therefore more opportunities for students to exercise conceptual agency and reactive autonomy than PI. With procedures with connections tasks, students will have opportunities to exercise disciplinary agency as well, due to the procedural aspects of the task. In Figure 4.2, tasks classified as procedures with connections are within the range of 0.2 to 0.4 for task weight. In both textbook series however, the cases where the solution variable was closed for the procedures with connections task were in the minority.

This overall numbers 0.38 and 0.19 belie the fact that each textbook series provided students with other opportunities to work on tasks that were *not* largely procedural. There *are* other tasks in both textbook series, of the kinds listed in Figure 4.2. To find out just how many of each kind and why the various other kinds of tasks are important, we need to take a closer look at the distribution of tasks in CPM and PI.

Differences in task Distribution: A closer look at the distribution of tasks within each textbook series as shown in Figures 4.3 and 4.4 tells a different story. Although the percentages of tasks in Figures 3 and 4 together show a greater skew toward procedural tasks (61% for CPM and 92% for PI), we also observe that CPM has a higher percentage of non-procedural tasks (39%) compared to PI's 8%. Additionally, if we include procedures with connections tasks among those tasks that students commonly work on that give them opportunities to exercise conceptual agency and intellectual autonomy, the percentage for CPM goes up from 39% to 85% and up for

PI from 8% to 49%. So not only does CPM offer more frequent opportunities for students to exercise their own thinking, CPM also does so with a wider range of tasks types requiring a higher cognitive demand. Out of the 8 task categories shown in Figure 2 that demand more of students' thinking (i.e. those with some open features), CPM had at 8 in Figure 3, while PI featured 4.

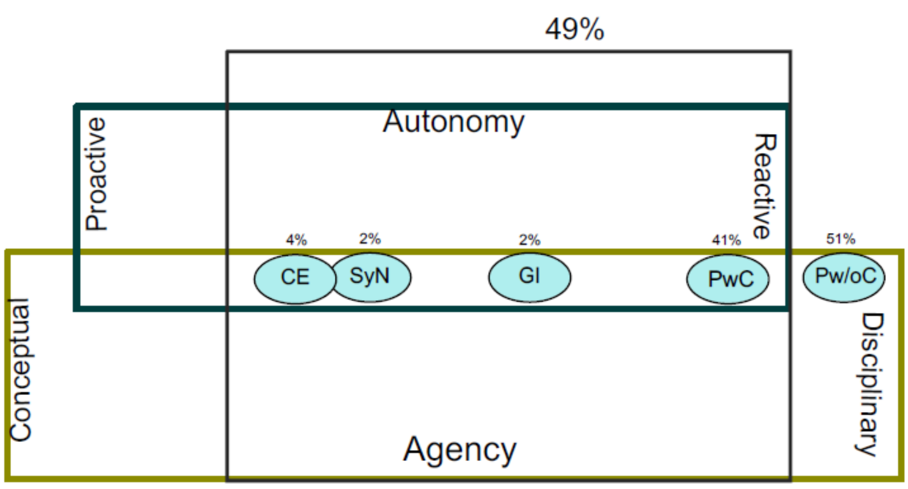
Figure 4.3: Distribution and frequencies of CPM tasks



Why are these findings important? They are important because they paint a more detailed picture, than the overall findings given earlier, of how CPM and PI each empower their students' learning of mathematics. These findings are important because having more tasks of higher cognitive demand, as in the case of CPM, gives students more opportunities for conceptual agency and intellectual autonomy, to engage in deeper mathematical thinking to arrive at meaning making and understanding. To arrive at meaning making and understanding requires students to discover patterns for themselves, to make connections between procedures and underlying concepts and to synthesize information regarding not only knowledge of key mathematical ideas but also an understanding of when, why and how to apply those ideas and associated methods in students' academic and regular lives. This greater proportion of challenging tasks gives students a wider range of opportunities to develop different forms of agency and autonomy. The procedural tasks allow for the development of some conceptual and disciplinary agency while the more open tasks allow students to exercise more conceptual agency and more intellectual (mostly reactive) autonomy.

With the opportunities that PI offer, given their more individualistic approach to learning, historically mathematically able students have the chance to develop their talents and interests in mathematics through their exercising of conceptual agency and intellectual autonomy while still benefiting from those problem solving skills that they can apply in everyday life. This is because even though they are comparatively fewer, PI still gives students opportunities to work on challenging tasks. However, because of the more active teaching role that teachers play in PI classrooms, it will fall to the teacher to promote those opportunities for students or to students who can benefit from extending their learning because they are able. On the other hand, learning mathematics for understanding within a group situation, as is often the case with CPM can also benefit those students who may not be as able or who may not have been positioned to think of themselves as being as able.

Figure 4.4: Distribution and frequencies of PI tasks



The greater frequency and variety of challenging tasks in CPM for topics on functions means that this latter kind of student can also be exposed to greater learning opportunities. For PI, this means fewer learning opportunities both for the more able students and those that may struggle with mathematics. If there are fewer challenging tasks and less variety among those, students' exposure to these tasks and the opportunities to develop conceptual agency and intellectual autonomy are less.

This finding that contrasts CPM and PI on the basis of learning opportunities is also in line with the philosophies of both curricula. CPM seeks to challenge students, to sometimes involve them in struggle, for students to understand their own concepts. PI positions students to reach fluency during the lesson, assessing them continuously with tasks that involve standard procedures and only adding in some challenging tasks in order to ensure that students have the basics covered. This also underscores why PI tasks almost entirely consist of procedural tasks. In the next section, I shall explore opportunities for agency and autonomy that each curriculum type gives at the level of the actual task features.

Opportunities based on task features: The percentages representing numbers of tasks in Figures 3 and 4 speak to the idea that students in CPM lessons are engaged in more guided investigation tasks, more synthesis tasks, and more guided real-life tasks. In short, having higher percentages, CPM students have more opportunities to exercise agency and autonomy during their work because these more challenging tasks demand more of students' thinking.

To gain a better understanding of what these findings mean in terms of the actual opportunities at the level of task types and their features, it is helpful to return to Figure 4.2. To recap, Figure 4.2 represents tasks categorized on the basis of five features: goal, method, complexity, solution and extension. The sections in red indicate where the feature was coded as closed for the given task. Conversely, the sections in blue indicate where the given feature was open. Students have opportunities to exercise conceptual agency and intellectual autonomy when task features are open. When all task features are closed, in the case of procedures without connections tasks, students can exercise disciplinary agency. When all features are open, they can experience conceptual agency and proactive autonomy – the greater extent of intellectual autonomy. When only one, two or three features are open, then students can experience some conceptual agency and reactive autonomy because of the constraints of the closed features.

Because 61% of CPM tasks are procedural versus 92% of PI, students in PI lessons have more opportunities to experience disciplinary agency than those in CPM classes, for the topic of functions. Considering the task distributions in Figures 4.3 and 4.4, and the portions of Figure 4.2 that are blue, the task features that offer students the highest proportions of opportunities for agency and autonomy, are in the complexity, solution, extension and method task variables in that order. The complexity task variable indicates in general how much the task is open in terms of leaving out steps for students to figure out. Task complexity also has implications for what prior knowledge and prior experience students have as they attempt to solve a task. What is complex for one student may not be for another. Some tasks too are complex because they require students to think about how to generalize and what to take into account for that. How students access the complexity variable depends on students' own abilities, experiences and how much prior knowledge they may have. This variable is more indirect than the solution, extension and method task variables for which it is possible to point out directly related task features. Of these latter 3, the solution variable is the most open. It offers the most opportunities for students to exercise agency and autonomy, based on the task distributions in Figures 4.3 and 4.4 which show the task types students have opportunities to work on in each textbook series. To get a better sense of what CPM and PI opportunities are for developing agency and intellectual autonomy, I shall present analyses of example tasks of the two kinds of procedural tasks that comprised the majority of tasks for both CPM and PI.

Opportunities in CPM and PI based on task types: Both CPM and PI had a high proportion of tasks classified as procedures with connections. Figure 4.5 shows a task classified as procedures with connections. For this task, the goal is included in the first sentence about Errol and Sandy. In this sentence, the student is told that Erol and Sandy aim to write an exponential equation in the form $y = ab^x$. In order for the student to also solve this task, Erol and Sandy's goal must be the student's goal as well.

Figure 4.5: Example of a CPM "procedures with connections" task

8-67. Errol and Sandy were working on writing an exponential equation in $y = ab^x$ form using this graph.

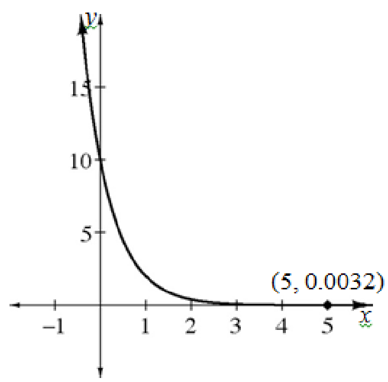
a. While trying to write the equation, Errol stated, *"I think 'a' must be 10 because the y-intercept is at (0, 10)."* Do you agree? Justify your answer.

b. *"But we still don't know what 'b' is"*, Errol noticed. His teammate, Sandy, had an idea. *"I think that $0.0032 = 10(b)^5$"* How did she get this equation? Is it valid? Explain.

c. If you have not done so already, use your calculator to help you solve $0.0032 = 10(b)^5$ for b. Explain how you solved this equation.

d. Use the values of a and b to write an equation for this graph.

Verify that your equation is correct, and explain how you did so.



The method students will use to solve this task is the form of the exponential $y = ab^x$ and in fact the task gives students an example $0.0032 = 10(b)^5$ framing it in terms of on the interaction between Errol and Sandy. So for this task, students are likely to use the method given as part of the statement of the task. In terms of the solution variable, students have several opportunities to give explanations and justifications (Stein et al., 1996; Selling, Garcia & Ball, 2016). The task asks them to justify their answer. It also asks them to explain how Errol got her equation and whether or not the equation is valid. Finally, the task asks students to explain how they determined the equation for the graph and to verify that their solution is correct. All of these explanations will feature in students' solutions. Students' solutions can also be judged to be mathematically correct or not, based on their reasoning. So the solution variable here is open because there are so many ways students can express themselves, in pursuit of correct answers. Students can exercise some conceptual agency for this task as they give solutions in the form of explanations, justifications and as they verify their solution because they will have to think about what aspects of the theory of exponential functions that they know or that they understand from the question will be relevant for the solution. They also exercise some intellectual autonomy, the reactive kind, because in deciding whether or not they agree with Errol and in deciding whether or not the equation is valid, they have control over which concepts on exponential functions to draw on in order to make their argument. Students are still constrained by the particular demands of the task so they have to work within those constraints. Finally, because this task is about a particular instance of an exponential equation, the task does not ask students to generalize or extend any findings. Students' main work is in understanding and explaining what is going on in the situation that Errol and Sandy find themselves in. So, the solution variable offers students the most opportunity in terms of conceptual agency and intellectual autonomy, in how they express their solutions. This task also has purely procedural elements. The task asks students to use their calculators to solve a numerical equation. So students can exercise some disciplinary agency in this task as well.

PI tasks classified as procedures with connections may also involve students in hypothetical situations as context for students to use when solving tasks. In Figure 6, students are asked to produce a graph of a falling acorn based on the function given in the task. They are also asked to work out the time when the acorn will hit the ground. Finally, students are asked to explain their reasoning regarding the domain and the range of a function.

Figure 4.6: Example of a PI "procedures with connections" task

Problem 5: An acorn drops from a tree branch 20ft above the ground. The function $h = -16t^2 + 20$ gives the height h of the acorn (in feet) after t seconds. What is the graph of this quadratic function? At about what time does the acorn hit the ground?

Got it? a. In problem 5, suppose the acorn drops from a tree branch 70ft above the ground. The function $h = -16t^2 + 70$ gives the height h of the acorn (in feet) after t seconds. What is the graph of this quadratic function? At about what time does the acorn hit the ground?

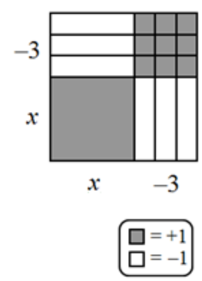
b. Reasoning What are a reasonable domain and range for the original function in Problem 5? Explain your reasoning.

Thus for this problem, in addition to giving two mathematical objects as part of the solution (a graph and a time value), students will have the opportunity to express their thinking involving domain and range. The solution variable is open for this aspect, as it affords students the opportunity to draw on and express their own reasoning, giving students some opportunity for exercising conceptual agency and intellectual autonomy. In this case as well, the opportunity is in reactive autonomy, as students' solutions are limited to what they think is relevant about domain and range in relation to the given function in the task.

In contrast to the opportunities that the two tasks in Figures 4.5 and 4.6, Figures 4.7 and 4.8 involve work primarily with procedures. Neither of these tasks asks students to express their thinking, or to make conceptual connections beyond the solutions they are asked to produce for the task. In Figure 4.7, students are asked to execute one standard procedure after another. First, they are asked to solve an equation. Then they are asked to write the solution in a very specific way, eliminating any chance of allowing students to express some conceptual agency. Students are then told that the solutions are irrational numbers, without asking them to make connections with this information. Finally, students write their solutions in yet another form, in order to complete the task.

Figure 4.7: Example of a CPM "procedures without connections" task

5-61. The quadratic equation $(x-3)^2 = 12$ is written in **perfect square form**. It is called this because the expression $(x-3)^2$ forms a square when built with algebra tiles.



a. Solve $(x-3)^2 = 12$. Write your answer in **exact form** (or **radical form**). That is, write it in a form that is precise and does not have any rounded decimals.

b. How many solutions are there?

c. The solution(s) from part (a) are **irrational**. That is, they are decimals that never repeat and never end. Write the solution(s) in **approximate decimal form**. Round your answers to the nearest hundredth.

So the task in Figure 4.7 is primarily procedural. Students are not required to make any conceptual connections that they can include in their solutions, as they work through the task. Such tasks have their place in building students' fluency with procedures. They allow students to exercise disciplinary agency.

PI has many more procedures without connections tasks than CPM, as shown in Figures 4.3 and 4.4. In PI, tasks classified as procedures without connections are stated plainly with little to no context. Consider the task in Figure 4.8. For the task in Figure 8, students need to work out the recursive formulas for the given arithmetic sequences. For that, they will likely use the method covered in class for determining arithmetic sequences. The task is stated more as an exercise in executing that method rather than a task that require deeper thinking and forming connections between procedures and concepts. For that matter, students will be exercising disciplinary agency in their work on this task.

Figure 4.8: Example of a PI "procedures without connections" task

Got it? Write a recursive formula for the arithmetic sequences below. What is the value of the 9th term?

a. 3, 9, 15, 21, …
b. 23, 35, 47, 59, …
c. 7.3, 7.8, 8.3, 8.8, …
d. 97, 88, 79, 70, …

e. **Reasoning** Is the recursive formula a useful way to find the value of an arithmetic sequence? Explain.

What this section highlights is that for the topic of functions, the most commonly occurring tasks in CPM and PI give students opportunities to exercise conceptual and disciplinary agency but with some differences. First of all, CPM offers students a greater proportion and a greater variety of tasks for exercising conceptual agency. CPM tasks for the topic of functions give twice as much opportunity to exercise conceptual agency and intellectual autonomy, and double the variety of the more challenging tasks than PI. PI on the other hand has a greater proportion of tasks that allow students to practice disciplinary agency. CPM tasks offer more content and context, while PI tasks give less content and context. These approaches to task structure confirm the approaches that each curriculum adopts. For each chapter selected for analysis, CPM had fewer tasks but they were longer because the CPM textbook authors frequently guided students through the tasks. PI has more tasks but they were shorter. This means that students will gain different mathematics learning experiences even when they are getting to experience agency and autonomy.

Findings from the linguistic perspective

In this section, I shall first give an overview of the differences between the two textbook series at the levels of lesson genre, stages, and clause. Next, I shall interpret similarities between the two textbook series at the same three levels. Finally, I shall compare and contrast CPM and PI similarities and differences and what they mean in terms of giving students opportunities to develop agency and autonomy.

The analysis from a linguistic perspective revealed both differences and similarities in how clauses in the declarative and in the imperative moods were used in the two textbook series. As a recap, clauses in the declarative mood give students information. Information can for instance be statements of mathematical theory for a given topic. Information can also consist of directions to solve a task. The analysis involved declarative clauses with modalization and with modulation. Modalization refers to the level of probability indicated in a clause, and can be seen in clauses with phrases such as “you may” and “you might”. Modulation refers to the implication of obligation in a clause, with phrases such as “you must” and “your group should” implying obligation. Modulation can also imply inclination, with phrases such as “you can”. In the sections below, I show how uses of clauses with modalization and modulation in the two textbook series position students differently.

I also study exclusive and inclusive imperatives. Exclusive imperatives position students to use standard mathematical procedures. An example of such a standard procedure is using the quadratic formula to find solutions to quadratic equations. Students need to substitute numbers into this formula and follow its steps to get a solution. Inclusive imperatives positions students to put forward their own thinking for instance through the explanations and justifications they give for a solution to a task.

Table 4.1 gives a snapshot of the linguistic perspective analysis findings by textbook series and by stage. For CPM, the stages are Lesson Orientation (denoted ‘LO’ in Table 1), Classroom Mathematical Tasks (CMT), and Math Notes (MN). For PI, they are Interactive Learning (IL), Guided instruction (GI) and Lesson Check (LC). These stages represent the main functions of the lesson text and those functions stay constant across different lesson texts. For instance the function of the LO stage in CPM is to introduce the learning goals for the lesson while the function of the LC stage in PI is to check that students have learned the concepts and procedures introduced in the lesson.

Table 4.1: Overview of linguistic perspective analysis

	Stages	Exclusive	Inclusive	Modalization	Modulation
CPM	LO			1	32
	CMT	67	62	5	9
	MN				1
PI	IL	3	14		2
	GI	119	22	4	43
	LC	18	41		

Analyzing the tasks by stages is useful because it lets us know how the textbook authors position students in each stage and the opportunities they afford them at that stage of the lesson text. To interpret what the numbers in the table mean, I will present more detailed findings and implications for analyses on declarative and imperative clauses in the sections below.

Single versus multi-stage approaches to problem-based learning (PBL): PBL involves students learning mathematics by solving problems during lesson. In PBL, each lesson consists of different problems all concerning the key mathematical ideas for the topic of study. The findings in this section will reveal how CPM and PI textbooks each interact with students through PBL. CPM promotes collaborative learning, where the responsibility for learning is primarily on the student, both as an individual and in collaboration with other students. With CPM, the teacher offers students assistance to support their learning process. PI brings the traditional textbook to life through active demonstration by the teacher and through multimedia interfaces. In PI, the teacher plays a more active role in teaching with examples and is more visible in directing learning. One main difference between PI and traditional textbooks is that because PI promotes PBL, the focus is primarily on solving mathematical tasks. Where in the traditional class, the teacher would go through theory before working examples, the PI lesson goes straight to the examples.

These findings will primarily reveal differences between the two textbook series. At the level of the lesson genre, the results in Table 4.1 above tell a story of two different approaches to students' work guided by lesson text. Let us first examine the case for CPM. For CPM, the lesson text has a single stage where students work on mathematical tasks. We can tell this by looking at which stages in Table 4.1 show imperatives. This is because imperatives direct students to carry out actions as they solve tasks. CPM has organized its lesson text to have all the mathematical tasks that students work on appear in stage 2, which is where all of CPM's imperatives appear. This stage is appropriately named CMT, for classroom mathematical tasks. This is also why there are 129 clauses in the imperative mood (exclusive and inclusive cases added together) directing student activity in the CMT stage and in no other. For CPM, as Table

4.1 shows, there are no imperatives in the LO and the MN stages, which are first and third stages respectively.

This is important for a number of reasons. First, the fact that all the classroom mathematics activity occurs continuously in one lesson stage is consistent with CPM's spiral curriculum design (Bruner, 1960). CPM curriculum designers believe that it is only through long term exposure that students can learn mathematical concepts deeply enough to remember and use them in their lives. Coupled with the fact that CPM lessons are student-centered, meaning that the teacher guides rather than leads, students can learn individually as well as through interactions with one another when necessary to share and contribute ideas for learning during this one continuous stage. Indeed, one of the three principles upon which the CPM curriculum is built (CPM, 2018c), is the idea in the initial stages of encountering a new mathematical idea, students learn best when working in collaboration with each other while being guided by a knowledgeable teacher. For this reason, CPM lesson texts give students ample time to investigate ideas, and patterns, to make conjectures and to allow room for students to interact with each other. It is worthy of note that related to students working in one continuous session is the fact that CPM frequently takes into consideration learning progression in their designs mathematical tasks. This means that earlier tasks sometimes inform later ones within the same lesson. On occasion, the textbook authors even direct students to review tasks outside of the lesson's tasks, in order to inform their work on tasks within the lesson. This feature of linking tasks within or even across lessons is distinctly absent in PI. Through this design choice, students can work by themselves, sometimes individually, sometimes in groups, as they progress through the lesson text, being guided from one task to the next. Students can experience individual as well as social autonomy. Thus by examining the CPM lesson text genre, it becomes possible to observe that

CPM achieves the spiraling curriculum by setting students up to work during one stage of continuous classroom work for each lesson. The main aim is for students to familiarize themselves with new mathematical ideas they will visit in a later lesson and to make connections with ideas they have already encountered in past lessons.

In contrast to CPM where students work on tasks only during the CMT stage, in PI lessons, students work on tasks in all three stages. This is shown in Table 4.1 in the imperatives that appear in all three stages of PI lesson texts. Each stage plays a different role. Collectively, students are not left to figure out mathematical patterns, learn from each other and to struggle through the process with minimal active teaching, as is the case with CPM. PI students more frequently experience elements of traditional teaching, since the classroom teacher helps students through example tasks that support students to then practice similar tasks on their own. During the interactive learning (IL) stage, students may interact with an online interface as they solve a mathematical task. They may choose to interact with one another but they are not required to. The IL stage is meant to introduce students to the lesson's mathematical ideas and the posed task also helps the teacher to differentiate student abilities. Although the GI stage covers the basic mathematical ideas for the lesson, the IL and LC stages have more challenging tasks that can serve the learning needs of more mathematically able students. This is why in Table 4.1 the tasks in the IL stage collectively have 14 inclusive imperatives as opposed to 3 exclusive imperatives. There are about five times as many inclusive imperatives as exclusive imperatives at this stage. This means that the mathematical tasks students encounter at this stage demand more of students' own thinking. These challenging tasks provide an opportunity for conceptual agency and intellectual autonomy for more mathematically able students while giving other students an idea of what the learning goals for the lesson will be.

Additionally, unlike CPM, the second stage of PI lessons, GI, positions students to learn mathematical procedures more than to figure out mathematical patterns and to make conjectures. This can also be seen in Table 4.1, which shows 119 exclusive imperatives as opposed to 22 inclusive imperatives. There are about five times as many exclusive imperatives as there are inclusive imperatives at this stage. Exclusive imperatives position students to execute standard mathematical methods. Students are thereby positioned to exercise disciplinary agency. This is because at this stage of the lesson, students learn how to solve tasks based on what the teacher shows. This is why this second stage is called Guided instruction (GI). This means that although PI claims students engage in PBL, the teacher-led element of traditional mathematics teaching remains a prevailing feature in PI lessons.

Finally, the third stage of PI lessons checks students' understanding of concepts and procedures in the lesson. Again this stage is an opportunity for the teacher to assess what students have learned for the purposes of differentiation, and for students to assess themselves. This is confirmed by Table 4.1, which shows that this stage has the highest number of inclusive imperatives. So students have the greatest opportunity to exercise intellectual autonomy. It also means that because students likely worked by themselves, it is likely those able ones who were able to grasp the concepts at the IL stage of the lesson and who were able to internalize the procedures during the GI stage that may benefit most from working on the more challenging tasks. This learning approach is in stark contrast to that of CPM, where student collaboration means that students can work with those in their groups or even with those from other groups.

In terms of differences, we also find that although CPM overall has fewer inclusive imperatives than PI, the variety and the spread of CPM's inclusive imperatives is wider than those of PI. This is true even though the number of the sampled CPM tasks is lower than those of

PI. Table 4.2 shows 12 different inclusive imperatives spread across the analyzed CPM lessons. This is in contrast to the 7 PI inclusive imperatives shown in Table 4.3.

Table 4.2: Frequencies of inclusive imperatives in sampled CPM lessons

Verb	**1-1.1.1**	**1-1.2.1**	**1-1.3.2**	**1-2.1.1**	**1-2.2.2**	**1-2.3.2**	**1-8.1.1**	**1-8.1.5**	**1-8.2.3**	**2-5.1.1**	**2-5.2.1**	**2-5.2.6**	**3-1.1.1**	**3-1.1.4**	**3-1.2.3**	**3-9.1.1**	**3-9.1.6**	**3-9.2.3**	**Total**
Compare												1		1	1				3
Confirm					1														1
Conjecture																		1	1
Create					1													1	2
Describe				1	2		2												5
Design																1			1
Explain	5	1	2		4		2	3			2	1		4		1		1	26
Explore												1		1					2
Generate									1										1
Investigate									1	1		1		2					5
Justify	1			1	2		1	1					1			1			8
Predict														5		2			7
Total	6	1	2	2	10	0	5	4	2	1	2	4	1	13	1	5	0	3	62
Proportion	0.38	0.07	0.11	0.08	0.45	0	0.31	0.4	0.14	0.07	0.11	0.15	0.1	0.43	0.14	0.26	0	0.17	

Average = 0.19

CPM has fewer overall inclusive imperatives because CPM has fewer tasks. CPM's tasks tend to be longer, more verbose and often more demanding of students than PI tasks. PI has more tasks so that students can get more practice solving similar kinds of tasks. This textbook style of having students work a lot of similar tasks is more akin to a traditional style of mathematics textbooks than it is to reform textbooks. In traditional style textbooks, it is common to have students frequently practicing similar kinds of mathematical tasks in order to master known procedures. In so-called reform-based textbooks, the emphasis is instead on engaging with a variety of tasks that exercise students' mathematical understanding rather than their fluency in using standard procedures. These differences once again reinforce the different orientations of each curriculum type. Because PI tasks are more like those found in a traditional textbook, where the statement of the task sticks to a structure with a narrow scope, they tend to be more generic. CPM on the other hand opts to be creative with its task, sometimes setting up the task context in the form of a story or a hypothetical situation involving students working together to solve a task.

Table 4.3: Frequencies of inclusive imperatives in sampled PI lessons

Verb	**1-2.1**	**1-2.4**	**1-2.7**	**1-3.1**	**1-3.4**	**1-3.7**	**1-5.1**	**1-5.5**	**1-5.9**	**2-12.1**	**2-12.6**	**2-12.12**	**3-5.1**	**3-5.4**	**3-5.7**	**3-8.1**	**3-8.6**	**3-8.11**	**Total**
Conjecture							1												1
Correct														1	1	1	1		4
Describe	6	1	4			1								2	2	1	1		18
Explain	3	5	3	7	2	2	1	2	3	3	2	2	5	1	1	1	2	3	48
Justify	1	1											1	1					4
Model					1														1
Prove												1							1
Total	10	7	7	7	3	3	2	2	3	3	2	3	6	5	4	3	4	3	77
Proportion	0.63	0.35	0.25	0.39	0.19	0.33	0.11	0.25	0.18	0.16	0.15	0.23	0.3	0.45	0.29	0.19	0.27	0.2	

Average: 0.27

The differences in the variety of exclusive imperatives are much smaller than those of the inclusive imperatives although the actual number difference is much larger. CPM has 17 different exclusive imperatives and 67 of them overall. PI has 16 different exclusive imperatives, and 140 of them overall. That is, CPM has almost as many types of exclusive imperatives as PI even where PI has twice as many counts of exclusive imperatives as CPM. This is confirmed in Tables 4.7 and 4.8.

Table 4.4: Frequencies of exclusive imperatives in sampled CPM lessons

Verb	**1-1.1.1**	**1-1.2.1**	**1-1.3.2**	**1-2.1.1**	**1-2.2.2**	**1-2.3.2**	**1-8.1.1**	**1-8.1.5**	**1-8.2.3**	**2-5.1.1**	**2-5.2.1**	**2-5.2.6**	**3-1.1.1**	**3-1.1.4**	**3-1.2.3**	**3-9.1.1**	**3-9.1.6**	**3-9.2.3**	**Total**
Calculate		1			3	1													5
Color				2															2
Draw											1						1		2
Evaluate	1																		1
Expand			1																1
Find		4																	4
Graph							1				1	1		1				1	5
Label																	3		3
Locate														1					1
Mark																	1		1
Make				1					1									1	3
Multiply														1					1
Record														1					1
Rewrite			1									3							4
Round											1								1
Sketch				1		1						3		2		1		3	11
Simplify												1							1
Solve											3	1							4
Write		2		2	1	1		3	1		4	1		1					16
Total	1	7	2	6	4	3	1	3	2	0	10	10	0	7	0	1	5	5	**67**
	0.06	0.63	0.11	0.25	0.24	0.15	0.06	0.30	0.14	0	0.53	0.38	0	0.24	0	0.05	0.50	0.28	

Average: 0.22

What these differences mean is that even with fewer tasks, CPM positions students to exercise a similar variety of exclusive imperatives that give them opportunities to exercise disciplinary agency. This is interesting because it really underscores a difference between the two curricula, which is that PI positions students to practice basic tasks repetitively whereas CPM does not. With fewer problems, CPM still gives students a similar range of experience in terms of variety.

Table 4.5: Frequencies of exclusive imperatives in sampled PI lessons

Verb	**1-2.1**	**1-2.4**	**1-2.7**	**1-3.1**	**1-3.4**	**1-3.7**	**1-5.1**	**1-5.5**	**1-5.9**	**2-12.1**	**2-12.6**	**2-12.12**	**3-5.1**	**3-5.4**	**3-5.7**	**3-8.1**	**3-8.6**	**3-8.11**	**Total**
Draw												1							1
Evaluate							6												6
Find			1	4			1	1			4	4			2	4	1	8	30
Give													1						1
Graph		7	6		1	2			11	7				1			2		37
Identify										1									1
Label	3																		3
Match	1																		1
Make		1							2										3
Rewrite					1														1
Round											1							1	2
Sketch	1																2		3
Simplify							3						1						4
Solve								2			3				2				7
State													10						10
Write		1	9		5	3			3			4		5					30
Total	5	9	16	4	7	5	10	3	16	8	8	9	12	6	4	4	5	9	**140**
	0.31	0.45	0.57	0.22	0.44	0.55	0.55	0.38	0.76	0.42	0.62	0.69	0.60	0.55	0.29	0.25	0.33	0.60	

Average: 0.48

To look a bit deeper, we can examine more closely what inclusive and exclusive imperatives look like for each textbook series and whether there are similarities and differences between the two. Starting with CPM, we find that the inclusive imperatives in each case of Table 4.6 demand that students share more of their own thinking. Whether it is confirming an answer, explaining the behavior of a mathematical object or predicting what happens when adding or subtracting two functions, the CPM inclusive imperatives in Table 4.6 are each asking students to think, and to share that thinking. Full sentences are given that describe the actions students are to take. PI also makes similar use of inclusive imperatives. As Table 4.3 shows, PI frequently asks students to explain their reasoning. This is also due to having students frequently practice mostly procedures with connections tasks. Frequently in analyzed PI tasks, the inclusive imperative consists only of the single word “explain”. PI however does have some variety in its use of inclusive imperatives, as Table 4.6 shows. In Table 4.6, students are asked to explain a mistake, to justify reasoning and to correct a friend’s error. These are all opportunities where a student will be able to express their own thinking. For both CPM and PI, inclusive imperatives give students the change to develop conceptual agency and intellectual autonomy. Depending on how they are used in a given task type with open features, students may have opportunities for more or less conceptual agency and for proactive or reactive autonomy.

Table 4.6: Examples of inclusive imperatives CPM and PI

CPM	"Confirm your answer algebraically" (lesson 2.2.2) "Explain how to use your graph to justify your answer" (lesson 2.2.2) "Explain the motion that the graph describes" (lesson 2.2.2) "Investigate the relationship between the original two functions and the different results you get from adding, subtracting, multiplying, and dividing the two functions" (lesson 1.1.4) "Predict what happens if you add or subtract any two linear functions" (lesson 1.1.4) CPM Copyrighted material. All rights reserved. Permission to reproduce within this dissertation permitted by CPM Educational Program.
PI	"Justify your reasoning" (lesson 2.1) "Explain the mistake" (lesson 3.1) "Make a conjecture about how the value of an exponential expression (an expression containing an exponent) changes when you decrease the exponent by 1" (lesson 5.1) "Correct your friend's error" (lesson 5.1) "Explain why there is no difference between the time travel to and from the store when there is no wind" (lesson 5.7) PRENTICE HALL, HIGH SCHOOL MATH 2014 COMMON CORE INTEGRATED MATH 2 WRITE-IN STUDENT EDITION 2-VOLUME + DIGITAL COURSEWARE 1-YEAR LICENSE (REALIZE) GRADE 9/10, 0, ©2014. Reprinted by permission of Pearson Education, Inc., New York, New York.

For clauses containing exclusive imperatives, CPM and PI have a similar style of stating the clause. This is shown in Table 4.7. These imperative clauses ask students to carry out standard mathematical procedures. These are procedures such as "calculate", "solve", "graph", or "draw". So not only do CPM and PI have a similar range of exclusive imperatives, the two textbook series also state exclusive imperatives in tasks similarly. This means that in spite of the different orientations CPM and PI adopt for their curricula, the two textbook series offer similar opportunities for students to exercise disciplinary agency when working on tasks. This finding too is important because it means that where CPM and PI differ is with the findings involving inclusive imperatives, which are those that give students more opportunities to develop conceptual agency and intellectual autonomy.

Table 4.7: Examples of exclusive imperatives in CPM and PI

CPM	"Calculate the unit rate for each situation" (lesson 2.2.2) "Write your answer as a **unit rate**" (lesson 2.2.2) "Calculate the speed in feet per second" (lesson 2.2.2) "Solve $(x - 3)^2 = 12$." (lesson 5.2.1) "Write your answer in **exact form** (or **radical form**)." (lesson 5.2.1)
PI	"Graph the function $y = -3x^2$." (lesson 12.1) "Find the vertex of $y = x^2 + 6x + 8$ by completing the square." (lesson 12.6) "Solve each equation by completing the square." (lesson 12.6) "Draw the graph" (lesson 12.12) "Write an equation for each translation" (lesson 12.12)

Modalization and Modulation in CPM and PI lesson texts: On the basis of analysis at the stage level, Table 4.1 shows that both textbook series have comparable and relatively low frequencies of declarative clauses with modalization. For CPM, they occur once in the LO stage and 5 times in the CMT stage. For PI, they only occur in the GI stage and only 4 times there. So for both textbooks, instances of modalization are more frequent in the second stage.

Before discussing what these findings mean, I shall explain some nomenclature that appear in Table 8. I have already covered the abbreviations for the stage names. The new abbreviations that appear in Table 8 are [EU] for "essential understanding" and [N] for "notes". "Essential understanding" and "notes" are two staple features in the PI GI stage, although the notes feature also appears on occasion in the IL stage, where it makes reference to tasks at that stage. "Essential understanding" gives the main mathematical idea of the lesson while "notes" gives mathematical ideas and suggestions to support students in their work.

From Table 4.8, one use of clauses with modalization common to both CPM and PI is to give students suggestions that they might use in their work. In the case of CPM, apart from the fifth example, the others give students suggestions that students may choose to act upon or not as they work. Similarly, apart from the first PI example in Table 4.8, the rest give students suggestions they may follow, or not, as they work. These few instances of clauses with modalization in both textbook series that give students suggestions can be seen as giving them choices and hence supporting students' conceptual agency and intellectual autonomy. The second use of clauses with modalization in both CPM and PI is to call students' attention to ideas encountered before, such as with the fifth CPM example or an idea just encountered, such as with the first PI example. In both cases, the textbook authors are giving students suggestions that students may then incorporate into their thinking and their work, or not, thereby supporting or limiting students' conceptual agency by contributing to the choices students consider for the task.

Table 4.8: Examples of uses of clauses with modalization in CPM and PI

CPM	"You may want to explore using the 2-1 Student eTool (lesson 2.1.1) [CMT] "You may refer to it later" (lesson 8.1.1) [CMT] "You may want to use a different color for each car" (lesson 8.2.3) [CMT] "You may make the shapes in any order you like" (lesson 1.1.1) [CMT] "You may recognize some functions you have previously investigated" (lesson 1.1.4) [LO] "You may need to move your table or desks out of the way" (lesson 9.1.1) [CMT]
PI	"As you may have noticed in the Solve it, the change in the height of the water as the volume increase is related to the shape of the container" (2.1) [N] [GI] "A graph may include solutions that do not appear in a table" (2.4)[EU] [GI] "Some graphs may be composed of isolated points" (2.4)[N] [GI] "You may need to complete the square" (12.12)[N] [GI]

The story is different for modulation. Because of the higher frequencies and differences in use, declarative clauses with modulation turn out to be more interesting in how they are used within each textbook series. On one hand, CPM lesson texts employ declarative clauses with high modulation (frequently using the phrase "you will…"). On the other hand, PI lesson texts use declarative clauses with low modulation (frequently using the phrase "you can…"), thereby positioning students to use the information in their work if they choose to. This positions students to exercise conceptual agency if they draw on the given information to solve challenging tasks. For CPM, there are much higher frequencies than PI of declarative clauses with modulation in the LO stage. For PI, there are much higher frequencies than CPM of the same type of clause in the GI & P stage. GI & P is the main learning stage during which the teacher demonstrates with examples and students subsequently work on tasks similar to those the teacher demonstrates.

Also in what they use these clauses for, the two textbook series differ and these differences reflect the underlying philosophies of each curriculum. In CPM, declarative clauses with modulation are used to set the lesson goals. Table 4.9 shows some examples.

Table 4.9: Examples of uses of declarative clauses with modulation in CPM

LO stage	"In this course, you will continue your study of linear functions, and extend these patterns to new kinds of functions" "Today you will look more closely at how equations that relate two variables help establish a function between the variables" "Throughout this chapter you will explore the multiple representations of a linear function" "Today you will focus on the meaning of "rate of change" in various situations" "In today's lesson, you will reverse the process used in Lesson 2.1.4 so that you can write the equation of a line from a table or graph"

As Table 4.9 shows, for CPM, what students will learn in a given lesson is clearly outlined at the beginning of each lesson text. This can be seen in the clausal examples above that start with

“Today, you will…”, “In today’s lesson, you will…”.Variations of this form of setting the lesson goal is a constant feature in the LO stage of each analyzed CPM lesson text. Students have little intellectual autonomy in changing the lesson goal. These are instances of declarative clauses with high modulation for which students are obliged to comply. Instead, they must work within the constraints of the goal set for a given lesson.

Table 4.10: Examples of uses of clauses with modulation in PI

PI Stage 2	[EU]“When you can identify a pattern in a sequence you can use it to extend the sequence” (lesson 2.7) “You can also model some sequences with a function rule that you can use to find any term of the sequence” (lesson 2.7) [EU] “You can find the value of any term of an arithmetic sequence using a recursive formula” (lesson 2.7) [N] “You can also write a sequence using an explicit formula” (lesson 2.7) [N] “You can write an explicit formula from a recursive formula and vice versa” (lesson 2.7) [EU] “You can use ratios to show a relationship between changing quantities, such as vertical and horizontal change” (lesson (3.1) [N] “You can use any two points on a line to find its slope” (lesson 3.1) PRENTICE HALL, HIGH SCHOOL MATH 2014 COMMON CORE INTEGRATED MATH 2 WRITE-IN STUDENT EDITION 2-VOLUME + DIGITAL COURSEWARE 1-YEAR LICENSE (REALIZE) GRADE 8/9, 0, ©2014. Reprinted by permission of Pearson Education, Inc., New York, New York.

Declarative clauses with modulation when they appear in either of these PI lesson text features served as hints for students, to aid them in their work. To elaborate further, we can consider some examples from Table 4.10. All in all, there are 43 instances of these declarative clauses with modulation across the PI GI & P stage, and they have the form “you can…” Let us take the first example in Table 3. In this example, the PI textbook authors are giving students suggestions outside of the statement of the mathematical task. This mathematical idea is designed to give students a way of thinking about how to go about a particular method or investigation. What this means is that the PI analyzed textbooks incorporate direct guiding of how they want students to think about mathematics. This is part of PI’s guided instruction approach. PI textbook authors position students to learn not only from the classroom teacher but also from pointed directions and suggestions the textbook provides. Traditional mathematics textbooks are typically set up in a similar fashion, to present theory, demonstrate application of theory through examples and engage students in tasks closely related to those examples.

In the LC stage of the lesson, students engage in more independent work and may need to draw on theory in order to solve more challenging tasks. This is where PI textbooks can play the role of "teacher" when giving students mathematical suggestions through the provision of "notes". These mathematical suggestions, in the form of declarative clauses with low modulation can serve as supports for students' conceptual agency and intellectual autonomy if students are able to take those suggestions and make use of them to support their work independent of the teacher. In so doing, such students will also be developing individual autonomy.

To instantiate with mathematical task examples, a main difference between CPM and PI in how each textbook series plays the role of 'teacher' involves how PI uses declarative clauses with modulation in a way that CPM does not. The PI textbook authors chose to give students information in relation to theory outside of mathematical tasks, while CPM mathematical tasks sometimes include such theory within the task itself. PI thus uses clauses with modulation as a means to guide students' thinking. When necessary, CPM gives students theoretical information directly using clauses in the declarative with no modulation. Consider the following example tasks:

Table 4.11: Differences in how CPM and PI direct students to use information in tasks

CPM	PI
1-54. CLOSED SETS Integers are said to be a closed set under multiplication: if you multiply two integers, the result is an integer. Integers are not a closed set under division: if you divide two integers, the result is not always an integer. For example, $2 \div 5$ is not an integer.	Write a recursive formula for the arithmetic sequence below. What is the value of the 9^{th} term? a. 3, 9, 15, 21, … b. 23, 35, 47, 59, … c. 7.3, 7.8, 8.3, 8.8, … d. 97, 88, 79, 70, … e. **Reasoning** Is the recursive formula a useful way to find the value of an arithmetic

a. Are one-variable polynomials a closed set under addition (or subtraction)? In other words, if you add (or subtract) two polynomials that both have the same variable, will the result always be a polynomial? If you think the set is closed, explain why. If, not, give counterexamples.

b. Are one-variable polynomials closed under multiplication? In other words, if you multiply two polynomials that both have the same variable, will the result always be a polynomial? If you think the set is closed, explain why. If, not, give counterexamples.

c. Consider whether polynomials are closed under division. What is your conclusion? Can your results from problems 1-49 and 1-53 help? Explain.

sequence? Explain.

Practice: Write a recursive formula for each sequence.

5. 2.3, 2.8, 3.3, 3.8, …

6. 4.6, 4.7, 4.8, 4.9, …

[Note] You can find the value of any term of an arithmetic sequence using a recursive formula. You can also write a sequence using an explicit formula. An explicit formula is a function rule that relates each term of a sequence to the term number.

*Words in parentheses included by the author

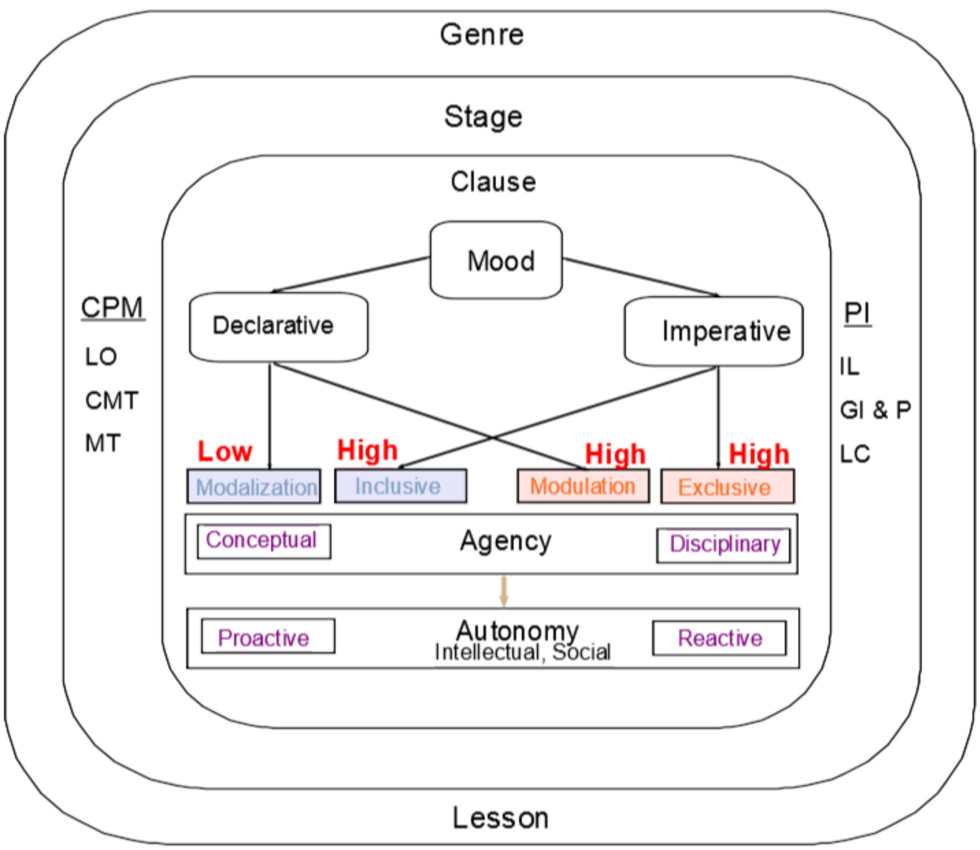
,

In the CPM task, the textbook authors give theoretical information concerning integers as a closed set. The first seven clauses in the task are all declaratives (i.e. statements) presenting mathematical terms and explanations. This information is given up front, before stating the parts of the task that students have to perform. Students therefore know that the given theoretical information pertains directly to the task. When CPM students then proceed in their investigations of whether one-variable polynomials form a closed set under addition or multiplication, they will extend their understanding of closed sets for addition and multiplication as they pertain to integers to the case of polynomials. It is worth noting that because CPM advocates learning progression, this CPM task follows earlier tasks in this very lesson that introduce work on addition and multiplication of polynomials. So the new information that students are getting in this task has to do with closed sets. With the given declarative clauses on mathematics theory, it is then up to students' own intellectual autonomy to make sense of the information and to apply it to the given situations in the task. This way, the CPM textbook authors act as teachers by first positioning students by giving them some theoretical information and then setting them off to apply that information. This approach also reinforces the idea that students bear the greater burden of learning and are supported in this regard by the textbook authors acting as “teachers”.

PI takes a different approach. Unlike CPM, PI does not give mathematical theory in declarative statements within the task. Instead, statements are given as suggestions in notes *outside* the statement of the task. For the example in Table 4.11, the statement of the task demands that students write a recursive formula for the given arithmetic sequences. Unlike CPM, PI gives students information on how to work with recursive formulas outside the statement of the task. This is a design choice that chooses not to integrate theory within the task. What this means is that students may or may not see the information in the note as being part of the task.

In order to have an overall summary of the findings for the linguistic perspective, we return to the manner in which the two textbook series organize their respective texts at the levels of lesson genre, stages, and clause. This is shown in Figure 4.9 below. Interpreting Figure 4.9 in these terms, and taking the findings just presented into account, we observe that both CPM and PI had low frequencies of clauses with modalization. Declarative clauses with modalization give students the choice of whether or not to apply mathematical suggestions as they solve tasks. They play different roles in CPM and in PI. In CPM, clauses with modalization give students options to consider as they work on methods to solve tasks. In PI, clauses with modalization give students suggestions on theory to consider when solving tasks. Declarative clauses with modulation also play different roles in both textbook series. In CPM, they are used to set the lesson’s goals while in PI they appear as suggestions to students primarily within the second stage of the lesson during which students work to solve the bulk of the tasks they work on for the lesson. Imperative clauses play a similar role in the two textbook series when considered at the clausal level. Exclusive imperatives give students opportunities to learn standard mathematical procedures and hence exercise disciplinary agency in the process. In this regard, CPM and PI have comparable levels of exclusive imperatives although PI has higher frequencies of exclusive imperatives compared to CPM because PI's learning orientation emphasizes repetition and practice of basic methods. Inclusive imperatives on the other hand give students the chance to include more of their own thinking, giving students opportunities to develop conceptual agency and intellectual autonomy. In this regard, CPM and PI have comparable frequencies of inclusive imperatives although CPM has a greater variety than PI because CPM emphasizes mathematical investigation and exploration more than PI.

Figure 4.9: Overview of results of analysis from the linguistic perspective



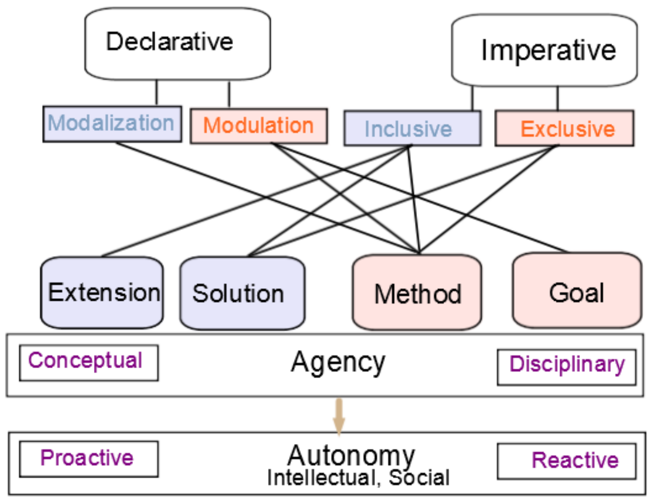
At the stage level, the findings in Table 4.1 showed that exclusive and inclusive imperatives appeared in different patterns across CPM and PI lesson texts. CPM lesson texts support learning progression, where earlier tasks within the lesson text are linked with later ones. Students work in one continuous session during the lesson, as is reflected in the appearance of exclusive and inclusive imperatives only in the second stage of the lesson text. In the second stage of CPM lesson texts, there are roughly the same number of inclusive and exclusive imperatives, as is shown in Table 4.1. PI lesson texts and learning orientation are organized differently from CPM. PI lesson texts are organized to have students encounter different task types at each stage of the lesson text. They work on a task at the IL stage of the lesson text, which is often challenging. This is shown in Table 4.1, where there are about five times as many inclusive imperatives compared to exclusive imperatives. Students work on several tasks at the second stage of the lesson text. These are tasks that largely emphasize procedures. In Table 4.1, we observe that there are about five times as many exclusive imperatives as inclusive imperatives at the second stage of PI lesson texts. At stage 3 of PI lesson texts, there were about twice as many inclusive imperatives as the exclusive imperatives. This finding shows that the tasks at this stage of the lesson text are also more challenging. In adopting this pattern in their lesson texts, PI aligns with the traditional teaching style of starting lessons with a warm up activity, continuing with the main lesson and finishing with a plenary.

As a final note on findings for the linguistic perspective, the overwhelming majority of analyzed declarative and imperative clauses were in the congruent mood. This was true for analyzed lesson texts in both CPM and PI. There were however some rare exceptions of clauses in non-congruent mood which are worth mentioning. One such, is a clause such as “Notice that $f(3) = 5$”. Although this clause appears to be an imperative, it is functioning as a statement. This clause is really stating that “$f(3) = 5$”. Clauses of this form are commonly used in formal mathematical parlance, for instance when writing a proof.

Findings from the combined perspective

In this final section, I first present a brief account of the findings from the cognitive and the linguistic perspectives to set the stage for presenting the findings for the combined perspective. With the review of the two perspectives as a basis, I next present the findings for the combined perspective. I show how instances of declarative clauses with modalization and modulation and imperative clauses with exclusive and inclusive verbs position students in relation goal, method, solution and extension task features. I link analyzed declarative and imperative clauses with each of these task features to reveal opportunities for students to exercise agency and autonomy. Findings from the cognitive perspective show that the majority of the tasks in CPM and in PI are procedural. There were important differences in this overall result, however. For instance CPM had a higher variety of tasks and proportionally more challenging tasks than PI. Findings from the linguistic perspective confirmed key differences in the way CPM and PI organize their respective lesson texts. This organization is influenced by the learning orientations each textbook series supports. CPM's reform-oriented approach supporting complex instruction and PI's traditional oriented approach come through in the ways each textbook series organize lesson texts. In CPM lessons, students work in one continuous problem solving class session. In PI, students work in three work periods that are reflected in the stages of the PI lesson texts. In the combined analysis of the cognitive and the linguistic perspectives, I focused on unpacking how analysis of clauses in the declarative and the imperative influenced the goal, method, solution and extension task variables. Figure 4.10 makes associations between aspects of clauses in tasks analyzed for the linguistic perspective that associate with task features analyzed for the cognitive perspective.

Figure 4.10: Linguistic elements of tasks linked with task features



In addition, Figure 4.10 also places those task features along the agency spectrum, in order to relate linguistic properties of clauses with task features that commonly assumed open or closed states for CPM and PI. In the sections that follow, I shall make more explicit the connections between the linguistic properties of clauses, the task features, and those opportunities for students.

Task variables related to declarative clauses with modalization: As shown in Figure 4.10, declarative clauses with modalization occurred in relation to the method variables. In CPM and PI texts, they played different roles. In the case of CPM, declarative clauses with modalization were used to direct students to carry out concrete actions. These are shown in Table 4.12. The first CPM example in the table gives students the option of using the 2-1 Student eTool, a CPM online resource that can aid students in their investigating patterns. Because the declarative clause uses modalization, the CPM textbook authors offer students the option of using this tool without making it mandatory. Students can thus use it if they choose to. In the second example, the CPM textbook authors give students the option of using different colors for the graphs of three different cars. Interestingly, the use of the phrase “you may want to…” read differently could be seen as a declarative in non-congruent imperative mood. In such a case, the first two examples can be read differently. The third CPM example also links with the method variable. In this case, the CPM textbook authors direct students on how to proceed with making different kinds of shapes when experimenting with a single loop of yarn. The declarative clause with modalization gives students the option of choosing whichever shapes they want to make for the investigation. So in all three CPM cases, students’ conceptual agency and intellectual autonomy is being supported by offering suggestions as choices students can make with aspects of task method.

Table 4.12: Examples of clauses with modalization in CPM and PI

CPM	PI
“You may want to explore using the 2-1 Student eTool” (CPM) (lesson 2.1.1) “You may refer to it later” (lesson 8.1.1) “You may want to use a different color for each car.” (lesson 8.2.3) “You may make the shapes in any order you like.” (lesson 1.1.1) “You may recognize some functions you have previously investigated” (lesson 1.1.4) “You may need to move your table or desks out of the way” (lesson 9.1.1) 	“As you may have noticed in the Solve it, the change in the height of the water as the volume increase is related to the shape of the container” (lesson 2.1) “A graph may include solutions that do not appear in a table”. (lesson 2.4) “Some graphs may be composed of isolated points” (lesson 2.4) “you may need to complete the square” (lesson 12.12)

For PI, the few examples of use of declarative clauses with modalization also related to the method variable. The first example informs students of a relationship between volume and shape. This first example establishes for the students the conceptual link between changes in water height and volume of containers. In effect, this first declarative clause with modalization suggests to students a relationship that students could have noticed but may have missed. The PI textbook authors are giving students information on how to think about the task. In so doing, they are scaffolding students' thinking processes but at the same time limiting their conceptual agency and intellectual autonomy. The second example about graphs and the fourth example about completing the square all take the form of giving students suggestions that serve as scaffolds for students' conceptual understanding. For this reason, these declarative clauses with modalization, though they are few, are examples of the PI textbook authors actually restricting students' conceptual agency and intellectual autonomy by offering students ways to think, in relation to method.

To summarize this section, what these examples mean in terms of agency and autonomy is that clauses with modalization function differently in CPM and in PI, in relation to method. CPM texts position students to make their own choices pertaining to specific actions within a given task. PI texts position students to think in specific ways in relation to a method. For the use of clauses with modalization, CPM supports students' conceptual agency and intellectual autonomy while PI restricts it. This finding also reinforces the differences in curriculum orientations the two textbook series assume. CPM's alignment with modern reform orients students to be more exploratory in their learning. This is why CPM supports students' in making their own choices. PI's alignment with traditional textbooks limits students' explorations when working on tasks and instead favors a more controlled approach to managing students' learning.

Task variables related to declarative clauses with modulation: Analysis via the linguistic perspective revealed that CPM's use of declarative clauses with modulation in mathematical tasks influences task goal and method variables in CPM and only the method variable in PI. CPM textbook authors use declarative clauses with modulation to state what the learning goal of a task is. This can be seen in the first and second examples in Table 4.13. The first example states a goal relating to linear equations whereas the second example states a goal in connection with exponential family functions. When CPM textbook authors set the goal for a given task using declarative clauses with modulation, students' conceptual agency is restricted. This is because students are guided to work on what the textbook authors plan for them to work on for that task. Students are constrained in that regard, and are positioned to exercise reactive autonomy, depending on other aspects of the task such as if the solution variable is open to allow students to engage in and express independent thinking. This feature of using declarative clauses with modulation to set task goals is absent in PI tasks.

Table 4.13: Examples of clauses with modulation in CPM

"In this problem, you will write the equation of the line that goes through the points in the table below." (task in lesson 2.3.2)
"Today you will begin to learn more about the exponential function family" (task in lesson 8.1.1)
"Each team member should write down three different numbers" (task in lesson 8.1.1)
"Your team will be assigned specific quadratic functions to study" (task in 5.1.1)

CPM textbook authors also use declarative clauses with modulation to give students direction on task method. The third and fourth examples in Table 4.13 relate to how CPM uses declarative clauses with modulation in relation to the method task variable. In the third example, the teacher gives students specific directions. They need to write down three different numbers. The statement does not direct students to write however many numbers they choose, neither does it direct them to think about what an appropriate number of numbers will be. Students' choices are restricted. In the fourth example, the textbook authors inform students that they will be given specific quadratic functions to work. The statement does not ask students to generate their own quadratic functions nor does it ask them to choose from a list. In both the third and the fourth examples in Table 13, students' options are restricted as they are required to carry out specific actions. In these latter two examples, students' conceptual agency and intellectual autonomy are restricted by the use of modulation. Thus for CPM the use of clauses with modulation both in relation to the goal and to the method task features restrict students' conceptual agency and intellectual autonomy.

With PI, the situation is different. First of all, unlike CPM, PI only uses clauses with modulation in relation to method. PI's use of declarative clauses with low modulation is purposefully to guide students on how to execute methods in the task. The clauses in Table 4.14 have words such as "find", "identify", "simplify", and "use", which are akin to the exclusive imperatives encountered in the section on the linguistic perspective. They refer to processes associated with methods in tasks.

Each of the examples in Table 4.14 begins with "you can…" and following that with a concrete suggestion about method. For instance the first example suggests to students how they can construct a graph with the aid of a table of values. The second example suggests to students that in order to be able to extend a sequence, they will need to determine the pattern that underlies the sequence. That is, how much it goes up by each time. In each of these examples, the PI textbook authors are restricting students' conceptual agency by directing students on what methods to use for their work. This use of clauses with low modulation is similar to PI's use of clauses with modalization. The difference is that PI's use of clauses with modulation suggests ways students should think about methods whereas PI's use of clauses with modalization suggests ways students should think about concepts. In both cases, directing students' thinking in these ways restricts their conceptual agency and intellectual autonomy.

Table 4.14: Examples of modulation in PI lessons

"You can use a table of values to help you make a graph in the Solve It" (lesson 2.4)
"When you can identify a pattern in a sequence you can use it to extend the sequence" (lesson 2.7)
"You can find the value of any term of an arithmetic sequence using a recursive formula" (lesson 2.7)
"You can use ratios to show a relationship between changing quantities, such as vertical and horizontal change" (lesson 3.1)

“You can use any two points on a line to find its slope” (lesson 3.1)
“To find the slope of AB, you can use the slope formula” (lesson 3.1)
“You can use a similar method to graph absolute value functions” (lesson 3.7)
“You can simplify the expression before substituting values for the variables” (lesson 5.1)

To summarize this section, although both textbook series use declarative clauses with modulation to direct students on method, each textbook series uses them differently. One difference between the two approaches is that CPM uses stronger modulation to set task goals and also to direct students on method. PI uses weaker modulation to direct students on method. This could be because in PI classrooms, the teacher plays a more active role. The “textbook as teacher” in PI lessons is auxiliary to the classroom teacher. In CPM, it is the other way around. CPM textbook authors assume a stronger role than PI textbook authors in their communication with students. Both textbook series’ use of clauses with modulation restricts students’ agency and autonomy in the ways described in this section.

Task variables related to exclusive imperatives: Exclusive imperatives in both CPM and PI direct students on what do to solve a task, that is, on method, and how to represent the solution. For CPM, in Table 4.15, the exclusive imperative verb “calculate” in the clause “calculate the unit rate for each situation” directs students to perform a mathematical operation. Likewise the exclusive imperative verb “solve” in the clause “Solve $(x – 3)^2 = 12$” directs students to execute a standard procedure for solving a quadratic equation. In both examples, the exclusive imperative verbs “calculate” and “solve” are both linked to a method. Also in relation to Table 4.15, the exclusive imperative verb “write” in the clauses “write your answer as a unit rate” and “write your answer in exact form (or radical form)” both connect to the solution variable. In the first case, students are told to represent their solution as a unit rate. In the second case, they are told to represent their solution in exact form. In both cases, the exclusive imperative verb in the clause is associated with the solution variable. Thus for CPM texts, the exclusive imperatives in clauses associated with the method or solution feature of the task together point to how the CPM textbook authors position students to exercise disciplinary agency.

Table 4.15: Examples of CPM and PI exclusive imperatives

CPM	"Calculate the unit rate for each situation" (lesson 2.2.2) "Write your answer as a **unit rate**" (lesson 2.2.2) "Calculate the speed in feet per second" (lesson 2.2.2) "Solve $(x - 3)^2 = 12$." (lesson 5.2.1) "Write your answer in **exact form** (or **radical form**)." (lesson 5.2.1)
PI	"Round your answer to a tenth of a degree" (lesson 8.11) "Graph the function $y = -3x^2$." (lesson 12.1) "Find the vertex of y = $x^2 + 6x + 8$ by completing the square." (lesson 12.6) "Solve each equation by completing the square." (lesson 12.6) "Draw the graph" (lesson 12.12) "Write an equation for each translation" (lesson 12.12)

For PI, exclusive imperative verbs can be associated with the method task feature in an identical way to the CPM case. We see in Table 4.15 that the exclusive imperative verb "find" in the clause "find the vertex of $y = x^2 + 6x + 8$ by completing the square" is associated with the method task feature because in this clause, the PI textbook authors direct students specifically to use completing the square". Similarly, the exclusive imperative verb "round", in the clause "round your answer to a tenth of a degree" clearly associates an exclusive imperative with the solution variable. As with CPM, exclusive imperatives are used to direct students on both method and solution. Both cases position students to exercise disciplinary agency. What this means is that when it comes to giving students opportunities to exercise disciplinary agency, CPM and PI are comparable. In other words, CPM and PI position students to learn the core mathematical methods and procedures in similar ways.

Task variables related to inclusive imperatives: Inclusive imperatives in both CPM and PI associate with the method task feature. The first example for CPM in Table 4.16 asks students to confirm an answer algebraically. In order to do this, students will need to work through the answer and then explain how their work confirms the answer. In this regard, the method variable comes into effect as students work out details needed for their thinking. These are cases where inclusive imperatives connect with the solution task feature. The last CPM example in Table 4.16 asks students to make a prediction that relates to any two linear functions. This demands that students go beyond the cases they investigated for the task in order to make a generalization. By making the generalization, students are extending the task. In this final example for CPM in Table 4.16, an inclusive imperative verb associates with the task extension feature:

Table 4.16: Examples of inclusive imperatives in CPM and PI

CPM	"Confirm your answer algebraically" (lesson 2.2.2) "Explain how to use your graph to justify your answer" (lesson 2.2.2) "Explain the motion that the graph describes" (lesson 2.2.2) "Investigate the relationship between the original two functions and the different results you get from adding, subtracting, multiplying, and dividing the two functions" (lesson 1.1.4) "Predict what happens if you add or subtract any two linear functions" (lesson 1.1.4) CPM Copyrighted material. All rights reserved. Permission to reproduce within this dissertation permitted by CPM Educational Program.
PI	"Justify your reasoning" (lesson 2.1) "Explain the mistake" (lesson 3.1) "Make a conjecture about how the value of an exponential expression (an expression containing an exponent) changes when you decrease the exponent by 1" (lesson 5.1) "Correct your friend's error" (lesson 5.1) "Explain why there is no difference between the time travel to and from the store when there is no wind" (lesson 5.7) PRENTICE HALL, HIGH SCHOOL MATH 2014 COMMON CORE INTEGRATED MATH 2 WRITE-IN STUDENT EDITION 2-VOLUME + DIGITAL COURSEWARE 1-YEAR LICENSE (REALIZE) GRADE 8/9, 0, ©2014. Reprinted by permission of Pearson Education, Inc., New York, New York.

PI lesson texts use inclusive imperatives in similar ways. For instance in Table 4.16, the PI inclusive imperative verb “correct” positioning students to correct a friend’s error prompts students to examine or modify the friend’s method to aid in the correction. Another PI example from Table 4.16 involves justifying reasoning. The inclusive imperative “justify” associates with the solution variable because the justification students give will be included in their solutions. The PI example in Table 4.16 involving making a conjecture associates an inclusive imperative with the task extension variable. Task extension involves opportunities the task gives students to attempt to generalize their thinking.

These three ways that inclusive imperatives associate with method, solution and extension task features all offer students ways to exercise conceptual agency and intellectual autonomy. In each case, the inclusive imperative associated either with method, solution or extension task features positions students to think and to express thoughts rather than merely execute a method, or give a solution that requires no demonstration of understanding. That CPM and PI give students these opportunities through the association of inclusive imperatives with the three task features has been shown through the findings. What is different between CPM and PI is the extent to which each textbook series gives students these opportunities. As shown in the cognitive perspective, CPM had a greater variety and proportionally more challenging tasks for students to work on. The variety associated with inclusive imperatives was confirmed in the findings for the linguistic perspective. Because of this, CPM students will have proportionally more opportunities to exercise conceptual agency and intellectual autonomy in terms of how inclusive imperatives are used to position students to think more when working on method, solution or task extension.

To summarize, CPM and PI both use clauses with modalization in relation to the method variable, but differently. CPM uses them to support students' conceptual agency and intellectual autonomy by giving students options for how to execute method. PI's use of them limits students' conceptual agency and intellectual autonomy, by suggesting to them ways to execute method. CPM and PI also use clauses with modulation. CPM uses them in relation to setting task goals and directing students on task method while PI uses them only in relation to task method. For clauses with modulation, both CPM and PI limit students' expression of conceptual agency and intellectual autonomy. CPM and PI use exclusive and inclusive imperatives in similar ways. Exclusive imperatives are used in both textbook series in relation to method and solution variables to give opportunities to exercise disciplinary agency. Inclusive imperatives are used by both textbook series to give students opportunities to develop conceptual agency and intellectual autonomy.

Chapter 5 Discussion and Conclusion

Summary of dissertation

At the onset of this dissertation, I set out to investigate ways that lesson texts and mathematical tasks in College Preparatory Mathematics (CPM) and Pearson Integrated (PI) position students to develop agency and autonomy. I was interested in carrying out this investigation because I am keen to learn how mathematics textbooks that are so common in the experiences of students and teachers position students to develop agency and autonomy during lessons. If mathematics textbooks are such a common feature of mathematics teaching and learning, and if it is now acknowledged that students need to rely more on their own thinking as they solve challenging tasks (Lester & Cai, 2016), why does there appear to be little research in mathematics education on how these textbooks position students to become more independent mathematics thinkers and doers through students' work on tasks? This concern prompted me to conduct the study in this dissertation.

In earlier chapters of this dissertation, I discussed aspects of traditional and so-called reform approaches to teaching. I argued that CPM through complex instruction positions students to learning mathematics through PBL in a reform-oriented manner while PI positions students for the same purpose but in a manner oriented toward a traditional approach to teaching and learning mathematics. Because of these orientations, there can be important differences in the opportunities each textbook series provides for students to develop forms of agency and autonomy. I then analyzed mathematical tasks and lesson texts from three perspectives:

cognitive, linguistic and combined. The cognitive perspective analyzed tasks in each textbook series based on different task types. These task types are in a hierarchy based on the degree of openness of each task type. The degree of openness depends on which of the task's five features are open or closed. The five features are the task goal, method, solution, complexity and extension. Depending on the tasks types students work on, and on whether the textbook series orientation is more traditional or more reform-based, students are given different opportunities to develop forms of agency and autonomy. The linguistic perspective analyzed declarative and imperative clauses in tasks and texts in different stages of a lesson text to determine ways in which the textbook authors used language choices to give students information and to direct them in their work. One kind of clause, the declarative, gives students information. Another kind of clause, the imperative, demands actions of students. By analyzing how each textbook series used clauses in each lesson stage, it was possible to determine how opportunities are made available to students to develop different forms of agency and autonomy. The combined perspective integrates the prior two perspectives. This perspective relates clause types to task features to show how declarative and imperative clauses are used by each textbook series to direct students in terms of task goal, method, solution, and extension variables. These various uses can point to opportunities for students to develop forms of agency and autonomy.

In this chapter, I shall discuss what has emerged in my study of opportunities that each textbook series offers students to develop forms of agency and autonomy during lessons. I shall relate the discussion back to literature discussed in chapter 2 of this dissertation and to other extant literature. I shall first present an overview of the findings and follow that with implications for those findings. I shall then state what contributions this dissertation can make to the research and practice of teaching and learning and follow that with limitations of the study. I shall complete this chapter with a discussion of possible future directions for research and the conclusion.

Overview of main findings

The findings revealed that both CPM and PI make opportunities available to students to develop forms of agency and autonomy when working on mathematical tasks. How each textbook series does so, however, differs in some important different ways. The cognitive perspective findings showed that even though the majority of tasks in both textbook series were procedural, CPM had a greater proportion and variety of the challenging mathematical tasks for the topic of functions than PI did. The challenging tasks are those that go beyond standard procedures to demand more of students' own thinking during classwork. This difference between CPM and PI is important because when students have a chance to work on challenging tasks, they can develop their mathematical thinking skills. They can make meaning of the mathematics they are learning and they can apply these skills to contexts outside of classroom learning situations. This difference between CPM and PI is also important because challenging tasks in each textbook series gave students different opportunities to experience agency and autonomy. In CPM, guided investigative and guided real-life tasks give students opportunities than PI to develop conceptual agency, intellectual and social autonomy. This happens because CPM positions students to explain and provide justifications for their solutions as they solve these challenging tasks. In PI, students can also develop conceptual agency and intellectual autonomy primarily within experiences of individual autonomy. This is because PI does not position students to work in groups. Students who work individually on challenging tasks will experience conceptual agency and intellectual autonomy.

The findings for CPM and PI differ from the findings in Kolovou et al. (2009). In Kolovou et al. (2009), most of the tasks they analyzed were standard tasks, which can be compared to the procedures without connections tasks discussed in this study. None of the six textbook series they analyzed had more than 10% of tasks in the challenging category (i.e. “gray-area” tasks and puzzle-like tasks). This finding contrasts with PI which had 49% of tasks on the topic of functions in the challenging category, that is, other than procedures without connections tasks. CPM had 85% of tasks in this category. Both CPM and PI had more challenging tasks than those textbooks analyzed in Kolovou et al. (2009). This contrast is important because Kolovou et al. (2009) is one of the few studies that has also analyzed textbook tasks for opportunities they offer students to exercise agency and autonomy. They analyzed mainstream textbook series, used in 85% of schools in the Netherlands, which were not PBL textbooks. It is also important to mention that Kolovou et al. (2009) studied textbook series at the primary level of education, whereas this dissertation studies textbook series at the secondary level of education. Differences in the two studies may be also accounted for by differences in and requirements at each school level.

That CPM gave students more opportunities to work on challenging tasks aligns with their philosophy of allowing students to be introduced to mathematical ideas that will engage them and possibly cause them to struggle as they work individually or in groups to solve those tasks. CPM supports complex instruction, a version of reform mathematics where students collaborate on mathematics tasks. This approach to learning bridges ability levels and different backgrounds. PI on the other hand supports a different philosophy that is also reflected in the cognitive perspective findings. Rather than having learning mathematics collaboratively, sometimes even through struggle, PI positions students to learn mathematics first by working through example tasks led by a teacher and then by individually practicing on simple, similar tasks. In PI lessons, the teacher plays an important role in leading learning. Rather than having the burden for learning fall more heavily on the student, in PI lessons it falls more heavily on the teacher. In other words, getting all students to achieve a basic level of understanding by practicing examples and solving simple tasks to reinforce learning is the role of the teacher. These two approaches to learning, student led and teacher led learning, serve as a distinct difference between the two textbook series.

The reform and traditional approaches to learning are also reflect in the findings for the linguistic perspective. For CPM lesson texts, all the mathematical tasks are located in one stage of the lesson text. PI lesson texts on the other hand show mathematical tasks in each of three stages. These differences in where mathematical tasks appear in the lesson text reflect actual classroom process. This means that in CPM classrooms, students solve tasks during one continuous lesson session. In PI classrooms, students solve tasks in three sessions of a lesson corresponding to the three stages in the lesson text. Analysis of inclusive and exclusive imperatives showed that CPM had roughly the same number of inclusive and exclusive imperatives in the second stage of the lesson text. This means that all work on tasks happens during this one stage. In PI lesson texts, the arrangement is different. There are about five times as many inclusive imperatives as exclusive imperatives in stage one, about five times as many exclusive imperatives as inclusive imperatives in stage two and about twice as many inclusive imperatives in stage three as exclusive imperatives. This is an interesting finding because what it shows is that the first and third stages of PI lesson texts have more challenging mathematical tasks. The second stage on the other hand has more procedural tasks. Therefore PI lesson texts start and end the lesson mostly with challenging mathematical tasks. The main section of the lesson has mostly procedural tasks, during which students work on more basic mathematics that reinforces procedural skills. This lesson structure aligns more with a traditional approach of instruction. These two approaches, one where students work mostly by themselves and the other where the teacher plays a more active role in students' learning are simply different approaches to teaching and learning mathematics. What is important are the opportunities each learning approach offers students to develop forms of agency and autonomy.

The linguistic perspective findings also corroborate the differences between CPM's refor-oriented approach and PI's traditional-oriented approach in what was revealed from the analysis of clauses with modalization and modulation. Both CPM and PI use clauses with modalization. However, where CPM positions students to have choices in the methods they use to solve tasks, PI positions students in clauses with modalization to think in particular ways by providing them with mathematical suggestions for solving tasks. Both CPM and PI also use modulation, however they do so in different ways. CPM guides students' work in a chapter by setting the lesson's goals using clauses with modulation. To a lesser extent, CPM uses clauses with modulation to set goals for particular tasks. When lesson or task goals are set, students must work within those constraints, meaning that their conceptual agency is constrained. Rather than being proactive, students become reactive when task goals are set for them, as is the case with CPM. Students must then work within the limits of the task goal set by textbook authors. Similarly, students become reactive when mathematical suggestions are given to them. Boaler (2002) however argues that assisting students working with reform curricula by helping them become aware of goals and purposes for the work they are involved in can actually help them make sense of and progress with their work. PI on their other hand uses clauses with modulation to give students suggestions on how they should solve mathematical tasks, thereby also limiting their ability to exercise conceptual agency. In other words, rather than thinking about those ideas by themselves, they may then use the suggestion given by the textbook in a reactive capacity.

Finally, the combined perspective also confirms the differences highlighted by the cognitive and linguistic perspectives with respect to the two learning orientations. Linking exclusive imperatives with method and solution task features showed that both CPM and PI provide students with opportunities to develop disciplinary agency in similar ways. Likewise, linking inclusive imperatives with method, solution and extension task features showed similarities with both textbook series for providing opportunities to experience conceptual agency and intellectual autonomy. So the analysis of imperative clauses linked with task features did not show any interesting differences for this analytic perspective. The findings for analysis of declarative clauses however *did* show differences reflecting the learning approaches each textbook series aligns with. Both textbook series use clauses with modalization in association with the method variable, but in different ways. CPM uses them to give students choices for method, thereby supporting their conceptual agency. PI uses them to suggest how students should think about method, thereby limiting students' choices and their conceptual agency. The CPM approach is more in line with reform-oriented mathematics where students investigate and discover on their own. The PI approach is more in line with traditional-oriented mathematics, where students are told how to do. Both textbook series also use clauses with modulation, and again in different ways. CPM employed clauses with modulation to set overall lesson goals and in relation to goal and method task features. Even though setting lesson and task goals and setting some constraints for method limits students' conceptual agency and intellectual autonomy, I argue that such positioning is in line with CPM's reform-orientation. This is because in this orientation, the textbook plays a more direct role in setting directions for students to assist them in their work independent of the classroom teacher. PI uses clauses with modulation to suggest ways for students to think about using mathematical theory and concepts, thereby again limiting students' conceptual agency and intellectual autonomy.

Implications of the findings

In this section, I discuss possible implications of the study for student learning, for teaching, for research and for the design of problem-based textbooks.

For student learning: Findings from the cognitive, the linguistic and the combined perspectives have important implications for empowering student learning. The overall findings show that CPM offers more opportunities than PI for students to develop conceptual agency and intellectual autonomy when working on the topic of functions. The important point to take note of, in regard to these findings, is that the opportunities for developing forms of agency and autonomy matter to student learning not only at the level of solving tasks but also at the level of learning orientations that suit different types of students. The findings imply that in terms of student learning, some types of students may be better suited to learning mathematics through the CPM approach whereas others may be better suited to learning mathematics through the PI approach.

With the CPM approach, learning through complex instruction (Cohen et al, 1999), there are opportunities for both historically able students and those who may not see themselves as very able and even those who belong to marginalized groups within society and who may have experienced inequitable access to educational experiences to all learn together for mutual success. Providing students in such a learning environment with opportunities for developing conceptual agency and intellectual autonomy creates the potential for a greater variety of students to be empowered in their learning. This has implications for education access and equity, in the sense that students who would normally be placed in higher tracks can engage with those who may normally be placed in lower tracks. In fact, Boaler and Staples (2008) found that urban students at “Railside School” had more integrated learning experiences because of their experience of learning through complex instruction. The students there achieved at higher levels across ability, ethnic and socioeconomic levels. Students who would normally be placed in higher tracks still achieved highly, despite the fact that they were in classrooms with those deemed of lower ability than them. Those who would normally have been placed in lower tracks also achieved highly in this mixed ability setting. Students across board were more motivated to work with each other and there were fewer incidences at Railside School of students forming cliques consisting only of members of their ethnic groups. So the learning orientation that CPM promotes has important consequences for student empowerment through the development of forms of agency and autonomy, especially in the case of a heterogeneous or diverse student population (Boaler & Staples, 2008; Nasir, 2014).

This is not to say that PI cannot not empower other kinds of students to develop forms of agency and autonomy when solving task on the topic of functions. PI provides opportunities that are suited to individual learning and progression. There are cases where the learning orientation promoted by PI may be appropriate for students. One such case is where students are more homogenous, in terms of having similar levels of ability or backgrounds. This could be a case where students are tracked. In such a case, students’ mathematical abilities would be more similar than different, so that the PI approach of providing individual opportunities for developing conceptual and disciplinary agency as well as intellectual autonomy can be appropriate for many similar kinds of students. This approach to empowering students still aligns with the traditional approach to teaching mathematics. In the case where students have diverse abilities and backgrounds, the PI approach can still position students to be empowered to develop forms of agency and autonomy. The approach and the outcomes would differ from the CPM approach. Where there is diversity among students, the PI approach could lead to greater differences in outcomes reflecting differences in students' abilities and other influences that impact students' success, such as home support. In short, the differences would reflect individual conditions among students.

Researchers in the United States have been aware for a while now that the curriculum students learn with can empower them with specific skills, learning orientations and attitudes to learning. For instance through The QUASAR Project (Silver & Stein, 1996), black students from disadvantaged backgrounds who experienced mathematics learning through problem-centered instruction developed stronger problem-solving skills than their peers of the same socioeconomic bracket. This group of higher performing students also underscores important equity and access issues related to empowerment. Not only must students learn from their mathematics lessons, they must ideally also have access to curriculum materials that extend learning beyond hypothetical situations encountered in textbooks to touch the lives of students, to engage their interest and to serve them with meaningful learning experiences. In this regard, both CPM and PI could provide carefully designed tasks with a higher degree of openness that go beyond guided tasks that engage students' own interests and that have relevance to students' lives. There are opportunities and implications for agency and autonomy here. Martin (2009) reported that when African-American students he worked with did well in mathematics, "[t]hey were in control instead of being controlled. They were planners, decision-makers and make-believe architects performing mathematical operations in context" (p. 323). He also explained that they worked on projects of personal interest to them and as a community of learners. On the other hand, other researchers such as Boaler (2002) stress that tasks that are overly open can pose challenges for students who have to figure out the purpose for the task. Perhaps there can be ways for textbook series such as CPM and PI to provide more tasks beyond the guided ones in such a way that takes into account students' own interests while empowering them to develop conceptual agency and intellectual autonomy while working on those tasks.

For teaching: Given that CPM promotes a reform-oriented approach to PBL, and PI promotes a traditional-oriented approach to the same and given the findings from this study, there are implications for teachers using curriculum of either kind in their teaching practice. Because of the nature of each curriculum orientation, teachers play different roles in relation to student learning. In CPM, teachers serve as a guide and a support for students as they lead themselves in learning. For the topic of functions, CPM teachers or others using comparable curricula who want to develop students' conceptual agency and intellectual autonomy can instill norms in their classrooms that encourage students to practice well mathematical thinking skills such as explaining thinking, giving justifications and making conjectures. Teachers can do this by instilling norms that facilitate group work and that guide students to produce solutions maximizing opportunities to draw on and develop thinking and problem-solving skills. Part of setting norms also includes preempting and addressing possible issues with regard to status frictions and to the fact that students of different ability groups would be working together working together (Cohen et al., 1999).

In PI, teachers play a more active role leading students in their learning than in CPM. For the topic of functions, PI teachers can also set norms that reinforce learning behaviors that will support students' development of conceptual agency and intellectual autonomy. As with CPM, PI teachers can emphasize that students pay attention to how they explain their solutions. If individual students practice thinking mathematically through the manner in which they express themselves in solutions to challenging mathematical tasks, they can develop their mathematical thinking skills. PI teachers can instill that norm. Another norm that PI teachers can instill in their students is the attitude of attempting challenging tasks to develop conceptual agency and intellectual autonomy, even if students struggle. For the topic of functions, such a norm can counterbalance students' frequent practice with the tasks in the middle stage of the lesson that give students practice with developing disciplinary agency.

The linguistic perspective also stresses the importance of knowledge concerning language choices. For instance, being aware of verbs that can be associated with exclusive imperatives versus those that can be associated with the inclusive kind can help teachers to better support students to take up opportunities in texts and tasks.

From the discussion in the section above on student learning, teachers would also benefit from stronger knowledge of the cultures and identities their students bring into the classroom so as to best support their students, especially in cases where textbooks might be tackled in ways that not only allow students to exercise conceptual agency and intellectual autonomy but that could also potentially engage students' curiosity and own life contexts to make the task more meaningful.

For research: The findings for this study imply that specific topics in mathematics textbooks can offer students opportunities to develop agency and autonomy. This study focused on the topic of functions because functions are such an important part of the mathematics curriculum from very early stages of learning right through to advanced mathematics study at the tertiary level. The findings from this study represent student learning during a three year span of that entire spectrum. Given the limited scope of the findings generated in this study, it has still been possible to gain a snapshot of what kinds of opportunities for developing forms of agency that PBL textbooks afford students in their learning. This is important because there appear not to be that many studies examining textbook tasks for opportunities for learning. Kolovou et al. (2009) is one of the few studies that does, and that study focused on non-PBL textbook series at the primary level.

Another important implication of the findings relevant to research is that I chose an approach to linguistic analysis of forms of agency and autonomy that focused in more detailed ways on language choices than other researchers have done. Morgan (1996) and Herbel-Eisenmann (2007) both conflate the notions of modalization and modulation in their analysis of modality. In this dissertation, for the analysis of lesson texts, I separate the two notions in order to better understand the ways in which each textbook series positions students to develop forms of agency and autonomy. By distinguishing modalization from modulation, it was possible to have a more nuanced analysis of CPM and PI lesson texts. This is because both textbook series use declarative clauses with modalization and with modulation in different ways that impact the ways each textbook series provides opportunities for students to develop agency and autonomy. Therefore, analyzing lesson texts both for declarative clauses with modalization and with modulation may reveal different ways that different textbook series position students to develop forms of agency and autonomy.

A third important implication of the findings relevant to research is the development of methodology in this study that allows the goal, method, solution and extension spaces of tasks to be linked with linguistic properties of clauses within the task to reveal opportunities for students to develop agency and autonomy. An earlier study that attempted to map goal, method and solution spaces to determine opportunities for students to exercise autonomy is Morgan and

Sfard (2016). Morgan and Sfard (2016) use linguistic tools to show opportunities for students to experience autonomy as a result of solving a task. Morgan and Sfard (2016)'s method involves preempting the decisions students will have to take in order to solve simple procedural tasks. This is in contrast to the approach I take in this dissertation. What they attempted solely with linguistic methods, this study does by combining cognitive and linguistic perspectives. In this study, by assigning open and closed states to specific task features, and by linking those task features with linguistic properties, it becomes possible to reveal opportunities for students to develop forms of agency and autonomy depending on how textbook authors' use language choices in relation to task features. The result is that even in the cases of more complex and challenging tasks with open features for which it may not be possible to determine all the ways students might think about the task, it is possible to reveal opportunities for students to develop agency and autonomy through their work on the task. Therefore the analytic approach I introduce in this dissertation may be a useful new research approach for analyzing different types of textbook tasks for opportunities they provide students to develop agency and autonomy based on linguistic and task features.

For textbook design*:* all the three perspectives have relevant implications for what textbooks can take into account when designing texts and tasks for students to work on. The main opportunities for students to develop forms of agency and autonomy are tied to the types of tasks they can work on. In order to develop more conceptual agency, textbook designers can have more challenging tasks that can empower students learning. In addition to procedural tasks, students can also work on guided investigative and real life tasks. They can be given opportunities to present conceptual explanations and to synthesize learning.

These are choices that textbook designers can make, which in turn depend on the learning orientation that a given textbook series supports. Depending on whether a given textbook series foregrounds investigation and exploration, as is the case with CPM, or repetition and practice, as is the case with PI, design choices can be made. The important point of note is that even with these two learning orientations, it is still possible to make strategic choices to develop students' conceptual agency and intellectual autonomy. The linguistic perspective analysis showed that it matters how textbook series are designed, because the structure and language of lesson texts can reflect learning orientations.

Findings from the linguistic and combined perspective also have implications for textbook design. From the linguistic perspective, this study revealed that the stages of a lesson text can indicate the manner in which students work during lessons. Textbook designers can choose to have students develop different forms of agency depending on the work they decide students should do at a given stage. Knowing that use of clauses with modalization and with modulation can support or restrict students' expression of conceptual agency, textbook designers can make strategic choices taking into mind the kinds of agency and autonomy they wish for students to develop forms of agency.

For the development of empowered identities: in this dissertation, I investigated ways that lesson texts position students to develop forms of agency and autonomy. Through the analysis of cognitive aspects of mathematical tasks and of linguistic aspects of mathematical tasks and lesson texts, I have shown how CPM and PI can position students to develop these attributes. The analysis included parsing text to generate frequencies for task features and linguistic attributes that indicate distinct forms of agency and autonomy. Developing the details of each analytic process and the distinct forms of agency and autonomy they reveal were centrally important in this study. The implications of knowing about how lesson texts position students to develop distinct forms of agency and autonomy which can then lead to particular learning outcomes are even more important.

The development of distinct forms of agency and autonomy during classroom learning serves the greater purpose of equipping students with critical thinking and problem-solving skills that they can internalize and use in their own lives. These critical thinking and problem-solving skills that students develop as they work on different kinds of mathematical tasks in class can contribute to the formation of students' empowered identities as thinkers and users of mathematics. For instance, students can be empowered into becoming educators. They can be empowered into becoming mathematicians. They can be empowered into becoming engineers, natural or social scientists. These constitute only a few possibilities. More importantly, they can be empowered into becoming thinkers and problem solvers who navigate their intellectual, cultural, economic and social realities having skill sets for which their ability to be agentic and autonomous thinkers serves the particular needs of their individual lives.

In this regard, the ultimate importance of focusing on students developing forms of agency and autonomy in this dissertation need not lay only on distinctions between for instance how many tasks allow students to develop conceptual and disciplinary agency or proactive and reactive autonomy. Neither must it lay only on the distributions of different kinds of declaratives and imperatives and the functions they play in lesson texts. Instead, it must emphasize the possibilities of developing empowered identities as mathematical thinkers and problem solvers as eventual outcomes of student learning. Phil Benson, a scholar on autonomy, posited that "agency can perhaps be viewed as a point of origin for the development of autonomy, while identity might be viewed as one of its more important outcomes" (Benson, 2007, p. 30).

Therefore awareness of the empowered identities that can result from students' development of forms of agency and autonomy while learning mathematics is the educational outcome of ultimate importance that this study brings attention to. That awareness depends on a better understanding of opportunities given in mathematical tasks and the instructional orientations within which students enact tasks. It depends on teachers' awareness of the importance of assisting students to master distinct forms of agency and autonomy during learning. It depends on textbook designers making such opportunities available through lesson texts in ways that are accessible and effective for student learning. It also benefits from researchers studying curriculum materials as designed and during enactment in order to provide feedback to practitioner, publisher and researcher communities both on the opportunities and on the outcomes related to those opportunities.

Limitations

One aspect of the study I conducted in the dissertation that limits the scope of the study is the fact that I focused primarily on analyzing some aspects of the lesson and not others. The analysis I carried out was largely limited to text. As a result of focusing on text for analysis, I did not adopt analytic tools for non-textual aspects within mathematical tasks. As non-textual aspects of lessons include mathematical objects such as graphs and tables that are closely linked to mathematical tasks, an analysis of these mathematical objects can only contribute to a more complete analysis of the lesson text. Another aspect of the study that was limited in scope is the use of a task classification framework that classifies mathematical tasks based on task features. This is the framework by Yeo (2017a), which includes the five task features: task goal, method, complexity, solution, and extension. These features limit the types of tasks that can be analyzed. Some tasks, for example those that involve proof, are more difficult to classify with this limited framework which is more process oriented. A third aspect entails not having analyzed teacher support materials. Having teacher materials analyzed in addition to students materials may have shed more light on some of the intentions the textbook authors had in their design of particular mathematical tasks and how those tasks were intended to be used in the lesson. These intentions could have revealed insights into opportunities for students to develop agency and autonomy. A fourth limitation involves the absence of in-class studies to test the concepts presented in this study. To better understand how the potential for students to develop agency and autonomy becomes reality, in-class studies can be conducted to show how students respond to the demands of tasks, which opportunities they capitalize on and those they do not.

Future research

One aspect of research that can extend the current study is to examine texts and tasks for topics other than functions and graphs for the same CPM and PI textbook series, to investigate whether similar findings will be generated with a different topic such as algebra. This study will replicate the one in this dissertation with a different topic, to determine whether similar opportunities are available for students with a different topic. Studying a different topic can also reveal differences and similarities in the distribution of tasks across textbook volumes. This distribution may hint at a possible developmental trajectory similar to the case of the sampled tasks in CPM.

Another aspect of research that can extend the current study is to vary the textbooks by studying material other than CPM and PI. These can be traditional or reform textbooks that support problem-based learning. For traditional textbooks, further research applying the same analytic approaches I developed for this dissertation could shed light on what findings emerge when the approaches adopted in this dissertation are applied to them.

A third aspect that can extend this work is to make comparisons between teacher guides and student textbooks to ascertain links between how the textbook authors are directing teachers to teach content in specific chapters and how the textbook authors direct students to learn content in those chapters. The question of teacher agency and autonomy can also be investigated in regard to studying teacher guides.

A fourth aspect for future research involves extending the study of agency and autonomy related to textbooks beyond the lesson tasks and texts to include working directly with students. This may possibly involve observing students work on tasks and conducting cognitive interviews of students to learn from them how they perceive and understand the opportunities made available to them in the lesson texts and mathematical tasks.

Conclusion

PBL textbooks provide students with different opportunities to develop forms of agency and autonomy through work on different task types and through other supporting lesson texts. This study analyzed CPM and PI, two mathematics textbook series that support PBL, to learn how each one positions students to develop forms of agency and autonomy with respect to topic on functions. The findings revealed that both textbook series provide students with those opportunities however CPM provides a wider range of opportunities for developing conceptual agency and intellectual autonomy than PI. PI provides proportionally more opportunities for students to develop disciplinary agency. These findings are not without context. They reflect the learning orientations of each textbook series. CPM supports student learning through a reform-oriented approach to PBL whereas PI does the same through traditional-oriented approach to PBL. Each approach provides opportunities for teaching different kinds of students.

Appendices

Appendix 1: Coded Examples of Task Features

Table A1.1: Task variables with examples to illustrate when and why tasks are closed or open

Variable	State	Task example with Comments
Goal	Closed [G:C]– Goal in task is clear, explicit and specific and there is no room for individuals to pursue their own goal.	**“What is the simplified form of each expression? a. 9^{-2} b.–$(3.6)^{0}$” Comments: the goal for this task is closed because the task is specific about what students should do.
Goal	Open, well-defined [G:OWD] – Task provides some guidance on possible ways to go about the task.	“Powers of 3 are 3^1, 3^2, 3^3, 3^4, …Find as many patterns as possible” (Yeo, 2017, p. 180). Comments: the goal for this task is open and well-defined because students are at liberty to explore any specific number of patterns they choose.
Goal	Open, ill-defined [G:ID] – goal is vague, such as asking for task to be investigated without specifying further.	“Powers of 3 are 3^1, 3^2, 3^3, 3^4, …Investigate” (Yeo, 2017, p. 178) Comments: the goal for this task is open and ill-defined because the task is vague about the exact nature of what students are expected to do.
Method	Closed [M:C] – when it is clear which procedure or procedures are needed to enact the task. (e.g. procedures without connections,	**“What is the simplified form of each expression? a. 9^{-2} b.–$(3.6)^{0}$” Comments: the method for this task is closed because students are expected to practice standard algorithms to solve the task.
Method	Open, well-defined [M:OWD] – teachable heuristics and strategies that when applied can lead to an answer and are repeatable when adopted by different students.	*First investigate the graphs for the sum and difference of your two functions. Predict what happens if you add or subtract any two linear functions. Can you think of any exceptions? Comments: as a standalone, the method needed to solve this task is open and well-defined, as students will have to come up with a strategy for investigating the task based on their understanding of sums and differences of linear functions. As the task appeared in CPM, earlier tasks within the same chapter set students up to work through a method that led to the task above, so the method was effectively scaffolded for students.
Method	Open, ill-defined [M:OID] – where it is not immediately clear which procedure or procedures are needed to enact the task but where students can draw on methods they know or come up with, as intermediate steps, as they work to solve the task.	“Choose a mathematics project to do. Submit a report at the end of the year.” (Yeo, 2017, p.181) Comments: the method needed to solve this task is open and ill-defined because the task is so underspecified that without any extra direction, students will be expected to devise a method from the ground up, in order to realize the project.
Method	Open, task-inherent – where it is impossible to rely on one method alone to come up with all the correct answers for	“Powers of 3 are 3^1, 3^2, 3^3, 3^4, … Investigate” (Yeo, 2017, p.178).

	the task.	Comments: in this case, the method is inherently open, because students can take different approaches to solving the task.
Complexity	Closed [C:C] – if the task provides all the information they need to solve the task. (e.g. procedures without connections,	*In 2011, a brand new SUV cost $35,000 to drive off the lot. In 2014, that same SUV was valued at $22,500. Write an exponential equation to represent this information. What is the rate of depreciation for the SUV? Comments: The task provides all the information needed to drive the method and to arrive at a solution.
Complexity	Open, student-dependent [C:OSD] – with provided scaffolding (e.g. investigative tasks, problem-solving tasks, procedures with connections), the task can become accessible	*What can the solutions to a quadratic equation tell you about the graph of the related quadratic function? Graph the functions below on your graphing calculator and sketch the graph of each function. a. $y = (5x - 2)^2 + 6$ b. $y = (4 + 2x)^2$ c. $y = (7 + 2x)^2 - 11$ d. How do your graphs relate to your answers from problem 5-63? Explain. Comments: The complexity variable for this task is open and student-dependent because the quality of solutions depends on each student's individual ability.
Complexity	Open, task-inherent [C: OTI] – nature of task is such that even with scaffolding, the task does not become closed.	"Choose a mathematics project to do. Submit a report at the end of the year." (Yeo, 2017, p.181). Comments: This task is inherently complex because it is very underspecified and thus needs all its features structured in order to solve the task.
Solution	Closed [S: C] – when it is possible to determine all correct answers to the task.	**"What is the simplified form of each expression? a. 9^{-2} b. $-(3.6)^0$" Comments: The solution variable is closed because it is possible to determine the correct numerical solutions.
Solution	Open, well-defined [S: OWD] – when it is clear which answers count as being correct or as being incorrect.	*First investigate the graphs for the sum and difference of your two functions. Predict what happens if you add or subtract any two linear functions. Can you think of any exceptions? Comments: The complexity variable for this task is open and well-defined because although there is room for students to work out details of method and solution, they will likely draw on the theory of linear functions. The solution can therefore be appraised objectively.
Solution	Open, ill-defined [S: OID] – when it is left to interpretation which solutions count as correct or incorrect.	"Choose a mathematics project to do. Submit a report at the end of the year." (Yeo, 2017, p.181). Comments: The solution variable is open and ill-defined because the task is underspecified. The only instruction students must follow is to submit a

		report at year's end.
Extension	Closed [E: C] – tasks that cannot be extended or that will lead to new tasks being formed.	**"What is the simplified form of each expression? a. 9^{-2} b.–$(3.6)^{0}$" <u>Comments</u>: This task is closed for extension because any changes to the task will result in a new task being formed.
Extension	Open, student-dependent [E:OSD] – whether or not the teacher or student can facilitate task extension. (e.g. problem-solving tasks)	*Look over your work from this lesson. What questions did you ask yourself as you were making observations and statements? How does changing the value of *b* affect a graph? What questions do you still have after this investigation? Write a Learning Log entry describing what mathematical ideas you developed during this lesson. Title this entry "Investigating $y = b^{x}$" and include today's date. <u>Comments</u>: The extension variable for this task is open and student-dependent because the quality of solutions depends on each student's individual ability.
Extension	Open, task-inherent [E:OTI] – the nature of the task as stated lends itself to being extended (e.g. investigative tasks)	*First investigate the graphs for the sum and difference of your two functions. Predict what happens if you add or subtract any two linear functions. Can you think of any exceptions? <u>Comments</u>: The complexity for this task is open and student-dependent because the quality of solutions depends on each student's individual ability.

Appendix 2: Task types and their features

Table A2.1: Task types ranging from the most closed to the most open

Task type	**Task features and states of openness**
Procedures without connections	Goal, method space, complexity, solution space & extension all closed
Procedures with connections	Goal, method space both closed; complexity, solution space & extension– closed or open (student-dependent)
Guided Real Life	Goal, method space – closed; complexity – open (task inherent); solution space – open (ill-defined); extension – closed
Guided Investigative	Goal, method space –closed; complexity – open (student-dependent); solution space – open (well-defined); extension – open (task-inherent)
Conceptual Explanations	Goal – closed; method space – open (well-defined); complexity – open (student-dependent); solution space – open (well-defined); extension – closed
Synthesis	Goal - closed, method space – open (guided); complexity – open (student-dependent); solution space – open (well-defined); extension – open (student-dependent)
Problem-solving	Goal – closed; method space – open (student-dependent or well-defined); complexity – open (student-dependent); solution space – closed; extension – open (student-dependent)
Real-life	Goal – closed or open (well-defined or ill-defined); method space – open (ill-defined and/or task-inherent); complexity – open (task-inherent); solution space – open (ill-defined); extension – closed
Investigative	Goal – open (well-defined or ill-defined) or closed; method space– open (well-defined and/or task-inherent); complexity – open (student-dependent); Solution – open (well-defined); Extension – open (task-inherent)

Appendix 3: Coding results for CPM and PI task types

Table A3.1: Coding for mathematical tasks in CPM lessons

Textbook	Vol	Lesson	Stage	Problem	Goal	Method	Complexity	Answer	Extension	Type	#	
CPM	1	1.1.1	MT	1-1	C	C	OSD	OWD	C	PwC	1	1
CPM	1	1.1.1	MT	1-2	C	C	OSD	OWD	C	PwC	2	2
CPM	1	1.1.1	MT	1-3	C	C	OSD	OWD	C	PwC	3	3
CPM	1	1.1.1	MT	1-4	C	C	OSD	OWD	C	PwC	4	4
CPM	1	1.1.1	MT	1-5	C	C	OSD	C	C	PwC	5	5
CPM	1	1.2.1	MT	1-33	C	C	OSD	C	C	PwC	6	6
CPM	1	1.2.1	MT	1-34	C	C	OSD	C	C	PwC	7	7
CPM	1	1.2.1	MT	1-35	C	C	OSD	C	C	PwC	8	8
CPM	1	1.2.1	MT	1-36	C	C	OSD	OWD	C	PwC	9	9
CPM	1	1.2.1	MT	1-37	C	C	OSD	C	C	PwC	10	10
CPM	1	1.3.2	MT	1-73	C	C	OSD	OWD	OSD	SyN	1	11
CPM	1	1.3.2	MT	1-74	C	C	OSD	OWD	OSD	SyN	2	12
CPM	1	1.3.2	MT	1-75	C	C	OSD	OWD	C	PwC	11	13
CPM	1	1.3.2	MT	1-76	C	C	OSD	OWD	OTI	GIT	1	14
CPM	1	1.3.2	MT	1-77	C	C	C	C	C	Pw/oC	1	15
CPM	1	1.3.2	MT	1-78	C	C	OSD	C	C	PwC	12	16
CPM	1	1.3.2	MT	1-79	C	C	OSD	C	C	PwC	13	17
CPM	1	1.3.2	MT	1-80	C	C	OSD	OWD	OSD	SyN	3	18
CPM	1	2.1.1	MT	2-1	C	C	OSD	OWD	OTI	GIT	2	19
CPM	1	2.1.1	MT	2-2	C	C	OSD	OWD	OTI	GIT	3	20
CPM	1	2.1.1	MT	2-3	C	C	OSD	C	C	PwC	14	21

CPM	1	2.1.1	MT	2-4	C	C	OSD	OWD	OSD	SyN	4	22
CPM	1	2.1.1	MT	2-5	C	C	OSD	OWD	OSD	SyN	5	23
CPM	1	2.2.2	MT	2-52	C	C	OSD	OWD	C	PwC	15	24
CPM	1	2.2.2	MT	2-53	C	C	OSD	OWD	C	PwC	16	25
CPM	1	2.2.2	MT	2-54	C	C	OSD	C	C	PwC	17	26
CPM	1	2.2.2	MT	2-55	C	C	OSD	OWD	C	PwC	18	27
CPM	1	2.2.2	MT	2-56	C	OWD	OSD	OWD	C	CE	1	28
CPM	1	2.2.2	MT	2-57	C	OWD	OSD	OWD	C	SyN	6	29
CPM	1	2.3.2	MT	2-99	C	C	OSD	C	C	PwC	19	30
CPM	1	2.3.2	MT	2-100	C	C	OSD	C	C	PwC	20	31
CPM	1	2.3.2	MT	2-101	C	C	OSD	OWD	OSD	SyN	7	32
CPM	1	8.1.1	MT	8-1	C	C	OSD	OWD	C	PwC	21	33
CPM	1	8.1.1	MT	8-2	C	C	OSD	OWD	OTI	GIT	4	34
CPM	1	8.1.1	MT	8-3	C	C	OSD	OWD	C	PwC	22	35
CPM	1	8.1.1	MT	8-4	C	C	OSD	OWD	OTI	GIT	5	36
CPM	1	8.1.1	MT	8-5	C	C	OSD	OWD	OTI	GIT	6	37
CPM	1	8.1.1	MT	8-6	C	C	OSD	OWD	OSD	SyN	8	38
CPM	1	8.1.5	MT	8-66	C	C	OSD	C	C	PwC	23	39
CPM	1	8.1.5	MT	8-67	C	C	OSD	OWD	C	PwC	24	40
CPM	1	8.1.5	MT	8-68	C	C	C	C	C	Pw/oC	2	41
CPM	1	8.1.5	MT	8-69	C	OWD	OSD	OWD	C	CE	2	42
CPM	1	8.2.3	MT	8-115	C	C	OSD	OID	C	GRLT	1	43
CPM	1	8.2.3	MT	8-116	C	C	OSD	OWD	OTI	GIT	7	44
CPM	1	8.2.3	MT	8-117	C	C	C	C	C	Pw/oC	3	45
CPM	1	8.2.3	MT	8-118	C	C	C	C	C	Pw/oC	4	46
CPM	2	5.1.1	MT	5-1	C	C	OSD	OWD	OTI	GIT	8	47
CPM	2	5.1.1	MT	5-2	C	OWD	OSD	OWD	C	CE	3	48
CPM	2	5.1.1	MT	5-3	C	C	OSD	OWD	OTI	SyN	9	49

CPM	2	5.2.1	MT	5-59	C	C	OSD	OWD	C	PwC	25	50
CPM	2	5.2.1	MT	5-60	C	C	OSD	OWD	C	PwC	26	51
CPM	2	5.2.1	MT	5-61	C	C	C	C	C	Pw/oC	5	52
CPM	2	5.2.1	MT	5-62	C	C	OSD	OWD	C	PwC	27	53
CPM	2	5.2.1	MT	5-63	C	C	OSD	C	C	PwC	28	54
CPM	2	5.2.1	MT	5-64	C	C	OSD	OWD	C	PwC	29	55
CPM	2	5.2.6	MT	5-124	C	C	C	C	C	Pw/oC	6	56
CPM	2	5.2.6	MT	5-125	C	C	OSD	OWD	C	PwC	30	57
CPM	2	5.2.6	MT	5-126	C	C	C	C	C	Pw/oC	7	58
CPM	2	5.2.6	MT	5-127	C	C	OSD	C	C	PwC	31	59
CPM	2	5.2.6	MT	5-128	C	C	OSD	OWD	C	PwC	32	60
CPM	2	5.2.6	MT	5-129	C	C	OSD	C	C	PwC	33	61
CPM	2	5.2.6	MT	5-130	C	C	OSD	OWD	OTI	GIT	9	62
CPM	3	1.1.1	MT	1-1	C	C	OSD	OWD	C	PwC	34	63
CPM	3	1.1.1	MT	1-2						N/A		64
CPM	3	1.1.1	MT	1-3	C	OSD	OSD	C	OSD	PST	1	65
CPM	3	1.1.4	MT	1-49	C	C	OSD	OWD	C	PwC	35	66
CPM	3	1.1.4	MT	1-50	C	OWD	OSD	OWD	OTI	IT	1	67
CPM	3	1.1.4	MT	1-51	C	OWD	OSD	OWD	OTI	IT	2	68
CPM	3	1.1.4	MT	1-52	C	C	C	C	C	Pw/oC	8	69
CPM	3	1.1.4	MT	1-53	C	C	OSD	OWD	OTI	GIT	10	70
CPM	3	1.1.4	MT	1-54	C	C	OSD	OWD	OTI	GIT	11	71
CPM	3	1.1.4	MT	1-55	C	C	OSD	OWD	C	PwC	36	72
CPM	3	1.2.3	MT	1-101	C	C	OTI	OID	C	GRLT	2	73
CPM	3	1.2.3	MT	1-102	C	C	OTI	OID	C	GRLT	3	74
CPM	3	9.1.1	MT	9-1	C	C	OTI	OID	C	GRLT	4	75
CPM	3	9.1.1	MT	9-2	C	C	OTI	OID	C	GRLT	5	76
CPM	3	9.1.6	MT	9-72	C	C	C	C	C	Pw/oC	9	77

CPM	3	9.1.6	MT	9-73	C	C	C	C	C	Pw/oC	10	78
CPM	3	9.1.6	MT	9-74	C	C	C	C	C	Pw/oC	11	79
CPM	3	9.16	MT	9-75	C	C	C	OWD	C	PwC	37	80
CPM	3	9.2.3	MT	9-116						N/A		81
CPM	3	9.2.3	MT	9-117	C	C	OSD	OWD	OTI	GIT	12	82
CPM	3	9.2.3	MT	9-118	C	C	OSD	OWD	C	PwC	38	83
CPM	3	9.2.3	MT	9-119	C	C	C	C	C	Pw/oC	12	84
CPM	3	9.2.3	MT	9-120	C	C	OSD	OWD	C	PwC	39	85

Table A3.2: Coding for mathematical tasks in PI lessons

Textbook	**Vol**	**Lesson**	**Stage**	**Task**	**Goal**	**Method**	**Complexity**	**Answer**	**Extension**	**Type**	**#**	
PI	1	2.1	IL	SolveIt	C	C	OSD	OWD	C	PwC	1	1
PI	1	2.1	GIP	Prob	C	C	OSD	OWD	C	PwC	2	2
PI	1	2.1	GIP	Prob	C	C	OSD	C	C	PwC	3	3
PI	1	2.1	GIP	Prob	C	C	OSD	OWD	C	PwC	4	4
PI	1	2.1	LC	DYKH	C	C	C	OWD	C	PwC	5	5
PI	1	2.1	LC	DYU	C	C	OSD	OWD	C	PwC	6	6
PI	1	2.4	IL	SolveIt	C	C	OSD	OWD	C	PwC	7	7
PI	1	2.4	GIP	Prob	C	C	C	C	C	Pw/oC	1	8
PI	1	2.4	GIP	Prob	C	C	OSD	OWD	C	PwC	8	9
PI	1	2.4	GIP	Prob	C	C	OSD	C	C	PwC	9	10
PI	1	2.4	GIP	Prob	C	C	C	C	C	Pw/oC	2	11
PI	1	2.4	LC	DYKH	C	C	OSD	C	C	PwC	10	12
PI	1	2.4	LC	DYU	C	C	OSD	OWD	C	PwC	11	13
PI	1	2.7	IL	SolveIt	C	C	OSD	OWD	C	PwC	12	14
PI	1	2.7	GIP	Prob	C	C	C	C	C	Pw/oC	3	15

PI	1	2.7	GIP	Prob	C	C	C	C	C	Pw/oC	4	16
PI	1	2.7	GIP	Prob	C	C	OSD	C	C	PwC	13	17
PI	1	2.7	GIP	Prob	C	C	OSD	C	C	PwC	14	18
PI	1	2.7	GIP	Prob	C	C	C	C	C	Pw/oC	5	19
PI	1	2.7	LC	DYKH	C	C	C	C	C	Pw/oC	6	20
PI	1	2.7	LC	DYU	C	C	OSD	C	C	PwC	15	21
PI	1	3.1	IL	SolveIt	C	C	OSD	OWD	C	PwC	16	22
PI	1	3.1	GIP	Prob	C	C	OSD	OWD	C	PwC	17	23
PI	1	3.1	GIP	Prob	C	C	C	C	C	Pw/oC	7	24
PI	1	3.1	GIP	Prob	C	C	OSD	OWD	C	PwC	18	25
PI	1	3.1	GIP	Prob	C	C	C	C	C	Pw/oC	8	26
PI	1	3.1	LC	DYKH	C	C	OSD	OWD	C	PwC	19	27
PI	1	3.1	LC	DYU	C	C	OSD	OWD	C	PwC	20	28
PI	1	3.4	IL	SolveIt	C	C	C	C	C	Pw/oC	9	29
PI	1	3.4	GIP	Prob	C	C	C	C	C	Pw/oC	10	30
PI	1	3.4	GIP	Prob	C	C	C	C	C	Pw/oC	11	31
PI	1	3.4	GIP	Prob	C	C	OSD	OWD	C	PwC	21	32
PI	1	3.4	GIP	Prob	C	C	OSD	OWD	C	PwC	22	33
PI	1	3.4	LC	DYKH	C	C	C	C	C	Pw/oC	12	34
PI	1	3.4	LC	DYU	C	C	OSD	OWD	C	PwC	23	35
PI	1	3.7	IL	SolveIt	C	C	OSD	OWD	C	PwC	24	36
PI	1	3.7	GIP	Prob	C	C	OSD	OWD	C	PwC	25	37
PI	1	3.7	GIP	Prob	C	C	C	C	C	Pw/oC	13	38
PI	1	3.7	GIP	Prob	C	C	C	C	C	Pw/oC	14	39
PI	1	3.7	GIP	Prob	C	C	C	C	C	Pw/oC	15	40
PI	1	3.7	GIP	Prob	C	C	C	C	C	Pw/oC	16	41
PI	1	3.7	LC	DYKH	C	C	C	C	C	Pw/oC	17	42
PI	1	3.7	LC	DYU	C	C	OSD	OWD	OTI	GIT	1	43

PI	1	5.1	IL	SolveIt	C	C	OSD	OWD	OTI	GIT	2	44
PI	1	5.1	GIP	Prob	C	C	C	C	C	Pw/oC	18	45
PI	1	5.1	GIP	Prob	C	C	C	C	C	Pw/oC	19	46
PI	1	5.1	GIP	Prob	C	C	C	C	C	Pw/oC	20	47
PI	1	5.1	GIP	Prob	C	C	OSD	OWD	C	PwC	26	48
PI	1	5.1	LC	DYKH	C	C	C	C	C	Pw/oC	21	49
PI	1	5.1	LC	DYU	C	C	OSD	C	C	PwC	27	50
PI	1	5.5	IL	SolveIt	C	C	C	C	C	Pw/oC	22	51
PI	1	5.5	GIP	Prob	C	C	C	C	C	Pw/oC	23	52
PI	1	5.5	GIP	Prob	C	C	C	C	C	Pw/oC	24	53
PI	1	5.5	GIP	Prob	C	C	C	C	C	Pw/oC	25	54
PI	1	5.5	GIP	LC	C	C	C	C	C	Pw/oC	26	55
PI	1	5.5	GIP	LC	C	C	OSD	OWD	OSD	SyN	1	56
PI	1	5.9	IL	SolveIt	C	C	OSD	OWD	C	PwC	28	57
PI	1	5.9	GIP	Prob	C	C	C	C	C	Pw/oC	27	58
PI	1	5.9	GIP	Prob	C	C	C	C	C	Pw/oC	28	59
PI	1	5.9	GIP	Prob	C	C	C	C	C	Pw/oC	29	60
PI	1	5.9	GIP	Prob	C	C	C	C	C	Pw/oC	30	61
PI	1	5.9	GIP	Prob	C	C	OSD	C	C	PwC	29	62
PI	1	5.9	LC	DYKH	C	C	OSD	OWD	C	PwC	30	63
PI	1	5.9	LC	DYU	C	C	OSD	OWD	C	PwC	31	64
PI	2	12.1	IL	SolveIt	C	C	OSD	OWD	C	PwC	32	65
PI	2	12.1	GIP	Prob	C	C	C	C	C	Pw/oC	31	66
PI	2	12.1	GIP	Prob	C	C	C	C	C	Pw/oC	32	67
PI	2	12.1	GIP	Prob	C	C	C	C	C	Pw/oC	33	68
PI	2	12.1	GIP	Prob	C	C	C	C	C	Pw/oC	34	69
PI	2	12.1	GIP	Prob	C	C	OSD	OWD	C	PwC	33	70
PI	2	12.1	LC	DYKH	C	C	C	C	C	Pw/oC	35	71

PI	2	12.1	LC	DYU	C	OWD	OSD	OWD	C	CE	1	72
PI	2	12.6	IL	SolveIt	C	C	OSD	OWD	C	PwC	34	73
PI	2	12.6	GIP	Prob	C	C	C	C	C	Pw/oC	36	74
PI	2	12.6	GIP	Prob	C	C	C	C	C	Pw/oC	37	75
PI	2	12.6	GIP	Prob	C	C	C	C	C	Pw/oC	38	76
PI	2	12.6	GIP	Prob	C	C	OSD	C	C	PwC	35	77
PI	2	12.6	LC	DYKH	C	C	C	C	C	Pw/oC	39	78
PI	2	12.6	LC	DYU	C	C	OSD	OWD	OSD	SyN	2	79
PI	2	12.12	IL	SolveIt	C	C	OSD	OWD	C	PwC	36	80
PI	2	12.12	GIP	Prob	C	C	C	C	C	Pw/oC	40	81
PI	2	12.12	GIP	Prob	C	C	C	C	C	Pw/oC	41	82
PI	2	12.12	GIP	Prob	C	C	OSD	OWD	C	PwC	37	83
PI	2	12.12	GIP	Prob	C	C	C	C	C	Pw/oC	42	84
PI	2	12.12	GIP	Prob	C	C	C	C	C	Pw/oC	43	85
PI	2	12.12	LC	DYKH	C	C	C	C	C	Pw/oC	44	86
PI	2	12.12	LC	DYU	C	OWD	OSD	OWD	C	CE	2	87
PI	3	5.1	IL	SolveIt	C	C	OSD	OWD	C	PwC	38	88
PI	3	5.1	GIP	Prob	C	C	C	C	C	Pw/oC	45	89
PI	3	5.1	GIP	Prob	C	C	C	C	C	Pw/oC	46	90
PI	3	5.1	GIP	Prob	C	C	C	C	C	Pw/oC	47	91
PI	3	5.1	GIP	Prob	C	C	OSD	OWD	C	PwC	39	92
PI	3	5.1	LC	DYKH	C	C	C	C	C	Pw/oC	48	93
PI	3	5.1	LC	DYU	C	C	OSD	OWD	OSD	SyN	3	94
PI	3	5.4	IL	SolveIt	C	C	OSD	OWD	C	PwC	40	95
PI	3	5.4	GIP	Prob	C	C	OSD	C	C	PwC	41	96
PI	3	5.4	GIP	Prob	C	C	C	C	C	Pw/oC	49	97
PI	3	5.4	GIP	Prob	C	C	OSD	C	C	PwC	42	98
PI	3	5.4	GIP	Prob	C	C	OSD	C	C	PwC	43	99

PI	3	5.4	GIP	Prob	C	C	OSD	OWD	C	PwC	44	100
PI	3	5.4	LC	DYKH	C	C	OSD	C	C	PwC	45	101
PI	3	5.4	LC	DYU	C	C	OSD	OWD	C	PwC	46	102
PI	3	5.7	IL	SolveIt	C	C	C	C	C	Pw/oC	50	103
PI	3	5.7	GIP	Prob	C	C	C	C	C	Pw/oC	51	104
PI	3	5.7	GIP	Prob	C	C	OSD	OWD	C	PwC	47	105
PI	3	5.7	GIP	Prob	C	C	C	C	C	Pw/oC	52	106
PI	3	5.7	LC	DYKH	C	C	C	C	C	Pw/oC	53	107
PI	3	5.7	LC	DYU	C	OWD	OSD	OWD	C	CE	3	108
PI	3	8.1	IL	SolveIt	C	C	OSD	OWD	C	PwC	48	109
PI	3	8.1	GIP	Prob	C	C	C	C	C	Pw/oC	54	110
PI	3	8.1	GIP	Prob	C	C	C	C	C	Pw/oC	55	111
PI	3	8.1	GIP	Prob	C	C	C	C	C	Pw/oC	56	112
PI	3	8.1	GIP	Prob	C	C	C	C	C	Pw/oC	57	113
PI	3	8.1	LC	DYKH	C	C	C	C	C	Pw/oC	58	114
PI	3	8.1	LC	DYU	C	C	OSD	OWD	C	PwC	49	115
PI	3	8.6	IL	SolveIt	C	OWD	OSD	OWD	C	CE	4	116
PI	3	8.6	GIP	Prob	C	C	C	C	C	Pw/oC	59	117
PI	3	8.6	GIP	Prob	C	C	C	C	C	Pw/oC	60	118
PI	3	8.6	GIP	Prob	C	C	C	C	C	Pw/oC	61	119
PI	3	8.6	LC	DYKH	C	C	C	C	C	Pw/oC	62	120
PI	3	8.6	LC	DYU	C	OWD	OSD	OWD	C	CE	5	121
PI	3	8.11	IL	SolveIt	C	C	OSD	OWD	C	PwC	50	122
PI	3	8.11	GIP	Prob	C	C	OSD	OWD	C	PwC	51	123
PI	3	8.11	GIP	Prob	C	C	C	C	C	Pw/oC	63	124
PI	3	8.11	GIP	Prob	C	C	C	C	C	Pw/oC	64	125
PI	3	8.11	LC	DYKH	C	C	C	C	C	Pw/oC	65	126
PI	3	8.11	LC	DYU	C	C	OSD	OWD	C	PwC	52	127

Appendix 4: Coded CPM and PI lesson texts

Table A4.1: Analysis of the stages in a CPM lesson

Stages for 1.1.1

[[**Lesson orientation**]]
Solving puzzles in teams

<d> In previous courses, you might have looked at patterns in tables, graphs, equations, and situations that were linear. <d> In this course, you will continue your study of linear functions, and extend these patterns to new kinds of functions. <d> Note that we will define a function formally in Section 1.2 of this chapter. <d> In this lesson, you and your team will examine the inputs and outputs of functions.

[[**Classroom mathematical tasks**]]

1-1. TEAM SORT
<d> Your teacher will give you a card with an algebraic expression on it.
<im> Evaluate the expression as instructed on the card.
<im> Then find the other students in your class who have the same value *after* evaluating it.
<d> These students will be your teammates, <im> so find a table and sit together.
<im> Justify to your teammates how you know your value matches their values.
<im> After your whole team is sitting together, introduce yourselves, <im> and then turn over the card at your table.
<im> Working together with your new team, use your new table number (at the bottom of the card) to evaluate the expression on the card.
A:C; Aut-Lmgmt-Social [L]; students may be autonomous when working on

1-2. FUNCTION MACHINES
<d> At the Function Factory, a number is put into the top of a function machine and <d> the machine puts out another number depending on how it is programmed.
<d> In the illustration at right, the worker input a “3” and the machine output an “8”.
<d> This machine is programmed to square the number and subtract 1.
<d> A diagram of the machine looks like the figure at right.
<d> A customer brings a box with a mix of integers (..., −3, −2, −1, 0, 1, 2, 3,…) as inputs to a function machine.
<d> She wants you to program a function machine so that the output is always negative.
<d> Your manager suggests − 10.
<in> Did the manager make a good suggestion?
<in> Are there any inputs that will not meet the customer’s needs?
<im> Explain

[[**Math Notes**]]

Absolute Value
<d> Absolute value is the numerical value of a number without its sign. <d> The symbol for absolute value is two vertical bars, | |. <d> Absolute value can represent the distance on a number line between a number and 0. <d> Since a distance is

always positive, the absolute value is *always* either a positive value or 0. <d> The absolute value of a number is *never* negative. <d> For example, the number –3 is 3 units away from 0, as shown on the number line at right. <d> Therefore, the absolute value of –3 is 3. <d> This is written |-3| = 3. <d> Likewise, the number 5 is 5 units away from 0. <d> The absolute value of 5 is 5, written |5| = 5.

Table A4.2: Analysis of the stages in a CPM lesson

Stages for 5.1

[[**Interactive Learning**]]

Solve it
<im> Copy and complete the table. <im> Make a conjecture about how the value of an exponential expression (an expression containing an exponent) changes when you decrease the exponent by 1. <in> What do you think the value of 5^{-2} is? <im> Explain your reasoning.

[*Note*] <d> The patterns you found in the Solve It illustrate the definitions of zero and negative exponents.

Essential understanding <d>You can extend the idea of exponents to include zero and negative exponents. <d> Consider 3^3, 3^2, 3^1. <d> Decreasing the exponents by 1 is the same as dividing by 3. <d> If you continue the pattern, 3^0 equals 1 and 3^{-1} equals 1/3.

Properties Zero and negative exponents
Zero as an exponent <d> For every nonzero number a, $a^0 = 1$

Examples $4^0 = 1$ $(-3)^0 = 1$ $(5.14)^0 = 1$

Negative exponent <d> For every nonzero number a and integer n,
$a^{-n} = 1/a^n$

Examples $7^{-3} = 1/7^3$ $(-5)^{-2} = 1/(-5)^2$

[*Note*] <im> Why can't you use 0 as a base with zero exponents? <d> The first property above implies the following pattern.

$3^0 = 1$ $2^0 = 1$ $1^0 = 1$ $0^0 = 1$

<d> However, consider the following pattern.
$0^3 = 0$ $0^2 = 0$ $0^1 = 0$ $0^0 = 0$

<d> It is not possible for 0^0 to equal both 1 and 0. <d> Therefore, 0^0 is undefined. <in> Why can't you use 0 as a base with a negative exponent? <d> Using 0 as a base with a negative exponent will result in division by zero, which is undefined.

[[**Guided instruction**]]

Problem 1 [Simplifying powers] <in> What is the simplified form of each expression?

a. 9^{-2}
b. $-(3.6)^0$

Got it? <in> What is the simplified form of each expression?
a. 4^{-3}
b. $(-5)^0$

Think <in> Can you use zero as an exponent when the base is a negative number?

Practice <im> Simplify each expression.

1. $1/2^0$
2. 1.5^{-2}

[Note] <d> An algebraic expression is in its simplest form when powers with a variable base are written with only positive exponents.

Problem 2 [Simplifying exponential expressions] – <in> What is the simplified form of the expression $5a^3b^{-2}$?

Got it? <in> What is the simplified form of each expression?

a. x-9
b. $1/n^{-3}$

Think – <in> what part of the expression do we need to rewrite?

Practice: <im> Simplify each expression

3. D
4. $7s^0t^{-5}/2^{-1}m^2$

[*Note*] <d> When you evaluate an exponential expression, you can simplify the expression before substituting values for the variables.

Problem 3 [Evaluating an exponential expression] – <in> what is the value of $3s^3t^{-2}$ for s=2 and t = –3?

Got it? <d> What is the value of each expression in parts (a) – (d) for n = – 2 and w = 5?

a. $n^{-4}w^0$
b. n^{-1}/w^2
c. n^0/w^6
d. $1/nw^{-1}$

[[**Lesson Check**]]

[Do you know how?]
<im> Simplify each expression
9. 2^{-5}
10. m^0
11. $5s^2t^{-1}$
12. $4/x^{-3}$

<im> Evaluate each expression for a = 2, and b = –4
13. a^3b^{-1}

14. $2a^{-4}b^0$

[Do you understand?]

- **15. Vocabulary** <d> A positive exponent shows repeated multiplication. <in> What repeated operation does a negative exponent show?

16. Error Analysis <d> A student incorrectly simplified $x^n/a^{-n}b^0$ as shown at the right. <im> Find and correct the student error.

Table C2: Analysis of the stages in a Pearson lesson

Appendix 5: Coding results for CPM and PI lesson texts

Table A5.1: Analysis of clause moods in CPM lessons (D = Declarative; IM = Imperative; IN = Interrogative)

Vol	Lesson	Lesson Orientation			Classroom Math Tasks			Math Notes		
		[D]	[IM]	[IN]	[D]	[IM]	[IN]	[D]	[IM]	[IN]
1	1.1.1	4	0	0	20	16	9	10	0	0
1	1.2.1	1	2	0	10	11	8	7	0	0
1	1.3.2	2	0	0	14	18	10	3	0	0
1	2.1.1	4	0	4	4	24	16	0	0	0
1	2.2.2	2	1	1	28	17	11	10	4	1
1	2.3.2	2	0	0	5	10	10	11	10	0
1	8.1.1	3	0	0	20	16	21	0	0	0
1	8.1.5	3	0	0	5	10	3	0	0	0
1	8.2.3	1	0	0	15	14	18	0	0	0
2	5.1.1	5	0	0	16	14	3	4	0	0
2	5.2.1	4	0	0	11	16	9	12	3	0
2	5.2.6	6	0	0	22	26	12	14	0	0
3	1.1.1	2	0	0	7	10	2	17	0	0
3	1.1.4	2	0	1	5	29	18	16	0	0
3	1.2.3	5	0	0	14	7	4	6	0	0
3	9.1.1	1	0	0	13	19	8	0	0	0
3	9.1.6	2	0	2	7	10	1	3	0	0
3	9.2.3	4	0	2	9	18	12	4	0	0
	Totals	53	2	10	225	285	175	117	17	1

Table A5.2: Analysis of clause moods in PI lessons (D = Declarative; IM = Imperative; IN = Interrogative)

Vol	Lesson	Interactive Learning			Guided Instruction			Lesson Check		
		[D]	**[IM]**	**[IN]**	**[D]**	**[IM]**	**[IN]**	**[D]**	**[IM]**	**[IN]**
1	2.1	3	1	1	12	11	9	0	4	1
1	2.4	5	2	1	20	14	10	5	4	1
1	2.7	6	2	1	28	19	15	1	7	1
1	3.1	4	0	2	19	14	13	1	4	6
1	3.4	7	0	1	5	10	13	0	6	2
1	3.7	7	2	2	5	5	12	7	2	0
1	5.1	7	3	2	11	12	11	2	3	1
1	5.5	8	2	0	7	3	10	3	3	2
1	5.9	11	3	1	25	16	7	1	2	7
2	12.1	19	1	1	18	16	17	0	2	3
2	12.6	13	1	1	9	9	10	0	3	1
2	12.12	9	1	2	3	9	13	1	3	1
3	5.1	6	2	1	11	12	16	2	6	5
3	5.4	8	2	2	27	6	25	1	3	2
3	5.7	10	2	1	14	7	7	0	5	1
3	8.1	6	2	1	9	9	16	4	5	1
3	8.6	9	2	2	21	10	7	3	3	1
3	8.11	19	2	1	8	7	10	8	6	3
	Totals	157	28	23	252	187	221	39	71	39

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