Teachers' Formative Assessment Practices for Early Addition and Subtraction: Is Teachers' Awareness of a Learning Trajectory Related to How They Respond to Students?

by

Kristi Haley Hanby

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Educational Studies)
in the University of Michigan
2018

Doctoral Committee:

Professor Patricio G. Herbst, Chair
Professor Deborah Loewenberg Ball
Professor Hyman H. Bass
Professor Mary Schleppegrell
Professor Edward Silver

Kristi L. Hanby

khanby@umich.edu

ORCID iD: 0000-0001-6477-6148

To my family . . .

ACKNOWLEDGEMENTS

I am thankful for the many family members and colleagues who have been a part of this journey with me. It is your confidence in me and your support along the way that made it possible for me to earn this degree. I thank some of you individually, but I share my gratitude with many others who took the time to mentor and support me in my studies or who relaxed with me and reminded me that those moments are important as well.

I am grateful to my husband Greg, who not only allowed me to start this journey, but also was willing to see it through to the end. Neither of us knew the abyss we were stepping into and yet he did not question my desire to venture into it. Also, to my father, who did not get to see me start graduate school, but from whom I often gathered strength by remembering his own hard work through long days or arduous projects. This skill of perseverance I learned through apprenticeship many years ago and it carried me through those moments when I most needed to endure. And, to my mother who has shared with me an ability to step back from it all, and laugh and sing and carry on another day, I'm grateful for the moments that spirit provides. Finally, to my children and sisters who believed in me long before I believed in myself.

I am also thankful to Patricio Herbst, who patiently read, listened, and provided guidance to shape this work, as well as other projects. Dr. Herbst has a great ability to get above the details and describe the broader landscape of emerging ideas. And, not only does he understand research in mathematics education at a profound level, his knowledge of topics related to this study were unexpected and extremely helpful. A thank you also to Kristin Frang who graciously helped ensure opportunities to gather data for the study; and to Nicolas Boileau who improved the analyses through his rigorous questioning.

I will always be grateful to everyone whose contributions during the last six years made this study possible because you provided the time, space, support, and intellect to see it through to the end.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF APPENDICES

Abstract

This dissertation is an investigation into the nature of teachers' formative assessment responses to students as they learn addition and subtraction. Teachers' background experiences, including classroom experience and professional learning opportunities, were considered as factors which could play a role in accounting for that variation, both when teachers responded to individual students' thinking and when they determined goals for group discussion based on students' thinking. In particular, this study investigates whether the responses from teachers who had been trained in a learning trajectory for early addition and subtraction reflected a quality that had the potential to extend student learning opportunities.

Data for the study came in the form of practicing elementary teachers' responses to a multimedia scenario-based survey. In a series of classroom scenarios, participant teachers were shown instances of students solving problems of early addition and subtraction. Those teachers were asked to describe those instances of student thinking, indicate how they would respond to the student, and what learning goal they would set forth for the student. After seeing two individual students' solutions, the teachers were also asked to choose a problem and set an instructional goal for a discussion of the problem with a group of students that included the two just observed. Twenty-two teachers teaching at the time in elementary schools in a Midwestern state participated; some of those teachers had previously participated in professional development related to a learning trajectory for early addition and subtraction.

The results of the study indicate that teachers' classroom and professional learning experiences were associated with higher rates of teachers interpreting student thinking. In

addition to this, those teachers who taught in an early elementary classroom and had training in a learning trajectory were more likely to describe responses to student thinking that showed a potential to extend learning opportunities. Some differences were found among the instructional goals set for the group discussion of addition and subtraction word problems: Some early elementary teachers were open to students' use of multiple methods, and a small number of early elementary teachers who had been trained in the learning trajectory discussed those multiple methods by connecting them in discussion in ways that attended to the mathematical sophistication of those methods.

The findings suggest that when supporting or studying teachers' formative assessment practices, a content-specific lens may be useful for informing and analyzing those practices. In addition, the findings may provide insight into teachers' mathematical knowledge for teaching and the measures used to determine quality of teaching responses.

CHAPTER 1: INTRODUCTION

As young learners share their early mathematical understandings of addition and subtraction, what might we expect teachers to say to and do with those students to extend their learning opportunities? And how do teachers attend to the learning needs of individual students while also supporting every other student in making progress towards understanding concepts of addition and subtraction? This dissertation takes addition and subtraction as a case in point to inquire on the more general question of how an awareness of students' learning of specific concepts might be observable in the formative assessment practices that a teacher uses to support students' learning. In doing so, it illustrates that our ways of accounting for how a teacher supports students' learning need not be limited to the generic (or content-independent) description of teaching actions. Instead, it suggests that a content-specific description of teaching actions may be key in helping us identify and promote teaching that supports students' learning.

I introduce this dissertation by describing a classroom vignette in which a second grade teacher has asked her students to solve a problem that involves the comparison of two quantities, 43 and 37. The vignette is taken from Carpenter and Fennema's (1992) study in which the researchers had supported elementary teachers, through professional learning, to use context-based problems as a means to engage young learners in problem solving, and to elicit and support students' informal mathematical thinking. The overarching purpose of the professional learning was to prepare teachers to be attentive to students' emergent thinking in mathematics and to encourage instructional practices that support students' intuitive problem solving abilities, a practice that has been established in the mathematics education research as productive for student learning (Hiebert & Grouws, 2007). As teachers engage in this form of instruction and elicit students' varied ideas, they have the opportunity to gather vital information regarding students' understanding and to leverage those moments to extend opportunities to learn. The variation that students exhibit in their understandings is a naturally complex aspect of teaching and yet this variation provides the information necessary for teachers to support students' mathematics learning. I introduce this study by considering the varied methods students use to solve problems and ask how teachers can make use of the information gathered during moments of instruction, and in order to extend students' learning opportunities.

A Vignette: How a Teacher Responds to Students' Informal Strategies

During the class period in which the vignette takes place, the featured teacher has asked the students in her class to determine the difference between the number of peanuts that have been eaten by two different elephants, by posing the comparison problem shown below. I provide a description of the group discussion that occurred in the classroom, which I synthesized from the research article in which it first appeared (Carpenter & Fennema, 1992). The teacher began the lesson by posing the following problem.

> *The African Elephant ate 37 peanuts. The Indian elephant ate 43 peanuts. How many fewer peanuts did the African elephant eat than the Indian elephant?*

After the teacher had given time to the students to solve the problem independently, she elicited strategies from three different students. The first student used blocks to build each of the quantities, 43 and 37, separately making multiple sets of tens with additional ones. Once each number had been built with blocks, the student found the difference by placing the two models next to each other and removing each unmatched block, counting them by ones. A second student solved the problem by counting on, from 37 to 43, stopping to make note when he reached 40 after 3 counts. The third student mentally added 10 to 37, but realized that would be too many and so subtracted 4 from the 10 to come to the correct solution.

The students, while varied in their use of strategies to solve the problem, were each successful in coming to the correct response—six. To each student in turn, the teacher responded by asking the class if what the student had done had worked well. Each time the teacher asked this of the class, the class responded that the students had used good methods, and in agreement with the students, the teacher stated that these strategies are good ways to solve the problem.

The pattern of instruction modeled in this vignette is aligned with recommendations that teachers engage students in solving mathematical tasks and subsequently elicit multiple methods for solving (Silver & Stein, 1996). Further, this is a classroom interaction that provides information to the teacher to guide instructional practice, such as problems to use during subsequent lessons and responses to provide to instances of student thinking. I highlight these practices as potential examples of formative assessment (Black & Wiliam, 1998), moments when the teachers elicit the thinking of students in the classroom in order to determine responses that support learning. I consider the vignette as a textual representation of teaching practice (Herbst, 2018) in which we can pose the question of what the teacher might have done to support students' further learning. Could the teacher have responded differently to the students' varied methods? What information about the students would be useful for the teacher to know in order to do so? These questions illustrate the more general questions about teacher knowledge and formative assessment practices this study addresses.

To answer those questions and consider how a teacher might respond, I focus first on the students' strategies as characterized by previous research. The first student used a strategy called *direct modeling by 10* (Carpenter, Fennema, & Franke, 1996) , utilizing blocks to make up each set and then comparing the quantities by matching one set to the other, counting each block in the difference individually. The second student used a counting strategy that is considered more efficient since he only *counted on* (Carpenter et al., 1996) the amount needed to add on to 37 in order to accumulate to 43. He would have tracked those counts in some manner to determine the difference between the two numbers. Furthermore, the student paused while counting to make note of the 40. Finally, the third student used a *heuristic* or *derived* (Carpenter et al., 1981) strategy categorized as an *incremental* (Carpenter et al., 1996) strategy, carried out in the context of multi-digit addition and subtraction, in which something that was known by the student was used to determine something that was unknown.

By eliciting these strategies from her students, the teacher had the opportunity to gather information about the varied understanding of her students when solving problems involving the comparison of two quantities. The differences between each of the solution methods used by the students in this vignette are distinguishable (Carpenter et al., 1996) and information of this type has been shown to be a valuable source of diagnostic assessment of students' understanding of concepts (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998). But could that knowledge be of use for the teacher to engage in formative assessment (Black & Wiliam, 1998), that is to support future learning? In this study, I contend that the teacher could use what she could learn about her students to inform her instructional decisions (Black & Wiliam, 1998b), such as what problem the students might solve next (Carpenter & Fennema, 1992), or to engage in a mathematical investigation of one of the strategies that was shared, encouraging reflection or justification (Chapin & O'Connor, 2007). In doing so, she would encourage students to continue to reflect on concepts related to the addition and subtraction of quantities, and choose, through her questions and prompts, the learning opportunities that would be extended to students.

In the following section, I describe in more detail the set of instructional practices meant to ensure teachers' instructional moves are responsive to students' current understandings, known collectively as formative assessment (Black & Wiliam, 1998). I discuss some of the varied ways in which researchers have supported and studied the formative assessment practices of mathematics teachers as well as what has been learned about teachers' implementation of the practices. Finally, I consider a learning trajectory as resource for supporting teachers' formative assessment responses and propose that a learning trajectory allows for an analysis of formative assessment that is content-specific.

Formative Assessment: The Object of This Study as Illustrated by the Vignette

In the relatively brief period of instruction represented in the vignette, the teacher has engaged in a number of productive instructional strategies meant to support the learning of the students. In addition to providing a context-based problem and some amount of time for the students to work on that problem independently, she actively elicited the understandings of a number of her students (Smith & Stein, 2011). In doing so, the teacher engaged in a number of teaching moves that can be the basis of a classroom assessment practice; a form of assessment used by teachers *for* learning rather than as a measure *of* learning. Assessment *for* learning is more commonly known as *formative assessment* and is a practice used by teachers to gather evidence regarding their students' understandings to inform the feedback they give to students. This practice aims at improving student learning over time (Black & Wiliam, 2009).

Classrooms are complex spaces in which teachers support a number of aspects of student development, cognitive as well as social emotional (Powell & Kalina, 2009). This means that in addition to those moments when teachers directly support a student's learning, there are also times when teachers make moves to support a student's confidence or to encourage students' social interactions with one another. Complexities like these can make the practice of formative assessment difficult to enact. Students will naturally be in different places in their learning of specific concepts, and likely, also making progress at different rates; further they may be in different places in their socio-emotional learning. A knowledgeable teacher may fully recognize what each of his or her students understands, and still have to manage dilemmas (Ball, 1993) when students of such varied understandings come together in a group and listen to each other share their strategies. Would the student who used the blocks in order to model the situation be able to understand the student who added ten and adjusted? Would it be appropriate or even needed for the latter student to understand the former? And, if not, what would be appropriate for the teacher to ask of the students?

In the vignette narrated above, the feedback the teacher gave consisted of validating that each of the students had used what she referred to as a "good" method, a response that evaluates the appropriateness of each of the students' methods (Smith & Gorard, 2005). The teacher could have enacted a number of other responses as well. She might have asked the other students to use one of the strategies that had been shared. She might also have asked if one of the methods used was a method that would always work when solving a problem like the one that had been presented. The options for how to respond can take on different forms, for example a teacher could ask a probing question or could instead ask the student to repeat the strategy with a new problem. The teacher might also ask a student to reflect on his or her own thinking or to reflect on the thinking of another student. These responses are a critical aspect of instruction because they can make a difference in what students take up next, what is considered more thoroughly, and even what is not explored further. For this reason, researchers have proposed that the feedback teachers provide in response to students' work plays a key role in students' learning (Black & Wiliam, 2009). However, observations of teachers' responses to student thinking indicate that the types of responses can vary (Crespo, 2002) and that teachers are not particularly adept at responding to student thinking (Heritage, Kim, Vendlinski, & Herman, 2009). So, while it can be productive practice to elicit student thinking within a group context (Carpenter & Fennema, 1992; Smith & Stein, 2011), as the teacher in the vignette did, without appropriate responses to these instances of student thinking, a teacher might be missing out on key opportunities to intentionally support the extension of students' understandings.

I raise this disparity, between the benefits of quality teaching responses and the reported lack of adequate implementation of this particular practice, as a tension within the literature that deserves further attention and exploration. Specifically, what kind of responding by teachers might help students learn? To begin to unpack this difficulty, I consider the recommendation by Sztajn, Confrey, Wilson, and Edgington (2012) that the research fields of formative assessment and teacher knowledge could be deepened by considering them through the lens of learning trajectories. They recommend both that teachers use learning trajectories to inform their instruction and further that researchers consider learning trajectories as a factor in the study of knowledge and practice in teaching. Learning trajectories are

> . . . descriptions of children's thinking and learning in a specific mathematical domain and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain. (Clements & Sarama, 2004, p. 83).

Investigations of teachers' formative assessment practice as studied through the lens of a learning trajectory have shown some promise for considering a learning trajectory as relevant teacher knowledge (Jacobs, Lamb, & Phillips, 2010; Ebby & Sirinides, 2015). Because a trajectory describes the expected progress of a student's understanding of a concept, it could provide useful information from which teachers could inform the responses provided to students. Thus informed, teachers could be intentional in providing responses that are likely to extend students' mathematical understanding of the target concept. Consider, for example, the three students' strategies from the earlier vignette. The teacher in the vignette provided an example of instruction that attended to students' understanding and allowed those students to construct their own strategies for comparing two amounts. The variation between those strategies could be useful information for thinking about the progress each student is making in understanding the concept of addition and subtraction. The teacher could now leverage that information, specific to the individual students, and provide feedback to the students to support their learning along a trajectory.

In this study, I consider whether teachers who have participated in professional study of a learning trajectory enact responses toward students that are more targeted at their individual learning need, or zone of proximal development (Vygotsky, 1987), and whether the response has the potential to extend student understanding. A better understanding of how teachers with awareness of a learning trajectory respond to students could lead to more content-specific ways of describing and enacting instructional responses. While asking a student to solve a problem in multiple ways can be beneficial for student understanding (Silver & Stein, 1996), knowing which of the multiple ways might be achievable by the student based on his or her current understanding is information that might better support the student to utilize a second method. For example, if a student had used a forward counting method to find a difference, as one student in the vignette did, the teacher could ask if there might be a more efficient method for counting, instead of generically asking the student to try another way. The student might recognize landmark numbers that were passed while counting by ones and be able to act on that recognition more readily than if he or she was asked to find another way without some direction to follow.

In this study, I consider a learning trajectory for early addition and subtraction based on the work of Steffe, vonGlasersfeld, Richards, and Cobb (1983; see also Clements, Sarama, & DiBiase, 2004) that describes the expected progression for students learning early addition and subtraction. I propose this investigation within the context of early addition and subtraction because this mathematical concept is taught universally in early elementary classrooms and because a learning trajectory has been well established in the literature. But addition and subtraction and this particular learning trajectory instantiate a more general research question. I ask whether having participated in a professional learning that supports a recognition of student development along a learning trajectory may inform instructional moves in ways that impact the quality of teaching responses in individual settings and whole group. I compare the responses of teachers who have participated in professional learning of a learning trajectory to responses of teachers who have not participated in such a professional learning opportunity. In doing so, I suggest that it is possible to consider means for appraising the quality of a teacher's formative assessment by analyzing teaching moves through the lens of learning trajectories, described at the level of concepts within mathematics. Whereas more generic guidance, such as the use of open-ended questioning, may provide teachers with ways of thinking about providing opportunities to students to explore concepts, guidance specific to a learning trajectory might support teachers to enact instructional moves that are intentional in extending those learning opportunities in specific ways.

Study Design

I investigate the instructional responses, or formative assessment practices, of teachers, related to the learning of early addition and subtraction strategies when engaged in individual and whole group instruction of early elementary students. I propose that a teacher's recognition of student development as described by a learning trajectory may bear a relationship with those responses and investigate both the instructional responses teachers suggest in response to individual students as well as within group settings. I ask the following questions:

1) Are there differences in what teachers notice from students' work on addition and subtraction when we compare teachers who have had training in a learning trajectory for addition and subtraction with those who have not had training in one such learning trajectory?

2) How are the learning goals of teachers who have had training in a learning trajectory for addition and subtraction different from the goals of teachers who have not had training in a learning trajectory as the teachers face the opportunity to respond to students? And, how does teachers' training in a learning trajectory relate to the extent to which teachers' responses have the potential to extend individual students' learning?

3) How do teachers manage the differences in student progress along a learning trajectory when discussing addition and subtraction problems in a group setting? Are those differences related to whether teachers have had training in a learning trajectory?

I conducted this study by utilizing a scenario-based teaching survey that simulates instances of student thinking in the context of a classroom environment. The representations of student thinking within classrooms allowed me to present teachers with specific student work detailed in the trajectory, while controlling for other classroom factors, and to study the responses teachers provide when students use any of the methods described within the trajectory used in the study. After observing each scenario in which an individual student had solved a problem, I asked teachers to describe what they considered to be important factors of what the student had done and in what way they would choose to respond to the student. In doing so, I was able to gather evidence from a sample of teachers about the ways in which they would interpret and respond to the array of methods commonly used when students solve problems that involve early addition and subtraction.

Beyond the responses to individual students, I also considered teachers' decisions when setting goals for group discussions. To do this, I asked teachers to design a problem to present to the group of simulated students observed individually earlier in the survey. I then asked teachers to describe what they would like to achieve by making use of that problem with the group of students. Because the participants had previously seen the individual work of some students within the group, I used this prompt to consider how teachers manage the complexity that comes with recognizing variation in student understanding. In this study, I consider whether the goals of teachers who have participated in professional learning that supports an understanding of a learning trajectory differ in their quality from those of teachers who have not participated in a similar professional learning. I do this by analyzing those responses in light of recommendations found in the literature meant to support students' deep understanding of mathematics.

Contributions of the Study

This study contributes to the broader field of knowledge and practice for elementary mathematics teaching by connecting recommendations from the formative assessment literature with more specific knowledge of a learning trajectory in elementary mathematics. Further, I propose that this study could help better understand the knowledge demands for teaching elementary mathematics. The results of the study would aid in understanding *if* teachers who have participated in professional learning that supports an understanding of a learning trajectory are likely to be able to leverage it in order to enable responses to students of higher quality than those of teachers who have not participated in a similar professional learning opportunity. The study also helps to understand how teachers negotiate the complexities that naturally arise when students, who vary in their understandings, are engaged in a group discussion of an addition and subtraction problem.

Based on a suggestion from Sztajn and colleagues (2012) that knowledge of a learning trajectory could add to the theories of formative assessment and pedagogical content knowledge, the results from studies like this one contribute to the extension of studies into teaching practices. Teachers' awareness of a learning trajectory could allow us to study the relationship between teaching and learning by focusing research on the interdependence between teacher knowledge, instructional practice, and student learning along a trajectory. Whereas research in education has developed recommendations for teachers' generic practices, for example selecting and sequencing student responses (Stein, Engle, Smith, & Hughes, 2008), studies of learning trajectories may provide greater specificity for informing and analyzing teachers' instructional moves when teaching particular mathematics concepts.

In the discussion section, I propose that training in a learning trajectory deepens a teachers' mathematical knowledge for teaching and supports teachers' formative assessment practices in ways that could add to the development of theory on formative assessment and on mathematical knowledge for teaching. The findings from this study and others like it could also inform policy decisions for teacher education and certification by better describing the knowledge demands that support teachers' appropriate formative assessment practices with students.

Organization of the Dissertation

This dissertation is organized into six chapters. Chapter 1 describes the research problem, provides an overview of the study, and outlines the dissertation. Chapter 2 includes the literature review, including the research on formative assessment, a description of a learning trajectory for early addition and subtraction, and research that documents the potential that learning trajectories have to support teachers' formative assessment practices. Chapter 3 outlines the methods of the study and describes the instrument used. Chapters 4 and 5 provide the results of the study, first of teachers' formative responses to individual students who vary in their understanding of addition and subtraction and then of teachers' goals when engaging students in a group discussion of an addition and subtraction problem. Chapter 6 discusses the findings of the study and the potential connection to the theories of formative assessment (Black & William, 1998) and mathematical knowledge for teaching (Ball, Thames & Phelps, 2008). I also include an appendix that provides a more in-depth description of the learning trajectory for addition and subtraction used in this study.

CHAPTER 2: LITERATURE REVIEW

This study is situated in the context of literature that argues for the benefits of formative assessment practices (Black & Wiliam, 1998) and that reports lack of adequate implementation of these practices (Heritage et al., 2009). I consider the merit of recommendations from researchers that suggest that a teacher may be better equipped to engage in productive formative assessment practices if he or she has knowledge of a learning trajectory for the intended concept (Sztajn et al., 2012). To operationalize the study, I investigate teachers' formative assessment practices for the concept of addition and subtraction. In carrying out this study, I look for any differences in formative assessment practices based on the background experiences of the teachers, including their classroom experience and exposure to professional development opportunities specific to a learning trajectory for addition and subtraction.

The practices known as formative assessment have been recommended to teachers to improve learning in a variety of school subjects, describing such instructional moves as eliciting and responding to student thinking and allowing for peer assessment in the classroom (Wiliam, 2007). Implicit in this advice is the presumption that teachers will recognize and be prepared to respond to students' understandings. At the same time, researchers have shown that teachers may notice other aspects of student behavior not related to students' understandings (Sherin, 2001). Professional learning has often been used to support teachers' attention to aspects of student thinking (e.g., Carpenter, Fennema, & Franke, 1996) as a means for improving teachers' instruction. In this literature review, I describe a number of studies that engaged teachers in learning meant to support the teachers' deepened understanding of student conceptions and their development over time, including learning trajectories. The evidence seems to indicate that recommendations to engage in formative assessment alone may not be sufficient to support teachers' responses to students. I propose that learning trajectories include information specific to particular content that could provide resources to support intentional formative assessment practices and thus be a key feature in enhancing teachers' responses to students.

Formative Assessment Research

In their seminal paper, Black and Wiliam (1998) reviewed over 250 studies from around the globe, conducted between 1987 and 1998, in which teachers had used assessment as a means to inform instruction. Their findings suggested that when teachers used assessment as a means to inform instruction, students showed marked improvement in their understanding. The findings were particularly strong for those students identified as low-attaining (Black & Wiliam, 1998a). Unlike the summative assessment *of* learning, the researchers described formative assessment as those assessment practices used *for* learning, to adjust instruction to the students' learning needs. These formative assessment practices carried out in the moments of instruction are dependent on a teacher's ability to make such adjustments. Thus, teachers play a key role in supporting the development of student understanding, in that the quality of a teacher's response to students "is a critical feature in determining the quality of learning activity, and is therefore a central feature of pedagogy" (Black & Wiliam, 2009, p. 100).

Formative assessment, then, was defined as "all those activities undertaken by teachers, and/or by their students, which provide information to be used as feedback to modify the teaching and learning activities in which they are engaged" (Black & Wiliam, 1998, p. 10). This definition does not describe how formative assessment is to be operationalized, but rather defines it by the outcome it is intended to accomplish (Dunn & Mulvenon, 2009). Because of this, there are numerous ways in which teachers can carry out the work of formative assessment, ranging from informal approaches, such as listening to and interpreting student responses in the midst of instruction, to more formalized approaches, such as conducting clinical interviews of student performance on particular tasks. A number of studies have set out to more clearly identify the practical work of teachers engaged in formative assessment. They identified practices such as questioning, feedback by marking, peer and self-assessment, and the formative use of summative assessments (Black et al., 2002, 2003). These descriptions of teaching practices related to formative assessment were meant to support teachers' implementation of the assessment practice. An important characteristic of these practices is that they are generic—nothing in their description seems dependent on the specific content being taught and learned.

Formative Assessment Implementation

Even when formative assessment practices have been described in such general ways that might support their observation across instruction, observations of teachers' feedback to students identified some difficulties in the implementation of formative feedback. Observations show a tendency toward evaluating student work rather than providing formative feedback (Smith & Gorard, 2005), difficulty choosing good questions (Black et al., 2002), and students working beside each other, but not working with each other, when they are in group settings (Dawes et al., 2004), which may inhibit their opportunity to learn from one another's feedback.

Others have pointed out that some responses chosen by teachers may not be timely for the students, making it difficult for them to understand or make use of the feedback. As Perrenoud (1998) and others have described, the effectiveness of teachers' responses is dependent on the extent to which those responses are appropriate for the students' current learning needs. Perrenoud (1998, p. 94) suggests that teaching responses should purposefully "optimize the activity and the learning process of each pupil within a given situation" and that there is an interdependence between the potential the activity holds for learning and each individual student's ability to assimilate that activity to their current understanding of the concept. From this perspective, teacher feedback needs to be seen as particular to both the situation and the learner, in regard to how targeted it is to each students' learning need. This would require the teacher to be knowledgeable about the student and the development of mathematics; to consider "the children in relation to the mathematics" (Ball, 1993, p. 394). As Vygotsky (1987) put it, a teacher should attend to the student's zone of proximal development (ZPD), or the distance between what a child can do independently and the "level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers" (Vygotsky, 1987, p. 86).

Because of the importance to attend to students' learning needs and the difficulties of implementation experienced by teachers, much work has been done to investigate teaching practice, along with attention to and understanding of students' mathematics thinking and its development. This work has taken on many forms, from investigation of teachers' implementation of learning tasks (as in Stein, Grover, & Henningsen, 1996) and attention to classroom questions during discussion (as in Chapin & O'Connor, 2007) to investigations of what teachers notice while watching videos of classrooms (as in Sherin, 2001, 2007) or when discussing student conceptions in mathematics with colleagues (as in Crespo, 2000). Other researchers have engaged teachers in professional learning related to student conceptions of particular mathematics concepts (as in Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996). For purposes of this review, I focus primarily on studies that have investigated teachers' noticing, interpreting, diagnosing, and responding to students' thinking. In the contexts addressed by those studies, teachers engaged in a variety of professional learning experiences meant to hone the teachers' attention to students' mathematical thinking.

In the sections that follow, I describe professional development opportunities in mathematics education that have supported teachers' attention to students' mathematical thinking. Some of those opportunities are generic in nature, in that the guidance provided to teachers through those professional development opportunities could be applied to the teaching of many mathematical concepts (as in Smith & Stein, 2011). Over many decades these studies have led to policy and practice guidance that describes generic teaching practices meant to support student learning, such as engaging in purposeful questioning and also describe students' habits when engaged in mathematical work, such as using mathematical justifications (Leinwand, Brahier, Huinker, Berry, Dilion, Larson et al., 2014). Other research and teacher learning has focused on particular concepts in mathematics, (Carpenter, Fennema, Franke, Levi, & Empson, 2000) in which teachers are exposed to the informal strategies that students use to solve mathematical problems and develop instructional methods for eliciting and developing students' understanding of the concepts. A subset of studies includes teacher professional learning meant to describe the development of students' strategies over time; for teachers to recognize students' strategies as described by a learning trajectory for a particular concept.

Professional learning that focuses on specific mathematics concepts has the potential for teachers to implement formative assessment practices in ways that are intentional and targeted at a student's current learning need. Whereas, general instructional supports may recommend that teachers engage in questioning that causes students to be reflective about mathematics (Boaler & Staples, 2008), the research into students' development of particular mathematical concepts would allow teachers to take up advice around questioning and utilize it in intentional ways, ways that are specific to particular content in mathematics.

Teachers' General Assessment Practices in Mathematics

The mathematical task framework (Stein, Grover, & Henningsen, 1996), originally designed to examine field note records of teachers' use of tasks, has also been used as a tool to support teachers in identifying features of a mathematical task that make it cognitively demanding for students and thus likely to lead to a more robust understanding of the mathematics (Boston & Smith, 2011). The framework is used by teachers to identify mathematics tasks that attend to student reasoning of mathematics and is intended to be used as a means for increasing student understanding of mathematics. In the original study, Stein et al. (1996) had collected a random sample of 140 tasks implemented in middle grades classrooms involved in the Quasar project and analyzed the demand of those tasks as well as the quality of implementation by teachers. The mathematical task framework used in professional development opportunities with teachers was associated with increases in the chances that formative assessment be more task-centric (Boston & Smith, 2011). Their findings also suggested that when teachers implement tasks with students, it can be difficult for teachers to maintain the cognitive demand of the task when engaged in discussion and questioning of the concepts. Thus, the potential for student engagement in cognitively demanding work has a tendency to decline, and teacher responses often focus on correctness more than mathematical reasoning.

Indeed, to carry out formative assessment in mathematics effectively, that is to notice a student's current understanding in order to respond accordingly, teachers need to look beyond the student's correctness and attend carefully to student thinking and reasoning related to the concept being learned (Boaler & Staples, 2008). To encourage teachers' deepened practice of noticing, teacher educators have used a variety of techniques from video clubs (Sherin & Han, 2004; Sherin & van Es, 2005) to letter writing between pre-service teachers and students (Crespo, 2000). Researchers have examined those teacher development practices and their outcomes on the participants' instruction. Qualitative analysis of teachers' noticing revealed that teachers can make a shift from focusing on the correctness of students' overt behaviors to interpretations of those behaviors that include inferences on student thinking (Sherin & Han, 2004; Sherin & van Es, 2005; van Es & Sherin, 2008). Further, this shift can be facilitated through ongoing interactions in which teachers are asked to notice or wonder about students' thinking from observing classroom video clips.

Teachers' Focus on Student Understanding

Interpretation of student thinking is a vital step toward being able to respond appropriately. But even when teachers do attend to students' mathematical thinking, there is no guarantee that the responses they provide will be able to further learning. Heritage and her colleagues (2009) used a generalizability (G) study (Brennan, 2001) to investigate the teachers' inferences and feedback to students' written work and found teachers to be more adept at interpreting student work than at responding to students' work. In other words, teachers' interpretations of student thinking may allow for a teacher to recognize the child's mathematics, but their ability to interpret such thinking does not entail that the teacher will choose a response that attends to how a child's mathematics may develop.

Perhaps for that reason, instead of supporting teachers to attend generically to student thinking, some professional development programs (such as Fennema et al's, 1996, CGI program) have supported teachers' more specific noticing of student conceptions within particular mathematics domains. These scholars have found some success in connecting attention to students' thinking with the teachers' response. In a four-year study of 21 teachers engaged in learning the CGI frameworks, researchers observed that teachers shifted their practices from providing examples of procedures to engaging students in problem solving and discussion of students' informal methods (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996). The shift in these teachers' practice went beyond noticing and interpreting student thinking, to also engage students in the sharing of their varied methods. In a similar investigation of teacher noticing and responses to student work, after learning about student thinking as described in the CGI frameworks, Jacobs and her colleagues (2010) showed that teachers, categorized into four varied levels of experience (from pre-service to teacher leaders) learned to notice and interpret student understandings. These teachers also improved in their responses to students as measured by the mathematical tasks they chose for upcoming lessons. The analysis of teacher responses to classroom scenarios and students' written work showed that while teachers could learn to target responses to students' understandings, the robustness of teachers' practices varied based on the extent of their professional experience.

Teacher Focus on Learning Trajectories

The CGI frameworks describe the students' strategies for solving addition and subtraction tasks; from the use of concrete materials to counting strategies to the use of derived facts (Fennema et al., 1996). Other cognitive research has described that development in greater detail by defining a learning trajectory. A learning trajectory not only describes the possible strategies students could use to solve problems within a particular domain, but also describes the hypothetical development of those strategies over time. This means that as teachers engage in professional development based on student learning as described by a trajectory, they are asked to attend to students' understandings while also considering what would be likely to expect the student to do in lessons in the near future. The longitudinal nature of a learning trajectory could be critical to supporting teachers' formative responses to students because it provides teachers with an objective for students to reach that is appropriate for the individual learner.

The development of learning trajectories has led to packaging that information in the form of curricula (Sarama & Clements, 2002) or as student diagnostic assessments (Wright, Martland, & Stafford, 2006). A cluster randomized trial of 1,375 preschoolers, whose teachers followed a curriculum based on a learning trajectory, showed students learned significantly more mathematics (effect size 0.72) than their peers in a control group (Clements, Sarama, Spitler, Lange, & Wolfe, 2011). And in three different professional learning opportunities in which teachers learned to utilize the diagnostic assessment from Math Recovery (Wright, Martland, & Stafford, 2006), also based on a learning trajectory, students had documented gains in achievement (Bobis, Clarke, Clarke, Thomas, Wright, Young-Loveridge, et al, 2005). In each of these studies, the argument is made that the teachers were not only trained to attend to student thinking, but they also learned the developmental path along which student thinking in mathematics develops. From these studies we know that materials that are written using a learning trajectory as a framework can have an effect on student learning, although these studies did not closely investigate the teaching practices associated with the implementation of the resources.

So, while we know that a teacher's response to his or her students is a key factor in students' opportunities to learn, the evidence indicates that on the whole the practice is not carried out in ways that engage students meaningfully. There are a number of factors that could influence the quality of teachers' responses to students, including an attention to students' understandings and knowledge of the ways in which students' conceptions in a particular domain develop over time. I argue that teachers who are trained in a learning trajectory may be more equipped to respond productively to students because the trajectory provides information specific to student development of the content over time. The specific nature of the learning trajectory related to students' understanding could be critical for being able to interpret and respond to students in ways that are intentional for the students' learning. I propose this investigation, then, to determine whether teachers' training in a learning trajectory is a factor that may play a role in teachers' responses to student learning addition and subtraction.

Informing the Formative Assessment Practice of Teachers

One key factor in quality teaching is the extent to which the teacher attends to the individual learning needs of his or her students and uses the knowledge of students' understandings to leverage learning opportunities (Popham, 2008). To be able to situate learning in this way, the question becomes how a teacher comes to recognize a student's current learning needs and how to leverage the learning activity to support that student within their zone of proximal development. Some researchers have suggested that to be able to recognize a student's current learning needs, the teacher would require knowledge of the ways in which students' understandings of particular concepts increase in sophistication over time. They recommend that teachers be able to recognize the development of student understandings and to pinpoint where, within the natural development of a concept, a student's understanding might lie (Sztajn, Confrey, Edgington, & Wilson, 2012). To do so, it is recommended that teachers have knowledge of a learning trajectory for the concept being taught. Heritage and her colleagues (2009) describe a learning trajectory, also known as learning progression, in this way:

> Learning progressions describe how concepts and skills increase in sophistication in a domain from the most rudimentary to the highest level, showing the trajectory of learning along which students are expected to progress. . . Teachers are able to connect formative assessment opportunities to short-term goals as a means to keep track of how their students' learning is evolving to meet the goal. (Heritage et al., 2009, p. 30)

Clements and Sarama (2004) define trajectories in terms of an interdependence between instruction and learning, as a teacher needs to choose and utilize instructional tasks to observe students' performance and support the student's progression of learning:

> . . . descriptions of children's thinking and learning in a specific mathematical domain and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain (Clements & Sarama, 2004, p. 83).

Because a learning trajectory identifies the hypothesized developmental path for learning, it can be used as a resource to inform instructional moves such that teachers' responses can meet the current learning needs of the student and extend their understanding along that progression. Certainly, if a teacher is asked to adjust his or her responses in order to meet the individual learning needs of each student, it would be useful to recognize when an individual student might be likely to carry out a particular performance based on his or her recent understandings. Teachers with knowledge of a learning trajectory would go beyond knowing what any student might be likely to do when working a problem to anticipating what individual students are likely to do as well as how the understanding of that concept is likely to progress. A learning trajectory, then, could inform the responses of a teacher in order to recognize a student's current learning and to challenge them to become more sophisticated in their mathematical reasoning (Clements & Sarama, 2004; Simon & Tzur, 2004; Steffe, 2004).

A Learning Trajectory for Early Addition and Subtraction

In order to investigate the influence of a learning progression on instructional practice, it is necessary to focus on a particular concept, since a learning progression defines the development of student conceptions within a mathematical domain. In this study, I focus on a learning trajectory for early addition and subtraction (Clements, Sarama, & DiBiase, 2004;

Steffe, vonGlasersfeld, Richards, & Cobb, 1983), meant to describe the highly probable progression of student learning for those concepts. I chose early addition and subtraction because it is a mathematical concept which is taught universally in early elementary classrooms and because a learning trajectory has been well-established in the literature, (Clements, Sarama, & DiBiase, 2004; Steffe et al., 1983). In this section, I provide a brief description of that trajectory with a more detailed description of a learning trajectory in Appendix A.

Research in students' mathematical thinking has described much about the relative difficulty of particular problem types and the strategies students use to solve problems of addition and subtraction (Carpenter et al., 1981; Carpenter & Moser, 1984). Some of the easiest problem types for children to solve are those in which children are able to "*directly model* the problem's actions" (Sarama & Clements, 2009, p. 121). These problem types tend to be more accessible for children because the actions needed to carry them out are directly translatable from the way in which the problem is stated (Carpenter, Ansell, Franke, Fennema, & Weisback, 1993). This includes problems in which two sets are joined and the sum is unknown, part-part-whole problems in which the total is unknown and problems in which one set is separated from another set and the difference is unknown (Sarama & Clements, 2009), each purposely formed so that the result of the sum or difference is unknown. As students develop ways of thinking about the quantities associated with numbers more abstractly, they also become able to solve *join* or *part-part-whole* problems (Carpenter et al., 1981) in which the change or part is unknown, sometimes called *missing addend* problems. Problems in which the sets are separated and the change is unknown, called *missing subtrahend* problems, are one of the more difficult problem types for young learners to solve (Sarama &

Clements, 2009). Typically, these problems are solved absent the use of materials once students are able to count on from a number other than one, a strategy often called *counting on* (Fuson, 1992). *Counting on* also allows students the ability to solve comparison problems without the use of objects to directly model the problem. *Comparison* problems (Carpenter et al., 1981) are those in which two quantities are compared and the student is asked to find how much more one quantity is than another. The most difficult problem type is that in which the starting number, either the first addend or the minuend, is unknown (Sarama & Clements, 2009).

This means that students develop from using objects in order to model addition and subtraction problems in which the result is unknown, toward manipulating or counting numbers mentally (Steffe et al., 1983) in order to solve more complex problems, such as those in which the change or start is unknown. Students who are able to solve the more complicated problem types are also able to move beyond the use of counting strategies and toward the use of derived or reasoning strategies (Sarama & Clements, 2009).

In describing children's mathematical thinking, the research into students' thinking when solving additive and subtractive problems (Carpenter et al., 1981; Steffe et al., 1983; Sarama & Clements, 2009) has provided the background knowledge needed to approach formative assessment in content-specific ways. It is these descriptions of students' common ways of thinking about addition and subtraction, along with the typical ways in which students' strategies progress that now provide valuable information for supporting student learning of addition and subtraction. Teachers' knowledge of such a learning trajectory could be utilized to more clearly understand students' mathematical thinking and to determine teaching responses that extend learning opportunities for students. In Table 2.1, I illustrate each additive and subtractive strategy utilized by students as they progress along a trajectory, and describe the conceptual development of students who make use of the strategy. A more complete description of this learning trajectory for addition and subtraction can be found in Appendix A.

Description of Conceptual Development	**Description of Student's Additive Strategy**	**Description of Student's Subtractive Strategy**
	Direct Modeling Strategies	
Using perceptual (Steffe et al., 1983) objects, student can model a simple experiential problem type (*join* or *separate*, result unknown) by counting out sets of objects (Steffe et al., 1983; Sarama & Clements, 2009).	To solve a problem that is modeled mathematically by the statement, 5 + 4, the student counts out five objects, then counts out four objects before counting the nine objects in total.	To solve a problem that is modeled mathematically by the statement 12 – 7, the student counts out twelve objects, then counts while removing seven objects, then counts the five that remain.
	Counting Strategies	
Using figurative (Steffe et al., 1983) counters student combines groups while tracking the counts that typically start at one and continue to the sum.	To solve a problem that is modeled mathematically by 5 + 4, the student counts from one to five, then *continues the count* to the sum by creating a representation of four, for example fingers, in the midst of the count, saying '1, 2, 3, 4, 5 - 6, 7, 8, 9' (Steffe et al., 1983). Sometimes called *counting all* (Fuson, 1982)	A student who counts from one, as is the case here, is likely to fall back on a direct modeling strategy in order to subtract.
Has established a numerical composite (Steffe et al., 1983) in which a number is recognized simultaneously as a composite unit and the counts that make up the unit. Student then can *count on* from a number other than one in order to solve a problem. To subtract, recognizing the numerical composite (Steffe et al., 1983) of the minuend, counts backward the number of units indicated by the subtrahend.	To solve a problem that is modeled mathematically by 5 + 4, the student recognizes 5 as a composite of 5 counts and then *counts on* to find the sum while saying, '6, 7, 8, 9"	To solve a problem that is modeled mathematically by 12 – 7, the student counts backward from twelve, enumerating each unit of the seven counts, saying, '11, 10, 9, 8, 7, 6, 5" Alternatively, the student may count, "12, 11, 10, 9, 8, 7, 6" leading to a response of 5. In doing so, the student tracks the ordinal nature of discrete objects as each is removed from the initial set of 12.

Derived or Reasoning Strategies		
Has an abstract composite unit (Steffe, Cobb, & vonGlasersfeld,1988) which allows the student to recognize a subtrahend as more than a number of units to be counted, but as a segment within the minuend. This allows the student to count in order to determine the remaining segment, rather than each countable unit. (Steffe et al., 1988)	Counts on from larger number (e.g., from 8 when solving 3 + 8) at least informally understanding the commutative property for addition and showing that the two quantities are interchangeable for the operation of addition.	To solve a problem that is modeled mathematically by 12 – 7, the student recognizes seven as a unit that makes up a part of twelve, counts back saying '11, 10, 9, 8, 7' and recognizing he/she has reached the segment size seven, which indicates the remainder of five, the number of counts that have occurred.
Student recognizes the relationship between problems related to the given problem. This is evidence of algebraic thinking (Schifter, 1999)	Student makes use of reasoning strategies in which he or she may make use of a known fact in order to determine an unknown fact. To solve 8 + __ = 11, the student may know that 8 + 2 makes 10 and so it would take 3 to make 11.	To solve 12 - __ = 7, the student could recognize that since 6 + 6 = 12 and since 7 is 1 more than 6, then the missing number would be 5, 1 less than 6.

Table 2.1: A learning trajectory for addition and subtraction

Formative Assessment Practices: The Learning Trajectories Approach

A learning trajectory, like the one described above, is a resource that has the potential to support teachers' formative feedback in ways that connect to a student's current learning need. The formative assessment literature recommends teachers support students' understanding of intended learning targets through questioning and responding to students' learning needs (Wiliam, 2007a), calling into question how one might determine a student's learning needs. Yet, while the literature in formative assessment recommends the use of questioning and responding, they do not differentiate strategies for such practices based on students' understandings. This could indicate that proponents of formative assessment either do not recognize the variations in student thinking that can be evidence of student understanding or there is not a belief that those variations deserve different treatments on the part of the teacher who enacts questions and responses to his or her students.

A learning trajectory not only describes how students solve problems, but also the likely path students will take in their learning, the "ordered expected tendencies . . . students follow as they develop their initial mathematical ideas into formal concepts (Sztajn et al., 2012, p. 148). This provides a content-specific way of identifying and responding to students learning needs. I argue that a teacher could enact any number of responses to student thinking that might seem appropriate given more generic advice in the mathematics education field, such as asking student to justify and reflect on their work (Boaler & Staples, 2008). Yet, these responses may still be inappropriate according to a learning trajectory, because the response may not be timely for the student (Perrenoud, 1998). A learning trajectory, then, provides a means for teachers to locate the student's current learning in the context of the development of their understanding of the concept and provide intentional feedback to students.

As an example, the three students featured in the vignette in chapter 1 had matched blocks representing the two quantities (a *direct modeling* strategy) counted from the smaller quantity to the larger quantity (a *counting on* strategy) and added a ten and adjusted (a *derived* strategy). The recommendations from researchers that suggest teachers situate their learning activities within the student's zone of proximal development become more viable to enact if we know that these students' common strategies are ones that can be identified as those along a path of expected learning that has been mapped through research. To enact formative assessment practices that are specific to the content of addition and subtraction, a teacher might ask if the first student could use his model to make sense of the counting strategy the second student had used and if this would support him to make use of this more efficient strategy in a future learning activity. Or, the teacher could highlight that the second student had noticed the number 40 as she counted and then support her continued attention to key numbers in the problems she solved, eventually making use of *derived* strategies. Further, seen in this way, a learning trajectory could provide a lens by which some teaching responses may be deemed inappropriate at particular moments because those responses might ask a student to work a problem in a way that has been learned in the past and doesn't provide a challenge for further learning. For example, asking a student who had used a *derived* strategy to solve a similar problem using individual cubes is not likely to support further learning for the concept of addition and subtraction for that student.

I extend beyond the example of the vignette to propose that it is possible to evaluate teaching responses to students who use the strategies that have been described in a learning trajectory. Because the research into student thinking has shown that students make use of a set of common informal strategies (Carpenter et al., 1981), found along a learning trajectory (Steffe et al., 1983; Clements & Sarama, 2004), teaching responses can be selected based on the understanding exhibited by the student for the particular content.

Sztajn and her colleagues (2012) propose that in addition to providing a resource with which teachers could make informed responses to students, teachers with knowledge of a learning trajectory have a depth of professional knowledge, specifically what Shulman (1986) had called *pedagogical content knowledge*, which can be the basis on which teachers could interpret and respond to student thinking. I take up this suggestion by focusing on teachers' formative assessment practices, in light of a learning trajectory for addition and subtraction, in order to describe aspects of what Ball, Thames, & Phelps (2008) have called a teacher's *mathematical knowledge for teaching*.

Mathematical Knowledge for Teaching

In their conceptualization of mathematical knowledge for teaching, Ball and her colleagues (2008) identify six domains of mathematical knowledge for teaching that describe the types of knowledge teachers draw on when engaged in mathematics instruction. The authors break mathematical knowledge for teaching into content knowledge and pedagogical content knowledge which are further broken down to more clearly describe the varied forms of knowledge that make the work of teaching mathematics unique. Of interest to this study are what they call *knowledge of content and students* and *knowledge of content and teaching*.

Knowledge of Content and Students The domain called *knowledge of content and students* (KCS) includes knowledge of students' common ways of thinking about problems, including their common conceptions about particular concepts as they are shown in problems they do in classrooms. KCS is a particular type of knowledge that allows one to understand how students "are likely to think" (Ball et al., 2008, p. 402) about given problems. KCS is a resource that enables teachers to anticipate student strategies. I argue that the formative assessment practice of interpreting student thinking is one context where KCS can be observed because it would serve as a resource in informing interpretations.

The anticipation and interpretation of student thinking is an important task of teaching that produces resources with which teachers can respond to students. These teaching tasks allow the teacher to prepare responses in advance of that moment when students first share their thinking. As an example, a teacher may anticipate what students might do when asked to find the difference between 13 and 4. The teacher may anticipate that some students will understand this difference problem in terms of its related addition problem, to find out the sum of 9 and 4, while others may count backward to solve the problem. The teacher may also anticipate common errors that occur when students solve problems of this nature, as when students erroneously attempt to make use of a *measurement* count (Fuson, 1984) starting the count on 13, rather than on 12, as would be appropriate when counting backward across intervals as on a number line. The importance of teacher anticipation and interpretation might be best illustrated by considering what might occur if a teacher does not anticipate a response that is then shared by a student. In that moment, the teacher will need to listen closely, make sense of the method mathematically, determine if the method is valid, and determine some response to the student or students who shared it; a fair amount of work to be done in the few moments that pass between teacher and student exchanges.

When KCS is examined through the lens of learning trajectories, Sztajn and her colleagues (2012) suggest that more than being able to recognize common student conceptions in relation to the particular problem type, teachers with knowledge of learning trajectories may also understand the relation the different conceptions have to each other in the progression along the trajectory. This would allow a teacher to do more than anticipate how any student in the class might respond; the teacher could also consider how particular students in the class might or might not respond because of what is known about the student's current progress along the trajectory. For example, if a student had been consistently using a *counting on* strategy, the teacher might soon expect that the student would be able to count backward to solve similar subtraction problems[1]. Or, if a student is using counters to *directly model* an addition problem, the teacher might surmise that the student is not likely to fully understand the use of a *counting back* method to subtract. Without the trajectory as a reference, a teacher may still recognize the student's strategy and consider it appropriate for a number of reasons, for example because it yields a correct answer or because it showcases the

[1] As described in the learning trajectory for addition and subtraction in Appendix A, students typically count forward, for example using a *counting on* strategy, to solve addition problems, before using a *backward count* in order to subtract.

unique thinking of one of her students, without regard to how the strategy provides evidence of development or the level of challenge it may or may not have presented to the individual students who used it. However, when a present strategy is placed in against the background of the trajectory, it can be understood as evidence of the development of a student's understanding. The observation of a present strategy against the background of the trajectory becomes an important means for recognizing the student's current learning needs because the strategy can be understood within the context of the expected progress of the mathematics concepts that are at stake in instruction. So, while all of the strategies used by students in the vignette led to a correct answer, a teacher with knowledge of a learning trajectory might have also recognized that the students, while together in the same class, were making progress in their mathematical understanding in varied ways, each of them with their own individual learning needs.

Knowledge of Content and Teaching In addition to understanding the set of common student methods and the progression along which children develop strategies for solving problems, teachers are called on to respond to students in ways that support and extend their learning. The knowledge that informs the choices teachers make in order to address the understandings or misunderstandings of students is associated with the domain of mathematical knowledge for teaching Ball et al. (2008) called *knowledge of content and teaching* (KCT). A teacher's KCT allows a teacher to consider "the instructional advantages and disadvantages of representations used to teach a specific idea" (Ball et al., 2008, p. 402), and whether or not those representations are useful for the needs of the students in the classroom. As Ball and her colleagues describe, the decisions teachers make when making such choices ". . . [require] coordination between the mathematics at stake and the instructional options and purposes at play" (Ball et al., 2008, p. 401). For example, if a student had made use of a *counting all* strategy, a relatively rudimentary strategy for solving addition problems, the teacher may respond differently than if a student had used a heuristic or derived strategy to solve the same problem. One could ask what informs the decisions teachers make when responding to these students who have shown evidence of having different learning needs. I consider the possibility that knowledge of a learning trajectory may influence responses to student thinking.

The decisions teachers make are meant to mediate between a student's current understanding and a determined learning target, and, at some point, to engage the student in opportunities to extend learning toward that target. The content of that learning target could come from a number of sources, such as a set of curricular materials or content standards that the teacher follows. On the other hand, the learning target could be situated within the progression of a student's thinking and in relation to a learning trajectory. If a learning trajectory becomes the grounds on which a teacher makes instructional decisions, those decisions would be made in order to challenge the learner to advance from his or her current mathematical understanding and take up what is known to be a strategy that is commonly invented later in a student's development. For example, if a teacher recognizes that some students have solved a subtraction problem by counting and others by *making ten*, he or she may prepare a representation that allows for the count to be recognized while also emphasizing the use of the ten by the second student. Similarly, the teacher may provide an explanation of the part-part-whole relationship between the addition and subtraction sentences that may be discussed by students and in doing so provide opportunities for students who are using more rudimentary strategies to connect their thinking to the more sophisticated reasoning abilities that are sought after as eventual outcomes of the learning.

Framed by the research in learning trajectories, teaching could include considering methods that not only teach a specific idea, but support learning that is timely for the particular students, as evidenced by a learning trajectory. In other words, teachers could consider the relationship between the responses given to students and the desired path for student learning, what others refer to as the interdependence between the task and the related student methods (Simon, 1995; Carpenter & Fennema, 1992). In light of a learning trajectory, the task and the way in which strategies are represented play the role of mediating the gap between a student's current understanding and the learning that is to be made along the trajectory (Sztajn et al., 2012).

I suggest the literature describing student conceptions for early addition and subtraction (Carpenter et al., 1981, Steffe, von Glasersfeld, Richard, & Cobb, 1983) can be used as a means to investigate the interpretations teachers make of student thinking, one aspect of a teacher's KCS for early addition and subtraction. Further, because for formative assessment practices to be effective they need to include quality responses from teachers, I investigate teachers' responses, in light of a learning trajectory, and relate those responses to the domain of KCT. I conjecture that there is a relationship between the knowledge a teacher has of student thinking and his or her responses to students in that teachers who have more in-depth knowledge of student thinking would implement higher quality responses to his or her students. I utilize such an analysis to consider the extent to which any variance in the quality of teacher's responses can be understood by considering the interpretations teachers make of student thinking.

Using a Learning Trajectory to Inform Instructional Goals in Group Settings

The research in student learning trajectories has based its development on the progression of student thinking that is probable to occur for any individual student. It may be argued that knowledge of a learning trajectory for a concept would be useful, if not essential, information for teachers to have. Yet, a classroom teacher will naturally encounter a range of student understandings for a given concept at any given time in the classroom, from an "emergence in the classroom of multiple, overlapping ZPDs" (Allal, & Pelgrims-Ducrey, 2000, p. 146), whose learning needs occur simultaneously. Meeting students' diverse learning needs concurrently would naturally be more complex than using a learning trajectory to assess and respond to the learning needs of a single student.

Recommendations from formative assessment scholars describe a number of practices meant to be supportive of teachers' practice. These include generic ones such as the use of productive questioning and integrating peer assessment into collaborative work (Wiliam, 2007a). Still, a teacher might ask him or herself which students should be asked to share, what the students should share, and what to include in the students' collaborative time. Further, given that students may have different understandings of the mathematics being discussed, the teacher may wonder if it is productive for all students to hear differing methods even though the methods of some learners may seem disparate to other learners. To address these questions, I consider the following synopsis of research in mathematics education. As described by Hiebert & Grouws (2007),

> *. . . students can acquire conceptual understandings of mathematics if teaching attends explicitly to concepts – to connections among mathematical facts, procedures, and ideas [. . .] includes asking questions about how different solution strategies are similar to or different from each other. . . consider the ways in which mathematical problems build on each other or are special (or general) cases of each other, attending to the relationships among mathematical ideas* (Hiebert & Grouws, 2007, p. 383).

This description of mathematics instruction, intended to support students' conceptual understandings, describes mathematics teaching and learning as a process in which students are engaged in understanding multiple solution methods (Silver, Ghousseini, Gosen, Charalambous, & Strawhun, 2005) for solving problems as well as the ways in which those methods are connected to each other mathematically. Teachers could take up the advice from this research by eliciting students varied thinking in the group and then using the time as an opportunity to discuss mathematical connections between the students' methods. Some researchers have described this as a process of sequencing and connecting student methods (Stein, Engle, Smith, & Hughes, 2008) and could be utilized to make evident the increasing sophistication of those methods as described by a trajectory (Steffe et al., 1983; Clements et al., 2004). A discussion of those methods and the connections between them would reflect attention to the recommendations from formative assessment as the teacher and students use this time to attend to understanding each other's methods as a means of more deeply understanding the concept being learned. Thus there are ways in which formative assessment practices can be informed by content-specific considerations, even if they are used to manage whole group interactions.

A learning trajectory could be a resource to inform the instruction carried out in groups, but will inform that instruction in ways that are different from the responses teachers might give to students individually. As Ball and Bass (2000) describe, teachers "need mathematical knowledge in ways that equip them to navigate these complex mathematical transactions flexibly and sensitively with diverse students in real lessons" (Ball & Bass, 2000, p. 94). While a learning trajectory can be a resource to inform the mathematical progress of students, the diversity of understanding in the group adds to the complexity of what teachers need to know or be able to do in order to teach well. The teacher may also need to develop instructional moves that engage a group of learners in ways that make the concept understandable to the diversity of individual students that make up the group, while simultaneously encouraging further extension of that learning.

I consider the complexity of attending to individual student understandings in the context of a group as a potential factor that may have an additional influence on instructional practice and propose the need to look more closely at the goals teachers set when working in group settings. What does this complexity mean for teachers and how would they manage the learning of the group of learners while also working to meet the needs of the individual students that make up that group? I ask whether teachers who have participated in training of a learning trajectory set goals that attend to students learning in groups in ways that are different from teachers without knowledge of a learning trajectory. I hypothesize that when setting instructional goals, teachers with and without an awareness of a learning trajectory, because of their differences in professional learning, will differ in the extent to which they allow individual student differences to be expressed within the context of group instruction, as peers interact with each other when solving addition and subtraction problems.

The Current Study

For this study, I investigate the formative assessment practices of teachers as it pertains to students' learning of early addition and subtraction, considering the potential role that teachers' background experiences may play in accounting for those responses. Related to the formative assessment practices of teachers, I study the interpretations teachers make of student strategies followed by the responses teachers provide to students. To conduct the analysis of teachers' responses to students, I draw on research that describes a learning trajectory for addition and subtraction (Clements, Sarama, & DiBiase, 2004; Steffe et al., 1983). After conducting the analysis for teachers' interpretations and responses to student thinking, I consider the possibility that a teacher's interpretation of student thinking may explain some amount of variance found in the teachers' responses to students. Further, I consider the goals teachers set when asking students to share their strategies in a group setting and the ways in which those goals provide an opportunity for the learners in the group to engage in learning that has the potential to deepen their understanding of the concept of addition and subtraction.

I collected data for this study using simulated teaching scenarios, in order to present teachers with all of the documented student strategies detailed in the trajectory. I investigate the potential relationship between teachers' self-reported background experiences, both in professional learning and in classroom settings, and their responses to the teaching scenarios. In these scenarios, I asked teachers to observe the work of a number of individual students, to interpret each student's work, and to describe what he or she considered to be an appropriate teaching response. These questions were designed after the formative assessment practices of eliciting, interpreting, and responding (Jacobs et al., 2010). I then asked the teacher to choose a problem for the same students to discuss in a group setting, to state the intended goal for the discussion, and to consider in what ways the teacher may manage the varied learning needs of the students within the group discussion.

In gathering these responses from teachers, related to teachers' formative assessment practices in teaching early addition and subtraction, I pursued the following questions:

1) Are there differences in what teachers notice from students' work on addition and subtraction when we compare teachers who have had training in a learning trajectory for addition and subtraction with those who have not had training in a learning trajectory?

2) How are the learning goals of teachers who have had training in a learning trajectory for addition and subtraction different from the goals of teachers who have not had training in a learning trajectory, and how does teachers' training in a learning trajectory relate to the extent to which responses have the potential to extend individual students' learning?

3) How do teachers manage the differences in student progress along a learning trajectory when discussing addition and subtraction problems in a group setting? Are those differences related to whether teachers have had training in a learning trajectory?

Chapter 3: METHODS

Introduction

The benefits of formative assessment (Black & Wiliam, 1998) can be outbalanced by the difficulties of implementation (Heritage, Kim, Vendlinski, & Herman, 2009). Based on suggestions that a learning trajectory may be a tool that serves to inform the responses made to students (Sztajn et al., 2012), I investigate teachers' instructional moves in response to individual students, as well as in groups, and consider if differences in teachers' practices relate to their background profession experiences related to early addition and subtraction. I study in what ways teachers attend to the learning needs of students both when interpreting and responding to individual thinking as well as when setting instructional goals for groups of learners.

Studies to investigate teaching practices related to the formative assessment of mathematics have included analysis of teachers' noticing during video clubs (Sherin & Ham, 2004), observations of teacher's practice in classrooms (Fennema et al., 1996; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989), and responses to student work in the form of a written response (Crespo, Oslund, & Parks, 2011; Jacob, Lamb, & Phillips, 2010; Ebby & Sirinides, 2015). Some of these studies analyzed teachers' responses to actual classroom events, either by the teachers themselves or by other teachers viewing the classroom events. Others have studied teacher responses to artifacts of instruction, in the form of students' written work (Kazemi & Franke, 2004). While video records provide real instances of students work and teachers' responses, they can also be limited in their capacity to capture how teachers respond to each of the student conceptions found within the trajectory, as not every strategy comes up during a given lesson. And, while the technique of asking teachers to respond to written student work does provide an opportunity to gather responses to multiple and diverse instances of student thinking to teachers, the responses generated are less likely to capture teachers' in-the-moment response. I looked for a way to systematically seek teachers' responses to an array of student conceptions and to do so in a way that would elicit their response in the moment.

For those reasons, I used storyboard representations of classroom scenarios (Herbst & Chazan, 2015) in order to represent the array of strategies used by students to solve addition and subtraction problems and depict those solution methods in the context of a classroom setting to simulate how teachers might respond in moments of classroom instruction. I used the storyboard representations to survey teachers by asking them to describe and respond to multiple instances of student thinking as well as to consider learning goals to support diverse groups of learners. I conducted this study in the context of early addition and subtraction because a learning trajectory for the concept has been well-established in the literature (Clements, Sarama, & DiBiase, 2004; Steffe et al., 1983) and the concept is one that is taught universally in elementary schools. Using scenarios to represent student thinking in early addition and subtraction, I was able to gather evidence of teachers' intended responses to each student conception identified in that learning trajectory. By considering teachers' self-reported training in a learning trajectory as a background factor, I investigated whether there are differences in teachers' formative assessment practices. Namely, I examined teachers' interpretations of and responses to students' individual thinking and did so by considering their responses to prompts associated with representations of student thinking that are described in the learning trajectory (Clements, DiBiase, & Sarama, 2004). In particular, I investigated the goals teachers set for individual students and the responses they provided as means to support those learning goals. Further, I examined the ways in which teachers chose to attend to the diversity of students' learning needs in the context of a group discussion and asked whether any differences in teachers' learning goals for group settings were associated with teachers' background experiences.

Data Collection

I conducted this study by gathering responses from elementary teachers using an online scenario-based questionnaire (Herbst & Chazan, 2015). In doing so, I was able to present multiple classroom moments that highlight each of the varied methods students utilize to come to solutions to addition and subtraction problems. Each item in the survey included a variety of open-ended questions, asking teachers to describe what they would do in response to students who share their thinking individually with the teacher, as well as to consider goals for learning when those individuals join together in a group setting. Each item also included a closed-ended question prompt in which respondents were asked to choose a word problem from a list, and to indicate the numbers to be used in the problem. The design of the survey is described more fully later in this section.

Participants

Because this was important to my research questions, I recruited teachers with differing professional learning experiences. Because there is a growing movement, in the mid-western state in which I recruited, to train teachers on learning trajectories, I contacted educational service agencies who had been involved in professional learning related to learning trajectories and intentionally recruited teachers who had been through the training. I also recruited another sample of teachers by sending out invitation emails to districts in regions where trainings had not yet occurred.

In total, I sampled 22 practicing elementary teachers who varied in a number of ways; from their current teaching assignment (e.g., general classroom instruction and grade level, small group interventionist) to years of experience in an elementary classroom, as well as in their professional learning experiences related to the mathematical thinking of students. Vital to the research questions I ask, the participants were asked to describe their professional experience in a background survey. In addition to gathering the teacher's current teaching assignment, years in that position, and years teaching elementary school more generally, I asked teachers to indicate any participation in professional learning related to the teaching of mathematics that had occurred in the last three years. Teachers were then asked to describe the purpose of that professional learning as well as the key take away from that learning. In addition, I asked if there had been any form of follow up support after the professional learning had ended, giving a number of choices that are often used to support teachers after a professional learning series. These included follow-up support from facilitators, coaching support, collaborative teacher meetings, or other. This was done in order to gather information that described the nature and extent of opportunity the teacher has had to engage in professional learning related to mathematics student thinking, and more specifically of a learning trajectory for addition and subtraction. Table 3.1 includes information related to the background of the teachers who participated in the survey.

Table 3.1

Background information of teachers participating in survey

ID	Teaching Assignment	Years of Teaching Experience	Professional Learning in Mathematics Teaching (in last 3 years)	Professional Learning in a Trajectory (in least 3 years)	Follow Up Support for Learning
2448	Math Specialist	12	Yes	Yes	None
4381	Special Education	17	Yes	Yes	Coaching
4411	4th grade	5	Yes	Yes	None
4422	Special Education	5	Yes	Yes	Coaching
4448	4th grade	5	Yes	Yes	Coaching
4493	5th grade	20	Yes	No	PLC
4520	Middle School	10	No	No	NA
4543	4th grade	31	Yes	Yes	PLC
4546	2nd grade	9	Yes	No	PLC
4552	4th grade	3	Yes	Yes	Coaching
4556	Special Education	31	Yes	Yes	Coaching
7110	Interventionist	12	No	No	NA
7150	Kindergarten	6	No	No	NA
7159	Kindergarten	14	Yes	No	None
7160	Early Childhood Specialist	15	Yes	Yes	NA
7161	3rd grade intervention	3	Yes	No	None
7165	Kindergarten	17	Yes	No	Coaching
7173	4th grade	10	No	No	NA
7174	3rd grade	4	Yes	No	Unknown
7195	First Grade	18	Yes	No	PLC
7228	Kindergarten	11	No	No	NA
7288	1st grade teacher	3	No	No	NA

I used the background information in later analyses to determine explore background characteristics of the teachers that might explain some amount of variation in the data. In previous studies, researchers have correlated teachers' experience and training in a learning trajectory teaching responses considered more ‘expert’ (Jacobs et al., 2010). As I will describe in the analyses, the categories that were most highly correlated with the responses that I coded as high quality were whether or not the teacher was responsible for teaching in the lower elementary grades (PreK through second grade) and whether or not the teacher had been trained in a learning trajectory for early addition and subtraction. For purposes of this study, I consider special education teachers in elementary buildings who teach in PreK-2 settings to be grouped into the group of early elementary teachers because of their interactions with children learning the targeted concepts. This created four groups of teachers used for comparisons of data in the analyses. The number of teachers that fell into each of the four categories are listed in Table 3.2.

Table 3.2

Number of teachers by grade level and training in a learning trajectory

	Trained in a Learning Trajectory for Addition and Subtraction (LT)	Not trained in a Learning Trajectory for Addition and Subtraction (non-LT)	Total
Teaches Early Elementary Mathematics (EE)	4	8	12
Teaches Upper Elementary Mathematics (UE)	5	5	10
Total	9	13	22

The Scenario-Based Survey: Responding to Students’ Addition and Subtraction Strategies

I created a scenario-based survey in order to gather teachers’ responses to the informal methods students use to solve addition and subtraction tasks; strategies which would be familiar to most elementary teachers and that were found along the target learning trajectory. I used the technique of a scenario-based instrument (Herbst & Chazan, 2015) because it allowed me to portray each of the student strategies and to do so in a manner that represented the students' work as occurring in the context of classroom interactions. These scenarios also allowed me to present each of the common student strategies in the form of a storyboard, using graphics to represent simulated classroom scenarios in which students make use of strategies such as those described in the target learning trajectory (Clements, Sarama, & DiBiase, 2004; Steffe et al., 1983).

I created the classroom scenarios by making use of the *Depict* tool in the Lesson*Sketch* online environment (www.lessonsketch.org). *Depict* is a cartoon storyboarding tool that has graphics designed to represent a typical elementary classroom environment (Herbst & Chieu, 2011). *Depict* enables the viewer to observe the classroom representations and its participants as interactions occur. Key to this study is the ability that *Depict* has to associate the language in the depicted representation with the particular actors who enact them, through the use of speech bubbles. Because the work of early elementary students is so often observed as students share verbally, *Depict* allows for the student work to be portrayed as it occurs over time, rather than projected onto a written artifact as in research that examines how teachers respond to students' written work (Kazemi & Franke, 2004). The storyboard also represents the multimodality of classroom interaction much better than a classroom transcript (Herbst, Chazan, Chen, Chieu, & Weiss, 2011). An example of a student sharing their strategy for solving a problem modeled as 8 + __ = 13 is shown in Figure

3.1 below. The depiction provides a reminder of the problem posed by the teacher, as it is written on the board, and the student shares her strategy using language that is typical of students in the early elementary grades.

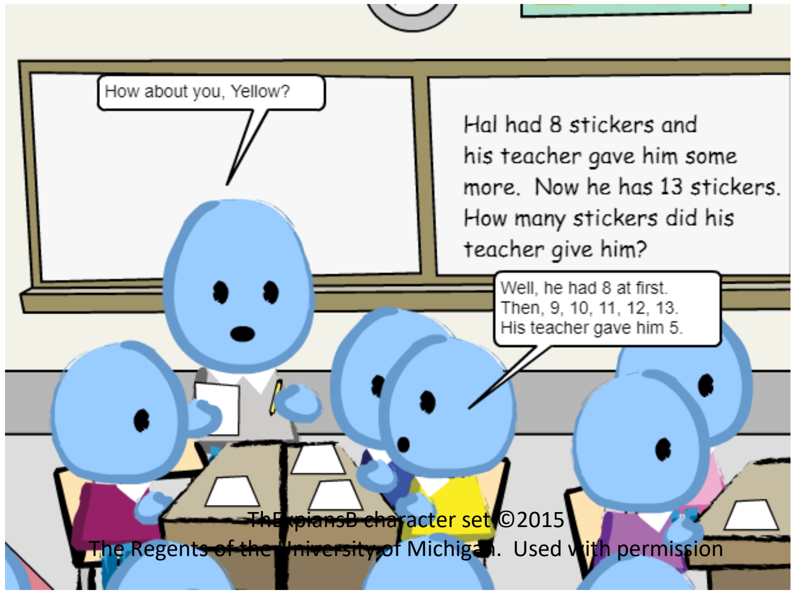


Figure 3.1: *Yellow shares a strategy for solving 8 + __ = 13.*

The scenarios provide a representation of a classroom that mutes many idiosyncrasies in the representation of individual students (Chazan and Herbst, 2011), while maintaining idiosyncrasies that represent particular conceptions of addition and subtraction that would be considered productive for teachers to recognize and attend to during instruction. Chazan and Herbst (2012) have observed that even though a storyboard or animation portrays a specific storyline, teachers are able to look beyond the specific storyboard to project themselves onto it or consider it akin to the teaching they have done. After the scripted scenarios were represented using *Depict*, I used the *Plan* tool in the LessonSketch platform to create an online questionnaire that asked participants to respond to variations of routine classroom interactions that occur in the context of teaching addition and subtraction in an early elementary classroom. In this way, I was able to present the scripted scenarios as a simulation of teaching practice rather than a self-reflection of instructional moves enacted in a teacher's classroom (Herbst, Aaron, & Chieu, 2013). I asked teachers to respond directly to those moments of instruction, giving what they consider to be an appropriate description and acceptable responses to typical enactments of student work. I delivered the questionnaires online, using the *Experience Manager* tool within the LessonSketch platform.

Appendix B includes screen shots of depictions found in the survey, along with the prompts that follow those depictions. In the following section, I describe both the design of each item within the instrument as well as the design of the set of items used across the instrument.

Design of an Item. Each item within the overall instrument was designed to elicit the formative assessment practices of teachers, specifically teachers' interpreting of and responding to the mathematical thinking of two individual students. After responding to the students working independently, the teachers were also asked to choose a problem to be posed during the group discussion and to state an instructional goal for the students in the classroom. The group to receive the problem was composed of the rest of the class as well as those two students who had been observed individually during the earlier portion of the item. This context set up the items such that participants were allowed to gather formative assessment information about individual students and respond to them individually, then also utilize that information to choose a problem and learning goal for the group of students who solve the problem during the discussion.

The design described above was realized in each of four items by depicting a teacher who has given the group of students an addition or subtraction problem, chosen from the framework of problem types for addition and subtraction (Carpenter & Fennema, 1992). The depicted teacher then asked the students to work on the problem independently while the teacher monitored the work of students in the classroom. An example of the frame that sets up this portion of the scenario is shown below in Figure 3.2. In this frame the teacher poses a problem, which is also written on the board, and the students are asked to solve the problem while the teacher monitors the students' work.

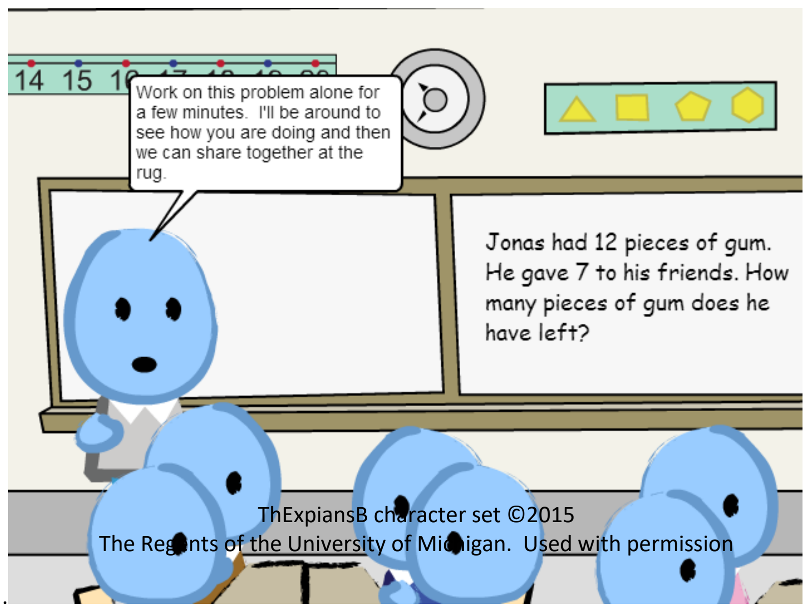


Figure 3.2: *The depicted teacher provides a problem for students to work on individually*

During the independent work time, the depicted teacher was represented monitoring the work of three different students in the class, one of the critical features of formative assessment, in which the teacher elicits student thinking. I depicted the teacher eliciting the thinking of three students in order to allow the participant to observe the work of each of these students. In the Figure 3.3 below, the teacher is represented observing the solution method a student named Blue has used to solve the subtraction problem that was presented to the class in the previous slide.

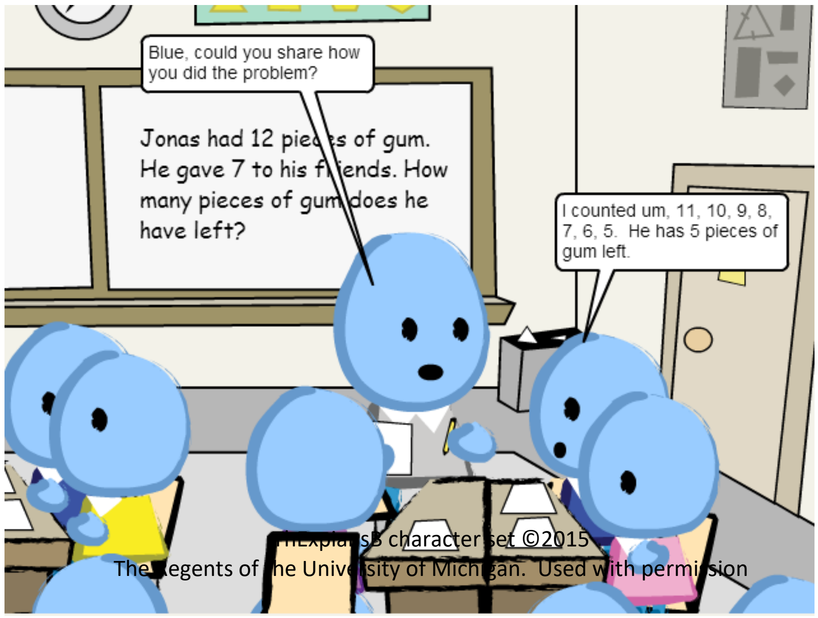


Figure 3.3: *The depicted teacher monitoring the work of Blue*

After each observation of student thinking, the participant is asked to respond to a series of prompts, in the form of an open-ended response. These prompts occur after each observation of a student in the scenario. To elicit formative assessment practices, the participant is asked the following 1) “Describe what you notice the student has done to solve the problem”, 2) “Describe what you would say and do in response to the work the student has done.” Then the participant is asked to describe a learning goal for the particular student, 3) “Describe what you would consider to be the next learning goal for the student.” In each case, the participants are provided an open text box in which to complete their response. The first prompt is meant to gather information regarding the details the participant takes into account when describing the student’s thinking while the two subsequent prompts ask the participant to describe what he or she would consider appropriate as next steps for the student.

After responding to the individual work of two students in each item, the depicted teacher is shown bringing the class together for a whole group discussion of a new problem. The depiction that represents this is shown in Figure 3.4 below, in which the depicted teacher sets up an expectation for the time spent in the group discussion, in which students will work a new problem and share their strategies in a group setting.

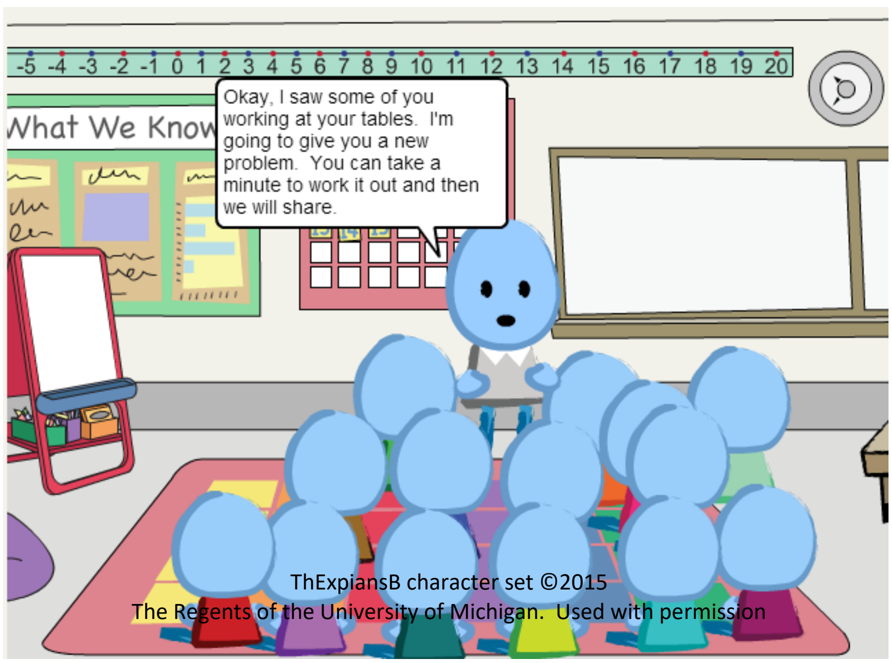


Figure 3.4: *The teacher brings students to rug to discuss a problem as a group*

At this point, the questionnaire asks the participants to choose a problem that would be used during this discussion with the whole group. Teachers are prompted to choose one out of five problems, each modeled after the addition and subtraction problem types, (Carpenter et al., 1981). The choices include an addition and subtraction problem in which the result is unknown, an addition and subtraction problem in which the change is unknown, and a comparison subtraction problem. The choices given to participants are stated as word problems in the following form and based on the particular problem type, *Jamis has ___ marbles and his friend gives him some more. Now he has ___ marbles. How many marbles did his friend give him?* Following this close-ended response, I ask the participant to choose the two numbers he or she would place in the problem. This allows me to determine which problem type the teacher would find appropriate, for example a *find result subtraction* or *missing addend*, as well as the values which the teacher considers appropriate for the students to work with. After participants have chosen a problem type and the numbers he or she would use to pose the problem to students, the participant is asked to indicate an instructional goal for the group of students. The participants are prompted to "describe what you plan to achieve with this group of students when the chosen problem is discussed." Participants are asked to give an open-ended response to this question. Because the problem is chosen as a learning opportunity for the whole class and participants have just viewed the individual work of students within the class, I designed this prompt in order to better understand how teachers might attend to the individual learning needs of students who are now engaged together in a whole group, or negotiate those individual needs with the needs of the class as a whole

Design of the Instrument. Across the whole instrument, the items depict conceptions of addition and subtraction of three students, identified orally by their differently colored vests, Blue, Yellow, and Green. Each student is making progress along the trajectory as viewed across the set of items found in the instrument. The four items that make up the instrument allow participants to view each of these three students within each class setting as well as the progress those students make across the academic year. Over the course of the instrument, I ensured that each informal method that is described in the trajectory appeared at least once, and that each method was carried out some times correctly and sometimes in error. A table showing which strategies were used by the students in each item and across the instrument is shown in Table 3.3 below.

Table 3.3:

Problem posed and student responses in survey.

	Item 1: Late Fall	Item 2: Early Winter	Item 3: Late Winter	Item 4: Early Spring
Problem Posed	Jonas had 12 pieces of gum. He gave 7 to his friends. How many pieces of gum does he have left?	Hal had 8 stickers and his teacher gave him some more. Now he has 13 stickers. How many stickers did his teacher give him?	Sadie had 11 markers and she left 8 of them at her friend's house. How many markers does Sadie have now?	Jack had 13 erasers. He gave some of them to his friends. Now he has 5 left. How many did he give to his friends?
Student A: Blue	I counted um, 11, 10, 9, 8, 7, 6, 5. He has 5 pieces of gum left.	Yeah, so 9, 10, that's 2 more. Then 11, 12, 13. That makes 5 from his teacher.	9 and 10 makes 2 and then 11 is 1 more. She has 3 left now.	5 and 10 more would make 15, but that's too many. I take off some then. He gave his friends 9.
Student B: Yellow	Well, he has 7 left. So, I can count to see how many more to get to 12. Um, 7, 8, 9, 10, 11, 12. He has 6 left.	Well, he had 8 at first. Then, 9, 10, 11, 12, 13. His teacher gave him 5.	10, 9, 8. She has 3 left.	I can go back like this. 12, 11, 10, 9, 8. And that means he gave 5 to his friends.
Student C: Green	Well, I counted 12 chips from my bucket. Then I put 7 of them away. That leaves 1, 2, 3, 4, 5.	Well, uh, 1, 2, 3, 4, 5, 6, 7, 8. Then 9, 10, 11, 12, 13. So, see, he got 6 from his teacher.	Okay, 10, 9, 8, 7, 6, 5, 4, 3. She has 3 now.	I counted. 5. Then 6, 7, 8, 9, 10, 11, 12, 13. That's 9 that he gave to his friends.

*Teachers were shown all solutions, but not asked to respond to those in gray cells.

Early in the survey, labeled for the teachers as occurring in late fall, the three students use strategies that are relatively rudimentary and without much variance in relation to each other on the trajectory. In fact, each student in Item 1 used a variation on a counting strategy --some using objects and some not using objects. As the students are depicted making progress throughout the school year (as the survey progresses), each student makes use of strategies that become increasingly more sophisticated, although the students make progress at different rates. In this way, the items that occur later in the survey present teachers with a classroom in which the students' abilities are more widely varied than what would have been observed in earlier items.

The design of one of the four items is unique in that none of the students have made an error. I consider this scenario unique since responding to a student's error is likely seen as a priority in classrooms in that a teacher should support the student in understanding the current method being used. However, responding to a group of students who have not made an error leaves the range of possible responses much more open since the teacher will need to choose how to extend each student's current understanding. The knowledge of learning trajectories not only provides a teacher with information regarding students' informal methods for solving problems, but also the order in which those informal methods are likely to develop in a child over time. This means that asking participants to respond to students who have carried out the strategy correctly provides an interesting space for teacher decision making. With knowledge of learning trajectories, a teacher has a framework on which to base decisions of this nature. Without the use of a learning trajectory, teachers might rely on other sources, such as a set of curricular materials or assessments used in their district, or perhaps considerations of a completely different nature.

Data Analysis

The design of the survey ensured that teachers respond to each instance of students' informal thinking described in the learning trajectory used in this study, as these are all instances of student thinking that teachers are likely to encounter in schools. I coded the data with the aim to define measures related to key formative assessment practices. In doing so, the analysis affords the ability to determine mean scores for key formative assessment practices across those instances of student thinking.

Teacher Interpretation of Students' Informal Methods

The first research question asks what details teachers take into account when interpreting student thinking. In accounting for that set of responses I consider whether teachers' interpretations differ in ways that can be associated with their background experiences. Later, I relate the ability to interpret student work to the knowledge of content and students (KCS) because, in part, KCS can be described as teachers' knowledge of student's common ways of thinking (Ball et al., 2008). To explore this first research question, I analyzed data gathered in response to the prompt, "Describe what you notice the student did to solve this addition or subtraction word problem." This prompt was posed each time the participant had an opportunity to observe individual student thinking: Twice in each item and a total of eight times across the four items that made up the instrument.

The analysis of the descriptions teachers gave in response to this prompt aimed to determine what teachers notice related to students' common ways of thinking and whether those responses include interpretations that go beyond a mere description of the student's work. While an objective description of a student's work is certainly a valid response to the question, further interpretation of the student's method is useful to support the practitioner's subsequent instructional responses. For example, a teacher could state that a student counted backward from 11 to 8 in order to determine that 3 is the difference between the two numbers. On the other hand, noticing where the student's count began, that the student counted by ones, or that the student counted only 3 counts when it might be more typical for the student to count 8 counts are all examples of a teacher's interpretation of the student's work. These details go beyond a description of what is seen in the depiction to attribute meaning to the work of the student. I posit that these interpretations can be essential for teachers to respond in ways that attend to the student's current understanding of addition and subtraction. I used dichotomous codes to code for description and interpretation as I analyzed teachers' responses to the prompt.

Each participant's response to a representation of student work was coded 0 or 1 for description and 0 or 1 for interpretation. I treated description and interpretation as separate coding decisions applied to the responses from the participants. It would be possible for a teacher to describe the work of a student without interpreting that work and vice versa. For example, stating that a student counted backward to subtract is an example of a description without interpretation and stating that a student recognizes the ten within a problem is an example of an interpretation without description.

I argue that an interpretation of student thinking, as opposed to only a description of the students' work, includes information useful for teachers when deciding how to respond to students. Because of this, the scores for description were used only to verify that participants were attending to the mathematical work of students depicted in the scenarios, as opposed to some other factor within the scenarios. I then used teachers' interpretation scores to determine whether differences in teachers' interpretations can be related to teachers' background experiences with the learning trajectory for addition and subtraction. I did this by finding each teacher's mean interpretation score across the eight representations of student thinking as a measure of the average extent to which they interpreted student responses using the learning trajectory. I then used the Wilcoxon rank-sum test, a non-parametric test of comparison between two samples, to determine if there was reason to believe the two samples came from different populations. The Wilcoxon rank-sum orders the individual scores (mean interpretation score across items) for the aggregate group of teachers and determines the probability that a randomly chosen score from the group of teachers trained in a learning trajectory has a higher rank than a randomly chosen score from the teachers not trained in a learning trajectory. To represent the groups, once there is evidence that they come from different populations, I found the median score for the teachers of each group. In this way, I proposed to first determine whether teachers' interpretations of student thinking differed based on whether or not they are characterized by these two professional experiences; Teaching in an early elementary setting and being trained in a learning trajectory.

The coding system described above reduced each item's response to a 1 (interpretation present) or 0 (interpretation absent). In turn, the coded data produced a measure (the average interpretation code across 8 items each coded 1 or 0), which I refer to as the *interpretation score* moving forward. This interpretation score serves two purposes for the analysis in the study. First, it provides a resource for responding to the first research question: Are there differences in what teachers notice from students' work in addition and subtraction and do those differences relate to teachers' participation in training of a learning trajectory? This could serve to describe one component that makes up a teacher's knowledge of content and students (KCS). Secondly, the interpretation code provides an independent variable that could be correlated with data from teachers' responses to students, as a teacher's response is likely to be related to what the teacher notices during an observation of the student's work. I discuss the correlation between interpretations of and responses to student thinking after I discuss the analysis of teacher responses.

Teacher Responses to Student Methods.

The second research question asks to what extent teachers' responses to students' common ways of thinking about addition and subtraction are of a quality that holds the potential to support student understanding along the trajectory. The question also asks whether teachers' responses to students, including the learning goals they set for students, differ based on the teachers' background. The design of the instrument included questions that focused participants' attention on each student's thinking and elicited a corresponding learning goal and response. The participants' attention was brought first to a particular conception by asking them to view a depiction of a student solving an addition and subtraction problem, then ask the teacher to describe the student's work. I developed a scalogram (Guttman, 1950) to code the quality of each teaching response, using the learning trajectory as a source of criteria for the scalogram's questions.

Other potential ways of coding the data, attending to the form of the response, could have been used to observe differences among teachers' responses. For example, the responses could have been coded according to whether they were open- or closed-ended. But I argue that such coding would not sufficiently distinguish responses in regard to their potential to extend a learning opportunity. To substantiate my argument, I provide illustrative responses from the data which are closed- and open-ended in Table 3.4 below. I also describe, based on the learning trajectory used in this study, whether the responses either hold the potential to extend student thinking or not.

The responses shown on Table 3.4 refer to a scenario in which the student had used a *counting on* strategy to solve a subtraction problem. The scenario provides evidence that the student is beginning to understand the inverse relationship between addition and subtraction. It would be in alignment with the trajectory for the student to consider this relationship further or begin to make use of *derived* strategies. The sample responses describe how a teacher's response could attend (or not) to the learning trajectory regardless of the open- or closed-ended nature of that response.

Table 3.4:

Examples of open- and closed-ended responses that differ in learning opportunity.

	Open-Ended	Closed-Ended
Potential to Extend	"Good! What math facts sentences can you share that could solve this problem?" Next Goal: "The student has an active strategy using structuring, but is not yet fluent."	"You made 10 by counting up from 8 then added one more. You used the number ten as a friendly number and counted up." Next Goal: "Remind her of the combinations of 10 so that she could have said 8 and 2 more is 10 then one more is 11."
Little Potential to Extend	"I like your thinking. Is there another strategy you could use?" Next Goal: "Just know the facts"	"I would say that they are right and that is one of the ways to solve the problem." Next Goal: "I would have the student work on solving similar problems with harder numbers."

I argue the open-ended responses shown in Table 3.4 differ in their potential to extend student thinking. The teacher's response shown on the first row and first column, can be described as having the potential to extend the student's thinking along the learning trajectory used in this study and is intentionally being used to ask the student to write number sentences that would model the relationship the teacher would like to establish. On the other hand, another teacher's open-ended response, shown on second row and first column, does little to give the student direction for the work he or she should do next and thus may have little potential to extend that student's thinking.

At the same time, the responses labeled as closed-ended differ in the potential to extend student thinking as well. Whereas one teacher, shown on the first row and second column of Table 3.4, describes a closed-ended response to specifically practice combinations of 10 in order to engender the use of a *derived* strategy, the other teacher's closed-ended response ends up evaluating the student's work and providing another set of problems to work. I argue this second response is not likely to press the student to extend his or her mathematical thinking, as much as to give the student additional time to practice his or her current strategy. So, while open ended questions are often highlighted as ones that can promote students' learning (Leinwand, Brahier, Huinker, Berry, Dilion, Larson et al., 2014), the data indicates that perhaps not all of those open-ended questions provide productive opportunities to extend a student's thinking. Instead of coding teachers' responses only on the basis of their generic linguistic characteristics, I developed a scalogram that would elicit information on the potential that teachers' responses had for extending thinking along the trajectory. After the participants viewed the student's thinking, they were asked to "describe what you would say and do in response to this student" and "describe what you consider to be the next learning goal for this student." The scalogram (Guttmann, 1950) relies on three nested questions, coding each with a 1 or 0 based on whether or not an indication is present. A scalogram is constructed in such a way that if a question is coded with a 0, the questions following it will not be coded with a 1. Because of this, the sum of the scores in the scalogram can be used as a rating of the quality of the teacher's response in an item.

The questions were designed to score the extent to which the teacher's response extends student thinking along the trajectory in a way that attends to the student's current use of a strategy. The questions in the scalogram are

1) Is the teacher's goal such that it would not be considered regressive according to the chosen learning trajectory and based on the student's current strategy?

2) Is the learning goal specific and aligned with what the student has yet to learn according to the learning trajectory, when taking into consideration the student's current strategy? and

3) Does the teacher describe an instructional move that could support the learning goal described by the teacher?

The coder, taking the student's current strategy into consideration, used the scalogram's questions to code each of teachers' responses. The scalogram first considers what goal the teacher intends to meet and then considers if the teacher suggests aligned means for meeting that goal. For example, if the response does not indicate a learning goal aligned with the learning trajectory used in this study, then it is not expected that the response will include a means by which to support the extension of the student's understanding either, because those extensions rely on such alignment.

For each of the questions, I drew on the literature describing a learning trajectory in addition and subtraction (Clements et al., 2004; Steffe et al., 1983) as well as Vygotsky's theory of the zone of proximal development (Vygotsky, 1987). In the former, researchers have described the development of student thinking for early addition and subtraction as one in which children develop more sophisticated thinking about the operation over time. Generally, this means that students move from the concrete use of objects to model additive and subtractive processes, to more sophisticated uses of counting patterns, and finally to reason about the properties of number and the operation itself. In the latter, Vygotsky describes the zone of proximal development as the distance between what a child can do independently and the "level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers" (Vygotsky, 1987, p. 86). The depiction is meant to represent the student's independent abilities, thus, I consider the depiction and the teacher's response in conjunction with each other to determine whether the teacher's response sets goals for the student that are attainable with the support of a teacher, as indicated by the learning trajectory being used in this study. Thus, each specific goal set by the teacher, was first coded as to its location on the learning trajectory used in this study relative to the depicted student's current strategy (e.g., +4 if the student was *direct modeling* and the teacher set a learning goal to learn a make ten strategy, or a -2 if the student was using a *counting on* strategy and a learning goal was set for the student to solve problems with counters). If this relative distance was +1 or +2, the response was coded with a 1 for the scalogram question, "Is the learning goal specific and aligned with what the student has yet to learn according to the learning trajectory, when taking into consideration the student's current strategy?"

The range of possible ratings for the quality of one teaching response is between 0 and 3. Based on the questions, a score of 0 would be an indication that the teacher proposed something in the response that would be considered to regress the student in their thinking (e.g., giving the student counters to do their next problem if they had previously used a counting method). A score of 1 would indicate that nothing in the response was considered to regress the student's thinking, but there was also no indication that the teacher proposed a learning goal aligned with the learning trajectory or was able to support that goal. Finally, a score of 2 would indicate that the response included a learning goal aligned with the learning trajectory, and a score of 3 would be given to a response that also had specific instructional moves for supporting that learning goal.

After coding the data, I found the mean response code for each teacher across all eight items, to represent the extent to which teachers' responses to all students have the potential to extend the students' learning opportunities—I refer to this average code as the response score. I compared the groups of participants by using the Wilcoxon rank-sum test, a non-parametric test that assesses the probability that a randomly chosen score from the group of teachers trained in a learning trajectory has a higher rank than a randomly chosen score from the teachers not trained in a learning trajectory. I did this to determine if teachers with particular classroom experience or training differed significantly in their responses to students. If these groups turned out to be associated from different populations, I represented them using the median response score of teachers within each group.

I use the scores from this coding to relate to the construct for knowledge of content and teaching (KCT). The scores provide a way of describing both the openness that teachers have to allow students to make sense of and develop informal methods for solving addition and subtraction problems as well as the skill of the teacher to do so.

Correlation Between Interpretation Scores and Response Scores

After analyzing teachers' interpretations and responses to students, it is reasonable to ask if a relationship exists between interpretation scores and response scores. I conjectured that response scores would correlate positively with interpretation scores. This correlation would suggest that there is a relationship between interpretation as a case of KCS in addition and subtraction, and teacher's responses as a case of KCT for addition and subtraction. It is clear that some teachers might choose an instructional approach that encouraged students to always have objects available in order to model a mathematics problem, regardless of the student's thinking. But, providing discrete, countable objects to a student who has an understanding of the relationship between quantities is not likely to be a helpful support for challenging that student's sophistication in mathematics, whereas a structure like a ten frame may be more supportive.[2] To find a positive and significant correlation would lend credence to the conjecture that a teacher who chooses and supports a response to a particular student which is aligned with the learning trajectory is more likely to have made an interpretation of that student's thinking.

By examining the data in this way, I propose to probe for a relationship between a teacher's KCS and KCT, using the variables defined above as proxies for each domain of mathematical knowledge for teaching (Ball et al., 2008). If a significant positive correlation existed, it would help document to what extent a teacher's

[2] There is evidence that giving objects to students who have progressed beyond the need for objects may revert to a basic counting strategy when provided objects (Steffe et al., 1983) and that a ten frame is a structure that supports student understanding of relationships between quantities (Clements & Sarama, 2014). See more about these teaching strategies in Appendix A.

interpretation of student's concepts is a helpful resource for teaching responses of a quality that can be appreciated from the perspective of the learning trajectory. Additionally, I compared the strength of that correlation for teachers with training on a learning trajectory with the strength of the correlation for teachers without training on a learning trajectory.

Up to this point, I have described how I processed the data to distinguish teachers in regard to their noticing of aspects of students' conceptions, and second to describe the nature of teachers' responses to students' informal methods. I have also described how I expected that the interpretations teachers make of student conceptions could play a role in the responses teachers give to those students and how a correlation between the interpretation and teaching response mean scores would support such conjectures.

Teachers' Goals for Instruction in Whole Group

The remaining piece of analysis considers that while a learning trajectory could be a beneficial tool for understanding individual student thinking, it is necessarily more complex for a teacher to consider learning goals for a group of learners because of the diversity of understandings that are naturally present within the group. While a teacher may make use of a learning trajectory to inform instructional moves for an individual's learning, each student will traverse through their learning beginning from different starting points (relative to the academic year) and at different paces. Yet, that they will all be learning in the same classroom is part of the complexity of his or her work a teacher needs to manage. The instructional goals that teachers set for groups of learners, then, could vary in significant ways from the goals set for individual students or from the goals set by other teachers when faced with the same complexity. I propose that when teachers engage with groups of students, who differ in their current understandings of a concept, their instructional goals may not be targeted at any one particular student's understanding, but instead serve to meet the needs of learners in different ways. For example, instead of setting a specific learning goal for students to make use of a derived strategy (which may not be attainable by all of the students), the teacher might instead set a goal for everyone to be able to communicate clearly about the problem that is posed (Yackel & Cobb, 1996). Or, a teacher might allow for the diversity of student thinking to be elicited within the group and have methods for that varied thinking to be utilized as a means for others to learn. Because of the multiple needs of students and the varied goals teachers might have for groups of students, I analyze the goals teachers write for the group of students and look for linguistic representations of how the teacher intends to engage the group in learning. Qualitative methods are used to analyze these goals because I conjectured that the group setting might be more likely to elicit instructional goals that varied more widely as compared to those responses to individual instances of student thinking.

I used systemic functional linguistics (Halliday, 1978; Halliday & Matthiessen, 2004) to discern differences in the varied ways that teachers approached group discussions. According to the theory of systemic functional linguistics, words in a text do not carry meaning in and of themselves. Instead, as speakers or writers make choices in the ways their words come together grammatically, the language "shapes, and is shaped by the contexts in which it is used" (Schleppegrell, 2012, p. 21). Through choices from lexical and grammatical systems, speakers and writers present meanings that are *functional* for their purposes in a particular social context. Through analysis of the ways speakers draw on these system, the theory of systemic functional linguistics affords the ability to interpret meaning in a text.

I used the SFL system of transitivity (Halliday and Mathiessen, 2004) to analyze how the language teachers used to respond to the prompt to “describe what you plan to achieve with this group of students when the chosen problem is discussed” reflected different choices in representing the experiential world. The tools of SFL permit to analyze those goal statements in more depth than other qualitative tools, such as grounded theory (Strauss & Corbin, 1994), because systemic functional linguistics organizes the systems of choice that are available to writers or speakers as they construct the text. In doing so, SFL provides the analyst with means to reveal the meaning potential of the actual choices participants made. Because the goals that teachers set could vary along many dimensions (e.g., the use of multiple methods or not, who engages in the discussion, what tools are used to aid the discussion, and so on), and because I conjectured that the learning trajectory might be less helpful to account for how teachers shape the work of the group, it was appropriate to use SFL as a tool to illuminate the qualities of the choices teachers made, analyzing text at the level of the clause.

Halliday describes language as having three metafunctions; the ideational, the interpersonal, and the textual (Halliday & Matthiessen, 2004, p. xiii). The content of a text is realized in the ideational metafunction—that is, the text present experiential and logical meanings through the choices in the text. The interpersonal metafunction refers to how the choices made in the text construe a relationship between writer and reader or speaker and listener. Finally, the textual metafunction refers to how choices in the text make it the kind of text it is and organize it as “a piece of writing or speech” (Eggins, 2004, p. 12). These metafunctions “are simultaneously realized in every clause we speak or write and relate our linguistic choices to the context that the language participates in” (Schleppegrell, 2012, p. 21).

The metafunctions of interest to the analysis of the teachers’ goals are the ideational and the textual because they support an analysis of the content and connections between content presented in the teacher’s language. I use transitivity analysis to extract the resources used in the text to represent ideational meaning and conjunction analysis to identify the textual resources teacher used to connect different components of the teachers’ representation of what they would do. These analyses allow me to more clearly describe multiple differences found across the goals.

Transitivity analysis examines how the choices of grammar of a clause represent the world. It requires identifying processes, participants, and circumstances of each clause. These are canonically indicated by the verbal groups, nominal groups, and prepositional phrases and adverbial groups, respectively in each clause. To do so, allowed me to distill from the text who or what participates, in which processes, and under which circumstances. Thus, transitivity analysis supports understanding how language represents experience: Implementing it helped discipline my reading of each goal statement, enabling me to classify, relate, and aggregate the goals of different individuals based on the experiences that they described being their goals for the class. Because such goals may include differentiating between possibilities, I complemented the transitivity analysis with a conjunction analysis so as to understand the nuances of the one or more goals that teachers hold for group instruction and the ways in which the teacher expected to manage differences within the classroom.

A *conjunction* analysis (Martin & Rose, 2005) permits to illuminate how conjunctions used in a text realize the logical relations that can be found among events represented in the text. Conjunctions can be analyzed in relation to four areas of meaning; addition, comparison, time, and consequence. An analysis of the conjunctions, then, allows me to be further descriptive of the purpose, intent, and composition of the teachers' goals. While the present account of transitivity and conjunction analysis is very brief, chapter 5 comes back to them as it reports on what the data showed.

CHAPTER 4: ANALYSIS

TEACHER RESPONSES TO STUDENTS

The practice of formative assessment has been described as the "extent to which evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction" (Black & Wiliam, 2009, p. 9). When carried out in skillful ways, this practice can connect the student's current ways of thinking about mathematical concepts to ever-more sophisticated ways of thinking and reasoning in mathematics. The general formative assessment literature (as in Black et al., 2002) advocate that teachers across all content areas can gather evidence of student understanding and make in-the-moment adjustments to instruction in ways that are advantageous to the learner. But, there is also evidence that teachers may need specific training in student cognition in order to hone their practices for responding (Jacobs et al, 2010). In this chapter, I consider two tasks of teaching, interpretation of and responding to student thinking, and consider if differences in those practices as carried out by teachers can be related to the background experiences of those teachers. At the conclusion of both of the analyses, I then study the relationship between scores for the interpretation of and responding to student thinking. I carry out the analyses in order to address my first two research questions.

1) Are there differences in what teachers notice from students' work on addition and subtraction when we compare teachers who have had training in a learning trajectory for addition and subtraction with those who have not had training in a learning trajectory?

2) How are the learning goals of teachers who have had training in a learning trajectory for addition and subtraction different from the goals of teachers who have not had training in a learning trajectory, and how does teachers' training in a learning trajectory relate to the extent to which responses have the potential to extend individual students' learning?

Teacher Classroom and Professional Learning Experiences

As each of the analyses I conducted are predicated on the hypothesis that teachers with training in a learning trajectory hold knowledge that could influence the quality of instruction, I first gathered relevant information related to the teachers' background, including teaching experience and professional learning the teachers had engaged in. In later analyses, I used these categories to determine if any of them was associated with the quality of instructional moves. Jacobs and her colleagues (2010) found that the depth of a teacher's experience impacted the responses to student work represented in scenarios. Similarly, in this study, once the data for interpretation and responding were analyzed, I compared these data as it related to different categorical groupings based on the teachers' background. In these comparisons, I asked whether grouping teachers by years of experience or type of professional learning showed trends in the data for interpretation and responding that consistently explained variations in that data. I also considered previous studies that have found that professional learning can impact teachers' professional noticing, but that a teacher's experience can also play a role in the robustness of that noticing (Jacobs, Lamb, & Phillips, 2010). Of the background information gathered in the survey (current teaching assignment, number of years teaching, professional learning, and professional learning in a learning trajectory), the two categories that became most relevant were teachers' current teaching assignment and their having had professional opportunities to learn a learning trajectory for addition and subtraction. With relative consistency across responses, teachers who taught in preschool through second grade, including special education or interventionists, and who had participated of a professional learning opportunity focused on learning trajectories had scores that were significantly different from the scores of teachers of upper elementary without training in a learning trajectory. Table 4.1 below shows the number of teachers in each of the categories.

Table 4.1: *Participating teachers, grouped by teaching assignment and training in a learning trajectory*

	Trained in a learning trajectory (LT)	Not trained in a learning trajectory (non-LT)	Total
Teaches early elementary mathematics (EE)	4	8	12
Teaches upper elementary mathematics (UE)	5	5	10
Total	9	13	22

Teachers' Descriptions and Interpretations of Students' Mathematical Thinking

In eight different instances across the survey, twice in each of four items, participants were shown a depiction of a student working out a context-based addition or subtraction problem. After each observation of individual student thinking, the participant was asked, “Describe what you noticed this student did to solve the addition or subtraction word problem.” Teachers were given a text box in which to type their open ended response.

Coding Teacher Observations of Student Thinking

The work of teaching requires that a teacher be able to notice and interpret the ways in which students are understanding the concepts (Hiebert & Grouws, 2007). The purpose of the analysis of these responses was to distinguish between the varying details included in teachers’ observations of student thinking and to describe any differences that might be noticed within those variations. Because of the nature of the prompt, I believed that most teachers would notice and describe details of the students’ thinking. However, I also conjectured that there would be variations in the interpretations teachers gave of those student responses. For example, some participants might notice what a student didn’t do, the potential meaning of what the student did, or the possible reasons why the student did what he or she did; but others might not make such inferences. Thus, responses from teachers were initially coded for description and interpretation separately, both using a dichotomous code, to determine, first, whether teachers were attending to the details of student thinking and secondly, to identify those responses that interpreted further details of the student’s work. A colleague and Icoded the data separately for description and interpretation and then reconciled each set of codes, keeping track of any justifications for coding each response.

To code for description and interpretation, we first considered what might distinguish the two codes from each other. We proposed that a participant who includes interpretation in his or her response would attribute meaning to the work of the student, whereas a response including a description would include objective details that could be noticed in the depiction. Thus, each of the coders first determined whether or not the participant included details in their responses that would be considered objective descriptions, which each of us then followed by coding for whether or not the participant had included details that might attribute meaning to that student's understanding. For example, in the depiction below, taken from the survey, we determined that an objective observer might notice that the student came to a correct answer of 5 or that the student counted in order to find an answer, as was the case in the response from one participant who stated, *"*The student counted on from 8.*"* (4411)

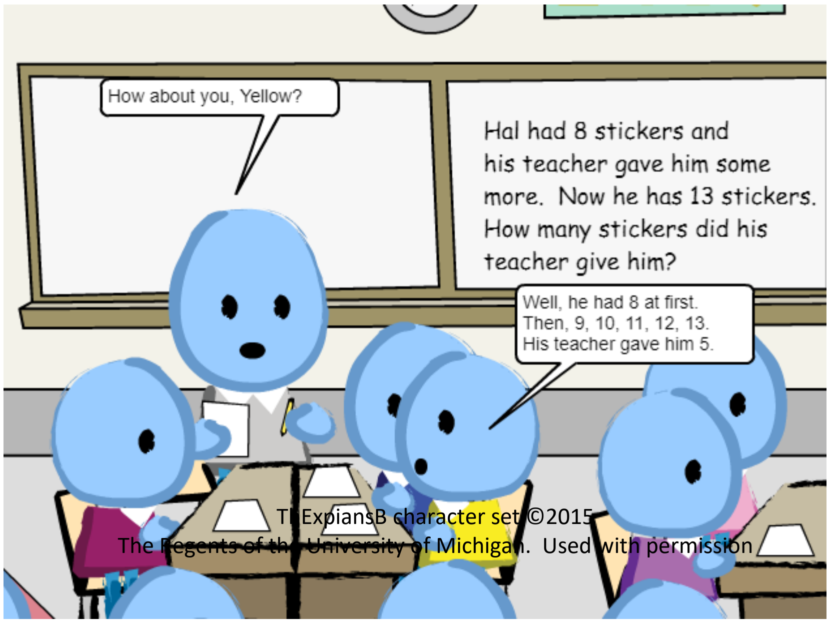


Figure 4.1: *A student counts in order to solve the missing addend problem*

Responses of this nature were coded as descriptive because the participant attended to details that were directly observable in the depiction, but not for interpretation because the participant responded without inferring anything about the student's understanding based on those details. As a comparison, after viewing the same depiction, another participant responded, "This student used counting up but already understands that they just needed to start at the 8 and then count up to 13*"* (7161). This response was coded as description because the participant mentions that the student counted up [from 8 to 13], and also for interpretation when the participant expands on this observation, and attributes that work to the student's understanding when the participant says that the student "already understands" the count can start at 8. Given that some students might solve this problem by starting the count at one (Secada, Fuson, & Hall, 1983), this participant's attention to the student's understanding provides evidence that the participant is accurately attending to details of student thinking that go beyond a surface recognition of the student's work.

Over time, some clear distinctions regarding some of the more common responses were found. For example, if a participant mentioned that the student had made an error, this was coded as description whereas the response was coded as interpretation if the participant indicated the source of the error (e.g., counted the 5 instead of starting on 6). Similarly, responses that mentioned that the student counted were coded as descriptive, but those in which the participant mentioned the nature of the count (e.g., counted by ones or counted efficiently) were coded as interpretation.

Common Themes in Teachers' Descriptions and Interpretations

In the following section, I briefly describe some of the more prevalent responses that were coded as description or interpretation. I share these details according to the method type: modeling, counting, and derived. I follow those descriptions by sharing the results of the coding.

Teachers' descriptions and interpretations of *direct modeling*. I first describe participants' responses to a student using a direct modeling strategy to solve the problem *Jonas had 12 pieces of gum. He gave 7 to his friend. How many pieces of gum does he have left?* (Item 2). In the depiction shown in Figure 4.2, Green is shown coming to a correct response using the *direct modeling* strategy (Carpenter et al., 1981).

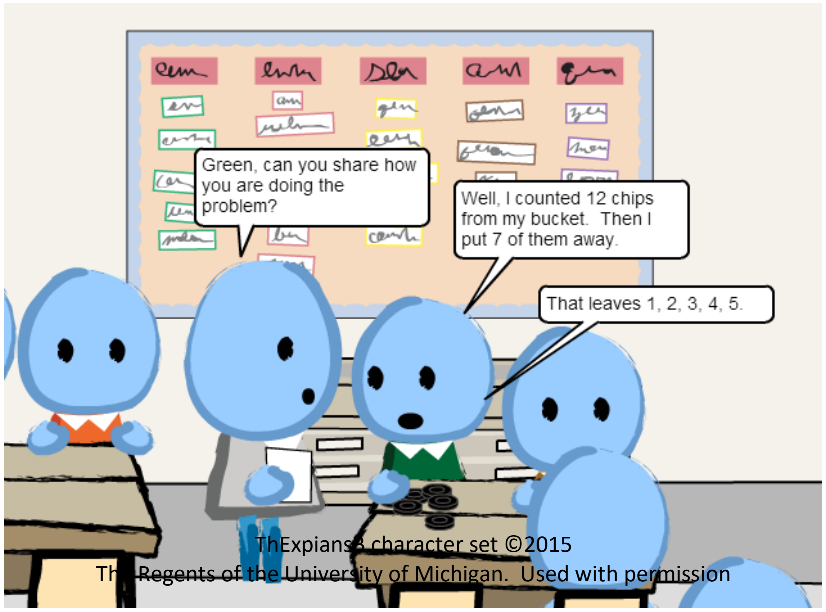


Figure 4.2: *Green uses a direct modeling strategy to solve*

Participants' responses to this item that were coded as *description* include mentions that Green used chips and counted to find the answer, details that would be recognizable through observation. Responses that were coded as *interpretation* went beyond description to infer the purpose of their use, for example that the chips were needed by the students to represent or model the actions in the problem. These interpretations are consistent with research that has shown that *direct modeling* strategies are those in which ". . . physical objects or fingers are used to represent each of the addends, and then the union of the two sets" (Carpenter & Moser, 1984, p. 180).

Teachers' descriptions and interpretations of counting strategies. There were five items in the survey that included variations of counting strategies, including counting back the amount indicated by the subtrahend, counting back until reaching the subtrahend in the count (counting back methods that differ in their efficiency), counting from one[3], counting on to add, and counting up to subtract. Two of these instances included an error.

The depiction below is one instance in which a student was represented using a counting method. In this case, the student made an error in doing so. The problem written on the board stated, *Hal had 8 stickers and his teacher gave him some more. Now he has 13. How many stickers does Hal have now?*

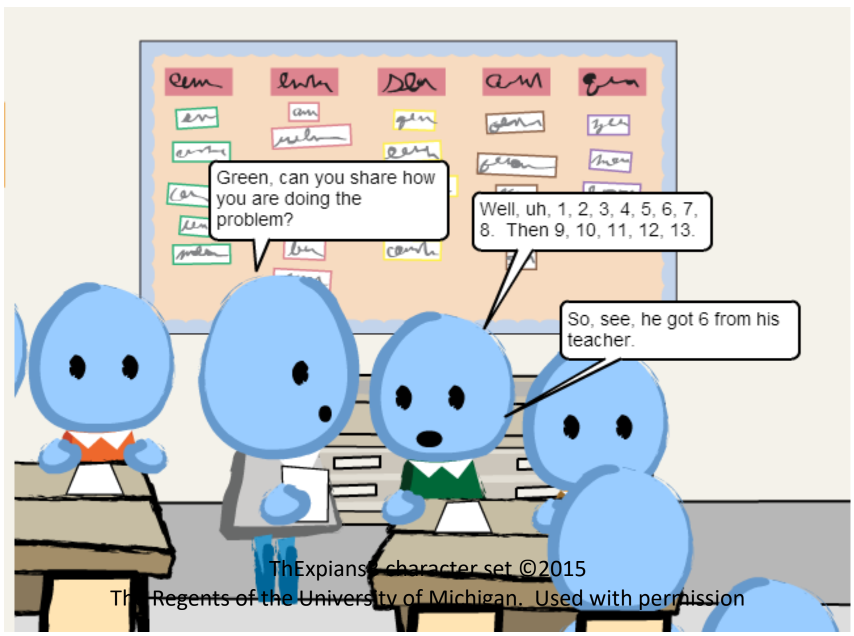


Figure 4.3: *A student incorrectly solves a problem using a counting strategy*

Of the responses coded as *description*, participants tended to give a description of the student's counting method, sometimes mentioning that the student came to an incorrect answer, as in the following response. "He counted to 8 which is the number she began with then kept counting to 13. However, he got the answer incorrect" (7173).

[3] Fuson (1984) has called this strategy *counting all*.

Responses coded as *interpretation* included references to the manner in which the student counted, including mentions of the student's efficiency, or lack thereof, when counting backward to subtract. Participants also made note of students' likely methods for tracking a count, often mentioning the possibility of fingers, which in the case of the depicted characters would not have been observable since the characters do not have fingers. Participants would also sometimes mention an inferred source of the student's error if one had occurred. For example, one participant described the work of the student in this way, "The student began counting from 1 instead of counting on from 8. Furthermore, when he gets to 8 he counts that as '1' instead of '0' as he continues to 14" (4381). Again, this interpretation is consistent with the literature that describes students as enacting counts that help them to understand how many have been accumulated or reserved when adding or subtracting (Carpenter & Moser, 1984).

Teachers' descriptions and interpretations of derived strategies. There were two items in which the student used a strategy which attended in some manner to the quantity ten within the problem: They did so either counting through the ten and recognizing the ten in doing so or adding a ten and adjusting to come to an answer.

The depiction below represents a student who has added ten and adjusted, although they made an error in doing so.

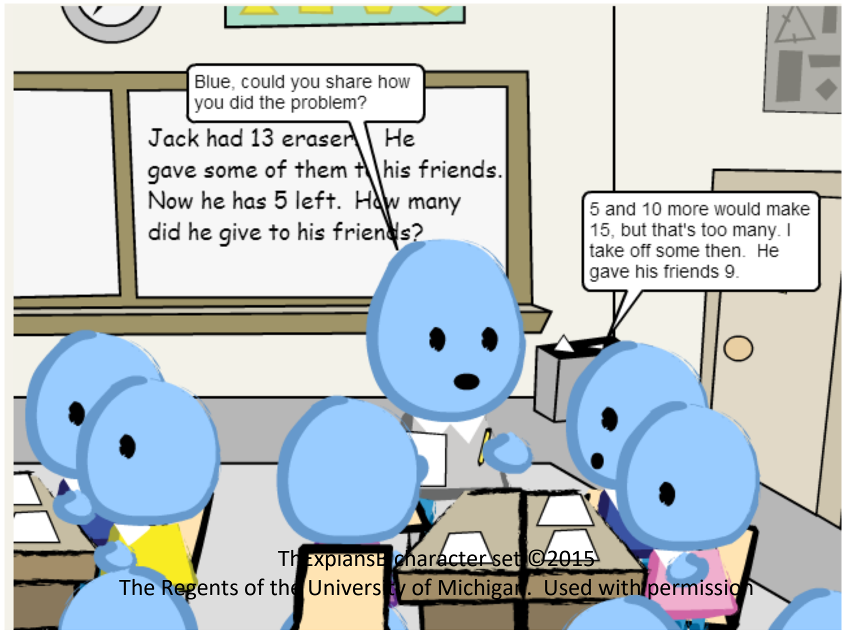


Figure 4.4: *A student incorrectly uses a derived strategy to solve*

When participants described the student’s work, they mentioned that the student had used a *make ten* strategy or that the student had added a ten and adjusted. In cases of interpretation, the participants mentioned a number of details regarding the student’s understanding of the number system, as in this response, *“*The student started chunky (sic) into multiples of 10, but completed the 2nd part of the process incorrectly. He stated he needed to take some off, but he only took 1. He didn’t identify how much ‘some’ was*”* (4422). In these responses, the participants typically mentioned that the student had been using known sums of ten (e.g., 8 + 2 and 7 + 3) to solve the problem, sometimes noting that the student is no longer counting by ones, but rather can make use of “chunks” or landmark numbers to solve more efficiently. This is consistent with the description of a derived fact in which a student uses a known addition or subtraction fact[4] in order to solve an unknown addition or subtraction fact or problem. The problems in which the students had used derived strategies were both subtraction problems which the students solved in additive ways, prompting some participants to mention that the student recognized that addition can be used to solve subtraction problems.

Results: Teachers' Descriptions and Interpretations of Student Thinking

The coding scheme for description and interpretation was applied first by each coder, a colleague and I, and then reconciled collectively until agreement was reached. I report the inter-rater reliability of the initial coding, using a Kappa statistic for each item, followed by a pooled Kappa statistic (DeVries, Elliott, Kanouse, & Teleki, 2008), in which the Kappa statistic for each item is pooled into a collective score across all items. The pooled Kappas are used as a measure of the reliability of each complete coding operation. These values are shown in Table 4.2 below.

Table 4.2:

Kappa reliability score for each item scored as description and interpretation

	Item 1	Item 2	Item 3	Item 4	Item 5	Item 6	Item 7	Item 8	Pooled
Description	N/A	0.45	1	N/A	0.46	N/A	1	0.63	0.55
Interpretation	1	0.65	0.56	0.72	0.73	0.58	0.90	0.70	0.77

The Kappa scores for interpretation fell between 0.56 and 1, indicating a moderate to strong amount of inter-rater reliability for the coding of interpretation. The Kappa scores

[4] In this study, a fact was the number model associated with the given mathematical task, representing sums or differences within 20.

could not be calculated for the coding of description for items 1, 4, and 6. In each of these items, both coders had full agreement and all teachers received a score of 1 for description. Because Kappa is a measure of reliability that takes chance agreement into account, the instance described here for items 1, 4, and 6 returns a Kappa score that is unable to be calculated. In other words, there is no way to determine the probability of chance agreement since there was full agreement. This difficulty highlights the value of the pooled Kappa as a summary measure of the quality of the description coding. The pooled Kappa could be calculated and it yielded a moderate reliability.

It was also important to consider whether each item in the survey was reliably measuring the same construct. Cronbach's Alpha and mean inter-item correlation (IIC were computed to assess the internal reliability of the items in the survey; to ensure that each item, as compared to others in the survey, were measuring the same construct. For participants' descriptions, Cronbach's Alpha equal to 0.09 and a mean inter-item correlation coefficient (IIC) of 0.02 indicated that the internal reliability of the items was very low, which may be due to the minimal amount of variation, which I report below, in the data for teachers' descriptions of student thinking. In contrast, the Cronbach's Alpha score for teachers' interpretations was 0.72, which indicates a moderate rate of internal consistency. I also calculated a mean inter-item correlation (IIC) for interpretation 0.024. An IIC below 0.20 suggests the items are not well related to each other and an IIC greater than 0.40 could indicate redundancy among the items (Piedmont, 2014). The IIC for teachers' interpretation of 0.24, suggests that the items are related to each other with little redundancy among them. Thus, if we presume all the items measure the same construct, the Cronbach's Alpha scores indicate a high internal consistency of those items.

Teacher descriptions of student thinking. Once all items from the survey were coded, I calculated the mean of each participant's description across all eight items to which they described the student's thinking. This mean description score is meant to describe the extent to which the participant described student thinking across all items in the survey. The mean score that measures the extent of each participant's description ranged from 0.75 to 1.0 (mean = 0.92, SD = 0.10, median = 1). Across the eight items, teachers would often, if not always, give a description of the student's thinking. While this meant that there was little variation in the data for description, it does provide validation that the participants were attending to details of students' mathematical thinking when responding to the depictions. After determining each participant's mean score for description across the items, I found a median of the mean scores, aggregated by teacher group, depending on whether the participant taught in early or upper elementary and whether or not the participant was trained in a learning trajectory. In Table 4.3 below, I show the median score for each teacher group.

Table 4.3:

The median score for teachers' descriptions within each group

	Trained in a learning trajectory (LT)	Not trained in a learning trajectory (non-LT)
Teaches early elementary mathematics (EE)	1.0	1.0
Teaches upper elementary mathematics (UE)	0.90	1.0

The median scores for descriptions of student thinking provide insight into the extent to which participants were attending to the details of student thinking when viewing the depictions. Participants in all groups were likely to be descriptive of what the student had done to solve a problem and noticed details related to the work the student had done.

Teacher interpretations of student thinking. Similar to the mean scores for participants' descriptions, I found the mean score for each participants' interpretations across all eight items, describing the extent to which each participant interpreted student thinking across all items. These scores ranged from 0 to 1 (mean = 0.52, SD = 0.28, median = 0.44), including participants who, in each instance, gave no indication of interpretation as well as participants who offered an interpretation for each instance of student thinking.

Teachers who had participated in professional learning opportunities based on a learning trajectory in mathematics had had exposure to interpretations of students' informal methods for solving addition and subtraction problem. Based on a conjecture that teachers who have been trained in a learning trajectory might be well-prepared to offer interpretation of student thinking, I compared the scores for participants with and without professional learning of a learning trajectory. Because of the small sample size, I used the non-parametric test of Wilcoxon (1945) rank-sum because it does not require the assumption of a normal distribution. The Wilcoxon rank-sum orders the mean scores for the group of teachers and determines if a randomly chosen score from the group of teachers trained in a learning trajectory has a higher rank than a randomly chosen score from the teachers not trained in a learning trajectory. In this sample, the distribution of scores for teacher groups based on teachers' training in a learning trajectory differed significantly at a significance level of 0.05.

I conducted further comparisons, breaking the teachers into the four groups based on teaching assignment and prior training in a learning trajectory. Comparisons of the four groups of teachers' interpretation scores revealed that there were significant differences ($\alpha = 0.05$) in interpretation scores found between participants who taught early elementary grades, depending on whether they had prior training in a learning trajectory or not, providing evidence that teachers with training in a learning trajectory may be better prepared to interpret student thinking than even their colleagues with similar teaching experience. There was also a significant difference ($\alpha = 0.05$) between early elementary teachers who had been trained in a learning trajectory and upper elementary teachers who had not been trained in a learning trajectory.

Thus, in this sample, teachers who taught in the early elementary grades showed to be more attentive to the interpretations of student thinking if they had also had training in a learning trajectory. And early elementary teachers without training in a learning trajectory were not more likely to interpret student thinking of addition and subtraction than those who taught in other grade levels. This finding is somewhat surprising considering that early elementary teachers are regularly engaged in the teaching of early addition and subtraction: this experience, by itself, was not associated with them being more likely to interpret the work of students than teachers who are not responsible for teaching early addition and subtraction.

After comparing teachers' interpretation scores by doing group comparisons, I found the median of the teachers' interpretation scores for each of the four teacher groups. The group median interpretation scores are shown in Table 4.4 below. As shown using the non-parametric tests, teachers who had training in a learning trajectory were more likely to offer interpretations of student thinking than teachers without training. Those who were trained in a learning trajectory interpreted student thinking approximately twice as often as their counterparts who had not engaged in a training focused on a learning trajectory for addition and subtraction.

Table 4.4

Median score for teachers' interpretations of student thinking depending on teaching assignment and training in a learning trajectory

	Trained in a learning trajectory (LT)	Not trained in a learning trajectory (non-LT)
Teaches early elementary (EE)	0.8	0.4
Teaches upper elementary (UE)	0.8	0.3
	0.8	0.4

The interpretations that teachers make of students' mathematical thinking play an important role in the ways in which teachers then respond to students in the classroom (Jacobs, Lamb, & Phillip, 2010). In the next section, I describe the coding for teachers' instructional responses to the students who were observed in the survey along with the results of that coding. In the section after next, I examine the correlation between teachers' interpretations of student thinking and their instructional responses.

Teachers' Responses to Students' Mathematical Thinking

In order to address the second research question, I consider the instructional moves that participants suggested as a response to the students they had just observed. In this analysis, I determine the extent to which the responses hold the potential to extend student understanding beyond what has been depicted as the student's current understanding of addition and subtraction. To conduct this analysis, I developed a scalogram (Guttman, 1950) to score the responses to each item in a way that assigns a scaled score to each instructional response. As noted in the methods chapter, a scalogram consists of a series of dependent questions, for which the response to each is binary. A positive response to one of those questions would imply a positive response for each question that came previously. Similarly, if a question is coded in the negative, this excludes any of the subsequent questions from being answered in the positive. The scalogram can then be accumulated into a scale score.

The Scalogram for Teaching Responses

After each observation of a student's strategy, the participant was asked to describe what they considered an appropriate response to the student. Specifically, the prompt asked the participant to "Describe what you would say and do in response to this student." As a follow up to this prompt, participants were asked to indicate a learning goal for the student. The prompt stated, "Describe what you consider to be the next learning goal for this student." Because learning occurs over periods of time and cannot be attributed to any particular instructional response, the prompts were meant to gather information regarding participants' immediate response to students as well as their longer-range goal for the student. I distinguish these two moments in the coding scheme by attending to the participant's instructional move and the participant's goal, respectively.

The scalogram included three nested questions, coded in this order,

1) Is the teacher's goal such that it would not be considered regressive according to the chosen learning trajectory and based on the student's current strategy?

2) Is the learning goal specific and aligned with what the student has yet to learn according to the learning trajectory, when taking into consideration the student's current strategy?, and

3) Does the teacher describe an instructional move that could support the learning goal they described?

If the first question in the scalogram was coded as 0, this indicates that the goal the teacher has set has the potential to cause the student to regress in their mathematical thinking. A learning goal of this nature might include the expectation that the student make use of counters in order to solve problems when the student had just shown he or she could solve the problem using a *counting on* strategy. There is evidence from cognitive research in mathematics education that given concrete materials, students are likely to use a *direct modeling* strategy even if the student is more advanced mathematically and can enact a more sophisticated strategy since the "search for perceptual input would [be] immediately satisfied through visual perception" (Steffe et al. 1983, p. 72). A teacher might ask a student to do such a thing on the grounds that it could boost the student's confidence when he or she recognizes that their answer had been correct using the *counting on* strategy. However, as one considers its alignment with the learning trajectory used in this study, it would have the effect of asking the student to use methods that are less sophisticated mathematically and thus less useful for the advancement of the student's learning. Thus, such a learning goal would be coded with a 0; and so would other instances in which the enactment of the learning goal would be aligned with a conception that, according to the learning trajectory, are enacted earlier in development. On the other hand, if nothing in the teacher's suggested goal would have the potential to regress the student's thinking, this question would be coded as 1.

The second coding question is whether the goal the teacher has set is one that not only does not have the potential to regress the student's current thinking, but is also one that has the potential to extend the student's understanding. Because the teacher has just observed an individual student's thinking and because the learning trajectory used in this study provides insight into how a student's learning progresses over time, this question is designed to determine if the teacher has clearly described a goal that is within the range of what is possible for the student's upcoming learning.

While some goals may have been considered not to regress the student's thinking, they might also be unspecified in ways that do not clearly target a learning goal that was specific and aligned to the learning trajectory used in this study. I also identified two categories of goals that were either not specific or not aligned with the learning trajectory. Goals that were non-specific were too broad to be identified, such as "solve without using counters", and goals that were not found on the trajectory typically attended to some indicator other than the student's strategy, such as asking a student to solve "another problem with larger numbers." In the former category, the goal may be valid and yet not specific enough to determine the teacher's intention for the student's learning as it pertains to the learning trajectory used in this study. In the example shown here, “solve without using counters,” there can be a number of student strategies along the trajectory that occur without the use of counters and so this goal is not specific enough to be able to determine what the teacher would consider to be the learning goal for the student moving forward. In the latter category, the goal cannot be located on the trajectory since the trajectory describes the solution methods of students and these goals are instead focused on other aspects of the problem or solution and thus do not clearly identify a learning goal for the student. As an example, a goal such as “solve another missing addend problem” would be coded as not on the trajectory. In the findings, I report the extent to which participants wrote non-specific goals.

If the teacher’s goal was coded as specific, for example “begin to count on,” I assessed the alignment of that goal with the learning trajectory by considering whether the target would be considered to emerge next or just after the next strategy in development. I propose this because Vygotsky (1987) describes the “zone of proximal development” as including what the student can do with the support of a more knowledgeable peer, for example a teacher who uses particular mathematical representations in order to support the student in thinking to shift from the counting of concrete objects to counting of mental or abstract images. To justify this decision, I have included a brief description of the learning trajectory used in this study and the type of strategy described along that trajectory.

Table 4.5:

Descriptions of strategies described in the learning trajectory

Strategy	Description	Strategy Type
Direct Modeling	Student uses objects to model the operation and determine the answer.	Concrete use of objects
Counting All	Student counts to find sum (not usually difference) and does so by starting at 1.	Counting
Counting On or Back	Student counts forward or backward to find sum or difference; can start from a number other than 1.	Counting
Counts Efficiently	Student counts forward or backward to find sum or difference; does so in the most efficient manner.	Recognizes relationships
Derived Facts	Students uses a known fact to determine a related unknown fact.	Recognizes relationships

Broadly speaking, there are five strategies in the trajectory for early addition and subtraction. The first includes the use of concrete objects for the students to represent a mathematical situation using counting to enumerate those objects. The two strategies that follow the *direct modeling* strategy are both based on a verbal count that might include the use of a tool to track that count, often their fingers. These counting strategies vary because a student who *counts all* may not recognize that the first addend in a sum represents a unit made up of that number of counts (Steffe et al., 1983) and thus may count from one to establish the unit. Students tend to make use of these counting strategies, in which they count forward to or backward from a number other than one in order to add on or subtract an amount from another. As they develop a further understanding of the number system, they may then begin to recognize relationships that allow them to work problems efficiently and solve unknown sums or differences with what is known to them. I break the trajectory then into three broader categories that include concrete use of objects, counting, and recognition of relationships. I use these categories to justify that if a student were to present as using a direct modeling strategy, it would be aligned with the learning trajectory to utilize instructional moves that support the use of a counting strategy of some sort. Similarly, if the student were using a counting strategy independently, it would be aligned with the learning trajectory to support the student's use of a derived strategy of some sort. Based on this, I conclude that it is within the student's zone of proximal development for a teacher to support the strategy that is either next in the trajectory or just after that strategy.

A distance of 0 was also considered to be in alignment with the learning trajectory, but only in those instances when the student had solved a problem and had an error, as the teacher would be providing further learning time to ensure fluency when using this strategy. In all other cases, the question was coded as 0. Note that the position of the learning goal in relation to the student's current development was also used to inform if the teacher's response was likely to regress the student in his or her thinking (e.g., the learning goal was negative in relation to the student's current progress).

The final question in the scalogram coded for whether or not the teacher suggested an instructional move meant specifically to support the learning goal coded as being aligned with the learning trajectory used in this study. If a teacher had suggested that a student begin to use a make ten strategy, he or she could support that goal by making use of tools that attend to the structure of ten, such as a ten frame, or the teacher could develop a string of problems that might elicit a tens strategy (9 + 2 followed by 9 + 3 and then 8 + 3). Similarly, if a teacher expected a student to begin using a backward counting strategy, the teacher might support that student by practicing backward counting sequences or by designing problems in which the subtrahend is easily within counting range. A suggestion that supports the aligned goal was coded as 1 and if the instructional move did not hold the potential to support the student's learning goal, the response was coded 0.

By the nature of the scalogram, the coding of the responses led to four categories, listed here. These include instructional goals which have the potential to regress the student's mathematical thinking (0,0,0), responses which do not regress the student's thinking, but also do not set a specific learning goal aligned with the learning trajectory (1,0,0), responses which indicate a specific learning goal aligned with the learning trajectory (1,1,0), and responses which also support the student in some activity that could lead to an understanding related to the learning goal, (1,1,1). The codes in the categories could then be aggregated into a single score that varied (from 0 to 3). In this way, a teacher who receives a score of 3 would be understood as having supported the student to achieve a learning goal that is specific and aligned with the learning trajectory when accounting for the student's current learning. And, responses scored as 2 are those that set a goal aligned with the learning trajectory, but do not include a suggestion for supporting that goal. A score of 1 is one that does not regress, but also does not extend student thinking and a response scored as a 0 has the potential to regress the student in their mathematical thinking.

Considerable time was taken to ensure the questions in the scalogram were mutually exclusive from each other. After coding each of the responses, the validity of the scalogram was verified by searching the scores for what would be considered errors in the scalogram. If, for example, the teacher's response was coded as a 0 for the first question in the scalogram, but then coded for a 1 for a later question, this would be considered an error. The scalogram developed for this study did not produce any errors.

It is also important to note that even though a subset of the teachers who responded to this survey had been trained in a learning trajectory, this did not necessarily mean that those teachers would always utilize that knowledge in their responses. For example, of those teachers who had been trained in a learning trajectory, some occasionally responded in ways that were not directly related to students' progress. In a few examples, this subset of teachers asked students to repeat the problem with a different tool or to begin to memorize their facts. And it's possible, if not likely, that in the context of a classroom in which teachers are attending to many competing demands, a teacher might have many reasons to respond in ways that don't necessarily extend student thinking. Because of this, I argue that the use of the learning trajectory as a resource to determine the depth of teachers' responses, is not an exercise in confirming that teachers use the knowledge they gain, but rather a way of showing what teachers could be capable of carrying out in regards to student thinking.

Example Responses and Teaching Response Scores

In this section, I provide some examples of responses that illustrate the range of the scalogram scores in teachers' responses to students. The examples were written by teachers in response to a student who had *counted on* in order to solve a *change unknown* subtraction problem (Carpenter et al., 1998), more commonly known as a *missing subtrahend* problem. The depiction of this student's strategy is shown in Figure 4.5 below.

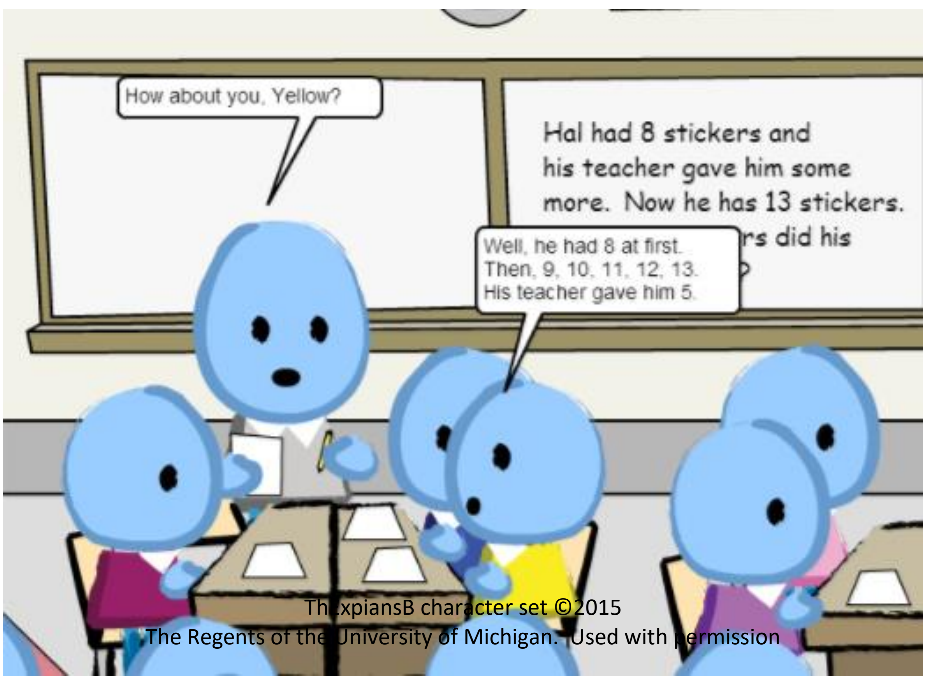


Figure 4.5: *Yellow uses a counting up strategy to solve a missing subtrahend problem*

The strategy used by the student provides some useful information for instruction. First, the student did not choose to use objects to represent the situation, but instead counted on from an amount that was already stated in the problem, showing the existence of a numerical composite (Steffe et al., 1983). Also, the student used an additive count to solve a subtractive statement, which could help one surmise that they have an understanding that addition is the inverse of subtraction. According to the learning trajectory used in this study, this student has a sufficient understanding of addition and subtraction, along with the ways in which quantities are joined together or separated that they might be able to begin using derived strategies. The use of those derived strategies could be supported in the classroom through the use of activities that support students' understanding of the ways in which numbers can be broken apart and reused, for example 8 + 3 could be done by breaking up the 3 into 2 and 1 to make 10 before adding on the extra 1. Similarly, a teacher could use strings of problems in which the relationship would be pronounced, for example by asking students to solve 8 + 2, followed by solving 8 + 3. I show four separate teacher responses in Table 4.6 below in order to describe the scalogram in more detail.

Table 4.6:

Sample responses for scalogram scores

Scalogram Score	Describe what you would say and do in response to this student.	What would you consider to be an appropriate learning goal for the student?
3	Teach this student how to anchor to ten. Then how to count up from the ten. I would use ten frames to illustrate making ten and how to use ten as a (sic) easy number to add to. (7165)	Use of number bonds and how to break apart numbers to get an easy number like 10 so that adding or subtracting is easier and faster. (7165)
2	Good thinking, how is your strategy different from green's strategy? (4411)	Next goal for this student would be to make a ten and add the rest. (4411)
1	I would tell him that was a good strategy (4493)	Subtracting without counting up (4493)
0	I would say can you draw me a picture or use counters to show me how you came up with that answer? (7173)	For this student to learn to subtract using counters (7173)

Recall that responses which receive a score between one and three are considered not to regress the student's thinking. The teaching response scored as a 3 is not regressive, as it includes a goal for the student to *anchor to* ten as well as support to do so, *ten frames to illustrate* and *number bonds.* The response scored as a 2 is not regressive, and it includes a goal for the student to *make a ten,* but it does not include supports to do so. In the response scored as 1, the teacher praises the student's current strategy, but then sets a goal that is unspecified. Rather than stating what the student should be expected to learn in coming days or weeks, the teacher describes instead what the student should not do next. Finally, the last response is coded as a 0 because the teacher sets a goal for the student to do subtraction *using counters*, an enactment of subtraction that represents work done by students earlier in the progression, that of *direct modeling*.

I assessed the reliability of the scalogram by comparing its application by two coders, a colleague and myself. For each item in which the teachers described their instructional move and a learning goal for the student, two coders applied the scalogram to two of the responses for a given item (approximately 10 percent of the data) and discussed the reasons for doing so. This was followed by coding two other responses for the same item, and determining whether there was agreement of the codes in this second subset of data. As the coding progressed, it became clear that because the coding of the teacher's goals and instructional moves were particular to each student's conception, it would be impractical to apply a general coding scheme to all items. As we would expect teachers to adjust their instructional moves to match the student, so our coding was recalibrated for each instance in which we encountered a new student strategy. With this in mind, we worked toward an in-depth understanding of instructional moves and goals that would be in alignment with the learning trajectory for each depicted student when coding the first two responses in an item, then coded the following two responses independently before coming together to determine agreement. Through this iterative process we were able to come to a shared understanding of the ways in which coding should be applied depending on the student conception and I felt confident that the process had sufficient interrater reliability. I then carried this forward and scored the remaining responses for each set of items in the survey. When the coding was complete, a calculation of Cronbach's Alpha was equal to 0.81, and indicated that the internal consistency of the items coded for teaching responses were relatively high. A mean inter-item correlation coefficient (IIC) of 0.35 provided additional support to that observation.

Before I share the findings from the coding scheme, I now return to describe those responses that included either inappropriate or unspecified learning targets. By the nature of the scalogram, these responses would be coded as either a 0 or 1.

Inappropriate or unspecified learning targets. In the analysis, I considered to be inappropriate learning goals those goals that, based on the learning trajectory used in this study, would be outside of the student's zone of proximal development (Vygotsky, 1987), with the latter coded as a distance that is outside the range of what a student could learn with the support of a knowledgeable peer. For example, if a teacher responded to a student who has just used a make ten strategy and set a learning target of using a counting back strategy, this would be coded not making progress according to the learning trajectory because any counting strategy, including counting back, is likely to have been enacted by the student earlier in their understanding of addition and subtraction.

In addition to learning targets that were not making progress according to the learning trajectory, some responses included learning targets that were either not on the learning trajectory, (e.g., if a teacher were to ask a student to use another tool to show their thinking, or not specific enough to be placed on the trajectory (e.g., when a student was asked to find other ways to solve a problem). These goals were coded as NT or NS respectively and led to a code of 0 for specific and aligned, in regard to the learning trajectory for addition and subtraction used in this study. If the goal had not been coded as regressive for the student’s thinking, a goal of this nature would receive a scalogram score of 1.

To better describe the learning targets coded as NS or NT, I went through all instances and created a descriptive list of what was stated as the learning target. I then tracked the number of times across all responses (each response for each item) the suggestion, or a similar suggestion, was made. There were 176 total responses, made up of the responses from 22 teachers who each responded to 8 instances of student thinking. The most frequently stated learning targets that were either not on the trajectory or not specific enough to be placed on the trajectory are listed in Table 4.7 below. The list is arranged by the number of responses from teachers across all items ($n = 176$).

Table 4.7:

Learning targets either not on trajectory or non-specific

Not on Trajectory	N	Proportion	Not Specific Enough to Place on Trajectory	N	proportion
Student should work different (harder, more) problems with (larger, smaller) numbers	17	0.10	Student should solve the problem in other ways	19	0.11
Student should use other tools (number line, counters), possibly to check the answer	12	0.07	Student should solve the problem using mental math	7	0.04
Student should communicate their thinking (sometimes with drawings)	8	0.05	Student should solve the problem without counting	4	0.02
Student should recognize small amounts without counting (subitize)	2	0.01	Student should solve the problem without using manipulatives	3	0.02
Teacher is unsure what goal should be set, student should work toward just knowing it	2	0.01			

Of these goals, the most often repeated were that the students should solve the problem in other ways (11%) or that the student should work on different problems of some sort (10%), followed by students using other tools to solve (7%). It is important to note that the learning targets listed in this table are not necessarily considered inappropriate for learners. Many of the recommendations would certainly be accepted as appropriate goals for students more generally. For example, it would not be considered poor teaching to ask a student to check their work using another method or to solve a problem in multiple ways. Teachers are likely to have participated in professional learning in which these recommendations are made. At the same time, the instructional moves suggested in these responses are ones that could be used at many moments in classrooms, in a wide variety of instructional settings, and with mathematical content of any grade level. I argue, however, that teachers who set these generic learning goals may not be prepared to ensure that specific students increase in their sophistication of mathematical thinking within a relevant time frame in order to be prepared for later topics.

I now share the median value for the number of times teachers in each group wrote goals that were coded as NS or NT. Each teacher in the sample responded to eight items across the survey. I first totaled across those responses the number of times each teacher stated a goal that was coded as NT or NS. The number of responses coded as NS or NT for each teacher ranged from 0 to 6. I then found the mean for the number of responses coded in this manner within each teacher group. The means for each teacher group are found in Table 4.8 below.

Table 4.8:

Median values for number of times (out of 8) teachers in each group stated learning goals coded as NS or NT

	Trained in a learning trajectory (LT)	Not trained in a learning trajectory (non-LT)
Teaches early elementary mathematics (EE)	0.5	1
Teaches upper elementary mathematics (UE)	3	5

In general, teachers who had been trained in a learning trajectory set learning goals that were not found on the learning trajectory used in this study or were unspecified less often than teachers who had not been taught a learning trajectory. And those teachers who also taught in early elementary averaged only one unspecified learning goal across the eight responses. Similarly, while all teachers who had not been trained in a learning trajectory wrote unspecified learning targets more often than those with training in a learning trajectory, early elementary teachers were less likely to do so than teachers who were not teaching in an early elementary setting. This may be due to the fact that early elementary teachers are generally more aware of the strategies students use to solve addition and subtraction problems. But, the findings also suggest that an awareness of those strategies is not likely to be enough to inform the intentional support of student understanding along the trajectory since teachers of early elementary classrooms set learning goals that were non-specific or not on the learning trajectory used in this study in nearly half of their responses.

Responses scored as 0 or 1. I now turn away from the discussion of NS and NT learning goals to consider the scalogram scores and discuss the findings from the analyses using the scalogram score. Each teacher in the sample (n = 22) responded to eight different depictions of student thinking, across four items in the survey, and described a learning goal and instructional response for each of the eight depictions. Each of these responses to student thinking was scored using the scalogram, with a sum of the three codes being used as the score for each response. Thus, each teacher had eight items for which he or she received a scalogram score.

While all goals labeled as NS or NT were considered inappropriate in the context of the scalogram because of their lack of specificity, goals that were inappropriate because of their location on the trajectory would also have been coded as 0 because those goals would be outside the student's zone of proximal development (Vygotzsky, 1987). In either case, responses that included these non-specific or inappropriate goals would, at best, receive a score of 1 for not being regressive. Because of the importance of supporting student development in learning, I now consider how frequently each teacher in the sample gave a response that was scored as a 0 or a 1. I chose this cutoff because such a score would indicate that the response is not likely to encourage mathematical thinking and may even regress it. The results of this analysis are shown in Table 4.9 below. Each cell indicates the median number of times, across eight items, teachers in each category had a response scored as a 0 or 1.

Table 4.9: *Median number of responses with response score of 0 or 1, by teacher group*

	Trained in a learning trajectory (LT)	Not trained in a learning trajectory (non-LT)
Teaches early elementary mathematics (EE)	2	5
Teaches upper elementary mathematics (UE)	5	8

In this sample, teachers without training in a learning trajectory were more likely than not to describe a teaching response that was coded as 0 or 1. Teachers with the combined experience that comes with teaching early elementary mathematics and having been trained in a learning trajectory were less likely to respond in these untargeted ways. For the sake of student learning, this could mean that early elementary teachers who had been trained in a learning trajectory utilize responses that are more likely to target a student's current learning needs and advance the sophistication of that student's understanding of mathematics.

Teachers' Responses to Students

Mean teaching response scores. The eight responses, scored using the scalogram, were averaged so that the set of responses by a given teacher to all 8 individual students was represented by a mean score. These mean scores, representing the extent to which an individual teacher responded in ways that extended students' understanding, ranged from 0.38 to 2.13 (mean = 1.38, SD = 0.47, median = 1.31), with scores above 1 being desirable.

I compared the mean scores for teachers with and without training in a learning trajectory, again using the Wilcoxon rank-sum test. The Wilcoxon rank-sum test orders the mean scores for the aggregate group of teachers and determines if a randomly chosen score from the group of teachers trained in a learning trajectory has a higher rank than a randomly chosen score from the teachers not trained in a learning trajectory. The only significant difference found between groups of teachers for responses provided to students was between early elementary teachers with training in a learning trajectory as compared to upper elementary teachers without training in a learning trajectory. Other pairwise comparisons turned out non significant.

I then found the median score (of the teachers' mean scores) for each group of teachers, because the distribution cannot be considered to be normal. The median score for each group is shown in Table 4.10 below.

Table 4.10:

Median score for responding to individual students, by teacher group

	Trained in a learning trajectory (LT)	Not trained in a learning trajectory (non-LT)
Teaches early elementary mathematics (EE)	1.9	1.3
Teaches upper elementary mathematics (UE)	1.4	1.0

In general, early elementary teachers had higher means than their counterparts who taught at other levels, regardless of whether they were trained in a learning trajectory. I consider the possibility that scores may not reflect teachers' full capabilities. Teachers were asked to describe what they would *say* and *do,* and to set a next learning goal. The former of these prompts may have been interpreted as a brief moment of feedback to the students, while the latter as a student learning objective. The scalogram, however, was used to determine if the teacher had instructional moves to support that next learning goal. In many responses, teachers would indicate what they would say to the student (e.g., *can you show me another way*) and subsequently set a next learning goal that was in alignment with the learning trajectory without describing how he or she would support that learning goal moving forward. Because of this, the responses were seldom scored as a 3. Knowing this, I consider the median aggregate score of 1.9 for early elementary teachers who had been trained in a learning trajectory to be a relatively strong measure of teachers' responses to students.

Correlation of Teachers' Interpretations and Responses to Student Thinking

A suitable interpretation of student thinking used to inform a teacher's response would seem necessary, although possibly not sufficient to ensure responses to students that support their learning. Heritage and her colleagues (2009) make the claim that "teachers do better at drawing reasonable inferences of student levels of understanding . . . [than] in deciding next instructional steps" (Heritage et al., 2009, p. 24). I investigated the correlation between the interpretation scores (obtained from the coding for interpretation) and the responding scores (obtained from the application of the scalogram). I acknowledge upfront that the interpretation coding scheme used in this study is evidence for whether or not the teacher interpreted, as opposed to if that interpretation was correct or supported by theory or evidence. I used each teacher's mean score (across all eight items) for interpretation and teaching response to measure the strength of a correlation. The graph that describes this correlation is shown in Figure 4.6 below.

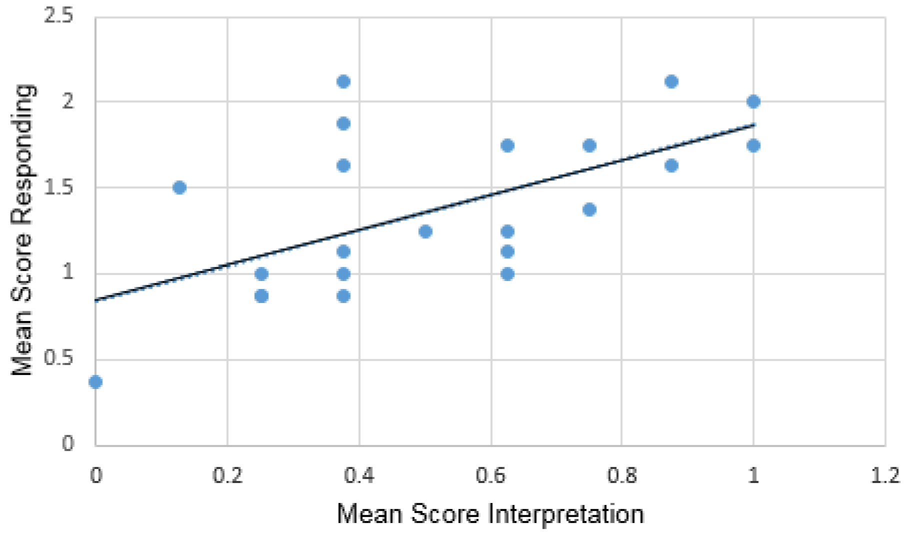


Figure 4.6: *Correlation of teachers' interpretation and instructional response*

There was a moderate positive correlation (r = 0.36), at a significance level of 0.003, between teachers' mean interpretation score and their mean response scores . This suggests that teachers who made inferences about students' understandings before responding might be better prepared to respond, based on the measures used in this study. I also investigated the correlations between teachers' interpretations and responses for only those teachers trained in a learning trajectory. The scatterplot in Figure 4.7 is a representation of that correlation.

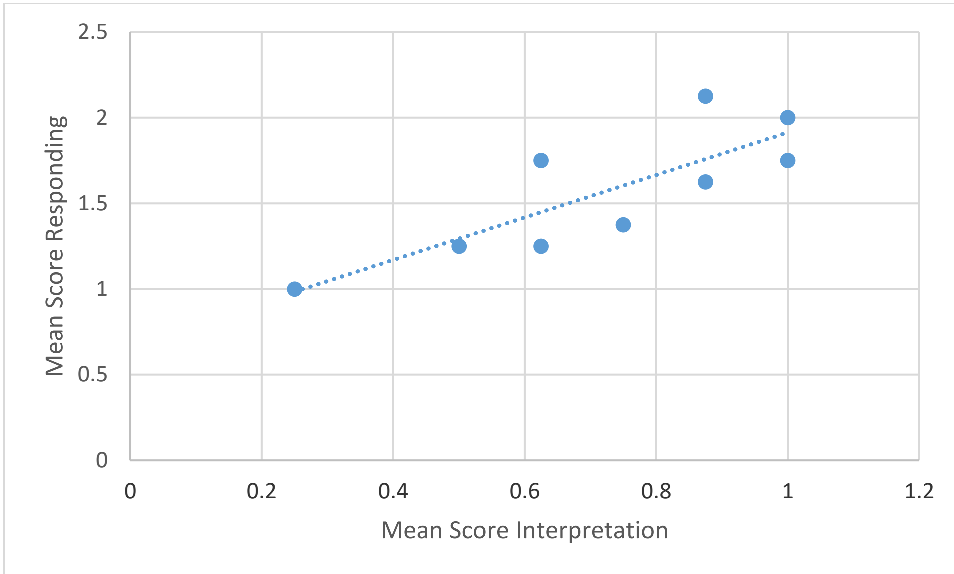


Figure 4.7: *Correlation of LT-trained teachers' interpretation and instructional response*

There was a significant (α = 0.01) strong positive correlation (r = 0.67), of mean interpretation scores and their mean responding score for teachers with training in a learning trajectory. The correlation representing teachers without training in a learning trajectory was not significant. That scatterplot is shown in Figure 4.8 below.

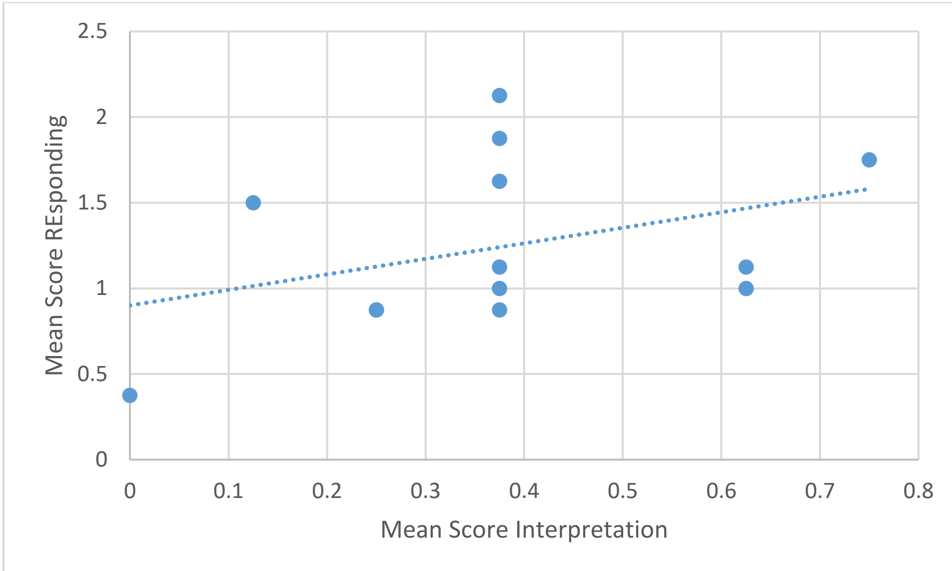


Figure 4.8: *Correlation of non-LT-trained teachers' interpretation and instructional response*

Each of these analyses, taken together, indicate that teachers with the combined experience of teaching in an early elementary classroom and having been trained in a learning trajectory are more likely to describe instructional moves that have the potential to benefit students' learning of early addition and subtraction. While experience teaching in an early elementary classroom does appear to impact teachers' interpretations, in this study it did not appear to be a factor, by itself, that significantly impacted teachers' responses.

As Heritage and her colleagues (2009) have noted, teachers can attend to the interpretation of student thinking and still not provide responses that are likely to the students' extended learning. And the evidence I have shown here supports the conjecture that training in a learning trajectory might better prepare teachers to make productive use of their interpretations of student thinking. Of course, the justification of that conjecture would require an experimental design that rules out other possible explanations.

CHAPTER 5: ANALYSIS

TEACHER GOALS FOR GROUP DISCUSSION

Up until this point, I have considered a learning trajectory as a resource that could be used by teachers to interpret and respond to the mathematical thinking of individual learners. And there are times in classroom instruction when a teacher needs to respond to an individual regarding his or her thinking about a mathematical concept. But because classrooms gather students who have many things in common and because the teacher needs to attend to them at the same time, those students will also engage in collective learning opportunities, in small groups or as a whole class. And when students do come together to share their work in a group, the decisions teachers make in choosing who will share, what will be shared, and what might be achieved during this group discussion will naturally be complex (Ball, 1993). A learning trajectory may still be useful in managing this complexity, but its use is likely to be different than when attending to the needs of a single student. While each student in the group may be making progress along the learning trajectory, all those learners will not always be in the same place in their learning. They also may be making progress at different rates. This means that to attend to the learning of all the students the teacher will need to simultaneously consider the learning needs of individual students while attending to the learning of all students within the group. Because of this complexity, I turn now from considering teachers' responses to individual students to better understand the instructional goals that teachers set when working with groups of students in a classroom in which the students have varied understandings of early addition and subtraction. I analyze the instructional goals that teachers set for working with a group of depicted students whose individual thinking had been seen earlier in the survey item. Based on evidence that student learning is improved when teachers invite student to solve problems and share their varied solution strategies with others in a group setting (Smith & Stein, 2011), I look for evidence of the extent to which teachers are open to students' methods and utilize them to deepen the students' understanding of addition and subtraction. Within the teachers' instructional goals, I look for indicators of the choices teachers make that illuminate the extent to which the teacher allows the variability in student understanding to play a role in the discussion. The third research question asks:

3) How do teachers manage the differences in student progress along a learning trajectory when discussing addition and subtraction problems in a group setting? Are those differences related to whether teachers have had training in a learning trajectory?

Recall that each item included work by two students, whom the teacher had been asked first to respond individually. At the end of each such item, the survey asked teachers to imagine that that the depicted students, observed individually during the earlier portion of the item, had been brought together for a group discussion. Participants were asked to address the group, assigning them work. I analyzed the responses to this prompt in order to better understand how teachers attend to the needs of students who share during a group setting. The depiction that goes along with the prompt is shown in Figure 5.1 below.

Figure 5.1: *The depicted teacher prompts the students before engaging in a group discussion*

Participants were asked to construct a problem that would be presented to the group of students in the classroom. I asked teachers to choose from one of five problems, modeled after the addition and subtraction problem types (*join result unknown, separate result unknown, comparison, join change unknown, and separate change unknown*; Carpenter et al., 1981). The different problem types were given in word form as in this example, *Jamis has ___ marbles and his friend gives him some more. Now he has ___ marbles. How many marbles did his friend give him?* Participants were asked to choose one of the word problems as a basis for a discussion within the group. Following this multiple-choice question, participants were asked to choose the two numbers that they would like to place in the problem. This gave teachers the freedom to design the learning opportunity they thought best suited the students who had been observed in the earlier portion of the item. Participants were then asked to describe their goals in leading this discussion by responding to the following prompt, "*What do you plan to achieve with this group of students when the problem is discussed?"* Because teachers had been exposed to some of the variability within the classroom as they observed the work of individual students, this prompt is meant to gather information as to teachers' decisions in light of the need to attend to the learning of all students.

Systemic Functional Linguistics

I focus the analysis on the content of the teachers' instructional goals and the ways in which teachers expected instruction to be carried out. To carry out such an analysis, I used systemic functional linguistics (Halliday, 1978; Halliday & Matthiessen, 2004). Systemic functional linguistics defines language as a resource for meaning-making, and sees grammar as a set of meaningful systems from which speakers and writers make choices. The choices that are made within a system enable us to analyze and interpret the ways in which speakers or writers present meanings of different kinds. Recognizing meanings in the grammatical systems enables the analyst to identify and interpret the ways speakers or writers present meanings of different kinds. According to the theory, words in and of themselves do not carry meaning in the text, but by making choices in the way that words come together grammatically, the language "shapes, and is shaped by the contexts in which it is used." (Schleppegrell, 2012, p. 21). It is through the use of these systems that the lexical and grammatical choices made by speakers and writers realize meaning in text. The theory of systemic functional linguistics, then, affords one the ability to interpret meaning from a text through analysis of those grammatical choices.

Analysis for Ideational Meaning: Transitivity Analysis

I used transitivity analysis to examine the responses in which participants stated their instructional goals for students sharing during a group discussion. Transitivity analysis is an analytic method that reveals the ideational meanings in a text. In analyzing choices from the system of transitivity one breaks grammatical clauses into three constituents: the processes, the participants, and the circumstances. These constituents are typically realized in the verbal group, the nominal group, and the adverbial or prepositional phrases respectively. For example, in the sentence, “The student solved the problem using counting chips*”,* the verb *solved* indicates the process, the participants in this case are both *the student* and *the problem* and the phrase *using counting chips* indicates the circumstances. I show this example, broken into its constituent parts in Table 5.1 below.

Table 5.1:

A clause broken into constituents in transitivity analysis

The student	Solved	the problem	using counting chips
Participant	*Process: Material*	*Participant*	*Circumstances*

In material processes, a participant can be presented as the person who carries out the process (*the teacher* or the *students*), while another participant can be that which the process is directed toward, for example a *subtraction* problem. In the clause above, the *student* is a participant engaged in a process that is directed toward the *problem*, another participant in the clause. Circumstances are indicated in the text by adverbial or prepositional phrases to describe the conditions under which those processes would take place, in this case by *using counting chips*.

Transitivity analysis requires identifying the process words used to represent the world: There are six types of possible processes realized in text. They include material, mental, verbal, behavioral, existential, and relational all of which refer to how language realizes processes of doing, sensing, saying, behaving, being, and relating, respectively to classify the way the clause represents what is going on in the world.[5] Of particular interest here are processes whose participants point to human beings such as the teacher or students. The type of processes can help identify the ways these clauses represent what the human participants do. In these clauses, typically the teacher, the students, or a subgroup of students are the agents of such processes, while other participants are present through words or nominal phrases as the targets of such processes: Participants such as *strategies, missing addend problems, the skill, the operation,* or *the answer.* The circumstances, while a more peripheral element of the clause and thus less frequent in the data, provide insight into the conditions under which the process is carried out by the agents. In this data set, some examples of circumstances include *using a visual representation, quickly, using subtraction or addition,* and *within a set of numbers.* The details described in these adverbial or prepositional phrases provides valuable insight into the goals of the teachers in that they describe the ways in which the teacher expects the goal to occur.

Analysis for Textual Metafunction: Conjunction Analysis

In addition to the transitivity analysis, I conducted a conjunction analysis in order to further understand the goals of the teachers. The conjunctions used in a text convey the logical connections between ideas found within the text. These conjunctions can be

[5] O'Halloran (1999) also includes operational processes when analyzing mathematical text.

used for adding, comparing, sequencing, or explaining cause or purpose between ideas in the text and use words such as *and, like, after,* or *so that* (Martin & Rose, 2003) as shown in Figure 5.2 below.

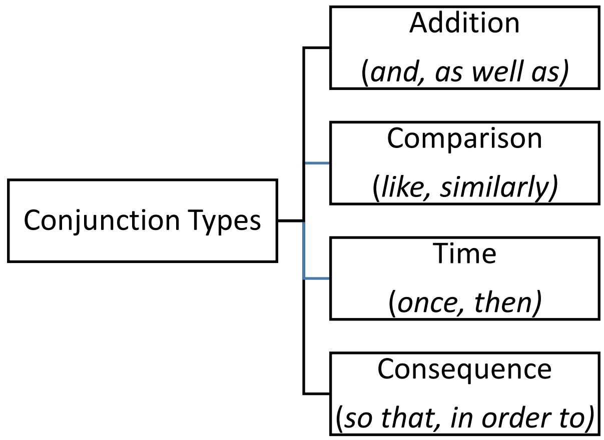


Figure 5.2: *Types of conjunctions and samples of each* (Martin & Rose, 2003)

I analyze the use of conjunctions and what those might mean about the goals teachers have set because of the suggestion in the literature in mathematics education that teachers should focus on connections among mathematics ideas and the comparisons between solution methods (Stein et al., 2008). I anticipated that the data might include additive conjunctions to indicate multiple sub-goals stated within the response, or conjunctions of time to set up a sequence of processes to be carried out. Likewise I expected that conjunctions of comparison and consequence might be there to link processes in relationships of contrast or similarity, or cause, reason, or purpose. Thus, I used the conjunction analysis to consider whether or not teachers were setting goals that support these kinds of connections among multiple solution methods or if the teachers would leverage the discussion to consider similarities or differences among ideas raised during the discussion of addition or subtraction concepts. The analyses of the conjunctions provide further details regarding the connections to be made between aspects of the mathematical content during the group discussion.

Analysis

To conduct each of the analyses, I first broke each response into clauses. I began by identifying clauses that had human participants, either the teacher or students, and classify the associated processes. This helped ascertain who the respondent[6] intended would participate in meeting the instructional goal they proposed and in what ways those participants would engage in the group discussion.

I followed the coding of human participants and processes by considering the target participant at which each process was directed (e.g., *the problem, the answer, a strategy)*. Because this set of participants was more varied than the human participants in the instructional goals, I listed each participant that was indicated and then searched across those participants for themes. Some of the themes that surfaced include the teacher's openness to students' varied methods or whether the goals were directed at solutions to the problem or toward an exploration of mathematical concepts.

I then searched through the responses for circumstances, indicated by adverbial or prepositional phrases, to further understand the manner in which the respondents expected the goals to be carried out. Because the circumstances varied quite widely, these adverbial or prepositional phrases were listed and used to add further understanding to the respondents' goals.

[6] In this chapter, whenever there is a chance to confuse the teacher in the scenario's classroom with the teachers who participated in the study, I call the latter *respondents*. In other chapters, the word *participant* was used for such purpose, but the word participant is here saved for its technical use in SFL's transitivity analysis

Finally, I searched for any conjunctions in the text which would indicate the ways in which respondents might be connecting subgoals within their response. These conjunctions ranged from respondents stating two subgoals and connecting them with an *and*, for example if the students should solve the problem using one solution method *and* then solve it again in another way, to connecting the subgoals with one being a consequence of the other, as in practicing combinations of ten *in order to* make use of derived strategies.

Across each of these analyses, I looked for themes that held across most of the goals and yet differed in ways that could illuminate the variation seen within the goals. I report these differences in the sections that follow.

An Analysis of Two Sample Goals

Each of the analytical tools, transitivity and conjunction analysis, allowed me to consider in what ways the goals set by respondents varied from each other. Before I describe the findings from those analyses in more detail, I compare two sample goals in order to illustrate the differences seen between respondents' goals. I chose these two goals from the set of responses because they differ in important ways and help to illustrate how the differences in respondents' goals might create different learning opportunities for students.

The two sample goals are included below and were both written in response to item one in the survey. In item one, the teacher had given the students a *result unknown* subtraction problem that could be modeled as $12 - 7 = \_\_$. One student had used *direct modeling* to solve the problem, counting out objects to represent the minuend and subtrahend, and another student had counted backward from twelve to five, in order to find the difference. When asked to write a goal for the full class of students, including the two students described here, two of the respondents in the sample wrote the sample goals below.

> Sample Goal 1:
>
> To show them the different ways to solve this problem by counting up and then how they may do it using manipulatives. (7161)
>
> Sample Goal 2:
>
> Find an efficient strategy for taking away a given amount. Then use student strategies in order from least efficient to most to show connection between strategies. (2448)

In Table 5.2 below, I show the transitivity and conjunction analysis of these two goals side by side. In the left column are the constituents found in the text, with the columns to the right identifying how those constituents were evident in the text.

Table 5.2:

Transitivity and conjunction analysis of two goals

	Goal 1	Goal 2
Participant (implied)	(Teacher)	(Students)
Process [participant]	To show [the different ways]	find [an efficient strategy (for taking away a given amount)]
Process [participant]	to solve [this problem]	
Circumstances	by counting up	
Conjunction (type)	and then (additive, time)	then (time)
Participant (implied)	they	(Teacher)
Process [participant]	do [it]	use [student strategies]
Circumstance	using manipulatives	from least efficient to most
Process [participant (Circumstance)]		to show [connection (between strategies)]

In some ways, both instructional goals share some commonalities. Both include a teacher participant who is open to the use of multiple methods for solving the problem;

in the first goal, by showing *the different ways* and in the second goal, by using *student strategies.* At the same time, there are some clear differences, including who participates and in what those actors participate, as well as the manner in which the respondent expects the work to be carried out. An analysis of the constituent parts within the respondents' instructional goals allowed me to make some closer comparisons. For example, the word *students* is a participant in the second sample, as agents of the process *find*, thus representing actual students as playing an active role in learning. But students are not represented as participants in the first sample, though presumably it is to them that the teacher will show the solution method. Also, the respondent who wrote the first response has two instructional goals joined by *and then,* indicating that there are two solution methods they would like the teacher to show. In the second sample, the respondent separates two instructional sub-goals by indicating one of those will occur before the other, in order for the students' methods to play a role in what the respondent would like the students to notice, indicating this with the term, *then*. Each of these features can be analyzed to understand the purpose the respondent has for the use of the problem during the group discussion. In the section that follows I describe in more detail the differences found in the sample statements and later share in what ways those variations were evident across all of the responses.

In these two statements, the first process in each, *to show* and *find* are carried out by different actors, the teacher and the student, an indication of the roles the teacher and students play within the classroom discussion being represented with these statements. In the first case, the teacher plays a more central role in the discussion, whereas the second respondent places the solution methods of her students as the focus of the discussion. In each statement, the process is directed at similar objectives, *the different ways* and *an efficient strategy* (later called *student strategies),* both an indication that the teacher is open to a variety of strategies being used. In both instances, this results in the introduction of multiple methods during the discussion. But those methods serve different instructional purposes. In the first sample the teacher goes on to explain the methods will be used *to solve* [*this problem*] and in the second sample the teacher uses the strategies *to show* [*connections*] *between strategies.* This indicates a difference in what students will have an opportunity to learn during this time: Either a procedure for solving a problem or the mathematical connection that can be made between different student methods. Each of the goals goes on to describe circumstances or conditions for the work being done. In the first goal the teacher shows students two methods for solving, *by counting up and [by] using manipulatives*, resources that would allow students to find a solution. In this phrase, the respondent also uses the conjunction *and* to indicate that these are two methods the teacher will share with the students in the class. In the second goal, the respondent connects the early portion of the goal in which the strategies were elicited in order to serve another purpose, to *then use* the strategies by placing them *from least efficient to most* for the students in order to be able to show a connection *between strategies.* Rather than focusing on the methods as solutions to problems, this respondent makes use of the varied methods to expose the students to a more conceptual understanding of addition and subtraction.

By analyzing the responses in these ways, I identified aspects of respondents' goals that differed from each other in ways that could play a role in describing what students would have opportunity to learn. In the analysis that follows, I focus on the variations in the articulation of the respondents' goals in order to illuminate the differences and discuss how these pertain to what students have opportunity to learn. I also use this analysis to reveal who is expected to participate in that learning, and the circumstances under which that learning could occur, or how students will engage in those opportunities. I chose these constructs because they reveal differences across the goals that are important indicators of the recommendation that teachers attend to the connections that can be made between students' methods, with explicit attention to concepts (Hiebert & Grouws, 2007) and to implement formative assessment practices such as questioning and peer assessment (Wiliam, 2007a)

As shown in earlier analyses, the most salient categories across which to describe differences among responses were whether respondents taught or did not teach in a PreK-2 setting and whether they had or had not had training in a learning trajectory. These groups and the number of teachers in each are shown in Table 5.3 below. In the sections that follow, I describe the analysis and what it revealed about teachers' instructional goals.

Table 5.3:

Teacher groups based on background experience

	Trained in a learning trajectory (LT)	Not trained in a learning trajectory (non-LT)
Teaches early elementary mathematics (EE)	4	8
Teaches upper elementary mathematics (UE)	5	5

Transitivity analysis: processes When conducting the analyses for processes, I found that the most frequent process types were material and mental, with a relatively small number of verbal, behavioral or existential clauses. With even less frequency, teachers would include a relational clause, which typically presented an explanation of some sort, (e.g., *the answer will be the same,* or *the sequence of numbers does not determine the equation)*. Because of this, I initially focused on the mental and material processes and the human actors involved in those processes; teachers, students, or some subgroup of students.

Material processes were presented with verbs such as *show, count,* or *model,* in which teachers or students would be engaged in processes of doing. Mental processes were presented with verbs such as *understand, determine,* or *figure* in which teachers or students would be engaged in processes of sensing. Identifying the types of processes presented by teachers' goals was useful for understanding whether the respondent expected students to be actively engaged in some learning activity or to be mentally engaged with an idea.

Table 5.4 below indicates the mean number of responses, across all four items, for each group of respondents, which included teachers or students engaged in material or mental processes.   The values represent the frequency with which respondents in each group referenced teachers or students engaged in material or mental processes across all responses. There were no statistically significant differences found between the proportion of teacher or student participants engaged in material or mental process within the instructional goals teachers wrote for their classroom discussion. These differences were not significant regardless of background experiences of the respondents group.   In other words, the goals, written by respondents of varied backgrounds, had similar inclusion of teachers and students engaged in processes of doing and sensing (or thinking) during the group discussion.

Table 5.4:

Material and mental processes in teachers' goals, mean score, across all items

	Teacher Material	Teacher Mental	Student Material	Student Mental
(UE, no LT)	0.40	0.10	0.55	0.45
(EE, no LT)	0.16	0.16	0.58	0.71
(UE, LT)	0.15	0.15	0.60	0.75
(EE, LT)	0.31	0.06	0.56	0.75

Transitivity analysis: objects of processes The analysis showed that the frequency with which teachers and students were expected to engage in either material or mental processes did not differ by teacher group. But, the measure of frequency of these processes does not describe the complexities of the interactions that were the object of those processes. To better understand the nuances of those material and mental processes, I identified, in each clause, the object of each process, the non-human grammatical participants in the processes presented by the respondents, by locating the nominal phrases associated with the processes. For example, in this portion of one response, <u>*to show*</u> [*the other ways*] <u>*to solve*</u> [*the problem*], includes two material processes, *to show* and *to solve.* These processes are directed at the objects [*the other ways*] and [*the problem*], respectively. In this way, I was able to begin to illuminate differences among respondents' goal statements. In these two clauses, the respondent indicates she is open to the use of multiple methods with a purpose of solving the problem, where the solution methods are seen as tools a student chooses in order to find solutions to problems. The variations of these phrases as seen across the responses provided insight into the purpose respondents had for engaging the students in mental and material processes in the discussion. I considered it important to analyze these phrases because they aid in understanding the opportunities extended to students during learning activities.

In the responses, some common nominal phrases included *strategies, missing addend problems, the skill, the operation,* and *the answer,* among other things. Because the variation in grammatical participants included in the responses was wide, I first listed each of the participants included in the goals and then searched for themes across the list. One important theme that arose was focused on the solution methods, as when a respondent asks that the students <u>*find*</u> [*other ways*] to solve the problem. In this section I describe examples from the responses which describe the variation in the goals respondents set for the use of solution methods. This variation includes goals in which the respondents prescribed a specific method, were open to the possibility of multiple methods, or did not mention the use of methods.

No mention of methods. Respondents in all groups wrote instructional goals that did not describe the use of a solution method within it, though respondents without training in a learning trajectory wrote goals of this nature more frequently (about a third of responses) as compared to respondents with training in a learning trajectory (only one tenth of responses). This is not entirely unexpected since a learning trajectory has a focus on the particular strategies that students utilize over time, exposing teachers in those learning sessions to closely consider students' methods for solving addition and subtraction problems. Goals that did not mention a solution method tended to focus either on mathematical vocabulary or on determining the mathematical operation indicated within the problem. As an example, the respondent does not attend to a solution method for the problem, but rather indicates a need *to listen [to the complete task]* and to *determine [what is known/determined].*

> First that the sequance (sic) of the numbers does not determine the equation, but it is important to listen to the complete task. Determine what is known and what is to be determined. (4520)

In this particular response, the respondent had chosen an additive problem in which the change was unknown, *Jonas has ___ trading cards. He gets some more from his friend. Now he has ___ trading cards. How many did his friend give him?* The focus in the goal is placed on gathering information from the problem in order to figure out what is to be determined. I would argue that while this is not an unnecessary aspect of student learning to attend to, the learning goal as written stops short of describing what students might be expected to learn about the mathematics once it has been determined what operation is represented within the problem.

There were also a number of responses in which the respondent made some mention of solution methods that could be used to solve the problem. I coded these according to whether the teacher prescribed the use of a particular method or methods in order to solve the problem or whether the teacher was open to the varied methods that students might utilize. While a prescribed method might be appropriate for a number of the students in the group to practice, the practice of eliciting student methods provides greater opportunity for students to build their independent problem solving abilities as well as provides valuable formative assessment information to the teacher (Jacobs et al., 2010). In the section that follows, I describe the types of responses in which the respondents either prescribed methods or remained open to the use of multiple methods on the part of students.

Prescribed Methods. Respondents with training in a learning trajectory were slightly more likely to state learning goals that included a mention of solution methods (44% of responses), as compared to teachers without training in a learning trajectory (32% of the time). Furthermore, upper elementary teachers were more likely to prescribe these methods (about half of responses), as compared to early elementary teachers (about one fourth of responses). A teacher who is trained in a learning trajectory may be made more aware of the solution methods that students use to solve problems and this might lead them to prescribe those methods more often. In the example below, from a respondent who had been trained in a learning trajectory but taught in a fourth grade classroom, the prescribed method of choice is *ten strategies.*

> How they can use [ten strategies] to help them figure out the answer. Take away 3 from 13 is ten and then take away two more to get to 8. They should be familiar with their ten addition families which makes it easier to figure out these subtraction problems (4552)

Across responses, teachers often prescribed the use of a *make tens* strategy. In the case of this particular response, the teacher had chosen a subtractive problem written as, *Hanna had ___ gumdrops and she ate ___ of them. How many does Hanna have now?,* choosing to place the numerals 13 and 8 in the blanks respectively. This problem would be modeled as 13 – 8 = __. In the goal, she describes how the *make ten* strategy would be carried out; that 13 minus 3 is 10 and subtracting another 2 would be 8, presumably choosing 5 as the correct answer. In doing so, the respondent presumes that the students would recognize that rather than removing 8 from 13, one could also remove a quantity to attain 8, an alteration of the way in which the original problem is structured. The recommendation of this particular method appears to overlook some aspects of mathematical learning that would be important for a teacher to consider.

Use of multiple methods. Early elementary teachers were more likely to remain open to the potential for students to utilize multiple methods (in half of responses) during the discussion than respondents teaching other grade levels (in one fifth of responses). This finding was true of PreK-2 teachers regardless of their training in a learning trajectory. At the same time, it's possible that teachers' awareness of a learning trajectory influenced in what ways students' methods for solving were leveraged in order to further the learning of the group.

As described by Hiebert and Grouws (2007), if students are to acquire conceptual understandings, the use of those multiple methods should be leveraged in order to show "how different solution strategies are similar to or different from each other" (p. 383). In the examples that follow I compare two example responses, both written by teachers who work in PreK-2 settings and who remained open to students using various solution methods. The first instructional goal is written by a teacher who did not indicate having been trained in a learning trajectory and the second by a teacher who had been trained in a learning trajectory.

> When using this problem during discussion I would hope to achieve the goal of students know how to use a variety of methods and strategies to solve addition and subtraction problems (7150)

In this first example, the phrase, *know how to* <u>*use*</u> *[a variety of methods and strategies]* is an indication of the use of multiple methods and the phrase <u>*to solve*</u> *[addition and subtraction problems]* indicates the purpose. While this instructional goal and purpose include the use of multiple methods, there is no indication that the teacher will work to support the connection between those strategies, choosing instead to ensure students are able to make use of those strategies in order to come to proper solutions to problems.

In the second example, on the other hand, the respondent is more intentional about the connection between strategies.

> Goal: Students understand this problem can be solved multiple ways. Give students a chance to share their strategies working them through discussion that leads from the least to most efficient. Also, using this as a way to check your work using more than one method. (2448)

In this second instructional goal, in addition to students' understanding of the varied methods for solving, the respondent has two purposes for the variation in strategies. First, to <u>*work*</u> [*them*] *. . . from the least to most efficient,* and *to* <u>*check*</u> [*your work*]*.* In this example, the respondent intends to make use of the multiple methods in order for students to meet two further objectives: to recognize a mathematical connection between the methods and to attend to the precision of their work verifying the answer using another method, a difference that could prove valuable for the learners in her classroom. The difference noticed here in the purpose the respondents had for making use of solution methods is another point of difference in teachers' goals. In the next section I describe how these differences are revealed in the teachers' responses and what those differences in purpose were across the goal statements.

The problem or the concept. A noticeable difference in the respondents' instructional goals for group discussion was related to the stated purpose the teacher had for engaging students in solving problems. Across the responses, the indications in the text revealed the teacher's response either had an explicit focus on finding a solution to the chosen problem, or that the teacher would be using the solution methods as a means to further understand the concept of addition and subtraction. In the sample goals (Table 5.2), this difference is apparent in the participant to which the processes *to solve* and *show* are directed. In the first, the process <u>*to solve*</u> is directed toward [*this problem*] and in the second, <u>*to show*</u> is directed toward [*the connection between strategies*]. Each of these would serve a different instructional purpose in the group discussion and lead to different understandings of mathematics—either a set of procedures for calculating or an interconnected system of operations meant to make sense of quantities as they are increased, decreased, or modified in some other way. In a few instances (14% of responses), an instructional purpose for the goal could not be determined from analyzing the response, as in this example, *Students will be able to model subtraction using counters.* (7173). While the respondent has a goal that the students will 'model subtraction using counters', the purpose she intends to meet (e.g., to find an answer, to show how subtraction relates as taking away, etc.) is not clear from the text.

The majority of responses had a focus on solving the problem (77% of responses). I include here an example response in which the teacher focused on the solution to the problem.

> My goal would be to have the whole class be able to do simple subtraction problems. While working on their own, some of the students were unable to come up with the correct answer. In doing a similar problem during the group discussion, it would be my hope that all students would be able to understand the difference between the two numbers. (7228).

In this case, the teacher repeats her desire to have students come to a correct solution to the problem in a number of different ways: <u>*do*</u> [*simple subtraction problems*]*,* <u>*come up with*</u> [*the correct answer*]*,* <u>*understand*</u> [*the difference (between the two numbers)*]*.*

In contrast, only a few responses included references to concepts and the connections between mathematical ideas (only about 8% of responses) and each of these instances was in a response from teachers who worked in early elementary settings. In these responses, the teachers were more likely to make use of the chosen problem in order to illustrate a concept in addition and subtraction, as in the following example,

> I would like the students to see that 5 and 7, in any order will be 12, and that when you take one of them away (ie 5), the other is the answer (7). I would refer to other sets of numbers we had worked with in the past. (7160).

In this example, the respondent uses the numbers from the problem to engage students in thinking about the relationship that numbers that make up addition or subtraction sentences have to each other when she states, *to* <u>*see*</u> *that* [*5 and 7, in any order will be*

12]. In addition to drawing out the relationship that 5, 7, and 12 have to each other in this word problem, the teacher chooses to *refer to* [*other sets of numbers*] that have been worked previously, building the idea that this is a relationship that holds true among other sets of numbers.

In the example focused on the solution, mathematics is portrayed as a tool to be applied to problems, a process that can be picked up and used when needed to find solutions to problems involving addition or subtraction; a way of determining answers. In the latter example, the teacher's goal is more conceptual and would lend itself to students recognizing relationships among numbers under the operations of addition and subtraction and that would allow them to work more efficiently, to build on these ideas later when they encounter new sets of numbers or different operations within mathematics.

Transitivity analysis: circumstances The circumstances, presented in the adverbial or prepositional phrases in each instructional goal, lend further evidence to understand the intentions teachers have for student work. As seen in the sample goals (Table 5.2), the respondent who wrote Sample 1 indicates the problem will be solved in two ways, *by counting up* and by *using manipulatives,* whereas in the Sample 2, the circumstance is to order the student strategies *from least to most efficient.* The differences in the circumstances in these two instructional goals could be important for learning in that they describe how teachers or students will carry out the work in the classroom and set the stage for what students will have opportunity to learn.

Because circumstances are more peripheral as an element in transitivity analysis, not every response includes a circumstance. Table 5.5 below lists each time a circumstance was indicated in the responses to Item One. To provide context for the circumstances included in the instructional goal, I also included the preceding process and participant in the clause. In the section that follows, I describe the importance of some types of circumstances mentioned in the responses.

Table 5.5:

Circumstances in teachers' goals for item one

	Process [participants]	Circumstances
UE, no-LT	model [subtraction]	[by] using counters
	use [more difficult numbers]	as they are ready
	know [how to regroup]	Correctly
EE, no-LT	solve [a problem]	with similar wording
	use [a different strategy]	after listening to others
	visualize [this problem]	much easier
	solve [missing addend problems]	in multiple ways
	show [how they understand]	Pictorially
	use [numbers]	with the drawing
	do [a similar problem]	during the group discussion
	understand [the difference]	between the two numbers
	apply [what was discussed]	following the last problem
	challenge [them]	with bigger numbers
UE, LT	make [the problem]	as real life as possible
	solve [it]	in whichever way is best for them
	count	past 10
	work	through that transitional number
	completing [the same concept]	with a different story
EE, LT	worked [other sets of numbers]	in the past
	use [student strategies]	in order from least to most efficient
	count	from a number >10 and <15
	begin	with smaller numbers as a model
	solve [the problem]	in pairs
	discuss [how they know their answer]	to the whole class
	figure out [the correct answer]	[by] using one of the other strategies
	extend [learning]	to the other numbers

I provide a list of all circumstances mentioned in response to the prompt for Item One in order to show the ways in which teachers design their instructional goals to be carried out under a number of different conditions. Here, I discuss some themes that characterized those conditions across all items. By looking at the circumstances in Item One, one can see that in some cases the circumstances prescribe the manner in which the work will be carried out (*using counters, pictorially, with a drawing),* further evidence that teachers will sometimes prescribe the way that students should carry out the work being done. Across all responses, goals of this nature (focused on *manner*) were much more likely to be written by respondents without training in a learning trajectory. In other cases, the response includes attention to correctness or speed (*correctly, quickly*). Again, these terms were more likely to be used by respondents without training in a learning trajectory.

Other circumstances describe how the respondent envisions the teacher setting up the problem in a way that connects with other learning moments, using circumstances of time or accompaniment (*following the last problem, in the past, with a different story),* presumably to extend what had been a part of work done earlier in the classroom. Because one benefit of learning in a group is to begin to understand and make use of the solution methods used by other students, I searched across the responses for indications of this type of engagement. There were a few instances in which the circumstances indicated students listening to or learning from each other, referenced by *after listening to others, in pairs,* and *to the whole class.* In general, the presence of this type of circumstance was limited. In the groups made up of early elementary teachers, respondents describe situations in which students are learning with and from each other. In the groups of teachers who do not work in early elementary there was no mention of learning that might occur between students in the group. In these cases, the goals might be carried out just as easily in an individual setting, similar to the finding of Dawes and his colleagues (2004) in which they observed students more likely to be working in groups than as groups.

Summary of Analysis of Goal Statements

In this data set, and based on their responses to the scenarios, it appears that early elementary teachers are more likely to engage students in learning from each other. Further, those early elementary teachers who had been trained in a learning trajectory were also more likely to attend to concepts of addition and subtraction. However, while each of these practices were present in some responses from early elementary teachers, they were not widespread across responses. To support students' deep understanding of mathematics, Hiebert & Grouws (2007) recommend an intentional focus on the connections made between strategies and concepts. Opportunities to engage in these connections between solution strategies, special or general cases, and relationships among ideas seem most advantageous to happen in a group setting since it is within this group setting that the variety in student thinking can be surfaced, and subsequently used to make such connections. This recommendation and the data in this sample point to a need to support teachers in implementing these instructional practices in all classrooms.

In the previous sections I have described some of the differences noticed between the instructional goals set by teachers for engaging in a group discussion. Those differences included whether or not the teacher was open to the use of multiple methods, whether the purpose was to determine a solution or consider some concept of addition and subtraction, and whether the teacher represented the group of students as learning from each other. In Table 5.6 below, I show a summary analysis for each goal set by a teacher for Item One.

Table 5.6: *Summary analysis of goals for group discussion, item one*

	Methods	Connections Made	Purpose: Solution or Concept	Engaging the Group
(UE, non-LT)	Prescribed	---	---	---
	No Mention	---	Solution	---
	No Mention	---	Solution	---
	Prescribed	---	Solution	---
	Prescribed	---	Solution	---
(EE, non-LT)	Multiple	Yes	Solution	Yes
	No Mention	---	Solution	---
	Multiple	---	Solution	---
	Prescribed	---	Solution	---
	No Mention	---	Solution	---
	Multiple	---	Solution	Yes
	Multiple	---	Solution	---
	Multiple	Yes	Concept	---
(UE, LT)	No Mention	---	---	---
	Prescribed	---	Solution	---
	Prescribed	---	---	---
	Multiple	---	Solution	---
	Prescribed	---	Solution	---
(EE, LT)	No Mention	Yes	Concept	---
	Multiple	Yes	Concept	Yes
	Prescribed	---	Solution	---
	Multiple	---	Solution	Yes

This summary table supports a conjecture that early elementary teachers or teachers who have been trained in a learning trajectory have experiences that lend these teachers to invite the use of multiple methods during their group discussions. In this sample, upper elementary teachers without training in a learning trajectory were the only group of teachers to not mention the use of multiple methods in their goals. Also, in this sample, a small number of early elementary teachers engaged students in the group with each other's thinking, leveraging learning opportunities for those in the group, but this practice was not evident in any responses from teachers in the upper elementary grades. Finally, a few early elementary teachers had responses that included an attention to the concepts of early addition and subtraction and the connections between the methods students use, whereas teachers of upper elementary focused their responses on solutions to the particular problem. In general, the purpose of the instructional goals represented in this table was to find the solution to the problem.

The data does not provide sufficient evidence to support a claim that knowledge of a learning trajectory supports teachers' instructional goals that pursue qualitatively different student understandings. Of the responses, a few teachers set goals to engage students in considering multiple methods and/or the connections between those strategies as recommended in Hiebert and Grouws (2007). These teachers were all early elementary teachers, but only some had been trained in a learning trajectory.

Conjunction Analysis

Conjunctions in a text convey the logical relations between clauses. The type of conjunction (addition, comparison, time, and consequence) is useful in understanding the design of a teacher's goal. For example, conjunctions of addition may indicate the teacher has two sub-goals, as in this response, *Review the use of landmark numbers <u>and</u> explain how to determine the number to subtract. . .* (4520). Conjunctions may also serve to compare, sequence in time, or indicate consequence. In the sample goals listed in Table 5.2 one respondent used the conjunction *and then* to communicate two subgoals sequenced in time, *solve this problem by counting up <u>and then</u> how they may* *do it using manipulatives.* In the second goal, the respondent also uses a conjunction to indicate time, <u>*then*</u> *use student strategies,* along with a clause indicating consequence, <u>*to*</u> *show connection between strategies*. In the latter use of *to,* the respondent indicates that the ordering of the students' strategies will be used in order to meet the purpose of demonstrating the *connection between strategies*.

I conducted a conjunction analysis because it could provide insight into the nature of the respondents' goals. I conjectured, for example, that goals with conjunctions indicating *addition* would differ from goals which include conjunctions of *consequence.* Whereas conjunctions of *addition* may indicate a number of sub goals, other conjunctions in the text, such as conjunctions of *time* or of *consequence* may indicate an attention to a design within the goal that allows for particular moves to occur before another or in order to facilitate the second. An example of this can be seen in Table 5.7 below in which the teacher *first* ensures students are able to make use of one strategy *before* exploring the use of others.

Table 5.7: *Conjunction types in responses*

Conjunction Type	Sample Responses from Data
Addition	"To clarify ways to solve this type of problem <u>and</u> show the other ways to solve this problem." (7161)
Comparison	"My goal would be for students to be able to see the problem <u>like</u> Blue did, making groups of 10 to get to the answer" (7110)
Time	"As a whole group review, I hope to get all students on board with being able to solve this subtraction problem use (sic) at least one strategy correctly. Peer tutoring may be used. <u>Once</u> all students can use a strategy proficiently, we will explore others." (7228)
Consequence	"Use similar problem (missing addend) <u>so</u> information from discussion can be applied to new problem. . . ." (4546)

Teachers in all groups used each type of conjunction. After analyzing the use of conjunctions across all responses, I found the mean number of times teachers in each group used any of the four conjunction types in any of their goal statements. The results are shown in Table 5.8 below.

Table 5.8: *Mean number of conjunction types by teacher group*

	Addition	Comparison	Time	Consequence
(UE, non-LT)	0.5	0.05	0.15	0.1
(EE, non-LT)	0.32	0.16	0.13	0.23
(UE, LT)	0.25	0.15	0.05	0.25
(EE, LT)	0.38	0.25	0.25	0.31

A few differences found between the groups are notable. Upper elementary teachers without training in a learning trajectory used conjunctions that indicated addition more than any other conjunction and used conjunctions to indicate consequence less frequently than all of the other groups of teachers. Since these teachers neither teach in an early elementary setting nor have had training in a learning trajectory, this evidence might be an indication that this group of teachers feels more secure stating a number of sub-goals than in considering how one sub-goal might serve a further purpose for the learning of the group. Teachers with training in a learning trajectory or who taught in an early elementary classroom used conjunctions in their goal statements in a higher proportion than teachers without these background factors. These teachers seemed more comfortable considering factors of the discussion as rationale for subsequent instructional moves. Because the number of goal statements in which conjunctions were found was minimal, it's difficult to draw conclusions from this particular set of data.

Conclusions

By looking across the variations in teachers' goals, each of these findings from the transitivity and conjunction analysis illustrate the ways in which teachers and students engage in group learning and the different purposes teachers have for group discussion, including what opportunity to learn students have. While those discussions could serve to engage students in learning with and from each other to connect the mathematical concepts of addition and subtraction, those discussions could also serve to ensure that students have time to practice strategies that help them come to correct answers for the problems posed. These differences could have significant implications for young learners of mathematics and the analysis shows potential methods for considering teachers' instructional goals. Transitivity, as a tool, provided ways of illuminating aspects of teachers' practice that have been found to be important for extending learning opportunities to students. Furthermore, transitivity allowed for an analysis that took a learning trajectory into account. In this way, I was able to show that not only do teachers' goal statements vary in ways that are important for student learning, those variations could be seen as providing different levels of access to students' learning needs. The findings from this small sample seem to indicate that teachers who had been trained in a learning trajectory may be more fully prepared to meet the demands of instruction that simultaneously engages students in thinking about the concepts of mathematics while attending to the individual learning needs of students.

CHAPTER 6

DISCUSSION AND CONCLUSIONS

Teachers' formative assessment practice, in particular those responses provided by teachers that take into account information from students and provide appropriate feedback, have been suggested as an effective instructional strategy (Black & Wiliam, 1998). At the same time, the implementation of formative assessment practice in mathematics classrooms varies in quality (Heritage et al., 2009) and may depend on a number of factors. In this study, I sought to identify some of those sources of variation in teachers' formative assessment practices. In particular, I investigated teachers' interpretation of and responses to student thinking as well as the goals teachers set for whole group discussions. I surveyed 22 practicing teachers and gathered their responses to students' emergent thinking in early addition and subtraction and I considered background factors that might account for the variation observed among teachers in those formative assessment practices.

An analysis of the data determined that at least two factors of teachers' background experiences could be related to teachers' formative assessment practices: teachers' present experience in early elementary grades and their prior training related to a learning trajectory for addition and subtraction. The teachers in the sample who were teaching early elementary grades and had had training related to the learning trajectory provided responses that differed significantly from those teachers who did not have experience in an early elementary classroom and who had not participated in a training related to a learning trajectory for addition and subtraction. Findings from this study broadly inform research on teachers' pedagogical knowledge, the benefits of teachers' classroom and professional learning experiences and the association of those variables with formative assessment practices.

I begin this chapter with a summary of the findings from the study. In it, I discuss the teachers' interpretations of and responses to student thinking while considering that teachers' awareness of a learning trajectory and a teachers' classroom experience in early elementary grades could influence those practices. I also discuss the goals that teachers set for whole group discussions and the differences that were noticeable within the data based on teachers' experiences related to the teaching and learning of early elementary addition and subtraction. I interpret these results and draw inferences from the data as a whole. Finally, I conclude this chapter by considering implications for further research into studies of teacher knowledge of learning trajectories as well as the ways in which knowledge of a learning trajectory may further inform the theory of mathematical knowledge of teaching (Ball et al, 2008) related to teachers' knowledge of content and students and knowledge of content and teaching. In particular, a learning trajectory may provide a knowledge base for considering teachers' practices in ways that are specific to concepts. The practical implications of this study include considerations for elementary instruction and teacher education or certification. Finally, I discuss limitations of this study and propose further research into studies of teacher knowledge and practice.

Summary and Interpretation of Findings

I discuss the findings of this study by first reviewing the aims of this dissertation. I reviewed research that allowed me to argue that a learning trajectory could be a knowledge resource teachers utilize to inform responses to students learning early addition and subtraction. I considered that teachers could encounter student thinking in individual as well as group settings and sought to analyze aspects of teachers' formative assessment practices from responses given to simulated forms of these two settings. Based on previous research that a teacher's professional learning and expertise has an impact on what is noticed in classroom scenarios (Jacobs et al., 2010), the design of the study permitted comparing the responses of teachers based on background factors such as classroom experience and prior training in a learning trajectory. An analysis of the data, accounting for these factors, allowed me to illuminate some of differences noticed in teachers' formative assessment practices and consider how a learning trajectory provides a lens for assessing teachers' content-specific practices.

To understand teachers' formative assessment practices related to individual students' thinking, I examined teachers' interpretations of and responses to simulations of individual student responses and developed a measure to assess the potential the teachers' responses have for extending learning along a trajectory. I also examined whether being trained in a learning trajectory was associated with differences in those responses. The analysis attended to the frequency of teachers' interpretations across eight instances of student thinking as well as to the extent to which teachers' responses to those instances held the potential to extend student learning along a learning trajectory for the concept. For those two measures, I was able to detect differences that could be accounted to the background factors named above. After discussing these findings separately, I discuss the correlation between the two teaching practices. I follow this discussion by considering the ways in which teachers manage the complex task of setting goals for groups of students that include individuals who vary in their current understanding of mathematics and discuss the findings of qualitative analyses of differences in teachers' goals.

My analyses of a relatively small sample of teachers (n = 22) showed that teachers' background experience teaching early elementary grades and their prior training in a learning trajectory may have a relation with teachers' formative assessment practices for teaching early addition and subtraction. There is less evidence to explain variations in the goals that teachers set for group discussions, but further analysis could hold potential to understand this better.

Teachers' Interpretations of Student Thinking

I asked teachers to describe the work of eight depicted students who used varying strategies to solve addition and subtraction problems and then examined their descriptions to determine whether or not the teacher inferred meaning from the student's response.

Regardless of their background experiences, teachers nearly always provided objective descriptions of the work of the depicted students, but they were not equally likely to interpret what the students had done. Teachers who taught in an early elementary setting were significantly more likely to interpret student thinking, as compared to their peers who taught in grades beyond second grade. This supports a claim that teachers' experience teaching elementary addition and subtraction provides some level of knowledge relevant to the teaching of addition and subtraction. The interpretations teachers made of student's common methods for solving addition and subtraction tasks included references to the particular type of count the student had conducted, indications of how that count might be unique, as well as the potential sources of student's errors. This would suggest that teachers who have regular interactions with early elementary students are able to make inferences regarding student work and attribute meaning to students' mathematical thinking. Furthermore, teachers with prior training in a learning trajectory were also significantly more likely to interpret student thinking than their peers without training in a learning trajectory. Overall, teachers who both taught in early elementary and had been trained in a learning trajectory were significantly more likely to interpret student thinking when describing the work of a student.

Given that teachers were prompted with the statement, *Describe what you noticed this student did to solve the addition or subtraction word problem*, I consider it important that some teachers chose to provide interpretive comments as well. But early elementary teachers with training in a learning trajectory did so more often than their counterparts in early elementary without knowledge of a learning trajectory. A teacher's experience in a classroom may allow him or her to recognize instances of student thinking, but teachers may need additional support to be able to make appropriate interpretations of students' common ways of thinking about addition and subtraction. In this study it seems that teachers who had been trained in a learning trajectory were more prepared to provide such interpretations or more confident on the value of their interpretations, as they were eager to provide them without being asked to do so.

Teachers' Responses to Individual Students

In the survey, teachers were asked to describe an instructional response to the student who had just been observed. The analysis of these responses assessed the extent to which a teacher's response to an individual student held the potential to extend student understanding, providing a measure of the quality of the responses. I utilized a set of dependent questions as a coding scheme to determine the quality of each response in regard to its alignment with the learning trajectory for addition and subtraction. The score scaled teachers' responses along a range from potentially regressing the student's thinking, to not regressing the student's thinking to setting a goal in alignment with the learning trajectory, to supporting that goal, a range from zero to three on the scale. I also identified goals that were either non-specific or not on the trajectory. For example, a goal to *solve the problem using different methods* was considered non-specific and to *solve a problem with different numbers* could not be placed on the trajectory. These goals might be considered appropriate in light of general recommendations that students use multiple methods to solve a variety of problems, as described by Hiebert and Grouws (2007). But, given the content-specific nature of the survey, such responses were not coded as being targeted at extending student understanding of addition and subtraction.

Teachers who had not been trained in a learning trajectory would often set a learning goal that was either non-specific or not on the learning trajectory. Similarly, teachers not trained in a learning trajectory were more likely to receive low scores, often because the goals they had set were non-specific. On the other hand, teachers trained in a learning trajectory and who taught in an early elementary setting were more likely to set goals that would support student thinking and be able to extend it. Across all eight items, the mean response score for teachers who both had experience in an early elementary classroom and been trained in a learning trajectory was significantly higher than for teachers who taught upper elementary and had not had training in a learning trajectory.

These findings are compelling considering that teachers had not been asked to choose an instructional response that would extend the student's learning. Rather, teachers were given a fairly general prompt that asked them to describe what they would *say and do* in response to the student and to determine a *next learning goal*. In that context, however, teachers who had been trained in a learning trajectory utilized that knowledge in presenting responses that had the potential to extend student understanding. Teachers who had not been trained in a learning trajectory, on the other hand, even those who teach in early elementary settings, offered extending responses less frequently.

Early elementary teachers are largely responsible for supporting students' development of the mathematics concepts described in this study. The findings from this group of teachers indicate that experience in an early elementary classroom can be an important factor for being attuned to children's mathematical thinking. However, training in a learning trajectory may have an added benefit for teachers and their interpretations of student thinking. It may be that a learning trajectory provides a greater level of detailed indicators for teachers to notice or it may be that professional learning provides detail regarding a larger range of strategies students use.

The Relation of Teachers' Interpretations to Responses

The practice of formative assessment involves an informed interpretation of and response to student thinking. After conducting an analysis for each of these separate practices, I investigated the correlation between interpretations of and responses to student thinking. In the sample, there was a moderate positive correlation between the teachers' mean scores for each variable. Higher mean scores for interpretation of student thinking were associated with higher mean score for responses that had the potential to support and extend student's mathematical thinking. In addition, there was a strong positive correlation between interpretation of and responses to student thinking that were provided by teachers who had been trained in a learning trajectory. For the teachers in the sample, it appeared that training in a learning trajectory may allow them to provide more targeted responses based on their interpretations of student thinking. On the other hand, there was no significant correlation between teachers' interpretations of and responses to student thinking for teachers not trained in a learning trajectory.

I consider this in the light of the observation from Heritage and her colleagues (2009) in which they describe teachers being more adept at interpreting than at responding to student thinking. The findings indicate that teachers with knowledge of a learning trajectory may be better prepared to make productive use of interpretations in order to respond to students' mathematical thinking in ways that teachers without knowledge of a learning trajectory were consistently unable to accomplish. Even though early elementary teachers regularly observe the work of students and sometimes interpret that work, this does not seem to necessarily prepare a teacher to determine responses meant to extend that thinking as described in a learning trajectory. The findings indicate that early elementary teachers were able to utilize the knowledge of a learning trajectory to make decisions about learning the concept in ways that may be unclear to teachers who were not trained in a learning trajectory. This may be because those teachers without the training do not recognize student thinking as indicators of milestones in a student's mathematical development.

Teachers' Goals for Group Discussions

In each survey item, I asked teachers to choose a problem and set an instructional goal for a group of students including the individual students observed immediately before in the scenario. My goal in doing so was to determine in what ways teachers would attend to the learning needs of students, given the natural complexity that one can surmise comes with engaging in a group discussion among students whose current understandings are diverse. While the trajectory is a resource that supports understanding the learning needs of individual students, teachers must also set goals for student learning in group settings and engage in group discussion in such a way to meet the needs of all the learners in the group. Through a qualitative analysis, I searched for themes across the goals and considered in what ways those themes varied across the responses.

The goals that teachers set for the groups of students varied in a number of relevant ways. One of those distinctions included the manner in which teachers proposed the use of solution methods--sometimes those methods were prescribed by the teacher but other times the teacher was open to the use of multiple methods by the students. In addition to research that has shown the benefits of allowing students to explore multiple solution methods (Silver & Stein, 1996), there would also be a benefit in the practice of formative assessment for teachers to allow students to share their varied solution methods and for teachers to use the evidence gathered during this time to inform feedback to students. Teachers who choose to prescribe a method to the students in the group instead close down opportunities for students to use varied methods and limit opportunities for the teacher to learn from his or her students.

As students made use of methods to solve problems, the teachers' purposes in solving those problems differed as well. In most cases, the purpose was to determine a solution to the problem. While this is a necessary goal, it is also one that may fall short of strengthening students' understanding of the concept of addition and subtraction. In other responses, the purpose went beyond finding the solution to understanding concepts of addition and subtraction, such as the relationship between the two operations or the mathematical connections that can be made between different solution methods. The number of goals in which the teacher's purpose was to develop concepts were relatively low, with most goals focused on finding a solution to the given problem.

Further, some goals indicated conditions for the work to be done. These conditions hold the potential to further define the work that is expected to be done in the group. Of the different conditions set on the work done in the group discussion, I focused on two types that were prevalent in the data. First, the way in which the work would be carried out and second, whether or not the students would engage with each other during the discussion. Teachers would sometimes indicate parameters for how the solution would be carried out, as in the student carrying out the work *correctly* or for students to show their solution *pictorially*. On a few occasions, teachers would describe how the group would engage with each other, by listening to others or responding to another student's strategy, leaving the discussion open for learning from peers. These different conditions held the potential to either narrow the possibilities for learning by indicating that work should be done in specific ways or to open up the possibilities for learning by allowing students to learn from each other and add to each other's thinking during the discussion. A group setting is a time when students would be well situated to learn varied methods by engaging in discussion with their peers, but most goals set by teachers in this survey did not indicate those conditions as part of their goal statements.

While the differences in goals described in this section appeared across all of the goals the teachers set for groups of students, few goals contained all of the positive indicators, such as allowing students' varied methods and providing opportunities to students to listen to the methods of other students. Instead, teachers of early elementary grades had a tendency to include students in the discussion in ways that would support their learning from each other. These teachers were also more likely to recognize that students might use multiple methods to solve the problems posed whereas teachers of the upper grades were more likely to prescribe methods. Teachers' background experiences related to a learning trajectory played a smaller role in the differences between group goals than it did in the responses made toward individual students. However, there may be reason to believe that teachers' background experiences play some role in the ways that group discussions are planned since early elementary teachers did include positive indicators more frequently than teachers of upper grades. It is also possible that the prompt in the survey did not elicit a full description of teachers' intended goals while carrying out a discussion. Finally, it is also possible that research on learning trajectories has not yet provided enough knowledge that is of use for thinking about how a class, as a collective, extends its public knowledge.

The findings from responses to individual students along with the variations presented in the goals for group settings lend evidence to the claim that early elementary teachers utilize the variation in student thinking to serve different purposes in the classroom. In general, early elementary teachers did recognize the methods used by students to solve addition and subtraction problems, but without training in a learning trajectory, this variation seemed to serve only as a set of choices for students in order to come to solutions to problems. Teachers who had been trained in a learning trajectory, on the other hand, seemed to be prepared to recognize the variation in student thinking as indicators of the students' conceptual development of addition and subtraction. These teachers leveraged that knowledge when responding to individual students and were somewhat more prepared to attend to students' conceptual understanding of the concept during whole group setting.

Responding to students' individual thinking is a less complex task than responding to students in a group setting. This conclusion is supported by the analyses that early elementary teachers who had training in a learning trajectory had more expert responses toward individual students, but differed less in their practice in whole group to their counterparts who had not been trained in a learning trajectory. Both groups of teachers, regardless of their professional learning experiences had similar patterns in their goals for groups, sometimes setting goals to encourage the use of multiple methods, but seldom describing moves to make connections between those representations of student thinking.

It may be that due to the complexity of working in groups teachers view this setting less as a time to respond to students' individual learning needs and more as a time to attend to broader or more general goals, such as ensuring students understand the problem and its associated operation (a response that was provided somewhat regularly). Another explanation would be that during group instruction, teachers' practice is more sensitive to other obligations, such as moving through the curriculum or ensuring a fair social environment in the classroom (see Herbst & Chazan, 2012).

Implications

The findings from this study have potential implications both for teacher professional learning and teachers' classroom practice. Tools, like the ones used in this study could serve a variety of purposes in education, such as to determine teachers' potential to implement formative assessment practices effectively. The findings also further inform the theory of mathematical knowledge for teaching (Ball et al., 2008) and other research related to teacher practices when teachers' learning experiences related to a learning trajectory is considered as a factor. In the sections that follow, I first describe how the findings could help further develop theory of knowledge for teaching and suggest further research. I go on to describe the ways in which the findings from this study inform classroom practice, both in service of classroom structure as well as design for professional learning.

Implications for Theory and Research

There has long been a debate about what knowledge teachers should have in order to be suitably prepared to teach. While it is clear that knowledge of mathematics alone is necessary but not sufficient knowledge to do the work of teaching, the work of describing the knowledge needed to teach well is ongoing. In their work to describe the pedagogical knowledge that mathematics teachers draw on in order to carry out the tasks of teaching, Ball and her colleagues define domains of this knowledge (Ball et al., 2008), including knowledge of content and students and knowledge of content and teaching. And, Sztajn and her colleagues (2012) suggest that knowledge of a learning trajectory may extend that description of mathematical knowledge for teaching and add to what makes up these domains of knowledge. In the section that follows, I discuss aspects of this study and the considerations that arise when including a learning trajectory as an aspect of teacher knowledge.

Mathematical Knowledge for Teaching

The theory of MKT (Ball et al., 2008) suggests that teachers use mathematical knowledge in ways that are unique to teaching. Knowledge of content and students (KCS) and knowledge of content and teaching (KCT) are domains of knowledge that purportedly support teachers' noticing of students' common solution methods and development of responses to those students.

Knowledge of content and students (KCS). Knowledge of content and students (KCS) allows a teacher to anticipate what students are likely to do when posed with a particular problem and to recognize common misconceptions that arise for students when learning a particular mathematics concept (Ball et al., 2008). The findings from this study indicate that teachers who regularly interact with early elementary students hold knowledge of students' common methods for solving addition and subtraction problems, and use that knowledge to make inferences about students' understandings of addition and subtraction. For example, in response to a student who had counted from one to solve a problem modeled as 13 - __ = 5, a teacher trained in a learning trajectory stated, "This student used the count from one strategy. They needed to count all of the objects to solve the problem. This is very early in the development." (4448). This early elementary teacher, who had also been trained in a learning trajectory, not only described the work of the students, but provided an interpretation related to the student's progress in learning the concept of addition and subtraction. Knowledge of this sort is associated with KCS, but viewing the interpretation through the lens of the learning trajectory allows one to recognize it as an observation that includes elements of the student's understanding as well as his or her development. If I consider teachers' interpretations of student thinking as a proxy for teachers' KCS, then knowledge of a learning trajectory may be one enabler of a teacher's KCS and could be an important factor in supporting teachers' KCS of specific mathematical content.

If knowledge of a learning trajectory is useful to support what teachers notice in their students' work, the definition of KCS could be broadened to include teachers' understanding of the relation different student conceptions have to each other in students' development. Beyond being able to anticipate how any student in the class might respond to a given problem or lesson, a teacher with knowledge of a learning trajectory may consider how particular students are likely to respond because of the teacher's awareness of the student's progress along the trajectory and how that learning is expected to progress over time. Further, knowledge of a learning trajectory may make teachers more aware when it might be appropriate for students to utilize more rudimentary methods as well as when it would no longer be considered suitable for the student, as movement along the trajectory would indicate that the student had progressed mathematically.

Knowledge of content and teaching (KCT). Knowledge of content and teaching (KCT) informs the teacher as to instructional decisions made on behalf of students, such as choosing examples for a lesson or evaluating which student contributions to highlight within a group. Since many have concluded that the quality of a teacher's response to students "is a critical feature in determining the quality of learning activity" (Black & Wiliam, 2009, p. 100), it may be useful to understand more about this domain of teachers' mathematical knowledge for teaching and how it can be developed.

The findings from this study indicate that being able to interpret students' common methods may not be enough to support productive responses that extend learning along a trajectory. The responses from early elementary teachers to students' methods for solving problems would often include a general comment such as asking the student to solve the problem in another way. But those who had been trained in a learning trajectory were better prepared to extend student thinking along the trajectory. For example, in response to a student who had used *direct modeling* to solve a subtraction problem, a teacher with training in a learning trajectory stated, "Green is ready to continue single digit addition and subtraction, but I would also work on concepts of numbers, so that Green does not have to count amounts of 5 or less"

(7160). This teacher stated a learning need for Green, recognizing that he or she counts objects through one-to-one correspondence and has yet to internalize amounts up to five, which might allow him or her to move away from tracking each count in tactile ways. The response from the teacher may be an example of a specialized response and indicate a difference in KCT between teachers who have varied professional learning experiences.

If we view KCT through the lens of a learning trajectory, this means that in addition to choosing examples that best represent a problem that is predetermined in a curriculum, a teacher might consider the problem and representation through the lens of the students' current learning needs. This could mean that a teacher evaluates his or her instructional moves based on the learning opportunity it would afford and that the needed learning would be informed by a learning trajectory. This broadened way of considering KCT allows for the teacher to consider problems, their associated representations, and students' current learning needs in conjunction with each other.

Relationship between KCS and KCT. In their study of teachers' formative assessment practices, Jacobs and her colleagues suggested three key aspects of formative assessment are teachers' elicitation of, interpretation of, and responses to student thinking (Jacobs et al., 2010). The findings from this study suggest that a correlation exists between teachers' interpretations and responses to student thinking, but that those correlations are stronger for teachers with the combined experience of teaching in an early elementary classroom and participating in professional learning of a learning trajectory for addition and subtraction. These teachers were more likely to make productive use of interpretations to provide targeted responses. This suggests that one aspect of a teacher's KCT (that which supports the composition of responses) may be somewhat dependent on a teachers' KCS, and that this knowledge may partially derive from their experiences in an early elementary classroom and in professional training.

To better understand the nature of KCS and KCT in light of learning trajectories, it could be productive to write items of the form used in tests of teachers' pedagogical content knowledge to be used for the additional certification of teachers who desire to specialize in the teaching of particular domains within mathematics. These items could include questions of the type used in this study, in which the teacher is first asked to interpret what a student has done to solve a problem and second to choose from a set of response that supports the student's continued progress learning the concept. Kersting (2008) provides one such example of assessment of teacher knowledge in which items were based on teacher responses to mathematical thinking in the form of video. This form of testing could be adapted to develop items that measure whether teachers have the MKT associated with the teaching and learning of specific mathematical content.

Implications for Further Research

This study relied on a small sample, but there is enough evidence found in the results to justify further studies of the instructional strategies of teachers who engage in different forms of professional learning. There is also evidence from a number of other studies, which had similar findings (Jacobs et al., 2010; Ebby & Sirinides, 2015) relating teachers' formative assessment practices with their professional learning and expertise. Further study could investigate different forms of professional learning about a variety of established learning trajectories. The teachers who participated in this study had learned about the learning trajectory by watching video clips of diagnostic assessments conducted with students (Bobis, et al., 2005). But teachers could also learn about a learning trajectory through other means, such as through investigations of student work that include sorting that work into conjectured learning trajectories or mapping the mathematical development that would need to occur to progress from one strategy to another. It would be useful to ascertain whether one or another way of providing training on the same learning trajectory is more useful for teachers to interpret and respond to students' work.

This study was designed to study the formative assessment practices of teachers when responding to students' early addition and subtraction strategies. Some of the teachers in the sample had been trained in a learning trajectory for that concept. Because there are many other mathematical concepts for which a learning trajectory could be described, it would be interesting to consider whether knowledge of one learning trajectory translates into improved teacher practice in other concept areas. To facilitate such a transfer, a learning trajectory may have to be understood as a set of strategies falling into particular categories related to student progress. One could consider the learning trajectory described in this study as describing students' strategies as falling into three categories, counting of objects, counting without objects, and use of derived strategies, with some variation in how strategies within those categories are carried out. Similarly, students learning multiplication and division make use of strategies that include the use of concrete objects to make sets of equal groups, counting strategies, and derived strategies (as in recognizing 6 times 7 is the same as finding the sum of 6 times 5 and 6 times 2), again with variation of strategies within each of these broader categories. Research on students development of multiplicative concepts (Steffe, 1994) has shown its dependence on the earlier counting schemes that mobilize their solution to addition and subtraction problems. This raises the question of whether teachers' knowledge of how students develop addition and subtraction concepts enables teachers to interpret and respond to students' solution of multiplication problems. Does teachers' capacity to recognize the indicators of student progress in learning addition and subtraction increase their capacity to recognize students' learning of multiplication? Whereas the specificity of a learning trajectory might lend toward professional learning that closely focuses on specific strategies within a particular learning trajectory, it may be useful to consider means by which teachers learn indicators of multiple learning trajectories.

Along these lines, I ask if facilitators of professional learning could consider methods to support teachers to make such transfer from KCS and KCT specific to one concept to KCS and KCT specific to another concept. Researchers could support teacher educators' work minimizing learning time for teachers if they studied what aspects of knowledge of a learning trajectory are amenable to transfer. Perhaps there is an underlying structure of all learning trajectories that can facilitate this transfer and that could be validated through research. While there are researchers who have studied achievement results for students whose teachers had been trained in a learning trajectory (Bobis et al, 2005) and teachers' responses to students whose strategies fall along a trajectory (Ebby & Sirinides, 2015), these studies have looked at results in the context of a learning trajectory for a particular concept, but have not considered teachers' responses across multiple concepts. While generic stage theories of cognitive development have been displaced by the attention to specific knowledge in the learning trajectories approach, the very fact that a learning trajectory exists for addition and subtraction concepts together suggests that the unit of analysis is not reducible to single topics. As we consider this from the perspective of teacher knowledge, one wonders whether more general developmental knowledge of early number and operations might be attainable by those teachers who have been trained in a learning trajectory for addition and subtraction.

Finally, the findings described in this study show a potential relationship between teachers' experience and professional training in a learning trajectory and their formative assessment practices. To be able to show that teacher learning of a learning trajectory has a positive effect on teachers' formative assessment practices, an experimental study would need to be conducted. One such study could be done by choosing a random sample of teachers who work in preschool to second grade settings. These two groups could then be randomly assigned to experimental and control groups, which receive professional learning of different types. The types of professional learning would vary in the specificity of the content. For example, the two groups of teachers could engage in professional learning related to interpreting and responding to students' mathematical work but one group learning generic practices such as noticing and questioning strategies while the other group learn specific practices related to a learning trajectory for addition and subtraction. Teachers in both groups could be asked to complete a survey similar to the one used in this study, as a pre- and post-assessment of teachers' formative assessment practices to determine if the professional learning can be considered to cause improvements in formative assessment practices (as determined by the gains in interpretation and response scores used in this study).

Implications for Classroom Practice and Professional Learning

The findings from this study indicate that teachers' awareness of a learning trajectory can be associated with differences in the quality of teachers' formative assessment practices. In this study, I did not ask teachers to implement formative assessment practices as a means for advancing student understanding. Instead, I asked teachers to respond to prompts that could naturally lend themselves to formative assessment practices. Teachers with experiences in teaching elementary addition and subtraction responded in ways that were more likely to advance students' understandings, as measured by a learning trajectory. I extrapolate from this finding that teachers without the relevant experience and knowledge for teaching elementary addition and subtraction are not as fully prepared to attend to students' progress in mathematics and are thus less likely to implement formative assessment. And, I take it as evidence that it may be useful to consider professional learning opportunities in which teachers engage with learning trajectories as a likely enabler of teachers' formative assessment practices.

Professional Learning and Teacher Certification. As we deepen our understanding of the mathematical knowledge needed for teaching, questions arise as to whether teachers should be learning about or receiving guidance regarding deepened generic questioning techniques or whether teachers require a more specific understanding of students' mathematical learning trajectories in order to inform instructional moves. The findings from this study suggest that teachers' awareness of a learning trajectory could be related to improved formative assessment practices. Yet, teachers are sometimes given too general a guidance regarding the responses they might enact when responding to student thinking, for example by suggesting teachers use more extension questions than questions used to gather information or explore terminology (Boaler & Brodie, 2004).

Because any particular learning trajectory includes deep and specific knowledge of a single concept learned in schools, then we must also consider that there are other learning trajectories teachers may need to learn in order to be fully prepared to teach their content well. This might mean that we reconsider the current structure of teacher preparation and certification. Under some states' certification structures, teachers are certified to teach grades Kindergarten through eighth grade on a single certificate. Only considering a few of the bigger concepts in number and operations across those grades (early addition and subtraction, place value addition and subtraction, multiplication and division, fraction sense, and proportionality), there are a number of learning trajectories teachers could potentially be asked to learn. Therefore, it may be prudent to consider altering certification structures, as the Michigan Department of Education is currently proposing (Jackson, 2017), such that teachers are certified for fewer grade levels (the current suggestion is PreK-3, 3-6, and 6-8). Narrower grade band certifications would allow those in teacher preparation programs to focus more deeply on learning trajectories that make up the content domains in those grade levels. In the absence of a change to certification structures, teacher preparation programs may consider knowledge of learning trajectories to be valuable content for teachers obtaining a master’s degree or additional certifications once the teacher has attained a teaching position and can focus on a smaller portion of the learning trajectories under consideration.

Teacher Responses with Potential to Extend Student Thinking

In this study, I considered the possibility that teachers’ formative assessment responses could be supportive of a student’s current learning need and those responses can be chosen intentionally to support the extension of learning along a trajectory. In Table 6.1 below, I share a summary of the responses provided by teachers in the sample that held that potential to support the students’ further learning. report those responses based on the current strategy used by the depicted student and include a summary of responses when the depicted student had made an error. I surmise that the responses themselves contain kernels of wisdom that might be useful implications for practitioners.

Table 6.1: *Summary of responses with potential to extend student progress*

Problem Posed and Depicted Student's Strategy	Summary of Teacher Responses Scored as 2 or 3
Jonas had 12 pieces of gum. He gave 7 to his friends. How many pieces of gum does he have left? Green responded, "Well, I counted 12 chips from my bucket. Then I put 7 of them away. That leaves 1, 2, 3, 4, 5"	Teachers indicated that Green used concrete objects to solve this subtraction problem. Since the counts he used consisted of forward counts, the teachers recommended Green work on backward counting sequences starting in the teen range so he is able to make use of counting strategies, absent the use of manipulatives, in the future.
Jonas had 12 pieces of gum. He gave 7 to his friends. How many pieces of gum does he have left? Blue responded, "I counted, um, 11, 10, 9, 8, 7, 6, 5. He has 5 pieces of gum left."	Teachers noted that Blue had counted backward to solve and seemed to have a method for tracking that count accurately. With this basic understanding of subtraction, he or she could likely begin to consider subtraction as an inverse of addition and possibly count up, which is a more efficient count in the case of this problem. Teachers also mentioned the possibility of Blue beginning to consider derived strategies to solve the problem. Instructional supports included providing problems in which a count up strategy would be efficient and using ten frames as a tool to develop derived strategies.
Sadie had 11 markers and she left 8 of them at her friend's house. How many markers does Sadie have now? Blue responded, "Um, so 9 and 10 makes 2 and 11 is 1 more. She has 3 left now."	Teachers indicated that Blue had used an efficient count and would be able to consider using derived strategies, such as those in which 10 is used as a landmark. Instructional strategies to do so included the use of ten frames and practicing combinations of 10.

Limitations

The findings from this study have shown that within a small sample of teachers there can be varied ways in which the practices of formative assessment are carried out in response to students learning of early addition and subtraction. The results of the analyses raise potential questions about placement and professional learning for teachers. At the same time, it's important to keep in mind the potential limitations of this study. In addition to the sample size being small, this study asks teachers to describe instructional goals and the ways in which those goals would be carried out in the context of simulated scenarios. The responses may not be authentically reflect what would occur when working with actual students given the added complexities of those encounters.

Some background data from teachers were taken into consideration when conducting this study, including years of teaching experience, current classroom assignment, and professional learning experiences. From the available information I was able to show that classroom assignment and prior training in a learning trajectory were associated with teachers' formative assessment practices, but the small size of the sample made it impossible to assess the size of the effect those predictors could have. It is therefore possible that a study with a larger sample might find the effects of experience and training to be small and that other factors are better bets at accounting for the variance in teachers interpretation and response scores.

Appendix A

A LEARNING TRAJECTORY FOR ADDITION AND SUBTRACTION

Through the use of cognitive interviews with young children, researchers in the early 1980s identified a set of strategies that students invent when asked to solve a context-based mathematics problem (Carpenter et al., 1981). After repeated investigations of this kind, the set of problems used to elicit the students' strategies was organized into a framework that places problems of lower complexity and greater ease for students in contrast with those problems that are more demanding and require students to have more sophisticated understandings in order to solve. The framework has been the foundation for professional learning of elementary teachers and its use is recommended in recent policy documents (CCSS, 2010). The framework, as it appears in the Common Core State Standards for Mathematics (CCSS, 2010) is shown in Figure A.1 in which the less complex problems can be found in the upper left corner with darker shading and the more demanding problems are in unshaded cells.

	Result Unknown	Change Unknown	Start Unknown
Add To	A bunnies sat on the grass. B more bunnies hopped there. How many bunnies are on the grass now? $A + B = \square$	A bunnies were sitting on the grass. Some more bunnies hopped there. Then there were C bunnies. How many bunnies hopped over to the first A bunnies? $A + \square = C$	Some bunnies were sitting on the grass. B more bunnies hopped there. Then there were C bunnies. How many bunnies were on the grass before? $\square + B = C$
Take From	C apples were on the table. I ate B apples. How many apples are on the table now? $C - B = \square$	C apples were on the table. I ate some apples. Then there were A apples. How many apples did I eat? $C - \square = A$	Some apples were on the table. I ate B apples. Then there were A apples. How many apples were on the table before? $\square - B = A$
	Total Unknown	Both Addends Unknown[1]	Addend Unknown[2]
Put Together /Take Apart	A red apples and B green apples are on the table. How many apples are on the table? $A + B = \square$	Grandma has C flowers. How many can she put in her red vase and how many in her blue vase? $C = \square + \square$	C apples are on the table. A are red and the rest are green. How many apples are green? $A + \square = C$ $C - A = \square$
	Difference Unknown	Bigger Unknown	Smaller Unknown
Compare	*"How many more?" version.* Lucy has A apples. Julie has C apples. How many more apples does Julie have than Lucy? *"How many fewer?" version.* Lucy has A apples. Julie has C apples. How many fewer apples does Lucy have than Julie? $A + \square = C$ $C - A = \square$	*"More" version suggests operation.* Julie has B more apples than Lucy. Lucy has A apples. How many apples does Julie have? *"Fewer" version suggests wrong operation.* Lucy has B fewer apples than Julie. Lucy has A apples. How many apples does Julie have? $A + B = \square$	*"Fewer" version suggests operation.* Lucy has B fewer apples than Julie. Julie has C apples. How many apples does Lucy have? *"More" version suggests wrong operation.* Julie has B more apples than Lucy. Julie has C apples. How many apples does Lucy have? $C - B = \square$ $\square + B = C$

Figure A.1: Addition and subtraction problem types. (CCSS, 2010), retrieved from www.corestandards.org

It has been recommended that teachers make use of this framework as a means for designing problem solving activities that are challenging enough to promote cognitive growth and yet still within the capabilities of students in the teacher's classroom. Earlier research has shown that the framework has been an effective tool for teacher decision making regarding the design of learning activities or the introduction of a new problem to a group of students (Carpenter & Fennema, 1992).

For this particular study, I focus on the trajectory that corresponds to learning early addition and subtraction (e.g., sums and differences with numbers that add up to

20). Before describing the strategies that students use and the trajectory along which those strategies tend to develop, I first discuss the framework of problem types used by teachers to choose tasks which are meant to deepen student understanding of addition and subtraction.

In the decades since the framework was first described, the field has understood more about the relative difficulty of the problem types as it relates to the problem solving methods of students. Some of the easiest problem types for children to solve are those in which children are able to "*directly model* the problem's actions" (Sarama & Clements, 2009, p. 121). These problem types tend to be more accessible for children because the actions needed to carry them out are directly translatable from the way in which the problem is stated (Carpenter, Ansell, Franke, Fennema, & Weisback, 1993). This includes problems in which two sets are joined and the sum is unknown, part-part-whole problems in which the total is unknown, and problems in which one set is separated from another set and the difference is unknown (Sarama & Clements, 2009), each purposely formed so that the result of the sum or difference is unknown. As students develop ways of thinking about the quantities associated with numbers more abstractly, they also become able to solve join or part-part-whole problems in which the change or part is unknown, sometimes called missing addend or missing subtrahend problems. Problems in which the sets are separated and the change is unknown, called missing subtrahend problems, are one of the more difficult problem types for young learners to solve and can be solved once he or she is able to count on from a number other than one, often called *counting on*. *Counting on* also allows students the ability to solve comparison problems without the use of objects to directly model the problem.

Comparison problems are those in which two quantities are compared and the student is asked to find how much more one quantity is than another. The most difficult problem type is when the starting number, either the first addend or the minuend, is unknown (Sarama & Clements, 2009). Broadly speaking, this means that students develop from using objects in order to model addition and subtraction problems in which the result is unknown, toward manipulating or counting numbers mentally (Steffe et al., 1983) in order to solve more complex problems, such as those in which the change or start is unknown. Students who are able to solve the more complicated problem types are also able to move beyond the use of counting strategies toward the use of derived or reasoning strategies (Sarama & Clements, 2009).

In the next section, I describe more specifically the trajectory that has been modeled by multiple researchers studying student methods for early addition and subtraction.

Using Objects to Model Addition and Subtraction Problems

Direct modeling. Early in their development, students solve problems by counting out sets of objects to model addition and subtraction situations in order to find the sum or difference (Carpenter et al., 1981), a strategy called *direct modeling* (Clements et al., 2004, p. 224). To do this, the student establishes, using perceptual objects (Steffe et al., 1983), each of the quantities that relate to each other mathematically as being joined, separated, or compared (Carpenter et al., 1981). Once the quantities have been established, the student then counts the sum or difference by manipulating them either through tactile means or by counting visually. This requires a student to count from one, three separate times, to model the given problem. For example, when solving an addition problem, a student would count out objects to model each of the addends before grouping them together to count the sum. For subtraction, the student would count the amount corresponding to the minuend then count to remove the amount indicated by the subtrahend, and finally count to determine the difference.

If objects are not available, then a student who still needs some form of perceptual marker for each of the quantities, would re-present (von Glasersfeld, 1995) each quantity on their fingers before counting. Because each quantity would need to be manipulated separately, this means that without the use of objects students are able to solve problems in which each of the addends are no greater than five or the minuend is ten or less, what is referred to as being 'within finger range' for the student. For example, when solving the following problem, *Sam has 5 racecars and his friend gives him 3 more racecars. How many does Sam have now?,* the student would re-present each quantity simultaneously. This means the student would put up five fingers on one hand and three fingers on the other before returning to counting all eight again.

Counting Strategies to Solve Addition and Subtraction

Counting all. As students develop more facility with the use of counting to solve problems, their conception of number and the associated quantities become progressively more interiorized (Steffe et al, 1983) in that students may no longer need perceptual objects in order to solve a problem, but may use figurative or replacement objects instead. This makes it possible for students to begin a transition away from the use of material objects for both addends and toward a counting procedure that includes abstracted quantities instead. While students might still need to count items, they would be able to count a re-presentation of the quantity rather than the perceptual items themselves, what is referred to as a figurative count (Steffe et al., 1983). When using figurative items, a student may count from one as he or she enumerates the initial amount then continue counting, without returning to one, in order to count to the solution, since a simultaneous model of each set is no longer needed for the student to be able to conceptualize the grouping of the objects. This strategy is sometimes called *counting all* (Secada, Fuson, & Hall, 1983). Using *count all* to solve 8 + 3, the student would count from one to eight, then continue by counting three more in order to find the sum, possibly tracking the counts on their fingers, particularly as markers of the second addend. Students have moved beyond the need to perceive each quantity of objects separately and are now able to represent each quantity as it is being counted, utilizing the fingers in order to track multiple quantities at different times during the enactment of the problem.

Counting on. As students develop, their abilities in counting become more sophisticated and allow the student to *count on* from a number other than one as he or she tracks the counts in the second amount, coming to a result. *Counting on* occurs when the student "is aware that the numerical structure designated by a number word is a composite of so many individual units" (Steffe et al., 1983, p. 66). This means that for the problem 8 + 3, the student who is able to *count on* recognizes 8 as a composite of 8 individual counts of one, and thus has the ability to begin *counting on* beginning at 9 while tracking the number of counts in the second addend in order to arrive at the correct response.

I identify a student's shift toward *counting on* as critical since it is this shift that allows the student to move away from the use of perceptual objects and instead to recognize the starting number as a composite of the counts that make up that quantity, enacting a mental counting strategy that does not depend on the use of perceptual objects. There is evidence that if objects are made available to a student who is just developing the *counting on* strategy, the "search for perceptual input would [be] immediately satisfied through visual perception" (Steffe et al. 1983, p. 72) leading the student to make use of a less sophisticated strategy, that of *direct modeling*. Instead, it is suggested that by hiding objects from view, the student develops an ability to count a figurative or abstract amount, one that can be re-presented on a student's fingers. Particularly when a student is initially beginning to use a figurative counting strategy, it may feel more comfortable to count objects instead, limiting the opportunity for a student to develop a *count on* strategy. This means that supporting a student as he or she develops the use of counting strategies would suggest limiting exposure to objects and carefully scaffolding the use of materials so that the student shifts away from counting perceptual items toward an abstract count.

Supporting the use of counting on strategies. In a teaching experiment to determine methods for teaching students who *count all* to be able to *count on*, Secada and his colleagues (1983) focused instruction on three subskills identified in a previous study (Fuson, 1982) that would be necessary for students to acquire in order to transition to a *count on* strategy and designed tasks to teach these subskills. The students in the teaching condition were children who were using the *count all* strategy in order to solve addition tasks, identified as not having the necessary subskills to *count on*. The use of resources to enact the teaching experiment is of interest. Researchers introduced students to two sets of dots as shown in Figure A.2. The dots represented the addition problem 13 + 6, but the researchers kept the dots in the first addend screened, thus prompting students to strengthen the use of skills that support *counting on* (Secada et al., 1983), rather than allowing the students to count each of the dots associated with both addends. The dots associated with the first addend were revealed only if students did not respond to prompts that asked the student to find how many altogether.

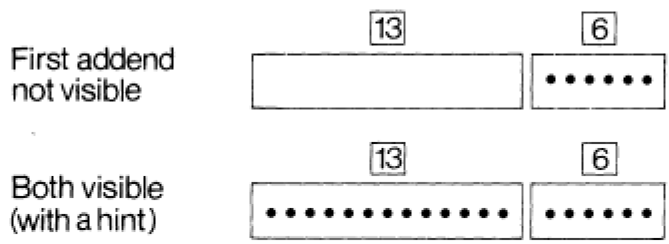


Figure A.2: *Tool used by Secada, Fuson, & Hall (1983), to teach the count on strategy.*

By hiding the dots in the first addend, the researchers encouraged students to consider the set of dots that represent the first addend as a numerical composite (Steffe et al., 1983) while allowing the student to *count on* from thirteen by coordinating the count with each object in the second addend, thus determining the sum by *counting on*. Of the eight students who were part of the teaching condition, seven transitioned to a *count on* strategy in one teaching session. Of those who were not taught the subskills, only one of the eight used the *count on* strategy in the post test. The evidence from the teaching experiment as well as the knowledge that students will tend to make use of objects when they are available, suggests that teaching practices that are supportive of students' development of a counting strategy that does not make use of objects would restrict the use of objects, rather than provide them. This makes for an interesting teaching case since the use of manipulatives is fairly common practice in early elementary classrooms and some would advocate that students should have access to multiple representations over a period of time and be allowed to reason flexibly within and between these representations so that over time students are able to understand the mathematics abstractly (Post, Wachsmuth, Lesh, & Behr, 1985; Star & Rittle-Johnson, 2008).

Thus, I identify the transition from using objects to not needing objects as one critical juncture in a learning trajectory since the use of objects in order to solve problems may be useful for students who need *direct modeling* in order to understand the context of the problem and what it means for the quantities being manipulated. At the same time, the objects themselves could serve as a hindrance to a student who is able to or will soon be able to count without the use of materials. The decisions that teachers make in this instance could play an influential role in whether or not students make conceptual shifts in thinking that lead to more sophisticated mathematical reasoning.

Counting back. With the development of the numerical composite and the use of a *counting on* strategy, students are then able to make use of counting strategies that include a backward count and are useful for solving problems of the *find result* type (Clements & Sarama, 2009) subtraction problems without the use of objects. This would mean that to solve a problem like 8 – 3, the student now has a numerical composite of eight, a unit of eight iterative counts, and can begin at eight and count backward three to find the difference. This is also indicative of the critical juncture that occurs with *counting on* since with objects available, a student would be likely to enact a *direct modeling* strategy in which the student first counts out a number of items that represent the minuend, then counts to remove the number of items corresponding to the subtrahend, followed by a count of the remaining items, which represent the difference. Fuson (1984) describes the *direct modeling* and *counting back* procedures as non-parallel strategies in that the action on the objects does not indicate what would occur when the student counts without the use of objects. The use of the objects allows the student to use a forward count to be able to find the difference, but does not make use of the student's conceptual understanding of number that would now allow the student to use an abstract backward count in order to solve the subtraction task. For this reason, the use of objects to enact *direct modeling* does not correspond in any parallel way (Steffe et al., 1983; Fuson, 1984) to the backward counting procedure and may be a hindrance to the student's development of a backward counting procedure in much the same way that it is a hindrance for *counting on.*

Reasoning Strategies for Addition and Subtraction

Distinguishing between counting back and reasoning. Steffe and his colleagues (1983) distinguish among students' development regarding subtraction by describing the different ways in which students may count when given a subtraction problem in which the subtrahend is larger than the difference, for example, the problem 8 – 6. Students who can count backward to subtract may begin a count on the number one less than the minuend and continue counting backward six counts until arriving at the difference of two, for example by counting 7, 6, 5, 4, 3, 2. This method for counting the subtraction task indicates the student has constructed a number sequence which is useful for counting units forward and backward and recognizes each unit as a cardinal object. This allows the student to count backward in order to determine how many units or objects would be left when six are removed from eight (Steffe et al., 1983) a literal interpretation of a problem like the following, *Josie had 8 stickers and gave 6 to her friends. How many stickers does Josie have now?* This is evidence that the student understands subtraction as taking away objects and is able to make use of a number sequence in order to count backward a number of units to determine the amount that remains when the subtrahend is removed and would be considered a counting strategy.

Part-Part-Whole. With further development, a student could move away from this literal interpretation of counting backward each of the units in the subtrahend as he or she recognizes the relationship between the quantities represented in the problem. In this way, rather than seeing the six as a number of units that must be counted backward from the minuend, six is a composite that is both embedded in eight and able to be dis-embedded (Steffe, Cobb, & Von Glasersfeld, 1988). Other researchers have referred to this as a *part-part-whole* relationship in which students make use of an understanding of the relationship that exists between the three numbers (Clements & Sarama, 2014). This understanding may present as a counting strategy, but the student would count only two counts, rather than six, indicating a reasoning strategy rather than a literal counting strategy. After counting two counts, the student would recognize that the two counts represent the solution since by counting back two counts, the student has arrived at six, another number in the relationship. For example, the student might say, '7, 6. The answer is 2' since arriving at six indicates the distance of two that six is from eight. The difference in how the student has counted marks a distinction between the student's understanding of quantities and a student who would count back six counts. For the first strategy, the student is able to count backward while keeping track of the six counts, and understands that by starting at eight and counting backward six, the student has tracked the number of objects that would still remain when the six are taken from a set. On the other hand, when using the *part-part-whole* strategy, while the problem may indicate eight remove six, the student instead counts only until having arrived at six, understanding that the two counts, the six, and the eight share an additive relationship and counting each removed item is not necessary.

Derived strategies. Once students have gained facility with the use of more and more sophisticated counting strategies, they begin to add and subtract by chunking numbers (Fuson, 2004), what other researchers have referred to as heuristic or derived strategies (Carpenter et al., 1981). The use of derived strategies, such as making use of a known fact in order to solve an unknown fact or using ten as an anchor within the problem to reason about the quantities nearby, represents an alteration in the conception a student has regarding number. Rather than counting an accumulation or separation of cardinal objects from sets, students recognize that a number can be broken into chunks, for example by finding groups of ten, the use of which may facilitate ease of computation. This shift toward derived or heuristic strategies allows the student to be more flexible in thinking about the problems that are presented and serves to foster learning goals in algebra later on (Schifter, 1999). A student that alters the numbers in a problem in order to facilitate the ease of computation and at the same time maintains the mathematical relationship between the values is beginning to make use of the properties of numbers that are algebraic in nature.

I identify this transition toward derived strategies as a second critical juncture for students progressing along a learning trajectory in addition and subtraction. In countries in which teachers specifically support the use of derived strategies, students on average have more success in mathematics throughout their schooling (Clements & Sarama, 2014; Fuson, 2004). Further, there is evidence that because reasoning strategies depend on students making sense of the properties of addition and subtraction, rather than just counting units forward and backward in order to determine a result, they do not “activate the same systems” (Clements & Sarama, 2014, p. 95) as counting strategies would. Because of this, it is important for teachers to expose students to, and expect them to make use of, derived or reasoning strategies.

For teachers to support the use of reasoning strategies in the classroom, he or she will need to use teaching practices that consider the mathematics that students do to be more algebraic in nature, rather than an accumulation or separation of objects. In comparison to practices that allow student opportunities to count out sets of objects, teachers will also need to support students’ use of materials that develop reasoning strategies, which would likely be different than loose sets of objects and would instead group numbers in some way. In addition, teachers will also have to negotiate this difference in the classroom since most classrooms would have students who need support to count objects as well as students who are ready to begin grouping.

Supporting the use of derived strategies. The learning of derived strategies can be supported in at least two important ways. First, by using representational tools that support students’ attention on the concept of ten, for example by using an arithmetic rack (similar to an abacus) or a ten frame to support the use of a *make ten* strategy. Second, derived strategies can be supported by asking students to solve a problem that is very similar to a problem already solved, for example asking the student to find the answer to 5 + 4 after 5 + 5 had already been solved.

In their description of instruction that specifically supports the learning of a *make ten* strategy, Clements and Sarama (2014) indicate specific steps that are taken to support the use of this strategy exclusively. In the first phase of this instructional sequence, teachers purposely choose problems in which the first addend is nine, for example 9 + 4, showing the problem in two ten frames as in Figure A.3 below, then asking how many more it would take to make 10, moving one item from the frame with four items to the frame with nine in order to make a ten before determining the sum.

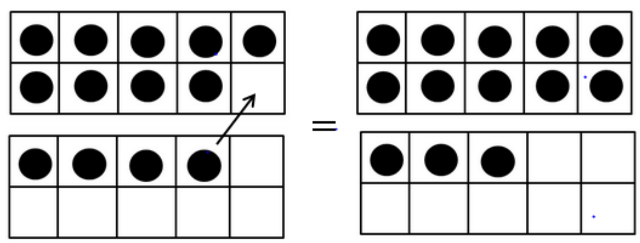


Figure A.3: *A model representing the transformation of 9 + 4 into 10 + 3; and the equivalency between the two sums*

The instructional sequence continues with the teacher then posing problems in which the first addend may be seven, eight, or nine, focusing on the properties of the solution type and switching to representations that are drawn rather than manipulated physically. After this, students are supported to become fluent with a *make ten* strategy (Clements & Sarama, 2014).

I identify the instruction focused on reasoning strategies as a second critical juncture in student learning of addition and subtraction. Since the use of reasoning strategies has been shown to be supportive of students' later understanding and learning in mathematics, it seems reasonable to consider that teachers would find methods for supporting students in making use of and understanding reasoning strategies.

I show the position of the critical junctures in Figure 4. While it is possible to describe the trajectory in a more fine-grained manner, as I did in Table 2.1, by zooming out and looking at just three major phases of development, I consider moments of instruction that are critical for students to be able to make significant shifts in their ways of doing and understanding mathematics and the relationship between the operations of addition and subtraction. Further, the three regions defined by the diagram indicating the crucial junctures are coarse enough to be more easily recognizable by teachers, whether or not they have had specific training in learning trajectories. With the exception of the *part-part-whole* strategy, which is categorized as a reasoning strategy even though the student could be counting, the diagram of critical junctures groups the *modeling* strategies separately from the *counting* strategies and also from the *reasoning* strategies.

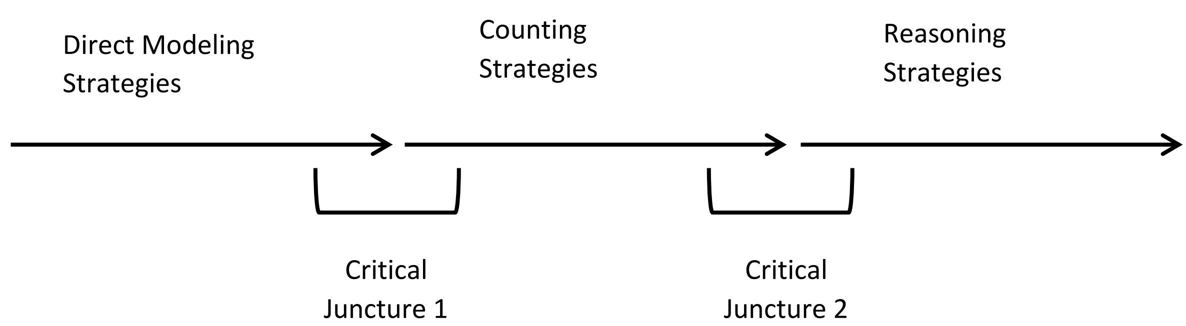
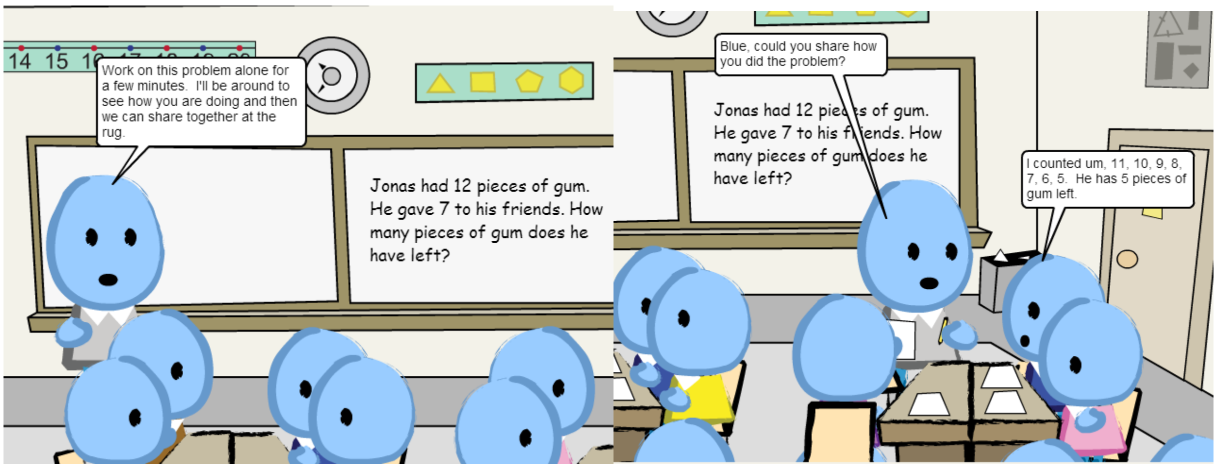


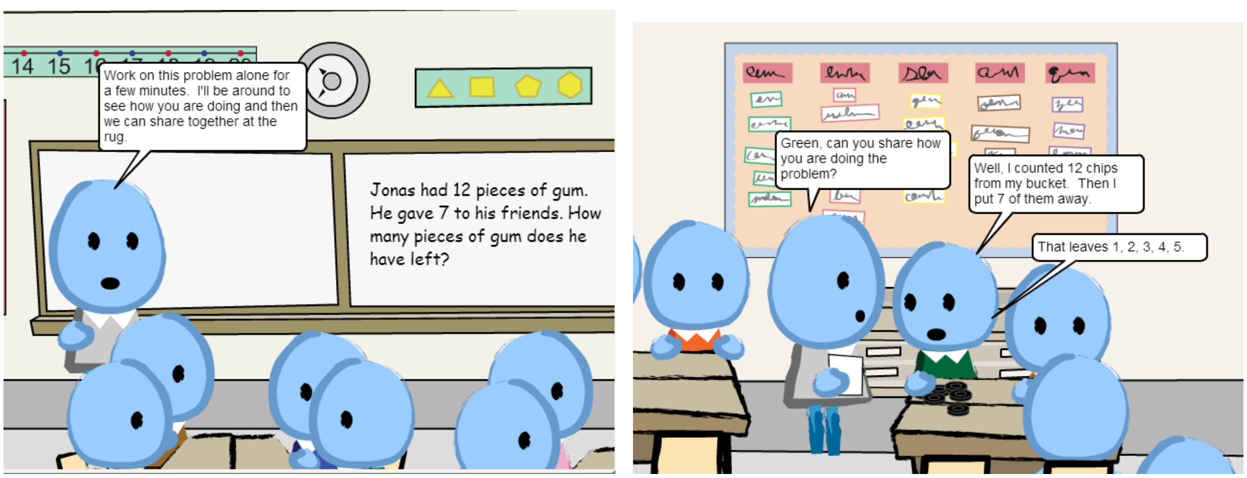
Figure A.4: *Critical Junctures in Relation to Types of Strategies Used by Students*

Appendix B

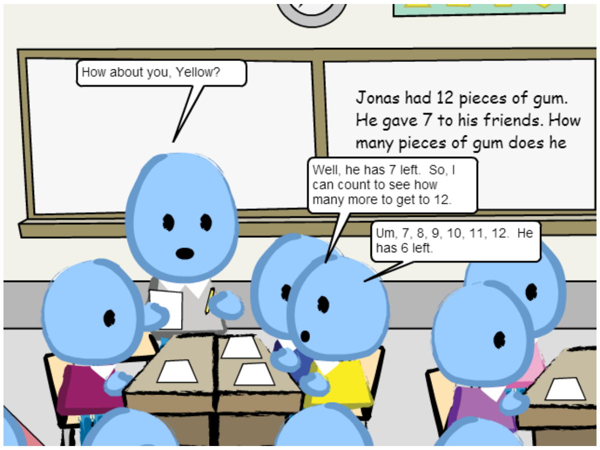
Item One

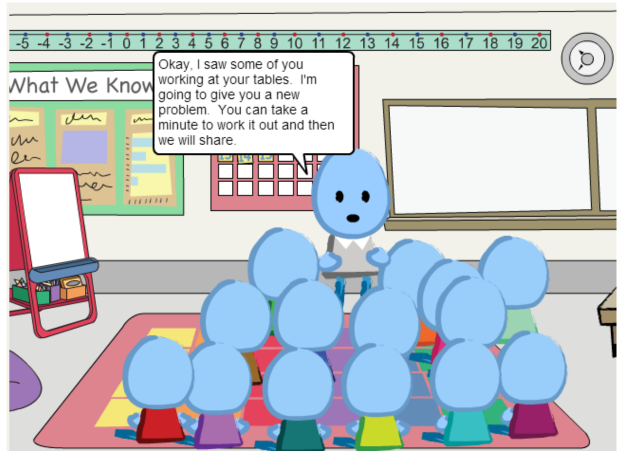


1) Describe what you noticed this student did to solve the addition or subtraction word problem.
2) Describe what you would say and do in response to this student.
3) Describe what you consider to be the next learning goal for this student.



1) Describe what you noticed this student did to solve the addition or subtraction word problem.
2) Describe what you would say and do in response to this student.
3) Describe what you consider to be the next learning goal for this student.

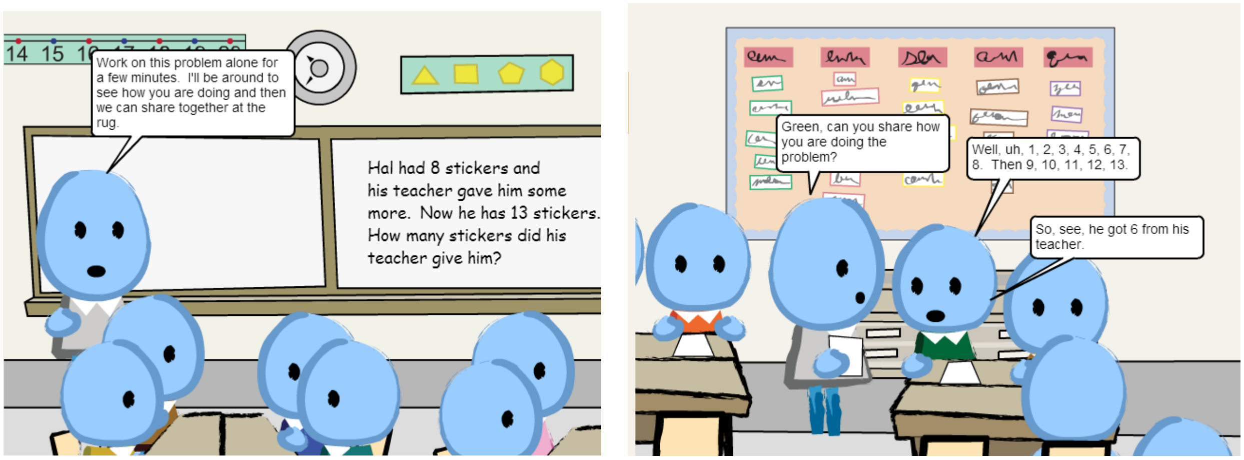




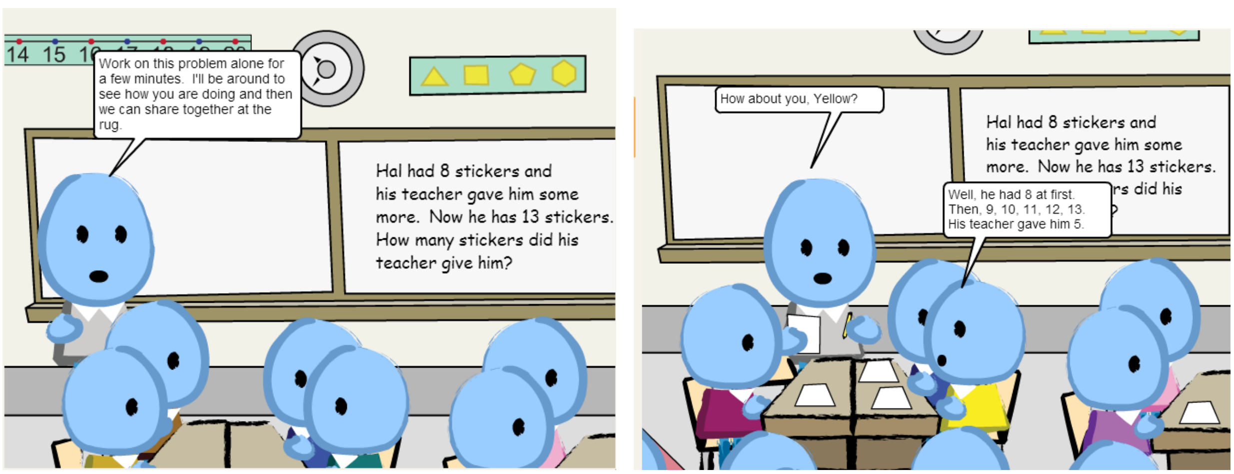
Participants are given an opportunity to view how one other student in the class had solve the problem.

4) Choose a problem type (multiple choice format) that you would consider most appropriate for the teacher to use when the students gather for group discussion. (Problem choices written in the form of a word problem with blanks left in order to fill in appropriate values)
5) Choose numbers that you would place in the blanks when using this problem.
6) Describe what you plan to achieve with this group of students when the chosen problem is discussed.

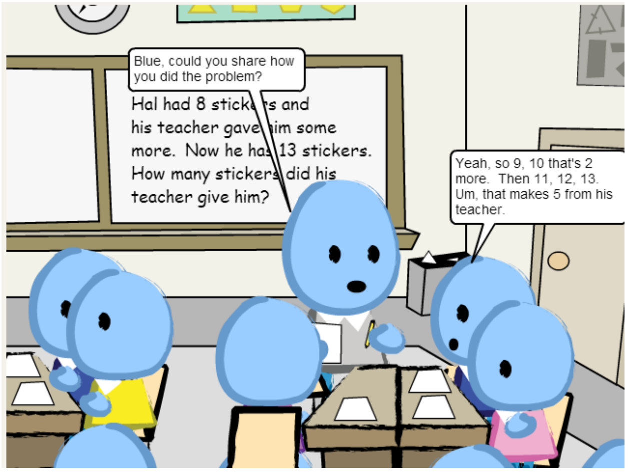
Item Two

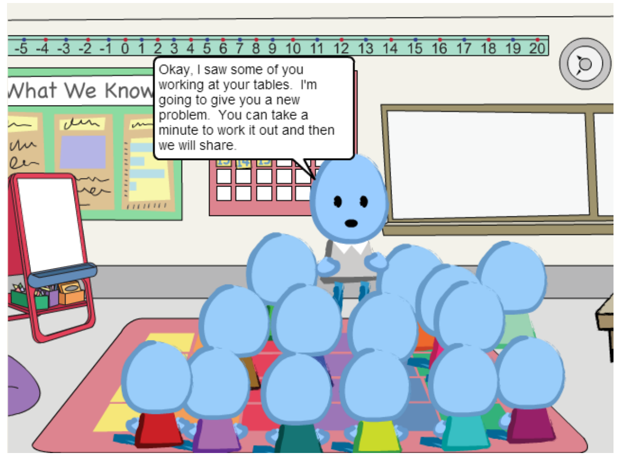


1) Describe what you noticed this student did to solve the addition or subtraction word problem.
2) Describe what you would say and do in response to this student.
3) Describe what you consider to be the next learning goal for this student.



1) Describe what you noticed this student did to solve the addition or subtraction word problem.
2) Describe what you would say and do in response to this student.
3) Describe what you consider to be the next learning goal for this student.

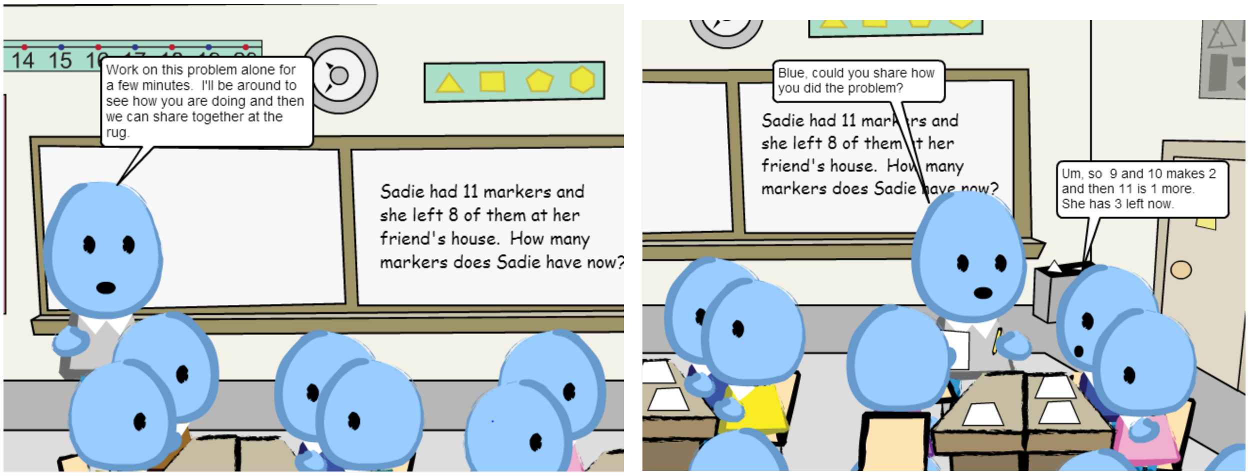




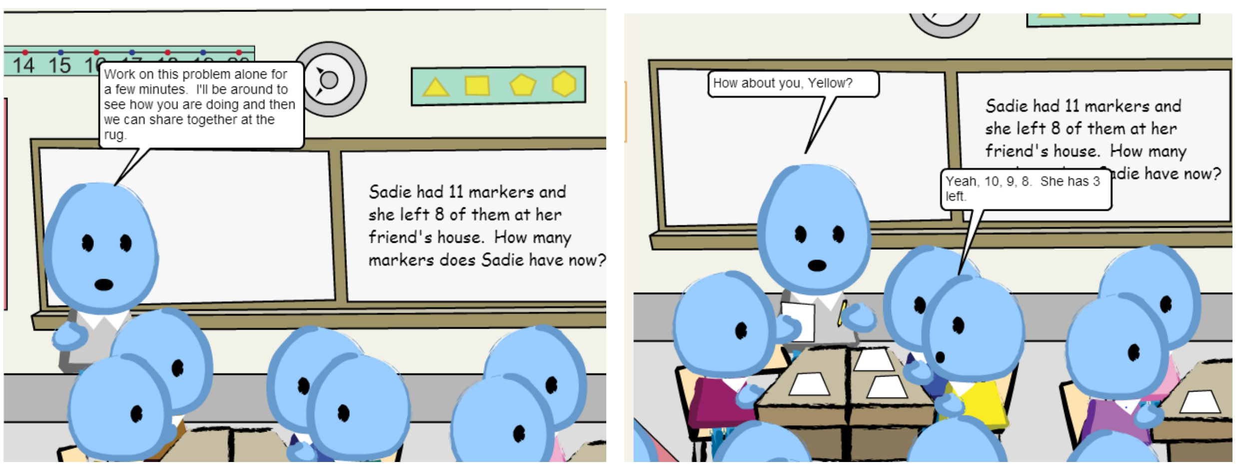
Participants are given an opportunity to view how one other student in the class had solve the problem.

1) Choose a problem type (multiple choice format) that you would consider most appropriate for the teacher to use when the students gather for group discussion. (Problem choices written in the form of a word problem with blanks left in order to fill in appropriate values)
2) Choose numbers that you would place in the blanks when using this problem.
3) Describe what you plan to achieve with this group of students when the chosen problem is discussed.

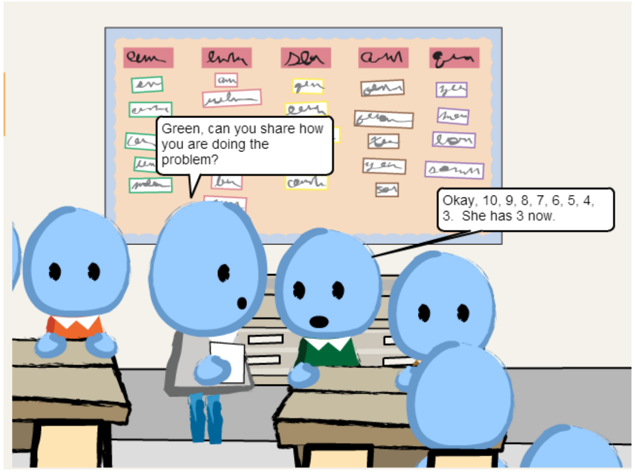
Item Three

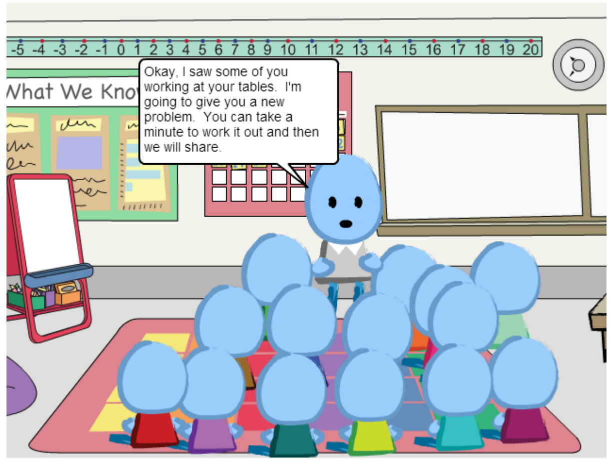


1) Describe what you noticed this student did to solve the addition or subtraction word problem.
2) Describe what you would say and do in response to this student.
3) Describe what you consider to be the next learning goal for this student.



1) Describe what you noticed this student did to solve the addition or subtraction word problem.
2) Describe what you would say and do in response to this student.
3) Describe what you consider to be the next learning goal for this student.

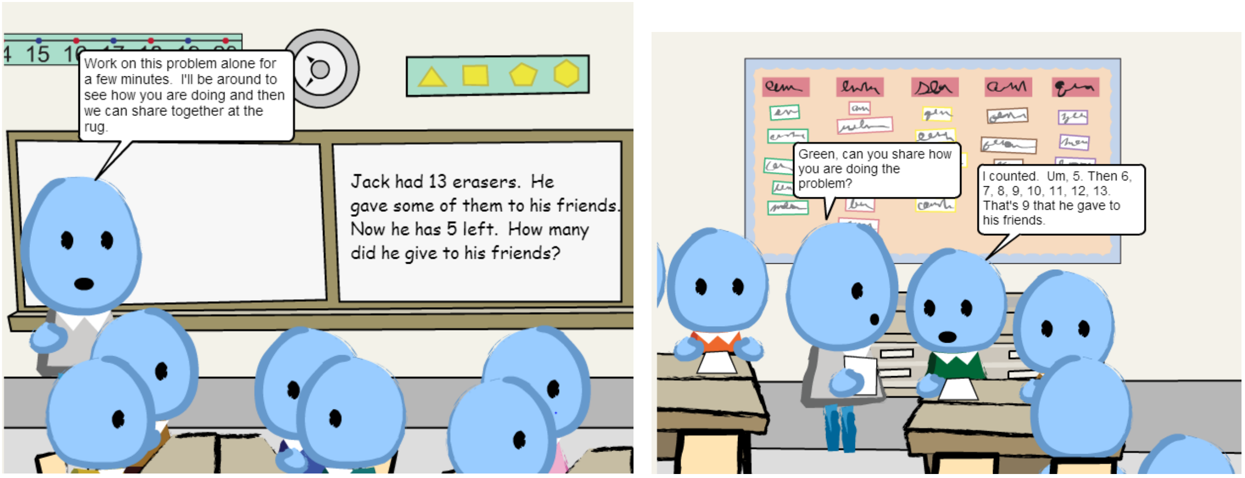




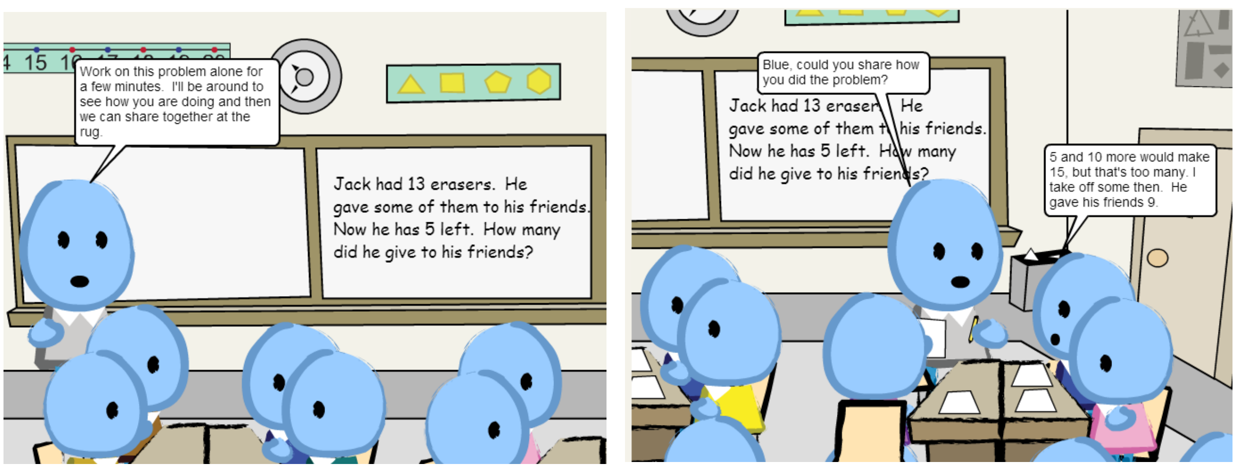
Participants are given an opportunity to view how one other student in the class had solve the problem.

1) Choose a problem type (multiple choice format) that you would consider most appropriate for the teacher to use when the students gather for group discussion. (Problem choices written in the form of a word problem with blanks left in order to fill in appropriate values)
2) Choose numbers that you would place in the blanks when using this problem.
3) Describe what you plan to achieve with this group of students when the chosen problem is discussed.

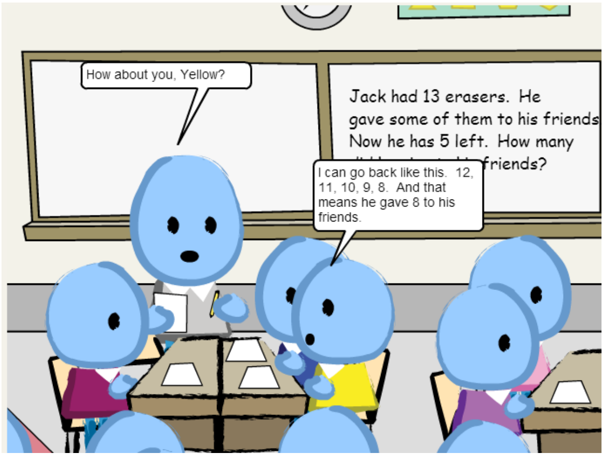
Item Four

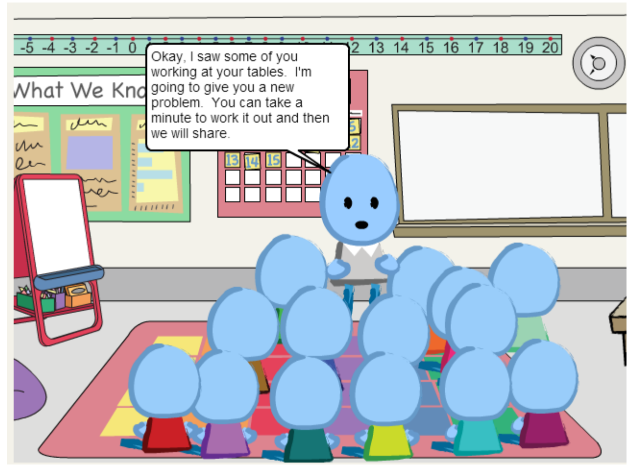


1) Describe what you noticed this student did to solve the addition or subtraction word problem.
2) Describe what you would say and do in response to this student.
3) Describe what you consider to be the next learning goal for this student.



1) Describe what you noticed this student did to solve the addition or subtraction word problem.
2) Describe what you would say and do in response to this student.
3) Describe what you consider to be the next learning goal for this student.





Participants are given an opportunity to view how one other student in the class had solve the problem.

1) Choose a problem type (multiple choice format) that you would consider most appropriate for the teacher to use when the students gather for group discussion. (Problem choices written in the form of a word problem with blanks left in order to fill in appropriate values)
2) Choose numbers that you would place in the blanks when using this problem.
3) Describe what you plan to achieve with this group of students when the chosen problem is discussed.

References

Allal, L., and Pelgrims-Ducrey, G. (2000). Assessment of–or in–the zone of proximal development. *Learning and Instruction, 10*, 137-152.

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal, 93*(4), 373-397

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*, 389-407.

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on the teaching and learning of mathematics* (pp. 83-104). Westport, CT: Ablex.

Baroody, A. J. (1987). The development of counting strategies for single-digit addition. *Journal for Research in Mathematics Education, 18*(2), 141-157.

Black, P., & Wiliam, D. (1998a). Assessment and classroom learning. *Assessment in education*: *Principles, Policy, and Practice*, *5*(1), 7-74.

Black, P., & Wiliam, D. (1998b). *Inside the black box: Raising standards through classroom assessment*. London: School of Education, King's College.

Black, P., & Wiliam, D. (2009). Developing the theory of formative assessment. *Educational Assessment, Evaluation and Accountability (formerly: Journal of Personnel Evaluation in Education)*, *21*(1), 5-31.

Boaler, J., & Staples, M. (2008). Creating mathematical futures through an equitable teaching approach: The case of Railside School. *Teachers College Record*, 110(3), 608-645.

Bobis, J., Clarke, B., Clarke, D., Thomas, G., Wright, R., Young-Loveridge, J., et al. (2005). Supporting teachers in the development of young children 's mathematical thinking: Three large-scale cases. *Mathematics Education Research Journal, 16*(3), 27-57.

Boston, M. D., & Smith, M. S. (2011). A 'task-centric approach' to professional development: Enhancing and sustaining mathematics teachers' ability to implement cognitively challenging mathematical tasks. *ZDM Mathematics Education, 4*3 , 965–977.

Bruce, C. D., & Ross, J. A. (2008). A model for increasing reform implementation and teacher efficacy: Teacher peer coaching in grades 3 and 6 mathematics. *Canadian Journal of Education, 31*, 346–370.

Carpenter, T. P., Hiebert, J., & Moser, J. M. (1981). First grade children's initial solution procedures for simple addition and subtraction problems. *Journal of Research in Mathematics Education, 12*, 27-39.

Carpenter, T. P., & Moser, J. M. (1982). The development of addition and subtraction problem- solving skills. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (9-24). Hillsdale, NJ: Lawrence Erlbaum Associates.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal, 26*(4), 499-531

Carpenter, T. P., & Fennema, E. (1992). Cognitively guided instruction: Building on the knowledge of students and teachers. *International Journal of Educational Research*, *17*(5), 457-470.

Carpenter, T. P., Ansell, E., Franke, M. L., Fennema, E., & Weisbeck, L. (1993). Models of problem solving: A study of kindergarten children's problem-solving processes. *Journal for Research in Mathematics Education*, *24*, 428-441.

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction*. Elementary School Journal, 97*, 3-20.

Carpenter, T. P., Franke, M. L., Jacobs, V. R., Fennema, E., & Empson, S. B. (1998). A longitudinal study of invention and understanding in children’s multidigit addition and subtraction. *Journal for Research in Mathematics Education*, 29(1), 3-20.

Carpenter, T.P., Fennema, E., Franke, M.L., Levi, L., & Empson, S.B. (2000, September). *Cognitively guided instruction: A research-based teacher professional development program for elementary school mathematics*. National Center for Improving Student Learning and Achievement in Mathematics and Science, Report No. 003. Madison, WI: Wisconsin Centre for Education Research, The University of Wisconsin-Madison.

Chapin, S. H., & O‘Connor, C. (2007). Academically productive talk: Supporting students' learning in mathematics. In W. G. Martin, M. Strutchens, & P. Elliot (Eds.), *The learning of mathematics* (pp. 113- 139). Reston VA: NCTM

Chazan, D., & Herbst, P. (2011). Challenges of particularity and generality in depicting and discussing teaching. *For the Learning of Mathematics,31*(1), 9-13.

Chazan, D., & Herbst, P. (2012). Animations of classroom interaction: Expanding the boundaries of video records of practice. *Teachers College Record*, *114*(3), 1-34.

Clements, D. H., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning*, *6*(2), 81-89.

Clements, D. H., Sarama, J., & DiBiase, A. M. (Eds.). (2004). *Engaging young children in mathematics: Standards for early childhood mathematics education*. New York: Routledge.

Clements, D. H., & Sarama, J. (2014). *Learning and teaching early math: The learning trajectories approach*. New York: Routledge.

Cohen, D. K., Raudenbush, S. W., & Ball, D. L. (2003). Resources, instruction, and research. *Educational evaluation and policy analysis*, *25*(2), 119-142.

Common Core State Standards Initiative. (2010). Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Crespo, S. (2000). Seeing more than right and wrong answers: Prospective teachers' interpretations of students' mathematical work. *Journal of Mathematics Teacher Education*, 3, 155–181.

Crespo, S. (2002). Praising and correcting: Prospective teachers investigate their teacherly talk. *Teaching and Teacher Education*, *18*(6), 739-758.

Crespo, S., Oslund, J. A., & Parks, A. N. (2011). Imagining mathematics teaching practice: Prospective teachers generate representations of a class discussion. *ZDM Mathematics Education, 43*, 119–131.

De Vries, H. Elliott, H., Kanouse, M.N., Kanouse, D.E., Teleki, S. (2008). Using pooled kappa to summarize interrater agreement across many items. *Field Methods, 20*(3) 272- 282.

Ebby C.B., & Sirinides P.M. (2015). Conceptualizing teachers' capacity for learning trajectory-oriented formative assessment in mathematics. In J. Middleton, J. Cai, & S. Hwang (Eds.). Large-scale studies in mathematics education. Cham, Switzerland: Springer.

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education*, 27(4), 403-434.

Fisher, D., & Frey, N. (2008). *Better learning through structured teaching: A framework for the gradual release of responsibility*. Alexandria, VA: Association for Supervision and Curriculum Development

Fuson, K.C. (1982). An analysis of the counting-on solution procedure in addition. In T.P. Carpenter, J.M. Moser, & T.A. Romberg (Eds,), *Addition and subtraction: A cognitive perspective* (pp. 67-81). Hillsdale, NJ: Erlbaum.

Fuson, K. (1984). More complexities in subtraction. *Journal for Research in Mathematics Education*, 15(3), 214-225.

Gravemeijer, K. (2004). Local instruction theories as means of support for teachers in reform mathematics education. *Mathematical Thinking and Learning*, *6*(2), 105-128.

Goodman, L. A. and Kruskal, William H. (1954). Measures of association for cross classifications*. Journal of the American Statistical Association, 49*(268), 732–764*.*

Guttman, L. (1950). The basis for scalogram analysis. In S. A. Stoufer et al. (Eds.), *Measurement and prediction* (pp. 60-90). Princeton, NJ: Princeton University Press.

Halliday, M. A. K. (1978). *Language as social semiotic.* London: Arnold.

Halliday, M. A., & Matthiessen, C. M. I. M. (2004). *Introduction to functional grammar*. New York: Edward Arnold.

Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support or inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education*, *28*, 524-549.

Herbst, P. (2018). On dialogue and stories as representations of practice: Introduction to the book. In R. Zazkis and P. Herbst (Eds.), *Scripting approaches in mathematics education: Mathematical dialogues in research and practice* (pp. 1-19). Cham, Switzerland: Springer.

Herbst, P., Aaron, W., and Chieu, V. M. (2013). Lesson*Sketch*: An Environment for Teachers to Examine Mathematical Practice and Learn about its Standards. In D. Polly (Ed.), *Common Core Mathematics Standards and Implementing Digital Technologies* (pp. 281-294). Hershey, PA: IGI Global.

Herbst, P., & Chazan, D. (2012). On the instructional triangle and sources of justification for actions in mathematics teaching. *ZDM Mathematics Education*, *44*(5), 601-612.

Herbst, P., & Chazan, D. (2015). Studying professional knowledge use in practice using multimedia scenarios delivered online. *International Journal of Research & Method in Education*, *38*(3), 272-287.

Herbst, P., & Chieu, V. M. (2011). Depict: A tool to represent classroom scenarios. Technical report. University of Michigan. Retrieved from http://hdl.handle.net/2027.42/87949

Herbst, P., Chazan, D., Chen, C., Chieu, V.M., and Weiss, M. (2011). Using comics-based representations of teaching, and technology, to bring practice to teacher education courses. *ZDM Mathematics Education*, *43*(1), 91–103.

Heritage, M. (2008). *Learning Progressions: Supporting instruction and formative assessment*. Washington DC: Chief Council of State School Officers.

Heritage, M., Kim, J., Vendlinski, T., & Herman, J. (2009). From evidence to action: A seamless process in formative assessment? *Educational Measurement: Issues and Practice*, *28*(3), 24-31.

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In F.K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 371-404). Greenwich, CT: Information Age.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*(2), 169-202.

Jackson, T. (2017, April 19). *Teacher Certification Changes Pending for Michigan.* Retrieved from www.mymassp.com

Kazemi, E., & Franke, M. L. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education, 7*, 203–235.

Kersting, N. (2008). Using video clips of mathematics classroom instruction as item prompts to measure teachers' knowledge of teaching mathematics. *Educational and Psychological Measurement, 68*(5), 845-861.

Kilpatrick, J. (2001). Understanding mathematical literacy: The contribution of research. *Educational Studies in Mathematics*, *47*(1), 101-116.

Leinwand, S., Brahier, D. J., Huinker, D., Berry, R. Q., III, Dillion, F. L., Larson, M. R., et al. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: National Council of Teachers of Mathematics.

Maher, C. A., & Davis, R. B. (1990). Chapter 6: Building Representations of Children's Meanings. *Journal for Research in Mathematics Education Monograph*, *4*, 79–210.

Martin, J. and Rose, D. 2003. *Working with discourse: Meaning beyond the clause*, London: Continuum.

National Governors Association Center for Best Practice (NGA Center) and the Council for Chief State School Officers (CCSSO) (2010), *Common Core State Standards for Mathematics*. Washington, DC: Author.

O'Halloran, K. L. (1999). Towards a systemic functional analysis of multisemiotic mathematics texts. *Semiotica, 124*(1/2), 1-29.

Piaget, J. (1970), *Genetic Epistemology.* New York: W. W. Norton.

Piedmont, R. L. (2014). Inter-item correlations. In A. C. Michalos (Ed.), *Encyclopedia of Quality of Life and Well-Being Research* (pp. 3303– 3304). Springer Netherlands. Retrieved from http://link.springer. com/referenceworkentry/10.1007/978-94-007-0753-5_1493.

Perrenoud, P. (1998). From formative evaluation to a controlled regulation of learning processes. Towards a wider conceptual field. *Assessment in Education: Principles, Policy & Practice*, *5*(1), 85-102.

Popham, W. J. (2008). *Transformative assessment*. Alexandria, VA: Association for Supervision and Curriculum Development.

Post, T. R., Wachsmuth, I., Lesh, R., & Behr, M. J. (1985). Order and equivalence of rational numbers: A cognitive analysis. *Journal for Research in Mathematics Education*, *16*, 18-37.

Powell, K.C., & Kalina, C.J. (2009). Cognitive and social constructivism: Developing tools for an i-effective classroom*. Education, 130*(2), 241-250.

Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. New York, NY: Routledge.

Sarama, J., & Clements, D. H. (2002). Building blocks for young children's mathematical development. *Journal of Educational Computing Research, 27*(1&2), 93-110

Schifter, D. (1999). Reasoning about operations: Early algebraic thinking in grades K–6. In L. Stiff & F. Curcio, (Eds.), *Mathematical reasoning, K-12* (1999 yearbook) (pp. 62-81). Reston, VA: National Council of Teachers of Mathematics

Schleppegrell, M. J. (2012). Academic language in teaching and learning. *The Elementary School Journal*, *112*(3), 409-418.

Secada, W. G., Fuson, K. C., & Hall, J. W. (1983). The transition from counting-all to counting-on in addition. *Journal for Research in Mathematics Education*, *14*, 47-57.

Sherin, M. G. (2001) Developing a professional vision of classroom events. In T. Wood, B. S. Nelson, and J. Warfield (Eds), *Beyond Classical Pedagogy: Teaching Elementary School Mathematics* (pp. 75–93). Hillsdale, NJ: Erlbaum.

Sherin, M. G. (2007). The development of teachers' professional vision in video clubs. In R. Goldman, R. Pea, B. Barron, & S. J. Derry (Eds.), *Video research in the learning sciences* (pp. 383–395). Mahwah, NJ: Erlbaum

Silver, E. A., Ghousseini, H., Gosen, D., Charalambous, C., & Strawhun, B. T. F. (2005). Moving from rhetoric to praxis: Issues faced by teachers in having students consider multiple solutions for problems in the mathematics classroom. *The Journal of Mathematical Behavior*, *24*(3), 287-301.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, *26*, 114-145.

Simon, M. A., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning*, *6*(2), 91-104.

Smith, E. & Gorard, S. (2005) 'They don't give us our marks': the role of formative feedback in student progress. *Assessment in Education, 12*(1), 21–38.

Star, J. R., & Rittle-Johnson, B. (2008). Flexibility in problem solving: The case of equation solving. *Learning and Instruction, 18*, 565–579.

Steffe, L. (1994). Children's multiplying schemes. In G. Harel and J. Confrey, *The development of multiplicative structures in the learning of mathematics* (pp. 3-34). Albany: SUNY Press.

Steffe, L. P. (2004). On the construction of learning trajectories of children: The case of commensurate fractions. *Mathematical Thinking and Learning*, *6*(2), 129-162.

Steffe, L. P., Cobb, P., & von Glasersfeld, E. (1988). *Construction of arithmetical meanings and strategies*. New York: Springer-Verlag.

Steffe, L.P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory, and application*. New York: Praeger.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, *33*(2), 455-488.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, *10*(4), 313-340.

Sztajn, P., Confrey, J., Wilson, P. H., & Edgington, C. (2012). Learning trajectory based instruction toward a theory of teaching. *Educational Researcher*, *41*(5), 147-156.

Vergnaud, G. (1982). A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. In T. P. Carpenter, J. M. Moser, & T. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 39-59). Hillsdale, NJ: Erlbaum.

Von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning.* Bristol, PA: Falmer Press.

Vygotsky, L.S., 1987. Zone of proximal development. In M. Cole, et al. (Eds.), *Mind in Society: The Development of Higher Psychological Processes* (pp. 52–91). Cambridge, MA: Harvard University Press.

Wilcoxon, F. (1945). Individual comparisons by ranking methods. *Biometrics Bulletin*,*1* (6), 80–83.

Wright, R. J., Martland, J., & Stafford, A. (2000). *Early numeracy: Assessment for teaching and intervention*. London: Paul Chapman Publications/Sage.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*, 458-477.