# Relationships Among Hilbert-Samuel Multiplicities, Koszul Cohomology, and Local Cohomology

by

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# GLOSSARY

Throughout this glossary, $R$ is a Noetherian ring, and $M$ is an $R$-module. Whenever we use $m$, we assume that $R$ is (Noetherian) local and $m$ its maximal ideal.

*local* means Noetherian local unless otherwise specified $\dim(M)$ is the Krull dimension of $R/\mathrm{Ann}(M)$
$e_I(R)$ is the Hilbert-Samuel multiplicity of $M$ with respect to the ideal $I$
$\mathrm{gr}_m(R)$ is the associated graded ring of $R$ with respect to $m$
$\ell(M)$ is the length of $M$
$H^i(I; M)$ is the $i^{th}$ Koszul cohomology of $M$ with respect to a minimal set of generators of $I$ and is only used when $R$ is local or nonnegatively graded
$\nu(M)$ is the least number of generators of $M$
$\mathrm{rank}(M)$ is the torsion-free rank of $M$ when $R$ is a domain
$\widehat{R}$ is the $m$-adic completion of $R$
$\widehat{M}$ is the $m$-adic completion of $M$
$M$ is *i-effaceable* if for every sequence of parameter ideals $(\underline{x}_n) = I_n \subseteq m^n$, we have $\lim\limits_{n\to\infty} \dfrac{\ell(H^i(\underline{x}_n, M))}{\ell(R/I_nR)} = 0$.
The *asymptotic depth* of $M$, denoted asydepth $M$, is $k$ if $M$ is i-effaceable for all $i < k$ and $M$ is not $k$-effaceable.
$M$ is *asymptotically Cohen-Macaulay* if asydepth $M = \dim M$.
$M$ is *Cohen-Macaulay on the punctured spectrum* if $M_P$ is Cohen-Macaulay for all primes $P \neq m$
$M$ is *equidimensional* if $\dim(R/P) = \dim(R)$ for every $P \in \min(M)$
$M$ is *quasi-unmixed* if $\widehat{M}$ is equidimensional over $\widehat{R}$
Lech-Stückrad-Vogel ratio is $\dfrac{e_I(M)}{\ell(M/IM)}$

## ABSTRACT

We consider relationships among Hilbert-Samuel multiplicities, Koszul cohomology, and local cohomology. In particular, we investigate upper and lower bounds on the ratio $\dfrac{e_I(M)}{\ell(M/IM)}$ for $m$-primary ideals $I$ of the local ring $(R, m)$ and finitely-generated quasi-unmixed $R$-modules $M$ and, in joint work with Linquan Ma, Pham Hung Quy, Ilya Smirnov, and Yongwei Yao, show that the ratio is bounded above by $\dim(R)!e(R/\operatorname{Ann}_R(M))$ for all finitely-generated $R$-modules $M$ and away from 0 whenever $M$ is quasi-unmixed. We also, as independent work, give a characterization of quasi-unmixed $R$-modules $M$ whose local cohomology is finite length up to some index in terms of asymptotic vanishing of Koszul cohomology on parameter ideals up to the same index. We show that if $M$ is an equidimensional module over a complete local ring, then $M$ is asymptotically Cohen-Macaulay if and only if $\sup\{\ell(H^i(f_1, \ldots, f_d; M)) \mid \sqrt{f_1, \ldots, f_d} = m, i < d\} < \infty$ if and only if $M$ is Cohen-Macaulay on the punctured spectrum.

# CHAPTER I

# Introduction

## 1.1 Outline

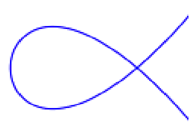
This preface consists of statements of the main results, the necessary definitions and notational conventions, and a brief sketch of others' previous work together with an explanation of how the stated results tie together. The summary of results includes statements of joint work with Linquan Ma, Pham Hung Quy, Ilya Smirnov, and Yongwei Yao. Proofs of those results can be found in Appendix A, and different proofs of special cases of those results, which give more or different information than the joint work, can be found in Chapter II of this thesis. Chapter III focuses on the proof of Theorem III.1, which is independent work.

## 1.2 An overview

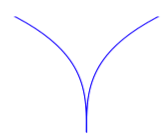
Broadly speaking, algebraic geometry is the study of solutions to polynomial equations, and commutative algebra is the study of polynomial functions on these sets of solutions. One of the primary goals of commutative algebra and algebraic geometry is detecting and managing *singularities*. In Calculus, we learn that $y = x^2$ is a very nice function, in large part because it is differentiable everywhere. We also learn that $y^2 = x^3$ is a much worse function because it has a singularity at the origin. We can see this different algebraically by the Implicit Function Theorem or geometrically by graphing the two functions and seeing that $y = x^2$ appears smooth at the origin while $y^2 = x^3$ has a sharp corner there.

Understanding singularities in broader settings is not only of theoretical interest to commutative algebraists and algebraic geometers but also has myriad applications, including phylogenetics [2, 19, 1], disclosure limitation [46, 14, 25], string theory [5], and statistics [23, 42, 22, 15, 54], for example. Mathematicians began to explore many of the tools of commutative algebra and algebraic geometry in the late nineteenth and early twentieth centuries with only the goal of understanding pure mathematics in mind. Then, in the same way that a rigorous notion of the limit suddenly found an application when Newton and Leibniz invented calculus and were then able to make vast gains in physics, the already well-developed fields of commutative algebra and algebraic geometry found a wealth of applications in the wake of the advent of computers, when the the reframing of certain biological, physical, and statistical questions in terms of algebra became computationally tractable. The goal of my research is to grow the understanding of the fields of commutative algebra and algebraic geometry so that they will be broad and deep enough to answer the questions of applied mathematicians and statisticians when they come asking.

Some primary objects of study in commutative algebra are called *rings*, which describe geometric *varieties*. For example, the ring $R = \mathbb{R}[x, y]/(y - x^2)$ refers to polynomials in two variables with real numbers as coefficients with the restriction that $y = x^2$. The solutions in the Cartesian plane to the equation $y = x^2$ cut out a parabola, and the ring of functions on this parabola is $R$. Some tools that we use in this thesis to study singularities of varieties are the *Hilbert-Samuel multiplicity* (or simply *multiplicity*) and *Koszul homology*, which can be used to compute the multiplicity. By computing multiplicity at different points on our variety, we can see whether or not the variety has a singularity at that point. The multiplicity of $R$ is 1 at the origin, and we know that the parabola does not have a singularity at the origin. This relationship is true quite broadly. Under mild hypotheses, a variety does not have a singularity at a certain point if and only if the multiplicity of its corresponding ring is 1 at that point. Large multiplicities indicate bad singularities. For example, the nodal curve graphed below has a multiplicity of 2, which captures that the graph has a mild-looking singularity at the origin, and the cusp, which is much pointier, has multiplicity 10.



nodal curve: $y^3 = x^3 + x^2$



cusp: $y^{31} = x^{10}$

Multiplicities have applications within math in areas ranging from Banach-space operators to Galois theory (see, for example, [29, 38, 11, 20, 32, 18, 21, 13]).

An *ideal* in a polynomial ring generated by some selected set $S$ of polynomials is the set of all finite sums of multiples of polynomials in $S$. Ideals can be thought of as a generalization of the notion of the set of multiples of a number in the integers. For example, the ideal $I$ generated by $\{x^5\}$ in the ring $R = \mathbb{R}[x,y]/(y-x^2)$ can be thought of as all polynomials in two variables that can be expressed as the product of $x^5$ and some other polynomial, where we are allowed to replace any $y$'s with $x^2$'s. The ring $R/I$, read "$R$ mod $I$", is the set of polynomials in two variables in which $y - x^2 = 0$, $x^5 = 0$, and all finite sums of multiples of $y - x^2$ or $x^5$ are equal to 0 as well. Setting all polynomials in $I$ equal to 0 generalizes modular arithmetic, or "clock arithmetic," in which we set some integer equal to 0, or start over counting the hours once we get to 12. Much of my research concerns the relationship between $e_I(R)$ and $\ell(R/I)$, where $e_I(R)$ denotes the multiplicity of the ideal $I$ on the ring $R$ and $\ell(R/I)$ is best thought of as the number of independent monomials after we set the polynomials in $I$ equal to 0. Precise definitions and a more sophisticated discussion appear in Subsection 1.3.

In our example, we compute $\ell(R/I) = 5$ because there are five independent monomials that do not use $y$ and are not 0 after we insist that $x^5 = 0$. Those monomials are 1, $x$, $x^2$, $x^3$, and $x^4$. It turns out that the multiplicity $e_I(R) = 5$ also. It is the multiplicity at the ideal generated by the variables (in our case $x$ and $y$) that tells us whether or not the variety corresponding to the ring has a singularity at the origin.

In this thesis, we will be particularly interested in the ratio $\dfrac{e_I(R)}{\ell(R/I)}$. In the example of the previous paragraph, we would compute $\dfrac{e_I(R)}{\ell(R/I)} = \dfrac{5}{5} = 1$. The outcome of 1 encodes the information that $R$ either has no singularity at the origin (which is actually the case) or that it has a relatively mild and well-understood sort of singularity. When $\frac{e_I(R)}{\ell(R/I)} = 1$, for certain ideals $I$, we say that $R$ is *Cohen–Macaulay*. In 1960, Lech [35] gave an upper bound on the ratio $\frac{e_I(R)}{\ell(R/I)}$ independent of $I$. A lower bound has been sought since at least 1996 [47]. Linquan Ma, Pham Hung Quy, Ilya Smirnov, Yongwei Yao, and I show that this ratio is bounded away from 0 in the greatest possible generality (Theorem I.10), solving the 21-year-old conjecture, and bounded above for all finitely-generated modules by a constant depending only on the ring (Theorem I.9), a result that extends Lech's result for rings. We also give an explanation for these bounds in terms of the Koszul homology (Theorem I.13) that computes the multiplicity. Independently, I give a new characterization of the property of being Cohen–Macaulay on the punctured spectrum in terms of Koszul homology. More generally, I describe a certain type of smallness of Koszul homology, which will be known as *asymptotic depth*, in terms of a finiteness condition on another type of cohomology known as *local cohomology* (Theorem I.14).

## 1.3 Preliminaries

All rings in this thesis will be commutative Noetherian rings with unity and all modules unital and finitely-generated, unless otherwise specified. The primary object of study throughout this thesis is *Koszul (co)homology*, especially finite-length Koszul (co)homology. Let $\underline{x} = x_1, \ldots, x_r$ be a sequence of $r$ elements of $R$, and let $G$ be a free $R$ module on $r$ free basis elements $u_1, \ldots, u_r$. Let $\wedge^i G$ denote the $i^{th}$ exterior power of $G$. The Koszul complex of $\underline{x} = x_1, \ldots, x_r$ on $R$, denoted $K_\bullet(\underline{x}; R)$ is

$$0 \to \wedge^r G \to \wedge^{r-1} G \to \cdots \to \wedge^1 G \to \wedge^0 G \to 0$$

where the map from $\wedge^n G$ to $\wedge^{n-1} G$ for $1 \leq n \leq r$ is given by

$$(u_{i_1} \wedge \cdots \wedge u_{i_n}) \mapsto \sum_{j=1}^{n} (-1)^{j+1} x_n (u_{i_1} \wedge \cdots \hat{u_{i_j}} \cdots \wedge u_{i_n}).$$

We may alternatively construct the Koszul complex $K_\bullet(\underline{x}; R)$ as

$$K_\bullet(x_1; R) \otimes_R \cdots \otimes_R K_\bullet(x_r; R)$$

where $K_\bullet(x_i; R)$ is the complex $0 \to R \xrightarrow{x_i} R \to 0$ for $1 \leq i \leq r$. The Koszul complex of $\underline{x}$ on an $R$-module $M$, denoted $K_\bullet(\underline{x}; M)$, is defined to be $K_\bullet(\underline{x}; R) \otimes M$.

We use $H_i(\underline{x}, M)$ to denote the $i^{th}$ Koszul homology module, i.e., the $i^{th}$ homology module of the Koszul complex, of a sequence $\underline{x}$ on $M$. We define the *cohomological Koszul complex* of $\underline{x}$ on $R$, denoted $K^\bullet(\underline{x}; R)$, to be $\mathrm{Hom}_R(K_\bullet(\underline{x}; R), R)$. Then the $i^{th}$ Koszul cohomology of $\underline{x}$ on the module $M$, denoted $H^i(\underline{x}; M)$, to be the $i^{th}$ cohomology of the complex $K^\bullet(\underline{x}; R) \otimes M$. Because the Koszul complex is self-dual, i.e., $K^\bullet(\underline{x}; M) \cong K_\bullet(\underline{x}; M)$, we have $H^i(\underline{x}, M) \cong H_{r-i}(\underline{x}, M)$. We make regular use throughout this thesis of the fact that the depth of $(\underline{\mathrm{x}})$ on $M$ can be characterized as the least $i$ such that $H^i(\underline{\mathrm{x}}; M) \neq 0$. These and other well-known properties of the Koszul complex and Koszul (co)homology can be found in [17, Chapter 17] and [9, Section 1.6]. When we are interested only in the isomorphism class of $H^i(\underline{x}, M)$, for example when we are interested in its length and we are working over a ring that is either local or graded, we will use $H^i(I, M)$ to mean $H^i(\underline{x}, M)$ for some $\underline{x}$ minimally generating the ideal $I$. Most often, $\underline{x}$ will be a system of parameters for the local ring $R$.

In Chapter II, we will also be interested in the *Hilbert-Samuel multiplicity* over the local ring $(R, m)$. We will use $\ell(N)$ to denote the length of an $R$-module $N$ and $\dim(M)$ to denote the Krull dimension of the $R$-module $M$, which is defined to be the Krull dimension of the ring $R/\mathrm{Ann}(M)$. If $M \neq 0$, then $\ell(M/IM) < \infty$ if and only if $\mathcal{V}(I)\cap$ Supp$(M) = \{m\}$ if and only if $\ell(M/I^tM) < \infty$ for all $t \geq 0$. If $d = \dim(M)$ and $\ell(M/IM) < \infty$, define the multiplicity of $I$ on $M$, denoted $e_I(M)$, by

$$e_I(M) = d! \lim_{t\to\infty} \frac{\ell(M/I^tM)}{t^d}.$$

In particular, $e_I(M)$ is the product of $d!$ and the leading coefficient of the Hilbert polynomial of $M$ with respect to $I$. If $I$ is generated by a system of parameters $x_1, \ldots, x_d$ on $M$, then Lech showed in [34] that it is equivalent to define the multiplicity in the following way:

$$e_I(M) = \lim_{t\to\infty} \frac{\ell(M/(x_1^t, \ldots, x_d^t)M)}{t^d}.$$

We will be particularly interested in the case where $\dim(M) = \dim(R)$ and $\sqrt{I} = m$. We will use two facts about multiplicity with great frequency. The first is that if $I$ and $J$ are two ideals with the same integral closure, then

$$e_I(M) = e_J(M). \tag{1.1}$$

The second is that if $I$ is generated by a system of parameters $x_1, \ldots, x_d$ on $M$, then for any $\alpha \geq 1$, there is an equality

$$e_{(x_1^\alpha, \ldots, x_d)}(M) = \alpha \cdot e_I(M). \tag{1.2}$$

More generally, if $(x_1, \ldots, x_d)$ and $(x_1', \ldots, x_d)$ are two systems of parameters on $M$, then $(x_1x_1', \ldots, x_d)$ is also a system of parameters on $M$, and $e_{(x_1x_1', \ldots, x_d)}(M) = e_{(x_1, \ldots, x_d)}(M) + e_{(x_1', \ldots, x_d)}(M)$. We will be particularly interested in characteristic $p > 0$ that for any $q$ a power of $p$, we have $e_I^{[q]}(M) = q^d \cdot e_I(M)$. Proofs of these and other facts about multiplicity can be found in [9, Sections 4.6-4.7] and [28, Chapter 11].

Our last major object of study, which will appear in Chapter III, will be local cohomology. We define

$$\Gamma_I(M) := \{m \in M \mid mI^t = 0 \text{ for some } t \geq 1\}.$$

One may verify that $\Gamma_I(M)$ is a submodule of $M$ and that the functor $\Gamma_I(\_\_)$ is left exact. It, therefore, has right derived functors. Call the $i^{th}$ right derived functor of $\Gamma_I(\_\_)$, denoted $H^i_I(M)$, the $i^{th}$ local cohomology module of $M$ with support in $I$. It is equivalent to define $H^i_I(M)$ in terms of Koszul cohomology. Again with $\underline{x} = x_1, \ldots, x_r$, let $I = (\underline{x})$ and $\underline{x}^t = (x_1^t, \ldots, x_r^t)$. Then

$$H^i_I(M) \cong \varinjlim_t H^i(K^\bullet(\underline{x}^t, M))$$

where the maps in the direct limit system are induced by the maps on complexes $K_\bullet(x_i; M)$ given below

$$\begin{array}{ccccccc}
0 & \longrightarrow & R & \xrightarrow{x_i^t} & R & \longrightarrow & 0 \\
 & & {\scriptstyle ID}\downarrow & & {\scriptstyle x_i}\downarrow & & \\
0 & \longrightarrow & R & \xrightarrow[x_i^{t+1}]{} & R & \longrightarrow & 0.
\end{array}$$

Local homology depends only on the radical of $I$. In particular, there is a map from each $i^{th}$ Koszul homology module on each system of parameters on $M$ to the local cohomology module $H^i_m(M)$. When $M$ is equidimensional (defined below), $\ell(H^i_m(M)) < \infty$ for all $i < \dim(M)$ if and only if $M$ is Cohen-Macaulay on the punctured spectrum. For proofs of these and other facts about local cohomology, see [30].

## 1.4 Definitions and Notation

We include below definitions and notation that are either particular to this thesis (Definitions I.1 and I.3) or about which there is not uniformity within the commutative algebra literature (Definitions I.4-I.6.). Unless otherwise specified, $R$ will denote the local (Noetherian) ring $(R, m)$, and $M$ a finitely generated $R$-module.

**Definition I.1.** We say that $M$ is *i-effaceable* if for every sequence of parameter ideals $(\underline{x}_n) = I_n \subseteq m^n$, we have $\lim_{n\to\infty} \dfrac{\ell(H^i(\underline{x}_n, M))}{\ell(R/I_nR)} = 0$.

*Remark* I.2. As a result of I.12, using $\ell(M/I_nM)$ in the denominator for the definition of i-effaceable in place of $\ell(R/I_n)$ yields an equivalent condition when $\dim(M) = \dim(R)$ and $R$ is a module-finite over a regular ring.

**Definition I.3.** The *asymptotic depth* of $M$, denoted asydepth $M$, is $k$ if $M$ is i-effaceable for all $i < k$ and $M$ is not $k$-effaceable. We say that $M$ is *asymptotically Cohen-Macaulay* if asydepth $M = \dim M$.

**Definition I.4.** We say that $M$ is *equidimensional* if $\dim(R/P) = \dim(R)$ for every $P \in \min(M)$. In particular, this condition forces $\dim(M) = \dim(R)$.

**Definition I.5.** We say that $M$ is *quasi-unmixed* if $\widehat{M}$ is equidimensional over $\widehat{R}$, where $\widehat{-}$ denotes $m$-adic completion. Again, this condition forces $\dim(M) = \dim(R)$.

**Definition I.6.** We say that $M$ is *Cohen-Macaulay on the punctured spectrum* if $M_P$ is Cohen-Macaulay for all primes $P \neq m$. This property is elsewhere called locally Cohen-Macaulay.

**Definition I.7.** We say that $M$ is *generalized Cohen-Macaulay* if $\ell(H^i_m(M)) < \infty$ for all $0 \leq i < \dim(M)$.

*Remark* I.8. Recall that whenever $M$ is equidimensional, which will quite often be an assumption throughout this thesis, the conditions *Cohen-Macaulay on the punctured spectrum* and *generalized Cohen-Macaulay* are equivalent.

### 1.5 Summary of the relevant literature and statements of main results

Lech's inequality states that for any local ring $(R, m, \kappa)$ of dimension $d$ and any $m$-primary ideal $I$, $\dfrac{e_I(R)}{\ell(R/I)} \leq d! \cdot e_m(R)$ [35, Theorem 3]. Lech's inequality is used as an invaluable tool in many areas of commutative algebra, such as in the study of the minimal number of generators of ideals [7], of first Hilbert coefficients [43], and of reduction numbers [51]. There has additionally been recent interest in improving Lech's inequality, most notably in [24] and [27].

In light of Lech's inequality, some natural questions to ask are (1) when the same or a similar inequality holds for finitely generated modules over local rings (2) when there is a lower bound for the ratio $\dfrac{e_I(M)}{\ell(M/IM)}$ for finitely generated modules $M$ over $R$, and (3) whether there is an explanation for such an inequality in terms of lengths of Koszul cohomology modules. Theorem I.9 gives an affirmative answer to question (1) in all cases.

**Theorem I.9.** *Let $M$ be a finitely generated module over the local ring $(R, m, \kappa)$ of dimension $d$. Then*

$$\frac{e_I(M)}{\ell(M/IM)} \leq d! \cdot e_m(R/Ann_R(M)).$$

Theorem I.9 is stated and proved as Theorem A.14 Appendix A.

Substantial work exists exploring questions (2) and (3). Study of the relationship between $\ell(M/IM)$ and $e_I(M)$ arose in the context of Buchsbaum modules and generalized Cohen-Macaulay modules and, separately, in generalizing Bezout's theorem [10, 3, 52, 48, 44]. It was originally conjectured by Stückrad and Vogel in 1996 that $\sup\limits_{\sqrt{I}=m} \left\{ \dfrac{\ell(M/IM)}{e_I(M)} \right\} < \infty$ whenever $M$ is quasi-unmixed, at which point they also showed that condition to be necessary [47]. In 2000, Allsop and Tuân Hoa proved the conjecture when $\dim(M) \leq 3$ or $M$ is generalized Cohen-Macaulay [3, 4].

Theorem I.10 is the general case.

**Theorem I.10.** *Let $M$ be a finitely generated quasi-unmixed module over the local ring $(R, m, \kappa)$. Then* $\sup_{\sqrt{I}=m} \left\{ \frac{\ell(M/IM)}{e_I(M)} \right\} < \infty$.

Theorem I.10 is stated and proved as Theorem A.5 in Appendix A.

We also obtain as a corollary of Theorems I.10 and I.9 control of the relationship between colengths of $m$-primary ideals and their integral closures:

**Theorem I.11.** *Let $M$ be a finitely generated quasi-unimxed module over the local ring $(R, m, \kappa)$. Then* $\sup_{\sqrt{I}=m} \left\{ \frac{\ell(M/IM)}{\ell(M/\bar{I}M)} \right\} < \infty$ *where $\bar{I}$ denotes the integral closure of $I$.*

Note that $\inf_{\sqrt{I}=m} \left\{ \frac{\ell(M/IM)}{\ell(M/\bar{I}M)} \right\} = 1$ and is achieved when $I = \bar{I}$. More generally, we may replace $\bar{I}$ with any ideal $J$ such that $\bar{I} = \bar{J}$ in the theorem statement above. We will, in particular, be interested in bounding $\frac{\ell(M/JM)}{\ell(M/IM)}$ from above when $J$ is a minimal reduction of $I$. In the more general formulation, Theorem I.11 is stated and proved as Lemma A.16 in Appendix A.

An additional corollary of Theorems I.10 and I.9 follows:

**Corollary I.12.** *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring that is either equal characteristic or in which char$(\kappa)$ is a parameter, and let $M$ be a finitely-generated $R$-module. Then* $\sup_{\sqrt{I}=m} \left\{ \frac{\ell(R/I)}{\ell(M/IM)} \right\} < \infty$.

Corollary I.12 is stated and proved as Corollary A.20.

Question (3) arises in light of the following equality due to Serre [45] and Lech's limit formula, which was generalized to the module case by Northcott [34, 40]. Fix a local ring $R$ of dimension $d$ and a parameter ideal $(x_1, \ldots, x_d)$ of $R$. Serre's result

[45] is that

$$(1.3) \qquad e_I(R) = \sum_{i=0}^{d} (-1)^i \ell(H^{d-i}(x_1, \ldots, x_d; R)).$$

The Lech-Northcott limit formula states that

$$\lim_{min_i\{n_i\}\to\infty} \frac{e_{(x_1^{n_1},\ldots,x_d^{n_d})}(M)}{\ell(M/(x_1^{n_1}, \ldots, x_d^{n_d})M)} = 1.$$

Because $H^d(x_1, \ldots, x_d; M) \cong M/(x_1, \ldots, x_d)M$, we are led to two possibilities: either the lengths of the higher Koszul cohomology modules on $(x_1^{n_1}, \ldots, x_d^{n_d})$ are all small relative to the length of $M/(x_1^{n_1}, \ldots, x_d^{n_d})M$ when the $n_i$ are large or the lengths of the cohomology modules are arbitrarily large relatively to the length of $M/(x_1^{n_1}, \ldots, x_d^{n_d})M$ but close in size to each other so that their lengths cancel in the alternating sum. A result of Kirby's is that when $M/(x_1, \ldots, x_r)M$ is finite length, $\sup_{\substack{r\geq d\\ i\leq r\\ n_j>0}} \left\{ \frac{\ell(H^i(x_1^{n_1}, \ldots, x_r^{n_r}; M))}{n_1 \cdots n_r} \right\} < \infty$ [31], which one might interpret as evidence for the former explanation. Theorem I.13 demonstrates that that explanation is, in fact, correct when $M$ is quasi-unmixed and represents a generalization of Kirby's result.

**Theorem I.13.** *[33] Let $(R, m, \kappa)$ be a Noetherian local ring of dimension $d$ and $M$ a finitely generated quasi-unmixed $R$-module. Then for every $k \geq 0$,*

$$\sup_{\substack{\sqrt{(x_1,\ldots,x_{d+k})}=m\\ 0\leq i\leq d+k}} \left\{ \frac{l(H_i(x_1, \ldots, x_d; M))}{l(M/(x_1, \ldots, x_{d+k})M)} \right\} < \infty.$$

Another natural question is whether the Lech-Northcott formula holds when the sequence of parameter ideals given by powers of a fixed system of parameters is replaced by any sequence of parameter ideals in increasingly high powers of the maximal ideal. Example III.9 demonstrates that it does not in general. Moreover, Theorem III.1 gives, under mild hypotheses, precise conditions on when $\lim_{n\to\infty} \frac{\ell(H^i(I_n, M))}{\ell(M/I_nM)} = 0$

for all sequences of parameter ideals $I_n \subseteq m^n$ and all $i < \dim(M)$, a condition that directly implies that $\lim_{n\to\infty} \dfrac{e_{I_n}(M)}{\ell(M/I_nM)} = 1$ by Serre's formula (1.3).

**Theorem I.14.** *If $M$ is an equidimensional module over the complete local ring $(R, m, \kappa)$ of dimension $d \geq 1$, then the following three conditions are equivalent:*

1. *asydepth$(M) \geq k$,*
2. $\sup\{\ell(H^i(f_1,\ldots,f_d;M)) \mid \sqrt{f_1,\ldots,f_d} = m,\ i < k\} < \infty$,
3. *$\ell(H^i_m(M)) < \infty$ for all $i < k$.*

Note: It is separately well-known that condition 3 is equivalent to $\mathrm{depth}_P M_P \geq \mathrm{height}(P) + k - d$ for all prime ideals $P \neq m$. It is surprising that requiring that certain Koszul cohomology modules grow slowly, as in the definition of aymptotic depth, actually forces it them to have finite length.

Theorem I.14 is stated and proved as Theorem III.1 in Chapter III.

# CHAPTER II

# Bounds on the Lech-Stückrad-Vogel Ratio

Let $(R, m, \kappa)$ be a local ring and $M \neq 0$ a finitely generated $R$-module. We call $\dfrac{e_I(M)}{\ell(M/IM)}$ the Lech-Stückrad-Vogel ratio. This chapter discusses upper and lower bounds on the Lech-Stückrad-Vogel ratio as well as the ratios $\dfrac{\ell(H^i(I;M))}{\ell(M/IM)}$ as $I$ varies among $m$-primary ideals. Lemma II.1 reduces the mixed characteristic case of Lech's inequality to the equal characteristic case. Hanes gave a novel argument in the equal characteristic case [24], and so this lemma combines with that work to recover Lech's result. The remainder of this chapter provides alternative proofs of the primary results from the paper "Lech's inequality and Stückrad-Vogel's conjecture," which is joint work with Linquan Ma, Pham Hung Quy, Ilya Smirnov, and Yongwei Yao, in special cases. The proofs of the special cases are either simpler than in the joint paper or give more information on the bounds of interest.

## 2.1 On Lech's inequality

**Lemma II.1.** *Let $(R, m, \kappa)$ be a local ring and $I$ an $m$-primary ideal. Let $S = \mathrm{gr}_m R$, and let $n$ be the maximal idea of $S$. Then for each $m$-primary ideal $I$ or $R$, there exists a homogeneous $n$-primary ideal $J$ such that* $\dfrac{e_I(R)}{\ell(R/I)} \leq \dfrac{e_J(S)}{\ell(S/J)}$.

*Proof.* Let

$$J := \frac{I+m^2}{m^2} \oplus \frac{(m^2 \cap I)+m^3}{m^3} \oplus \frac{(m^3 \cap I)+m^4}{m^4} \oplus \cdots ,$$

and set $n = mS$. We note that $\dim(R) = \dim(S)$. We also note that $e_m(R) = e_n(S)$ because for each $t$, $\ell(S/n^t) = \ell(R/m \oplus m/m^2 \oplus \cdots m^{t-1}/m^t) = \ell(R/m^t)$. We further claim that $\ell(R/I) = \ell(S/J)$. We observe that

$$\frac{S}{J} = \frac{R}{m} \oplus \frac{m}{(I \cap m)+m^2} \oplus \frac{m^2}{(I \cap m^2)+m^3} \oplus \cdots \oplus \frac{m^r}{(I \cap m^r)+m^{r+1}}$$

where $r$ is the least positive integer such that $m^{r+1} \subseteq I$. It is clear that $\ell(S/J)$ is the sum of the lengths of the direct summands given above. But $\ell(R/I)$ is also the sum of the lengths of those summands because they are the factors in an $m$-adic filtration of $R/I$. Each such factor is naturally described as $\dfrac{m^k}{(I+m^{k+1}) \cap m^k}$, but $(I \cap m^k) + m^{k+1} = (I + m^{k+1}) \cap m^k$.

Lastly, we must show that $e_I(R) \leq e_J(S)$. To do this, we observe that for each $t$, $\ell(R/I^t)$ is given by the sum

$$\ell\left(\frac{R}{I^t}\right) = \ell\left(\frac{R}{m}\right) + \ell\left(\frac{m}{I^t+m^2}\right) + \ell\left(\frac{m^2}{(m^2 \cap I^t)+m^3}\right) + \cdots$$

because these are the factors of an $m$-adic filtration of $M/I^tM$. Meanwhile,

$$[J^t]_k = \frac{(I+m)^{a_1}(m^2 \cap I + m^3)^{a_2}(m^3 \cap I + m^4)^{a_3} \cdots (m^f \cap I + m^{f+1})^{a_f}}{m^{k+1}}$$

some $f \in \mathbb{N}$ where $a_1 + 2a_2 + \cdots + fa_f = k$. Each term of $[J^t]_k$ has degree at least $k$ because of the constraint on the sum of the $a_i$. Every term $b$ of the numerator of the right-hand side above has $f$ factors of the form $i + p_u$ for $i \in I \cap m^{u-1}$ and $p_u \in m^u$. If any such factor has $p_u = 0$ for all $u$, then $b \in I^t$. But if any $p_u \neq 0$, then every term in the expansion of $b$ divisible by $p_u$ is in $m^{k+1}$. It follows that $b \in I^t + m^{k+1}$. Therefore, $(I+m)^{a_1}(m^2 \cap I + m^3)^{a_2}(m^3 \cap I + m^4)^{a_3} \cdots (m^f \cap I + m^{f+1})^{a_f} \subseteq (I^t + m^{k+1}) \cap m^k =$

$(m^k \cap I^t) + m^{k+1}$. It follows that $\ell([S/J^t]_k) \geq \ell\left(\frac{m^k}{(m^k \cap I^t) + m^{k+1}}\right)$ for every $t \geq 0$ and every $k \geq t$, from which it follows that $\ell(S/J^t) \geq \ell(R/I^t)$ and so that $e_J(S) \geq e_I(R)$, completing the proof. $\square$

Because $\mathrm{gr}_m R$ contains the field $R/m$, the above lemma reduces Lech's inequality to the equal characteristic case. We now work towards special cases of Lech's inequality for finitely generated $R$-modules in II.2 through II.4.

Let $R$ be any ring, $M$ be a finitely-generated $R$-module, and $I$ an ideal of $R$ such that $\ell(R/I) < \infty$. Let $\nu(M)$ denotes the minimal number of generators of $M$ as an $R$-module. Then by tensoring the surjection $R^{\nu(M)} \to M$ with $R/I$, it is immediate from the right exactness of the tensor that $\ell(M/IM) \leq \nu(M) \cdot \ell(R/I)$. When $R$ is a domain, let $\mathrm{rk}_R(M)$ denote the torsion-free rank of $M$ as an $R$-module.

**Lemma II.2.** *Let $(R, m, \kappa)$ be a complete local domain of dimension $d$, $M$ a finitely generated $d$-dimensional module over $R$, and $I$ an $m$-primary ideal of $R$. Suppose either that*

1. *$M = \mathcal{A} \subseteq R$ is generated by a regular sequence or*

2. *$R$ is unramified regular.*

*Then $rk_R(M) \cdot \ell(R/I) \leq \ell(M/IM)$.*

*Proof.* If $d = 0$, then $R$ is a field, in which case the claim is trivial, and so we assume $d > 0$. We first reduce to the case where $M$ is torsion-free. Let $T$ be the torsion submodule of $M$. Because $M \twoheadrightarrow M/T$ and $\mathrm{rk}_R(M) = \mathrm{rk}_R(M/T)$, it is sufficient to show that $\ell(M/T \otimes_R R/I) \geq h \cdot \ell(R/I)$. Let $h = \mathrm{rk}_R(M)$. Assuming $M$ is torsion-free, we now consider the short exact sequence

$$0 \to M \to R^h \to C \to 0,$$

where $C$ is a torsion module. We now tensor the above sequence with $R/IR$ to obtain

$$0 \to \mathrm{Tor}_1(C, R/IR) \to M/IM \to (R/IR)^h \to C/IC \to 0. \tag{2.1}$$

Serre showed that $\dim(C) < d$ implies that $\chi(C, R/I) = 0$ [45], , and theorems of Lichtenbaum's [36, Theorems 1 and 2] imply that if either $C = R/\mathcal{A}$ with $\mathcal{A}$ generated by a regular sequence or if $R$ is unramified regular, then $\chi_i(M, R/I) \geq 0$ for $i > 0$. It follows that

$$\ell(C/IC) - \ell(\mathrm{Tor}_1(C, R/IR)) = \chi_0(C, R/IR) - \chi_2(C, R/IR) = -\chi_2(C, R/IR) \leq 0$$

or, equivalently, that $\ell(C/IC) \leq \ell(\mathrm{Tor}_1(C, R/IR)$. Hence, using 2.1

$$h \cdot \ell(R/IR) \leq \ell(M/IM),$$

as desired. □

**Lemma II.3.** *If $(R, m, \kappa)$ is any local ring, then $\ell(R/I) \leq \ell(m/Im)$ for all $m$-primary ideals $I \leq 0$ of $R$. The restriction $I \neq 0$ is only needed when* $\dim(R) = 0$.

*Proof.* When we tensor short exact sequence $0 \to m \to R \to \kappa \to 0$ with any $m$-primary ideal $I$, we obtain $0 \to \mathrm{Tor}_1(R/I, R/m) \to m/Im \to R/I \to \kappa \to 0$, from which it follows that $\ell(m/Im) = \ell(R/I) + \ell(\mathrm{Tor}_1(R/I, R/m)) - \ell(\kappa) \geq \ell(R/I)$ because $\mathrm{Tor}_1(R/I, R/m) \neq 0$. □

**Theorem II.4.** *If the conditions of II.2 are satisfied or if $M = m$, then*

$$\frac{e_I(M)}{\ell(M/IM)} \leq d! \cdot e_m(R).$$

*Proof.* We compute $e_I(M) = h \cdot e_I(R) \leq d! \cdot e_m(R) \cdot h \cdot \ell(R/I) \leq d! \cdot e_m(R) \cdot \ell(M/IM)$ where the last inequality follows from II.2 or II.3. □

## 2.2 Upper bounds on the ratios $\dfrac{\ell(H^i(I;M))}{\ell(M/IM)}$ and $\dfrac{\ell(M/IM)}{e_I(M)}$

For the remainder of Chapter II, we will give bounds on the above ratios in special cases. Recall from Appendix A that both are bounded whenever $M$ is quasi-unmixed. In all of the questions described in the remainder of this chapter, we may replace the local ring $(R, m)$ by $R(t)$ so that the residue field of $R$ is infinite. This replacement can only increase the suprema we study, and so we will always assume in the proofs throughout the remainder of this chapter that the residue field of $R$ is infinite, which guarantees that every minimal reduction of every $m$-primary ideal $R$ is generated by $\dim(R)$ elements [28].

We reproduce from [33] a key lemma for the convenience of the reader.

**Lemma II.5.** *[33] Let $(R, m)$ be a local ring and $M$ a finitely generated $R$-module. If $(y_1, \ldots, y_n) \subseteq (x_1, \ldots, x_d)$ are $m$-primary ideals of $R$, then for all $0 \leq i \leq d$,*

$$l(H_i(x_1, \ldots, x_d; M)) \leq \sum_{k=0}^{d} \binom{d}{k} l(H_{i-k}(y_1, \ldots, y_n; M))$$

*Proof.* If $\underline{f} = f_1, \ldots, f_s$ is any sequence of elements of $R$ and $\underline{f}^- = f_1, \ldots, f_{s-1}$, then there is a short exact sequence for each $0 \leq i \leq s-1$

$$0 \to \frac{H_i(\underline{f}^-; M)}{f_s H_i(\underline{f}^-; M)} \to H_i(\underline{f}; M) \to \mathrm{Ann}_{H_{i-1}(\underline{f}^-;M)}(f_s) \to 0.$$

Using that each $y_j \in (x_1, \ldots, x_d)$ so that $y_j H_i(x_1, \ldots, x_d, y_1, \ldots, y_{j-1}; M) = 0$ for $1 \leq j \leq n$ and $y_1 H_i(x_1, \ldots, x_d; M) = 0$, it follows from the first injection in the short exact sequence above that

$$l(H_i(x_1, \ldots, x_d; M)) \leq l(H_i(x_1, \ldots, x_d, y_1; M)) \leq s \leq l(H_i(x_1, \ldots, x_d, y_1, \ldots, y_n; M)).$$

Now using that for $1 \leq j \leq d-1$,

$$l\left(\frac{H_i(x_1, \ldots, x_j, y_1, \ldots, y_n; M)}{x_{j+1} H_i(x_1, \ldots, x_j, y_1, \ldots, y_n; M)}\right) \leq l(H_i(x_1, \ldots, x_j, y_1, \ldots, y_n; M))$$

and that

$$l\left(\frac{H_i(y_1,\dots,y_n;M)}{x_1H_i(x_1,y_1,\dots,y_n;M)}\right) \le l(H_i(x_1,y_1,\dots,y_n;M)),$$

$$\begin{aligned} l(H_i(x_1,\dots,&x_d,y_1,\dots,y_n;M)) \\ &\le l(H_i(x_1,\dots,x_{d-1},y_1,\dots,y_n;M)) + l(H_{i-1}(x_1,\dots,x_{d-1},y_1,\dots,y_n;M)). \end{aligned}$$

and that, by iterating, the previous expression is bounded above by

$$\sum_{k=0}^{d}\binom{d}{k} l(H_{i-k}(y_1,\dots,y_n;M)),$$

completing the proof. □

**Lemma II.6.** *Fix a quasi-unmixed local ring $(R,m,\kappa)$ of dimension $d$. If $\dfrac{\ell(H_i(J;R))}{\ell(R/J)}$ is bounded for all $0\le i\le d$ and all parameter ideals $J$, then $\dfrac{\ell(H_i(I;R))}{\ell(R/I)}$ is bounded for all $0\le i\le d+k$ and all $m$-primary ideals $I$ with at most $d+k$ generators.*

*Proof.* Fix an $m$-primary ideal $I$ minimally generated by at most $d+k$ generators and an index $i$. Let $I_0$ be a minimal reduction of $I$. By Lech's inequality, 1.3, and our hypotheses,

$$\ell(R/I) \ge \frac{e_I(R)}{d!e_m(R)} = \frac{e_{I_0}(R)}{d!e_m(R)} \ge C_R\cdot \ell(R/I_0)$$

for some positive constant $C_R$, which exists by I.10. A bound from above for $\dfrac{\ell(H_i(I;R))}{\ell(H_i(I_0;R))}$ for each $0\le i\le d+k$ independent of $I$ follows from II.5. It is clear that if $I$ is generated by fewer than $d+k$ elements, the bound is only smaller, so it is sufficient to consider the case in which $I$ is generated by exactly $d+k$ elements. We combine the two bounds described above to see

$$\frac{\ell(H^i(I;R))}{\ell(R/I)} \le C_R\cdot\frac{\ell(H^i(I;R))}{\ell(R/I_0)} \le C_R\cdot\frac{\sum_{j=0}^{k}\binom{k}{j}\ell(H^{i+j}(I_0;R))}{\ell(R/I_0)},$$

which is bounded independent of $I_0$ by assumption. □

**Lemma II.7.** *[16, discussion following the Special Lemma] Let $R$ be a ring of characteristic $p > 0$ and $N$ a finitely generated $R$-module of dimension $d$. There exist prime cyclic filtrations of the $N^{1/p^e}$ for $j \geq 0$ and a constant $C_N$ such that the only primes occurring in any of these filtrations are a finite family $P_1, \ldots, P_k$ and such that $R/P_i$ occurs at most $C_N p^{ed}$ times in the filtration of $N^{1/p^e}$ for $1 \leq i \leq k$. Furthermore, whenever $\dim(R/P_i) = d$, $R/P_i$ occurs at most $p^{ed}$ times in the prime cyclic filtration of $N^{1/p^e}$ for each $e \geq 0$.*

An precise explanation of how this statement follows from Dutta's discussion can be found in [6].

**Theorem II.8.** *If $M$ is a finitely generated module over the local ring $(R, m, \kappa)$ of equal characteristic $p > 0$, then for each $k \geq 0$,*

$$\sup_{\substack{\sqrt{I}=m \\ \nu_R(M)\leq d+k \\ 0\leq i\leq d}} \left\{ \frac{\ell(H^i(I;M))}{\ell(R/I)} \right\} < \infty.$$

*Proof.* The question is unaffected by completion, and so we assume that $M = \hat{M}$. Let $R/P_1, \ldots, R/P_n$ be the (not necessarily distinct) factors appearing in a prime cyclic filtration of $M$. We note that for each $0 \leq i \leq d$ and every $m$-primary ideal $I$,

$$\frac{\ell(H^i(I;M))}{\ell(R/I)} \leq \frac{\sum_{j=1}^{n} \ell(H^i(I;R/P_j))}{\ell(R/I)} \leq \sum_{j=1}^{n} \frac{\ell(H^i(I;R/P_j))}{\ell(R/(I+P_j))}$$

we may assume that $M = (R, m, \kappa)$, a $d$-dimensional local domain, which we have taken to be complete, and that every module of dimension $< d$ satisfies the outcome of the theorem. By II.6, it suffices to consider the case $k = 0$, and so we fix a parameter ideal $I$.

We consider the map $0 \to R^{p^d} \to R^{1/p} \to N \to 0$ with $\dim(N) < d$. The long exact sequence for Koszul cohomology yields for each $i \geq 0$

$$H^{i-1}(I;N) \to H^i(I;R^{p^d}) \to H^i(I;R^{1/p})$$

from which we see

$$\ell(H^i(I;R)) \leq \frac{\ell(H^{i-1}(I;N))}{p^d} + \frac{\ell(H^i(I;R^{1/p})}{p^d} = \frac{\ell(H^{i-1}(I;N))}{p^d} + \frac{\ell(H^i(I^{[p]};R))}{p^d}.$$

By iterating, we find that for each $e \geq 1$

$$\begin{aligned}\ell(H^i(I;R)) &\leq \left(\sum_{j=0}^{e} \frac{\ell(H^{i-1}(I^{[p^j]};N))}{p^{(j+1)d}}\right) + \frac{\ell(H^i(I^{[p^e]};R))}{p^{ed}} \\ &= \left(\sum_{j=0}^{e} \frac{\ell(H^{i-1}(I;N^{1/p^j}))}{p^{(j+1)d}}\right) + \frac{\ell(H^i(I^{[p^e]};R))}{p^{ed}}.\end{aligned}$$

Because $d' \leq d-1$, it follows from II.7

$$\begin{aligned}\sum_{j=0}^{e} \frac{\ell(H^{i-1}(I;N^{1/p^j}))}{p^{(j+1)d}} &\leq \sum_{r=1}^{k} \left(C_N \cdot \sum_{j=0}^{e} \frac{p^{j(d-1)} \cdot \ell(H^{i-1}(I;R/P_r))}{p^{(j+1)d}}\right) \\ &= \sum_{r=1}^{k} \left(C_N \cdot \sum_{j=0}^{e} \frac{\ell(H^{i-1}(I;R/P_r))}{p^{j+d}}\right).\end{aligned}$$

Because each $\dim(R/P_r) < d$, by the inductive hypothesis there exists some $B_r$ such that $\frac{\ell(H^{i-1}(I;R/P_r)}{\ell(R/I)} \leq \frac{\ell(H^{i-1}(I;R/P_r)}{\ell(R/(I+P_r))} \leq B_r$ independent of $I$. Set $B = \max_r B_r$. Separately, $\lim_{e\to\infty} \frac{\ell(H^i(I^{[p^e]};R))}{p^{ed}} = 0$, which is shown in forthcoming work of Bhatt, Hochster, and Ma [6]. Hence, for each parameter ideal $I$ and each $0 \leq i < d$,

$$\begin{aligned}\frac{\ell(H^i(I;R))}{\ell(R/I)} &\leq \lim_{e\to\infty} \left(C_N \cdot \left(\sum_{j=0}^{e} \frac{k \cdot B}{p^{d+j}}\right) + \frac{\ell(H^i(I^{[p^e]};R))}{p^{ed}}\right) \\ &= \frac{C_n \cdot k \cdot B}{p^d(1-1/p)} + 0 = \frac{C_N \cdot k \cdot B}{p^d(1-1/p)},\end{aligned}$$

where $k \cdot B$ depends only on the cokernel of the map $R^{p^d} \hookrightarrow R^{1/p}$ and not on $I$. $\square$

**Corollary II.9.** *If $i = 1$, then the same proof, together with the fact that $H^0(I;N) \hookrightarrow H^0_m(N)$ for each parameter ideal $I$, shows that*

$$\ell(H^1(I;R)) \leq \sum_{j=0}^{\infty} \left(\frac{\ell(H^0_m(N))}{p^{d+j}}\right) = \frac{\ell(H^0_m(N))}{p^d(1-1/p)}.$$

**Lemma II.10.** *[3] Fix a module $M$ over $(R, m, \kappa)$ with* $\dim(R) = \dim(M)$ *and* $|\kappa| = \infty$. *Then* $\sup_{\sqrt{I}=m} \left\{ \frac{\ell(M/IM)}{e_I(M)} \right\} = \sup_{I \text{ parameter}} \left\{ \frac{\ell(M/IM)}{e_I(M)} \right\}$.

*Proof.* The inequality $\sup_{\sqrt{I}=m} \left\{ \frac{\ell(M/IM)}{e_I(M)} \right\} \geq \sup_{I \text{ parameter}} \left\{ \frac{\ell(M/IM)}{e_I(M)} \right\}$ is automatic because parameter ideals are $m$-primary. To see the other inequality, fix an $m$-primary ideal $I$ and a minimal reduction $J$ of $I$, which will be parameter because $\kappa$ is infinite. If $\frac{\ell(M/JM)}{e_J(M)} \leq C$, then using 1.3

$$\ell(M/IM) \leq \ell(M/JM) \leq C \cdot e_J(M) = C \cdot e_I(M).$$

□

**Corollary II.11.** *With notation as above, if* $\frac{C_n \cdot k \cdot B}{1 - 1/p} < 1$, *then* $\inf_{\sqrt{I}=m} \frac{e_I(R)}{\ell(R/I)} > 0$.

*Proof.* If $I$ is parameter, then if follows from Serre's expansion of multiplicity as the alternating sum of the lengths of Koszul homology modules 1.3 and the positivity of $\chi_1(I; R) = \sum_{i=1}^{d} (-1)^{i+1} \ell(H_1(I; R))$ [45] that, with the same notation as above, $\frac{e_I(R)}{\ell(R/I)} \geq 1 - \frac{C_n \cdot k \cdot B}{1 - 1/p}$. If $\frac{C_n \cdot k \cdot B}{1 - 1/p} < 1$, then we obtain a bound on $\frac{e_I(R)}{\ell(R/I)}$ from below whenever $I$ is parameter. By II.10, this bound in the case when $I$ is parameter is also a bound whenever $I$ is $m$-primary. □

# CHAPTER III

## Characterizing finite length local cohomology in terms of bounds on Koszul homology

The purpose of this chapter is to prove Theorem III.1:

**Theorem III.1.** *If $M$ is an equidimensional module over the complete local ring $(R, m, \kappa)$ of dimension $d \geq 1$, then the following three conditions are equivalent:*

1. *asydepth($M$) $\geq k$,*
2. $\sup\{\ell(H^i(f_1, \ldots, f_d; M)) \mid \sqrt{f_1, \ldots, f_d} = m,\ i < k\} < \infty$,
3. $\ell(H^i_m(M)) < \infty$ *for all* $i < k$.

We note that condition 2, which requires an absolute bound on the lengths of Koszul homology modules, is prima facie much stronger than condition 1, which merely requires that these lengths grow somewhat slowly. For that reason, their equivalence is quite surprising.

### 3.1 The regular case

We first address the case in which either $R$ contains a field or in which the characteristic of $\kappa$ is a parameter in $R$. We will then separately discuss the modifications necessary for the remaining mixed-characteristic case. In the former case, we will begin by showing that over a large class of complete rings if $M$ is asymptotically

Cohen-Macaulay, then $M$ is generalized Cohen-Macaulay. We first reduce to the case where $R$ is regular and then show that if $M$ is asymptotically Cohen-Macaulay, then $M$ must be torsion-free. Once we have that $M$ is torsion-free, we show that we may quotient by a non-zero element of $M$ and preserve the asymptotically Cohen-Macaulay property, at which point we are prepared to show that $M$ is generalized Cohen-Macaulay by induction.

**Lemma III.2.** *If it is true that all $M$ that are asymptotically Cohen-Macaulay over a complete unramified regular local ring are also generalized Cohen-Macaulay, then the same holds over any complete local ring $R$ of mixed characteristic with the characteristic of the residue field a parameter in $R$ or of equal characteristic.*

*Proof.* Suppose $M$ is asymptotically Cohen-Macaulay over $R$ satisfying the hypotheses of the theorem. By Cohen's structure theorem, $R$ is module finite over an unramified complete regular local ring $S$. Every parameter ideal $I_n$ of $S$ is a parameter ideal in $R$, and every finitely generated $R$-module $M$ is also a finitely generated $S$-module, and $H^i_R(I_n; M) = H^i_S(I_n; M)$. Because $\ell(R/I_n) \leq \nu_S(R) \cdot \ell(S/I_n)$ for each $n \geq 1$, where $\nu_S(R)$ denotes the minimal number of generators of $R$ as an $S$ module, and so $\dfrac{\ell(H^i(I_n; M))}{\ell(S/I_n)} \leq \nu_S(R) \cdot \dfrac{\ell(H^i(I_n; M))}{\ell(R/I_n)} \xrightarrow{n\to\infty} 0$, which is to say that $M$ is asymptotically Cohen-Macaulay over $S$. By assumption, then, $M$ is generalized Cohen-Macaulay over $S$, from which it follows that $M$ is generalized Cohen-Macaulay over $R$. $\square$

**Lemma III.3.** *If $R = k[[x_1, \ldots, x_d]]$ with $d \geq 3$, then $M = (x_1, \ldots, x_{d-1})R$ is not asymptotically Cohen-Macaulay. If $R = V[[x_1, x_2, \ldots, x_{d-1}]]$ where $V = (V, p, k)$ is a complete discrete valuation ring, then neither of $M = (p, x_1, \ldots, x_{d-2})$ nor $M = (x_1, \ldots, x_d)$ is asymptotically Cohen-Macaulay. Furthermore, if $M = (p^s, x_1, \ldots, x_{d-2})$*

*for some $s > 1$ is not asymptotically Cohen-Macaulay. In all cases, $N = R/M$ is also not asymptotically Cohen-Macaulay.*

*Proof.* Before giving the general proof, we will compute in detail the case of $d = 3$ with $R = k[[x, y, z]]$ as a guiding example: Let

$$I_n = (z^{n^4} - z^n x^n, y^n - z^n x, x^{n+1} - xz^{n^4-n} + yz^n).$$

It is easy to see that $\ell(R/(I_n + (x, y))) = n^4$, and the computation below will show that $\ell(R/I_n) \leq (n^4 + 2n) + (2n+1)^2(3n)$. We claim that $x^{2n+1}, y^{2n+1}, yz^{2n}, xz^{3n}$, and $z^{n^4+2n}$ are elements of $I_n$, and so the elements $z^i$ with $i < n^4 + 2n$ and $z^i y^j z^k$ with $i < 3n$; $j$, $k < 2n + 1$ span the quotient $R/I_n$ as a $k$ vector space (though they will not in general form a basis).

The claimed inclusions can be seen in the following identities:

$$yz^{2n} = z^n(x^{n+1} - xz^{n^4-n} + yz^n) + x(z^{n^4} - z^n x^n)$$

$$xz^{3n} = -z^{2n}(y^n - z^n x) + y^n z^{2n}$$

$$x^{n+1} + yz^n = (x^{n+1} - xz^{n^4-n} + yz^n) + z^{n^4-4n}(xz^{3n})$$

$$z^n x^{n+1} = z^n(x^{n+1} + yz^n) - yz^{2n}$$

$$x^n y^n = x^n(y^n - z^n x) + z^n x^{n+1}$$

$$y^{2n+1} = y^{n+1}(y^n - z^n x) + y^n x(yz^n + x^{n+1}) - x^2(x^n y^n)$$

$$x^n yz^n = -y(z^{n^4} - z^n x^n) + yz^{n^4}$$

$$x^{2n+1} = x^n(x^{n+1} - xz^{n^4-n} + yz^n) + x^{n+1}z^{n^4-n} - x^n yz^n$$

$$z^{n^4+2n} = z^{2n}(z^{n^4} - z^n x^n) - x^n z^{3n}$$

Elements on the left can be seen to be in $I_n$ as they are put in terms of elements already known to be in $I_n$ on the righthand side of each equation.

Because $R$ is regular, $H^i(I_n; R) = 0$ for $i < 3$, and so the long exact sequence of Koszul cohomology yields

$$0 \to H^1(I_n; k[[z]]) \to H^2(I_n; M) \to 0.$$

Because $z^{n^4}$ is not a zerodivisor on $k[[z]]$ and $I_n = (z^{n^4}, 0, 0)$, we have that $H^1(I_n; N) \cong H^0(0, 0; k[[z]]/(z^{n^4}) \cong R/(I_n + (x, y))$, whose length is $n^4$. From the long exact sequence above, $n^4$ must also be the length of $H^2(I_n; M)$. Now because

$$\lim_{n\to\infty} \frac{\ell(H^2(I_n; M))}{\ell(R/I_n)} \geq \lim_{n\to\infty} \frac{n^4}{n^4 + 2n + (2n+1)^2(3n)} = 1 > 0,$$

$M$ is not 2-effaceable and so in particular is not asymptotically Cohen-Macaulay.

We now prove the cases in which $p^s$ with $s > 1$ is not a generator of $M$ in all dimensions $\geq 3$. Let $R = k[[x, y, z, v_1, \ldots, v_{d-3}]]$, $M = (x, y, v_1, \ldots, v_{d-3})R$, or $R = V[[y, z, v_1, \ldots, v_{d-3}]]$ and $M = (p, y, v_1, \ldots, v_{d-3})$, in which case we will denote $p$ by $x$ below, or $R = V[[x, y, v_1, \ldots, v_{d-3}]]$ and $M = (x, y, v_1, \ldots, v_{d-3})$ in which case we will denote $p$ by $z$ below. In all cases, take $N = R/M$. From the short exact sequence $0 \to M \to R \to N \to 0$ and the fact that $R$ is regular, we know $H^i(I_n; R) = 0$ for all $i < d$, and so $H^i(I_n; M) \cong H^{i-1}(I_n; k[[z]])$ or $\cong H^{i-1}(I_n; V)$ for all $i \leq d-1$. We aim to show that $N$ is not 1-effaceable and so that $M$ is not 2-effaceable.

We define $I_n = (f_1, \ldots, f_d)$ where $f_1 = z^t - z^n x^n$, $f_2 = x^{n+1} - xz^{t-n} + yz^n$, $f_3 = y^n + v_1 z^n - v_2 z^n + \cdots + (-1)^{i+1} v_i z^n + \cdots + (-1)^{d-2} v_{d-3} z^n + (-1)^{d-3} xz^n$, and $f_{i+3} = v_i^n - v_i z^{t-n} + (v_{i+1} - v_{i+2} + \cdots + (-1)^{d+i}(v_{d-3}) + (-1)^{d+i+1}(x))^n - v_i z^n + x^n$ for $1 \leq i \leq d-3$ and some $t \in \mathbb{N}$.

As in the 3-dimensional case, we may use the first two equation to show that $yz^{2n}$ and $x^{n+1}z^n - xz^t \in I_n$. We then find

$$(\sum_{j=1}^{d-3}(-1)^{j+1}v_iz^{3n}) + (-1)^{d-1}(xz^{3n}) = z^{2n}f_3 \in I_n.$$

We will now show by induction on $i$ that $v_iz^{3n^{i+1}+n^i+\cdots+n}$ and

$$(\sum_{j=i+1}^{d-3}(-1)^{j+1}v_jz^{3n^2+n}) + (-1)^{d-j}(xz^{3n^2+n})$$

are elements of $I_n$ for all $1 \leq i \leq d-3$. If $i = 1$, then, using that modulo $I_n$

$$v_1z^{3n} \equiv (\sum_{j=2}^{d-3}(-1)^{j+1}v_jz^{3n}) + (-1)^{d-1}(xz^{3n})$$

implies that modulo $I_n$,

$$(v_1z^{3n})^n \equiv ((\sum_{j=2}^{d-3}(-1)^{j+1}v_iz^{3n}) + (-1)^{d-1}(xz^{3n}))^n.$$

We compute

$$v_1z^{3n^2+n} = vz^{3n^2-n}(z^t-z^nx^n)+z^{3n^2}f_4+((v_1z^{3n})^n-(\sum_{j=2}^{d-3}(-1)^{j+1}v_iz^{3n})+(-1)^{d-1}(xz^{3n})^n),$$

where the right-hand side consists of elements known to be in $I_n$. It follows that $(\sum_{j=2}^{d-3}(-1)^jv_jz^{3n^2+n}) + (-1)^d(xz^{3n^2+n}) = -z^{3n^2}f_3 + v_1z^{3n^2+n} \in I_n$.

For the inductive step, we compute

$$\begin{aligned}
&v_{i+1}z^{3n^{i+2}+n^{i+1}+\cdots+n^2+n}\\
&= z^{3n^{i+2}+n^{i+1}+\cdots+n^2-n}(z^t - z^nx^n) + z^{3n^{i+2}+n^{i+1}+\cdots+n^2}f_{i+1} + (v_{i+1}z^{3n^{i+2}+n^{i+1}+\cdots+n^2+n})^n\\
&- \left((\sum_{j=i+2}^{d-3}v_jz^{3n^{i+2}+n^{i+1}+\cdots+n^2+n}) + (-1)^{d-1}xz^{3n^{i+2}+n^{i+1}+\cdots+n^2+n}\right)^n \in I_n,
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&(\sum_{j=i+2}^{d-3}(-1)^jv_jz^{3n^{i+2}+n^{i+1}+\cdots+n^2+n}) + (-1)^d(xz^{3n^{i+2}+n^{i+1}+\cdots+n^2+n})\\
&\qquad = -z^{3n^{i+2}+n^{i+1}+\cdots+n^2+n}f_3 + v_1z^{3n^{i+2}+n^{i+1}+\cdots+n^2+n} \in I_n.
\end{aligned}$$

In particular, when $i+1=d-3$, then we have that

$$v_{d-3}z^{3n^{d-2}+n^{d-1}+\cdots+n^2+n} - xz^{3n^{d-2}+n^{d-1}+\cdots+n^2+n} \in I_n.$$

We are now prepared to see

$$\begin{aligned} xz^{3n^{d-1}+n^{d-1}+\cdots+n^2+n} &= v_{d-3}z^{3n^{d-2}+n^{d-1}+\cdots+n^2-n}(z^t - z^n x^n) \\ &+ z^{3n^{d-1}+n^d+\cdots+n^2}(v_{d-3}^{n+1} - v_{d-3}z^{t-n} + xz^n) \\ &+ v_{d-3}((xz^{3n^{d-2}+n^{d-1}+\cdots+n^2+n})^n - (v_{d-3}z^{3n^{d-2}+n^{d-1}+\cdots+n^2+n})^n), \end{aligned}$$

which shows that both $xz^{3n^{d-1}+n^{d-1}+\cdots+n^2+n}$ and $v_{d-3}z^{3n^{d-1}+n^{d-1}+\cdots+n^2+n}$ are elements of $I_n$. In particular, the elements $xz^{3n^{d-1}+n^{d-1}+\cdots+n^2+n}, yz^{3n^{d-1}+n^{d-1}+\cdots+n^2+n}$ and $v_i z^{3n^{d-1}+n^{d-1}+\cdots+n^2+n}$ are in $I_n$ for all $1 \le i \le d-3$. It follows that modulo $I_n$

$$(x^{n+1})^{3n^{d-1}+n^{d-1}+\cdots+n^2+n} \equiv (xz^{t-n} - yz^n)^{3n^{d-1}+n^{d-1}+\cdots+n^2+n} \equiv 0$$

and then that

$$\begin{aligned} & z^{t+(3n^{d-1}+n^{d-1}+\cdots+n^2+n)} \\ & = z^{(3n^{d-1}+n^{d-1}+\cdots+n^2+n)}(z^t - z^n x^n) + (x^{n-1}z^n)(xz^{3n^{d-1}+n^{d-1}+\cdots+n^2+n}) \in I_n. \end{aligned}$$

We also note that modulo $I_n$

$$(y^n)^{3n^{d-1}+n^{d-2}+\cdots+n^2+n} \equiv ((\sum_{j=1}^{d-3}(-1)^{j+1}v_i z^n) + (-1)^{d-3}xz^n)^{3n^{d-1}+n^{d-1}+\cdots+n^2+n} \equiv 0.$$

We will now show by induction on $k$ that

$$v_{d-3-k}^{(n+1)(3n^{d-1}+n^{d-1}+\cdots+n^2+n)} \in I_n.$$

When $k=0$, we use $f_d$ to see that

$$v_{d-3}^{(n+1)(3n^{d-1}+n^{d-1}+\cdots+n^2+n)} \equiv (v_{d-3}z^{t-n}+x^n-v_{d+3}z^n+x^n)^{(n+1)(3n^{d-1}+n^{d-1}+\cdots+n^2+n)} \equiv 0 \mod I_n.$$

For the inductive step, we see using $f_{d-k}$ that modulo $I_n$

$$
\begin{aligned}
&v_{d-3-k}^{(n+1)(3n^{d-1}+n^{d-1}+\cdots+n^2+n)} \equiv (v_{d-3-k}z^{t-n} + v_{d-2-k} - v_{d-1-k} + \cdots \\
&\cdots + (-1)^{3+k}(v_{d-3}) + (-1)^{d+i+1}(x)^n - v_i z^n + x^n)^{(n+1)(3n^{d-1}+n^{d-1}+\cdots+n^2+n)} \equiv 0.
\end{aligned}
$$

The attentive reader will notice that efforts have not been made to keep minimal the degrees of polynomials appearing in the above expressions. It is now clear that $R/I_n$ is spanned by $xz^j$, $yz^j$, and $v_iz^j$ for $j < (3n^{d-1} + n^{d-2} + \cdots + n^2 + n)$ and $1 \leq i \leq d-3$ together with $z^j$ for $j < t + (3n^{d-1} + n^{d-2} + \cdots + n^2 + n)$ and $x_x^{\alpha} y^{\alpha_y} v_1^{\alpha_1} \cdots v_{d-3}^{\alpha_{d-3}}$ with $\alpha_y < n(3n^{d-1} + n^{d-2} + \cdots + n^2 + n)$ and $\alpha_x, \alpha_i <$ $(n+1)(3n^{d-1} + n^{d-1} + \cdots + n^2 + n)$. Hence,

$$\ell(R/I_n) \leq t + (3n^{d-1} + n^{d-2} + \cdots + n^2 + n) + ((n+1)(3n^{d-1} + n^{d-2} + \cdots + n^2 + n))^d.$$

The term $t + (3n^{d-1} + n^{d-1} + \cdots + n^2 + n)$ counts powers of $z$, and

$$((n+1)(3n^{d-1} + n^{d-2} + \cdots + n^2 + n))^d$$

bounds ways to pick an allowable monomial that is not a power of $z$. One sees directly from the Koszul complex that $\ell(H^1(I_n; N)) = t$. Any choice of $t$ much larger than $((n+1)(3n^{d-1} + n^{d-2} + \cdots + n^2 + n))^d$, for example $t = n^{d^d+1}$, will give

$$
\begin{aligned}
\lim_{n\to\infty} \frac{\ell(H^1(I_n; N))}{\ell(R/I_n)} &= \lim_{n\to\infty} \frac{\ell(H^2(I_n; M))}{\ell(R/I_n)} \\
&= \lim_{n\to\infty} \frac{t}{t + ((n+1)(3n^{d-1} + n^{d-2} + \cdots + n^2 + n))^d} = 1.
\end{aligned}
$$

This computation demonstrates that neither $N$ nor $M$ is asymptotically Cohen-Macaulay.

We now consider the final case: $R = V[[x_1, \ldots, x_{d-1}]]$, $M = (p^s, \ldots, x_{d-2})$, and $N = R/M = \dfrac{V}{(p^s)}[[x_{d-1}]]$. Using the short exact sequence $0 \to p^{s-1}N \to N \to$

$\frac{V}{(p^{s-1})} \to 0$ and the fact that $\frac{V}{(p^{s-1})}[[x_{d-1}]]$ has depth 1, we obtain an injection $H^1(I; p^{s-1}N) \hookrightarrow H^1(I; p^{s-1}N)$ from the long exact sequence of Koszul cohomology for every parameter ideal $I$. The result now follows from the isomorphism $p^{s-1}N \cong k[[x_{d-1}]]$ and the previous cases.

□

**Proposition III.4.** *Suppose $R = k[[x_1, \ldots, x_d]]$ with $d \geq 3$ and $M = (x_1, \ldots, x_{d-h})R$ or that $R = V[[x_1, \ldots, x_{d-1}]]$ and $M = (x_1, \ldots, x_{d-h})$ or $M = (p, x_1, \ldots, x_{d-h-1})$ or $M = (p^s, x_1, \ldots, x_{d-h-1})$ for some $s > 1$ and $1 \leq h < d-1$. Then neither $M$ nor $N = R/M$ is asymptotically Cohen-Macaulay. In particular, there exists a sequence of parameter ideals $I_n$ such that $\lim_{n\to\infty} \frac{\ell(H^{h+1}(I_n; M))}{\ell(R/I_nR)} \neq 0$ and $\lim_{n\to\infty} \frac{\ell(H^h(I_n; N))}{\ell(R/I_nR)} \neq 0$.*

*Proof.* We will proceed by induction on $h$. The base case $h = 1$ is Lemma III.3. For the inductive step, we consider the following short exact sequences

$$0 \to (x_1, \ldots, x_{d-(h+1)})R \to (x_1, \ldots, x_{d-h})R \to (x_{d-h})k[[x_{d-h}, \ldots, x_d]] \to 0$$

or

$$0 \to (x_1, \ldots, x_{d-(h+1)})R \to (p^s, x_1, \ldots, x_{d-(h+1)})R \to (p^s)V[[x_{d-(h+1)}, \ldots, x_{d-1}]] \to 0$$

or

$$0 \to (x_1, \ldots, x_{d-(h+1)})R \to (x_1, \ldots, x_{d-h})R \to (x_{d-h})V[[x_{d-h}, \ldots, x_{d-1}]] \to 0$$

or

$$0 \to (p^s, x_1, \ldots, x_{d-h})R \to$$
$$(p^s, x_1, \ldots, x_{d-(h-1)})R \to (x_{d-(h-1)})V[[x_{d-(h-1)}, \ldots, x_d]] \to 0$$

for some $s \geq 1$. Call the module appearing as the cokernel in each of these sequences $aC$ where $a = x_{d-h}$, $p^s$, or $x_{d-(h-1)}$ and $C = k[[x_{d-h}, \ldots, x_d]]$, $V[[x_{d-(h+1)}, \ldots, x_{d-1}]]$, or $V[[x_{d-h}, \ldots, x_{d-1}]]$, the middle term of each sequence $D$, and the left-hand term $E$. We observe that the depth of $E$ is $h+2$, and the depths of $D$ and $aC$ are $h+1$. We, therefore, have

$$0 \to H^{h+1}(I_n; D) \to H^{h+1}(I_n; aC)$$

which shows that $(x_{d-h})C \cong C$ is not $(h+1)$-effaceable since $D$ is not by the inductive hypothesis and the latter injects into the former.

We now consider another short exact sequence:

$$0 \to E \to R \to C \to 0$$

for each $C$ and $E$ defined above. Because $R$ has depth $d > h+2$ by assumption, the long exact sequence of Koszul homology yields

$$0 \to H^{h+1}(I_n; C) \to H^{h+2}(I_n; E) \to 0,$$

and so $E$ is not $(h+2)$-effaceable because $C$ is not $(h+1)$-effaceable, completing the proof.

$\square$

**Proposition III.5.** *In the same setting as above,*

$$\frac{k[[x_1, \ldots, x_d]]}{(x_d)} \cong k[[x_1, \cdots, x_{d-1}]],$$

$$V[[x_1, \ldots, x_{d-1}]]/(p^s) \cong \frac{V}{(p^s)}[[x_1, \ldots, x_{d-1}]] \textit{ for } s \geq 1,$$

*and*

$$V[[x_1, \ldots, x_{d-1}]]/(x_{d-1}) \cong V[[x_1, \ldots, x_{d-2}]]$$

*are not $(d-1)$-effaceable for $d \geq 2$.*

*Proof.* We call the modules we aim to show are not $(d-1)$-effaceable $N$ and the generator of the principle ideal by which we are quotienting in each case $a$. We note that the techniques of the previous lemma do not work here because $aR \cong R$. Using the short exact sequence $0 \to R \xrightarrow{a} R \to N \to 0$, we get the long exact sequence

$$0 \to H^{d-1}(I_n; N) \to R/I_n \to R/I_n \to N/I_nN \to 0$$

from which we see that $\ell(H^{d-1}(I_n; N)) = \ell(N/I_nN)$. It is therefore sufficient to give a family of ideals $I_n$ such that $\lim_{n\to\infty} \dfrac{\ell(R/((a)+I_n))}{\ell(R/I_n)} \neq 0$. We have previously discussed such families for $d \geq 3$ since

$$\ell(R/((a, x_1, \ldots, x_{d-2}) + I_n)) \leq \ell(R/((a) + I_n)).$$

For $d = 2$, we may use the family $I_n = (x_1^{n+1} - x_1x_2^n, x_2^{n^3} + x_1^n)$ with $x_2 = p$ when $R = V[[x_1]]$. It is clear that $\ell(R/((x_1)+I_n)) = n^3$, and because neither $x_1$ nor $x_1^n - x_2^n$ is a zerodivisor on $R/(x_2^{n^3} + x_1^n)$, we may compute

$$\ell(R/I_n) = \ell(R/(x_1, x_2^{n^3})) + \ell(R/(x_1^n - x_2^n, x_2^{n^3} + x_1^n)) = n^3 + n^2.$$

Of course, $\lim_{n\to\infty} \dfrac{n^3}{n^3+n^2} = 1 \neq 0$.

□

**Lemma III.6.** *Suppose $(R, m, \kappa)$ is a complete unramified regular local ring of dimension $d \geq 2$. If $M$ is dimension $d$ and asymptotically Cohen-Macaulay, then $M/H^0_m(M)$ is torsion-free.*

*Proof.* By Cohen's structure theorem, $R = k[[x_1, \ldots, x_d]]$ or $R = V[[x_1, \ldots, x_{d-1}]]$ where $V = (V, p, k)$, a complete discrete valuation ring. Because $H^0_m(M)$ is finite length, either $M$ and $M/H^0_m(M)$ are both asymptotically Cohen-Macaulay or neither is because the difference between any $\ell(H^i(I_n; M))$ and $\ell(H^i(I_n; M/H^0_m(M)))$

is bounded by $(\binom{d}{i}+\binom{d}{i-1})\cdot\ell(H^0_m(M))$ from the long exact sequence of Koszul cohomology while $\ell(R/I_n)\to\infty$. We may, therefore, replace $M$ by $M/H^0_m(M)$. Suppose $T\neq 0$ is the torsion submodule of $M$. $T$ cannot be supported only at the maximal ideal because then $T\subseteq H^0_m(M)=0$. Let $P\subseteq R$, a prime ideal of height $h<d$, be minimal in $\mathrm{Supp}(T)$. We will show that $M$ is not $h$-effaceable.

Let $(y_1,\dots,y_h)$ or $(p,y_1,\dots,y_{h-1})$ be local generators of $P$ that extend to a system of parameters $y_1,\dots,y_d$ or $p,y_1,\dots,y_{d-1}$ of $R$, and form the regular ring $S=k[[y_1,\dots,y_d]]$ if $R$ is equal characteristic or $S=V[[y_1,\dots,y_{d-1}]]$ if $R$ is mixed characteristic. (By clearing denominators, we may without loss of generality assume that the local generators of $P$ are elements of $R$.) This can be done in equal characteristic by prime avoidance by choosing each $y_i$ a minimal generator of $P$ not in the minimal primes of $(y_1,\dots,y_{i-1})$ for each $h+1\leq i\leq d$. In mixed characteristic, we choose $y_i$ to avoid $(p,y_1,\dots,y_{i-1})$ and the minimal primes of $((y_1,\dots,y_{i-1})+P^2)R_P\cap R$ for $h+1\leq i\leq d-1$ so that $p$ will also be a parameter in $S$. Because $P$ was a minimal prime of $T$, we have $S/(P\cap S)\hookrightarrow R/P\hookrightarrow T\hookrightarrow M$. More concretely, we have $\bar{S}=S/(P\cap S)=k[[y_{h+1},\dots,y_d]]\hookrightarrow M$ or $\bar{S}=S/(P\cap S)=V[[y_{h+1},\dots,y_{d-1}]]$.

We aim to use these injections to split off as a direct summand of $M$ over a smaller regular ring a torsion module of the form we have studied in earlier lemmas. Let $M'$ be a maximal submodule of $M$ disjoint from $\bar{S}$ and $N=M/M'$. Then $\bar{S}\hookrightarrow N$ is an essential extension, and a retraction of the inclusion of $\bar{S}$ into $N$ lifts to a retraction of the map to $M$.

We begin with the equal characteristic case. For each $i\leq h$, there exists $k\geq 1$ such that $(y_i)^kN\cap k[[y_{h+1},\dots,y_d]]=(y_i)^{n-k}(y_i^kN\cap k[[y_{h+1},\dots,y_d]])=0$ by the Artin-Rees Lemma. But $(y_i)^kN\cap k[[y_{h+1},\dots,y_d]]=0$ implies that $(y_i)^kN=0$ because the extension is essential. Therefore, after replacing each $y_i$ with $y_i^{k_i}$, we may assume without loss of generality that $y_iN = 0$ for all $i \leq h$ and view $N$ as a module over $k[[y_{h+1}, \ldots, y_d]]$. Now because $\mathrm{Frac}(k[[y_{h+1}, \ldots, y_d]])$ is a maximal essential extension of $k[[y_{h+1}, \ldots, y_d]]$, we may view $N$ as a finitely generated submodule of $\mathrm{Frac}(k[[y_{h+1}, \ldots, y_d]])$, i.e. $N \cong k[[y_{h+1}, \ldots, y_d]][\frac{1}{f}]$ for some $f \in k[[y_{h+1}, \ldots, y_d]]$. Equivalently, the essential extension we have been studying may be described as $k[[y_{h+1}, \ldots, y_d]] \xrightarrow{f} k[[y_{h+1}, \ldots, y_d]]$. Choosing $t \gg 0$, we may take $f$ to be part of a basis of $k[[y_{h+1}, \ldots, y_d]]$ over $k[[y_{h+1}^t, \ldots, y_d^t]]$, which means that our map splits as a maps of $A = k[[y_{h+1}^t, \ldots, y_d^t]]$ modules.

In mixed characteristic, we consider the case of $p \in P$ and $p \notin P$ separately. We first assume $p \notin P$. Call $\bar{A} = V[[y_{h+1}, \ldots, y_{d-1}]]$. Because $\bar{A}$ injects into $M$, $\bar{A}_{(p)}$ injects into $M_{(p)}$. Because $\bar{A}_{(p)}$ is a discrete valuation ring, $M_P$ must be free over $\bar{A}_{(p)}$, and so we may choose an element $u$ of a free basis of $M_{(p)}$ over $A_{(p)}$ and note that $u \notin pM_{(p)}$. Then $A_{(p)} \hookrightarrow M_{(p)}$ given by $1 \mapsto u$ is a splitting. Now because $\mathrm{Hom}(M_{(p)}, \bar{A}_{(p)}) \cong \mathrm{Hom}(M, \bar{A})_{(p)}$, the retraction $M_{(p)} \twoheadrightarrow \bar{A}_{(p)}$ with $u \to 1$ gives a map $\alpha : M \to \bar{A}$ with $\alpha(u) \notin (p)\bar{A}$. Therefore, there exists a map $\theta : M \hookrightarrow F$ where $F$ is a free $\bar{A}$ module and $\theta(u) \notin (p)F$. Now because $\bigcap_t (p, y_{h+1}^t, \ldots, y_{d-1}^t)F = (p)F$ and $\theta(u) \notin (p)F$, we may choose $t$ sufficiently large that $\theta(u) \notin (p, y_{h+1}^t, \ldots, y_{d-1}^t)F$, which is to say that $\theta(u)$ is not in the maximal ideal of $B := V[[y_{h+1}^t, \ldots, y_{d-1}^t]]$ expanded to $F$. Because $F$ is free over $\bar{A}$ and $\bar{A}$ is free over $B$, $F$ is free over $B$. It follows that there is a retraction $F \twoheadrightarrow \bar{A}$ as $B$ modules. Composing with a retraction $\bar{A} \twoheadrightarrow B$ and restriction to $M$, we obtain a splitting of $B \hookrightarrow M$ as $B$ modules.

Lastly, we suppose $p \in P$, in which case $\bar{A} = k[[y_h, \ldots, y_{d-1}]]$. Fix $k$ so that $p^kM = 0$ but $p^{k-1}M \neq 0$. As in the previous cases, we replace each $y_i$ with some $y_i^{k_i}$ for $h < i \leq d-1$ so that each $y_iM = 0$, and think of $M$ as a module over $B = \dfrac{V}{(p^k)}[[y_h, \ldots, y_{d-1}]]$. For each $t \geq 0$, set $B_t = \dfrac{V}{(p^k)}[[y_h^t, \ldots, y_{d-1}^t]]$. We aim to find a $t$ such that a copy of $B_t$ splits from $M$ as a $B_t$-module. Because $B_{(p)}$ is a 0-dimensional Gorenstein ring, it splits from $M_{(p)}$. As in the previous case, this gives a map $\alpha : M \to B$ with an element $u \in M$ such that $\alpha(u) \notin (p)B$. Again, choose $t$ sufficiently large that $\alpha(u) \notin (p, y_1^t, \ldots, y_s^t)B$. Now $B$ is free over $B_t$ and $\alpha(u)$ is not in the expansion of the maximal ideal of $B_t$ to $B$, so there is a $B_t$ module map $B \to B_t$ such that $\alpha(u) \mapsto 1$, and so the composite map $M \to B \to B_t$ sends $u$ to 1, which gives a splitting of $B_t$ from $M$ as a $B_t$ module.

We now have a module of the form of of III.4 or III.5 as a direct summand of $M$ over an unramified regular ring, which we will rename $A$. Because $R$ is module finite over $A$, a system of parameters in $A$ is a system of parameters in $R$, and so it is sufficient to find a sequence of parameter ideals $I_n$ in $A$ such that $\lim_{n\to\infty} \dfrac{\ell(H^{d-h}(I_n; M))}{\ell(R/I_nR)} \neq 0$. Because $R$ is module finite over $A$, by II.2, it is sufficient to show that $\lim_{n\to\infty} \dfrac{\ell(H^{d-h}(I_n; M))}{\ell(A/I_nA)} \neq 0$, and we know that $\lim_{n\to\infty} \dfrac{\ell(H^{d-h}(I_n; M))}{\ell(A/I_nA)} \neq 0$ because by III.4 and III.5 and the fact that Koszul homology splits over direct sums. It follows that $M$ is not $(d-h)$-effaceable and in particular is not asymptotically Cohen-Macaulay, a contradiction.

$\square$

**Lemma III.7.** *Let $(R, m, \kappa)$ be a complete Cohen-Macaulay local ring of dimension $d$ that is either equal characteristic or in which char$(k)$ is a parameter, $M$ a finitely generated $d$ dimensional module over $R$, and $x \in R$ a non-unit, non-zerodivisor on $M$. If $M$ is asymptotically Cohen-Macaulay over $R$, then $M/xM$ is asymptotically Cohen-Macaulay over $(R/x, \mu)$.*

*Proof.* Fix $R$, $M$ and $x$ as in the theorem statement. Using the short exact sequence

$0 \to \mathrm{syz}^1(M) \to R^h \to M \to 0$, from the long exact sequence of cohomology we have

$$0 \to H^i(I + (x^N); M) \to H^{i+1}(I + (x^N); \mathrm{syz}^1(M)) \to 0$$

for every $i < d-1$ and every parameter $I$ of $R$. It follows that $\mathrm{syz}^1(M)$ is asymptotically Cohen-Macaulay whenever $M$ is. Similarly, from

$$0 \to \mathrm{syz}^1(M/xM) \to (R/x)^g \to M/xM \to 0$$

with $g \leq h$, we have

$$0 \to H^i(J; M/x) \to H^{i+1}(J; \mathrm{syz}^1(M/xM)) \to 0$$

for all $i < d-2$. It follows that if $M/xM$ is not i-effaceable for some $i < d-2$, then $\mathrm{syz}^1(M/xM) \cong \mathrm{syz}^1(M)/x \cdot \mathrm{syz}^1(M)$ is not $i+1$-effaceable. But $\mathrm{syz}^1(M)$ must be asymptotically Cohen-Macaulay because $M$ is. Therefore, we may assume by induction that $i = d-2$.

Let $\varepsilon > 0$. We aim to show that there exists $N' \in \mathbb{N}$ such that for all parameter ideals $I' \subseteq \mu^{N'}$, $\dfrac{\ell(H^{d-2}(I'; M/xM))}{\ell((R/x)/I'(R/x))} < \varepsilon$. We claim that $\mathrm{syz}^1(M)$ is quasi-unmixed. Fix $P \in \min(\mathrm{syz}^1(M))$ and fix an unramified regular ring $A$ over which $R$ is module finite. By Lemma III.2, $\mathrm{syz}^1(M)$ is also asymptotically Cohen-Macaulay over $A$ and so by Lemma III.6 torsion-free. Hence, $A \cap P$ must also be an associated prime of $A$, which is to say $A \cap P = 0$ because $A$ is regular. But then $d = \dim(A) = \dim(A/(A \cap P)) = \dim(R/P)$, as desired. It, therefore, follows from Theorems 1.1 and 1.4 that there exists a constant $c_{\mathrm{syz}^1(M)}$ such that $\ell(R/I) \leq c_{\mathrm{syz}^1(M)} \cdot e_I(\mathrm{syz}^1(M))$ for any $m$-primary ideal $I$ of $R$. Now because $M$ is asymptotically Cohen-Macaulay over $R$, we may fix $N \in \mathbb{N}$ such that for all parameter ideals $I \subseteq m^N$, we have $\dfrac{\ell(H^{d-1}(I; M))}{\ell(R/I)} < \dfrac{\varepsilon \cdot c_{\mathrm{syz}^1(M)}}{2 \cdot \nu_R(\mathrm{syz}^1(M))}$ where $\nu_R$ denotes the least number of generators as an $R$ module. Fix an arbitrary parameter ideal $\bar{J} \subseteq \mu^N$ and fix a $(d-1)$-generator lift $J$ of $\bar{J}$ to $R$ with $J \subseteq m^N$. Note that for any $t \geq 1$, $J + (x^t)$ is a parameter ideal of $R$. For each $t \geq N$, we observe

$$\begin{aligned}
\frac{\ell(H^{d-2}(\bar{J}; M/xM))}{\ell(R/(J+(x)))} &= \frac{t \cdot \ell(H^{d-1}(J+(x);M))}{t \cdot \ell(R/(J+(x)))} \\
&\leq \frac{t \cdot \ell(H^{d-1}(J+(x);M))}{\ell(R/(J+(x^t)))} \\
&= \frac{t \cdot \ell(H^{d-1}(J+(x);M))}{\ell(H^{d-1}(J+(x^t);M))} \cdot \frac{\ell(H^{d-1}(J+(x^t);M)}{\ell(R/(J+(x^t)))} \\
&< \frac{t \cdot \ell(H^{d-1}(J+(x);M))}{\ell(H^{d-1}(J+(x^t);M))} \cdot \frac{\varepsilon \cdot c_{\mathrm{syz}^1(M)}}{2 \cdot \nu_R(\mathrm{syz}^1(M))}.
\end{aligned}$$

Hence, it suffices to show that there exists $t \geq N$ such that

$$\frac{\ell(H^{d-1}(J+(x);M))}{\ell(H^{d-1}(J+(x^t);M))/t} \leq \frac{2 \cdot \nu_R(\mathrm{syz}^1(M))}{c_{\mathrm{syz}^1(M)}}.$$

From $0 \to \mathrm{syz}^1(M) \to R^h \to M \to 0$ and the fact that $R$ is Cohen-Macaulay, the long exact sequence of Koszul cohomology gives

$$0 \to H^{d-1}(J+(x);M) \to \frac{\mathrm{syz}^1(M)}{(J+(x))\mathrm{syz}^1(M)}$$

from which it follows that $\ell(H^{d-1}(J+(x);M)) \leq \nu_R(\mathrm{syz}^1(M)) \cdot \ell(R/(J+(x)))$. We now consider for each $t \geq 1$ the short exact sequence

$$0 \to \frac{H^{d-2}(J;M)}{x^t \cdot H^{d-2}(J;M)} \to H^{d-2}(J; M/(x^t)M) \to \mathrm{Ann}_{M/JM}(x^t) \to 0,$$

from which we see that

$$\ell(H^{d-1}(J+(x^t);M)) = \ell(H^{d-2}(J;M/(x^t)M)) \geq \ell\left(\frac{H^{d-2}(J;M)}{x^t \cdot H^{d-2}(J;M)}\right).$$

Because $H^{d-1}(J;M)$ is a one-dimensional $R/J$ module, there exists some $T \in \mathbb{N}$ such that for all $t \geq T$,

$$\ell\left(\frac{H^{d-2}(J;M)}{x^t \cdot H^{d-2}(J;M)}\right)/t \geq e_{(x)}(H^{d-2}(J;M))/2.$$

Now $H^{d-2}(J;M) \cong H^{d-1}(J;\mathrm{syz}^1(M)) \cong \dfrac{\mathrm{syz}^1(M)}{J\mathrm{syz}^1(M)}$ because $R$ is Cohen-Macaulay and $J$ is generated by $(d-1)$ elements. It follows that

$$e_{(x)}(H^{d-2}(J;M)) = e_{(x)}\frac{\mathrm{syz}^1(M)}{J\mathrm{syz}^1(M)} \geq e_{(J+(x))}(\mathrm{syz}^1(M))) \geq c_{\mathrm{syz}^1(M)} \cdot \ell(R/(J+(x)).$$

Hence,

$$\frac{\ell(H^{d-1}(J+(x);M))}{\ell(H^{d-1}(J+(x^t);M))/t} \leq \frac{2 \cdot \nu_R(\mathrm{syz}^1(M)) \cdot \ell(R/(J+(x)))}{c_{\mathrm{syz}^1(M)} \cdot \ell(R/(J+(x))} = \frac{2 \cdot \nu_R(\mathrm{syz}^1(M))}{c_{\mathrm{syz}^1(M)}}$$

as desired. □

**Theorem III.8.** *If $(R,m,\kappa)$ is any complete (Noetherian) local ring that is either equal characteristic or mixed characteristic with char$(k)$ a parameter in $R$ and $M$ is dimension $d$ and asymptotically Cohen-Macaulay, then $M$ is generalized Cohen-Macaulay.*

*Proof.* We assume that $M$ is asymptotically Cohen-Macaulay. By III.2, we may assume that $R$ is regular and unramified. Having replaced $M$ by $M/H^0_m(M)$, we may assume that $M$ is torsion-free by III.6. Because all torsion-free modules of dimension 1 are Cohen-Macaulay, we may assume that $d-1>0$. Fix a prime $P$ of $R$ of height $d-1$. We aim to show that some system of parameters on $M_P$ is a regular sequence on $M_P$. We claim that depth$M_P > 0$. If $M_P$ has depth 0, then $P$ is an associated prime of $M$ (equivalently, of $M_P$). Then because $M$ is torsion-free, $P$ must be an associated prime of $R$, but because $R$ is regular, $P=0$, but $P$ has height $d-1>0$. Therefore, we may fix $x_1,\ldots,x_h$ a system of parameters of $M_P$ with $x_1$ not a zero-divisor and, by prime avoidance, $x_1 \notin (\mathrm{char}(k))$ in the mixed characteristic case. By III.7, $M/x_1M$ is asymptotically Cohen-Macaulay. Because $x_1 \notin (\mathrm{char}(k))$, char$(k)$ remains a parameter in $R/x_1$ in the mixed characteristic case, and so $M/x_1M$

is locally Cohen-Macaulay by induction. Hence $(M/x_1M)_P \cong M_P/x_1MP$ is Cohen-Macaulay. Now because $x_2, \dots, x_h$ is a system of parameters on $R_P/x_1R_P$, it must also be a regular sequence. It follows that $x_1, \dots, x_h$ is a regular sequence on $M_P$.

□

**Example III.9.** We now give an application of the above theorem. In particular, we give an example showing that $\dfrac{\ell(R/I_n)}{e_{I_n}(R)}$ need not approach 1 as $I_n \to \infty$. Let $R = (\frac{k[x,y,z]}{x^3+y^3+z^3} \circledS k[u,v])[w]$ localized at the homogeneous maximal ideal. Because $R$ is normal, it is in particular $S_2$, and so its only nonvanishing Koszul homology modules are $i = 3, 4$. It is not generalized Cohen-Macaulay and so not asymptotically Cohen-Macaulay. Therefore, we may pick a sequence of parameter ideals $I_n$ such that $\dfrac{H^2(I_n;R)}{\ell(R/I_n)} \not\to 0$. It follows that $\dfrac{e_{I_n}(R)}{\ell(R/I_n)} = \dfrac{H^3(I_n;R) - H^2(I_n;R)}{H^3(I_n;R)} \not\to 1$.

We will now undertake the backward direction of one of this chapter's main theorems. In particular, we will now show that if $M$ is equidimensional, then $M$ generalized Cohen-Macaulay implies $M$ asymptotically Cohen-Macaulay.

**Lemma III.10.** *Let $(R, m, \kappa)$ be a complete (Noetherian) local ring of dimension $d$, and let $M$ be at finitely generated $R$-module. Then for each $0 \le s < d$,*

$$\ell(H^s(I;M)) \le \sum_{r=0}^{s} \binom{d}{s+r} \ell(H^r_m(M)).$$

*Proof.* By Cohen's structure theorem, $R \cong T/J$ for some regular local ring $(T, n)$ of dimension $d+h$ and ideal $J$ of $T$ of height $h$. Let $\mathcal{A}^\bullet$ be a dualizing complex for $R$ over $T$. Fix a parameter ideal $I$ of $R$, fix a set of parameters that generate $I$. By tensoring the Koszul complex on our set of generators of $I$ with $\mathrm{Hom}_R(M, \mathcal{A}^\bullet)$, we get the grid below.

$$
\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Hom}_R(M, \mathcal{A}^0) & \longrightarrow & \cdots & \longrightarrow & \operatorname{Hom}_R(M, \mathcal{A}^d) & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Hom}_R(M^{\binom{d}{i}}, \mathcal{A}^0) & \longrightarrow & \cdots & \longrightarrow & \operatorname{Hom}_R(M^{\binom{d}{i}}, \mathcal{A}^d) & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Hom}_R(M, \mathcal{A}^0) & \longrightarrow & \cdots & \longrightarrow & \operatorname{Hom}_R(M, \mathcal{A}^d) & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}
$$

where the horizontal maps are the standard maps induced by applying $\operatorname{Hom}_R(M, \_\_)$, and the vertical maps are the maps in the Koszul complex. Taking this double complex as the $E^0$ page of a spectral sequence, we will first compute the homology of the rows first using local duality and obtain the $E^1$ page below:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{H}^d_n(M)^\vee & \cdots & \mathrm{H}^0_n(M)^\vee \\
\downarrow & & \downarrow \\
\vdots & \vdots & \vdots \\
\downarrow & & \downarrow \\
\mathrm{H}^d_n(M^{\binom{d}{i}})^\vee & \cdots & \mathrm{H}^0_n(M^{\binom{d}{i}})^\vee \\
\downarrow & & \downarrow \\
\vdots & \vdots & \vdots \\
\downarrow & & \downarrow \\
\mathrm{H}^d_n(M)^\vee & \cdots & \mathrm{H}^0_n(M)^\vee \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Each $\mathrm{H}^j_n(M)^\vee$ is finite length for $j < d$ since $\mathrm{H}^j_n(M) = \mathrm{H}^j_m(M)$ and $M$ is generalized Cohen-Macaulay as an $R$ module. Because the $E^0$ page is only possibly nonzero in a $(d+1)\times(d+1)$ grid, the $E^\infty = E^d$. We will use $\mathcal{B}^k_{ij}$ to indicate the module in the $i^{th}$ row and the $j^{th}$ column of the the $k^{th}$ page of the spectral sequence with homology computed first by rows. For $k \geq 2$, $\mathcal{B}^k_{ij} = \ker(\mathcal{B}^{k-1}_{ij} \to \mathcal{B}^{k-1}_{i+k-1,j-(k+2)})/\mathrm{im}(\mathcal{B}^{k-1}_{ij} \to \mathcal{B}^{k-1}_{i-(k-1),j+k+2})$. In particular, for each $k > 0$, $\ell(\mathcal{B}^k_{ij}) \leq \ell(\mathcal{B}^{k-1}_{ij})$, and so $\ell(\mathcal{B}^k_{ij}) \leq \binom{d}{i}\ell(H^{d-j}_m(M))$.

We now run the spectral sequence by taking homology of the columns first. The $E_1$ page is below:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_0(I;\mathrm{Hom}_R(M,\mathcal{A}^0)) & \longrightarrow & \cdots & \longrightarrow & H_0(I;\mathrm{Hom}_R(M,\mathcal{A}^d)) & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & H_i(I;\mathrm{Hom}_R(M,\mathcal{A}^0)) & \longrightarrow & \cdots & \longrightarrow & H_i(I;\mathrm{Hom}_R(M,\mathcal{A}^d)) & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & H_d(I;\mathrm{Hom}_R(M,\mathcal{A}^0)) & \longrightarrow & \cdots & \longrightarrow & H_d(I;\mathrm{Hom}_R(M,\mathcal{A}^d)) & \longrightarrow & 0
\end{array}$$

Because $\mathcal{A}^j = \mathrm{Hom}_T(R, E^{j+h})$ where $E^{j+h}$ is the sum of the injective hulls of the $T/P$ for $P$ height $j+h$ in $T$ and $I$ is height $d+h$, for every $j < d$ one of the generators of $I$ is not an element of any of the $P$ of height $j+h$ by prime avoidance and so acts invertible on $\mathcal{A}^j$ and so on $\mathrm{Hom}_R(M, \mathcal{A}^j)$. It follows that the grid above is 0 except in column $d$. It follows that the $E^1 = E^\infty$.

We now fix some $0 < s \leq d$. For each $I$, we must have

$$\ell(H_s(I;\mathrm{Hom}(M,\mathcal{A}^d))) = \sum_{r=0}^{d-s} \ell(\mathcal{B}^d_{s+r,d-r}) \leq \sum_{r=0}^{d-s} \binom{d}{s+r} \ell(H^r_m(M)). \tag{3.1}$$

We note that this bound in independent of $I$.

Now let $E = E(T/n)$, the injective hull of $T/n$. Because $\mathcal{A}^d \cong \mathrm{Hom}_T(R, E)$, we have $\mathrm{Hom}_R(M, \mathcal{A}^d) \cong \mathrm{Hom}_R(M, E)$, and so for each $1 \leq i \leq d$,

$$\begin{aligned}
\ell(H_i(I; \mathrm{Hom}(M, \mathcal{A}^d))) &= \ell(H_i(I; \mathrm{Hom}(M, E))) \\
&= \ell(\mathrm{Hom}(H^i(I; \mathrm{Hom}(M, E)), E))) \\
&= \ell(\mathrm{Hom}(H_i(K^\bullet(I; R) \otimes (M^\vee)), E)) \\
&= \ell((H_i(K^\bullet(I; R) \otimes M^\vee))^\vee) \\
&= \ell(H_i(K^\bullet(I; R) \otimes M^{\vee\vee})) \\
&= \ell(H_i(K^\bullet(I; R) \otimes M)) \\
&= \ell(H_i(I; M)) \\
&= \ell(H^{d-i}(I; M))
\end{aligned}$$

where $\_\_^\vee := \mathrm{Hom}(\_\_, E)$ and $K^\bullet$ indicates the Koszul complex. We may, therefore, rewrite 3.1 for $0 \leq s < d$ as

$$\ell(H^s(I; M)) \leq \sum_{r=0}^{s} \binom{d}{r} \ell(H^r_m(M)). \tag{3.2}$$

□

**Theorem III.11.** *Let $(R, m, \kappa)$ be a complete (Noetherian) local ring of dimension $d$, and let $M$ be at finitely generated $R$-module. If $M$ is generalized Cohen-Macaulay and equidimensional, then $M$ is asymptotically Cohen-Macaulay. Furthermore, in this case*

$$\sup\{\ell(H^s(I; M)) \mid I \textit{ parameter }, 0 \leq s < d\} < \infty.$$

*Proof.* Whenever $M$ is generalized Cohen-Macaulay and equidimensional, then for each $0 \leq i \leq d-1$, $\ell(H^r_m(M)) < \infty$ and so the result follows from 3.2. □

**Theorem III.12.** *If $M$ is an equidimensional finitely generated module over the complete local ring $(R, m, \kappa)$ where either $R$ is equicharacteristic or char$(k)$ is a parameter in $R$, then the following three conditions are equivalent:*

1. *$M$ is asymptotically Cohen-Macaulay,*

2. $\sup\{\ell(H^i(f_1,\ldots,f_d;M)) \mid \sqrt{f_1,\ldots,f_d} = m,\ i < d\} < \infty$,

3. *$M$ is generalized Cohen-Macaulay.*

*Proof.* This theorem follows immediately from III.8 and III.11 together with the fact that 2 obviously implies 1. $\square$

**Corollary III.13.** *If $M$ is an equidimensional module over the complete local ring $(R, m, \kappa)$ of dimension $d \geq 1$ and either $R$ is equicharacteristic or char$(\kappa)$ is a parameter in $R$, then the following three conditions are equivalent:*

1. *asydepth$(M) \geq k$,*

2. $\sup\{\ell(H^i(f_1,\ldots,f_d;M)) \mid \sqrt{f_1,\ldots,f_d} = m,\ i < k\} < \infty$,

3. *$\ell(H^i_m(M)) < \infty$ for all $i < k$.*

*Proof.* As above, it is clear that condition (2) implies condition (1). To see that (1) implies (3), fix a regular ring $(A, n)$ over which $(R, m, \kappa)$ is module finite, and take

$$0 \to \mathrm{syz}^1_A(M) \to A^\nu \to M \to 0$$

for $v = \nu_A(M)$. We first establish the result over $A$. If $k < d-1$, then for any parameter ideal $I$ of $A$, $H^k(I;M) \cong H^{k+1}(I;M)$, and $H^i_n(M) \cong H^{i+1}_n(\mathrm{syz}^1_A(M))$. We may, then, without loss of generality assume $k = d-1$, which is the result of Theorem III.1, using the fact that $\mathrm{syz}^1_A(M)$ is also equidimensional. If condition (1) is satisfied over $R$, then it is certainly satisfied over $A$ because every system of parameters in $A$ is a system of parameters in $R$. It then follows that $\ell(H^i_m(M)) = \ell(H^i_n(M)) < \infty$ for all $i < k$, which is to say that condition (3) is satisfied over $R$. Lastly, it is immediate from equation 3.2 that (3) implies (2). $\square$

## 3.2 The Gorenstein case

In this section, we will assume that $(R, m, \kappa)$ is mixed characteristic and that $\mathrm{char}(\kappa)$ is not a parameter in $R$. This section will both complete the proof of III.1 and also will give a roadmap for an alternative proof in the equal characteristic case than the one described in the previous section. We first notice that is characteristic independent, and so the we alredy know that $\ell(H^i_m(M)) < \infty$ for all $i < k$ implies $\sup\{\ell(H^i(f_1, \ldots, f_d; M)) \mid \sqrt{f_1, \ldots, f_d} = m,\, i < k\} < \infty$ implies $\mathrm{asydepth}(M) \geq k$ in all characteristics. The proof that we need to alter is the one that shows that $\mathrm{asydepth}(M) \geq k$ implies that $\ell(H^i_m(M)) < \infty$. We first reduce to the case where $R$ is Gorenstein by modifying Lemma III.2.

**Lemma III.14.** *If it is true that all finitely-generated modules $M$ that are asymptotically Cohen-Macaulay over a Gorenstein ring of the form $\dfrac{V[[x_1, \ldots, x_d]]}{(p^s x_1)}$ where $p$ generates the maximal ideal of the discrete valuation ring $V$ and $s \geq 1$ are generalized Cohen-Macaulay, then the same holds over any complete local ring $R$ in which the characteristic of the residue field of $R$ is not a parameter in $R$.*

*Proof.* Suppose $M$ is asymptotically Cohen-Macaulay over $R$ satisfying the hypotheses of the lemma. By Cohen's structure theorem, $R$ is a module finite extension of a ring $S = \dfrac{V[[x_1, \ldots, x_d]]}{(p^s x_1)}$ as described in the lemma statement. Every parameter ideal $I_n$ of $S$ is a parameter ideal in $R$, every finitely generated $R$-module $M$ is also a finitely generated $S$-module, and $H^i_R(I_n; M) = H^i_S(I_n; M)$. Because $\ell(R/I_n) \leq \nu_S(R) \cdot \ell(S/I_n)$ for each $n \geq 1$, where $\nu_S(R)$ denotes the minimal number of generators of $R$ as an $S$-module, $\dfrac{\ell(H^i(I_n; M))}{\ell(S/I_n)} \leq \nu_S(R) \cdot \dfrac{\ell(H^i(I_n; M))}{\ell(R/I_n)} \xrightarrow{n \to \infty} 0$, which is to say that $M$ is asymptotically Cohen-Macaulay over $S$. By assumption, then, $M$ is generalized Cohen-Macaulay over $S$, from which it follows that $M$ is generalized Cohen-Macaulay over $R$. $\square$

**Lemma III.15.** *Let* $R = \dfrac{V[[x_1, \ldots, x_d]]}{(p^s x_1)}$ *where* $(V, pV)$ *is a discrete valuation ring and* $s \geq 1$*, and let* $M$ *be a finitely-generated* $R$*-module. Then there exists a sequence of parameter ideals* $I_n \subseteq m^n$ *such that*

$$\frac{\ell(M/I_nM)}{\ell(R/I_n)} \not\to 0.$$

*Proof.* Because $\ell(R/I_n) \leq s\ell(R/I_n + (px_1))$ and $\ell(M/I_nM) \geq \ell(M/(I_n + (px_1))M)$, we may assume that $s = 1$. By taking a prime cyclic filtration of $M$, there exists some prime $P$ of $R$ such that $M \twoheadrightarrow R/P$. By possibly quotienting further, we may replace $M$ by $R/P$ and assume that $\dim(R/P) = 1$. Let $z$ be an element of $R$ whose image is a parameter in $R/P$, call $x = x_1$, and extend $z$ to $z, x - p, y, v_1, \ldots, v_{d-3}$ so that $z, x, y, v_1, \ldots, v_{d-3}$ is a system of parameters in $R_a = V/(p)[[x_1, \ldots, x_d]]$ and $p, y, v_1, \ldots, v_{d-3}$ is a system of parameters in $R_b = V[[x_2, \ldots, x_d]]$. Following the argument of III.14, we may replace $R$ by $\dfrac{V[[x, y, z, v_1, \ldots, v_{d-3}]]}{(px)}$. Set $I_n = (f_1, \ldots, f_d)$ where $f_1 = z^t - z^n(x-p)^n$, $f_2 = (x-p)^{n+1} - (x-p)z^{t-n} + yz^n$,

$$f_3 = y^n + v_1 z^n - v_2 z^n + \cdots + (-1)^{i+1} v_i z^n + \cdots + (-1)^{d-2} v_{d-3} z^n + (-1)^{d-3}(x-p)z^n,$$

and

$$f_{i+3} = v_i^n - v_i z^{t-n} + (v_{i+1} - v_{i+2} + \cdots + (-1)^{d+i} v_{d-3} + (-1)^{d+i+1}(x-p))^n - v_i z^n + x^n$$

for $1 \leq i \leq d-3$ and some $t \in \mathbb{N}$.

First suppose that $p \in P$. Then set $A = k[[z]] \hookrightarrow R/P$, a regular ring over which $R/P$ is module finite. If $p \notin P$, then $R/P$ is a module-finite extension of the regular ring $A = V$. In either case, it follows from II.2 that $\ell(R/(P + I_n)) \geq \ell(A/I_n) = t$. We now follow identically the computations of Lemma III.3 with $x$ replaced by $x - p$

to see that that

$$((x-p)^{n+1})^{3n^{d-1}+n^{d-1}+\cdots+n^2+n},$$

$$z^{t+(3n^{d-1}+n^{d-1}+\cdots+n^2+n)},$$

$$(y^n)^{3n^{d-1}+n^{d-2}+\cdots+n^2+n},$$

and

$$v_i z^{3n^{i+1}+n^i+\cdots+n}$$

for $1 \leq i \leq d-3$ are elements of $I_n$. Multiplying the first element by $x$ or by $p$, we have

$$(x^{n+2})^{3n^{d-1}+n^{d-1}+\cdots+n^2+n}$$

and

$$(p^{n+2})^{3n^{d-1}+n^{d-1}+\cdots+n^2+n}$$

elements of $I_n$ as well. Similarly,

$$(x-p)z^{3n^{d-1}+n^{d-1}+\cdots+n^2+n}$$

$$yz^{3n^{d-1}+n^{d-1}+\cdots+n^2+n}$$

and

$$v_i z^{3n^{d-1}+n^{d-1}+\cdots+n^2+n}$$

for $1 \leq i \leq d-3$ are elements of $I_n$, and, multiplying the first of these three elements by $x$ or $p$, we have that

$$x^2 z^{3n^{d-1}+n^{d-1}+\cdots+n^2+n}$$

and

$$p^2 z^{3n^{d-1}+n^{d-1}+\cdots+n^2+n}$$

are elements of $I_n$.

Hence, counting spanning elements of $R/I_n$ as in Lemma III.3, $R/I_n$ is spanned by $x^2z^j$, $p^2z^j$, $yz^j$, and $v_iz^j$ for $j < (3n^{d-1}+n^{d-2}+\cdots+n^2+n)$ and $1 \leq i \leq d-3$ together with $z^j$, $pz^j$, and $xz^j$ for $j < t+(3n^{d-1}+n^{d-2}+\cdots+n^2+n)$ and $x_{xp}^{\alpha}y^{\alpha_y}v_1^{\alpha_1}\cdots v_{d-3}^{\alpha_{d-3}}$ and $p_{xp}^{\alpha}y^{\alpha_y}v_1^{\alpha_1}\cdots v_{d-3}^{\alpha_{d-3}}$ with $\alpha_y < n(3n^{d-1}+n^{d-2}+\cdots+n^2+n)$ and $\alpha_{xp}, \alpha_i < (n+2)(3n^{d-1}+n^{d-1}+\cdots+n^2+n)$. Therefore,

$$\ell(R/I_n) \leq 3(t+(3n^{d-1}+n^{d-2}+\cdots+n^2+n))+((n+2)(3n^{d-1}+n^{d-2}+\cdots+n^2+n))^d,$$

and so, if $\lim_{I_n\to\infty}\dfrac{\ell(M/I_nM)}{\ell(R/I_n)}$ exists, then

$$\begin{aligned}
&\lim_{I_n\to\infty}\frac{\ell(M/I_nM)}{\ell(R/I_n)}\\
&\geq \frac{t}{3(t+(3n^{d-1}+n^{d-2}+\cdots+n^2+n))+((n+2)(3n^{d-1}+n^{d-2}+\cdots+n^2+n))^d}\\
&= 1/3
\end{aligned}$$

for every $t >> n^{d^2}$. $\square$

**Theorem III.16.** *Let $(R, m, \kappa)$ be a Gorenstein ring of dimension $d$ in which char$(\kappa)$ is not a parameter in $R$ and $M$ a finitely-generated quasi-unmixed $R$-module. Then for all $0 \leq k \leq d$, asydepth$(M) \geq k$ implies that $\ell(H_m^i(M)) < \infty$ for all $0 \leq i < k$.*

*Proof.* Suppose that asydepth$(M) \geq k$ but that $\ell(H_m^i(M)) = \infty$ for some $0 \leq k \leq d$, and assume that $k$ is minimal with respect to this property. Notice that for each parameter ideal $I$, $H^{i+1}(I; \mathrm{syz}^1(M)) \cong H^i(I; M)$, and so by replacing $M$ by $\mathrm{syz}^{d-k-1}(M)$, we have a counterexample when $k = d-1$ and $\ell(H_m^i(M)) < \infty$ for all $0 \leq i < d-1$. We now return to the spectral sequence from Lemma 3.2 in order to improve equation 3.2. Below is the $E_2$ page of the spectral sequence run by taking homology of columns first omitting maps, which will not be of interest to us:

$$
\begin{array}{ccc}
0 & & 0 \\
H^0(I; \mathrm{H}^d_n(M)^\vee) & \cdots & H^0(I; \mathrm{H}^0_n(M)^\vee) \\
\vdots & \vdots & \vdots \\
H^i(I; \mathrm{H}^d_n(M)^\vee) & \cdots & H^i(I; \mathrm{H}^0_n(M)^\vee) \\
\vdots & \vdots & \vdots \\
H^d(I; \mathrm{H}^d_n(M)^\vee) & \cdots & H^d(I; \mathrm{H}^0_n(M)^\vee) \\
0 & & 0
\end{array}
$$

We may from here improve equation 3.2 in the case of $s = d - 1$ to

$$|\ell(H^{d-1}(I; M)) - H^d(I; H^{d-1}_m(M))| \leq \sum_{r=0}^{d-1} \ell(H^r(I; H^{r-1}_m(M))),$$

which is to say that controlling the lengths of the $H^{d-1}(I; M)$ is the same task as controlling the lengths of the $H^d(I; H^{d-1}_m(M))$ because each of the $\ell(H^i_m(M)) < \infty$. More precisely, the task remaining to us is to show that if $\dim(H^{d-1}_m(M)) > 0$, then there exists a sequence of parameter ideals $I_n$ such that $\dfrac{\ell(H^{d-1}_m(M)/I_n H^{d-1}_m(M))}{\ell(R/I_n} \not\to 0$ as $I_n \to \infty$. Because $\dim(H^{d-1}_m(M)) \leq d - 1$, the result follows from lemma III.14 and III.15. $\square$

Combining III.13 and III.16, we have now shown the entirety of III.1.

# APPENDIX

# APPENDIX A

# Lech's inequality and Stückrad-Vogel's conjecture

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## A.1 Abstract

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d$ and let $M$ be a finitely generated $R$-module. We prove that the set $\left\{\frac{l(M/IM)}{e(I,M)}\right\}_{\sqrt{I}=\mathfrak{m}}$ is bounded below by $1/d!e(\overline{R})$ where $\overline{R} = R/\mathrm{Ann}_R M$. Moreover, when $M$ is quasi-unmixed (i.e., $\widehat{M}$ is equidimensional), this set is bounded above by a finite constant depending only on $M$. The lower bound extends a classical inequality of Lech to all finitely generated modules, and the upper bound answers a question of Stückrad-Vogel in the affirmative.

## A.2 Introduction

In [35], Lech proved a simple inequality relating the Hilbert-Samuel multiplicity and the colength of an ideal. It states that if $(R, \mathfrak{m})$ is a Noetherian local ring of dimension $d$ and $I$ is any $\mathfrak{m}$-primary ideal of $R$, then we have

$$e(I, R) \leq d!e(R)l(R/I),$$

where $e(I, R)$ denotes the Hilbert-Samuel multiplicty of $I$ and $e(R) = e(\mathfrak{m}, R)$. In the same paper Lech conjectured that for every flat local extension $(R, \mathfrak{m}) \to (S, \mathfrak{n})$

of Noetherian local rings, one has $e(R) \leq e(S)$. This conjecture is wide open in general, and Lech obtained a partial estimate, using the above inequality, that we have $e(R) \leq d!e(S)$ where $d = \dim R$ [35]. We refer to [27] for some generalizations of Lech's inequality and to [37] for recent progress on Lech's conjecture.

If we consider the set $\left\{\frac{l(R/I)}{e(I,R)}\right\}_{\sqrt{I}=\mathfrak{m}}$ of positive numbers, then the above Lech's inequality is simply saying that this set is bounded below by $\frac{1}{d!e(R)}$ (and thus is bounded away from 0). The infimum of this set was investigated by Mumford in his study of local stability [39]. In a different direction, Stückrad and Vogel studied the supreme of $\left\{\frac{l(R/I)}{e(I,R)}\right\}_{\sqrt{I}=\mathfrak{m}}$ in [47]. A fundamental question they asked is that whether the supreme is finite, and they conjectured the following [47, Theorem 1 and Conjecture]:

**Conjecture A.1** (Stückrad-Vogel). *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let $M$ be a finitely generated $R$-module. Let $e(I, M)$ be the Hilbert-Samuel multiplicity*[1] *of $M$ with respect to $I$. Set*

$$n(M) = \sup_{\sqrt{I}=\mathfrak{m}} \left\{\frac{l(M/IM)}{e(I,M)}\right\}.$$

*Then $n(M) < \infty$ if and only if $M$ is quasi-unmixed (i.e., $\widehat{M}$ is equidimensional).*

Stückrad and Vogel proved the "only if" direction in general and the graded case of the "if" direction [47, Theorem 1]. In this paper we settle this conjecture in the affirmative. Furthermore, motivated by Conjecture A.1, it is quite natural to ask that whether the classical Lech's inequality can be extended to all finitely generated modules, i.e., whether there is a lower bound on the set $\left\{\frac{l(M/IM)}{e(I,M)}\right\}_{\sqrt{I}=\mathfrak{m}}$ for a finitely generated $R$-module $M$. We also answer this question in the affirmative. In sum, our main result is the following:

[1] In this paper, we define the Hilbert-Samuel multiplicity of a finitely generated module $M$ with respect to $I$ to be $e(I, M) = \lim_{n\to\infty} t!\frac{l_R(M/I^nM)}{n^t}$ where $t = \dim M$. This is always a positive integer even when $\dim M < \dim R$.

**Main Theorem.** *(Theorem A.5 and Theorem A.14) Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d$ and let $M$ be a finitely generated $R$-module. Set*

$$m(M) = \inf_{\sqrt{I}=\mathfrak{m}} \left\{ \frac{l(M/IM)}{e(I, M)} \right\} \quad \text{and} \quad n(M) = \sup_{\sqrt{I}=\mathfrak{m}} \left\{ \frac{l(M/IM)}{e(I, M)} \right\}.$$

*Then we have*

$$m(M) \geq \frac{1}{d! e(\overline{R})}$$

*where $\overline{R} = R/Ann_R M$. Moreover, if $M$ is quasi-unmixed, then we also have*

$$n(M) < \infty.$$

We remark that, if $\dim M = d$, then our Main Theorem implies that the set $\left\{ \frac{l(M/IM)}{e(I,M)} \right\}_{\sqrt{I}=\mathfrak{m}}$ is bounded below by $\frac{1}{d! e(R)}$, which is independent of $M$ (and in general, the lower bound $m(M)$ only depends on $\mathrm{Ann}_R M$). One cannot expect the same for the upper bound $n(M)$. For example, take $R = k[[x, y]]$ and let $M_t = \mathfrak{m}^t = (x, y)^t$, then $M_t$ are all faithful $R$-modules of rank one, but clearly

$$n(M_t) \geq \frac{l(M_t/\mathfrak{m}M_t)}{e(\mathfrak{m}, M_t)} = \frac{l(\mathfrak{m}^t/\mathfrak{m}^{t+1})}{e(R)} = t + 1.$$

Therefore there cannot exist a constant $c$ such that $n(M_t) \leq c$ works for all $M_t$.

This paper is organized as follows: in Section 2 we prove Conjecture A.1, which is the second part of the Main Theorem, and in Section 3 we extend the classical Lech's inequality and prove the first part of the Main Theorem. Some applications will be given in Section 4.

## A.3 Finiteness of $n(M)$: resolving Stückrad-Vogel's conjecture

To prove the Stückrad-Vogel's conjecture, we need the concept of extended degree of a finitely generated module introduced by Vasconcelos in [50] [49].

**Definition A.2.** Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let $\mathcal{M}(R)$ denote the category of finitely generated $R$-modules. An *extended degree* on $\mathcal{M}(R)$ with respect to an $\mathfrak{m}$-primary ideal $I$ is a numerical function

$$\mathrm{Deg}(I, \bullet) : \mathcal{M}(R) \to \mathbb{R}$$

satisfying the following conditions:

1. $\mathrm{Deg}(I, M) = \mathrm{Deg}(I, \overline{M}) + l(H^0_{\mathfrak{m}}(M))$, where $\overline{M} = M/H^0_{\mathfrak{m}}(M)$;
2. $\mathrm{Deg}(I, M) \geq \mathrm{Deg}(I, M/xM)$ for every generic element $x \in I - \mathfrak{m}I$ of $M$;
3. If $M$ is Cohen-Macaulay then $\mathrm{Deg}(I, M) = e(I, M)$.

The original definition in [49] only deals with the case $I = \mathfrak{m}$. The above definition was taken from [12, Definition 5.3]. The first question is that, given a Noetherian local ring $(R, \mathfrak{m})$, whether an extended degree function exist. This was settled in the affirmative by Vasconcelos [50],[49], who showed that the following *homological degree* is an example of extended degree.[2]

**Definition A.3.** Let $(R, \mathfrak{m})$ be a homomorphic image of a Gorenstein local ring $(S, \mathfrak{n})$ of dimension $n$, and let $M$ be a finitely generated $R$-module of dimension $d$. Then the *homological degree*, $\mathrm{hdeg}(I, M)$, of $M$ with respect to an $\mathfrak{m}$-primary ideal $I$ is defined by the following recursive formula

$$\mathrm{hdeg}(I, M) = e(I, M) + \sum_{i=n-d+1}^{n} \binom{d-1}{i-n+d-1} \mathrm{hdeg}(I, \mathrm{Ext}^i_S(M, S)).$$

We note that the above definition is recursive on dimension since $\dim \mathrm{Ext}^i_S(M, S) \leq n - i < d = \dim M$ for all $i = n-d+1, \ldots, n$. For a long time, the homological degree is the only explicit extended degree found in general. Until quite recently

[2] Here again, Vasconcelos's papers [50],[49] focus on the case $I = \mathfrak{m}$, and in fact the main case Vasconcelos considered is the graded case. However the proofs in [50],[49] work in the general set up, and we refer to [12] for more details.

in [12], Cuong and the third author discovered another extended degree, defined in terms of their *Cohen-Macaulay deviated sequence* $\{U_i(M)\}_{i=0}^{d-1}$. Roughly speaking, $U_i(M)$ is the unmixed component of $M/(x_{i+2},\dots,x_d)M$ for certain carefully chosen system of parameters $x_1,\dots,x_d$ of $M$ (note that $U_{d-1}(M)$ is just the unmixed component of $M$), it is shown in [12, Theorem 4.4] that this is independent of the choice of $x_1,\dots,x_d$ as long as $x_1,\dots,x_d$ is a *$C$-system of parameters of $M$*, which always exists when $R$ is a homomorphic image of a Cohen-Macaulay local ring. Thus $\{U_i(M)\}_{i=0}^{d-1}$ are a sequence of finitely generated $R$-modules depending only on $M$. We refer to [12, Section 4] for more details on this.

**Definition A.4.** Let $(R,\mathfrak{m})$ be a homomorphic image of a Cohen-Macaulay local ring. Let $M$ be a finitely generated $R$-module of dimension $d$ and let $U_i(M)$, $0\leq i\leq d-1$, be the Cohen-Macaulay deviated sequence of $M$. We define the *unmixed degree* of $M$ with respect to an $\mathfrak{m}$-primary ideal $I$, $\operatorname{udeg}(I,M)$, as follows:

$$\operatorname{udeg}(I,M)=e(I,M)+\sum_{i=0}^{d-1}\delta_{i,\dim U_i(M)}e(I,U_i(M)).$$

It was shown in [12, Theorem 5.18] that $\operatorname{udeg}(I,\bullet)$ is an extended degree. We make an elementary but important observation that, for a fixed finitely generated module $M$, $\operatorname{hdeg}(I,M)$ (resp. $\operatorname{udeg}(I,M)$) is a *finite* sum $\sum_i e(I,M_i)$, where $\{M_i\}$ only depends on $M$ (this is clear from definition for $\operatorname{udeg}(I,M)$, and is easily seen by induction for $\operatorname{hdeg}(I,M)$). Therefore by the associativity formula for multiplicities, for a fixed finitely generated $R$-module $M$, there exists a *finite* collection of prime ideals $\Lambda(M)=\Lambda$ (allowing repetition) such that

$$\operatorname{hdeg}(I,M)=\sum_{P\in\Lambda}e(I,R/P),\ \text{and similarly for } \operatorname{udeg}(I,M). \tag{A.1}$$

Now we are ready to state and prove our main result in this section:

**Theorem A.5.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let $M$ be a finitely generated quasi-unmixed $R$-module. Then we have*

$$n(M) = \sup_{\sqrt{I}=\mathfrak{m}} \left\{ \frac{l(M/IM)}{e(I,M)} \right\} < \infty.$$

*Proof.* First of all by [47, Theorem 1], it is enough to prove $n(R) < \infty$ where $R$ is a complete local domain with infinite residue field. We now consider $\mathrm{Deg}(I, M) = \mathrm{hdeg}(I, M)$ (or $\mathrm{Deg}(I, M) = \mathrm{udeg}(I, M)$), this is an extended degree and thus by Definition A.2 (2), we know that for every generic element $x \in I - \mathfrak{m}I$ we have

$$\mathrm{Deg}(I, R) \geq \mathrm{Deg}(I, R/xR).$$

Therefore for a generic sequence of elements $x_1, \ldots, x_d$ of $I$ (we may choose $x_i$ sufficiently general such that $x_1, \ldots, x_d$ is a system of parameters of $R$), we have

$$\begin{aligned} \mathrm{Deg}(I, R) \geq \mathrm{Deg}(I, R/x_1R) \geq \cdots \geq \mathrm{Deg}(I, R/(x_1, \ldots, x_d)) \\ = l(R/(x_1, \ldots, x_d)) \geq l(R/I), \end{aligned}$$

where the equality is because $R/(x_1, \ldots, x_d)$ is Cohen-Macaulay and thus

$$\mathrm{Deg}(I, R/(x_1, \ldots, x_d)) = e(I, R/(x_1, \ldots, x_d)) = l(R/(x_1, \ldots, x_d)).$$

Thus it is enough to prove that

$$\sup_{\sqrt{I}=\mathfrak{m}} \left\{ \frac{\mathrm{Deg}(I,R)}{e(I,R)} \right\} < \infty.$$

At this point we invoke (A.1), it is enough to prove that for every $P \in \mathrm{Spec}R$,

$$\sup_{\sqrt{I}=\mathfrak{m}} \left\{ \frac{e(I,R/P)}{e(I,R)} \right\} < \infty. \tag{A.2}$$

In order to prove (A.2), we use induction on $\dim R$. If $\dim R = 0$, (A.2) is obvious. In the general case, if $\dim R/P = \dim R$ (i.e., $P = 0$) then (A.2) is again obvious.

Now we assume $\dim R/P < \dim R$, we pick $0 \neq x \in P$ and a minimal prime $Q$ of $(x)$ such that $Q \subseteq P$. Since $R$ is a complete local domain, $R/(x)$ is equidimensional, in particular $\dim R/(x) = \dim R/Q$ and thus $e(I, R/(x)) \geq e(I, R/Q)$. Now we write:

$$\frac{e(I,R/P)}{e(I,R)} = \frac{e(I,R/P)}{e(I,R/Q)} \cdot \frac{e(I,R/Q)}{e(I,R/(x))} \cdot \frac{e(I,R/(x))}{e(I,R)} \leq \frac{e(I,R/P)}{e(I,R/Q)} \cdot \frac{e(I,R/(x))}{e(I,R)}.$$

Since $\dim R/Q < \dim R$, $\sup_{\sqrt{I}=\mathfrak{m}} \left\{\frac{e(I,R/P)}{e(I,R/Q)}\right\} < \infty$ by induction, which means there exists a constant $c_1$ such that $\frac{e(I,R/P)}{e(I,R/Q)} \leq c_1$ for all $\mathfrak{m}$-primary ideal $I$. Since $x$ is a nonzerodivisor in a complete local ring $R$, by Lemma A.6 below, we know that there exists a constant $c_2$ such that $\frac{e(I,R/(x))}{e(I,R)} \leq c_2$ for all $\mathfrak{m}$-primary ideal $I$. Thus putting $c = c_1c_2$ we see that

$$\frac{e(I,R/P)}{e(I,R)} \leq c$$

for all $\mathfrak{m}$-primary ideals $I$. This finishes the proof. □

**Lemma A.6.** *Let $(R, \mathfrak{m})$ be a Noetherian complete local ring and let $x$ be a nonzerodivisor on $R$. Then there exists a constant $k$ such that for all $\mathfrak{m}$-primary ideals $I$, we have*

$$e(I, R/(x)) \leq k \cdot e(I, R).$$

*Proof.* We consider the short exact sequence:

$$0 \to \frac{R}{I^n : x} \xrightarrow{\cdot x} \frac{R}{I^n} \to \frac{R}{I^n + (x)} \to 0$$

Note that if $y \in I^n : x$, then $xy \in I^n \cap (x)$. By Huneke's uniform Artin-Rees lemma [26], there exists a constant $k$ such that for all $I \subseteq R$, $I^n \cap (x) \subseteq I^{n-k}x$. Thus $xy \in I^{n-k}x$ and hence $y \in I^{n-k}$ since $x$ is a nonzerodivisor. This shows that $I^n : x \subseteq I^{n-k}$ for all $\mathfrak{m}$-primary ideals $I$. By the short exact sequence above, we know that

$$l\left(\frac{R}{I^n + (x)}\right) \leq l\left(\frac{R}{I^n}\right) - l\left(\frac{R}{I^{n-k}}\right)$$

Now we let $n \to \infty$ and compute the corresponding Hilbert function, we see that

$$e(I, R/(x)) \leq k \cdot e(I, R)$$

for all $\mathfrak{m}$-primary ideal $I$. □

### A.3.1 Localization and local flat extension

**Theorem A.7.** *Let $(R, )$ be a Noetherian ring and $M$ be a finite $R$-module. Assume that $R/Ann(M)$ is catenary and locally equidimensional. Then for any $\mathfrak{p} \in \operatorname{Supp}(M)$, then $\mathrm{n}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq \mathrm{n}_R(M)$.*

*Proof.* Without loss of generality, we assume that $M$ is faithful, and by easy induction we may assume that $\dim(R/\mathfrak{p}) = 1$. As $R_\mathfrak{p}$ has infinite residue field, it suffices to show

$$\mathrm{n}(M) \geq \frac{l_{R_\mathfrak{p}}(M_\mathfrak{p}/IM_\mathfrak{p})}{\mathrm{e}(I, M_\mathfrak{p})}$$

for all ideals $I$ generated by a system of parameters in $R_\mathfrak{p}$.

For any such ideal, by prime avoidance, we can find elements $x_1, \ldots, x_{d-1} \in R$ such that they form a part of a system of parameters in $R$ and $I$ is generated by their images in $R_\mathfrak{p}$. So, perhaps abusing notation, we will call $I = (x_1, \ldots, x_{d-1}) \subseteq R$.

Suppose $x \in R$ is such that $(I, x)$ is $\mathfrak{m}$-primary. Since $x_1, \ldots, x_{d-1}, x$ form a system of parameters, we have

$$l(M/(I, x)M) \geq \mathrm{e}(x, M/IM) = \sum_{Q \in \operatorname{Min}(M/IM)} \mathrm{e}(x, R/Q) l(M_Q/IM_Q),$$

where the last equality holds by the associativity formula. By Lech's associativity formula for parameter ideals

$$\mathrm{e}((I, x), M) = \sum_{Q \in \operatorname{Min}(M/IM)} \mathrm{e}(x, R/Q)\, \mathrm{e}(IR_Q, M_Q).$$

Therefore, by definition,

$$\begin{aligned}\mathrm{n}(M) &\geq \frac{l(M/(I,x)M)}{\mathrm{e}((I,x),M)}\\ &\geq \frac{\sum_{Q\in\mathrm{Min}(M/IM)} \mathrm{e}(x,R/Q) l_{R_Q}(M_Q/IM_Q)}{\sum_{Q\in\mathrm{Min}(M/IM)} \mathrm{e}(x,R/Q)\,\mathrm{e}(IR_Q,M_Q)}.\end{aligned}$$

Now, let $y \notin \mathfrak{p}$ be an element that belongs to all $\mathfrak{p} \neq Q \in \operatorname{Min} M/IM$ and $z \in \mathfrak{p}$ but not in any other element of $\mathrm{Min}(M/IM)$. Observe that for any $n$ we can use $x = y^n + z$ to complete $I$ to a system of parameters. In this case

$$\begin{aligned}&\frac{\sum_Q \mathrm{e}(x,R/Q) l_{R_Q}(M_Q/IM_Q)}{\sum_Q \mathrm{e}(x,R/Q)\,\mathrm{e}(IR_Q,M_Q)}\\ &\qquad= \frac{\mathrm{e}(y^n,R/\mathfrak{p}) l_{R_\mathfrak{p}}(M_\mathfrak{p}/IM_\mathfrak{p}) + \sum_{\mathfrak{p}\neq Q} \mathrm{e}(z,R/Q) l_{R_Q}(M_Q/IM_Q)}{\mathrm{e}(y^n,R/\mathfrak{p})\,\mathrm{e}(IR_\mathfrak{p},M_\mathfrak{p}) + \sum_{\mathfrak{p}\neq Q} \mathrm{e}(z,R/Q)\,\mathrm{e}(IR_Q,M_Q)}.\end{aligned}$$

Since $\mathrm{e}(y^n, R/\mathfrak{p}) = n\,\mathrm{e}(y, R/\mathfrak{p})$, if we pass to the limit as $n$ approaches infinity we obtain that

$$\mathrm{n}(M) \geq \frac{l_{R_\mathfrak{p}}(M_\mathfrak{p}/IM_\mathfrak{p})}{\mathrm{e}(IR_\mathfrak{p},M_\mathfrak{p})}. \qquad \square$$

As a corollary, we show that the invariant $\mathrm{n}(-)$ is non-decreasing under local flat extension.

**Corollary A.8.** *Let $(R,\mathfrak{m})$ be a local ring and $(S,\mathfrak{n})$ be a faithfully flat extension of $R$. Suppose $M$ is a finite $R$-module such that $R/Ann(M)$ is locally equidimensional and catenary. Then $\mathrm{n}(M) \leq \mathrm{n}(M \otimes_R S)$.*

*Proof.* Let $P$ be a minimal prime of $\mathfrak{m}S$. By the theorem

$$\mathrm{n}_{S_P}((M\otimes_R S)_P) \leq \mathrm{n}_S(M\otimes_R S),$$

so we may assume that $S$ is local and $\mathfrak{m}S$ is $\mathfrak{n}$-primary. For any $\mathfrak{m}$-primary ideal its extension $IS$ is $\mathfrak{n}$-primary ideal and tensoring the composition series with $S$ we see that

$$l_R(M/IM) l_S(S/\mathfrak{m}S) = l_S(M/IM \otimes_R S)$$

for any finite $R$-module $M$. Thus $\mathrm{e}(I)l_S(S/\mathfrak{m}S) = \mathrm{e}(IS)$ and

$$\mathrm{n}(M) = \sup_I \frac{l_R(M/IM)}{\mathrm{e}(I)} = \sup_I \frac{l_S\left((M \otimes_R S)/I(M \otimes_R S)\right)}{\mathrm{e}(IS)} \leq \mathrm{n}(M \otimes_R S). \qquad \square$$

**Question A.9.** *Let* $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ *be a local flat map and* $M$ *a finitely generated* $R$*-module. Is it true that* $\mathrm{n}(M \otimes_R S) \leq \mathrm{n}(M)\,\mathrm{n}(S/\mathfrak{m}S)$*? (Here we may either assume infinite residue fields or just define* $\mathrm{n}(-)$ *with parameter ideals.)*

For example, is it true when $S$ is module-finite and free over $R$. A concrete case could be $S = R[x]/(x^2)$. The above question has a positive answer if both $M$ and $S/\mathfrak{m}S$ are CM, as $1 = 1 \cdot 1$. We also have the following

**Lemma A.10.** *Let* $(R, \mathfrak{m})$ *be a local ring and* $(S, \mathfrak{n})$ *be a faithfully flat extension of* $R$*. Suppose* $M \neq 0$ *is an* $R$*-module with* $l_R(M) < \infty$*. Then* $\mathrm{n}(M \otimes_R S) \leq \mathrm{n}(S/\mathfrak{m}S) = \mathrm{n}(M)\,\mathrm{n}(S/\mathfrak{m}S)$.

*Proof.* Say $l(M) = c$, hence $M$ has a filtration of length $c$ with each consecutive quotient isomorphic to $R/\mathfrak{m}$. Thus there is an induced filtration of $M \otimes S$ of length $c$ with each consecutive quotient isomorphic to $S/\mathfrak{m}S$.

Let $I$ be any $\mathfrak{n}$-primary ideal of $S$. Via the above filtration, we see

$$\begin{aligned} \frac{l((M \otimes S)/I(M \otimes S))}{\mathrm{e}(I, M \otimes S)} &\leq \frac{c \cdot l((S/\mathfrak{m}S)/I(S/\mathfrak{m}S))}{c \cdot \mathrm{e}(I, S/\mathfrak{m}S)} \\ &= \frac{l((S/\mathfrak{m}S)/I(S/\mathfrak{m}S))}{\mathrm{e}(I, S/\mathfrak{m}S)} \leq \mathrm{n}(S/\mathfrak{m}S). \end{aligned}$$

Therefore $\mathrm{n}(M \otimes_R S) \leq \mathrm{n}(S/\mathfrak{m}S) = \mathrm{n}(M)\,\mathrm{n}(S/\mathfrak{m}S)$. $\square$

**Lemma A.11.** *Let* $(R, \mathfrak{m})$ *be a local ring and* $(S, \mathfrak{n})$ *be a faithfully flat extension of* $R$*. Suppose* $M \neq 0$ *is a finitely generated* $R$*-module, and denote* $M_S = M \otimes_R S$*. Then for all (full) sop* $x$ *of* $M$ *and all (full) sop* $y$ *of* $S/\mathfrak{m}S$*,*

$$\frac{l_S(M_S/(x, y)M_S)}{\mathrm{e}((x, y), M_S)} \leq \mathrm{n}(M)\,\mathrm{n}(S/\mathfrak{m}S).$$

*Proof.* Denote $N = M/xM$, so that $N$ has finite length over $R$. By Lemma A.10 and associativity formula, we see

$$\begin{aligned}
l_S(M_S/(x,y)M_S) &= l_S(N_S/(y)N_S) \\
&\leq \mathrm{e}((y), N_S)\, \mathrm{n}(S/\mathfrak{m}S) \\
&= \sum_P \mathrm{e}((y), S/P) l(N_{S_P})\, \mathrm{n}(S/\mathfrak{m}S) \quad \text{(associativity formula)} \\
&= \sum_P \mathrm{e}((y), S/P) l(M_{S_P}/(x)M_{S_P})\, \mathrm{n}(S/\mathfrak{m}S) \\
&\leq \sum_P \mathrm{e}((y), S/P)\, \mathrm{e}((x), M_{S_P})\, \mathrm{n}(M)\, \mathrm{n}(S/\mathfrak{m}S) \\
&= \mathrm{e}((x,y), M_S)\, \mathrm{n}(M)\, \mathrm{n}(S/\mathfrak{m}S) \quad \text{(associativity formula)}
\end{aligned}$$

Therefore $\frac{l_S(M_S/(x,y)M_S)}{\mathrm{e}((x,y),M_S)} \leq \mathrm{n}(M)\, \mathrm{n}(S/\mathfrak{m}S)$. $\square$

### A.3.2 Modding out by a sop

We say that a sequence $\underline{x}$ is a (partial or full) sop on an $R$-module $M$ if $\underline{x}$ is a (partial or full) sop on $R/\mathrm{Ann}(M)$.

**Theorem A.12.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring with infinite residue field and $M$ be a finite $R$-module. Then for any $x = x_1, \ldots, x_c$ that form a (partial) sop of $M$, we have $\mathrm{n}(M/(x)M) \leq \mathrm{n}(M)$.*

*Proof.* Without loss of generality, we assume $M$ is a faithful $R$-module. Hence $x$ is a partial sop on $R$ as well. Choose any $y = y_1, \ldots, y_{d-c}$ such that $x, y$ together form a (full) sop of $R$, where $d = \dim(R)$.

As $\mathrm{e}((y), M/xM) \geq \mathrm{e}((x,y), M)$, we have

$$\frac{l(M/(x,y)M)}{\mathrm{e}((y), M/xM)} \leq \frac{l(M/(x,y)M)}{\mathrm{e}((x,y), M)} \leq \mathrm{n}(M).$$

Since $|R/\mathfrak{m}| = \infty$, we see $\mathrm{n}(M/(x)M) \leq \mathrm{n}(M)$. $\square$

## A.4 The lower bound: a generalization of Lech's inequality

Our goal in this section is to generalize the classical Lech's inequality to all finitely generated $R$-modules, thus proving the first part of our Main Theorem in the introduction. We first prove a lemma.

**Lemma A.13.** *Let $(R, \mathfrak{m}, k)$ be a complete local domain with an algebraically closed residue field. Let $M$ be a finitely generated $R$-module with $\dim(R) = \dim(M)$ and let $J$ be an integrally closed $\mathfrak{m}$-primary ideal. Then we have*

$$l(M/JM) \geq l(R/J) \dim_K(M \otimes_R K),$$

*where $K$ denotes the fraction field of $R$.*

*Proof.* First of all, if we let $T(M)$ denote the torsion part of $M$, then we have

$$0 \to T(M) \to M \to M' \to 0$$

where $M'$ is torsion-free. Since $l(M/JM) \geq l(M'/JM')$ while $\dim_K(M \otimes_R K) = \dim_K(M' \otimes K)$, if the lemma holds for $M'$ then it also holds for $M$. Thus in the rest of the proof we assume $M$ is torsion-free. In this case $\dim_K(M \otimes K) = \mathrm{rank} M$.

By [41, Corollary 2.2], we have

$$l(M/JM) \geq \bar{l}(R/J) \cdot \mathrm{rank} M,$$

where $\bar{l}(R/J)$ denote the length of the longest chain of integrally closed ideals between $J$ and $R$. Therefore it is enough to show $\bar{l}(R/J) = l(R/J)$. To prove this it is enough to find an integrally closed ideal $J' \supseteq J$ in $R$ such that $l(J'/J) = 1$, because then $\bar{l}(R/J) = l(R/J)$ follows from an easy induction. Let $R \to S$ be the normalization of $R$. Since $R$ is a complete local domain, $S$ is local by [28, Proposition 4.8.2] and thus

$S = (S, \mathfrak{n})$ is a normal local domain with $R/\mathfrak{m} = S/\mathfrak{n} = k$ since $k$ is algebraically closed. Now by [53, Theorem 2.1], there exist a chain

$$\overline{JS} = J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n = \mathfrak{n}$$

such that

1. Each $J_i$ is integrally closed in $S$;

2. $l(J_{i+1}/J_i) = 1$ for every $i$.

Since $J$ is integrally closed in $R$ and $S$ is integral over $R$, by [28, Proposition 1.6.1] we know

$$J_0 \cap R = \overline{JS} \cap R = \bar{J} = J.$$

Let $t = \max\{i | J_i \cap R = J\}$, obviously $0 \leq t < n$. Set $J' = J_{t+1} \cap R$, it is easy to see that $J' \supseteq J$ is integrally closed in $R$ (one can use [28, Proposition 1.6.1] again). Moreover, $l(J'/J) > 0$ by our choice of $t$ while $J'/J \hookrightarrow J_{t+1}/J_t$ shows that $l(J'/J) \leq l(J_{t+1}/J_t) = 1$. Thus we have $l(J'/J) = 1$. $\square$

We are ready to state and prove the following generalization of Lech's inequality.

**Theorem A.14.** *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of dimension $d$. Then for every finitely generated $R$-module $M$ and every $\mathfrak{m}$-primary ideal $I$, we have*

$$e(I, M) \leq d! e(\overline{R}) l(M/IM)$$

*where $\overline{R} = R/Ann_R M$.*

*Proof.* First of all we can replace $R$ by $\overline{R}$, this does not change the left hand side of the inequality, and the right hand side only possibly decreases because $\dim \overline{R} \leq d$. Therefore we may assume $\mathrm{Ann}_R M = 0$ and thus $\dim M = d$. We next take a flat local extension $(R, \mathfrak{m}, k) \to (R', \mathfrak{m}', k')$ such that $\mathfrak{m}' = \mathfrak{m}R'$ and $k' = R'/\mathfrak{m}'$ is the algebraic closure of $R/\mathfrak{m} = k$ (such $R'$ always exists: it is a suitable gonflement of $R$; see [8]). Then $R \to R' \to \widehat{R'}$ is a faithfully flat extension with $\mathfrak{m}\widehat{R'} = \mathfrak{m}_{\widehat{R'}}$, so passing from $R$ to $\widehat{R'}$ and replacing $M$ by $M \otimes_R \widehat{R'}$ do not affect both sides of the inequality. Therefore without loss of generality, we may assume $(R, \mathfrak{m}, k)$ is a complete local ring of dimension $d$ with $k = \bar{k}$ and $\dim M = d$.

By the associativity formula of multiplicity, we have

$$e(I, M) = \sum_{\dim R/P = d} l_{R_P}(M_P) e(I, R/P) = \sum l_{R_P}(M_P) e(\overline{IR/P}, R/P).$$

Using Lech's inequality [35] for each $R/P$, we have

$$e(I, M) \le \sum d! e(R/P) l\left((R/P)/(\overline{IR/P})\right) l_{R_P}(M_P). \tag{A.3}$$

**Claim A.15.** *For every minimal prime $P$ of $R$, we have*

$$l\left((R/P)/(\overline{IR/P})\right) \cdot l_{R_P}(M_P) \le l(M/IM) \cdot l_{R_P}(R_P). \tag{A.4}$$

*Proof of Claim.* Clearly we have $l_{R_P}(M_P) \le l_{R_P}(R_P) \cdot l_{R_P}(M_P/PM_P)$, because $l_{R_P}(M_P/PM_P)$ is the minimal number of generators of $M_P$ as an $R_P$-module. Therefore

$$l\left((R/P)/(\overline{IR/P})\right) \cdot l_{R_P}(M_P) \le l\left((R/P)/(\overline{IR/P})\right) \cdot l_{R_P}(M_P/PM_P) \cdot l_{R_P}(R_P).$$

Now $R/P$ is a complete local domain with algebraically closed residue field $k = \bar{k}$ and $M/PM$ is a finitely generated $R/P$-module. Applying Lemma A.13 and noting that $\dim_{\kappa(P)}(M/PM) \otimes \kappa(P) = l_{R_P}(M_P/PM_P)$, we have

$$\begin{aligned} l\left((R/P)/(\overline{IR/P})\right) \cdot l_{R_P}(M_P/PM_P) &\le l\left(\frac{M/PM}{(\overline{IR/P})(M/PM)}\right) \\ &\le l\left(\frac{M}{(I+P)M}\right) \le l(M/IM). \end{aligned}$$

Putting the above two inequalities together we get

$$l\left((R/P)/(\overline{IR/P})\right)\cdot l_{R_P}(M_P)\leq l(M/IM)\cdot l_{R_P}(R_P).$$

This finishes the proof of the Claim. $\square$

Finally, we plug in (A.4) to (A.3) and apply the associativity formula of multiplicity to get:

$$\begin{aligned} e(I,M) &\leq \sum d!e(R/P)l_{R_P}(R_P)l(M/IM) \\ &= d!l(M/IM)\left(\sum_{\dim R/P=d} l_{R_P}(R_P)e(R/P)\right) \\ &= d!e(R)l(M/IM). \end{aligned}$$

This finishes the proof. $\square$

## A.5 Some applications

**Lemma A.16.** *If $(R,\mathfrak{m})$ a Noetherian local ring and $M$ is a finitely generated quasi-unmixed $R$-module, then*

$$\sup_{\substack{\sqrt{I}=\mathfrak{m}\\ I\subseteq J\subseteq \bar{I}}}\left\{\frac{l(M/IM)}{l(M/JM)}\right\}<\infty$$

*where $\bar{I}$ denotes integral closure of $I$.*

*Proof.* With notation as in (A.5), we use (A.5) and (A.14) to see

$$l(M/IM)\leq n(M)e(I,M)=n(M)e(J,M)\leq n(M)d!e(R)l(M/JM).$$

$\square$

In the above lemma, we particularly have in mind the case where $J$ is a minimal reduction of $I$ or in characteristic $p>0$ to show that

$$\sup_{\substack{\sqrt{I}=\mathfrak{m}\\ J=I*}}\left\{\frac{l(M/IM)}{l(M/JM)}\right\}<\infty$$

where $I^*$ is the tight closure of $I$.

**Lemma A.17.** *Let $(R, m)$ be a local ring and $M$ a finitely generated $R$-module. If $(y_1, \ldots, y_n) \subseteq (x_1, \ldots, x_d)$ are $m$-primary ideals of $R$, then for all $0 \le i \le d$,*

$$l(H_i(x_1, \ldots, x_d; M)) \le \sum_{k=0}^{d} \binom{d}{k} l(H_{i-k}(y_1, \ldots, y_n; M))$$

*Proof.* If $\underline{f} = f_1, \ldots, f_s$ is any sequence of elements of $R$ and $\underline{f}^- = f_1, \ldots, f_{s-1}$, then there is a short exact sequence for each $0 \le i \le s-1$

$$0 \to \frac{H_i(\underline{f}^-; M)}{f_s H_i(\underline{f}^-; M)} \to H_i(\underline{f}; M) \to \mathrm{Ann}_{H_{i-1}(\underline{f}^-; M)}(f_s) \to 0.$$

Using that each $y_j \in (x_1, \ldots, x_d)$ so that $y_j H_i(x_1, \ldots, x_d, y_1, \ldots, y_{j-1}; M) = 0$ for $1 \le j \le n$ and $y_1 H_i(x_1, \ldots, x_d; M) = 0$, it follows from the first injection in the short exact sequence above that

$$l(H_i(x_1, \ldots, x_d; M)) \le l(H_i(x_1, \ldots, x_d, y_1; M)) \le s \le l(H_i(x_1, \ldots, x_d, y_1, \ldots, y_n; M)).$$

Now using that for $1 \le j \le d-1$,

$$l\left(\frac{H_i(x_1, \ldots, x_j, y_1, \ldots, y_n; M)}{x_{j+1} H_i(x_1, \ldots, x_j, y_1, \ldots, y_n; M)}\right) \le l(H_i(x_1, \ldots, x_j, y_1, \ldots, y_n; M))$$

and that

$$l\left(\frac{H_i(y_1, \ldots, y_n; M)}{x_1 H_i(x_1, y_1, \ldots, y_n; M)}\right) \le l(H_i(x_1, y_1, \ldots, y_n; M)),$$

$$\begin{aligned} &l(H_i(x_1, \ldots, x_d, y_1, \ldots, y_n; M)) \\ &\qquad \le l(H_i(x_1, \ldots, x_{d-1}, y_1, \ldots, y_n; M)) + l(H_{i-1}(x_1, \ldots, x_{d-1}, y_1, \ldots, y_n; M)). \end{aligned}$$

and that, by iterating, the previous expression is bounded above by

$$\sum_{k=0}^{d} \binom{d}{k} l(H_{i-k}(y_1, \ldots, y_n; M)),$$

completing the proof. $\square$

**Theorem A.18.** *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring of dimension $d$ and $M$ a finitely-generated quasi-unmixed $R$-module. Then for every $k \geq 0$,*

$$\sup_{\substack{\sqrt{(x_1,\ldots,x_{d+k})}=\mathfrak{m} \\ 0\leq i\leq d+k}} \left\{ \frac{l(H_i(x_1, \ldots, x_{d+k}; M))}{l(M/(x_1, \ldots, x_{d+k})M)} \right\} < \infty.$$

*Proof.* Using A.17 and A.16, we may assume $k = 0$. As in the proof of (A.14), we may make a flat local extension so that $R$ is complete with algebraically closed residue field. Let $R/P_1, \ldots, R/P_k$ be the (not necessarily distinct) factors appearing in a prime cyclic filtration of $M$. We note that for each $0 \leq i \leq d$,

$$\begin{aligned} \frac{l(H_i(x_1, \ldots, x_d; M))}{l(M/(x_1, \ldots, x_d)M)} &\leq \frac{\sum_{j=1}^k l(H_i(x_1, \ldots, x_d; R/P_j))}{l(M/(x_1, \ldots, x_d)M)} \\ &\leq \sum_{j=1}^k \frac{l(H_i(x_1, \ldots, x_d; R/P_j))}{l(M/((x_1, \ldots, x_d) + P_j)M)}. \end{aligned}$$

Now each $M/P_jM$ is a finitely generated faithful module over the complete local domain $R/P_j$ with infinite residue field. It then follows from (A.16) and (A.13) that we may replace each term $l(M/((x_1, \ldots, x_d) + P_j)M)$ by $l(R/((x_1, \ldots, x_d) + P_j)R)$ without impacting whether the supremum we study is finite. We have now reduced to the case of $M = R$ a complete local domain of dimension $n \leq d$.

Fix $A$ a regular local ring of which $R$ is a module-finite extension. We will first prove the claim for any finitely generated $A$-module $N$. Let $(y_1, \ldots, y_n)$ any system of parameters in $A$. We note that for each $i > 1$ and each $j \geq 0$,

$$H_i(y_1, \ldots, y_n; \mathrm{syz}^{j-1}(N)) \cong H_{i-1}(y_1, \ldots, y_n; \mathrm{syz}^j(N)),$$

and so by taking syzygies repeatedly, may assume that $i = 1$. But then the long exact sequence for homology gives

$$0 \to H_1(y_1, \ldots, y_n; N) \to \mathrm{syz}^1(N)/(y_1, \ldots, y_n)\mathrm{syz}^1(N),$$

and we know that $l(\mathrm{syz}^1(N)/(y_1,\ldots,y_n)\mathrm{syz}^1(N)) \leq \nu_A(\mathrm{syz}^1(N))\cdot l(A/(y_1,\ldots,y_n))$ where $\nu$ denotes the minimal number of generators. With $R$ playing the role of $N$, we have now bounded the desired ratio for systems of parameters coming from $A$.

We now scaffold the general case. Let $(y_1,\ldots,y_n)$ be a minimal reduction of $(x_1,\ldots,x_d)\cap A$ in $A$. Because $(y_1,\ldots,y_n) \subseteq (x_1,\ldots,x_d)\cap A \subseteq \overline{(y_1,\ldots,y_n)}$ in $A$, we may use (A.16) to bound the $\dfrac{l(R/(y_1,\ldots,y_n)R)}{l(R/(x_1,\ldots,x_d))} \leq \dfrac{\nu_A(R)\cdot l(A/(y_1,\ldots,y_n))}{l(A/(x_1,\ldots,x_d)\cap A)}$ independent of the choices of the $x_i$ and $y_j$. It, therefore, suffices to bound the $l(H_i(x_1,\ldots,x_d;R))$ for $0\leq i\leq d$ from above in terms of the $l(H_j(y_1,\ldots,d_n;R))$ for $0\leq j\leq n$. The result now follows from (A.17) using that each $y_j\in(x_1,\ldots,x_d)$ for $1\leq j\leq n$ so that $l(H_i(x_1,\ldots,x_d;R)) \leq \sum_{k=0}^{n}\binom{n}{k} l(H_{i-k}(y_1,\ldots,y_n;R))$ completing the proof. $\square$

In [6], it is shown that whenever $R$ is a complete local domain of characteristic $p>0$ and dimension $d\geq 1$ with perfect residue field, for every parameter ideal $I=(x_1,\ldots,x_d)$ of $R$, $\dfrac{H_i(x_1,\ldots,x_d;R^{1/p^e})}{l(R/I^{[p^e]})} \xrightarrow{e\to\infty} 0$ for each $1\leq i\leq d$. The characteristic-free application below together with the fact that $H_i(x_1^{p^e},\ldots,x_d^{p^e};R)\cong H_i(x_1,\ldots,x_d;R^{1/p^e})$ for $0\leq i\leq d$ shows that the convergence to 0 occurs independent of the parameter ideal $I$.

**Theorem A.19.** *Let $R$ be a quasi-unmixed local ring of dimension $d\geq 1$. For every $\varepsilon>0$, there exists $t_0$ such that for all $t\geq t_0$, all parameter ideals $I=(x_1,\ldots,x_d)$ of $R$, and all $1\leq i\leq d$,*

$$\frac{H_i(x_1^t,\ldots,x_d^t;R)}{l(R/(x_1^t,\ldots,x_d^t))} < \varepsilon$$

*Proof.* Fix $R$ as in the statement of the theorem and $\varepsilon>0$. We replace $R$ by $R(s)$ so that the residue field of $R$ is infinite. Because we consider only finitely many $i$, it is sufficient to fix some $1\leq i\leq d$. By tensoring the dualizing complex for $R$ with the

Koszul complex on $I_t = (x_1^t, \dots, x_d^t)$, we obtain a spectral sequence that bounds the lengths of the Koszul homology modules $H_i(x_1^t, \dots, x_d^t; R)$ in terms of the lengths of the Koszul homology modules of the duals of the local cohomology modules $H_m^i(R)$ with $1 \le i < d$. More precisely,

$$l(H_i(x_1^t, \dots, x_d^t; R)) \le \sum_{j=0}^{d-i} l(H_j(x_1^t, \dots, x_d^t; H_m^{d-i-j}(R)^\vee)).$$

The details of this computation can be found in [6]. The duals $H_m^{d-i-j}(R)^\vee$ for $1 \le i \le d$ and $0 \le j \le d-i$ have strictly lower dimension than $R$. By taking a prime cyclic filtration of each $H_m^{d-i-j}(R)^\vee$, it suffices to show that

$$\frac{l(H_j(x_1^t, \dots, x_d^t; D))}{l(R/I_t)} < \varepsilon/d.$$

for $0 \le j \le d-1$ and $(D, \mathfrak{m})$ a domain of dimension $d' \le d-1$ that is a proper quotient of $R$. By A.18, we fix

$$C_{D,d} = \sup_{\substack{\sqrt{(x_1,\dots,x_d)}=\mathfrak{m} \\ 0 \le i \le d}} \left\{ \frac{l(H_i(x_1, \dots, x_d; D))}{l(D/(x_1, \dots, x_d)D)} \right\} < \infty,$$

and by A.16 we fix

$$B_D = \sup_{\substack{\sqrt{I}=\mathfrak{m} \\ I \subseteq J \subseteq \bar{I}}} \left\{ \frac{l(D/I)}{l(D/J)} \right\} < \infty.$$

Let $J = (y_1, \dots, y_{d'})$ be a minimal reduction of $I$ in $D$ and $J_t = (y_1^t, \dots, y_{d'}^t)$, and compute for every $t \ge 1$

$$\begin{aligned} \frac{l(H_j(x_1^t, \dots, x_d^t; D))}{l(R/I_t)} &\le \frac{C_{D,d} l(D/I_t)}{l(R/I_t)} \le \frac{C_{D,d} l(D/J_t)}{l(R/I_t)} \le \frac{t^{d'} C_{D,d} l(D/J)}{l(R/I_t)} \\ &\le \frac{t^{d'} C_{D,d} l(D/J)}{e(I_t; R)} \le \frac{t^{d'} C_{D,d} l(D/J)}{t^d e(I; R)} \le \frac{B_D C_{D,d} l(D/I)}{t e(I; R)} \le \frac{B_D C_{D,d} l(D/I)}{t n(R) l(R/I)} \le \frac{B_D C_{D,d}}{t n(R)}. \end{aligned}$$

We may, therefore, take $t_0 \ge \dfrac{d B_D C_{D,d}}{n(R)\varepsilon}$. $\square$

Lastly, we return to A.13 to give a result that omits the hypothesis that the ideal of interest be integrally closed at the cost of precision in the inequality. Define

$\text{Assh}(M) = \{P \in \text{Ass}(M) \mid \dim(R/P) = \dim(R)\}$ for any finitely-generated $R$-module.

**Corollary A.20.** *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring that is either equal characteristic or in which char$(\kappa)$ is a parameter, and let $M$ be a finitely-generated $R$-module of dimension $d$. Then $\sup_{\sqrt{I}=m} \left\{ \frac{\ell(R/I)}{\ell(M/IM)} \right\} < \infty$.*

*Proof.* Because completing does not change any of the length computations, we may assume that $R$ and $M$ are complete. We may also replace $R$ by $R(t)$ so that the residue field of $R$ is infinite. Because $R$ is either equal characteristic or $\text{char}(\kappa)$ is a parameter, $R$ is module finite over some regular local ring $A$. Because $l(R/I) \leq \nu_A(R) l(A/(I \cap A))$, we may without loss of generality assume that $R$ is a regular local ring. Let $T$ be the torsion submodule of $M$ and $\overline{M} = M/T$. Because $l(M/IM) \geq l(\overline{M}/I\overline{M})$, we may assume that $M$ is a torsion-free $R$-module. Recall that because $R$ is a regular local ring with infinite residue field, $\sup_{\sqrt{I}=m} \left\{ \frac{\ell(R/I)}{e(I,R)} \right\} = 1$. It then follows from Theorem A.14 (to obtain the first inequality) that for every $m$-primary ideal $I$ of $R$

$$l(M/IM) \geq \frac{e(I,M)}{d!e(R)} \geq \text{rk}_R(M) \frac{e(I,R)}{d!e(R)} \geq \text{rk}(M) \frac{l(R/I)}{d!e(R)},$$

as desired. □

The same cannot be hoped for if $R$ is not module-finite over a regular ring. If $R = V[[x]]/(px)$ where $V = (V, p)$ a complete discrete valuation ring, $M = R/(x) \cong V$, and $I_n = (p - x^n)$ for each positive integer $n$, then $l(R/I_n) = n + 1$ while $l(M/I_n M) = 1$ for all $n \geq 1$.

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