Pseudoeffective Cones and Morphisms of Projective Varieties

by

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ABSTRACT

The cycles on an algebraic variety contain a great deal of information about its geometry. This thesis is concerned with the pseudoeffective cone obtained by taking the closure of the cone of numerical classes of effective cycles on algebraic varieties. Our interest, motivated by different existing lines of research, is in showing when the pseudoeffective cone is not polyhedral in specific examples. We do this by first proving a sufficient criterion for non-polyhedral pseudoeffective cone (also known as Mori cone) for the case of surfaces. We apply this to the case of $C \times C$ where C is a smooth projective curve of genus at least 2. Using induction, we prove that all intermediate cones of cycles on $C \times \ldots \times C$ are not polyhedral. Finally, we study the case of surfaces fibered over curve and give a sufficient criterion for when its pseudoeffective cone is not polyhedral.

CHAPTER 1

Introduction

1.1 Overview

A *cycle* on an algebraic variety X is a linear combination (say, with real coefficients) of algebraic subvarieties of X. If all these subvarieties have dimension k (or codimension k), the cycle has dimension (respectively, codimension k). Cycles are usually considered up to a suitable equivalence relation such that the resulting quotient space for k-dimensional cycles (respectively codimension k) is a finite-dimensional real vector space $N_k(X)$ (respectively[1] $N^k(X)$) . A cycle is *effective* if all its coefficients are nonnegative. Effective cycles in $N_k(X)$ (respectively $N^k(X)$) generate cones whose closures are denoted $\overline{\mathrm{Eff}}_k(X)$ (respectively $\overline{\mathrm{Eff}}^k(X)$). Suppose that X is a smooth projective variety of dimension n. In this case, we have a perfect pairing between $N_k(X)$ and $N^k(X)$ given by intersecting cycles of complimentary dimensions. Therefore $N_k(X)$ and $N^k(X)$ have the same dimension.

The purpose of this thesis is to study the shape of $\overline{\mathrm{Eff}}_k(X)$ in a variety of situations. It turns out that the shape of these cones contains a lot of information about the geometry of X and its subvarieties. Historically, most of the results about the shape have focused on when $\overline{\mathrm{Eff}}_k(X)$ is polyhedral – in other words, when it is generated as a cone by finitely many elements of $N_k(X)$. Our perspective is the opposite – we seek to prove that $\overline{\mathrm{Eff}}_k(X)$ is *non-polyhedral* for certain classes of X. Let us take for example the case when X is a surface i.e. $\dim(X) = 2$. In this case, $\overline{\mathrm{Eff}}_1(X)$ is the well-known *Mori cone* (usually denoted $\overline{\mathrm{NE}}(X)$) and there has been a great deal of study of the shapes of this cones. Consider the following pair of well-known examples which illustrate the dichotomy between polyhedral and non-polyhedral Mori cones. In what follows, we work over the complex numbers $\mathbb{C}$.

Example. When $X = \mathbb{P}^1 \times \mathbb{P}^1$ we have that

$$\overline{\mathrm{Eff}}_1(X) = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}.$$

[1]Note that $N^k(X) = N_{\dim(X)-k}(X)$.

In this case, the Mori cone is polyhedral and is generated in $\mathbb{R}^2$ by $(1, 0)$ and $(0, 1)$ – it's the first quadrant.

Example. An elliptic curve E is a smooth projective curve of genus 1 with a marked point. There's a well-known group action on E where the marked point serves as the zero element. When $X = E \times E$ where E is a general elliptic curve over the complex numbers, $\overline{\mathrm{Eff}}_1(X)$ is not a polyhedral cone. In fact, in $\mathbb{R}^3$ it can be described by the equations

$$\begin{aligned} xy + yx + xz &\geq 0 \\ x + y + z &\geq 0 \end{aligned}$$

which describe a circular cone $\mathcal{K} \subset \mathbb{R}^3$.

The central goal of this thesis is to show that many interesting examples folllow the pattern in the second case above. One can't expect a necessary and sufficient criterion for when $\overline{\mathrm{Eff}}_k(X)$ is not polyhedral. Instead, we establish a sufficient criterion in the case of surfaces which turns out to be quite powerful. In particular, this allows us to analyze certain surface fibrations. We discuss the special case of $C \times C$ when C has genus $g \geq 2$ – it follows the example of $E \times E$ in not having polyhedral Mori cone. We also seek to understand higher codimension cycles in the case of $C \times \ldots \times C$. Using our result about $C \times C$, we prove that intermediate pseudoeffective cones of $C \times \ldots \times C$ are not polyhedral.

1.2 Historical Context

1.2.1 Mori Cones and Nef Cones

Let X be a smooth projective variety. The study of effective cones begins with the study of the nef cone of divisors. An integral divisor D is *ample* if some multiple mD gives a closed embedding into projective space. The convex cone generated by ample divisors in $N^1(X)$ is said to be the *ample cone*. Its closure in $N^1(X)$ is called the *nef cone*. The nef cone turns out to be the cone dual of the Mori cone $\overline{\mathrm{NE}}(X)$.

In his seminal paper [21], Mori proved that the structure of $\overline{\mathrm{Eff}}_1(X)$ (also denoted $\overline{\mathrm{NE}}(X)$) can be broken up into two parts as described as follows. Assume that X is a smooth projective variety and D is a divisor on X. Recall that we have a pairing between divisors and curves given by intersections. Write

$$\overline{\mathrm{NE}}(X)_{D\geq 0} := \overline{\mathrm{NE}}(X) \cap D_{\geq 0}$$

to be the subset of $\overline{\text{NE}}(X)$ whose intersection with D is nonnegative.

Theorem (Cone Theorem, [21])**.** Assume that $\dim(X) = n$ and the canonical divisor K_X fails to be nef.

1. There are countably many rational curves $C_i \subset X$ with

$$0 \leq -(C_i \cdot K_X) \leq n+1,$$

   that together with $\overline{\text{NE}}(X)_{K_X \geq 0}$ generate $\overline{\text{NE}}(X)$. In other words,

$$\overline{\text{NE}}(X) = \overline{\text{NE}}(X)_{K_X \geq 0} + \sum_i \mathbb{R}_{\geq 0} \cdot C_i.$$

2. Fix an ample divisor H. Then, given $\varepsilon > 0$, there are only finitely many of these curves – say $C_1, \dots, C_t$ – whose classes lie in the region $(K_X + \varepsilon H \leq 0)$. Therefore,

$$\overline{\text{NE}}(X) = \overline{\text{NE}}(X)_{(K_X+\varepsilon H)\geq 0} + \sum_{i=1}^{t} \mathbb{R}_{\geq 0} \cdot C_i.$$

Although we do not prove the cone theorem, we emphasize the importance of C_i being rational curves. Mori proves this using his famous 'bend and break' argument that produces a rational curve L_x through a given point $x \in X$ with a bound on its degree $\deg_H(L_x) = (L_X \cdot H)$ with respect to a fixed ample H. We apply the second part of the cone theorem to the following example which gives us a large class of varieties where $\overline{\text{NE}}(X)$ is polyhedral.

Example (Fano varieties, [24])**.** Let X be a Fano variety, i.e. a smooth projective variety such that $-K_X$ is ample. Then $\overline{\text{NE}}(X) \subset N_1(X)$ is a polyhedral cone generated by finitely many rational curves. This is essentially because for $\varepsilon > 0$ small enough, $-K_X - \varepsilon H$ is ample and therefore $\overline{\text{NE}}(X)_{(K_X+\varepsilon H)\geq 0} = 0$.

For example, we have that $\mathbb{P}^2$ blown up at $r \leq 6$ points is Fano and therefore has polyhedral Mori cone. However when r is taken to be larger than 10, this breaks down as seen in the following example.

Example ($\mathbb{P}^2$ blown up at $r \geq 10$ very general points, [24])**.** Let X be the blowing up of $\mathbb{P}^2$ at $r \geq 10$ very general points. Let $e_i \in N^1(X)$ be the classes of the exceptional divisors and let ℓ be the pullback to X of the hyperplane class on $\mathbb{P}^2$. We may fix $0 < \varepsilon \ll 1$ such that $h := \ell - \varepsilon \cdot \sum e_i$ is ample. It's known (see [17] Ex. V.4.15) that there exists a sequence

$C_i \subset X$ of smooth rational curves with

$$(C_i \cdot C_i) = -1 \text{ and } (C_i \cdot h) \to \infty.$$

Therefore each C_i generates an extremal ray of $\overline{\text{NE}}(X)$. Recall that an *extremal ray* τ of a cone σ is of the form $\mathbb{R}_{\geq 0} \cdot \alpha$ such that if $v, w \in \sigma$ and $v + w \in \tau$, then v and w lie in τ. The condition $(C_i \cdot h) \to \infty$ tells us that C_i are eventually numerically distinct and thus $\overline{\text{NE}}(X)$ has infinitely many extremal rays and therefore it is not polyhedral. Let K_X be the canonical divisor on X. The adjunction formula tells us that

$$\begin{aligned} -2 &= (C_i \cdot C_i) + (C_i \cdot K_X) \\ &= -1 + (C_i \cdot K_X), \end{aligned}$$

and therefore $(C_i \cdot K_X) = -1$. Therefore as $i \to \infty$, since $(C_i \cdot K_X) = -1$ but $(C_i \cdot h) \to \infty$, the rays $\mathbb{R}_{\geq 0} \cdot [C_i]$ generated by C_i cluster in $N_1(X)$ towards the plane $K_X^{\perp}$. This is illustrative of the cone theorem since the C_i generate the region $\overline{\text{NE}}(X) \cap (K_X)_{\leq 0}$. Harbourne in [16] and Hirschowitz in [18] have conjectured that the region $\overline{\text{NE}}(X) \cap (K_X)_{>0}$ is circular and consists of curves of nonnegative self intersection but this still open in this case.

The cone theorem stops being useful once K_X is nef because in this case $\overline{\text{NE}}(X) = \overline{\text{NE}}(X)_{K_X \geq 0}$. Therefore, different tactics are needed. There has been a great deal of study of various examples of this nature. We discuss a few of these, beginning with the case of when $K_X = 0$. When X is a surface, we are dealing with either K3 surfaces or abelian surfaces. We begin with the case where X an abelian variety. Recall that an abelian variety is a smooth projective variety X with the structure of an abelian group. Note that a *rational polyhedral* cone is one generated by finitely many vectors with rational coefficients in the standard basis.

Proposition 1 (Bauer, [3])**.** Let X be an abelian variety with $\dim(X) = n$. The following are equivalent.

1. $\overline{\text{NE}}(X)$ is rational polyhedral.

2. $\text{Nef}(X)$ is rational polyhedral.

3. X is isogenous to a product
$$X_1 \times \ldots \times X_r$$
where the X_i are mutually non-isogenous varieties and $\text{NS}(X_i) = \mathbb{Z}$ for $1 \leq i \leq r$.

From the example of $X = E \times E$ where E is a general elliptic curve we know that for an abelian variety $\overline{\text{NE}}(X)$ can fail to be polyhedral, not just rational polyhedral. We note that the methods used by Bauer in [3] do not allow us to say whether $\overline{\text{NE}}(X)$ is not polyhedral in the absence of condition 3 of Proposition 1.

Recall that a K3 surface is a smooth projective surface with trivial canonical bundle ω_X and $H^1(X, \mathcal{O}_X) = 0$. When X is a K3 surface, the situation is similar. Kovács has studied this case extensively and showed that in some cases $\overline{\text{NE}}(X)$ is polyhedral and in others it is not polyhedral. We discuss this case in greater detail in Chapter 6 in connection with elliptic fibrations. Let $\rho(X) := \dim_{\mathbb{R}} N^1(X) = \dim_{\mathbb{R}} N_1(X)$ be the *Picard rank* of X.

Proposition 2 (Kovács, [23])**.** Let X be a K3 surface with $\rho(X) \geq 3$.[2] One of the following mutually exclusive conditions is satisfied.

1. X does not contain any curve of negative self intersection. In this case, we have

$$\overline{\text{NE}}(X) = \text{Conv}(\mathcal{D}(X))$$

where for h an ample class,

$$\mathcal{D}(X) := \{\xi \in N_1(X) : (\xi \cdot h) > 0, (\xi \cdot \xi) = 0\}.$$

In particular, $\overline{\text{NE}}(X)$ is not polyhedral.

2. X contains curves of negative self intersection, in particular smooth rational curves. We have

$$\overline{\text{NE}}(X) = \sum_{\ell} \mathbb{R}_{\geq 0} \cdot \ell$$

where ℓ runs over all smooth rational curves in X. If there are finitely many smooth rational curves, $\overline{\text{NE}}(X)$ is polyhedral. If not, $\overline{\text{NE}}(X)$ is not polyhedral.

When K_X is ample, the situation once again depends on the Picard rank. Consider the case of a general hypersurface of large degree $d > n + 1$ in $\mathbb{P}^n$. The usual calculation by adjunction gives $\omega_X = \mathcal{O}_X(d - n - 1)$ and hence ω_X is ample when $d > n + 1$. We rely on the following theorem.

Theorem 1 (Grothendieck-Lefschetz, [15])**.** Let $S_d \in \mathbb{P}^n$ be a smooth degree d hypersurface. When $d > n + 1$, the restriction map

$$\mathbb{Z} = \text{Pic}(\mathbb{P}^n) \to \text{Pic}(S_d)$$

[2] Of course, when $\rho(X) \leq 2$, $\overline{\text{NE}}(X)$ is automatically polyhedral since a cone in $\mathbb{R}^2$ is generated by 2 vectors.

is surjective.

Therefore for a smooth hypersurface X of large degree, we are automatically in the realm of $\rho(X) = 1$ and therefore $\overline{\text{NE}}(X)$ is polyhedral.

Some other interesting examples include various moduli spaces. Take for instance the simple case of $\text{Sym}^n(C)$ which parametrizes effective divisors of degree n on a curve C. When C is general, $N^1(X)$ is spanned by two classes θ and x. Here θ is the class of the pullback of the theta divisor from the Jacobian along the Abel Jacobi map $\text{Sym}^n(C) \to J(C)$ which takes a divisor D to $D - nP$ for some point P. The class x represents the inclusion $\text{Sym}^{n-1}(C)$ in $\text{Sym}^n(C)$ via $D \mapsto D + P$. Since $\rho(\text{Sym}^n(C)) = 2$, the Mori cone is polyhedral. For $n \geq 2g + 1$, $\text{Sym}^n(C)$ is a projective bundle over the Jacobian and Pacienza gives a complete description of $\overline{\text{NE}}(\text{Sym}^n(C))$ in [32]. For small values of n, there has been a great deal of work by Kouvidakis, Pacienza, Mustopa and others studying both $\overline{\text{NE}}(\text{Sym}^n(C))$ and $\overline{\text{Eff}}^1(\text{Sym}^n(C))$. Further details can be found in [22], [32], [27] and [26]

Similar work has been done in the case of the Hilbert scheme of points on $\mathbb{P}^2$ denoted by $(\mathbb{P}^2)^{[n]}$ which parametrizes length n subschemes of $\mathbb{P}^2$. This is more complicated than $\text{Sym}^n(\mathbb{P}^2)$ because it includes additional tangent vector data. In fact, $\text{Sym}^n(\mathbb{P}^2)$ is singular for $n > 2$ but $(\mathbb{P}^2)^{[n]}$ is smooth. We have the Hilbert-Chow morphism

$$\pi : (\mathbb{P}^2)^{[n]} \to \text{Sym}^n(\mathbb{P}^2)$$

taking a length n subscheme to the associated 0-cycle. The map π resolves the singularities of $\text{Sym}^n(\mathbb{P}^2)$. Work by Huizenga and others tells us that $N_1((\mathbb{P}^2)^{[n]})$ and $N^1((\mathbb{P}^2)^{[n]})$ (being dual to each other) are two-dimensional and thus $\overline{\text{NE}}((\mathbb{P}^2)^{[n]})$ and $\overline{\text{Eff}}^1((\mathbb{P}^2)^{[n]})$ are both polyhedral. These cones are described in greater detail in the papers [1] and [19].

1.2.2 Higher Codimension Cycles

In recent years, there has been a great deal of interest in not merely understanding the Mori cone but all other cones of the form $\overline{\text{Eff}}_k(X)$ as well. Of course, when $k = \dim(X) - 1$ we are dealing with the usual pseudoeffective cone which is obtained by taking the closure of the cone generated by effective divisors. Its dual is somewhat mysterious and was only recently understood by work of Boucksom-Demailly-Paun-Peternell in [5]. They describe it as the cone of *movable curves*. Of course when $\dim(X) = 2$, we have

$$\overline{\text{Eff}}_1(X) = \overline{\text{NE}}(X) = \overline{\text{Eff}}^1(X).$$

Definition (Movable Curve)**.** A curve C on a smooth projective variety X is said to be movable if it is a member of a family $(C_t)_{t\in S}$ of curves in X such that $\bigcup_{t\in S} C_t$ is a dense open subset of X.

The closure of the cone generated in $N_1(X)$ by movable curves is denoted by Mov(X) and by the results in [5] is the dual of $\overline{\text{Eff}}^1(X)$.

As discussed in section 1.2.1, $\overline{\text{Eff}}^1(X)$ has been studied in the case of Abelian varieties. Since any effective divisor on an Abelian variety is nef, $\overline{\text{Eff}}^1(X) = \text{Nef}(X)$ in this case and the cone is completely described by Proposition 1. In general whenever $\rho(X) \leq 2$, both Nef(X) and $\overline{\text{Eff}}^1(X)$ are necessarily polyhedral. This happens for both Sym$^n(C)$ and $(\mathbb{P}^2)^{[n]}$ which are also discussed in section 1.2.1. Recently, Mihai Fulger has computed $\overline{\text{Eff}}_k(\mathbb{P}(E))$ for any projective bundle $\mathbb{P}(E)$ over a smooth curve C in [10]. Since $N_k(\mathbb{P}(E))$ is two-dimensional for all values of k, $\overline{\text{Eff}}_k(\mathbb{P}(E))$ is polyhedral.

Some of the most exciting work about higher codimension cycles has been in the study of the moduli space of curves. Consider the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves of genus g with n marked points. Although it is believed that the results below hold for $\overline{\text{Eff}}^k(X)$, we focus on the cone Eff$^k(X)$ in $N^k(X)$ generated by effective codimension cycles. [3] Chen and Coskun proved in [7] that for $n \geq 5$, Eff$^2(\overline{\mathcal{M}}_{1,n})$ has infinitely many extremal rays and for $n \geq 2$, Eff$^2(\overline{\mathcal{M}}_{2,n})$ has infinitely many extremal rays. Both these cones are therefore not polyhedral. This has been generalized in recent work by Mullane to higher codimension as well as higher genus.

Proposition 3 (Mullane, [25])**.** . In what follows, $\overline{\mathcal{M}}_{g,n}$ is the moduli space of stable curves of genus g with n marked points.

1. For $g \geq 3$ and $n \geq g-1$, Eff$^2(\overline{\mathcal{M}}_{g,n})$ has infinitely many extremal rays and is therefore not polyhedral.

2. Eff$^k(\overline{\mathcal{M}}_{g,n})$ is not polyhedral for $g \geq 2$ and $k \leq \min(n-g, g)$.

3. Eff$^k(\overline{\mathcal{M}}_{1,n})$ is not polyhedral for $k \leq n-2$.

1.3 Motivating Conjectures

Our work is connected to two motivating conjectures – the Bounded Negativity Conjecture for complex surfaces and the Strong Conjecture of Debarre-Jiang-Voisin on kernels of pushforwards. We discuss these conjectures and their connection to our work.

[3]Note that Eff$^k(X)$ is not necessarily closed and is a subset of $\overline{\text{Eff}}^k(X)$.

The first conjecture is the Bounded Negativity Conjecture. Its origins are shrouded in mystery and it is often referred to as a folklore conjecture.

Conjecture 1 (Bounded Negativity Conjecture)**.** For everysmooth projective surface S, there exists a lower bound $b(S)$ such that for all integral curves $Y \subset S$,

$$(Y \cdot Y) \geq b(S).$$

Over the complex numbers, this conjecture remains open in the general case. However, when X has a polyhedral Mori cone, it has finitely many extremal rays and therefore finitely many curves with negative self-intersection. In this case, the conjecture follows trivially. However as discussed in [4], there are all manner of examples over the complex numbers where $\overline{\mathrm{NE}}(X)$ is not polyhedral but bounded negativity is not provably false.

Over a field of positive characteristic, the conjecture is false as seen in the following example.

Example. When C is defined over $k = \bar{k}$ where $\mathrm{char}(k) = p > 0$, the graph $\Delta_e \in C \times C$ of the e^{th} power of Frobenius on C is an integral curve and

$$(\Delta_e \cdot \Delta_e) = p^{2e}(2 - 2g).$$

When e tends towards infinity, $(\Delta_e \cdot \Delta_e)$ tends towards negativity infinity and thus the Bounded Negativity Conjecture is false. In particular, the various Δ_e lie on distinct extremal rays of $\overline{\mathrm{Eff}}_1(C \times C)$ and therefore $\overline{\mathrm{Eff}}_1(C \times C)$ has infinitely many extremal rays and is not polyhedral.

If the Mori cone of $C \times C$ when C is a smooth projective curve over the complex numbers of genus $g \geq 2$ were polyhedral, the Bounded Negativity Conjecture would be true. However, one of our main results is that this cone is not polyhedral. We exhibit a general sufficient criterion for when the Mori cone of a surface is not polyhedral and apply this in the case of $C \times C$ to prove that $\overline{\mathrm{Eff}}_1(C \times C)$ is not polyhedral. This application involves the construction of certain special ample divisors called *Vojta divisors* on $C \times C$. These divisors were introduced in the work of Paul Vojta in his study of the Mordell conjecture in [36]. Once we have the result for $C \times C$, we can use induction to prove that $\overline{\mathrm{Eff}}_k(C^{\times n})$ is not polyhedral for $1 \leq k \leq n - 1$.

Another question about the behavior of $\overline{\mathrm{Eff}}_k(X)$ is the *Strong Conjecture* of Debarre-Jiang-Voisin which is introduced in [8]. Let X and Y be smooth complex projective varieties and let $\alpha : X \to Y$ be a morphism. There's an induced map $\alpha_* : N_k(X) \to N_k(Y)$ which is the usual pushforward. The conjecture attempts to relate $\ker(\alpha_*)$ with the classes of subvarieties contracted by α. One might hope that these classes generated the kernel, but that is easily seen to not be the case. Take for example, the projection $\pi_1 : E \times E \to E$ for E an elliptic curve and see that the class $\delta := \Delta - e_1 - e_2 \in \ker((\pi_1)_*)$ where $\Delta \subset E \times E$ is the diagonal and e_i are the fibers for $i = 1, 2$. However, the subvarieties contracted by π_1 are all generated by e_1, which is not sufficient to get δ. Restricting to only the effective classes gives us the following conjecture.

Conjecture 2 (Strong Conjecture, [8]). If $\overline{\mathrm{Eff}}_k(\alpha)$ is defined to be the closed cone in $N_k(X)$ generated by k-dimensional subvarieties on X that are contracted by α, then

$$\ker(\alpha_*) \cap \overline{\mathrm{Eff}}_k(X) = \overline{\mathrm{Eff}}_k(\alpha).$$

Conjecture 2 is proved for $k = 1, \dim(X) - 1$ in [8]. We use this result to prove that for a smooth surface S with a smooth morphism π to smooth curve C and $\rho(S) \geq 3$, that $\overline{\mathrm{Eff}}_1(S)$ is not polyhedral. When π is not smooth, we need to impose an upper bound on the total number of irreducible components of the reducible singular fibers of π to ensure that $\overline{\mathrm{Eff}}_1(S)$ is not polyhedral. We note that Fulger and Lehmann have made a serious study of Conjecture 2 in [12] and [13] and have proved the case when α maps a 4-fold to a 3-fold as well as some weaker results in this direction.

1.4 Main Theorems

We summarize our main results in the following section. We begin by proving the following criterion for when the Mori cone of a surface is not polyhedral. This is the crux of this thesis from which all other results flow.

Theorem 2. For X a smooth projective surface with $\rho(X) \geq 3$, let $\{h, f_1, \ldots, f_{\rho(X)-1}\}$ be an orthonormal basis for $N_1(X)$ such that h is ample, $(h \cdot h) = 1$ and $(f_i \cdot f_i) = -1$ for $1 \leq i \leq \rho(X) - 1$. If there exist e and f such that

1. $0 \neq e$ is a boundary class of $\overline{\mathrm{Eff}}_1(X)$ such that $(e \cdot e) = 0$,
2. $0 \neq f$ is a class in the linear span of $\{f_1, \ldots, f_{\rho(X)-1}\}$ such that $(e + \mathbb{R}f) \cap \overline{\mathrm{Eff}}_1(X) = \{e\}$,

then $\overline{\mathrm{Eff}}_1(X)$ is not polyhedral.

We first apply this to the case of $C \times C$, using the Vojta construction discussed in Chapter 12. This gives us the following.

Proposition 4. For $X = C \times C$ where C is a smooth projective curve of genus $g \geq 2$, $\overline{\mathrm{Eff}}_1(X)$ is not polyhedral.

We use the previous result as the base case for an inductive argument which gives the following result.

Theorem 3. Let $C^{\times n} := C \times \ldots C$ where C is a smooth projective curve of genus $g \geq 2$ and the self-product comprises n terms. We have that the cone $\overline{\mathrm{Eff}}_k(C^{\times n})$ for $1 \leq k \leq n-1$ is not polyhedral.

We then turn to the case of a surface fibration over a curve. We handle the smooth case separately from the singular case, but the main theorem is the following.

Theorem 4. Suppose $\pi : S \to C$ is a morphism from smooth surface S to smooth curve C. Let N be the total number of irreducible components of the various reducible singular fibers of π. If $N + 2 < \rho(S)$, then $\overline{\mathrm{Eff}}_1(S)$ is not polyhedral.

CHAPTER 2

Background

2.1 Convex Geometry of Cones

In this section, we give some background material on the convex geometry of cones. Much of this exposition is drawn from Appendix A of [9]. Let V be a finite dimensional real vector space. Denote by V^* to be the vector space $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. We have a canonical pairing

$$\langle -, - \rangle : V \times V^* \to \mathbb{R}$$

such that $\langle f, g \rangle = g(f)$. Note that we have $(V^*)^* = V$. For a subset S of V^*, define

$$S^{\perp} = \{v \in V : \langle v, w \rangle = 0 \ \forall w \in S\}.$$

We begin with the definition of a (convex) cone $\sigma \subset V \cong \mathbb{R}^n$.

Definition. Let $\sigma \subset V$ be a subset of a finite dimensional real vector space. It follows that that σ is a *cone* if $tv \in \sigma$ whenever $v \in \sigma$ for all $t \geq 0$. We say that a set $T \subset V$ is *convex* if

$$tv + (1-t)w \in T \text{ for any } v, w \in T \text{ and } t \in [0,1].$$

We say that σ is a *convex cone* if

$$t_1 v + t_2 w \in \sigma \text{ when } v, w \in \sigma \text{ and } t_1, t_2 \geq 0.$$

From the definition, it's clear that the intersection of two convex sets (respectively cones) is a convex set (respectively cone). Suppose S is an arbitrary subset of V. The *convex hull* of S, denoted $\mathrm{Conv}(S)$ is the intersection of all convex sets containing S and is hence the smallest convex set containing S. The *convex cone generated by* S is the intersection of all convex cones containing S and is hence the smallest convex cone containing

S. Denote it by $\Sigma(S)$. If a cone σ is equal to $\Sigma(S)$, we say that S is a system of generators of σ.

If σ is a convex cone, then its closure $\overline{\sigma}$ is again a convex cone. It follows that if S is a non-empty subset of V, then the closed convex cone generated by S is the closure of $\Sigma(S)$. We make the convention that all closed convex cones are non-empty, thus they contain 0.

A *polytope* is the convex hull of finitely many vectors in V. A convex cone is *polyhedral* if it is the convex cone generated by a finite set. A cone is *strongly convex* if $v, -v \in \sigma$ if and only if $v = 0$. In other words σ is strongly convex if and only if it contains no nonzero linear subspaces of V. It's easy to see that any polytope and any polyhedral cone is closed. On the other hand, the set

$$\{(x, y) : x \cdot y > 0\} \cup \{(0, 0)\},$$

it is a convex cone but is not closed (and not polyhedral).

Definition. If σ is a closed convex cone, the *dual cone* $\sigma^\vee$ is the subset of V^* such that

$$\sigma^\vee = \{u \in V^* : \langle u, v\rangle \geq 0 \text{ for all } v \in \sigma\}.$$

The dual of a closed convex cone is a closed convex cone. A key fact about cone duals is that $(\sigma^\vee)^\vee = \sigma$. Proving this is nontrivial, but details can be found in A.2 in [9].

Definition. If σ is a closed convex cone, a *face* of σ is a subset of σ of the form

$$\sigma \cap u^\perp = \{v \in \sigma : \langle u, v\rangle = 0\}$$

for some $u \in \sigma^\vee$.

Note that σ itself is a face of σ we can simply take $u = 0$. A *proper face* is a face of σ different from σ. It's clear that any face τ of σ is a closed convex cone. Furthermore if $v_1, v_2 \in \sigma$, then $v_1 + v_2 \in \tau$ if and only if $v_1, v_2 \in \tau$. If $\tau = \bigcap_{i=1}^{r} \tau_i$ is the intersection of a finite family of faces, where $\tau_i = \sigma \cap u_i^\perp$ for $u_i \in \sigma^\vee$, then defining $u := \sum_{i=1}^{r} u_i \in \sigma^\vee$, we see that $\tau = \sigma \cap u^\perp$. Therefore the intersection of a finite set of faces is a face. In fact, one can extend this to show that the intersection of an arbitrary family of faces is face.

Definition. A subcone τ of σ is said to be *extremal* if $v + w \in \tau$ if and only if $v, w \in \tau$. It follows from the above discussion that all faces are extremal. An *extremal ray* is an extremal subcone of the form $\mathbb{R}_{\geq 0} \cdot v$ for some element $v \in V$ or in other words, it's an extremal subcone which generates a one-dimensional vector space in V.

The following proposition turns out to be very useful.

Proposition 5. If σ is a closed, strongly convex cone, then σ is generated as a convex cone by its extremal rays.

It's clear from the definition that to obtain any element in an extremal ray as a linear combination of generating elements, one needs an element in these extremal rays. Therefore if σ is generated by finitely many elements(that is to say, polyhedral), there must necessarily be finitely many extremal rays. Furthermore if there are infinitely many extremal rays, then σ cannot be polyhedral and a common strategy to show that a closed convex cone is not polyhedral is to exhibit infinitely many extremal rays.

We also have the following result.

Proposition 6 (Farkas)**.** If σ is polyhedral, then $\sigma^{\vee}$ is also polyhedral. Equivalently, a cone σ is polyhedral if and only if it is the intersection of finitely many half-spaces.

We have the following corollary.

Corollary 1. The intersection of finitely many polyhedral cones is polyhedral.

Finally, we need the following lemma.

Lemma 1. *If $\sigma \subset V$ is a closed convex polyhedral cone and H is a linear subspace of V, then $\sigma \cap H$ is a closed convex polyhedral cone.*

Proof. Since both σ and H are closed and convex sets, $\sigma \cap H$ is closed and convex. H is a polyhedral cone since if $\mathfrak{B}$ is a basis for H as a vector space, $\mathfrak{B} \cup -\mathfrak{B}$ generates H as a cone in V. Therefore by Corollary 1, we know that $\sigma \cap H$ is polyhedral. We can thus conclude that $\sigma \cap H$ is a closed convex polyhedral cone. $\square$

2.2 Cycles and Intersection Theory

2.2.1 Rational Equivalence and Chow Groups

Let X be a smooth projective variety over an algebraically closed field K of characteristic zero. In what follows, a variety is an irreducible reduced scheme of finite type over K. Much of the exposition below is drawn from [14].

A *cycle* on a projective variety X is a finite formal linear combination $Z = \sum_i a_i V_i$ of closed subvarieties of X, with $a_i \in \mathbb{R}$. When all V_i have dimension k, we say that Z is a k-cycle. When for all i, we have $a_i \geq 0$ we say that the cycle Z is *effective*. When $a_i \in \mathbb{Z}$, we say that the cycle is *integral*. The group of integral cycles is denoted by $Z_k(X)$. Its rank is usually infinite and is hard to calculate. In order to understand the geometry better, we introduce equivalence relations on X; one such is rational equivalence.

To understand rational equivalence, we need to define the *order of vanishing* of a rational function on a variety W along a codimension 1 subvariety Z. Consider the local ring of W along Z, $\mathcal{O}_{W,Z}$, denoted A for convenience. For a non-zero element $f \in K(W)$, we write $f = \dfrac{a}{b}$ for $a, b \in A$ and define

$$\text{ord}_Z(f) = \text{ord}_Z(a) - \text{ord}_Z(b).,$$

where $\text{ord}_Z(r) := \ell_A(A/(r))$ for $r \in A$. Therefore

$$\text{ord}_Z : K(W)^* \to \mathbb{Z}$$

is well defined. Note that

$$\text{ord}_Z(fg) = \text{ord}_Z(f) + \text{ord}_Z(f),$$

and therefore $\text{ord}_Z : K(W)^* \to \mathbb{Z}$ is a group homomorphism.

Definition (k-cycle of a rational function on a $(k+1)$-dimensional subvariety)**.** For a variety $W \subset X$ of dimension $k+1$ and a nonzero rational function f on W, we set

$$\text{div}_W(f) = \sum_{Z \subset X} \text{ord}_Z(f) \cdot [Z],$$

where the sum runs over all k-dimensional subvarieties Z of W. Note that $\text{ord}_Z(f)$ is negative if f has a pole along Z.

Definition (Rational Equivalence)**.** For a variety X, we define the group of *rationally trivial cycles* $B_k(X)$ to be the group in $Z_k(X)$ generated by (f) where $f \in K(W)^*$ and W is any $(k+1)$-dimensional subvariety of X. Define $A_k(X) := Z_k(X)/B_k(X)$ to be the *Chow group* of k-cycles on X. [1] We say that cycles α and β in $Z_k(X)$ are *rationally equivalent* if their images in $A_k(X)$ are the same.

[1]We will write $A^k(X)$ for the Chow group of codimension k-cycles on X.

Definition (Fundamental Cycle)**.** To any closed subscheme $Z \subset X$ of $\dim(Z) = k$, we associate the *fundamental cycle* $[Z]$ as follows. Let the Z_i be the irreducible components of Z of dimension k and let n_i be the length of the local ring of Z at the generic point of Z_i. We define the k-cycle associated to Z to be k-cycle $[Z] = \sum_i n_i[Z_i]$.

Definition (Proper Pushforward)**.** Suppose that $f : X \to Y$ is a proper morphism between smooth projective varieties. Let $V \subset X$ be a k-dimensional closed subvariety. We define the *pushforward* $f_*[Z]$ to be 0 if $\dim(f(Z)) < k$. If $\dim(f(Z)) = k$, we define $f_*[Z] := d \cdot [f(Z)]$ where $d = [K(Z) : K(f(Z))]$. Here, $K(W)$ is the function field of W. Now for $\alpha = \sum n_i Z_i$ a k-cycle, the pushforward $f_*(\alpha) = \sum n_i f_*[Z_i]$, where $f_*[Z_i]$ is defined above.

Definition (Flat Pullback)**.** Suppose that $f : X \to Y$ is a flat morphism of smooth projective varieties of relative dimension r i.e. all fibers have dimension r. Note that the condition that all fibers have the same dimension is guaranteed by flatness. Let $Z \subseteq Y$ be a k-dimensional closed subvariety. We define $f^*[Z]$ to be the $(k+r)$-cycle associated to the scheme theoretic inverse image $f^{-1}(Z)$. Now for $\alpha = \sum n_i Z_i$ a k-cycle, the *pullback* $f^*(\alpha)$ is given by $\sum n_i f^*[Z_i]$ where $f^*[Z_i]$ is defined above.

Proposition 7. The maps f^* and f_* descend to the Chow group. In other words, proper pushforward and flat pullback of rationally trivial cycles are rationally trivial inducing a new proper pushforward and flat pullback on the quotients, namely the Chow groups.

2.2.2 Intersection Theory and Numerical Equivalence

In general, $A_k(X)$ is not finite dimensional and so we need to work with a weaker equivalence relation. We do this by studying the intersection theory on $A_*(X)$ when X is a smooth projective variety. The idea here is to have a notion of *numerical equivalence* which captures the intersection theory on $A_k(X)$. In other words, two cycles α and β are said to be *numerically equivalent* if their intersection numbers with all elements of $A^k(X)$ are equal. To make sense of this, we set up a general framework

$$A^p(X) \times A^q(X) \to A^{p+q}(X)$$

taking a pair of codimension p and codimension q cycles (up to rational equivalence) and producing a codimension $(p+q)$-cycle (up to rational equivalence).

We follow the construction of Serre from [35]. Let X be a smooth projective variety of dimension n. Suppose that V and W are closed subvarieties of X with $\dim(V) = r$ and

$\dim(W) = s$. Assume that $\dim(V \cap W) = r + s - n$. We say that V and W *intersect properly* if this condition holds. Let I, J be the ideal sheaves of V and W in $\mathcal{O}_X$. Let Z be an irreducible component of the set-theoretic intersection $V \cap W$ and z be its generic pont. The *intersection multiplicity* of Z in the product $V \cdot W$ is given by

$$\mu(Z; V, W) = \sum_{i=0}^{\infty} (-1)^i \ell_{\mathcal{O}_{X,z}} (\mathrm{Tor}^i_{\mathcal{O}_{X,z}} (\mathcal{O}_{X.z}/I_z, \mathcal{O}_{X.z}/J_z)),$$

or the alternating sum over the length over the local ring of z in X of the Tor groups of the factor rings corresponding to V and W. This is often referred to as Serre's Tor-formula. Note that the first term when $i = 0$ is the length of

$$(\mathcal{O}_{X.z}/I_z) \otimes_{\mathcal{O}_{X,z}} (\mathcal{O}_{X.z}/J_z)$$

which is a naive guess for intersection multiplicity. However this isn't sufficient which is why the derived functors Tor of the tensor product are necessary. This sum is finite because the ring $\mathcal{O}_{X,z}$ is a regular local ring and has finite Tor-dimension. If the intersection of V and W is not proper, the above multiplicity is zero. If it is proper, the above multiplicity is positive. Neither of these is obvious and require detailed arguments. Finally, one uses a spectral sequence argument to prove that $\mu(Z; V, W) = \mu(Z; W, V)$. In other words, intersection multiplicities are commutative. We can now construct an *intersection product* for subvarieties V and W which intersect properly. We define

$$V \cdot W := \sum_i \mu(Z_i; V, W) \cdot [Z_i]$$

where $V \cap W = \cup Z_i$ is the decomposition of the set-theoretic intersection into irreducible components. With this notation $V \cdot W$ is a formal linear combination $\sum e_i Z_i$ of the irreducible components of $V \cap W$.

If $r + s = \dim(X)$, we see that $W \cdot V$ is a 0-cycle class on X. It's also easy to see that $A_0(\mathrm{Spec} K) = \mathbb{Z}$.

Definition (Degree of a 0-cycle)**.** The *degree* of a 0-cycle (up to rational equivalence) on X is given by the proper pushforward

$$p_* : A_0(X) \to A_0(\mathrm{Spec}(K)),$$

where $p : X \to \mathrm{Spec}(K)$ is the structure map. Combined with the natural isomorphism

$A_0(\text{Spec}(K)) = \mathbb{Z}$, we get a map

$$\deg : A_0(X) \to \mathbb{Z}.$$

Let α be a k-cycle. If $\deg(\alpha \cdot \beta) = 0$ for any $(n-k)$-cycle β (up to rational equivalence), we say that α is *numerically trivial*. The quotient of $A_k(X)$ modulo numerically trivial cycles is $N_k(X)_{\mathbb{Z}}$ which is said to be the numerical group of k-cycles on X. We define $N_k(X) := N_k(X)_{\mathbb{Z}} \otimes \mathbb{R}$ as the numerical group with $\mathbb{R}$-coefficients. Denote $N^k(X) := N_{n-k}(X)$ where $n = \dim(X)$.

Theorem 5. *The rank of $N_k(X)$ is finite.*

We have the following result on pushforwards and pullbacks.

Theorem 6. If $f : X \to Y$ is a proper morphism between smooth varieties X and Y, the maps f_* and f^* descend to $N_k(X)$. [2] We have a perfect pairing

$$N_k(X) \times N^k(X) \to \mathbb{R}$$

which maps (α, β) to $\deg(\alpha \cdot \beta)$ which we denote as $(\alpha \cdot \beta)$ for convenience.

To connect f_* and f^* we have the *projection formula* when the morphism $f : X \to Y$ is proper and flat. For α on X and β on Y we have

$$f_*\alpha \cdot \beta \equiv_{\text{num}} \alpha \cdot f^*\beta,$$

where $\equiv_{\text{num}}$ denotes numerical equivalence.

2.3 Positive Cones

We now discuss some properties of positive cones. This exposition is drawn from [11]. Let X be a smooth projective variety as in previous sections.The group $N_k(X)$ is the natural ambient space for any notion of positivity one might wish to impose upon cycles since it is finite dimensional. A class α is effective if it is the image of an effective cycle (with real cofficients). We define the *pseudoeffective cone* $\overline{\text{Eff}}_k(X)$ to be the closure in $N_k(X)$ of the convex cone generated by classes of k-dimensional subvarieties in X. The interior of $\overline{\text{Eff}}_k(X)$ is called the *big cone*. We mention the following features of $\overline{\text{Eff}}_k(X)$:

[2] When the target is not smooth, this can be complicated and it isn't always clear if flat pullbacks descend. It is known (see [14], Example 19.2.3) that l.c.i pullbacks descend.

- $\overline{\text{Eff}}_k(X)$ is a closed, convex cone.

- $\overline{\text{Eff}}_k(X)$ generates $N_k(X)$.

- If $\pi : X \to Y$ is a proper morphism, we have

$$\pi_*(\overline{\text{Eff}}_k(X)) \subseteq \overline{\text{Eff}}_k(Y).$$

  If π is surjective, then equality holds for both the pseudoeffective cone as well as the big cone.

- If $\pi : X \to Y$ is a flat morphism between smooth varieties of relative dimension d, we have

$$\pi^*(\overline{\text{Eff}}_k(Y)) \subseteq \overline{\text{Eff}}_{k+d}(X).$$

- If h is the class of an ample divisor in $N^1(X)$, then h^k lives in the interior of $\overline{\text{Eff}}^k(X)$. The same is true of any complete intersection of ample classes.

- (Kleiman) For any $\alpha \in \overline{\text{Eff}}_k(X)$ and ample class h, we define $\deg_h(\alpha) := (h^k \cdot \alpha)$. We have that $\alpha = 0$ if and only if $\deg_h(\alpha) = 0$.

Recall that the dual of a cone $\sigma \subset V$ is defined as

$$\sigma^\vee := \{\beta \in V : \beta(x) \geq 0, \forall x \in \sigma\}.$$

In our case the dual is called the *nef cone*. It lives in $N^k(X)$ which is the dual of $N_k(X)$ and is defined as

$$\text{Nef}^k(X) := \{\beta \in N^k(X) : (\alpha \cdot \beta) \geq 0, \forall \alpha \in \overline{\text{Eff}}_k(X)\}.$$

We mention the following features of $\text{Nef}^k(X)$.

- $\text{Nef}^k(X)$ generates $N_k(X)$.

- $\text{Nef}^k(X)$ is a closed, convex cone.

- $\text{Nef}^k(X)$ contains complete intersections of ample classes in its interior.

- If $\pi : X \to Y$ is a flat morphism between smooth varieties,

$$\pi^*(\text{Nef}^k(Y)) \subseteq \text{Nef}^k(X).$$

- If π is dominant, then $\pi^*\alpha$ is nef if and only if α is nef.

- If $h_1, \ldots, h_k \in \mathrm{Nef}^1(X)$, then $h_1 \cdot \ldots \cdot h_k \in \mathrm{Nef}^k(X)$. Note that this is not true in codimension 2. There are examples in [31] of codimension 2 nef classes whose intersection product is not nef. Note that the same reference contains examples of nef classes that are not pseudoeffective. However it's clear that

$$\mathrm{Nef}^1(X) \subseteq \overline{\mathrm{Eff}}^1(X).$$

2.4 Divisors and Curves

When $k = 1$, we are dealing with the case of curves and divisors. The story here is more classical. $\overline{\mathrm{Eff}}_1(X)$ is also denoted as $\overline{\mathrm{NE}}(X)$ and is called the *Mori cone*. $\mathrm{Nef}^1(X)$ is the classical nef cone defined as the closure of the cone generated by ample divisors.

The following criterion due to Nakai and Moishezon tells us when a line bundle $\mathcal{L}$ is ample.

Proposition 8 (Nakai-Moishezon)**.** If X is a smooth projective variety, a line bundle $\mathcal{L}$ on X is ample if and only if for $0 \leq k \leq n$,

$$(\mathcal{L}^k \cdot Z) > 0$$

for all subvarieties Z of dimension k. When $k = n$ this condition tells us that $(\mathcal{L}^n) > 0$.

The following criterion due to Kleiman describes the relationship between $\overline{\mathrm{Eff}}_1(X)$ and ample divisors.

Proposition 9 (Kleiman's Criterion)**.** Let X be a smooth projective variety. The following are equivalent.

1. D is ample.

2. D defines a positive linear functional on

$$\overline{\mathrm{Eff}}_1(X) - \{0\} \to \mathbb{R}_{>0}$$

   defined by extending the map $[C] \to [C \cdot D]$ linearly.

In particular, $\overline{\mathrm{Eff}}_1(X)$ does not contain a line and if D is ample, the set

$$\{\alpha \in \overline{\mathrm{Eff}}_1(X) : \alpha \cdot D \leq K\}$$

is compact for any positive real constant K.

We also have a relative version of Kleiman's criterion. Recall that a divisor is *relatively ample* for a morphism $\pi : X \to Y$ if the restriction of D to the fibers of π is ample.

Proposition 10. If $\pi : X \to Y$ is a proper morphism and $\overline{\text{Eff}}_1(\pi)$ is the subcone of $\overline{\text{Eff}}_1(X)$ generated by classes of curves contracted by π, a divisor D is π-ample if and only which tells us that a divisor D is π-ample if and only if it defines a positive linear functional on $\overline{\text{Eff}}_1(\pi) - \{0\}$.

A substantial portion of this thesis is dedicated to the study of surfaces in which case $\dim(X) = 2$. We'll now discuss some useful facts that are specific to the surface case. Since $N^1(X) \cong N_1(X)$, the intersection pairing is actually a bilinear form on $N_1(X)$. Such a form has a signature which is described by the Hodge index theorem. We denote the rank of $N_1(X)$ as $\rho(X)$. We mention the following facts from Pg. 65 of [24].

- It's clear that in this case $\text{Nef}(X) \subseteq \overline{\text{NE}}(X)$. However equality only holds when $(C^2) \geq 0$ for all irreducible curves $C \subset X$.

- If $C \subset X$ is an irreducible curve with $C^2 \leq 0$, then $\overline{\text{NE}}(X)$ can be spanned by $[C]$ and the subcone

  $$\overline{\text{NE}}(X)_{C \geq 0} := C_{\geq 0} \cap \overline{\text{NE}}(X)$$

  where $C_{\geq 0}$ is the halfplane in $N_1(X)$ of classes whose intersection with C is non-negative.

- In the above case, $[C]$ must lie on the boundary of $\overline{\text{NE}}(X)$. If $C^2 < 0$, then $[C]$ must lie on an extremal ray of $\overline{\text{NE}}(X)$.

The following is the Hodge Index Theorem.

Proposition 11 ([17], Ex V.1.9)**.** The signature of the intersection form on $N_1(X)$ is $(1, \rho(X) - 1)$. More specifically, we can pick an orthonormal basis $\{h, f_1, \ldots, f_{\rho(X)-1}\}$ where h is ample and for $i = 1, \ldots \rho(X) - 1$,

$$(h \cdot h) = 1, (f_i \cdot f_i) = -1, (h \cdot f_i) = 0.$$

We will also need the following lemma.

Lemma 2 ([9], Ex 1.4.21)**.** If X is a smooth projective surface and $\alpha \in N^1(X)$ satisfies $(\alpha \cdot \alpha) > 0$, either α or $-\alpha$ is big.

CHAPTER 3

Criterion for non-polyhedral cones

In this chapter we explain the core result – a criterion for when $\overline{\mathrm{Eff}}_1(X)$ is not polyhedral when X is a smooth projective surface. As discussed in chapter 1.1, the standard approach to proving that a pseudoeffective cone is not polyhedral relies on either constructing infinitely many extremal rays or giving a complete description of the cone that proves that it is round. For example, we used the first approach to analyze $\mathbb{P}^2$ blown up at ≥ 9 general points in section 1.2.1. The work of Mullane in [25] and Chen-Coskun in [7] also follow the first approach. The second approach is used in [24] to study the case of $E \times E$ where E is a general elliptic curve over $\mathbb{C}$. Kovács uses both approaches in [23] to analyze the case of K3 surfaces.

Our approach is fundamentally different. It relies on an analysis of a portion of the Mori cone and not the entire cone. This allows us in Chapter 4 to prove that for $C \times C$ where C is a smooth projective curve of genus $g \geq 2$, the Mori cone has regions that are locally polyhedral and regions that are locally not polyhedral.

Let X be a smooth projective surface. Recall that in this case $N^1(X) \cong N_1(X)$. Suppose that the Picard rank $\rho(X)$ is at least 3. We can pick an orthonormal basis $\{h, f_1, \ldots, f_{\rho(X)-1}\}$ of $N_1(X)$ such that h is ample, $(h \cdot h) = 1$ and $(f_i \cdot f_i) = -1$ for $1 \leq i \leq \rho(X) - 1$. The existence of such a basis follows from Proposition 11.

Theorem 7. For X as above, if there exist e and f such that

1. $0 \neq e$ is a boundary class of $\overline{\mathrm{Eff}}_1(X)$ such that $(e \cdot e) = 0$,
2. $0 \neq f$ is a class in the linear span of $\{f_1, \ldots, f_{\rho(X)-1}\}$ such that

$$(e + \mathbb{R}f) \cap \overline{\mathrm{Eff}}_1(X) = \{e\},$$

then $\overline{\mathrm{Eff}}_1(X)$ is not polyhedral.

Proof. The idea of the proof goes roughly as follows: we take a 2-dimensional slice of $\overline{\mathrm{Eff}}_1(X)$ containing e. If $\overline{\mathrm{Eff}}_1(X)$ were polyhedral, this slice would be a convex polygon as seen in Figure 3.1. By hypothesis, we have a line $(e + \mathbb{R}f)$ passing through e pointing in the direction of f such that $(e + \mathbb{R}f) \cap \overline{\mathrm{Eff}}_1(X) = \{e\}$. Now we consider rays emanating from e on the same side of $e + \mathbb{R}f$ as the Mori cone but lying outside the convex polygon. Using Lemma 2, we find points on them which are very close to e and are big. This is a contradiction because big classes must lie in the interior of $\overline{\mathrm{Eff}}_1(X)$. Therefore $\overline{\mathrm{Eff}}_1(X)$ can't be polyhedral.

We now formalize this. Consider the lines $\ell_1 = \{e + sf : s \in \mathbb{R}\}$ and $\ell_2 = \{te + (1 - t)(h \cdot e)h : t \in \mathbb{R}\}$. These lines are distinct because otherwise $e + sf$ would equal h for some value of s, which is impossible since $(e + \mathbb{R}f) \cap \overline{\mathrm{Eff}}_1(X) = \{e\}$. The affine 2-plane P spanned by ℓ_1 and ℓ_2 is contained in the affine hyperplane

$$H := \{v \in N_1(X) : (v \cdot h) = (e \cdot h)\}.$$

Since $0 \neq e \in \overline{\mathrm{Eff}}_1(X)$, we know that $(e \cdot h) > 0$ by Proposition 9. The image of $\overline{\mathrm{Eff}}_1(X) - \{0\}$ in $\mathbb{P}(N_1(X)_{\mathbb{R}})$ is closed, hence compact. Since H maps homeomorphically onto its image in $\mathbb{P}(N_1(X)_{\mathbb{R}})$, we conclude that $H \cap \overline{\mathrm{Eff}}_1(X)$ is compact. It follows that $P \cap \overline{\mathrm{Eff}}_1(X)$ is compact, being a closed subset of $H \cap \overline{\mathrm{Eff}}_1(X)$.

Assume that $\overline{\mathrm{Eff}}_1(X)$ is a polyhedral cone. It follows that $P \cap \overline{\mathrm{Eff}}_1(X)$ must be a convex polygon as see in 3.1. Since ℓ_1 intersects this convex polygon at precisely one point, e must be a vertex. The class $h' := (e \cdot h)h$ lies in the interior of this polygon, being an ample class. Since $(e + \mathbb{R}f) \cap \overline{\mathrm{Eff}}_1(X) = \{e\}$, neither edge of the polygon emanating from e is contained in $(e + \mathbb{R}f)$. Hence $(h' + \mathbb{R}f)$ is not parallel to either of these edges and it must intersect both edges at precisely one point each, say $h' + \chi_i f$ for $i = 1, 2$. Picking m such that $|m| > \max(|\chi_1|, |\chi_2|)$, we see that the segment joining e and $h' + mf$ lies entirely outside $\overline{\mathrm{Eff}}_1(X)$ except for e. A general point on this segment is

$$P_t := te + (1 - t)(h' + mf)$$

for $0 \leq t \leq 1$. We compute $(P_t \cdot P_t)$ using the fact that $(e \cdot e) = 0$, and get

$$\begin{aligned}(P_t \cdot P_t) = (1 - t)\Big[&(1 - t)(h' \cdot h') + 2(1 - t)m(h' \cdot f) \\ &+ (1 - t)m^2(f \cdot f) + 2t(e \cdot h') + 2mt(e \cdot f)\Big]\end{aligned}$$

Observe that for when $t = 1$, the interior term reduces to $2(e \cdot h') + 2m(e \cdot f)$. Pick the sign of m so that $2m(e \cdot f) \geq 0$ and observe that $(e \cdot h') > 0$. It follows that for t slightly

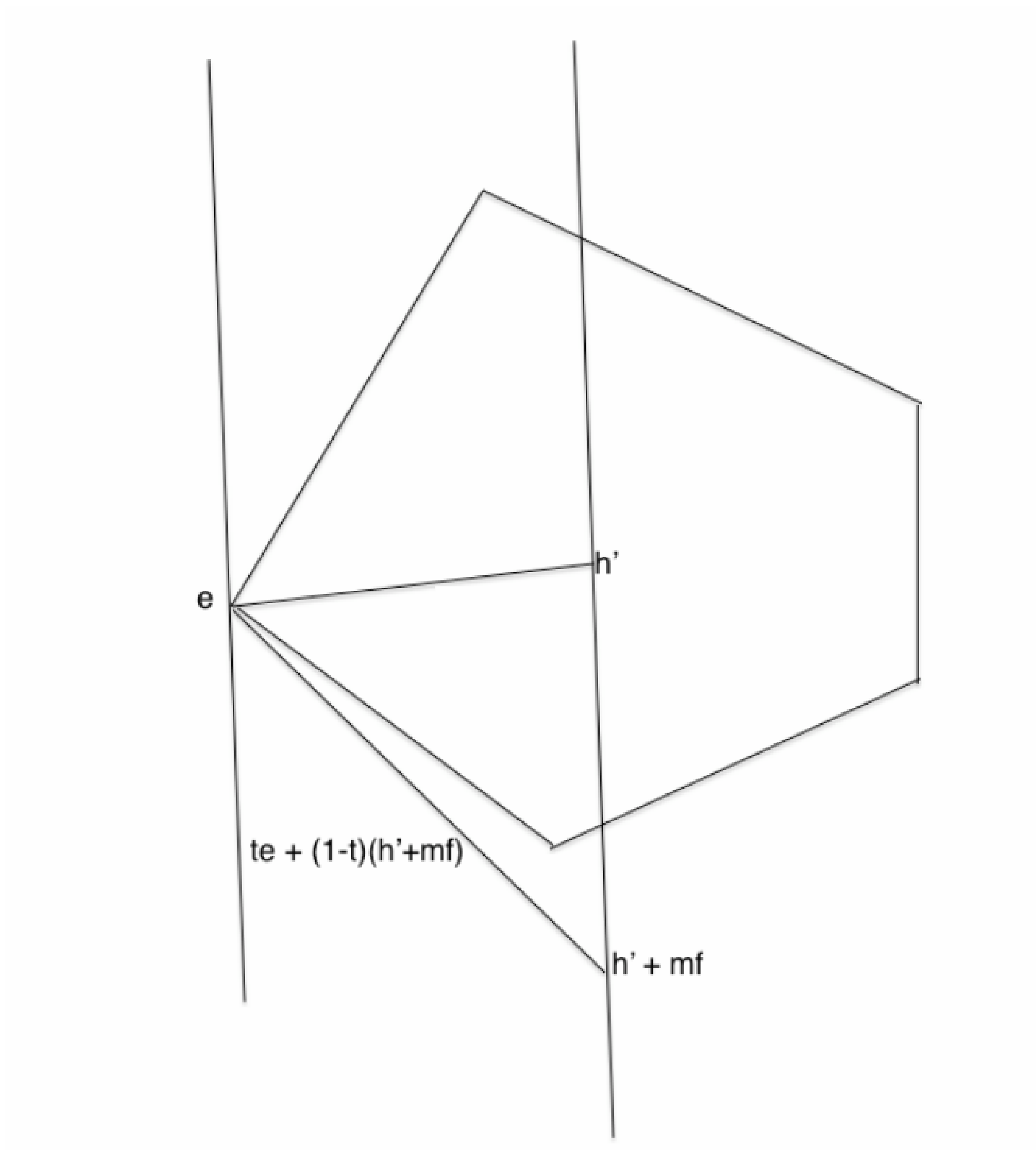


Figure 3.1: $P \cap \overline{\mathrm{Eff}}_1(X)$

less than 1, the interior term is positive, forcing $(P_t \cdot P_t)$ to be positive. Lemma 2 tells us that either P_t or $-P_t$ is big. But since $(P_1 \cdot h) = (e \cdot h) > 0$, it follows that $(P_t \cdot h) > 0$ for t slightly less than 1, making P_t big and contained in the interior of $\overline{\mathrm{Eff}}_1(X)$, a contradiction! We thus conclude that $\overline{\mathrm{Eff}}_1(X)$ is not polyhedral. □

In the following chapters, we apply Theorem 7 to give concrete cases of surfaces where $\overline{\mathrm{Eff}}_1(X)$ is not polyhedral. Each application requires different insights, but a clear pattern emerges for condition 1. Constructing e where $(e \cdot e) = 0$ requires us to think of some kind of fiber of a morphism from a surface to a curve. The hard question is the construction of the class f such that $(e + \mathbb{R}f) \cap \overline{\mathrm{Eff}}_1(X) = \{e\}$, sp that we satisfy satisfy condition 2. We can think of this as a kind of tangent line to $\overline{\mathrm{Eff}}_1(C \times C)$ and it is here that the geometric insight is needed.

CHAPTER 4

Mori Cone of $C \times C$

In this chapter, we focus on a fixed smooth projective curve C of genus $g \geq 2$. This is the first application of Theorem 7 and it generalizes what is known about $E \times E$ where E is a general elliptic curve. We begin by fixing some notation.

Let $\Delta \subset C \times C$ be the diagonal and let J be the Jacobian of C. Let $p_1, p_2 : C \times C \to C$ be the projection morphisms. Let e_i be the numerical class of a fiber of p_i and $\delta := \Delta - e_1 - e_2$. It is easy to see that that

$$(e_1 \cdot e_1) = (e_2 \cdot e_2) = (e_1 \cdot \delta) = (e_2 \cdot \delta) = 0,\ (e_1 \cdot e_2) = 1,\ \text{and } (\delta \cdot \delta) = -2g. \tag{4.1}$$

Recall that the last of the intersection numbers is computed using the adjunction formula by obsering that $\Delta = e_1 + e_2 + \delta$ and unwinding the equality

$$(\Delta \cdot \Delta) + (\Delta \cdot K_{C \times C}) = 2g - 2.$$

We use the fact that

$$K_{C \times C} = (2g - 2)(e_1 + e_2).$$

Furthermore, we have that $\rho(C \times C) \geq 3$ since e_1, e_2 and δ are all linearly independent. It is well known that $\overline{\text{Eff}}_1(C \times C)$ is a full-dimensional cone in $N_1(C \times C)_{\mathbb{R}}$.

4.1 Vojta's Construction

The core of this proof revolves around Vojta's construction of a 2-parameter family of divisors $Y(r, s)$ which are ample. Vojta constructed these divisors in [36] to study the Mordell conjecture. The key insight here is that these divisors move in a family where the coefficients of δ are ± 1 but the coefficients of e_1 and e_2 vary. This family is therefore rather non-linear and offers a keen insight into the structure of $\overline{\text{Eff}}_1(C \times C)$. Our purpose is to use a strand of these divisors when $s = 1$ and $r \to \infty$ to to prove that

$$(e_1 + \mathbb{R} \cdot \delta) \cap \overline{\mathrm{Eff}}_1(C \times C) = e_1,$$

to get condition 2 of Theorem 7. Because these divisors cluster in specific ways, we are able to do this.

Remark 1. In [36], Vojta mentions that this is a geometric incarnation of Dyson's Lemma on diophantine approximation. Although we do not explore this connection in this thesis, it's worth asking if further strengthenings of Dyson's lemma such as Roth's theorem on diophantine approximation can offer deeper insight on the structure of $\overline{\mathrm{Eff}}_1(C \times C)$.

Proposition 12 (Vojta, [36])**.** Let $Y(r,s) := a_1 e_1 + a_2 e_2 + a_3 \delta$ where $a_1 = \sqrt{\dfrac{g+s}{r}}$, $a_2 = \sqrt{(g+s)r}$ and $a_3 = \pm 1$, for $r, s \in \mathbb{R}_{>0}$. If

$$r > \frac{(g+s)(g-1)}{s},$$

then $Y(r,s)$ is nef.

In his paper, Vojta only considers the case $a_3 = 1$. For completeness, we sketch (with suitable modifications) the proof of Proposition 12 below.

Proof due to Vojta. Assume, arguing by contradiction, that there exists a curve C_0 (not necessarily smooth) on $C \times C$ such that $(C_0 \cdot Y(r,s)) < 0$. We may assume that C_0 is irreducible. Note that it is not a fiber of p_i for $i = 1, 2$ since $(e_i \cdot Y(r,s)) \geq 0$. Applying the adjunction formula, we get

$$\begin{aligned}
(C_0^2) + (2g-2)((C_0 \cdot e_1) + (C_0 \cdot e_2)) &= (C_0^2) + (C_0 \cdot K_{C \times C}) \\
&= 2p_a(C_0) - 2 \\
&\geq 2p_g(C_0) - 2 \\
&\geq (2g-2)(C_0 \cdot e_1),
\end{aligned}$$

where $p_a(C_0)$ and $p_g(C_0)$ are the arithmetic and geometric genera[1] of C_0. Note that the last inequality follows by applying Riemann-Hurwitz to $p_1 \circ \eta : \widetilde{C_0} \to C$, where $\eta : \widetilde{C_0} \to C_0$ is the normalization. Note that the composition $p_1 \circ \eta$ is a finite morphism because C_0 is

[1] Recall that the geometric genus of a singular curve is defined as the genus of its normalization.

not a fiber of either projection. We can then conclude that

$$(C_0^2) + (2g-2)(C_0 \cdot e_2) \geq 0. \tag{4.2}$$

Write $C_0 \equiv b_0\delta + b_1 e_1 + b_2 e_2 + \nu$, where ν is orthogonal to δ, e_1 and e_2 in $N^1(C \times C)_{\mathbb{R}}$. The Hodge index theorem forces $(\nu \cdot \nu) \leq 0$. Using this and (4.2), we compute

$$2b_1 b_2 + (2g-2)b_1 \geq 2g b_0^2.$$

Since $b_1 \geq 0$ and is an integer (being equal to $(C_0 \cdot e_2)$) we have $b_1^2 \geq b_1$ and can write

$$2b_1 b_2 + (2g-2)b_1^2 \geq 2g b_0^2. \tag{4.3}$$

Now we apply $(C_0 \cdot Y(r,s)) < 0$ which gives

$$b_1\sqrt{(g+s)r} + b_2\sqrt{\frac{g+s}{r}} < 2a_3 g b_0. \tag{4.4}$$

Since $b_1, b_2 \geq 0$, the left hand side of (4.4) is nonnegative. Thus we can square (4.4) [2] and combine it with (4.3) to get

$$(g+s)(b_2^2/r + 2b_1 b_2 + b_1^2 r) < 4g(b_1 b_2 + (g-1)b_1^2).$$

Rearranging this, we get

$$b_2^2(g+s)/r + 2b_1 b_2(s-g) + b_1^2((g+s)r - 4g(g-1)) < 0.$$

This is a quadratic form in b_1, b_2 and therefore its discriminant must be nonnegative. Solving for r then gives

$$r \leq \frac{(g+s)(g-1)}{s}.$$

However this contradicts the hypothesis about r. Hence no such C_0 can exist and $Y(r,s)$ must be nef. □

We use Proposition 12 to prove the following result.

Proposition 13. If $\nu = e_2 + q\delta$ and $q \neq 0$ then

$$\nu \notin \overline{\mathrm{Eff}}_1(C \times C).$$

[2] This is the only step where a_3 makes an appearance and it is immediately being squared.

Proof. If we pick a_3 so that $a_3 q = |q|$, then

$$\begin{aligned}(Y(r,s)\cdot\nu) &= a_1 - 2gqa_3 \\ &= \sqrt{\frac{g+s}{r}} - 2|q|g.\end{aligned}$$

Now letting $s = 1$ and r tend to ∞, we get that $\sqrt{\frac{g+s}{r}}$ approaches 0. This forces $(Y(r,1)\cdot\nu)$ to approach $-2|q|g < 0$, implying that for $r \gg 0$, $(Y(r,s)\cdot\nu) < 0$. We conclude that ν is not pseudoeffective, since its intersection with a nef divisor is negative. □

4.2 $\overline{\mathrm{Eff}}_1(C \times C)$ is not polyhedral

Having constructed the Vojta divisors, we use them to prove the following result.

Theorem 8. If C is a smooth projective curve of genus $g \geq 2$, then $\overline{\mathrm{Eff}}_1(C \times C)$ is not polyhedral.

We are now ready to prove Theorem 8.

Proof of Theorem 8. It suffices to apply Theorem 7 with $h = \dfrac{e_1 + e_2}{2}$, $e = e_2$ and $f = \delta$. Proposition 13 tells us that condition (2) in Theorem 7 is satisfied. □

Remark 2. Observe that for C/k, where $k = \bar{k}$ is a field of characteristic $p > 0$, Theorem 8 is easily seen to be true because the graph of the e^{th} power of Frobenius, denoted by Δ_e, is irreducible and $(\Delta_e \cdot \Delta_e) < 0$. It follows that $\overline{\mathrm{Eff}}_1(C \times C)$ has infinitely many extremal rays, hence is not polyhedral. This is the standard counterexample to Conjecture 1.

Remark 3. We note that over $\mathbb{C}$, the cone $\overline{\mathrm{Eff}}_1(C \times C)$ has both polyhedral parts and non-polyhedral parts. We sketch the ideas behind this claim as follows.

- The region near Δ is polyhedral. We consider the Zariski decomposition of a class ν in $\overline{\mathrm{Eff}}_1(C \times C)$ such that $\Delta \cdot \nu < 0$. Recall that Δ is extremal since $\Delta \cdot \Delta < 0$. The decomposition must be of the form $A + \lambda\Delta$ where $A \in \Delta^{\perp}$. Here A is playing the role of the positive part of the Zariski decomposition and $\lambda\Delta$ plays the role of the negative part of the Zariski decomposition. Therefore the region

$$\overline{\mathrm{Eff}}_1(C \times C)_{\Delta \leq 0} := \overline{\mathrm{Eff}}_1(C \times C) \cap (\Delta \leq 0)$$

is a cone with $\rho(X)$ generators.

- The region near e_2 is not polyhedral. This is seen in the proof of Theorem 8.

CHAPTER 5

Cycles on $C \times \ldots C$

In this chapter, we use the results of Chapter 4 to prove that all intermediate effective cones of cycles on $C \times \ldots C$ are not polyhedral for C a smooth projective curve of genus $g \geq 2$. The proof relies on a few general lemmas.

We begin with the following lemma relating cycles on X to cycles on $X \times Y$ via proper pushforward.

Lemma 3. Let $i : X \to X \times Y$ be given by $i(x) = (x, y_0)$ for some $y_0 \in Y$. Then we have

$$i_* \overline{\text{Eff}}_k(X) = \overline{\text{Eff}}_k(X \times Y) \cap i_* N_k(X)$$

Proof. Since the proper pushforward of an effective cycle is effective, it's clear that

$$i_* \overline{\text{Eff}}_k(X) \subseteq \overline{\text{Eff}}_k(X \times Y) \cap i_* N_k(X).$$

We simply have to prove the reverse inclusion. Pick $\alpha \in \overline{\text{Eff}}_k(X \times Y)$ such that $\alpha = i_*(\beta)$ for $\beta \in N_k(X)$. If π_1 is the projection to the first component, $\pi_1 \circ i = \text{id}_X$. We therefore have

$$\beta = (\pi_1)_*(i_*(\beta)) = \pi_*(\alpha).$$

Thus β must be effective, being the proper pushforward of an effective cycle α. This gives us the reverse inclusion

$$i_* \overline{\text{Eff}}_k(X) \supseteq \overline{\text{Eff}}_k(X \times Y) \cap i_* N_k(X),$$

and hence the equality. $\square$

The following lemma proves a similar result in the case of pullbacks of divisors.

Lemma 4. Let $\pi_1 : X \times Y \to X$ be the projection map. Then we have

$$\pi_1^* \overline{\mathrm{Eff}}^1(X) = \overline{\mathrm{Eff}}^1(X \times Y) \cap \pi_1^* N^1(X)$$

Proof. Pick $D = \sum a_i D_i$ a divisor class on X where D_i are irreducible divisors. We know that $\pi_1^*(D) = \sum a_i(D_i \times Y)$. Since the product of irreducible varieties is irreducible, we see that $\pi_1^*(D)$ is effective if and only if $a_i \geq 0$ for all i. Therefore $\pi_1^*(D)$ is effective if and only if D is effective. Taking limits tells us that $\pi_1^*(D)$ is pseudoeffective if and only if D is pseudoeffective. An element of $\overline{\mathrm{Eff}}^1(X \times Y) \cap \pi_1^* N^1(X)$ is pseudoeffective and of the form $\pi_1^*(D)$ and is psuedoeffective, forcing it to lie in $\pi_1^*(\overline{\mathrm{Eff}}^1(X))$. Conversely, an element of $\pi_1^*(\overline{\mathrm{Eff}}^(X))$ must be pseudoeffective and therefore lies in $\overline{\mathrm{Eff}}^1(X \times Y)$ as well as $\pi_1^* N^1(X)$ and hence in the intersection. The result follows. □

The following theorem uses Theorem 8 as a base case and proceeds via induction. The key insight is Lemma 1 which tells us that linear cross-sections of polyhedral cones are polyhedral.

Theorem 9. Let $C^{\times n} := C \times \ldots C$ where C is a smooth projective curve of genus $g \geq 2$ and the self-product comprises n terms. The cone $\overline{\mathrm{Eff}}_k(C^{\times n})$, for $1 \leq k \leq n-1$, is not polyhedral.

Proof. We proceed using induction on $n \geq 2$. The base case $n = 2$ follows from Theorem 8. Suppose we have the result for $n = r$, we need to prove it for $n = r+1$. Let

$$i : C^{\times r} \to C^{\times r+1}$$

be the map taking x to (x, p) for some fixed point p. Lemma 3 tells us that for $1 \leq k \leq r-1$,

$$i_* \overline{\mathrm{Eff}}_k(C^{\times r}) = \overline{\mathrm{Eff}}_k(C^{\times r+1}) \cap i_* N_k(C^{\times r})$$

and therefore $\overline{\mathrm{Eff}}_k(C^{\times r})$ is a linear cross-section of $\overline{\mathrm{Eff}}_k(C^{\times r+1})$. It follows that $\overline{\mathrm{Eff}}_k(C^{\times r+1})$ for $1 \leq k \leq r-1$ is not polyhedral by Lemma 1 since its cross-section is not polyhedral by hypothesis. For $k = r$, Lemma 4 implies that $\overline{\mathrm{Eff}}^1(C^{\times r})$ is a linear cross-section of $\overline{\mathrm{Eff}}^1(C^{\times r+1})$. Therefore $\overline{\mathrm{Eff}}^1(C^{\times r+1}) = \overline{\mathrm{Eff}}_r(C^{\times r+1})$ is not polyhedral by Lemma 1 since its cross-section is not polyhedral by hypothesis. So by induction, the result follows. □

cross-sections

We can make a more general statement following Lemma 3.

Corollary 2. If $\overline{\mathrm{Eff}}_k(X)$ is not polyhedral, then $\overline{\mathrm{Eff}}_k(X \times Y)$ is not polyhedral.

Proof. Lemma 3 tells us that

$$i_*\overline{\mathrm{Eff}}_k(X) = \overline{\mathrm{Eff}}_k(X \times Y) \cap i_* N_k(X),$$

where i is the usual inclusion taking x to (x, y_0). From Lemma 1, we know that since $\overline{\mathrm{Eff}}_k(X)$ is not polyhedral, $\overline{\mathrm{Eff}}_k(X \times Y)$ is also not polyhedral. □

This has several applications, for example we can conclude that $X \times \mathbb{P}^N$ has non-polyhedral higher-codimension pseudoeffective cones whenever X has higher codimension pseudoeffective cones. We end this chapter with the following question.

Question. What conditions must we impose on X and Y to guarantee that $\overline{\mathrm{Eff}}_k(X \times Y)$ is polyhedral for all $1 \leq k \leq \dim(X) + \dim(Y) - 1$?

Certainly it must follow that all pseudoeffective cones of X and Y must be polyhedral, but this is hardly sufficient. An interesting condition to consider might be

$$(\pi_1)^* N_k(X) \oplus (\pi_2)^* N_k(Y) = N_k(X \times Y)$$

since this limits the kind of behavior we see in $C \times C$ where C is a smooth projective curve of genus $g \geq 2$. However this doesn't work in general, especially in the upper range of values of k and one imagines that this condition will be very rarely satisfied.

CHAPTER 6

Surface fibered over curve

In this chapter, we study the case of a smooth projective surface S with a dominant map to a smooth projective curve C. When $\rho(S) \geq 3$, we have the potential to apply Theorem 7. The key insight here is a result of Debarre-Jiang-Voisin on curve classes and kernels of numerical pushforwards. This is part of a circle of ideas on kernels of numerical pushforwards which we now introduce.

6.1 Kernels of Numerical Pushforwards

Let $\pi : X \to Y$ be a morphism of projective varieties over an algebraically closed field. The pushforward of cycles induces a map $\pi_* : N_k(X) \to N_k(Y)$ on numerical classes with $\mathbb{R}$-coefficients. We want to understand how ker(π_*) reflects the geometry of the map π. For example, if $\alpha \in N_k(X)$ is the class of a closed subvariety Z, then α lies in ker(π_*) precisely when $\dim(\pi(Z)) < \dim(Z)$. A similar statement holds when α is the class of an effective cycle. However in general the kernel can be bigger. Consider the following example.

Example. Let E be a general elliptic curve over $\mathbb{C}$ and $X = E \times E$. We know that $\rho(X) = 3$ and $N_1(X)$ is spanned by the fibers e_1, e_2 and the diagonal class Δ. We know that $\overline{\text{Eff}}_1(X)$ is round. Consider the projection to the first component $\pi : X \to E$. The curves contracted by π are all numerically equivalent to e_1. However the kernel is 2-dimensional and is generated by e_1 and $\Delta - e_2$. In particular, the kernel is not spanned by the subvarieties that are contracted by π.

In [8], Debarre, Jiang and Voisin suggest that instead of studying the entire kernel, we focus on the portion of the kernel comprising of pseudoeffective classes. They make the following conjecture.

Conjecture 3. Let $\pi : X \to Y$ be a morphism of projective varieties over an algebraically closed field. Suppose that $\alpha \in \overline{\mathrm{Eff}}_k(X)$ satisfies $\pi_*\alpha = 0$. Then

- **(Weak version)** α is in the vector space generated by k-dimensional subvarieties of X that are contracted by π.
- **(Strong version)** α is in the closure of the cone generated by k-dimensional subvarieties of X that are contracted by π.

We make the following definition.

Definition. Let $\pi : X \to Y$ be a morphism of projective varieties. Let

- $\overline{\mathrm{Eff}}_k(\pi)$ be the closed convex cone generated by effective k-classes of X contracted by π.
- $N_k(\pi)$ denote the subspace of $N_k(X)$ generated by effective k-classes of X contracted by π.

We can therefore restate the conjecture of Debarre-Jiang-Voisin as

Conjecture 4. Let $\pi : X \to Y$ be a morphism of projective varieties over an algebraically closed field. Suppose that $\alpha \in \overline{\mathrm{Eff}}_k(X)$ satisfies $\pi_*\alpha = 0$. Then

- **(Weak version)** $\ker(\pi_*) \cap \overline{\mathrm{Eff}}_k(X) \subseteq N_k(\pi)$.
- **(Strong version)** $\ker(\pi_*) \cap \overline{\mathrm{Eff}}_k(X) = \overline{\mathrm{Eff}}_k(\pi)$.

In their original paper, Debarre, Jiang and Voisin prove the strong version in the case $k = 1$ which we discuss below.

Theorem 10 (Debarre-Jiang-Voisin, [8])**.** Let X and Y be smooth complex projective varieties and let $\alpha : X \to Y$ be a morphism. Then we have that $\ker(\alpha_*) \cap \overline{\mathrm{Eff}}_1(X) = \overline{\mathrm{Eff}}_1(\alpha)$.

Proof. In $N_1(X)_{\mathbb{R}}$, we have $\mathcal{C}_1 = \overline{\mathrm{Eff}}_1(X) \cap \ker(\alpha_*)$ and the closed convex cone $\mathcal{C}_2$ spanned by classes of curves contracted by α. We know for sure that $\mathcal{C}_2 \subset \mathcal{C}_1$, but we have to prove the cones are equal. Suppose for purposes of contradiction that they are not. We can then pick $d \in N^1(X)_{\mathbb{R}}$ such that d is positive on $\mathcal{C}_2 - \{0\}$ but $(d \cdot \beta) < 0$ for some $\beta \in \mathcal{C}_1$. We can even pick for d to be the class of a divisor D. D is α-ample by the relative Kleiman theorem. If H is ample divisor on Y, the divisor $m\alpha^* H + D$ is ample for $m \gg 0$. But we have

$$\begin{aligned}(m\alpha^* H + D) \cdot \beta &= m(H \cdot \alpha_*\beta) + (d \cdot \beta) \\ &= 0 + (d \cdot \beta) < 0\end{aligned}$$

which by the usual Kleiman criterion, contradicts $\beta \in \overline{\text{Eff}}_1(X)$. Hence the cones $\mathcal{C}_1$ and $\mathcal{C}_2$ are equal.

□

We will apply this to study the case of fiberd surfaces.

6.2 The cone of curves of fibered surfaces

In this section, we analyze the case of surface fibrations. We begin with the case where π is smooth. We then consider an example where π is not smooth to illustrate how the proof method fails. Finally we introduce an additional condition on the total number of irreducible components of the various reducible singular fibers to overcome the failure of the original method. We note that if $\rho(S) = 2$, then $\overline{\text{Eff}}_1(S)$ is automatically polyhedral being a closed, convex cone in $\mathbb{R}^2$.

Theorem 11. Suppose that $\pi : S \to C$ is a smooth morphism from a smooth projective surface S to smooth projective curve C. Let $\rho(S) \geq 3$. Then $\overline{\text{Eff}}_1(S)$ is not polyhedral.

Proof. Let e be the class of a generic fiber of π. Note that this is independent of the choice of fiber. Fix an ample class h on S such that $(h \cdot h) = 1$. It's clear that $\pi_*(h) \neq 0$ Since $\rho(S) \geq 3$, it follows that if $V := \ker(\pi_*) \cap h^{\perp}$, V is the intersection of two codimension-1 subspaces of $N_1(S)$ and is therefore nonzero. Pick f in V. If $Z \subset S$ is an irreducible curve contracted by π, it must lie in a fiber. Since all fibers of π are irreducible, Z must be equal to a fiber and thus $Z \equiv e$. It follows that

$$\overline{\text{Eff}}_1(\pi) = \mathbb{R}_{\geq 0} \cdot e.$$

By Theorem 10, we know that

$$\overline{\text{Eff}}_1(S) \cap \ker(\pi_*) = \mathbb{R}_{\geq 0} \cdot e.$$

We therefore can conclude that

$$\overline{\text{Eff}}_1(S) \cap (e + \mathbb{R}f) = \{e\}$$

which satisfies condition (2) of Theorem 7. Since $(e \cdot e) = 0$, condition (1) is also satisfied. Thus by Theorem 7, we see that $\overline{\text{Eff}}_1(S)$ is not polyhedral. □

Note that this extends the previous result about $C \times C$. One might hope to remove the smoothness condition for π but this turns out to be subtle. To understand this, we use the following example.

Example (Blow-up of a projective bundle over C)**.** Suppose that C is a smooth projective curve and we have $\pi_0 : S_0 \to C$, a projective bundle over C. In this case, we know that $\rho(S_0) = 2$ and therefore $\overline{\mathrm{Eff}}_1(S_0)$ is polyhedral. Blow up a general point $p \in S_0$ and let $S = \mathrm{Bl}_p(S_0)$. Let the blowup map be denoted by χ. The induced map $\pi : S \to C$ has general fiber isomorphic to $\mathbb{P}^1$ but if $x = \pi_0(p)$, the fiber of S lying over x is not irreducible and is composed of two distinct $\mathbb{P}^1$ curves, say C_1 and C_2 which intersect at a point. C_1 is the proper transform of the fiber $(S_0)_x$ of π_0 lying over x and C_2 is the exceptional divisor of the blowup over p. It's clear that $C_2 \cdot C_2 = -1$. We know that

$$\chi^*((S_0)_x) = C_1 + C_2.$$

The projection formula tells us that

$$(\chi^*((S_0)_x) \cdot C_1) = ((S_0)_x \cdot (S_0)_x) = 0.$$

Since we already know $C_1 \cdot C_2 = 1$, we get

$$\begin{aligned} 0 &= (\pi^*((S_0)_x) \cdot C_1) \\ &= (C_1 \cdot C_1) + (C_2 \cdot C_1) \\ &= (C_1 \cdot C_1) + 1. \end{aligned}$$

Therefore, $(C_1 \cdot C_1) = -1$.

Let the classes of C_1 and C_2 be denoted as e_1 and e_2 and let the generic fiber of π be e. It's clear that $e \equiv e_1 + e_2$. However if we try to run the argument of Theorem 11, we soon find that

$$\overline{\mathrm{Eff}}_1(S) \cap \ker(\pi_*) = \mathbb{R}_{\geq 0} \cdot e_1 + \mathbb{R}_{\geq 0} \cdot e_2.$$

In this case, there's no way to cook up an f such that

$$\overline{\mathrm{Eff}}_1(S) \cap (e + \mathbb{R}f) = \{e\}$$

because e lies in a codimension-1 face of $\overline{\mathrm{Eff}}_1(S)$ spanned by e_1 and e_2 and thus any line through e must also lie in said face. Thus this argument fails. In fact, one expects $\overline{\mathrm{Eff}}_1(S)$ to be polyhedral in this case.

To make up for this, we need to impose a condition on the total number of irreducible components of the singular fibers of π.

Theorem 12. Suppose that $\pi : S \to C$ is a morphism from smooth projective surface S to smooth projective curve C. Let N be the total number of irreducible components of the various reducible singular fibers of π. If $N + 2 < \rho(S)$, then $\overline{\text{Eff}}_1(S)$ is not polyhedral.

Proof. The proof is very similar to that of Theorem 11 with the suitable changes. Let $e := e_0$ be the class of a smooth fiber of π and let $e_1, \dots, e_N$ be the classes of the various irreducible components of the reducible singular fibers of S. Note that if f is the class of an irreducible singular fiber, it is numerically equivalent to e. Fix an ample class h on S such that $(h \cdot h) = 1$. It's clear that $\pi_*(h) \neq 0$. If $V := \ker(\pi_*) \cap h^\perp$, V is the intersection of two codimension-1 subspaces of $N_1(S)$ and is therefore codimension 2 in $N_1(S)$. If $Z \subset S$ is an irreducible curve contracted by π, it must lie in a fiber and thus $Z \equiv e_i$ for some $i \in \{0, \dots N\}$.

It follows that

$$\overline{\text{Eff}}_1(\pi) = \sum_{i=0}^{N} \mathbb{R}_{\geq 0} \cdot e_i = \sum_{i=1}^{N} \mathbb{R}_{\geq 0} \cdot e_i.$$

Note that e is a positive linear combination of some subset of $\{e_1, \dots, e_N\}$ thus giving us the second equality in the above formula. The hypothesis tells us that

$$N < \rho(S) - 2 \leq \dim(V).$$

Therefore we can find $f \in V$ such that f is not in the linear span of e_i. By Theorem 10, we know that

$$\overline{\text{Eff}}_1(S) \cap \ker(\pi_*) = \sum_{i=1}^{N} \mathbb{R}_{\geq 0} \cdot e_i.$$

We can therefore deduce that

$$\overline{\text{Eff}}_1(S) \cap (e + \mathbb{R}f) = \{e\}$$

which satisfies condition (2) of Theorem 7. Since $(e \cdot e) = 0$, condition (1) is also satisfied. Thus by Theorem 7, we see that $\overline{\text{Eff}}_1(S)$ is not polyhedral. $\square$

6.3 K3 surfaces

We discuss the case of K3 surfaces to illustrate our result. This section is largely expository and is drawn from [20].

Definition. A K3 surface over the complex numbers is a smooth projective surface such that

$$\omega_X \cong \mathcal{O}_X \text{ and } H^1(X, \mathcal{O}_X) = 0.$$

For a K3 surface, $h^0(X, \mathcal{O}_X) = 1$ and $h^1(X, \mathcal{O}_X) = 0$. Since $\omega_X = \mathcal{O}_X$, Serre duality tells us that $h^2(X, \mathcal{O}_X) = 1$ and therefore $\chi(X, \mathcal{O}_X) = 2$. Applying Riemann-Roch for a line bundle L, we have that

$$\chi(X, L) = 2 + \frac{(L^2)}{2}.$$

Recall that $h^0(X, L)$ and $h^0(X, L^{-1})$ are both nontrivial precisely when L is trivial. Furthermore for L ample,

$$h^1(X, L) = h^1(X, \omega_X \otimes L) = 0$$

by the Kodaira vanishing theorem. Therefore for L ample, we have that

$$h^0(X, L) = 2 + \frac{(L^2)}{2}.$$

In what follows, we use additive and multiplicative notation for line bundles/divisors as needed.

6.3.1 Fibrations on K3 surfaces

We begin by asking what kind of fibrations can a K3 surface have. Recall that a fibration is a surjective morphism with connected fibers.

Lemma 5. Let X be a K3 surface. The following statements hold.

1. If $\pi : X \to C$ is a fibration to a smooth projective curve C, then $C \cong \mathbb{P}^1$.
2. Any smooth irreducible fiber of a fibration $\pi : X \to \mathbb{P}^1$ must be an elliptic curve.
3. If $\pi : X \to \mathbb{P}^1$ has smooth fibers isomorphic to elliptic curves, not all fibers can be smooth.

Proof.

1. Since C is smooth and π has connected fibers, we have that $\pi_*(\mathcal{O}_X) = \mathcal{O}_C$. The Leray spectral sequence tells us that $H^1(C, \mathcal{O}_C) \hookrightarrow H^1(X, \mathcal{O}_X) = 0$ and hece $C \cong \mathbb{P}^1$.
2. Let $t \in \mathbb{P}^1$ be such that X_t is a smooth fiber. Using $K_X \equiv 0$ and the adjunction

formula, we have

$$
\begin{aligned}
0 &= (X_t \cdot X_t) \\
&= (X_t \cdot X_t) + (X_t \cdot K_X) \\
&= 2g(X_t) - 2.
\end{aligned}
$$

Thus $g(X_t)$ = 1 and since X_t is smooth, it must be an elliptic curve.

3. If all fibers were smooth, by the well-known multiplicativity of the topological Euler number,

$$
\begin{aligned}
24 &= e(X) \\
&= e(X_t) \cdot e(\mathbb{P}^1) = 0,
\end{aligned}
$$

a contradiction. Thus all fibers can't be smooth.

□

Lemma 5 tells us that any surjective map from a smooth K3 surface X to a smooth curve C must be an elliptic fibration with singular fibers. The next question is whether such a fibration exists.

Proposition 14. Let X be a K3 surface. Let L be a non-trivial nef line bundle on X such that $L^2 = 0$ Then L is base-point free and there exists a smooth irreducible genus 1 curve $E \subset X$ such that $dE \in |L|$ for some $d > 0$.

Proof. Since $L^2 = 0$ and $K_X = 0$ by the Riemann-Roch theorem we know that

$$
\begin{aligned}
h^0(X, L) - h^1(X, L) + h^2(X, L) &= h^0(X, L) - h^1(X, L) + h^0(X, -L) \\
&= 2 + \frac{(L^2)}{2} \\
&= 2
\end{aligned}
$$

which tells us that either L or $-L$ is effective. By assumption L is nef and therefore $-L$ cannot be effective. Therefore L is effective and so

$$h^2(X, L) = h^0(X, -L) = 0$$

implying that $h^0(X, L) \geq 2$. Let F be the fixed part of L and define the mobile part $M := L - F$. By definition, the base locus of M contains at most points. Note that M is nontrivial since $h^0(X, M) = h^0(X, L) \geq 2$. Finally M is nef, $M^2 \geq 0$ and $M \cdot F \geq 0$. From the hypothesis, we have that $L^2 = 0$ and since $L = M + F$ we get $L \cdot M + L \cdot F = 0$. Notice that L, M and F are all effective and so $L \cdot M \geq 0$ and $L \cdot F \geq 0$ since L is nef. It follows that

$$L \cdot M = L \cdot F = 0.$$

We can write

$$0 = L \cdot M = M \cdot M + F \cdot M.$$

Also in this case, we have that M^2 and $F \cdot M$ are both nonnegative since M is nef and so $M^2 = 0$ and $M \cdot F = 0$. Putting these $F \cdot M = 0$ and $L \cdot F = 0$ together, we get that $F \cdot F = 0$. Assume now that F is non-trivial. Then again by Riemann-Roch we get that $h^0(X, F) \geq 2$. However since F is the fixed part of L, we know that $h^0(X, F) \leq 1$ by definition. It follows that F must be trivial and hence L has at most fixed points. But $L^2 = 0$ and therefore it has no fixed points.

Let $\phi_L : X \to \mathbb{P}^m$ be the map defined by $|L|$. Since L is base-point free, ϕ_L is a morphism. We claim that $D := \phi_L(X)$ is a curve. Suppose D is a surface, and consider a general hyperplane $H \subset \mathbb{P}^m$. The section $H \cap D$ is a curve in D and its pullback is linearly equivalent to L. But a hyperplane section is an ample divisor which must have positive self-intersection. This contradicts the projection formula,xw since $L^2 = 0$. Therefore D must be a curve. Now applying the first part of Lemma 5, we find that D is rational. Consider the Stein factorization

$$C \to \tilde{D} \to D$$

where the first map ϕ_1 has connected fibers and the second map π is finite. Note that $\tilde{D}$ is also rational. Let E be the general fiber of ϕ_1. Since $\tilde{D}$ is rational, all its points are linearly equivalent and therefore the fibers of ϕ_1 form a linear system. The theorem on generic smoothness tells us that the general fiber is smooth and by the second part of Lemma 5, we know that it has genus 1. Since all the fibers of ϕ_1 are connected, it follows that E is irreducible. As $\tilde{D} \cong \mathbb{P}^1$, all the fibers are linearly equivalent and in particular

$$L \cong \phi_L^*(D \cap H) = (\pi \circ \phi_1)^*(D \cap H) = \phi_1^*(\pi * (D \cap H)) \cong (\deg \pi)(\deg D)E$$

and therefore if $d := (\deg \pi)(\deg D)$, we have $dE \in |L|$. $\square$

We have the following consequence.

Proposition 15. Let X be a K3 surface defined over the complex numbers.

1. X admits an elliptic fibration if and only if it contains a divisor class L such that

$L^2 = 0$.

2. If $\rho(X) \geq 5$, then X admits an elliptic fibration.

Sketch of Proof. If we have an elliptic fibration, it's clear that the fiber satisfies the role of L in the first part. By Proposition 14, if L is nef, then we are done. If L is not nef, we can find L' such that $(L')^2 = 0$ and L' is nef. Roughly, this is done by passing successively from L to the reflection $L_1 := L + (L \cdot C)C$ for a (-2)-curve C with $(L \cdot C) < 0$. This process stops because for a fixed ample H, Full details can be found in Ex 8.2.13 of [20].

$$(L_1 \cdot H) = (L \cdot H) + (C \cdot H) \cdot (C \cdot L) < (L \cdot H)$$

and therefore eventually $(L_i \cdot H) < 0$, a contradiction.

For the second part, we know from Corollary IV.2.3.2 in [34] that an indefinite lattice of rank greater than or equal to 5 represents 0. □

6.3.2 Singular Fibers

We now examine the singular fibers in detail. We recall the following lemma of Zariski.

Lemma 6 ([2], Lemma III.8.2)**.** If $X_t = \sum_{i=1}^{r} n_i C_i$, for $n_i > 0$ and $C_i \subset X$ irreducible, is a fiber of an elliptic K3 surface $\pi : X \to \mathbb{P}^1$, we have

1. $(C_i \cdot X_t) = 0$.
2. If $D = \sum_{i=1}^{r} m_i C_i$ then $(D \cdot D) \leq 0$ with equality holding if and only if

$$\frac{m_1}{n_1} = \ldots = \frac{m_r}{n_r}.$$

Applying this to the case of an elliptic fibration, we observe that if $r > 1$, we can rewrite $(C_i \cdot X_t) = 0$ as

$$(C_i \cdot C_i) = \frac{-1}{n_i} \sum_{j \neq i} n_j (C_i \cdot C_j) < 0.$$

The sum above is negative because fibers are connected and so we have $(C_i \cdot C_j) > 0$ for some value of $j \neq i$.

If $C \subset X$ is a smooth rational curve, using $K_X \equiv 0$, the adjunction formula tells us that

$$(C \cdot C) + (C \cdot K_X) = 2g(C) - 2$$

which implies that $(C \cdot C) = -2$. Conversely if $(C \cdot C) < 0$ for irreducible C, adjunction tells us that $(C \cdot C) = -2$. Therefore we have that

$$0 = p_a(C) \geq p_g(C) \geq 0$$

where $p_a(C)$ is the arithmetic genus and $p_g(C)$ is the geometric genus. Since $p_a(C) = p_g(C) = 0$, it follows that $C \cong \mathbb{P}^1$. In particular we have the following corollary.

Proposition 16. If $X \to \mathbb{P}^1$ is an elliptic fibration, the singular fibers are of the form $\sum n_i C_i$ where $C_i \cong \mathbb{P}^1$.

Figure 6.1 shows us the various kinds of singular fibers that can occur for an elliptic K3 surface $\pi : X \to \mathbb{P}^1$. The vertices are labelled by the coefficients n_i and the last column gives the topological Euler number.

Remark 4. The possible configurations of singular fibers are further restricted by the topology of the K3 sufaces. In fact by additivity of the Euler number, we find that

$$24 = e(X) = \sum_{t \in \mathbb{P}^1} e(X_t).$$

All but finitely many fibers are smooth and therefore $e(X_t) = 0$ for all but finitely many values of t.

The following formula connects the Picard rank with the Mordell-Weil rank of the elliptic surface.

Proposition 17 (Shioda-Tate formula)**.** If $\pi : X \to \mathbb{P}^1$ is an elliptic fibration, we have

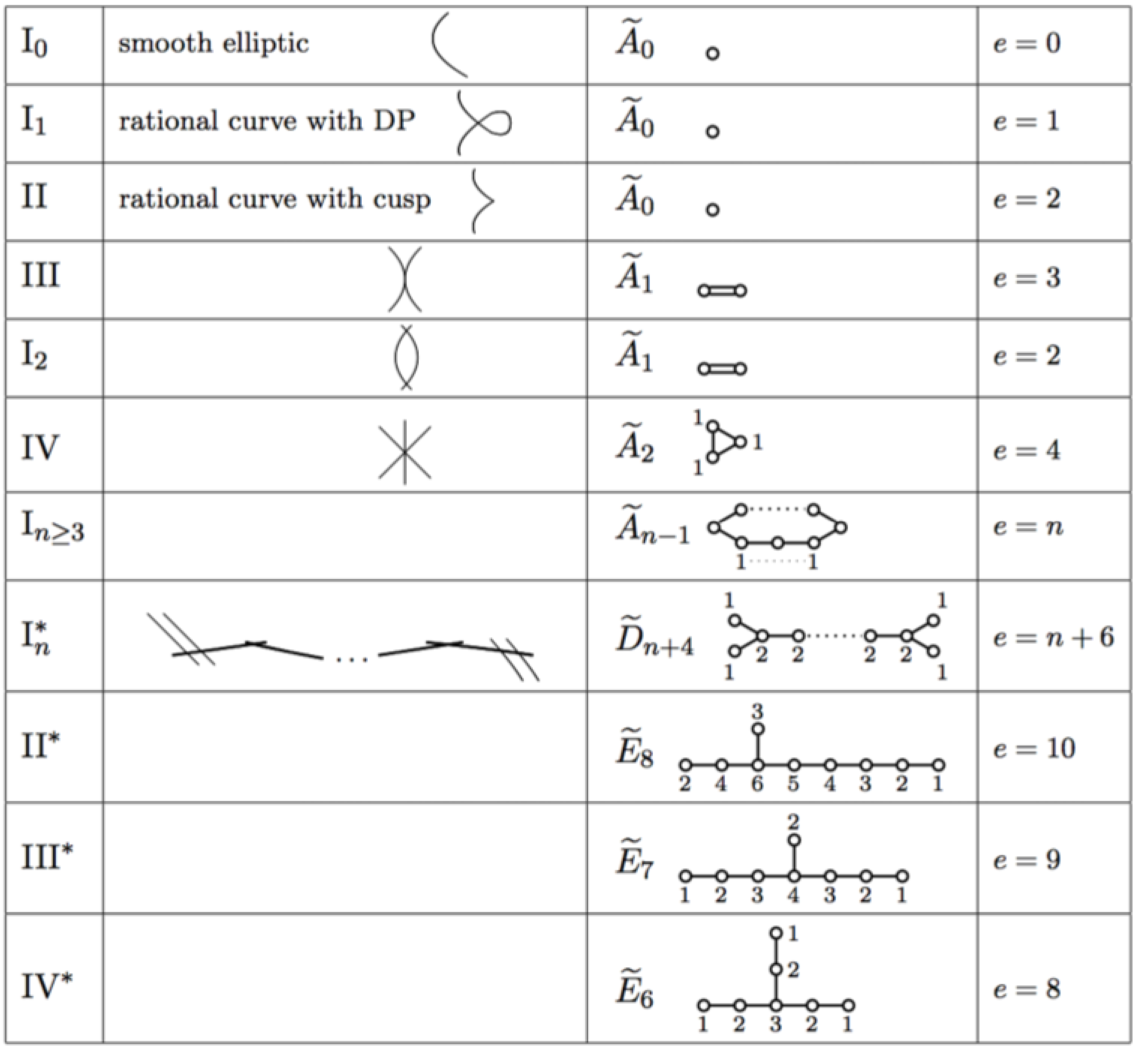
$$\rho(X) = \text{rankMW}(X/\mathbb{P}^1) + 2 + \sum_{v \in \mathbb{P}^1} (n_v - 1)$$

where n_v is the number of irreducible components of the fiber over v and $\text{MW}(X/\mathbb{P}^1)$ is the Mordell-Weil group of the generic fiber of π over the function field of $\mathbb{P}^1$.

6.3.3 Mori cone of K3 surfaces

Kovács has studied $\overline{\text{Eff}}_1(X)$ for a K3 surface. He proves the following result. In what follows, $\text{Conv}(S)$ will denote the convex hull of a set S.

Theorem 13 ([23], Theorem 7)**.** Let X be a K3 surface with $\rho(X) \geq 3$. One of the following mutually exclusive conditions is satisfied.



I_0	smooth elliptic	$\widetilde{A}_0$	$e=0$
I_1	rational curve with DP	$\widetilde{A}_0$	$e=1$
II	rational curve with cusp	$\widetilde{A}_0$	$e=2$
III		$\widetilde{A}_1$	$e=3$
I_2		$\widetilde{A}_1$	$e=2$
IV		$\widetilde{A}_2$	$e=4$
$\mathrm{I}_{n\geq 3}$		$\widetilde{A}_{n-1}$	$e=n$
I_n^*		$\widetilde{D}_{n+4}$	$e=n+6$
II^*		$\widetilde{E}_8$	$e=10$
III^*		$\widetilde{E}_7$	$e=9$
IV^*		$\widetilde{E}_6$	$e=8$

Figure 6.1: Singular fibers of elliptic K3 surfaces

1. X does not contain any curve of negative self intersection. In this case, we have

$$\overline{\text{Eff}}_1(X) = \text{Conv}(\mathcal{D}(X))$$

where for h an ample class,

$$\mathcal{D}(X) := \{\xi \in N_1(X) : (\xi \cdot h) > 0, (\xi \cdot \xi) = 0\}.$$

In particular, $\overline{\text{Eff}}_1(X)$ is not polyhedral.

2. X contains curves of negative self intersection, in particular smooth rational curves. We have

$$\overline{\text{Eff}}_1(X) = \sum \mathbb{R}_{\geq 0} \cdot \ell$$

where ℓ runs over all smooth rational curves in X. If there are finitely many smooth rational curves, $\overline{\text{Eff}}_1(X)$ is polyhedral. If not, $\overline{\text{Eff}}_1(X)$ is not polyhedral.

Proposition 16 tells us that for X an elliptic K3 surface, we are in the second case of Kovács' theorem. However to distinguish between when X has finitely many (-2)-curves and infinitely many (-2)-curves requires us to study the singular fibers. A result of Piatetskii-Shapiro from [33] tells us that $\overline{\text{Eff}}_1(X)$ is polyhedral precisely when the automorphism group $\text{Aut}(X)$ is finite. Nikulin has classified these cases in his seminal works [29], [30] and [28]. There are too many cases to enumerate here, but a quick inspection tells us that when $\rho(S)$ is relatively small and the number of components of singular fibers are relatively large, we have finite $\text{Aut}(X)$ while when $\rho(S)$ is relatively large and the the number of components of singular fibers are relatively small we have infinite $\text{Aut}(X)$. This fits in with the general theme of Theorem 12.

Example. *If $X \to \mathbb{P}^1$ is an elliptic K3 surface with Mordell-Weil rank*

We discuss the following example introduced in [6]. This illustrates a situation where Theorem 12 does not apply.

Example (Bryan-Leung surface)**.** Let S be an elliptic K3 surface with a section whose singular fibers are 24 nodal curves. Following the notation of Theorem 12, we see that $N = 24$. A simple calculation in [6] tells us that $\rho(S) = 2$. In particular, since $N + 2$ is far bigger than $\rho(S)$, the condition is not satisfied. Since $\rho(S) = 2$, $\overline{\text{Eff}}_1(S)$ is naturally polyhedral.

We also discuss an example where the conditions for Theorem 12 fail but $\overline{\text{Eff}}_1(S)$ is still not polyhedral. This shows us that Theorem 12 is merely sufficient and not necessary for $\overline{\text{Eff}}_1(S)$ to not be polyhedral.

Example (Fermat quartic)**.** Let $S \subset \mathbb{P}^3$ be the Fermat quartic $x_0^4 - x_1^4 + x_2^4 - x_3^4 = 0$. The map given by the parameter

$$t := \frac{x_0^2 + x_1^2}{x_2^2 - x_3^2}$$

is an elliptic fibration. When $t = 0, \pm 1, \pm i, \infty$, the fiber degenerates into a cycle of type I_4 as in 6.1. The number N of components of the singular fibers is therefore 24. On the other hand it is well-known that $\rho(S) = 20$. Since $N + 2 > \rho(S)$, we see that the condition for Theorem 12 is not satisfied. However by a well-known theorem of Shioda-Inose, S has infinite automorphism group and therefore $\overline{\mathrm{Eff}}_1(S)$ is not polyhedral. Thus, Theorem 12 is merely sufficient and not necessary.

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