Geometric Lifting of Affine Type A Crystal Combinatorics

by

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ABSTRACT

In the first part of this thesis, we construct a type $A_{n-1}^{(1)}$ geometric crystal on the variety $\mathbb{X}_k := \mathrm{Gr}(k, n) \times \mathbb{C}^\times$, and show that it tropicalizes to the disjoint union of the Kirillov–Reshetikhin crystals corresponding to rectangular semistandard Young tableaux with $n-k$ rows. A key ingredient in our construction is the $\mathbb{Z}/n\mathbb{Z}$ symmetry of the Grassmannian which comes from cyclically shifting a basis of the underlying vector space. We show that a twisted version of this symmetry tropicalizes to combinatorial promotion.

In the second part, we define and study the geometric R-matrix, a birational map $R : \mathbb{X}_{k_1} \times \mathbb{X}_{k_2} \to \mathbb{X}_{k_2} \times \mathbb{X}_{k_1}$ which tropicalizes to the combinatorial R-matrix on pairs of rectangular tableaux. We show that R is an isomorphism of geometric crystals, and that it satisfies the Yang–Baxter relation. In the case where both tableaux have one row, we recover the birational R-matrix of Yamada and Lam–Pylyavskyy. Most of the properties of the geometric R-matrix follow from the fact that it gives the unique solution to a certain equation of matrices in the loop group $\mathrm{GL}_n(\mathbb{C}(\lambda))$.

CHAPTER 1

Introduction

1.1 Affine crystals and the combinatorial R-matrix

In the early 1990s, Kashiwara introduced the theory of crystal bases [Kas90, Kas91]. This groundbreaking work provides a combinatorial model for the representation theory of semisimple (and more generally, Kac–Moody) Lie algebras, allowing many aspects of the representation theory to be studied from a purely combinatorial point of view. In type A, crystal bases can be realized as a collection of combinatorial maps on semistandard Young tableaux, and many previously studied combinatorial tableau algorithms turned out to be special cases of crystal theory. For example, the Robinson–Schensted–Knuth correspondence is the crystal version of the decomposition of the GL_n-representation $(\mathbb{C}^n)^{\otimes d}$ into its irreducible components [Shi05]; Lascoux and Schützenberger's symmetric group action on tableaux is a special case of the Weyl group action on any crystal [BS17]; Schützenberger's promotion map, restricted to rectangular tableaux, is the crystal-theoretic manifestation of the rotation of the affine type A Dynkin diagram [Shi02].

Tableau algorithms are traditionally described as a sequence of local modifications to a tableau, such as bumping an entry from one row to the next, or sliding an entry into an adjacent box. These combinatorial descriptions are quite beautiful, but for some purposes, one might want a formula that describes the local transformations in terms of a natural set of coordinates on tableaux, such as the number of j's in the i^{th} row (or the closely related Gelfand–Tsetlin patterns). Kirillov and Berenstein discovered that the Bender–Knuth involutions, which are the building blocks for algorithms such as promotion and evacuation, act on a Gelfand–Tsetlin pattern by simple piecewise-linear transformations [KB96]. This discovery sparked a search for piecewise-linear formulas for other combinatorial algorithms.

This thesis is centered around the problem of finding piecewise-linear formulas for combinatorial maps coming from affine crystal theory. Quantum affine algebras admit a class of finite-dimensional, non-highest-weight representations called Kirillov–

Reshetikhin (KR) modules. The crystal bases of these representations, which we call KR crystals, have received a lot of attention for several reasons. Kang et al. showed that the crystal bases of highest-weight modules for quantum affine algebras can be built out of infinite tensor products of KR crystals, and they used this construction to compute the 1 point functions of certain solvable lattice models coming from statistical mechanics [KKM+92]. Kirillov–Reshetikhin crystals have also played a central role in the study of a cellular automaton called the box-ball system and its generalizations [TS90, HHI+01].

Unlike the tensor product of representations of Lie algebras and finite groups, the tensor product of representations of quantum algebras (and thus of crystals) is not commutative. In the case of KR crystals, however, there is a unique crystal isomorphism

$$\widetilde{R} : B_1 \otimes B_2 \to B_2 \otimes B_1.$$

This isomorphism is called the combinatorial R-matrix, and it plays an essential role in both of the applications mentioned in the preceding paragraph. For example, the states of the box-ball system can be represented as elements of a tensor product of KR crystals, and the time evolution is given by applying a sequence of combinatorial R-matrices.

In (untwisted) affine type A, Kirillov–Reshetikhin modules correspond to partitions of rectangular shape (L^k), and their crystal bases, which we denote by $B^{k,L}$, are modeled by semistandard Young tableaux of shape (L^k). If one ignores the affine crystal operators $\widetilde{e}_0, \widetilde{f}_0$, then $B^{k,L}$ is the crystal associated to the irreducible $\mathfrak{sl}_n$-module of highest weight (L^k). Shimozono showed that the affine crystal operators are obtained by conjugating the crystal operators $\widetilde{e}_1, \widetilde{f}_1$ by Schützenberger's promotion map [Shi02]. He also gave a combinatorial description of the action of the combinatorial R-matrix on pairs of rectangular tableaux, which we now explain.

Let $*$ denote the associative product on the set of semistandard Young tableaux introduced by Lascoux and Schützenberger (see §2.2.3 for the definition). If $T \in B^{k_1,L_1}$ and $U \in B^{k_2,L_2}$, then there are unique tableaux $U' \in B^{k_2,L_2}$ and $T' \in B^{k_1,L_1}$ such that $T * U = U' * T'$, and the combinatorial R-matrix is realized by the map $\widetilde{R} : T \otimes U \mapsto U' \otimes T'$. For example, suppose

$$T = \boxed{1\,|\,2\,|\,2\,|\,3\,|\,4\,|\,4\,|\,4\,|\,5} \in B^{1,8} \quad \text{and} \quad U = \boxed{1\,|\,2\,|\,2\,|\,4\,|\,5\,|\,5} \in B^{1,6}.$$

The product $T * U$ can be computed by using Schensted's row bumping algorithm to insert the entries of U into T, starting from the left end of U; the result is

$$T * U = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 2 & 4 & 4 & 4 & 5 & 5 \\ \hline 2 & 3 & 4 & 5 \\ \cline{1-4} \end{array}.$$

The reader may verify that the tableaux

$$U' = \boxed{1}\boxed{2}\boxed{2}\boxed{3}\boxed{4}\boxed{5} \qquad \text{and} \qquad T' = \boxed{1}\boxed{2}\boxed{2}\boxed{4}\boxed{4}\boxed{4}\boxed{5}\boxed{5}$$

satisfy $U' * T' = T * U$, so $\widetilde{R}(T \otimes U) = U' \otimes T'$.

There is a combinatorial procedure for pulling $T * U$ apart into U' and T', so the whole process is algorithmic. It is nevertheless natural to ask if the map $\widetilde{R}$ can be computed in one step, without first passing through the product $T * U$. In the case where T and U are both one-row tableaux, there is an elegant piecewise-linear formula for $\widetilde{R}$, due to Hatayama et al.

Proposition 1.1 ([HHI+01, Prop. 4.1])**.** *Suppose T and U are one-row tableaux, with entries at most n, and suppose $\widetilde{R}(T \otimes U) = U' \otimes T'$. Let a_j, b_j be the numbers of j's in T and U, respectively. Define*

$$b'_j = b_j + \widetilde{\kappa}_{j+1} - \widetilde{\kappa}_j, \qquad a'_j = a_j + \widetilde{\kappa}_j - \widetilde{\kappa}_{j+1},$$

where

$$\widetilde{\kappa}_j = \min_{0 \le r \le n-1} \left(b_j + b_{j+1} + \cdots + b_{j+r-1} + a_{j+r+1} + a_{j+r+2} + \cdots + a_{j+n-1} \right),$$

and all subscripts are interpreted modulo n. Then b'_j, a'_j are the numbers of j's in U' and T', respectively.

The motivating goal of this thesis was to generalize Proposition 1.1 to a formula for the combinatorial R-matrix on pairs of arbitrary rectangular tableaux.

1.2 Geometric lifting

How does one find—and work with—piecewise-linear formulas for complicated combinatorial operations? A very useful method is to use tropicalization and geometric lifting. Tropicalization is the procedure which turns a positive rational function (i.e., a function consisting of the operations $+, \cdot, \div$, but not $-$; such functions are often called "subtraction-free" in the literature) into a piecewise-linear function by making the substitutions

$$(+, \cdot, \div) \mapsto (\min, +, -).$$

A geometric (or rational) lift of a piecewise-linear function $\widetilde{h}$ is any positive rational function h which tropicalizes to $\widetilde{h}$. Rational functions are often easier to work with than piecewise-linear functions, since one may bring to bear algebraic and geometric techniques. Furthermore, identities proved in the lifted setting can be "pushed down," via tropicalization, to results about the piecewise-linear functions and the corresponding combinatorial maps.

For example, the formula for $\widetilde{R}$ in Proposition 1.1 turns out to be the tropicalization of a rational map which solves a certain matrix equation. Given $x = (x_1, \ldots, x_n) \in (\mathbb{C}^\times)^n$, define

$$g(x) = \begin{pmatrix} x_1 & & & & & \lambda \\ 1 & x_2 & & & & \\ & 1 & x_3 & & & \\ & & & \ddots & & \\ & & & & x_{n-1} & \\ & & & & 1 & x_n \end{pmatrix}. \tag{1.1}$$

Here λ is an indeterminate, and we view $g(x)$ as an element of the loop group $\mathrm{GL}_n(\mathbb{C}(\lambda))$.

Proposition 1.2 ([Yam01], [LP12, Th. 6.2]). *If $x, y \in (\mathbb{C}^\times)^n$ are sufficiently generic, then the matrix equation*

$$g(x)g(y) = g(y')g(x') \tag{1.2}$$

has two solutions: the trivial solution $y'_j = x_j, x'_j = y_j$, and the solution

$$y'_j = y_j \frac{\kappa_{j+1}}{\kappa_j}, \quad x'_j = x_j \frac{\kappa_j}{\kappa_{j+1}}, \quad \textit{where} \quad \kappa_j = \sum_{r=0}^{n-1} y_j \cdots y_{j+r-1} x_{j+r+1} \cdots x_{j+n-1}, \tag{1.3}$$

and subscripts are interpreted modulo n. The solution given by (1.3) *is the unique solution to* (1.2) *which satisfies the additional constraint*

$$\prod x_j = \prod x'_j \qquad \textit{and} \qquad \prod y_j = \prod y'_j. \tag{1.4}$$

Note that the piecewise-linear map $\widetilde{R}$ in Proposition 1.1 is the tropicalization of the rational map $R : (x, y) \mapsto (y', x')$, where y', x' are defined by (1.3)[1] (note also that (1.4) tropicalizes to the condition $\sum a_j = \sum a'_j, \sum b_j = \sum b'_j$, which says that the tableaux T and T' (resp., U and U') have the same length). Thus, the map R is a geometric lift of the combinatorial R-matrix on pairs of one-row tableaux.

Upon learning of Propositions 1.1 and 1.2, we were deeply impressed that the solution to a matrix equation could also describe a combinatorial procedure for swapping pairs of tableaux. In fact, this example is just one instance of a larger

[1]In the tropicalization, we replace the "rational variables" x_j and y_j, which can be thought of as generic nonzero complex numbers, or indeterminates, with the "combinatorial variables" a_j and b_j, which take on integer values.

phenomenon. Since Kirillov–Berenstein's work on the Bender–Knuth involutions, many other combinatorial algorithms have been lifted to rational maps, including the Robinson–Schensted–Knuth correspondence, the Lascoux–Schützenberger symmetric group action, and rowmotion on posets [KB96, Kir01, NY04, DK07, EP14].

One of the crowning achievements of the geometric lifting program is Berenstein and Kazhdan's theory of geometric crystals, which provides a framework for lifting the entire combinatorial structure of crystal bases [BK00, BK07a]. Roughly speaking, a geometric crystal is a complex algebraic variety X, together with rational actions $e_i : \mathbb{C}^\times \times X \to X$, which are called geometric crystal operators. The geometric crystal operators are required to satisfy rational lifts of the piecewise-linear relations satisfied by (combinatorial) crystal operators. In many cases, the geometric crystal operators are positive, and they tropicalize to piecewise-linear formulas for the combinatorial crystal operators $\widetilde{e}_i$ on a corresponding combinatorial crystal B_X; when this happens, we say that X tropicalizes to B_X. For each reductive group G, Berenstein and Kazhdan [BK07a] constructed a geometric crystal on the flag variety[2] of G which lifts the crystals associated to all the irreducible representations of $G^\vee$, the Langlands dual group (see Remark 2.27(3)). These geometric crystals provide a new method for constructing and studying crystals; in addition, they have proved useful beyond combinatorics, with applications to quantum cohomology and mirror symmetry, Brownian motion on Lie groups, and the local Langlands conjectures [LT17, Chh13, BK07b].

Nakashima [Nak05] extended the definition of geometric crystal to the setting of Kac–Moody (and in particular, affine) Lie algebras. There has been a concerted effort to construct geometric lifts of Kirillov–Reshetikhin crystals, and to find compatible lifts of the associated combinatorial R-matrices. In the case of the one-row affine type A crystals mentioned above, it is straightforward to define a corresponding geometric crystal, and Yamada's rational map from Proposition 1.2 turns out to be an isomorphism of geometric crystals (see the introduction of [KOTY03]). When we began work on this project, Kuniba–Okado–Takagi–Yamada and Kashiwara–Nakashima–Okado had constructed a geometric crystal for the analogue of one-row KR crystals in all non-exceptional affine types, and a compatible geometric R-matrix in types $D_n^{(1)}, B_n^{(1)}, D_{n+1}^{(2)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}$ [KOTY03, KNO08, KNO10]. Beyond the one-row case, Misra and Nakashima had constructed a geometric crystal for two-row tableaux in affine type A (i.e., type $A_{n-1}^{(1)}$) [MN13].

In this thesis, we construct a geometric crystal on $\mathrm{Gr}(n-k, n) \times \mathbb{C}^\times$ which tropicalizes to the disjoint union of the KR crystals $B^{k,L}$, $L \geq 0$, and a compatible geometric R-matrix on products of these geometric crystals. In the next two sections, we give

[2]The geometric crystal is actually constructed on $G/B \times T$, where T is a maximal torus.

an overview of some of the key ideas in our constructions.

1.3 Cyclic symmetry and the Grassmannian

We saw above that the geometric R-matrix in the one-row case is the solution to a matrix equation. The same is true in the general case, and in fact, the full geometric crystal structure is determined from the appropriate generalization of the matrix $g(x)$ in (1.1). Before describing this matrix, we introduce coordinates on semistandard rectangular tableaux with k rows (and entries at most n). The entries in the i^{th} row of such a tableau must lie in the interval $\{i, i+1, \ldots, i+n-k\}$. If we fix the row length L, then the i^{th} row is determined by the $n-k$ integers $b_{ii}, b_{i,i+1}, \ldots, b_{i,i+n-k-1}$, where b_{ij} is the number of j's in the i^{th} row. Thus, a k-row rectangular tableau is determined by $k(n-k)$ integers b_{ij}, plus the row length L. (These integers must satisfy certain inequalities, such as non-negativity, but we ignore the inequalities in this discussion; see §2.2.4 for full details.)

To lift the combinatorial R-matrix in the one-row case, we replaced the integer coordinates $b_1, \ldots, b_n$ with rational coordinates $x_1, \ldots, x_n$. In the k-row case, we replace b_{ij} with x_{ij}, and the row length L with the rational coordinate t. It turns out that the coordinates (x_{ij}, t) are not well-suited to defining the analogue of the matrix (1.1) in the general case. In the $k = 1$ case, the coordinates $x_1, \ldots, x_n$ have a simple cyclic symmetry of order n, which is reflected in the matrix (1.1).[3] For $k > 1$, the coordinates (x_{ij}, t) do not have any obvious cyclic symmetry. There is, however, a "hidden" cyclic symmetry coming from Schützenberger promotion, which has order n on rectangular tableaux with entries at most n. The key to defining the analogue of (1.1) is to use an alternative set of coordinates which makes the action of promotion transparent. This alternative set of coordinates comes from the Grassmannian.

Let $\mathrm{Gr}(n-k, n)$ denote the Grassmannian of $(n-k)$-dimensional subspaces in $\mathbb{C}^n$. Borrowing a construction from the work of Lusztig and Berenstein–Fomin–Zelevinsky on total positivity [Lus94, BFZ96], we define a birational isomorphism from the $k(n-k)$ rational coordinates x_{ij} to a subspace $N \in \mathrm{Gr}(n-k, n)$. Let Θ_{n-k} denote the birational map from $\mathbb{C}^{k(n-k)+1} \to \mathrm{Gr}(n-k, n) \times \mathbb{C}^\times$ given by

$$(x_{ij}, t) \mapsto (N, t) =: N|t.$$

(See Definition 4.1 for the definition of Θ_{n-k}.) The Grassmannian has a natural cyclic symmetry induced by rotating a basis of the underlying n-dimensional vector space. By "twisting" this symmetry by the parameter t, we define a map $\mathrm{PR} : N|t \mapsto N'|t$,

[3] To see this symmetry in the matrix, one must "unfold" $g(x)$ into an infinite periodic matrix which repeats the sequence $x_1, \ldots, x_n$ along the main diagonal, and has an infinite diagonal of 1's just below the main diagonal. See §2.4 for the precise definition of "unfolding."

and we show in Theorem 4.24 that the composition $\Theta_{n-k}^{-1} \circ \mathrm{PR} \circ \Theta_{n-k}$ tropicalizes to a piecewise-linear formula for promotion on k-row rectangular tableaux. Since Θ_{n-k} is a birational isomorphism (and there is a simple formula for its inverse), we may do computations in terms of Plücker coordinates on the Grassmannian, and then translate back to the coordinates (x_{ij}, t) at the end.

The analogue of (1.1) in the general k-row case is a matrix in $\mathrm{GL}_n(\mathbb{C}(\lambda))$ filled with ratios of Plücker coordinates. When $n = 4$ and $k = 2$, the matrix looks like this:

$$g(N|t) = \begin{pmatrix} \frac{P_{14}}{P_{34}} & 0 & \lambda & \lambda\frac{P_{13}}{P_{23}} \\ \frac{P_{24}}{P_{34}} & \frac{P_{12}}{P_{14}} & 0 & \lambda \\ 1 & \frac{P_{13}}{P_{14}} & t\frac{P_{23}}{P_{12}} & 0 \\ 0 & 1 & t\frac{P_{24}}{P_{12}} & t\frac{P_{34}}{P_{23}} \end{pmatrix}.$$

Here P_I is the I^{th} Plücker coordinate of the two-dimensional subspace N. See Definition 3.2 for the general definition of $g(N|t)$; note that the one-row case corresponds to $\mathrm{Gr}(n-1, n) \times \mathbb{C}^\times$.

Suppose $M|s \in \mathrm{Gr}(\ell, n) \times \mathbb{C}^\times$ and $N|t \in \mathrm{Gr}(k, n) \times \mathbb{C}^\times$. As in the one-row case, we seek a solution to the matrix equation

$$g(M|s)g(N|t) = g(N'|t)g(M'|s), \tag{1.5}$$

where $N' \in \mathrm{Gr}(k, n)$ and $M' \in \mathrm{Gr}(\ell, n)$. Using properties of the Grassmannian and linear algebra, we show that for sufficiently generic M, N, s, t, there is a unique candidate for the solution to (1.5) (Lemma 5.8, Corollary 5.9). We define the geometric R-matrix to be the map

$$R : (M|s, N|t) \mapsto (N'|t, M'|s)$$

given by this unique candidate. The main technical results of this thesis are

- Theorem 5.4, which states that R does in fact give a solution to (1.5);
- Theorem 5.3, which states that R is positive, in the sense that the map

  $$(\Theta_k^{-1} \times \Theta_\ell^{-1}) \circ R \circ (\Theta_\ell \times \Theta_k) : ((y_{ij}, s), (x_{ij}, t)) \mapsto ((x'_{ij}, t), (y'_{ij}, s))$$

  is given by positive rational functions in y_{ij}, x_{ij}, s, and t.

The latter result shows that the geometric R-matrix can be tropicalized, and the former result is the key to showing that R commutes with the geometric crystal operators.

1.4 Unipotent crystals and the loop group

Why should the geometric R-matrix satisfy a matrix equation? One explanation comes from the notion of unipotent crystals. Let G be a reductive group, B^- a fixed Borel subgroup, and U the unipotent subgroup of the opposite Borel. In the case $G = \mathrm{GL}_n(\mathbb{C})$, one can take B^- to be the lower triangular matrices and U the upper uni-triangular matrices. Berenstein and Kazhdan [BK00] gave B^- a geometric crystal structure in which the geometric crystal operator e_i is given by simultaneous left and right multiplication by certain elements of the one parameter subgroup in U corresponding to the i^{th} simple root. They defined a unipotent crystal to be a pair (X, g), where X is a variety which carries a rational action of U, and $g : X \to B^-$ is a rational map which is "compatible" with the U-action (see §2.4 for details). A unipotent crystal (X, g) induces a geometric crystal on X, in such a way that g intertwines the geometric crystal operators on X and B^- (i.e., $ge_i = e_i g$). Furthermore, if (X, g) and (Y, g) are unipotent crystals, then $(X \times Y, g)$ is a unipotent crystal, where

$$g(x, y) = g(x)g(y). \tag{1.6}$$

This unipotent crystal induces a geometric crystal on the product $X \times Y$, and if X and Y tropicalize to crystals B_X, B_Y, then $X \times Y$ tropicalizes to the tensor product $B_X \otimes B_Y$.

In our affine type A setting, the appropriate analogue of the reductive group G is the loop group $\mathrm{GL}_n(\mathbb{C}(\lambda))$, which consists of invertible $n \times n$ matrices over the field of rational functions in an indeterminate λ. We take B^- to be a certain "lower triangular" submonoid of G, and U an "upper uni-triangular" subgroup (this triangularity refers to the "unfolded" version of the matrices; see the discussion preceding Definition 2.33). Berenstein and Kazhdan's theory of unipotent crystals extends essentially unchanged to this setting.

Let $\mathbb{X}_k := \mathrm{Gr}(k, n) \times \mathbb{C}^\times$. In §3.1, we show that the map $g : \mathbb{X}_k \to B^-$ discussed above makes $\mathbb{X}_k$ into an affine type A unipotent crystal. This explains why the geometric R-matrix ought to provide a solution to the matrix equation (1.5). Indeed, the geometric R-matrix is supposed to be a map $R : \mathbb{X}_{k_1} \times \mathbb{X}_{k_2} \to \mathbb{X}_{k_2} \times \mathbb{X}_{k_1}$ which commutes with the geometric crystal operators. Equation (1.5) says that $g \circ R = g$; if this is satisfied, then since g commutes with the geometric crystal operators, we have

$$g \circ e_i R = g \circ R e_i. \tag{1.7}$$

By the uniqueness of the solution to (1.5), (1.7) implies that R commutes with e_i.

(There is an alternative explanation, based on the combinatorial description of $\widetilde{R}$, for why the geometric lift of $\widetilde{R}$ should satisfy a matrix equation; see Remark 5.15).

1.5 Related work

Misra and Nakashima recently constructed a geometric crystal which tropicalizes to a certain limit of the crystals $B^{k,L}$ [MN16]. Their construction is based on a description of the affine crystal operators $\widetilde{e}_0, \widetilde{f}_0$ in terms of lattice paths, rather than promotion.

The idea of relating the cyclic symmetry of the Grassmannian to promotion of rectangular tableaux is not new. In Rhoades' work on the cyclic sieving phenomenon, he showed that a natural cyclic shift in a certain realization of the irreducible $\mathrm{GL}_n(\mathbb{C})$-representation corresponding to a rectangular partition permutes the dual canonical basis according to promotion (up to a sign) [Rho10, Prop. 5.5]. Lam translated this result into a statement about cyclic shifting in the homogeneous coordinate ring of the Grassmannian [Lam16, Th. 12.2(4)]. Our result that the twisted cyclic shift map tropicalizes to promotion (Theorem 4.24) was inspired by Rhoades' result; we do not, however, know of any direct connection between the two. More recently, Grinberg and Roby [GR15] used the cyclic symmetry of the Grassmannian to prove that birational rowmotion on the $k \times (n-k)$ rectangle has order n, a result essentially equivalent to Theorem 4.24. In fact, our proof is similar to theirs, although ours arose from the parametrization Θ_k (which comes from the theory of total positivity), whereas theirs was inspired by Volkov's proof of the Zamolodchikov periodicity conjecture in type $A \times A$ [Vol07].

Our formalism is similar in some important respects to that of [KOTY03, KNO10]. In particular, the matrix $g(N|t)$ is the analogue of the "M-matrix" (or "matrix realization") in those works, and our use of the uniqueness of the solution to (1.5) to prove properties of the geometric R-matrix in Theorem 5.11 is identical to Kuniba–Okado–Takagi–Yamada's use of [KOTY03, Th. 3.13] to prove [KOTY03, Prop. 4.6, 4.7, 4.8].

1.6 Applications and future directions

Perhaps the most important property of the combinatorial R-matrix is that it satisfies the Yang–Baxter relation. Akasaka and Kashiwara proved this result by analyzing the poles of the R-matrix on tensor products of Kirillov–Reshetikhin modules [AK97], and Shimozono gave a combinatorial proof using a generalization of Lascoux and Schützenberger's cyclage poset [Shi01]. In §5.1.2, we show that the Yang–Baxter relation for the geometric R-matrix follows immediately (using a bit of linear algebra) from the fact that the geometric R-matrix satisfies the matrix equation (1.5), thereby giving a new proof of the corresponding result for the combinatorial R-matrix.

As discussed above, the twisted cyclic shift map on the Grassmannian, which is a geometric lift of promotion, plays a crucial role throughout this work. This map has order n by definition, so by tropicalizing, we obtain a proof that promotion on rectangular tableaux has order n.[4] Two additional geometric symmetries play an important role: transposition of the matrix $g(N|t)$ over the anti-diagonal, which turns out to be a geometric lift of the Schützenberger involution, and the map from a subspace to its orthogonal complement, which is related to a lift of the "column complementation" map on rectangular tableaux (that is, the map which replaces each column with its complement in $\{1, \ldots, n\}$, and reverses the order of the columns). We show that these symmetries are compatible with the geometric crystal operators, which implies that the corresponding combinatorial symmetries are compatible with the crystal operators on rectangular tableaux. In the case of the column complementation map, this compatibility seems to be a new result (see Remark 2.25).

The one-row geometric R-matrix of Proposition 1.2 has proved to be an interesting map. It induces a birational action of the symmetric group S_m on the field of rational functions in mn variables. Lam and Pylyavskyy called the polynomial invariants of this action loop symmetric functions, and they showed that these invariants have many properties analogous to those of symmetric functions [LP12, Lam12]. We expect that the more general geometric R-matrix constructed here will have applications to loop symmetric functions.

In fact, our original motivation for lifting the combinatorial R-matrix comes from a conjectural connection between loop symmetric functions and the above-mentioned box-ball system. The box-ball system exhibits soliton behavior; that is, regardless of the initial configuration, the balls in the system eventually form themselves into several connected blocks, or solitons, each of which moves as a unit. Lam–Pylyavskyy–Sakamoto [LPS16] conjectured a formula, in terms of the tropicalization of loop symmetric functions, for determining the lengths and internal composition of the solitons from the initial configuration of balls. Using the one-row geometric R-matrix, they were able to prove the first case of their conjecture. To extend their method to prove the full conjecture, one needs a lift of the combinatorial R-matrix in the case where one of the tableaux has more than one row. We are optimistic that our general geometric R-matrix can be used to prove the conjecture in full generality.

The (co)energy function on tensor products of Kirillov–Reshetikhin crystals is another interesting feature of affine crystal theory which plays an important role in the study of the box-ball system. Lam and Pylyavskyy showed that a certain

[4] This proof also follows from the above-mentioned result of Grinberg and Roby [GR15].

"stretched staircase" loop Schur function tropicalizes to the coenergy function on tensor products of arbitrarily many one-row tableaux [LP13]. As an application of our setup, we show that a minor of the matrix $g(M|s)g(N|t)$ tropicalizes to the coenergy function on tensor products of two arbitrary rectangular tableaux. It would be interesting to find a geometric coenergy function on tensor products of more than two rectangular tableaux which simultaneously generalizes this minor and the "stretched staircase" loop Schur function.

We hope that our methods can be extended to lift Kirillov–Reshetikhin crystals and their combinatorial R-matrices in other affine types (beyond the analogue of the one-row case). One potential difficulty is that most Kirillov–Reshetikhin crystals outside of type $A_{n-1}^{(1)}$ are reducible as classical crystals. We suspect that this will make it necessary to use "isotropic partial flag varieties," rather than just "isotropic Grassmannians," in the other types.

1.7 Organization

Chapter 2 surveys much of the required background for this thesis. In §2.1 we discuss notation. In §2.2 we review the combinatorics of Kirillov–Reshetikhin crystals and the translation into piecewise-linear maps on Gelfand–Tsetlin patterns; in §2.3 we review the definition of geometric crystals; in §2.4 we discuss the loop group and the affine version of unipotent crystals that we use; in §2.5 we review several important results about the Grassmannian; in §2.6 we discuss the Lindström Lemma, which enables one to compute minors of matrices in terms of paths in planar networks.

Chapter 3 revolves around the unipotent crystal map $g : \mathbb{X}_k = \mathrm{Gr}(k,n) \times \mathbb{C}^\times \to \mathrm{GL}_n(\mathbb{C}(\lambda))$. In §3.1, we define this map, show that it makes $\mathbb{X}_k$ into a unipotent crystal, and present explicit formulas for the induced geometric crystal structure on $\mathbb{X}_k$. In §3.2, we prove several important properties of the map g. In §3.3, we study the cyclic symmetry of $\mathbb{X}_k$, the geometric lift of the Schützenberger involution, and the map from a subspace to its orthogonal complement in the dual Grassmannian. Using the relationship between these symmetries and the map g, we show that the symmetries are compatible with the geometric crystal structure. These symmetries play an indispensable role in the proofs of later results.

Chapter 4 explains how to tropicalize the geometric crystal on $\mathbb{X}_k$ to obtain piecewise-linear formulas for the affine crystal structure on rectangular tableaux with $n-k$ rows. The first step is to introduce the map Θ_k, which parametrizes $\mathbb{X}_k$ by a complex torus of dimension $k(n-k)+1$. In §4.1, we define this parametrization, give an explicit formula for its inverse, and use the Lindström Lemma to derive formulas for Plücker coordinates in terms of the parameters. §4.2 discusses a general notion of positive rational maps, and defines the tropicalization of such maps. In §4.3, we consider the tropicalization of the geometric crystal structure on (products of) $\mathbb{X}_k$. We show that the tropicalization of a function called the decoration defines a polyhedron whose integer points are precisely the rectangular tableaux with $n-k$ rows (the decoration is Berenstein and Kazhdan's ingenious solution to the problem of "lifting" the inequalities of the piecewise-linear setting to the geometric setting). We then prove that the tropicalizations of the geometric crystal maps, when restricted to the integer points of this polyhedron, agree with their combinatorial counterparts. The key step in this proof is Theorem 4.24, which states that the cyclic shift map PR tropicalizes to promotion. We also consider the tropicalization of the other two symmetries, and we work out some small examples.

Chapter 5 is devoted to the geometric R-matrix $R : \mathbb{X}_{k_1} \times \mathbb{X}_{k_2} \to \mathbb{X}_{k_2} \times \mathbb{X}_{k_1}$. In §5.1, we define this map and state its two most important properties, namely that it is positive (Theorem 5.3), and that it satisfies the identity $g \circ R = g$ (Theorem 5.4). We show that these two results almost immediately imply that R is an isomorphism of geometric crystals, an involution, and a solution to the Yang–Baxter equation. Using the uniqueness of the combinatorial R-matrix $\widetilde{R}$, we deduce that the geometric R-matrix tropicalizes to a piecewise-linear formula for $\widetilde{R}$. In §5.2, we define a rational function $E : \mathbb{X}_{k_1} \times \mathbb{X}_{k_2} \to \mathbb{C}$ which tropicalizes to the coenergy function on the tensor product of two rectangular tableaux. In §5.3, we work out explicit formulas for R (and thus for $\widetilde{R}$) in the case where the first tableau has one row. We show that when both tableaux have one row, we recover the map of Yamada and Lam–Pylyavskyy from Proposition 1.2. Finally, we prove Theorems 5.3 and 5.4 in §5.4 and §5.5, respectively.

CHAPTER 2

Preliminaries

2.1 Notation

Throughout this thesis, we fix an integer $n \geq 2$. For two integers i and j, we write

$$[i, j] = \{m \in \mathbb{Z} \,|\, i \leq m \leq j\}.$$

We often abbreviate $[1, j]$ to $[j]$. We write $\binom{[n]}{k}$ for the set of k-element subsets (or *k-subsets*) of $[n]$, and $|J|$ for the cardinality of a set J.

Given a matrix X and two subsets I, J, we write $X_{I,J}$ to denote the submatrix using the rows in I and the columns in J. If $|I| = |J|$, we write

$$\Delta_{I,J}(X) = \det(X_{I,J}).$$

We use the term *upper (resp., lower) uni-triangular* to refer to matrices with zeroes below (resp., above) the main diagonal, and 1's on the main diagonal.

Given a subset $J \subset [n]$, we write $w_0(J)$ for the set obtained by replacing each $j \in J$ with $n - j + 1$; $\overline{J}$ for the complement $[n] \setminus J$; and J^* for $\overline{w_0(J)}$. For an integer c, we write $J - c$ for the subset of $[n]$ obtained by subtracting c from each element of J, and then taking the residues of the resulting integers mod n.

By affine type A, we mean the untwisted affine root system $A_{n-1}^{(1)}$, whose Dynkin diagram is a cycle with n nodes. Type A refers to the root system A_{n-1}, whose Dynkin diagram is a path with $n - 1$ nodes. The simple Lie algebra $\mathfrak{sl}_n$ has root system A_{n-1}, and the affine Lie algebra $\widehat{\mathfrak{sl}}_n$ has root system $A_{n-1}^{(1)}$.

We write $\mathbb{C}^\times$ for the multiplicative group of nonzero complex numbers, and GL_n for $\mathrm{GL}_n(\mathbb{C})$. Almost all the maps between algebraic varieties appearing in this thesis are rational, so we write them with solid arrows (e.g., $h : X \to Y$), rather than dotted arrows. We apologize to any algebraic geometers who are annoyed by this choice.

2.2 Crystals

In §2.2.1, we present Kashiwara's axioms for (abstract) crystals. In §2.2.2, we present the type A crystal structure on semistandard Young tableaux of shape λ. We then introduce Schützenberger's promotion map, which allows us to extend the type A crystal structure to an affine type A crystal structure in the case where λ is a rectangle. In §2.2.3, we review the definition of the tensor product of crystals, and we describe Shimozono's realizations of the combinatorial R-matrix and coenergy function on tensor products of rectangular tableaux. In §2.2.4, we review the notion of Gelfand–Tsetlin pattern, which identifies semistandard tableaux with the integer points of a polyhedron, and we translate several maps on tableaux into maps on Gelfand–Tsetlin patterns given by piecewise-linear formulas. We introduce "k-rectangles" as the subset of Gelfand–Tsetlin patterns which correspond to rectangular tableaux with k rows.

2.2.1 Crystal axioms

Kashiwara introduced the crystal basis as the $q \to 0$ limit of a special basis of a module for the quantized universal enveloping algebra $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a Kac–Moody Lie algebra [Kas91]. The crystal basis can be viewed as a combinatorial skeleton of the corresponding representation of $\mathfrak{g}$. Kashiwara's theory gives rise to the following paradigm for studying representations of $\mathfrak{g}$: find a model for the crystal bases in terms of a combinatorial object (tableaux, Littelmann paths, Mirkovic–Vilonen polytopes, rigged configurations, etc.), and then analyze the combinatorics of this model.

Kashiwara abstracted several properties of crystal bases into axioms for (abstract) crystals, which we now state, following the presentation in [BS17]. To streamline the presentation, we specialize the definition to types A_{n-1} and $A_{n-1}^{(1)}$, which are the only types considered in this thesis. In both cases, we use the weight lattice $\Lambda = \mathbb{Z}^n$. Let $\{v_1, \ldots, v_n\}$ be the standard basis of Λ, and for $i \in [n-1]$, let $\widetilde{\alpha}_i = v_i - v_{i+1}$ be the i^{th} simple root in Λ. Let $\widetilde{\alpha}_i^\vee : \Lambda \to \mathbb{Z}$ be the i^{th} simple coroot, the map sending $(a_1, \ldots, a_n) \mapsto a_i - a_{i+1}$. Let $\widetilde{\alpha}_0 = v_n - v_1$ be the affine simple root, and let $\widetilde{\alpha}_0^\vee : (a_1, \ldots, a_n) \mapsto a_n - a_1$ be the affine simple coroot. We identify the index set of the affine simple roots and coroots with $\mathbb{Z}/n\mathbb{Z}$.

Definition 2.1. A *(Kashiwara, or abstract) crystal of type A_{n-1} (resp., type $A_{n-1}^{(1)}$)* consists of a set $\mathcal{B}$, together with

- a *weight map* $\mathrm{wt} : \mathcal{B} \to \Lambda$;
- for each $i \in [n-1]$ (resp., $i \in \mathbb{Z}/n\mathbb{Z}$), maps

$$\widetilde{\varepsilon}_i, \widetilde{\varphi}_i : \mathcal{B} \to \mathbb{Z}_{\geq 0} \qquad \text{and} \qquad \widetilde{e}_i, \widetilde{f}_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\}.$$

Here $0 \notin \mathcal{B}$ is an auxiliary element. The maps $\widetilde{e}_i$ and $\widetilde{f}_i$ are called *crystal (or Kashiwara) operators*. We say that $\widetilde{e}_i$ is *defined* on an element $b \in B$ if $\widetilde{e}_i(b) \neq 0$, and *undefined* if $\widetilde{e}_i(b) = 0$ (and similarly for $\widetilde{f}_i(b)$). The maps must satisfy the following three axioms:[1]

1. If $a, b \in \mathcal{B}$, then $\widetilde{e}_i(b) = a$ if and only if $\widetilde{f}_i(a) = b$. In this case,

$$\mathrm{wt}(a) = \mathrm{wt}(b) + \widetilde{\alpha}_i, \qquad \widetilde{\varepsilon}_i(a) = \widetilde{\varepsilon}_i(b) - 1, \qquad \widetilde{\varphi}_i(a) = \widetilde{\varphi}_i(b) + 1.$$

2. For $b \in \mathcal{B}$,

$$\widetilde{\varphi}_i(b) - \widetilde{\varepsilon}_i(b) = \widetilde{\alpha}_i^\vee(\mathrm{wt}(b)).$$

3. For $b \in \mathcal{B}$, $\widetilde{e}_i(b)$ is defined if and only if $\widetilde{\varepsilon}_i(b) > 0$, and $\widetilde{f}_i(b)$ is defined if and only if $\widetilde{\varphi}_i(b) > 0$.

Given crystals $\mathcal{A}$ and $\mathcal{B}$ of the same type, a map $\psi : \mathcal{A} \to \mathcal{B} \cup \{0\}$ is a *strict*[2] *morphism of crystals* if

$$\mathrm{wt}(\psi(a)) = \mathrm{wt}(a) \qquad \widetilde{\varepsilon}_i(\psi(a)) = \widetilde{\varepsilon}_i(a) \qquad \widetilde{\varphi}_i(\psi(a)) = \widetilde{\varphi}_i(a)$$

whenever $\psi(a) \in \mathcal{B}$, and ψ commutes with $\widetilde{e}_i$ and $\widetilde{f}_i$ (using the convention $\psi(0) = \widetilde{e}_i(0) = \widetilde{f}_i(0) = 0$). The map ψ is an *isomorphism* if in addition, ψ is a bijection $\mathcal{A} \to \mathcal{B}$.

It is common to visually represent a crystal by its *crystal graph*; this is the graph on the vertex set $\mathcal{B}$, with a directed i-labeled edge from a to b whenever $\widetilde{f}_i(a) = b$. Figure 1 shows an example of a crystal graph.

Remark 2.2. The crystal basis of any $U_q(\mathfrak{g})$-module is a crystal in the sense of Definition 2.1 (where the definition is adapted to the appropriate root system), but the converse is false. Stembridge introduced additional axioms that characterize the class of crystals which come from highest-weight $U_q(\mathfrak{g})$-modules when the root system of $\mathfrak{g}$ is simply-laced [Ste03]. For non-simply-laced types, and for non-highest weight modules (such as Kirillov–Reshetikhin modules), there is no known axiomatic characterization of the class of crystals arising from modules.

2.2.2 Crystal structure on tableaux

Let λ be a partition with at most n parts. A *semistandard Young tableau (SSYT) of shape* λ is a filling of the Young diagram of λ with entries in $[n]$, such that the rows are weakly increasing, and the columns are strictly increasing. We will often

[1] The definition in [BS17] allows $\widetilde{\varepsilon}_i$ and $\widetilde{\varphi}_i$ to take values in $\mathbb{Z} \cup \{-\infty\}$, and slightly modifies the second axiom. Furthermore, the third axiom is not assumed, and crystals with this property are called *seminormal*.

[2] There is a weaker notion of morphism that we will not need; see [BS17, p.19].

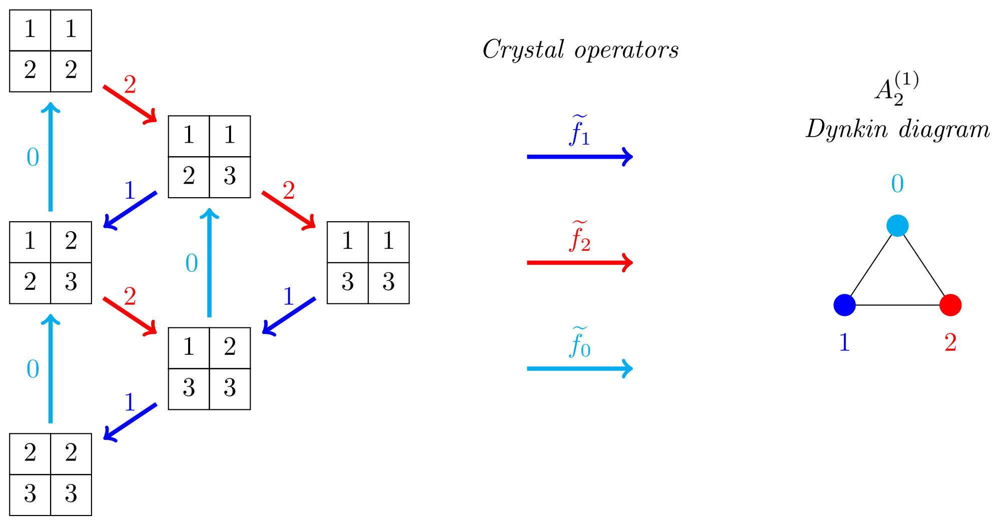


Figure 1: The Kirillov–Reshetikhin crystal $B^{2,2}$ of type $A_2^{(1)}$.

refer to these objects simply as *tableaux*. We write $B(\lambda)$ to denote the set of SSYTs of shape λ.

For each partition λ, there is an irreducible $\mathfrak{sl}_n$-representation whose basis is indexed by $B(\lambda)$, and a corresponding type A_{n-1} crystal on the vertex set $B(\lambda)$. The weight map wt is the *content* of a tableau, i.e., $\mathrm{wt}(T) = (a_1, \ldots, a_n)$, where a_i is the number of i's in T. We now describe the standard procedure for computing the maps $\widetilde{\varepsilon}_i, \widetilde{\varphi}_i, \widetilde{e}_i$, and $\widetilde{f}_i$.

Definition 2.3. For $i \in [n-1]$, the maps $\widetilde{\varepsilon}_i, \widetilde{\varphi}_i, \widetilde{e}_i$, and $\widetilde{f}_i$ are defined on $T \in B(\lambda)$ as follows. To begin, let w be the *(row) reading word* of T, i.e., the word formed by concatenating the rows of T, starting with the bottom row.[3] Now apply the following algorithm to w:

1. Cross out all letters not equal to i or $i+1$.
2. For each consecutive pair of (non-crossed out) letters of the form $i+1, i$, cross out both letters.
3. Repeat the previous step until there are no remaining pairs to cross out.
4. Let w' be the resulting subword, which is necessarily of the form

$$w' = i^{\alpha}\ (i+1)^{\beta}.$$

The functions $\widetilde{\varepsilon}_i$ and $\widetilde{\varphi}_i$ are defined by

$$\widetilde{\varepsilon}_i(T) = \beta \qquad\qquad \widetilde{\varphi}_i(T) = \alpha.$$

[3]We follow the English convention, where the rows of a Young diagram decrease in length from top to bottom.

If $\beta = 0$, then the crystal operator $\widetilde{e}_i(T)$ is undefined; if $\beta > 0$, then $\widetilde{e}_i(T)$ is the tableau of shape λ whose reading word is obtained from w by changing the left-most $i+1$ in w' into an i (it is clear that there is such a tableau). Similarly, if $\alpha = 0$, then $\widetilde{f}_i(T)$ is undefined, and if $\alpha > 0$, then $\widetilde{f}_i(T)$ is the tableau of shape λ whose reading word is obtained from w by changing the right-most i in w' into an $i+1$.

Example 2.4. Let $T = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&2&3&3&3\\ \hline 2&3&3&3\\ \cline{1-4}\end{array}$. The subword of 2's and 3's in w is

$$2 \quad 3 \quad 3 \quad 3 \quad 2 \quad 2 \quad 2 \quad 3 \quad 3 \quad 3$$

and after (recursively) crossing out consecutive pairs of the form 3 2, we are left with

$$w' = 2 \quad \cancel{3} \quad \cancel{3} \quad \cancel{3} \quad \cancel{2} \quad \cancel{2} \quad \cancel{2} \quad 3 \quad 3 \quad 3 \quad = \quad 2 \quad 3 \quad 3 \quad 3.$$

Thus, we have $\widetilde{\varepsilon}_2(T) = 3$, $\widetilde{\varphi}_2(T) = 1$, and

$$\widetilde{e}_2(T) = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&2&2&3&3\\ \hline 2&3&3&3\\ \cline{1-4}\end{array} \qquad \widetilde{f}_2(T) = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&2&3&3&3\\ \hline 3&3&3&3\\ \cline{1-4}\end{array}.$$

The reader may easily verify that the maps defined above make $B(\lambda)$ into a crystal of type A_{n-1}. A much deeper result is that this crystal arises as the crystal basis of a $U_q(\mathfrak{sl}_n)$-module; this was proved by Kashiwara and Nakashima [KN94].

Affine crystal structure on rectangular tableaux

For $k \in [n-1]$ and $L \geq 0$, define $B^{k,L} := B(L^k)$, the set of SSYTs (with entries in $[n]$) whose shape is the $k \times L$ rectangle. (By convention, $B^{k,0}$ consists of a single "empty tableau.") The type A_{n-1} crystal structure on $B^{k,L}$ can be extended to a type $A_{n-1}^{(1)}$ crystal structure. This affine crystal is the crystal basis of a *Kirillov–Reshetikhin module*, a finite-dimensional representation of $U_q'(\widehat{\mathfrak{sl}}_n)$. Furthermore, the Kirillov–Reshetikhin crystals in type $A_{n-1}^{(1)}$ are precisely the $B^{k,L}$.

We now present Shimozono's combinatorial description of the affine crystal operators $\widetilde{e}_0, \widetilde{f}_0$ on $B^{k,L}$ in terms of promotion [Shi02]. Let $\widetilde{\sigma}_i$ be the *Bender–Knuth involution* which interchanges the numbers of i's and $i+1$'s in a semistandard tableau. Given $T \in B(\lambda)$, $\widetilde{\sigma}_i(T) \in B(\lambda)$ is obtained by applying the following procedure to each row of T:

In a given row, suppose there are α boxes containing i which are not directly above a box containing $i+1$, and β boxes containing $i+1$ which are not directly below a box containing i. Thus, this row contains a consecutive subword of the form $i^\alpha(i+1)^\beta$. Replace this subword with $i^\beta(i+1)^\alpha$.

Promotion is the map $\widetilde{\mathrm{pr}} : B(\lambda) \to B(\lambda)$ defined by

$$\widetilde{\mathrm{pr}} = \widetilde{\sigma}_1 \widetilde{\sigma}_2 \cdots \widetilde{\sigma}_{n-1}. \tag{2.1}$$

Remark 2.5. It is well-known that promotion as defined here is equivalent to the following algorithm based on Schützenberger's jeu-de-taquin: remove the n's; slide the remaining entries outward (start by sliding into the left-most hole); fill the vacated boxes with 0; increase all entries by 1.

Example 2.6. If T is the tableau in Example 2.4, then

$$\widetilde{\sigma}_2(T) = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\ \hline 2 & 2 & 3 & 3 \\ \cline{1-4} \end{array}$$

and if $n = 3$, we have

$$\widetilde{\mathrm{pr}}(T) = \widetilde{\sigma}_1(\widetilde{\sigma}_2(T)) = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 \\ \hline 2 & 2 & 3 & 3 \\ \cline{1-4} \end{array}.$$

Definition 2.7. On $B^{k,L}$, define

$$\widetilde{\varepsilon}_0 = \widetilde{\varepsilon}_1 \circ \widetilde{\mathrm{pr}} \qquad \widetilde{\varphi}_0 = \widetilde{\varphi}_1 \circ \widetilde{\mathrm{pr}}$$
$$\widetilde{e}_0 = \widetilde{\mathrm{pr}}^{-1} \circ \widetilde{e}_1 \circ \widetilde{\mathrm{pr}} \qquad \widetilde{f}_0 = \widetilde{\mathrm{pr}}^{-1} \circ \widetilde{f}_1 \circ \widetilde{\mathrm{pr}}$$

where we set $\widetilde{e}_0(T) = 0$ if $\widetilde{e}_1 \circ \widetilde{\mathrm{pr}}(T) = 0$ (equivalently, if $\widetilde{\varepsilon}_0(T) = 0$), and $\widetilde{f}_0(T) = 0$ if $\widetilde{f}_1 \circ \widetilde{\mathrm{pr}}(T) = 0$ (equivalently, if $\widetilde{\varphi}_0(T) = 0$).

The reader may verify that the crystal operators in Figure 1 are computed by Definitions 2.3 and 2.7.

Proposition 2.8. *We have the following identities of maps on* $B^{k,L}$*:*

1. $\widetilde{\mathrm{pr}}^n = \mathrm{Id}$*;*
2. $\mathrm{wt} \circ \widetilde{\mathrm{pr}} = \widetilde{\mathrm{sh}} \circ \mathrm{wt}$*, where* $\widetilde{\mathrm{sh}}(a_1, \ldots, a_n) = (a_n, a_1, \ldots, a_{n-1})$*;*
3. $\widetilde{\varepsilon}_i \circ \widetilde{\mathrm{pr}} = \widetilde{\varepsilon}_{i-1}$ *and* $\widetilde{\varphi}_i \circ \widetilde{\mathrm{pr}} = \widetilde{\varphi}_{i-1}$ *for* $i \in \mathbb{Z}/n\mathbb{Z}$*;*
4. $\widetilde{e}_i \circ \widetilde{\mathrm{pr}} = \widetilde{e}_{i-1} \circ \widetilde{\mathrm{pr}}$ *and* $\widetilde{f}_i \circ \widetilde{\mathrm{pr}} = \widetilde{\mathrm{pr}} \circ \widetilde{f}_{i-1}$ *for* $i \in \mathbb{Z}/n\mathbb{Z}$*.*

Part (1) is well-known (see, e.g., [Shi02, Rho10]). Part (2) is immediate from the definitions. Parts (3) and (4) are due to Shimozono [Shi02, §3.3].

Remark 2.9. The definitions of $\widetilde{\varepsilon}_0, \widetilde{\varphi}_0, \widetilde{e}_0$, and $\widetilde{f}_0$ make sense for any partition λ, and in fact they define an affine crystal structure on $B(\lambda)$ (in the sense of Definition 2.1). When λ is not a rectangle, however, this crystal does not arise from a $U_q'(\widehat{\mathfrak{sl}}_n)$ module. This is related to the fact that the identity $\widetilde{\mathrm{pr}}^n = \mathrm{Id}$ holds only for rectangular tableaux [Shi02, §3.3].

2.2.3 Tensor product of crystals

One of the most important features of crystal theory is the tensor product, which corresponds to the tensor product of modules.

Definition 2.10. Given two crystals $\mathcal{A}, \mathcal{B}$ of the same type (e.g., type A_{n-1} or type $A_{n-1}^{(1)}$), their *tensor product* $\mathcal{A} \otimes \mathcal{B}$ is defined as follows. The underlying set is the Cartesian product $\mathcal{A} \times \mathcal{B}$, whose elements we denote by $a \otimes b$. The crystal structure is defined by[4]

$$\mathrm{wt}(a \otimes b) = \mathrm{wt}(a) + \mathrm{wt}(b)$$

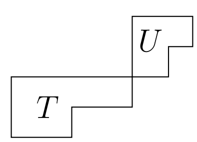
$$\widetilde{\varepsilon}_i(a \otimes b) = \widetilde{\varepsilon}_i(b) + \max(0, \widetilde{\varepsilon}_i(a) - \widetilde{\varphi}_i(b)) \qquad \widetilde{\varphi}_i(a \otimes b) = \widetilde{\varphi}_i(a) + \max(0, \widetilde{\varphi}_i(b) - \widetilde{\varepsilon}_i(a))$$

$$\widetilde{e}_i(a \otimes b) = \begin{cases} \widetilde{e}_i(a) \otimes b & \text{if } \widetilde{\varepsilon}_i(a) > \widetilde{\varphi}_i(b) \\ a \otimes \widetilde{e}_i(b) & \text{if } \widetilde{\varepsilon}_i(a) \leq \widetilde{\varphi}_i(b) \end{cases} \qquad \widetilde{f}_i(a \otimes b) = \begin{cases} \widetilde{f}_i(a) \otimes b & \text{if } \widetilde{\varepsilon}_i(a) \geq \widetilde{\varphi}_i(b) \\ a \otimes \widetilde{f}_i(b) & \text{if } \widetilde{\varepsilon}_i(a) < \widetilde{\varphi}_i(b). \end{cases}$$

In the definition of $\widetilde{e}_i$ and $\widetilde{f}_i$, we use the convention $0 \otimes b = a \otimes 0 = 0$.

It is straightforward to verify that $\mathcal{A} \otimes \mathcal{B}$ satisfies the axioms of Definition 2.1, and that the tensor product is associative. Kashiwara proved that if $\mathcal{A}$ and $\mathcal{B}$ are crystal bases of modules V and W, then $\mathcal{A} \otimes \mathcal{B}$ is the crystal basis of the tensor product $V \otimes W$ [Kas91].

It turns out that the tensor product of the type A_{n-1} crystals $B(\lambda)$ corresponds to an associative product of semistandard tableaux that was introduced by Lascoux and Schützenberger [LS81]. Given two tableaux T and U, the *tableau product* $T * U$ may be defined as the rectification of the skew-tableau obtained by placing U to the northeast of T, as shown here:



The rectification can be computed using Schützenberger's jeu-de-taquin slides or Schensted's row insertion (we refer the reader to [Ful97] for details).

The following result states that the tableau product is compatible with the tensor product of type A_{n-1} crystals.

Proposition 2.11. *If $T \in B(\lambda)$ and $U \in B(\mu)$ and s is one of the maps* $\mathrm{wt}, \widetilde{\varepsilon}_i, \widetilde{\varphi}_i$, *then $s(T \otimes U) = s(T * U)$. If $s = \widetilde{e}_i, \widetilde{f}_i$, and $s(T \otimes U) = (T' \otimes U')$ in $B(\lambda) \otimes B(\mu)$, then $s(T * U) = T' * U'$ in $B(\nu)$, where ν is the shape of $T * U$.*

[4]We use the convention of [Shi02, BS17]; Kashiwara's original convention interchanges the roles of a and b.

Using Proposition 2.11 and the crystal structure on tableaux defined above, one can derive the Littlewood–Richardson rule for the decomposition of a tensor product of $\mathfrak{sl}_n$-modules. For the proof of Proposition 2.11 and the derivation of the Littlewood–Richardson rule, see, e.g., [Shi05] or [BS17, Ch. 9].

The combinatorial R-matrix

The tensor product of crystals is not commutative. In the case of the Kirillov–Reshetikhin crystals $B^{k,L}$, however, there is a unique affine crystal isomorphism $\widetilde{R} : B^{k_1,L_1} \otimes B^{k_2,L_2} \to B^{k_2,L_2} \otimes B^{k_1,L_1}$, called the *combinatorial R-matrix*. The existence and uniqueness of this isomorphism is proved using quantum groups; see [Shi02, Th. 3.19]. We now describe how this map acts on tableaux, following Shimozono [Shi02].

Proposition 2.12. *Suppose $(T, U) \in B^{k_1,L_1} \otimes B^{k_2,L_2}$.*

1. *There is a unique pair $(U', T') \in B^{k_2,L_2} \times B^{k_1,L_1}$ such that $T * U = U' * T'$.*
2. *The combinatorial R-matrix is given by $\widetilde{R}(T, U) = (U', T')$.*

Proof (sketch). The Littlewood–Richardson coefficient $c^{\nu}_{\lambda\mu}$ is equal to the number of pairs $(T, U) \in B(\lambda) \times B(\mu)$ such that $T * U = V$, where V is a fixed element of $B(\nu)$ (see [Ful97, §5.1, Cor. 2]). If λ and μ are rectangles, then the product of Schur functions $s_\lambda s_\mu$ is multiplicity-free (see [Ste01]). Thus, there is exactly one pair $(U', T') \in B^{k_2,L_2} \times B^{k_1,L_1}$ such that $U' * T' = T * U$. This proves (1).

By [Shi02, Lem. 3.8], if λ and μ are arbitrary partitions and $\psi : B(\lambda) \times B(\mu) \to B(\mu) \times B(\lambda)$ is a bijection which commutes with the classical crystal operators $\widetilde{e}_1, \ldots, \widetilde{e}_{n-1}$, then $\psi(T, U) = (U', T')$ implies that $T * U = U' * T'$ (this is essentially a converse to Proposition 2.11). Thus, (2) follows from (1) and the existence of $\widetilde{R}$. ∎

Example 2.13. If

$$T = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 3 & 3 & 3 & 3 & 4 \\ \hline \end{array} \in B^{1,7} \qquad \text{and} \qquad U = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 4 & 4 & 4 \\ \hline \end{array} \in B^{2,5},$$

then

$$T * U = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 2 & 3 & 4 & 4 & 4 \\ \hline 2 & 2 & 3 & 3 & 4 \\ \cline{1-5} 3 & 3 \\ \cline{1-2} \end{array} .$$

The reader may verify that the tableaux

$$U' = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 & 4 \\ \hline \end{array} \qquad \text{and} \qquad T' = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 3 & 4 & 4 & 4 \\ \hline \end{array}$$

satisfy $U' * T' = T * U$, so $\widetilde{R}(T \otimes U) = U' \otimes T'$.

Proposition 2.14. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be Kirillov–Reshetikhin crystals of type $A^{(1)}_{n-1}$.*

1. *The map $\widetilde{R}^2 : \mathcal{A} \otimes \mathcal{B} \to \mathcal{A} \otimes \mathcal{B}$ is the identity.*
2. *The combinatorial R-matrix satisfies the Yang–Baxter relation. That is, if $\widetilde{R}_1 : \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C} \to \mathcal{B} \otimes \mathcal{A} \otimes \mathcal{C}$ is the map which applies $\widetilde{R}$ to the first two factors and does nothing to the third factor, and $\widetilde{R}_2 : \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C} \to \mathcal{A} \otimes \mathcal{C} \otimes \mathcal{B}$ is the map which applies $\widetilde{R}$ to the last two factors and does nothing to the first factor, then*

$$\widetilde{R}_1 \circ \widetilde{R}_2 \circ \widetilde{R}_1 = \widetilde{R}_2 \circ \widetilde{R}_1 \circ \widetilde{R}_2$$

as maps from $\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C} \to \mathcal{C} \otimes \mathcal{B} \otimes \mathcal{A}$.

The first statement follows immediately from the description of $\widetilde{R}$ in Proposition 2.12. There are several proofs of the Yang–Baxter relation. For instance, the Yang–Baxter relation is a consequence of Akasaka–Kashiwara's result that every tensor product $B^{k_1,L_1} \otimes \cdots \otimes B^{k_d,L_d}$ is connected (as an affine crystal), which in turn is proved using quantum groups [AK97]. Shimozono gave a purely combinatorial proof of the Yang–Baxter relation using a generalization of Lascoux and Schützenberger's cyclage poset [Shi01, Th. 8(A3)]. In §5.1.3, we give a new proof using the geometric R-matrix.

The coenergy function

Another important element of affine crystal theory is the coenergy function.

Definition 2.15. Let $\mathcal{A}$ and $\mathcal{B}$ be Kirillov–Reshetikhin crystals. A function $\widetilde{H} : \mathcal{A} \otimes \mathcal{B} \to \mathbb{Z}$ is a *coenergy function* if $\widetilde{H} \circ \widetilde{e}_i = \widetilde{e}_i$ for $i = 1, \ldots, n-1$, and $\widetilde{H}$ interacts with $\widetilde{e}_0$ as follows: if $a \otimes b \in \mathcal{A} \otimes \mathcal{B}$ and $\widetilde{R}(a \otimes b) = b' \otimes a'$, then

$$\widetilde{H}(\widetilde{e}_0(a \otimes b)) = \widetilde{H}(a \otimes b) + \begin{cases} 1 & \text{if } \widetilde{\varepsilon}_0(a) > \widetilde{\varphi}_0(b) \text{ and } \widetilde{\varepsilon}_0(b') > \widetilde{\varphi}_0(a') \\ -1 & \text{if } \widetilde{\varepsilon}_0(a) \leq \widetilde{\varphi}_0(b) \text{ and } \widetilde{\varepsilon}_0(b') \leq \widetilde{\varphi}_0(a') \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Remark 2.16. A function $\widetilde{H}$ is a coenergy function if and only if $-\widetilde{H}$ is an *energy function*, in the sense of [KKM+92, Shi02]. We have chosen to work with coenergy instead of energy because the coenergy function $\widetilde{E}$ defined below naturally arises as the tropicalization of a certain rational function on our geometric crystals.

Given $T \in B^{k_1,L_1}$ and $U \in B^{k_2,L_2}$, define $\widetilde{E}(T \otimes U)$ to be the number of boxes in the tableau $T * U$ which are not in the first $\max(k_1, k_2)$ rows. It's clear from the nature of Schensted insertion that if T_0 is the classical highest weight element of B^{k_1,L_1} (that is, the tableau whose i^{th} row is filled with the number i), then

$$\widetilde{E}(T_0 \otimes U) = 0 \text{ for all } U \in B^{k_2,L_2}. \tag{2.3}$$

Example 2.17. Let T and U be the tableaux in Example 2.13. There are two boxes outside the first $\max(1,2)$ rows of $T * U$, so $\widetilde{E}(T \otimes U) = 2$.

Proposition 2.18.

1. *Up to a global additive constant, there is a unique (co)energy function on* $B^{k_1,L_1} \otimes B^{k_2,L_2}$.
2. $\widetilde{E}$ *is a coenergy function on* $B^{k_1,L_1} \otimes B^{k_2,L_2}$.

Proof (sketch). For part (1), see [KKM+92, §4] and [Shi02, §3.6]. For (2), define $\widetilde{F}(T \otimes U)$ to be the number of boxes in $T * U$ that are not in the first $\max(L_1, L_2)$ columns. By [Shi02, Prop. 4.5 and (2.4)], $\widetilde{F}$ is an energy function. Using the properties of jeu-de-taquin and Schensted insertion, it is straightforward to show that

$$\widetilde{E}(T \otimes U) + \widetilde{F}(T \otimes U) = \min(k_1, k_2) \min(L_1, L_2),$$

so $\widetilde{E}$ is a coenergy function. □

2.2.4 Piecewise-linear translation

We now translate many of the combinatorial maps on tableaux from the previous section into piecewise-linear maps on arrays of integers subject to certain inequalities, or in other words, integer points of polyhedra.

Gelfand–Tsetlin patterns

A *Gelfand–Tsetlin pattern* (GT pattern) is a triangular array of nonnegative integers $(A_{ij})_{1 \le i \le j \le n}$ satisfying the inequalities

$$A_{i,j+1} \ge A_{ij} \ge A_{i+1,j+1} \tag{2.4}$$

for $1 \le i \le j \le n-1$. Gelfand–Tsetlin patterns can be represented pictorially as triangular arrays, where the j^{th} row in the triangle lists the numbers A_{ij} for $i \le j$. For example, if $n = 3$, then a Gelfand–Tsetlin pattern looks like:

$$\begin{array}{ccccc} & & A_{11} & & \\ & A_{12} & & A_{22} & \\ A_{13} & & A_{23} & & A_{33} \end{array}$$

There is a natural bijection between Gelfand–Tsetlin patterns and SSYTs with entries in $[n]$. Given a Gelfand–Tsetlin pattern (A_{ij}), the associated tableau T is described as follows: the number of j's in the i^{th} row of T is $A_{ij} - A_{i,j-1}$ (we use the convention that $A_{i,i-1} = 0$). Equivalently, the j^{th} row of the pattern is the shape of

$T_{\leq j}$, the part of T obtained by removing numbers larger than j. In particular, the last row of the pattern is the shape of T. Here is an example of a Gelfand–Tsetlin pattern and the corresponding SSYT:

$$
\begin{array}{ccccccccc}
 & & & & 2 & & & & \\
 & & & 4 & & 2 & & & \\
 & & 6 & & 3 & & 1 & & \\
 & 6 & & 6 & & 1 & & 0 & \\
6 & & 6 & & 6 & & 0 & & 0
\end{array}
\quad \longleftrightarrow \quad
\begin{array}{|c|c|c|c|c|c|}
\hline
1 & 1 & 2 & 2 & 3 & 3 \\ \hline
2 & 2 & 3 & 4 & 4 & 4 \\ \hline
3 & 5 & 5 & 5 & 5 & 5 \\ \hline
\end{array}\,. \tag{2.5}
$$

Many maps on tableaux can be described by piecewise-linear formulas in the entries of the corresponding Gelfand–Tsetlin pattern. In general, we will use the same notation for a combinatorial map and its piecewise-linear translation, and we'll rely on context to determine which is meant. Here is a simple example.

Example 2.19. We describe how the maps $\widetilde{\varepsilon}_1, \widetilde{\varphi}_1, \widetilde{e}_1$, and $\widetilde{f}_1$ act on Gelfand–Tsetlin patterns. Let (A_{ij}) be a Gelfand–Tsetlin pattern with corresponding tableau T. When we apply the algorithm of Definition 2.3, every 2 in the second row of T pairs with a 1 in the first row, so the subword of unpaired 1's and 2's is

$$w' = 1^{A_{11}-A_{22}} \quad 2^{A_{12}-A_{11}}.$$

Thus, $\widetilde{\varepsilon}_1(A_{ij}) = A_{12} - A_{11}$, and when $\widetilde{\varepsilon}_1(A_{ij}) > 0$, $\widetilde{e}_1(A_{ij})$ is obtained by increasing A_{11} by 1, and leaving the other entries unchanged. Similarly, $\widetilde{\varphi}_1(A_{ij}) = A_{11} - A_{22}$, and $\widetilde{f}_1(A_{ij})$ is obtained by decreasing A_{11} by 1 (if the result is still a GT pattern).

There is a simple piecewise-linear formula for the Bender–Knuth involutions.

Lemma 2.20 (Kirillov–Berenstein [KB96])**.** *Let (A_{ij}) be a Gelfand–Tsetlin pattern. For $r \in [n-1]$, we have $\widetilde{\sigma}_r(A_{ij}) = (A'_{ij})$, where*

$$A'_{ij} = \begin{cases} \min(A_{i-1,r-1}, A_{i,r+1}) + \max(A_{i,r-1}, A_{i+1,r+1}) - A_{ir} & \text{if } j = r \\ A_{ij} & \text{if } j \neq r \end{cases} \tag{2.6}$$

and we use the convention that $A_{0,j} = \infty$ and $A_{i,i-1} = 0$.

Note that $\widetilde{\sigma}_r$ changes only the r^{th} row of the Gelfand–Tsetlin pattern, and for each i, $\widetilde{\sigma}_r(A_{ir})$ depends only on A_{ir} and the four entries diagonally adjacent to A_{ir} in the Gelfand–Tsetlin pattern (some of which may be "missing" if A_{ir} is on the upper boundary of the triangle):

$$
\begin{array}{ccc}
A_{i-1,r-1} & & A_{i,r-1} \\
 & A_{ir} & \\
A_{i,r+1} & & A_{i+1,r+1}
\end{array}\,.
$$

Since promotion is a composition of Bender–Knuth involutions ((2.1)), Lemma 2.20 gives a recursive piecewise-linear description of promotion.

k-rectangles

Gelfand–Tsetlin patterns can be thought of as coordinates for SSYTs of arbitrary shape. Here we consider the restriction of these coordinates to the subset of rectangular tableaux.

For $k \in [n-1]$, set

$$R_k = \{(i,j) \mid 1 \leq i \leq k,\ i \leq j \leq i+n-k-1\}, \tag{2.7}$$

and define $\widetilde{\mathbb{T}}_k = \mathbb{Z}^{R_k} \times \mathbb{Z} \cong \mathbb{Z}^{k(n-k)+1}$. We will denote a point of $\widetilde{\mathbb{T}}_k$ by $b = (B_{ij}, L)$, where (i,j) runs over R_k.

Given $(B_{ij}, L) \in \widetilde{\mathbb{T}}_k$, define a triangular array $(A_{ij})_{1 \leq i \leq j \leq n}$ by

$$A_{ij} = \begin{cases} B_{ij} & \text{if } (i,j) \in R_k \\ L & \text{if } j > i+n-k-1 \\ 0 & \text{if } j < i. \end{cases}$$

Definition 2.21. Define B^k to be the set of $(B_{ij}, L) \in \widetilde{\mathbb{T}}_k$ such that (A_{ij}) is a Gelfand–Tsetlin pattern. We call an element of B^k a *k-rectangle*, and we say that (A_{ij}) is the *associated Gelfand–Tsetlin pattern.*

For example, if $n = 5$ and $k = 3$, then we may pictorially represent a 3-rectangle (B_{ij}, L) and its associated GT pattern as follows:

$$\left(\begin{array}{ccccc} & B_{11} & & & \\ B_{12} & & B_{22} & & \\ & B_{23} & & B_{33} & , \quad L \\ & & B_{34} & & \end{array} \right) \quad \longleftrightarrow \quad \begin{array}{ccccccccc} & & & & B_{11} & & & & \\ & & & B_{12} & & B_{22} & & & \\ & & L & & B_{23} & & B_{33} & & \\ & L & & L & & B_{34} & & 0 & \\ L & & L & & L & & 0 & & 0 \end{array} . \tag{2.8}$$

As (2.5) and (2.8) illustrate (and the reader may easily verify), the bijection between GT patterns and SSYTs restricts to a bijection between k-rectangles and rectangular tableaux with k rows, with the coordinate L giving the number of columns in the tableau. Thus, we identify

$$B^k = \bigsqcup_{L=0}^{\infty} B^{k,L}.$$

Sometimes it will be more convenient to work with the following alternative set of coordinates on $B^{k,L}$. For $1 \leq i \leq k$ and $i \leq j \leq i+n-k$, define

$$b_{ij} = B_{ij} - B_{i,j-1}, \tag{2.9}$$

where we use the convention that $B_{i,i-1} = 0$ and $B_{i,i+n-k} = L$ for all i. Thus, b_{ij} is the number of j's in the i^{th} row of the k-row rectangular tableau corresponding to $b = (B_{ij}, L)$.

Symmetries of k-rectangles

Throughout this section, fix $k \in [n-1]$ and $L \geq 0$.

Definition 2.22. Define *rotation* $\widetilde{\mathrm{rot}} : B^k \to B^k$ by $\widetilde{\mathrm{rot}}(B_{ij}, L) = (B'_{ij}, L)$, where

$$B'_{ij} = L - B_{k-i+1,n-j}.$$

Define *reflection* $\widetilde{\mathrm{refl}} : B^k \to B^{n-k}$ by $\widetilde{\mathrm{rot}}(B_{ij}, L) = (B''_{ij}, L)$, where

$$B''_{ij} = L - B_{j-i+1,j}.$$

The first map rotates the rectangular Gelfand–Tsetlin pattern 180 degrees, and then replaces each entry a with $L - a$; the second map reflects the rectangular Gelfand–Tsetlin pattern over a vertical axis, and then replaces each entry a with $L - a$.

The operations $\widetilde{\mathrm{rot}}$ and $\widetilde{\mathrm{refl}}$ have simple effects on rectangular tableaux.

Lemma 2.23. *Suppose $b = (B_{ij}, L) \in B^k$ and let T, U, V be the rectangular tableaux corresponding to $b, \widetilde{\mathrm{rot}}(b), \widetilde{\mathrm{refl}}(b)$, respectively. Then*

1. *U is obtained by rotating T 180 degrees and replacing each entry i with $n-i+1$.*
2. *V is obtained by replacing each column of T with the complement in $[n]$ of the entries in that column (arranged in increasing order), and then reversing the order of the columns.*

Proof. First we prove (1). Set $(B'_{ij}, L) = \widetilde{\mathrm{rot}}(B_{ij}, L)$, and let U' be the SSYT obtained by rotating T 180 degrees and replacing each entry i with $n - i + 1$. Let W_{ij} be the number of j's in the i^{th} row of the tableau W. Clearly $U'_{ij} = T_{k-i+1,n-j+1}$. For $i \in [k]$ and $j \in [i, i+n-k]$, we have (using the convention of (2.9))

$$U_{ij} = B'_{ij} - B'_{i,j-1} = B_{k-i+1,n-j+1} - B_{k-i+1,n-j} = T_{k-i+1,n-j+1} = U'_{ij},$$

so $U = U'$, as claimed.

To prove (2), first consider the case $L = 1$. In this case, the tableau corresponding to b is a single column of length k, or in other words, a subset $S = \{s_1 < \cdots < s_k\} \subset [n]$. We must show that if b corresponds to S, then $\widetilde{\mathrm{refl}}(b)$ corresponds to $[n] \setminus S$.

Identify the k-rectangle $b = (B_{ij}, 1)$ with a partition λ inside the $k \times (n-k)$ rectangle by setting $\lambda_i = |\{j \mid B_{ij} = 1\}|$ for $i = 1, \ldots, k$. The entries $s_1 < \cdots < s_k$ of the corresponding tableau are related to λ by

$$\lambda_i = i + n - k - s_i.$$

$$b = \begin{array}{cccccc} & & 1 & & & \\ & 1 & & 0 & & \\ 1 & & 1 & & 0 & \\ & 1 & & 1 & & 0 \\ & & 1 & & 0 & \\ & & & 1 & & \end{array} \qquad \qquad \widetilde{\mathrm{refl}}(b) = \begin{array}{cccccc} & & & 0 & & \\ & & 1 & & 0 & \\ & 1 & & 0 & & 0 \\ 1 & & 0 & & 0 & \\ & 1 & & 0 & & \\ & & 0 & & & \end{array}$$

Figure 2: An example of $\widetilde{\mathrm{refl}}$ in the $L = 1$ case (with $n = 7, k = 4$). Here b corresponds to the partition $(3, 2, 2, 1)$ and the subset $\{1, 3, 4, 6\}$; $\widetilde{\mathrm{refl}}(b)$ corresponds to $(3, 1, 0)$ and $\{2, 5, 7\}$.

Equivalently, s_i is the position of the i^{th} vertical step in p_λ, the lattice path from the top-right corner of the $k \times (n - k)$ rectangle to the bottom-left corner which traces out the lower boundary of the Young diagram of λ.

Now identify the $(n - k)$-rectangle $\widetilde{\mathrm{refl}}(b)$ with a partition $\widetilde{\lambda}$ inside the $(n - k) \times k$ rectangle in the same manner. From the definition of $\widetilde{\mathrm{refl}}$, one sees that the positions of the vertical steps in p_λ are precisely the positions of the horizontal steps in $p_{\widetilde{\lambda}}$, so $\widetilde{\lambda}$ corresponds to the $(n-k)$-subset $[n] \setminus S$, as claimed. (See Figure 2 for an example.)

Now suppose $L > 1$. The rectangle b is equal to the entry-wise sum of the rectangles corresponding to the individual columns of T, and the same is true of $\widetilde{\mathrm{refl}}(b)$ and its corresponding tableau V. Let V' be the array obtained by replacing each column of T by its complement in $[n]$, and reversing the order of the columns. Using the $L = 1$ case, we see that $\widetilde{\mathrm{refl}}(b)$ is also equal to the entry-wise sum of the rectangles corresponding to the individual columns of V'. To conclude that $V = V'$, it remains to show that V' is semistandard, i.e., that its rows are weakly increasing.

Let S and S' be the subsets of entries in two consecutive columns of V'. The condition for V' to be semistandard is that $s_i \leq s'_i$ for $i = 1, \ldots, n - k$. If this condition holds, write $S \preccurlyeq S'$. Let λ, λ' be the partitions associated to S, S', respectively. From the proof of the $L = 1$ case, one sees that

$$S \preccurlyeq S' \iff \lambda \supseteq \lambda' \iff \widetilde{\lambda}' \supseteq \widetilde{\lambda} \iff [n] \setminus S' \preccurlyeq [n] \setminus S,$$

where $\subseteq$ denotes inclusion of Young diagrams. Thus, V' is semistandard because T is semistandard. □

In §4.3, we will use geometric crystals to prove the following compatibility of $\widetilde{\mathrm{rot}}$ and $\widetilde{\mathrm{refl}}$ with the affine crystal structure on rectangular tableaux (see Remark 4.27).

Proposition 2.24. *For $i \in \mathbb{Z}/n\mathbb{Z}$, we have the identities*

$$\widetilde{e}_i \circ \widetilde{\mathrm{rot}} = \widetilde{\mathrm{rot}} \circ \widetilde{f}_{n-i} \qquad \text{and} \qquad \widetilde{e}_i \circ \widetilde{\mathrm{refl}} = \widetilde{\mathrm{refl}} \circ \widetilde{f}_i.$$

Remark 2.25. Let $\widetilde{S}$ denote the *Schützenberger involution* (also known as *evacuation*) on semistandard tableaux. It is well-known that the restriction of $\widetilde{S}$ to rectangular tableaux is equal to $\widetilde{\mathrm{rot}}$ (see, e.g, [Ful97]). In general, one has $\widetilde{e}_i \circ \widetilde{S} = \widetilde{S} \circ \widetilde{f}_{n-i}$ for $i \in [n-1]$ (see, e.g., [LLT95, §3]).

The "column complementation" map has been studied by Stembridge in his work on rational tableaux [Ste87], but the compatibility of this map with the classical crystal operators does not seem to have been investigated. We conjecture that this compatibility holds for all shapes; it would be interesting to find a combinatorial proof, even in the rectangular case.

2.3 Geometric crystals

A geometric crystal is an analogue of a Kashiwara crystal, where the underlying set is replaced by an algebraic variety, and the maps associated to the crystal are replaced by rational maps on the algebraic variety. We present the definition in type $A_{n-1}^{(1)}$. For more general definitions, see [BK00, BK07a, Nak05].

Let $T = (\mathbb{C}^\times)^n$ be an n-dimensional complex torus. For $i \in \mathbb{Z}/n\mathbb{Z}$, let $\alpha_i : T \to \mathbb{C}^\times$ be the *character* sending $(z_1, \ldots, z_n)$ to z_i/z_{i+1} (indices interpreted mod n), and let $\alpha_i^\vee : \mathbb{C}^\times \to T$ be the *cocharacter* sending z to $(1, \ldots, z, z^{-1}, \ldots, 1)$, where z is in the i^{th} component and z^{-1} is in the $(i+1)^{th}$ component (mod n).

Definition 2.26. A *geometric pre-crystal of type* $A_{n-1}^{(1)}$ consists of an irreducible complex algebraic (ind-)variety X, together with

- a rational map $\gamma : X \to T$;
- for each $i \in \mathbb{Z}/n\mathbb{Z}$, rational functions $\varepsilon_i, \varphi_i : X \to \mathbb{C}^\times$ which are not identically zero,[5] and a rational unital[6] action $e_i : \mathbb{C}^\times \times X \to X$.

We call e_i a *geometric crystal operator*, and we usually denote its action by $e_i^c(x)$ instead of $e_i(c, x)$. These rational maps must satisfy the following identities (whenever both sides are defined):

1. For $x \in X$ and $c \in \mathbb{C}^\times$,

$$\gamma(e_i^c(x)) = \alpha_i^\vee(c)\gamma(x), \quad \varepsilon_i(e_i^c(x)) = c\varepsilon_i(x), \quad \varphi_i(e_i^c(x)) = c^{-1}\varphi_i(x). \tag{2.10}$$

2. For $x \in X$,

$$\frac{\varepsilon_i(x)}{\varphi_i(x)} = \alpha_i(\gamma(x)). \tag{2.11}$$

[5] In [BK07a], some of the ε_i and φ_i are allowed to be zero, but we will not need this more general setting.
[6] This means that $e_i(1, x)$ is defined (and thus equal to x) for all $x \in X$.

Remark 2.27. The identities (2.10) and (2.11) are analogous to the first and second crystal axioms in Definition 2.1. There are, however, several important subtleties to this analogy:

1. The true analogues of $\widetilde{\varepsilon}_i$ and $\widetilde{\varphi}_i$ are the maps $1/\varepsilon_i$ and $1/\varphi_i$, and the analogue of $\widetilde{f}_i$ is the rational action $e_i^{c^{-1}}$. This is made precise in Theorem 4.23.

2. The precise connection between the geometric crystal maps and the combinatorial crystal maps comes from tropicalization, which treats rational maps as formal algebraic expressions rather than actual functions. Thus, the partially-defined nature of rational maps is <u>not</u> analogous to the partially-defined nature of the crystal operators. The geometric analogue of the partially-defined nature of the crystal operators (and thus of the third axiom in Definition 2.1) is the notion of decoration introduced in Definition 2.30. This is made precise in Proposition 4.21 and Theorem 4.23.

3. The role of α_i and $\alpha_i^\vee$ in the geometric crystal axioms is opposite that of $\widetilde{\alpha}_i$ and $\widetilde{\alpha}_i^\vee$ in the crystal axioms. For this reason, geometric crystals corresponding to a given root system are analogues of combinatorial crystals for the Langlands dual root system, which is obtained by interchanging roots and coroots. For example, simply-laced root systems are self-dual, while the root system of type B_n is Langlands dual to that of type C_n. Since the root system $A_{n-1}^{(1)}$ is simply-laced, we may ignore the Langlands duality in this thesis.

A geometric pre-crystal is the analogue of a Kashiwara crystal. To upgrade a geometric pre-crystal to a geometric crystal, one requires an additional axiom, which can be thought of as an analogue of Stembridge's additional crystal axioms (see Remark 2.2). We remark, however, that the geometric crystal axiom is weaker than the Stembridge axioms, in the sense that a geometric crystal does not necessarily tropicalize to a crystal satisfying the Stembridge axioms.

Definition 2.28. A *geometric crystal of type* $A_{n-1}^{(1)}$ is a geometric pre-crystal of type $A_{n-1}^{(1)}$ which satisfies the following *geometric Serre relations*:

If $n \geq 3$, then for each pair i, j of distinct elements of $\{0, \ldots, n-1\}$, and $c_1, c_2 \in \mathbb{C}^\times$, the actions e_i, e_j satisfy

$$\begin{aligned} &e_i^{c_1} e_j^{c_2} = e_j^{c_2} e_i^{c_1} && \text{if } |i-j| > 1 \\ &e_i^{c_1} e_j^{c_1 c_2} e_i^{c_2} = e_j^{c_2} e_i^{c_1 c_2} e_j^{c_1} && \text{if } |i-j| = 1. \end{aligned} \tag{2.12}$$

If $n = 2$, there is no Serre relation for e_0 and e_1, so a geometric pre-crystal of type $A_1^{(1)}$ is automatically a geometric crystal.

Remark 2.29. One of Berenstein and Kazhdan's motivations for introducing geometric crystals was to obtain rational actions of Weyl groups [BK07b]. The geometric Serre relations imply that the rational maps $s_i : X \to X$ defined by

$$s_i(x) = e_i^{\frac{1}{\alpha_i(\gamma(x))}}(x)$$

generate a rational action of the type $A_{n-1}^{(1)}$ Weyl group, which is the affine symmetric group $\widetilde{S}_n$ (see Prop. 2.3 and the subsequent remark in [BK00]).

Definition 2.30. A *decorated geometric (pre-)crystal of type* $A_{n-1}^{(1)}$ is a geometric (pre-)crystal X equipped with a rational function $f : X \to \mathbb{C}$ such that

$$f(e_i^c(x)) = f(x) + \frac{c-1}{\varphi_i(x)} + \frac{c^{-1}-1}{\varepsilon_i(x)} \tag{2.13}$$

for all $x \in X$ and $i \in \mathbb{Z}/n\mathbb{Z}$. The function f is called a *decoration*.

Definition 2.31. A *morphism* of geometric (pre-)crystals X and Y is a rational map $h : X \to Y$ such that $e_i h = h e_i$, and $\rho h = \rho$ for $\rho = \gamma, \varepsilon_i, \varphi_i$.

Product of geometric crystals

Berenstein and Kazhdan defined a product of decorated geometric pre-crystals [BK00, BK07a].

Definition/Proposition 2.32. Suppose X and Y are decorated geometric pre-crystals (of type $A_{n-1}^{(1)}$). Define the following rational maps on $(x, y) \in X \times Y$:

$$\gamma(x, y) = \gamma(x)\gamma(y)$$

$$\varepsilon_i(x, y) = \frac{\varepsilon_i(y)(\varepsilon_i(x) + \varphi_i(y))}{\varphi_i(y)} \qquad \varphi_i(x, y) = \frac{\varphi_i(x)(\varepsilon_i(x) + \varphi_i(y))}{\varepsilon_i(x)}$$

$$e_i^c(x, y) = (e_i^{c_1}(x), e_i^{c_2}(y)) \quad \text{where} \quad c_1 = \frac{c\varepsilon_i(x) + \varphi_i(y)}{\varepsilon_i(x) + \varphi_i(y)}, \quad c_2 = \frac{\varepsilon_i(x) + \varphi_i(y)}{\varepsilon_i(x) + c^{-1}\varphi_i(y)}$$

$$f(x, y) = f(x) + f(y).$$

These maps make $X \times Y$ into a decorated geometric pre-crystal, which we call the *product* of X and Y. This product is associative.

Proof. The proof of [BK07a, Lemma 2.34] shows that the decoration on $X \times Y$ satisfies (2.13). The remainder of this Proposition is stated as [BK07a, Claim 2.16], and the proof is left to the reader. The various assertions are indeed straightforward (if tedious) to verify from the definitions. Here we show that (2.10) and (2.11) hold for $X \times Y$.

First, using (2.11) for the geometric pre-crystals X and Y and the fact that α_i is multiplicative, we have

$$\frac{\varepsilon_i(x,y)}{\varphi_i(x,y)} = \frac{\varepsilon_i(y)}{\varphi_i(y)}\frac{\varepsilon_i(x)}{\varphi_i(x)} = \alpha_i(\gamma(x)\gamma(y)) = \alpha_i(\gamma(x,y)),$$

so (2.11) holds for $X \times Y$. Now let $(x', y') = e_i^c(x,y) = (e_i^{c_1}(x), e_i^{c_2}(y))$. Using (2.10) for X and Y and the identity $c_1c_2 = c$, we have

$$\gamma(x',y') = \gamma(x')\gamma(y') = \alpha_i^\vee(c_1)\gamma(x)\alpha_i^\vee(c_2)\gamma(y) = \alpha_i^\vee(c)\gamma(x,y),$$

so $\gamma(e_i^c(x,y)) = \alpha_i^\vee(c)\gamma(x,y)$. Likewise, we compute

$$\begin{aligned} \varepsilon_i(x',y') &= \frac{\varepsilon_i(y')(\varepsilon_i(x') + \varphi_i(y'))}{\varphi_i(y')} = \frac{c_2\varepsilon_i(y)(c_1\varepsilon_i(x) + c_2^{-1}\varphi_i(y))}{c_2^{-1}\varphi_i(y)} \\ &= c_2\frac{\varepsilon_i(y)(c\varepsilon_i(x) + \varphi_i(y))}{\varphi_i(y)} = c\frac{\varepsilon_i(y)(\varepsilon_i(x) + \varphi_i(y))}{\varphi_i(y)}, \end{aligned}$$

so $\varepsilon_i(e_i^c(x,y)) = c\varepsilon_i(x,y)$. Finally, using the preceding identities and the fact that $\alpha_i(\alpha_i^\vee(c)) = c^2$, we have

$$\varphi_i(x',y') = \frac{\varepsilon_i(x',y')}{\alpha_i(\gamma(x',y'))} = \frac{c\varepsilon_i(x,y)}{c^2\alpha_i(\gamma(x,y))} = c^{-1}\varphi_i(x,y).$$

Similar computations show that $e_i^c(x,y)$ is an action of $\mathbb{C}^\times$ (clearly it is unital), and that the product is associative. □

If X and Y are geometric crystals, their product is not necessarily a geometric crystal ([BK07a, Remark 2.21]). To get around this problem, Berenstein and Kazhdan introduced unipotent crystals, and showed that if X and Y are induced from unipotent crystals, then their product is a geometric crystal.

2.4 Unipotent crystals

The definition of geometric pre-crystal given in the previous section makes sense for any reductive group G; one simply replaces the torus $T = (\mathbb{C}^\times)^n$ by a maximal torus in G, and $\alpha_i, \alpha_i^\vee$ with the corresponding simple characters and cocharacters. Given a geometric pre-crystal, it is in general quite difficult to verify the geometric Serre relations, and, as mentioned above, the fact that the Serre relations hold for X and Y does not guarantee that they hold for $X \times Y$. Berenstein and Kazhdan invented unipotent crystals to get around these difficulties [BK00]. The intuitive idea behind a unipotent crystal is that a geometric pre-crystal which "comes from" G itself will automatically satisfy the Serre relations, and will automatically behave nicely under products.

Nakashima extended the notions of geometric and unipotent crystals to affine (and even Kac–Moody) groups [Nak05]. The (minimal) Kac–Moody group which corresponds to the affine Lie algebra $\widehat{\mathfrak{sl}}_n$ is closely related to $\mathrm{SL}_n(\mathbb{C}[\lambda, \lambda^{-1}])$, the group of $n \times n$ matrices of determinant 1 with entries in the Laurent polynomial ring $\mathbb{C}[\lambda, \lambda^{-1}]$. For our purposes, however, we have found it necessary to allow determinants other than 1, so we work with the bigger group $\mathrm{GL}_n(\mathbb{C}(\lambda))$, which consists of $n \times n$ matrices with entries in the field of rational functions in the indeterminate λ, and nonzero determinant. We call $\mathrm{GL}_n(\mathbb{C}(\lambda))$ the *loop group*,[7] and λ the *loop parameter*.

Before giving the definition of unipotent crystals, we pause to discuss a correspondence between $n \times n$ matrices with entries in $\mathbb{C}((\lambda))$ and "infinite periodic" matrices with entries in $\mathbb{C}$. This construction generalizes the correspondence between formal Laurent series and Toeplitz matrices (which is the $n = 1$ case), and plays an important role in [LP12].

Unfolding

Let $\mathbb{C}((\lambda))$ be the field of formal Laurent series in the indeterminate λ, that is, expressions of the form

$$\sum_{m=m_0}^{\infty} a_m \lambda^m$$

where m_0 is an integer, and each a_m is in $\mathbb{C}$. Let $M_n[\mathbb{C}((\lambda))]$ denote the ring of $n \times n$ matrices with entries in this field.

An *n-periodic matrix* (over $\mathbb{C}$) is a $\mathbb{Z} \times \mathbb{Z}$ array of complex numbers $(X_{ij})_{(i,j) \in \mathbb{Z}}$ such that $X_{ij} = 0$ if $j - i$ is sufficiently large, and $X_{ij} = X_{i+n,j+n}$ for all i, j. Say that the entries X_{ij} with $i - j = k$ lie on the k^{th} *diagonal* of X, or that k indexes this diagonal. Thus, the main diagonal of X is indexed by 0, and higher numbers index lower diagonals. We add these matrices entry-wise, and multiply them using the usual matrix product: if $X = (X_{ij})$ and $Y = (Y_{ij})$, then

$$(XY)_{ij} = \sum_{k \in \mathbb{Z}} X_{ik} Y_{kj}.$$

The hypothesis that $X_{ij} = 0$ for $j - i$ sufficiently large ensures that each of these sums is finite, and it is clear that the product of two n-periodic matrices is n-periodic. Denote the ring of n-periodic matrices by $M_n^{\infty}(\mathbb{C})$.

Given a matrix $A = (A_{ij}) \in M_n[\mathbb{C}((\lambda))]$, where $A_{ij} = \sum a_m^{i,j} \lambda^m$, define an n-periodic matrix $X = (X_{ij})$ by[8]

$$X_{rn+i,sn+j} = a_{r-s}^{i,j}$$

[7] The term "loop group" does not have a fixed meaning in the literature.

[8] The definition in [LP12] uses $s - r$ instead of $r - s$. This is equivalent to interchanging λ and λ^{-1}.

for $r, s \in \mathbb{Z}$ and $i, j \in [n]$. For example, if $n = 2$ and

$$A = \begin{pmatrix} 2\lambda^{-1} + 3 + 4\lambda + 5\lambda^2 & \lambda^{-1} + 7 + 8\lambda \\ -3\lambda^{-1} + 1 + \lambda^2 & -2\lambda^{-1} + 5 + 6\lambda \end{pmatrix}$$

then

$$X = \left(\begin{array}{c|cc|cc|cc|c} \ddots & & & & & & & \iddots \\ \hline & 3 & 7 & 2 & 1 & 0 & 0 & \\ & 1 & 5 & -3 & -2 & 0 & 0 & \\ \hline & 4 & 8 & 3 & 7 & 2 & 1 & \\ & 0 & 6 & 1 & 5 & -3 & -2 & \\ \hline & 5 & 0 & 4 & 8 & 3 & 7 & \\ & 1 & 0 & 0 & 6 & 1 & 5 & \\ \hline \iddots & & & & & & & \ddots \end{array}\right)$$

where the row (resp., column) indexed by 1 is the upper-most row (resp., left-most column) whose entries are shown. The vertical and horizontal lines partition the matrix into 2×2 blocks whose entries are the m^{th} coefficients of the entries of A, for some m.

It is straightforward to check that the map $A \mapsto X$ is an isomorphism of rings. We will refer to the $n \times n$ matrix A as a *folded matrix*, and the n-periodic matrix X as an *unfolded matrix*. We call X the *unfolding* of A, and A the *folding* of X. When it is important to distinguish between folded and unfolded matrices, we will try to use letters near the beginning of the alphabet for folded matrices, and letters near the end of the alphabet for unfolded matrices.

Definition of unipotent crystals

Every rational function in λ has a Laurent series expansion, so $\mathrm{GL}_n(\mathbb{C}(\lambda))$ is a subset of $M_n[\mathbb{C}((\lambda))]$, and we may talk about the unfoldings of its elements.

In what follows, we will work with the submonoid $G \subset \mathrm{GL}_n(\mathbb{C}(\lambda))$ consisting of matrices whose entries are Laurent polynomials in λ, and whose determinant is a nonzero Laurent polynomial in λ. The purpose of restricting to this monoid is that it is an ind-variety, so we may talk about rational maps to and from this space. (For our purposes, an ind-variety is simply an infinite-dimensional object that admits rational maps. We refer the reader to [Kum02] for more information about ind-varieties.)

Let $B^- \subset G$ be the submonoid of matrices whose unfolding is lower triangular with nonzero entries on the main diagonal. In terms of folded matrices, this means that all entries are (ordinary) polynomials, with the entries on the diagonal having nonzero constant term, and the entries above the diagonal having no constant term.

B^- is naturally an ind-variety, where the m^{th} piece consists of unfolded matrices which are supported on diagonals $0, \ldots, m$.

For $a \in \mathbb{C}$, define the folded matrices

$$\widehat{x}_i(a) = Id + aE_{i,i+1} \quad \text{for } i \in [n-1], \qquad \text{and} \quad \widehat{x}_0(a) = Id + a\lambda^{-1}E_{n1},$$

where Id is the $n \times n$ identity matrix, and E_{ij} is an $n \times n$ matrix unit. For $i \in \mathbb{Z}$, set $\widehat{x}_i(a) = \widehat{x}_{\bar{i}}(a)$, where $\bar{i}$ is the residue of i mod n (in $\{0, \ldots, n-1\}$). Let $U \subset G$ be the subgroup generated by the elements $\widehat{x}_i(a)$. Note that the unfolding of each element of U is upper uni-triangular.

The usual definition of unipotent crystals ([BK00, Nak05]) is based on rational actions of U. We work here with a slightly weaker notion.

Definition 2.33. Let V be an irreducible complex algebraic (ind-)variety, and let $\alpha : U \times V \to V$ be a partially-defined map. Let $u.v := \alpha(u, v)$. We will say that α is a *pseudo-rational U-action* if it satisfies the following properties:

1. $1.v = v$ for all $v \in V$;
2. If $u.v$ and $u'.(u.v)$ are defined, then $(u'u).v = u'.(u.v)$;
3. For each $i \in \mathbb{Z}/n\mathbb{Z}$, the partially defined map from $\mathbb{C} \times V \to V$ given by $(a, v) \mapsto \widehat{x}_i(a).v$ is rational.

Remark 2.34. We suspect that it is possible to give U an ind-variety structure so that a pseudo-rational U-action is actually a rational U-action. The difficulty is that U is <u>not</u> the full set of matrices whose unfolding is upper uni-triangular, and whose folding has determinant 1 (it is not possible to generate all the one-parameter subgroups corresponding to positive real roots using only the $\widehat{x}_i(a)$). Fortunately, pseudo-rational U-actions suffice for our purposes.

Definition 2.35. Define $\alpha_{B^-} : U \times B^- \to B^-$ by $u.b = b'$ if $ub = b'u'$, with $b' \in B^-, u' \in U$. If ub does not have such a factorization, then $u.b$ is undefined.

Note that if $b_1u_1 = b_2u_2$, then $b_2^{-1}b_1 = u_2u_1^{-1}$ is both lower triangular and upper uni-triangular (as an unfolded matrix), so it must be the identity matrix, and thus $b_1 = b_2$ and $u_1 = u_2$. This shows that α_{B^-} is well-defined (as a partial map). Observe that if $X \in B^-$ is an unfolded matrix and $i \in \mathbb{Z}$, then

$$\widehat{x}_i(a) \cdot X \cdot \widehat{x}_i\left(\frac{-aX_{i+1,i+1}}{X_{ii} + aX_{i+1,i}}\right) \in B^-,$$

so we have

$$(2.14) \quad \widehat{x}_i(a).X = \widehat{x}_i(a) \cdot X \cdot \widehat{x}_i(\tau_i(a,X)) \qquad \text{where} \qquad \tau_i(a,X) = \frac{-aX_{i+1,i+1}}{X_{ii} + aX_{i+1,i}}.$$

This shows that α_{B^-} satisfies property (3) of Definition 2.33. It's clear that the first two properties are satisfied as well, so α_{B^-} is a pseudo-rational U-action.

Definition 2.36. A *U-variety* is an irreducible complex algebraic (ind-)variety X together with a pseudo-rational U-action $\alpha : U \times X \to X$. A *morphism of U-varieties* is a rational map which commutes with the U-actions (when they are defined).

For example, the ind-variety B^- with the pseudo-rational U-action α_{B^-} is a U-variety.

Definition 2.37. A *unipotent crystal (of type $A_{n-1}^{(1)}$)* is a pair (V, g), where V is a U-variety, and $g : V \to B^-$ is a morphism of U-varieties, such that for each $i \in [n]$ (equivalently, each $i \in \mathbb{Z}$), the rational function $v \mapsto g(v)_{i+1,i}$ is not identically zero (here $g(v)$ is viewed as an unfolded matrix).

Note that the pair (B^-, Id) is a unipotent crystal.

The following result, which is essentially due to Berenstein and Kazhdan ([BK00, Theorem 3.8]) shows how to obtain a geometric crystal from a unipotent crystal.

Theorem 2.38. *Let (V, g) be a unipotent crystal. Suppose $v \in V$, and let $X = g(v)$ be an unfolded matrix. Define*

$$\gamma(v) = (X_{11}, \ldots, X_{nn}) \qquad \varepsilon_i(x) = \frac{X_{i+1,i}}{X_{i+1,i+1}} \qquad \varphi_i(v) = \frac{X_{i+1,i}}{X_{ii}}$$

$$e_i^c(v) = \widehat{x}_i\left(\frac{c-1}{\varphi_i(v)}\right).v$$

where . is the pseudo-rational action of U on V. These maps define a type $A_{n-1}^{(1)}$ geometric crystal on V.

We say that the geometric crystal on V is *induced* from the unipotent crystal (V, g).

Proof. We first show that these maps define a geometric pre-crystal on V. The rational functions ε_i and φ_i are not identically zero due to the assumption about $g(v)_{i+1,i}$ in Definition 2.37. The identity (2.11) is immediate. Given $v \in V$, set $X = g(v)$, $v' = \widehat{x}_i\left(\frac{c-1}{\varphi_i(v)}\right).v$, and $X' = g(v')$. View X and X' as unfolded matrices. By (2.14) and the assumption that g is a morphism of U-varieties, we have

$$X' = \widehat{x}_i(a).X = \widehat{x}_i(a) \cdot X \cdot \widehat{x}_i(\tau_i(a, X)),$$

where $a = \frac{c-1}{\varphi_i(v)}$. A short computation shows that the principal two-by-two submatrix of X' using rows and columns i and $i+1$ is

$$\begin{pmatrix} cX_{ii} & 0 \\ X_{i+1,i} & c^{-1}X_{i+1,i+1} \end{pmatrix},$$

and the other entries on the main diagonal of X' are equal to those of X. This proves the identities (2.10).

To see that e_i is an action, compute

$$\begin{aligned} e_i^{c_1}(e_i^{c_2}(v)) &= \widehat{x}_i\left(\frac{c_1-1}{\varphi_i(e_i^{c_2}(v))}\right).\widehat{x}_i\left(\frac{c_2-1}{\varphi_i(v)}\right).v \\ &= \widehat{x}_i\left(\frac{c_1-1}{c_2^{-1}\varphi_i(v)}+\frac{c_2-1}{\varphi_i(v)}\right).v = \widehat{x}_i\left(\frac{c_1c_2-1}{\varphi_i(v)}\right).v = e_i^{c_1c_2}(v) \end{aligned}$$

where the second equality uses (2.10).

It remains to prove the geometric Serre relations (2.12). Suppose $i, j \in \{0, \ldots, n-1\}$. If $|i-j| > 1$, then $\widehat{x}_i(a)$ and $\widehat{x}_j(b)$ commute, and it is not hard to check that the values of ε_i, φ_i (resp., ε_j, φ_j) are unchanged by applying e_j^c (respectively, e_i^c), so the Serre relation for i and j holds. The case $|i-j| = 1$ is a somewhat lengthy calculation inside GL_3, which is worked out in [BK00, §5.2, Proof of Theorem 3.8]. □

The unipotent crystal (B^-, Id) induces a geometric crystal on B^-. A short computation using (2.14) shows that for $X \in B^-$,

$$e_i^c(X) = \widehat{x}_i\left(\frac{c-1}{\varphi_i(X)}\right)\cdot X \cdot \widehat{x}_i\left(\frac{c^{-1}-1}{\varepsilon_i(X)}\right), \tag{2.15}$$

where e_i, ε_i, and φ_i are the induced geometric crystal maps on B^-. Note that for any unipotent crystal (V, g), we have by definition the formal identities

$$\gamma = \gamma g, \qquad \varepsilon_i = \varepsilon_i g, \qquad \varphi_i = \varphi_i g, \qquad g e_i = e_i g, \tag{2.16}$$

where the geometric crystal maps on the left-hand side come from the induced geometric crystal on B^-, and those on the right-hand side come from the induced geometric crystal on V.

Product of unipotent crystals

We now define the product of unipotent crystals, following [BK00]. Given $u \in U$ and $b \in B^-$, define $\beta(u, b) = u'$ if $ub = b'u'$, with $b' \in B^-$ and $u' \in U$. If ub does not have such a factorization, then $\beta(u, b)$ is undefined (cf. Definition 2.35).

The following result is essentially the combination of Theorem 3.3 and Lemma 3.9 in [BK00]. Although Berenstein and Kazhdan work with rational actions of the unipotent subgroup of a reductive group and we work with pseudo-rational actions of an infinite-dimensional group, the proof is identical.

Theorem 2.39. *Suppose (V, g) and (W, g) are unipotent crystals. Define $g : V \times W \to B^-$ by $g(v, w) = g(v)g(w)$, and equip $V \times W$ with the pseudo-rational U-action*

$$u.(v, w) = (u.v, \beta(u, g(v)).w).$$

Then $(V \times W, g)$ is a unipotent crystal. Furthermore, the geometric crystal induced from $(V \times W, g)$ is the product of the geometric crystals induced from (V, g) and (W, g).

2.5 The Grassmannian

Here we recall some basic facts and notation concerning Grassmannians. For more details we refer the reader to [Ful97]. As a set, the Grassmannian $\mathrm{Gr}(k, n)$ consists of the k-dimensional subspaces in $\mathbb{C}^n$. We view the Grassmannian as a projective algebraic variety in its Plücker embedding, and for $J \in \binom{[n]}{k}$, we write $P_J(N)$ for the J^{th} Plücker coordinate of the subspace N. Plücker coordinates are projective—that is, they are only defined up to a common nonzero scalar multiple. We represent a point $N \in \mathrm{Gr}(k, n)$ as the column span of a (full-rank) $n \times k$ matrix N', so that $P_J(N)$ is the maximal minor of N' using the rows in J. When there is no danger of confusion, we treat a subspace and its matrix representatives interchangeably. For example, we may speak of the Plücker coordinates of a full-rank $n \times k$ matrix.

There is a natural (left) action of $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})$ on $\mathrm{Gr}(k, n)$ given by matrix multiplication. We denote the action of $A \in \mathrm{GL}_n$ on $N \in \mathrm{Gr}(k, n)$ by $(A, N) \mapsto A \cdot N$; this is the subspace spanned by the columns of $A \cdot N'$, where N' is an $n \times k$ matrix representative of N.

To reduce the number of special cases needed in various arguments, we make the following convention.

Convention 2.40. Let N' be a full-rank $n \times k$ matrix representing a point in $N \in \mathrm{Gr}(k, n)$.

- Unless otherwise indicated (see the last bullet point), we label Plücker coordinates of M by sets, not by ordered lists. That is, if $I \in \binom{[n]}{k}$, then $P_I(N)$ means the determinant of the $k \times k$ submatrix of N' using the rows indexed by the elements of I, taken in the order in which they appear in N'. Thus, $P_{\{1,2\}}(N) = P_{\{2,1\}}(N)$. We will often write $P_{12}(N)$ or $P_{1,2}(N)$ instead of $P_{\{1,2\}}(N)$.
- If $I \subset [n]$ does not contain exactly k elements, then we set $P_I(N) = 0$.
- If I is any set of integers, we set $P_I(N) = P_{I'}(N)$, where I' is the set consisting of the residues of the elements of I modulo n, where we take the residues to lie in $[n]$.
- We use the notation $P_{<i_1,\ldots,i_k>}(N)$ for the determinant of the $k \times k$ matrix whose j^{th} row is row i_j of N'. We will only use this notation when $i_1, \ldots, i_k$ are (not

necessarily distinct) elements of $[n]$. Note that $P_{<1,2>}(N) = -P_{<2,1>}(N) = P_{12}(N)$.

A proof of the following classical result can be found in, e.g., [Ful97].

Proposition 2.41 (Grassmann–Plücker relations)**.** *Let $i_1, \ldots, i_{k+1}$ and $j_1, \ldots, j_{k-1}$ be elements of $[n]$. For $N \in \mathrm{Gr}(k,n)$, we have*

$$\sum_{r=1}^{k+1} (-1)^r P_{<i_1,\ldots,i_{r-1},i_{r+1},\ldots,i_{k+1}>}(N) P_{<i_r,j_1,\ldots,j_{k-1}>}(N) = 0. \tag{2.17}$$

Corollary 2.42 (Three-term Plücker relation)**.** *Fix $k \geq 2$. If $I \in \binom{[n]}{k-2}$ and a, b, c, d are elements of $[n]$ satisfying $a \leq b \leq c \leq d$, then for $N \in \mathrm{Gr}(k,n)$, we have*

$$P_{I\cup\{a,b\}}(N)P_{I\cup\{c,d\}}(N) + P_{I\cup\{a,d\}}(N)P_{I\cup\{b,c\}}(N) = P_{I\cup\{a,c\}}(N)P_{I\cup\{b,d\}}(N). \tag{2.18}$$

Note that the subscripts in (2.17) are ordered lists, whereas the subscripts in (2.18) are sets.

Basic Plücker coordinates

Here we introduce a distinguished class of Plücker coordinates that plays an important role throughout this thesis.

Say that a Plücker coordinate P_J is *cyclic* if the elements of J are consecutive mod n, and let $\mathrm{Gr}^\circ(k,n)$ denote the *open positroid cell*, the open subset of $\mathrm{Gr}(k,n)$ where the cyclic Plücker coordinates do not vanish. We start by introducing a canonical matrix representative for subspaces in $\mathrm{Gr}^\circ(k,n)$.

Say that an $n \times k$ matrix N has *diagonal form* if its first k rows are lower triangular with nonzero entries on the main diagonal, and its last k rows are upper uni-triangular. For example, if $n = 7$ and $k = 3$, then a matrix of diagonal form looks like

$$\begin{pmatrix} a_1 & 0 & 0 \\ * & a_2 & 0 \\ * & * & a_3 \\ * & * & * \\ 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

where a_1, a_2, a_3 are nonzero, and the $*$'s are arbitrary.

Lemma 2.43. *Every subspace in $\mathrm{Gr}^\circ(k,n)$ is the column span of a unique $n \times k$ matrix of diagonal form.*

Proof. Suppose $N \in \mathrm{Gr}^\circ(k,n)$. Since $P_{[n-k+1,n]}(N) \neq 0$, N can be represented by an $n \times k$ matrix N' whose bottom k rows are the identity matrix. Clearly $\Delta_{[1,i],[1,i]}(N') = \dfrac{P_{[1,i]\cup[n-k+i+1,n]}(N)}{P_{[n-k+1,n]}(N)}$ for $i \leq k$, so the principal minors $\Delta_{[1,i],[1,i]}(N')$ are nonzero for $i \leq k$. We may therefore use Gaussian elimination on the columns of N' to make the first k rows lower triangular with nonzero entries on the main diagonal. The last k rows will still be upper uni-triangular, so we obtain a diagonal form representative of the subspace N.

If N' and N'' are diagonal form representatives of N, and $A \in \mathrm{GL}_k$ is the change of basis matrix, then A must be lower triangular to preserve the form of the first k rows, and upper uni-triangular to preserve the form of the last k rows. This proves uniqueness. □

Definition 2.44. A subset $J \subset [n]$ is a *basic subset* if it consists of a single interval of $[n]$, or it consists of two disjoint intervals, one of which contains n. A subset $J \subset [n]$ is a *reflected basic subset* if it is of the form $w_0(J)$, where J is basic (and w_0 replaces each $i \in J$ with $n-i+1$). Every basic k-subset is of the form

$$J_{i,j} = [i,j] \cup [n-k+j-i+2, n]$$

for some $i \in [n-k+1]$ and $j \in [i-1, i+k-1]$.[9] We refer to $P_{J_{i,j}}$ (resp., $P_{w_0(J_{i,j})}$) as a *basic* (resp., *reflected basic*) *Plücker coordinate.* Define U_k to be the open subset of $\mathrm{Gr}(k,n)$ consisting of subspaces whose basic Plücker coordinates are all nonzero.

Cyclic Plücker coordinates are basic, so every element of U_k has a diagonal form representative by Lemma 2.43. If N' is the diagonal form representative of N, then

$$\Delta_{[i,j],[1,j-i+1]}(N') = \frac{\Delta_{J_{i,j},[k]}(N')}{\Delta_{[n-k+1,n],[k]}(N')} = \frac{P_{J_{i,j}}(N)}{P_{[n-k+1,n]}(N)}. \tag{2.19}$$

This observation leads to the following result.

Lemma 2.45. *Every element of U_k is uniquely determined by its basic Plücker coordinates.*

Proof. Suppose $N \in U_k$, and let N' be its diagonal form representative. We inductively show that all the entries of N' are determined by the basic Plücker coordinates of N. Consider an entry N'_{ab} which is not automatically 0 or 1, and assume that $N'_{a'b'}$ is known for $a' < a$, and for $a' = a, b' < b$. Expand the determinant $\Delta_{[a-b+1,a],[1,b]}(N')$ along its last column. This gives an equation

$$N'_{ab} \cdot \Delta_{[a-b+1,a-1],[1,b-1]}(N') = \Delta_{[a-b+1,a],[1,b]}(N') + \text{a polynomial in known entries of } N'.$$

[9]There is intentionally some redundancy in this notation: if $i = n-k+1$ or $j = i-1$, then $J_{i,j} = [n-k+1, n]$.

By (2.19), the determinant on the left-hand side of this equation is a ratio of basic Plücker coordinates of N, and since these are nonzero, the entry N'_{ab} is determined. □

The following result significantly strengthens Lemma 2.45, and plays a crucial role in the arguments about positivity in later sections.

Proposition 2.46. *Every Plücker coordinate can be expressed as a Laurent polynomial in the basic (resp., reflected basic) Plücker coordinates, with non-negative integer coefficients.*

This result is proved at the end of §4.1.2.

Remark 2.47. Proposition 2.46 is a special case of the (positive) Laurent phenomenon in the theory of cluster algebras. Indeed, the $k(n-k)+1$ basic (resp., reflected basic) Plücker coordinates are a cluster in the homogeneous coordinate ring of $\mathrm{Gr}(k,n)$ (see [MS16, Figure 18]).

The dual Grassmannian

Given a subspace $N \subset \mathbb{C}^n$, let $N^\perp$ be the orthogonal complement of N with respect to the non-degenerate bilinear form given by $\langle v_i, v_j \rangle = (-1)^{i+1}\delta_{i,j}$, where $v_1, \ldots, v_n$ is the standard basis. Note that if $N \in \mathrm{Gr}(k,n)$, then $N^\perp$ is in the "dual Grassmannian" $\mathrm{Gr}(n-k,n)$. The Plücker coordinates of $N^\perp$ are closely related to those of N.

Lemma 2.48. *If $N \in \mathrm{Gr}(k,n)$, then for $J \in \binom{[n]}{k}$, we have*

$$P_J(N) = P_{\overline{J}}(N^\perp)$$

(as projective coordinates), where $\overline{J}$ denotes the complement $[n] \setminus J$.

The proof relies on Jacobi's identity for complementary minors of inverse matrices, which states that

$$\Delta_{I,J}(X^{-1}) = (-1)^{\sum I + \sum J} \frac{1}{\det(X)} \Delta_{\overline{J},\overline{I}}(X), \tag{2.20}$$

where $\sum S$ is the sum of the elements of S (see [GJPS12] for several proofs of this classical identity). Let X^c be the matrix obtained from X by scaling the i^{th} row and column by $(-1)^i$ (so $(X^c)_{ij} = (-1)^{i+j}X_{ij}$). If X is invertible, define $X^{-c} = (X^{-1})^c = (X^c)^{-1}$. It follows immediately from (2.20) that

$$\Delta_{I,J}(X^{-c}) = \frac{1}{\det(X)} \Delta_{\overline{J},\overline{I}}(X). \tag{2.21}$$

Proof of Lemma 2.48. Let N' be an $n \times k$ matrix whose column span is N. Choose a k-subset I so that $P_I(N) \neq 0$, and suppose $\overline{I} = \{i_1 < \ldots < i_{n-k}\}$. Let X be the $n \times n$ matrix whose j^{th} column is the standard basis vector e_{i_j} for $j = 1, \ldots, n-k$, and whose last k columns are the matrix N'. Clearly X is invertible. Let N'' be the $(n-k) \times n$ matrix consisting of the first $n-k$ rows of X^{-c}. Since $X^{-1}X = Id$, we have

$$0 = \sum_{r=1}^{n} (-1)^{i+r} N''_{ir} N'_{rj} = (-1)^{i-1} \sum_{r=1}^{n} (-1)^{r+1} N''_{ir} N'_{rj}$$

for $i = 1, \ldots, n-k$ and $j = 1, \ldots, k$. Thus, every row of the matrix N'' is orthogonal to every column of the matrix N' with respect to the bilinear form defined above, and since these rows are linearly independent, they span the $(n-k)$-dimensional subspace $N^\perp$.

By (2.21), we have $P_{\overline{I}}(N^\perp) = \Delta_{[n-k],\overline{I}}(X^{-c}) = \dfrac{1}{\det(X)} \Delta_{I,[n-k+1,n]}(X) \neq 0$. Combining this with another application of (2.21), we obtain

$$\frac{P_J(N)}{P_I(N)} = \frac{\Delta_{J,[n-k+1,n]}(X)}{\Delta_{I,[n-k+1,n]}(X)} = \frac{\Delta_{[n-k],\overline{J}}(X^{-c})}{\Delta_{[n-k],\overline{I}}(X^{-c})} = \frac{P_{\overline{J}}(N^\perp)}{P_{\overline{I}}(N^\perp)}$$

for all $J \in \binom{[n]}{k}$. Thus, $P_J(N) = P_{\overline{J}}(N^\perp)$ as projective coordinates, as claimed. □

2.6 Planar networks and the Lindström Lemma

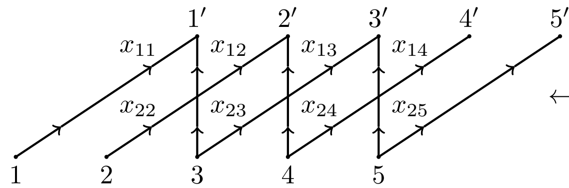
By *planar network*, we mean a finite, directed, edge-weighted graph embedded in a disc, with no oriented cycles. The edge weights are nonzero complex numbers (or indeterminates which take values in $\mathbb{C}^\times$). We assume there are r distinguished source vertices, labeled $1, \ldots, r$, and s distinguished sink vertices, labeled $1', \ldots, s'$. To each such network Γ, we associate an $r \times s$ matrix $M(\Gamma)$, as follows. Define the weight of a path to be the product of the weights of the edges in the path. The (i,j)-entry of $M(\Gamma)$ is the sum of the weights of all paths from source i to sink j', that is,

$$M(\Gamma)_{ij} = \sum_{p\,:\, i \to j'} \mathrm{wt}(p).$$

We say that $M(\Gamma)$ is the matrix associated to Γ, and that Γ is a network representation of M. For an example of a network and its associated matrix, see Figure 3.

The *gluing* of networks is compatible with matrix multiplication, in the sense that if a planar network Γ is obtained by identifying the sinks of a planar network Γ_1 with the sources of a planar network Γ_2, then

$$M(\Gamma) = M(\Gamma_1) \cdot M(\Gamma_2). \tag{2.22}$$



$$\longleftrightarrow \begin{pmatrix} x_{11} & 0 & 0 & 0 & 0 \\ x_{22} & x_{12}x_{22} & 0 & 0 & 0 \\ 1 & x_{12}+x_{23} & x_{13}x_{23} & 0 & 0 \\ 0 & 1 & x_{13}+x_{24} & x_{14}x_{24} & 0 \\ 0 & 0 & 1 & x_{14} & x_{25} \end{pmatrix}$$

Figure 3: A planar network and its associated matrix. Unlabeled edges have weight 1.

Let $I = \{i_1 < \ldots < i_m\} \subset [r]$ and $J = \{j_1 < \ldots < j_m\} \subset [s]$ be two subsets of cardinality m. A *family of paths from I to J* is a collection of m paths $p_1, \ldots, p_m$, such that p_a starts at source i_a and ends at sink $j'_{\sigma(a)}$, for some permutation $\sigma \in S_m$. We denote such a family by $\mathcal{F} = (p_a; \sigma)$, and we define the weight of the family by $\mathrm{wt}(\mathcal{F}) = \prod_{a=1}^m \mathrm{wt}(p_a)$. If no two of the paths share a vertex, we say that the family is *vertex-disjoint.*

We refer to the following result as the Lindström Lemma.

Proposition 2.49 (Lindström [Lin73])**.** *Let Γ be a planar network with r sources and s sinks, and let $I \subset [r], J \subset [s]$ be two subsets of the same cardinality. Then the minor of $M(\Gamma)$ using rows I and columns J is given by*

$$\Delta_{I,J}(M(\Gamma)) = \sum_{\mathcal{F}=(p_a;\sigma)\,:\,I\to J} \mathrm{sgn}(\sigma)\,\mathrm{wt}(\mathcal{F}),$$

where the sum is over vertex-disjoint families of paths from I to J.

For example, let Γ be the network in Figure 3. There are three vertex-disjoint families of paths from $\{3,4\}$ to $\{2',3'\}$. The weights of these families are $x_{12}x_{13}, x_{12}x_{24}$, and $x_{23}x_{24}$, and in all three cases σ is the identity permutation. From the matrix, one computes

$$\Delta_{34,23}(M(\Gamma)) = x_{12}x_{13} + x_{12}x_{24} + x_{23}x_{24},$$

in agreement with the Lindström Lemma.

With a single exception (in §5.3), our networks will have the property that every vertex-disjoint family of paths is of the form $(p_a; \mathrm{Id})$, so the Lindström Lemma expresses every minor of the associated matrix as a polynomial in the edge weights with non-negative integer coefficients.

CHAPTER 3

Geometric and unipotent crystals on the Grassmannian

3.1 Main definitions

For $k \in [n-1]$, let $\mathbb{X}_k$ denote the variety $\mathrm{Gr}(k,n) \times \mathbb{C}^\times$.[1] We denote a point of $\mathbb{X}_k$ by $N|t$, where $N \in \mathrm{Gr}(k,n)$ and $t \in \mathbb{C}^\times$. We begin by introducing an order n cyclic symmetry of $\mathbb{X}_k$ that plays a central role in everything that follows.

Definition 3.1. Define the *cyclic shift map* $\mathrm{PR} : \mathbb{X}_k \to \mathbb{X}_k$ by $\mathrm{PR}(N|t) = N'|t$, where N' is obtained from N by shifting the rows down by 1 (mod n), and multiplying the new first row by $(-1)^{k-1}t$. We write PR_t to denote the map $N \mapsto N'$.

For example, when $n = 4$ and $k = 2$, we have

$$\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \\ z_{31} & z_{32} \\ z_{41} & z_{42} \end{pmatrix} \overset{\mathrm{PR}_t}{\mapsto} \begin{pmatrix} -t \cdot z_{41} & -t \cdot z_{42} \\ z_{11} & z_{12} \\ z_{21} & z_{22} \\ z_{31} & z_{32} \end{pmatrix}.$$

It's easy to see that PR is well-defined (i.e., it does not depend on the choice of matrix representative for the subspace N), and that PR has order n. Note that the Plücker coordinates of $N' = \mathrm{PR}_t(N)$ are given by

$$P_J(N') = \begin{cases} P_{J-1}(N) & \text{if } 1 \notin J \\ t \cdot P_{J-1}(N) & \text{if } 1 \in J \end{cases} \tag{3.1}$$

where $J - 1$ is obtained from J by subtracting 1 from each element (mod n).

There is also a natural order n symmetry on the loop group $\mathrm{GL}_n(\mathbb{C}(\lambda))$. Recall from §2.4 the unfolding construction, which identifies $\mathrm{GL}_n(\mathbb{C}(\lambda))$ with a subset of n-periodic matrices. Define the *shift map* sh on an n-periodic matrix X by

$$\mathrm{sh}(X)_{ij} = X_{i-1,j-1}.$$

[1] Since n is fixed throughout, we suppress the dependence on n in the notation $\mathbb{X}_k$.

This map is easily seen to be an automorphism of order n which preserves both B^- and U.

Now we make $\mathbb{X}_k$ into a unipotent crystal. For $A \in \mathrm{GL}_n(\mathbb{C}(\lambda))$ and $z \in \mathbb{C}$, let $A|_{\lambda=z}$ denote the matrix obtained by evaluating the loop parameter λ at z. This is defined as long as z is not a pole of any entry of A; the resulting matrix is invertible if z is not a root of the determinant of A. Define a U-action $U \times \mathbb{X}_k \to \mathbb{X}_k$ by

$$u.(N|t) = (u|_{\lambda=(-1)^{k-1}t} \cdot N)|t. \tag{3.2}$$

Note that $u.(N|t)$ is always defined, since every element of U has Laurent polynomial entries and determinant 1. This action makes $\mathbb{X}_k$ into a U-variety.

Definition 3.2. Define a rational map $g : \mathbb{X}_k \to B^-$ by $g(N|t) = A$, where A is the folded matrix defined by

$$A_{ij} = c_{ij} \frac{P_{[j-k+1,j-1]\cup\{i\}}(N)}{P_{[j-k,j-1]}(N)}, \qquad c_{ij} = \begin{cases} 1 & \text{if } j \leq k \\ t & \text{if } j > k \text{ and } i \geq j \\ \lambda & \text{if } j > k \text{ and } i < j. \end{cases}$$

For example, if $N|t \in \mathrm{Gr}(2,5) \times \mathbb{C}^\times$, then setting $P_J = P_J(N)$, we have

$$g(N|t) = \begin{pmatrix} \frac{P_{15}}{P_{45}} & 0 & \lambda & \lambda\frac{P_{13}}{P_{23}} & \lambda\frac{P_{14}}{P_{34}} \\ \frac{P_{25}}{P_{45}} & \frac{P_{12}}{P_{15}} & 0 & \lambda & \lambda\frac{P_{24}}{P_{34}} \\ \frac{P_{35}}{P_{45}} & \frac{P_{13}}{P_{15}} & t\frac{P_{23}}{P_{12}} & 0 & \lambda \\ 1 & \frac{P_{14}}{P_{15}} & t\frac{P_{24}}{P_{12}} & t\frac{P_{34}}{P_{23}} & 0 \\ 0 & 1 & t\frac{P_{25}}{P_{12}} & t\frac{P_{35}}{P_{23}} & t\frac{P_{45}}{P_{34}} \end{pmatrix}. \tag{3.3}$$

Note that g is defined if and only if the cyclic Plücker coordinates of N do not vanish, that is, if and only if N is in the open positroid cell $\mathrm{Gr}^\circ(k,n)$.

Lemma 3.3.

1. *If $N|t \in \mathbb{X}_k$ with $N \in \mathrm{Gr}^\circ(k,n)$, then*

$$g \circ \mathrm{PR}(N|t) = \mathrm{sh} \circ g(N|t).$$

2. *For $i \in \mathbb{Z}/n\mathbb{Z}$, $a \in \mathbb{C}$, $N|t \in \mathbb{X}_k$, and $X \in B^-$, we have*

$$\mathrm{PR}^{-1}(\widehat{x}_i(a).\,\mathrm{PR}(N|t)) = \widehat{x}_{i-1}(a).(N|t),$$
$$\mathrm{sh}^{-1}(\widehat{x}_i(a).\,\mathrm{sh}(X)) = \widehat{x}_{i-1}(a).X.$$

Proof. Let $X = g(N|t)$ and $X' = g \circ \mathrm{PR}(N|t)$ be unfolded matrices. By definition, $X_{ij} = 0$ if $i - j \notin [0, n-k]$, and for $i - j \in [0, n-k]$, X_{ij} is given by

$$X_{ij} = t^{b_{ij}} \frac{P_{[j-k+1,j-1]\cup\{i\}}(N)}{P_{[j-k,j-1]}(N)}, \qquad b_{ij} = \begin{cases} 0 & \text{if } \overline{j} \in [1,k] \text{ or } \overline{i} < \overline{j} \\ 1 & \text{otherwise} \end{cases}$$

where $\overline{i}$ denotes the residue of i mod n in the interval $[1, n]$. By (3.1), we have (for $i - j \in [0, n-k]$)

$$X'_{ij} = t^{b_{ij}+b'_{ij}} \frac{P_{[j-k,j-2]\cup\{i-1\}}(N)}{P_{[j-k-1,j-2]}(N)}, \qquad b'_{ij} = \begin{cases} 1 & \text{if } \overline{j} \notin [2,k+1] \text{ and } \overline{i} = 1 \\ -1 & \text{if } \overline{j} = k+1 \text{ and } \overline{i} \neq 1 \\ 0 & \text{otherwise.} \end{cases}$$

By considering several cases, one verifies that $b_{ij} + b'_{ij} = b_{i-1,j-1}$, so $X'_{ij} = X_{i-1,j-1} = \mathrm{sh}(X)_{ij}$, proving (1).

Let $v_i \in \mathbb{C}^k$ be the i^{th} row of N (more precisely, of a fixed matrix representative of N). Acting on N by $\widehat{x}_i(a)$ replaces v_i with $v_i + av_{i+1}$ if $i \neq 0$, and it replaces v_n with $v_n + \frac{(-1)^{k-1}}{t} av_1$ if $i = 0$. The map PR_t replaces v_i with v_{i-1} for $i \neq 1$, and it replaces v_1 with $(-1)^{k-1} t v_n$; the inverse map PR_t^{-1} replaces v_i with v_{i+1} for $i \neq n$, and it replaces v_n with $\frac{(-1)^{k-1}}{t} v_1$. From this description, it's clear that the first identity of (2) holds.

For the second identity of (2), let $X' = \mathrm{sh}(X)$. By (2.14) and the fact that sh is multiplicative, we have

$$\begin{aligned} \mathrm{sh}^{-1}(\widehat{x}_i(a).X') &= \mathrm{sh}^{-1}(\widehat{x}_i(a)) \cdot \mathrm{sh}^{-1}(X') \cdot \mathrm{sh}^{-1}(\widehat{x}_i(\tau_i(a, X'))) \\ &= \widehat{x}_{i-1}(a) \cdot X \cdot \widehat{x}_{i-1}(\tau_{i-1}(a, X)) \\ &= \widehat{x}_{i-1}(a).X, \end{aligned}$$

where for the second equality we use

$$\tau_i(a, X') = \frac{-aX'_{i+1,i+1}}{X'_{ii} + aX'_{i+1,i}} = \frac{-aX_{ii}}{X_{i-1,i-1} + aX_{i,i-1}} = \tau_{i-1}(a, X).$$

□

Proposition 3.4. *The pair $(\mathbb{X}_k, g)$ is a unipotent crystal.*

Proof. It's clear that the rational functions $v \mapsto g(v)_{i+1,i}$ are not identically zero. We must show that g commutes with the U-actions. Since U is generated by $\widehat{x}_i(a)$, we need only show that

$$g(\widehat{x}_i(a).v) = \widehat{x}_i(a).g(v) \tag{3.4}$$

for all i. In fact, if we know that (3.4) holds for a particular value of i, then Lemma 3.3 allows us to deduce that it holds for all i, so it suffices to consider the case $i = 1$.

Suppose $N|t \in \mathbb{X}_k$ and $a \in \mathbb{C}$. Set $N'|t = \widehat{x}_1(a).(N|t)$, and write $P_J = P_J(N)$ and $P'_J = P_J(N')$. The matrix $\widehat{x}_1(a)$ does not depend on λ, so for any t, the matrix N' is obtained from N by adding a times row 2 to row 1. Thus, we have

$$P'_J = \begin{cases} P_J + aP_{(J\setminus\{1\})\cup\{2\}} & \text{if } 1 \in J \text{ and } 2 \notin J \\ P_J & \text{otherwise} \end{cases}. \tag{3.5}$$

Set $A = g(N|t)$, $A' = g(N'|t)$, and $A'' = \widehat{x}_1(a).A$ (view these as folded matrices). We must show that $A' = A''$. By (2.14),

$$A'' = \widehat{x}_1(a) \cdot A \cdot \widehat{x}_1(\tau_1(a, A)).$$

In words, A'' is obtained from A by adding a times row 2 to row 1, and then adding $\tau_1(a, A)$ times column 1 to column 2. Thus, A'' and A differ only in the first row and the second column. There are four cases to consider.

Case 1: $i \neq 1, j \neq 2$. In this case, $A''_{ij} = A_{ij}$, and by (3.5) and the definition of g, we see that $A'_{ij} = A_{ij}$ as well.

Case 2: $i = 1, j = 2$. By definition, A_{12} and A'_{12} are equal to λ if $k = 1$, and 0 otherwise. The quantity $\tau_1(a, A)$ is defined so that A''_{12} has no constant term, so $A''_{12} = \delta_{k,1}\lambda$ as well.

Case 3: $i = 1, j \neq 2$. In this case, we have

$$\begin{aligned} A''_{1j} = A_{1j} + aA_{2j} &= \lambda^{1-\delta_{j,1}} \frac{P_{\{1\}\cup[j-k+1,j-1]} + aP_{\{2\}\cup[j-k+1,j-1]}}{P_{[j-k,j-1]}} \\ &= \lambda^{1-\delta_{j,1}} \frac{P'_{\{1\}\cup[j-k+1,j-1]}}{P'_{[j-k,j-1]}} = A'_{1j}. \end{aligned}$$

Case 4: $i \neq 1, j = 2$. Since the matrix entries A_{11}, A_{12}, and A_{22} do not depend on λ, we have $\tau_1(a, A) = \frac{-aA_{22}}{A_{11}+aA_{21}}$. We compute

$$\begin{aligned} A''_{i2} &= A_{i2} + \tau_1(a, A)A_{i1} \\ &= t^{\delta_{k,1}} \frac{P_{\{1,i\}\cup[n-k+3,n]}}{P_{\{1\}\cup[n-k+2,n]}} + \frac{-at^{\delta_{k,1}} \dfrac{P_{\{1,2\}\cup[n-k+3,n]}}{P_{\{1\}\cup[n-k+2,n]}}}{\dfrac{P_{\{1\}\cup[n-k+2,n]}}{P_{[n-k+1,n]}} + a\dfrac{P_{\{2\}\cup[n-k+2,n]}}{P_{[n-k+1,n]}}} \frac{P_{\{i\}\cup[n-k+2,n]}}{P_{[n-k+1,n]}} \\ &= t^{\delta_{k,1}} \frac{P_{\{1,i\}\cup[n-k+3,n]}(P_{\{1\}\cup[n-k+2,n]} + aP_{\{2\}\cup[n-k+2,n]}) - aP_{\{1,2\}\cup[n-k+3,n]}P_{\{i\}\cup[n-k+2,n]}}{P_{\{1\}\cup[n-k+2,n]}(P_{\{1\}\cup[n-k+2,n]} + aP_{\{2\}\cup[n-k+2,n]})}. \end{aligned}$$

If $k > 1$ and $i \leq n-k+2$, apply a three-term Plücker relation (Corollary 2.42) to the terms in the numerator containing a to obtain

$$A''_{i2} = \frac{P_{\{1,i\}\cup[n-k+3,n]} + aP_{\{2,i\}\cup[n-k+3,n]}}{P_{\{1\}\cup[n-k+2,n]} + aP_{\{2\}\cup[n-k+2,n]}} = \frac{P'_{\{1,i\}\cup[n-k+3,n]}}{P'_{\{1\}\cup[n-k+2,n]}} = A'_{i2}.$$

If $i > n-k+2$, then $A_{i1} = A_{i2} = A''_{i2} = A'_{i2} = 0$, and if $k = 1$, then

$$A''_{i2} = t\frac{P_i(P_1 + aP_2) - aP_2P_i}{P_1(P_1 + aP_2)} = t\frac{P_i}{P_1 + aP_2} = t\frac{P'_i}{P'_1} = A'_{i2}.$$

□

By Theorem 2.38, the unipotent crystal on $(\mathbb{X}_k, g)$ induces a geometric crystal on $\mathbb{X}_k$. Unraveling the definitions, we obtain the following formulas for the geometric crystal structure on $\mathbb{X}_k$.

- The map $\gamma : \mathbb{X}_k \to (\mathbb{C}^\times)^n$ is given by $\gamma(N|t) = (\gamma_1, \ldots, \gamma_n)$, where

$$\gamma_i = \begin{cases} \dfrac{P_{[i-k+1,i]}(N)}{P_{[i-k,i-1]}(N)} & \text{if } 1 \leq i \leq k \\[2ex] t\dfrac{P_{[i-k+1,i]}(N)}{P_{[i-k,i-1]}(N)} & \text{if } k+1 \leq i \leq n. \end{cases}$$

- For $i \in \mathbb{Z}/n\mathbb{Z}$, the functions $\varepsilon_i, \varphi_i : \mathbb{X}_k \to \mathbb{C}^\times$ are given by

$$\varepsilon_i(N|t) = t^{-\delta_{i,k}} \frac{P_{[i-k+1,i-1]\cup\{i+1\}}(N)P_{[i-k+1,i]}(N)}{P_{[i-k,i-1]}(N)P_{[i-k+2,i+1]}(N)}.$$

$$\varphi_i(N|t) = t^{-\delta_{i,0}} \frac{P_{[i-k+1,i-1]\cup\{i+1\}}(N)}{P_{[i-k+1,i]}(N)},$$

- For $i \in \mathbb{Z}/n\mathbb{Z}$, the rational action $e_i : \mathbb{C}^\times \times \mathbb{X}_k \to \mathbb{X}_k$ is given by $e_i^c(N|t) = N'|t$, where

$$N' = \begin{cases} x_i\left(\dfrac{c-1}{\varphi_i(N|t)}\right) \cdot N & \text{if } i \neq 0 \\[2ex] x_0\left(\dfrac{(-1)^{k-1}}{t} \cdot \dfrac{c-1}{\varphi_0(N|t)}\right) \cdot N & \text{if } i = 0. \end{cases}$$

  Here $x_i(a) = Id + aE_{i,i+1}$ for $i \in [n-1]$, and $x_0(a) = Id + aE_{n1}$, where E_{ij} is an $n \times n$ matrix unit.

Finally, we make $\mathbb{X}_k$ into a decorated geometric crystal. Say that an n-periodic matrix X is *m-shifted unipotent* if $X_{ij} = 0$ when $i - j > m$, and $X_{ij} = 1$ when $i - j = m$. If X is m-shifted unipotent, define

$$\chi(X) = \sum_{j=1}^{n} X_{j+m-1,j}.$$

It is easy to see that if X is m-shifted unipotent and Y is m'-shifted unipotent, then XY is $(m+m')$-shifted unipotent, and

$$\chi(XY) = \chi(X) + \chi(Y). \tag{3.6}$$

If $N|t \in \mathbb{X}_k$, then $g(N|t)$ is $(n-k)$-shifted unipotent. For example, the matrix $g(N|t)$ for $N \in \mathrm{Gr}(2,5)$ is shown above in (3.3). This matrix is 3-shifted unipotent, and

$$\chi(g(N|t)) = \frac{P_{35}(N)}{P_{45}(N)} + \frac{P_{14}(N)}{P_{15}(N)} + t\frac{P_{25}(N)}{P_{12}(N)} + \frac{P_{13}(N)}{P_{23}(N)} + \frac{P_{24}(N)}{P_{34}(N)}.$$

Definition 3.5. Define $f : \mathbb{X}_k \to \mathbb{C}$ by

$$f(N|t) = \chi(g(N|t)) = \sum_{i \neq k} \frac{P_{\{i-k\}\cup[i-k+2,i]}(N)}{P_{[i-k+1,i]}(N)} + t\frac{P_{[2,k]\cup\{n\}}(N)}{P_{[1,k]}(N)}.$$

Lemma 3.6. *The function f satisfies* (2.13)*, so it is a decoration on $\mathbb{X}_k$.*

Proof. Using (2.15) and (2.16), we compute

$$\begin{aligned} f(e_i^c(N|t)) = \chi(g(e_i^c(N|t))) &= \chi(e_i^c(g(N|t))) \\ &= \chi\left(\widehat{x}_i\left(\frac{c-1}{\varphi_i(N|t)}\right) \cdot g(N|t) \cdot \widehat{x}_i\left(\frac{c^{-1}-1}{\varepsilon_i(N|t)}\right)\right) \\ &= \frac{c-1}{\varphi_i(N|t)} + f(N|t) + \frac{c^{-1}-1}{\varepsilon_i(N|t)}, \end{aligned}$$

so (2.13) holds. □

Products

For $k_1, \ldots, k_d \in [n-1]$, define

$$\mathbb{X}_{k_1,\ldots,k_d} = \mathbb{X}_{k_1} \times \cdots \times \mathbb{X}_{k_d}.$$

Since each $\mathbb{X}_{k_j}$ is a unipotent crystal, the product $\mathbb{X}_{k_1,\ldots,k_d}$ is also a unipotent crystal by Theorem 2.39, and the map $g : \mathbb{X}_{k_1,\ldots,k_d} \to B^-$ is given by

$$g(x_1, \ldots, x_d) = g(x_1) \cdots g(x_d). \tag{3.7}$$

By Theorem 2.38, the unipotent crystal $(\mathbb{X}_{k_1,\ldots,k_d}, g)$ induces a geometric crystal on $\mathbb{X}_{k_1,\ldots,k_d}$. By Definition/Proposition 2.32, the map $f : \mathbb{X}_{k_1,\ldots,k_d} \to \mathbb{C}$ defined by $f(x_1, \ldots, x_d) = f(x_1) + \ldots + f(x_d)$ is a decoration. Note that Definition 3.5 and equations (3.6), (3.7) imply that

$$f(x_1, \ldots, x_d) = \chi(g(x_1, \ldots, x_d)). \tag{3.8}$$

3.2 Properties of the matrix $g(N|t)$

Here we prove several important properties of the matrix $g(N|t)$.

Proposition 3.7. *Suppose N is in the open positroid cell $\mathrm{Gr}^\circ(k,n)$, and let $A = g(N|t)$, viewed as a folded matrix.*

1. *The first k columns of A span the subspace N.*
2. *The matrix $A|_{\lambda=(-1)^{k-1}t}$ has rank k.*
3. *The determinant of A is $(t+(-1)^k\lambda)^{n-k}$.*

Proof. By Lemma 2.43, the subspace N has a diagonal form representative N'. It follows from the definition of diagonal form that for $j = 1, \ldots, k$,

$$N'_{ij} = \frac{P_{[1,j-1]\cup\{i\}\cup[n-k+j+1,n]}(N)}{P_{[1,j-1]\cup[n-k+j,n]}(N)} = \frac{P_{[j-k+1,j-1]\cup\{i\}}(N)}{P_{[j-k,j-1]}(N)}$$

(recall Convention 2.40) if $i \in [j, j+n-k]$, and $N'_{ij} = 0$ otherwise. Comparing with the definition of g, we see that N' is equal to the first k columns of A, which proves (1).

For (2), set $A_t = A|_{\lambda=(-1)^{k-1}t}$. We claim that $\Delta_{I,[1,k]\cup\{j\}}(A_t) = 0$ for all $(k+1)$-subsets $I \subset [n]$, and $j \in [k+1, n]$. To see this, suppose $I = \{i_1 < \cdots < i_{k+1}\}$, and expand the determinant along column j:

$$\Delta_{I,[1,k]\cup\{j\}}(A_t) = \sum_{r=1}^{k+1} (-1)^{k+1+r} (A_t)_{i_r,j} \Delta_{I\setminus\{i_r\},[1,k]}(A_t). \tag{3.9}$$

By part (1) (and the fact that $\Delta_{[n-k+1,n],[1,k]}(A_t) = 1$), we have

$$\Delta_{I\setminus\{i_r\},[1,k]}(A_t) = \frac{P_{I\setminus\{i_r\}}(N)}{P_{[n-k+1,n]}(N)}.$$

By the definition of g, we have

$$\begin{aligned}(A_t)_{i_r,j} &= \begin{cases} t\dfrac{P_{[j-k+1,j-1]\cup\{i_r\}}(N)}{P_{[j-k,j-1]}(N)} & \text{if } i_r \geq j \\ (-1)^{k-1}t\dfrac{P_{[j-k+1,j-1]\cup\{i_r\}}(N)}{P_{[j-k,j-1]}(N)} & \text{if } i_r < j \end{cases} \\ &= t\frac{P_{<j-k+1,j-k+2,\ldots,j-1,i_r>}(N)}{P_{[j-k,j-1]}(N)}\end{aligned}$$

where in the last line, the angle brackets indicate that we are taking the columns inside the brackets in the order in which they appear in the sequence, rather than sorting them in increasing order (see Convention 2.40). Now (3.9) becomes

$$\Delta_{I,[1,k]\cup\{j\}}(A_t) = \sum_{r=1}^{k+1} (-1)^{k+1+r} t \frac{P_{<j-k+1,j-k+2,\ldots,j-1,i_r>}(N)}{P_{[j-k,j-1]}(N)} \frac{P_{I\setminus\{i_r\}}(N)}{P_{[n-k+1,n]}(N)} = 0$$

by the Grassmann–Plücker relations (Proposition 2.41).

We have shown that each of the last $n-k$ columns of A_t is in the span of the first k, and since the first k columns have rank k by part (1), this proves (2).

For (3), let A_t be as above. By part (2), it is possible to add linear combinations of the first k columns of A_t to the last $n-k$ columns to obtain a matrix with zeroes in the last $n-k$ columns. Let A' be the matrix obtained by adding the same linear combinations of the first k columns of A (which are equal to the first k columns of A_t) to the last $n-k$ columns of A. Then we have

$$(A')_{ij} = c'_{ij} \frac{P_{[j-k+1,j-1]\cup\{i\}}(N)}{P_{[j-k,j-1]}(N)}, \qquad c'_{ij} = \begin{cases} 1 & \text{if } j \leq k \\ (-1)^k t + \lambda & \text{if } i \leq n-k \text{ and } j > i \\ 0 & \text{otherwise.} \end{cases}$$

For example, if $n = 5$ and $k = 2$, then A' is of the form

$$\begin{pmatrix} * & 0 & t+\lambda & (t+\lambda)* & (t+\lambda)* \\ * & * & 0 & t+\lambda & (t+\lambda)* \\ * & * & 0 & 0 & t+\lambda \\ 1 & * & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

where the $*$'s are certain ratios of Plücker coordinates. Thus, we have

$$\det(A) = \det(A') = (-1)^{k(n-k)}((-1)^k t + \lambda)^{n-k} = (t + (-1)^k \lambda)^{n-k},$$

proving (3). □

Combining Proposition 3.7 with some simple linear algebra, we obtain two statements that play an important role in the study of the geometric R-matrix in §5.1.

Corollary 3.8. *Suppose* $N|t \in \mathbb{X}_k$ *and* $B \in M_n(\mathbb{C}[\lambda, \lambda^{-1}])$.

1. *The first* k *columns of* $(g(N|t) \cdot B)|_{\lambda=(-1)^{k-1}t}$ *are contained in the subspace* N.
2. *If* $B|_{\lambda=(-1)^{k-1}t}$ *is invertible, then the matrix* $(B \cdot g(N|t))|_{\lambda=(-1)^{k-1}t}$ *has rank* k. *Furthermore, the first* k *columns have full rank, and they span the subspace* $B|_{\lambda=(-1)^{k-1}t} \cdot N$.

Proof. By parts (1) and (2) of Proposition 3.7, the column span of the matrix $g(N|t)|_{\lambda=(-1)^{k-1}t}$ is the subspace N. Multiplication of this matrix by $B|_{\lambda=(-1)^{k-1}t}$ on the right is equivalent to performing a sequence of (possibly degenerate) column operations, so all columns of the resulting matrix are contained in N, proving (1).

Part (2) follows from parts (1) and (2) of Proposition 3.7, and the fact that invertible linear transformations preserve dimension. □

3.3 Symmetries

In §3.1, we introduced the cyclic shift map PR : $\mathbb{X}_k \to \mathbb{X}_k$, and the shift map sh on the loop group, and we showed that the unipotent crystal map g intertwines these two maps. In this section, we study these $\mathbb{Z}/n\mathbb{Z}$ symmetries in a bit more detail, and then we study two additional symmetries of the geometric crystals $\mathbb{X}_k$: a geometric analogue of the Schützenberger involution, and the duality map from a subspace to its orthogonal complement. In both cases, we show that g intertwines the symmetry of the Grassmannian with a natural map on the loop group, and from this we deduce that the symmetries are compatible with the geometric crystal structure. This compatibility allows us to prove analogous results about (combinatorial) crystals in §4.2. Furthermore, these symmetries play an indispensable role in proving the main results of Chapter 5.

3.3.1 $\mathbb{Z}/n\mathbb{Z}$ symmetry

Recall from §3.1 the maps PR and sh. Recall also that $(\mathbb{X}_{k_1,\ldots,k_d}, g)$ is a unipotent crystal, where $g(x_1, \ldots, x_d) = g(x_1) \cdots g(x_d)$. Extend PR to a map $\mathbb{X}_{k_1,\ldots,k_d} \to \mathbb{X}_{k_1,\ldots,k_d}$ by

$$\mathrm{PR}(x_1, \ldots, x_d) = (\mathrm{PR}(x_1), \ldots, \mathrm{PR}(x_d)).$$

Since sh is an automorphism, Lemma 3.3(1) extends to the identity

$$g \circ \mathrm{PR} = \mathrm{sh} \circ g \tag{3.10}$$

on any product $\mathbb{X}_{k_1,\ldots,k_d}$.

Proposition 3.9. *The map* PR *interacts with the geometric crystal structure on* $\mathbb{X}_{k_1,\ldots,k_d}$ *as follows:*

1. $\gamma \circ \mathrm{PR} = \widetilde{\mathrm{sh}} \circ \gamma$, *where* $\widetilde{\mathrm{sh}}(z_1, \ldots, z_n) = (z_n, z_1, \ldots, z_{n-1})$*;*
2. $\varepsilon_i \circ \mathrm{PR} = \varepsilon_{i-1}$ *and* $\varphi_i \circ \mathrm{PR} = \varphi_{i-1}$ *for* $i \in \mathbb{Z}/n\mathbb{Z}$*;*
3. $e_i^c \circ \mathrm{PR} = \mathrm{PR} \circ e_{i-1}^c$ *for* $i \in \mathbb{Z}/n\mathbb{Z}$*;*
4. $f \circ \mathrm{PR} = f$.

Proof. These identities are essentially a formal consequence of (3.10) and basic properties of unipotent crystals. We will use the same technique to prove analogous results for the symmetries introduced in §3.3.2 and §3.3.3, as well as for the geometric R-matrix in §5.1.2.

Recall that the unipotent crystal (B^-, Id) induces a geometric crystal on B^- (Theorem 2.38). It is immediate from the definitions that the induced geometric crystal maps on B^- satisfy

$$\gamma \circ \mathrm{sh} = \widetilde{\mathrm{sh}} \circ \gamma \qquad \varepsilon_i \circ \mathrm{sh} = \varepsilon_{i-1} \qquad \varphi_i \circ \mathrm{sh} = \varphi_{i-1}. \tag{3.11}$$

Combining (3.11) with (2.16) and (3.10), we have

$$\gamma \circ \mathrm{PR} = \gamma \circ g \circ \mathrm{PR} = \gamma \circ \mathrm{sh} \circ g = \widetilde{\mathrm{sh}} \circ \gamma \circ g = \widetilde{\mathrm{sh}} \circ \gamma,$$

proving (1). Part (2) is proved in the same way. Similarly, the function χ defined on m-shifted unipotent n-periodic matrices clearly satisfies $\chi \circ \mathrm{sh} = \chi$, so part (4) follows from (3.8) and (3.10).

The action e_i on B^- is defined by $e_i^c(X) = \widehat{x}_i\left(\frac{c-1}{\varphi_i(X)}\right).X$, so by Lemma 3.3(2) and (3.11), we have

$$e_i^c \circ \mathrm{sh} = \mathrm{sh} \circ e_{i-1}^c. \tag{3.12}$$

Combining (3.12) with (2.16) and (3.10), we compute

$$\begin{aligned} g \circ e_i^c \circ \mathrm{PR} &= e_i^c \circ g \circ \mathrm{PR} \\ &= e_i^c \circ \mathrm{sh} \circ g \\ &= \mathrm{sh} \circ e_{i-1}^c \circ g \\ &= \mathrm{sh} \circ g \circ e_{i-1}^c \\ &= g \circ \mathrm{PR} \circ e_{i-1}^c. \end{aligned}$$

If $d = 1$, then the identity $e_i^c \circ \mathrm{PR} = \mathrm{PR} \circ e_{i-1}^c$ follows from Proposition 3.7(1) by "projecting" both sides of the preceding identity onto the first k columns. To prove the general case, we will show that if X and Y are products of the geometric crystals $\mathbb{X}_{k_j}$ such that part (3) holds on X and Y separately, then it holds on the product $X \times Y$. By Definition/Proposition 2.32, the action e_i on $X \times Y$ is given by

$$e_i^c(x,y) = (e_i^{c_1}(x), e_i^{c_2}(y)) \quad \text{where} \quad c_1 = \frac{c\varepsilon_i(x) + \varphi_i(y)}{\varepsilon_i(x) + \varphi_i(y)}, \quad c_2 = \frac{\varepsilon_i(x) + \varphi_i(y)}{\varepsilon_i(x) + c^{-1}\varphi_i(y)}.$$

Since we have already shown that part (2) holds on any product of the $\mathbb{X}_{k_j}$, we have

$$e_i^c \circ \mathrm{PR}(x,y) = (e_i^{c'_1}(\mathrm{PR}(x)), e_i^{c'_2}(\mathrm{PR}(y))) = \mathrm{PR}(e_{i-1}^{c'_1}(x), e_{i-1}^{c'_2}(y)) \tag{3.13}$$

where

$$c'_1 = \frac{c\varepsilon_{i-1}(x) + \varphi_{i-1}(y)}{\varepsilon_{i-1}(x) + \varphi_{i-1}(y)}, \quad c'_2 = \frac{\varepsilon_{i-1}(x) + \varphi_{i-1}(y)}{\varepsilon_{i-1}(x) + c^{-1}\varphi_{i-1}(y)}.$$

The right-most expression of (3.13) is $\mathrm{PR} \circ e_{i-1}^c(x,y)$, so we are done. □

Remark 3.10. It is possible to deduce part (3) directly from the identity $g \circ e_i^c \circ \mathrm{PR} = g \circ \mathrm{PR} \circ e_{i-1}^c$ by appealing to Corollary 5.9. Indeed, this is how we will prove that the geometric R-matrix commutes with the geometric crystal operators. We have chosen to use a more elementary approach here because the only proof we know of Corollary 5.9 relies on two difficult results about the geometric R-matrix (Theorems 5.3 and 5.4), and we want to emphasize that the simple fact proved here does not depend on those results.

The following result plays an important role in the proof of the positivity of the geometric R-matrix in §5.4.

Lemma 3.11. *Suppose $N|t \in \mathbb{X}_k$. Let $A = g(N|t)$ and $A' = g(\mathrm{PR}(N|t))$, and view these as folded matrices. Then for $I, J \in \binom{[n]}{r}$, we have*

$$\Delta_{I,J}(A') = \begin{cases} \Delta_{I-1,J-1}(A) & \text{if } 1 \in I \cap J \text{ or } 1 \notin I \cup J \\ (-1)^{r-1}\lambda \cdot \Delta_{I-1,J-1}(A) & \text{if } 1 \in I \setminus J \\ (-1)^{r-1}\lambda^{-1} \cdot \Delta_{I-1,J-1}(A) & \text{if } 1 \in J \setminus I \end{cases} \tag{3.14}$$

where $S - 1$ is obtained from S by subtracting 1 from each element (mod n).

Proof. By (3.10), we have $A' = \mathrm{sh}(A)$. Observe that the submatrix $\mathrm{sh}(A)_{I,J}$ is obtained from the submatrix $A_{I-1,J-1}$ by the following two steps:

- If $1 \in I$, multiply the last row by λ and interchange it with the other $r-1$ rows.
- If $1 \in J$, multiply the last column by λ^{-1} and interchange it with the other $r-1$ columns.

This implies (3.14). □

3.3.2 The geometric Schützenberger involution

For $z \in \mathbb{C}^\times$, define $\pi_z^k : M_n(\mathbb{C}[\lambda, \lambda^{-1}]) \to \mathbb{X}_k$ by

$$\pi_z^k(A) = N|z \tag{3.15}$$

where N is the subspace spanned by the first k columns of the $n \times n$ matrix $A_z = A|_{\lambda=(-1)^{k-1}z}$. This map is undefined if the first k columns of A_z do not have full rank. Proposition 3.7(1) states that for $N \in \mathrm{Gr}^\circ(k,n)$ and $t \in \mathbb{C}^\times$, we have

$$\pi_t^k \circ g(N|t) = N|t. \tag{3.16}$$

This shows that the matrix $A = g(N|t)$ is determined by the subspace spanned by its first k columns (and the value of t). Now we consider what happens if we "project" onto the last k rows instead of the first k columns.

Define the *flip map* fl on an $n \times n$ matrix A by

$$\mathrm{fl}(A)_{ij} = A_{n-j+1,n-i+1}. \tag{3.17}$$

In words, fl reflects the matrix over the anti-diagonal. It is easy to see that fl is an anti-automorphism, and that it satisfies

$$\mathrm{fl}^2 = \mathrm{Id} \qquad \text{and} \qquad \mathrm{fl} \circ \mathrm{sh} = \mathrm{sh}^{-1} \circ \mathrm{fl}\,. \tag{3.18}$$

Definition 3.12. Define the *geometric Schützenberger involution* $S : \mathbb{X}_k \to \mathbb{X}_k$ by

$$S(N|t) = \pi_t^k \circ \mathrm{fl} \circ g(N|t).$$

This is a rational map which is defined when N is in the open positroid cell $\mathrm{Gr}^\circ(k,n)$. Continuing the notation used for PR, we write S_t to denote the map $N \mapsto N'$, where $N'|t = S(N|t)$. Extend S to a map $\mathbb{X}_{k_1,\dots,k_d} \to \mathbb{X}_{k_d,\dots,k_1}$ by

$$S(x_1, \dots, x_d) = (S(x_d), \dots, S(x_1)). \tag{3.19}$$

Note that the order of the factors is reversed.

Remark 3.13. This definition was inspired by work of Noumi and Yamada on a geometric[2] lift of the Robinson–Schensted–Knuth correspondence, in which they observed that the anti-transposition map fl plays the role of the Schützenberger involution [NY04].

For example, if $N|t \in \mathrm{Gr}(2,5) \times \mathbb{C}^\times$, then setting $P_J = P_J(N)$, we have (cf. (3.3))

$$S_t(N) = \begin{pmatrix} t\dfrac{P_{45}}{P_{34}} & 0 \\ t\dfrac{P_{35}}{P_{23}} & t\dfrac{P_{34}}{P_{23}} \\ t\dfrac{P_{25}}{P_{12}} & t\dfrac{P_{24}}{P_{12}} \\ 1 & \dfrac{P_{14}}{P_{15}} \\ 0 & 1 \end{pmatrix}.$$

By definition (and the fact that $\Delta_{[n-k+1,n],[k]}(g(N|t)) = 1$), the Plücker coordinates of $N' = S_t(N)$ are given by

$$\frac{P_J(N')}{P_{[n-k+1,n]}(N')} = \Delta_{[n-k+1,n],w_0(J)}(g(N|t)), \tag{3.20}$$

[2]Noumi and Yamada use the term "tropical" for what we call "geometric" or "rational," and the term "ultradiscretization" for what we call "tropicalization." This terminology is common in the literature coming from Japan.

where $w_0(J)$ is the subset obtained from J by replacing each $i \in J$ with $n-i+1$. In general, it is not so easy to express the right-hand side of (3.20) in terms of the Plücker coordinates of N. When J is a basic subset, however, there is a simple expression. Recall from §2.5 the notation $J_{i,j} = [i,j] \cup [n-k+j-i+2, n]$ for basic subsets, and $U_k \subset \mathrm{Gr}(k,n)$ for the open subset where the basic Plücker coordinates do not vanish.

Lemma 3.14. *Suppose $N|t \in \mathbb{X}_k$, and $N'|t = S(N|t)$. If $N \in U_k$, then so is N', and the basic Plücker coordinates of N' are given by*

$$\frac{P_{J_{i,j}}(N')}{P_{[n-k+1,n]}(N')} = t^{\min(j,n-k)-i+1} \frac{P_{J_{n-k-i+2,n-j}}(N)}{P_{[n-j-k+1,n-j]}(N)}. \tag{3.21}$$

Proof. Set $A = g(N|t)$, and fix a basic subset $J_{i,j}$. Choose a and b so that $w_0(J_{i,j}) = [1,a] \cup [b+a+1, b+k]$ (explicitly, $a = k-j+i-1$ and $b = n-i-k+1$). Consider the $k \times k$ submatrix of A using the rows $[b+1, b+k]$ and the columns $[1,a] \cup [b+a+1, b+k]$. The last $k-a$ columns of this submatrix consist of a rows of zeroes followed by a lower triangular $(k-a) \times (k-a)$ block, so

$$\begin{aligned}
\Delta_{[b+1,b+k],[1,a]\cup[b+a+1,b+k]}(A) &= \Delta_{[b+1,b+a],[1,a]}(A) \prod_{r=b+a+1}^{b+k} A_{rr} \\
&= \Delta_{[b+1,b+a],[1,a]}(A) \prod_{r=b+a+1}^{b+k} t^{c_r} \frac{P_{[r-k+1,r]}(N)}{P_{[r-k,r-1]}(N)}
\end{aligned}$$

where $c_r = 0$ if $r \le k$, and $c_r = 1$ if $r > k$. Using (2.19) and canceling terms in the product, we obtain

$$\Delta_{[b+1,b+k],[1,a]\cup[b+a+1,b+k]}(A) = t^{\min(k-a,b)} \frac{P_{J_{b+1,b+a}}(N)}{P_{[n-k+1,n]}(N)} \frac{P_{[b+1,b+k]}(N)}{P_{[b+a+1-k,b+a]}(N)}. \tag{3.22}$$

Note that the Plücker coordinates appearing here are nonzero because $N \in U_k$.

Let $A_t = A|_{\lambda=(-1)^{k-1}t}$. By Proposition 3.7, all the columns of A_t are in the span of the first k columns (which is N), so if a single minor using a given set of k columns is nonzero, then those k columns also span the subspace N. Thus, the non-vanishing and λ-independence of the right-hand side of (3.22) implies that columns $[1,a] \cup [b+a+1, b+k]$ of A_t span N, so we have

$$\frac{\Delta_{[n-k+1,n],[1,a]\cup[b+a+1,b+k]}(A_t)}{\Delta_{[b+1,b+k],[1,a]\cup[b+a+1,b+k]}(A_t)} = \frac{P_{[n-k+1,n]}(N)}{P_{[b+1,b+k]}(N)}. \tag{3.23}$$

Both minors appearing in the left-hand side of (3.23) are independent of λ, so this equation still holds if we replace A_t with A. The lemma follows from combining (3.20), (3.22), and (3.23), and replacing a, b with $k-j+i-1, n-i-k+1$, respectively. $\square$

Specializing (3.21) to the case of cyclic Plücker coordinates, we obtain

$$\frac{P_{[i,i+k-1]}(N')}{P_{[n-k+1,n]}(N')} = t^{|[i,i+k-1]\cap[n-k]|}\frac{P_{[n-k+1,n]}(N)}{P_{[n-i-2k+2,n-i-k+1]}(N)} \tag{3.24}$$

for $i \in \mathbb{Z}/n\mathbb{Z}$. This shows that when $N \in \mathrm{Gr}^\circ(k,n)$, $S_t(N) \in \mathrm{Gr}^\circ(k,n)$ as well, so $g \circ S(N|t)$ is defined.

Proposition 3.15. *Suppose* $(N_1|t_1,\ldots,N_d|t_d) \in \mathbb{X}_{k_1,\ldots,k_d}$*, with each* $N_j \in \mathrm{Gr}^\circ(k_j,n)$*. Then*

$$g \circ S(N_1|t_1,\ldots,N_d|t_d) = \mathrm{fl} \circ g(N_1|t_1,\ldots,N_d|t_d).$$

Proof. Since fl is an anti-automorphism, it suffices to prove the $d=1$ case. Suppose

$$N|t \overset{g}{\mapsto} A \overset{\mathrm{fl}}{\mapsto} A' \overset{\pi_t^k}{\mapsto} N'|t.$$

We must show that $g(N'|t) = A'$. By definition, the first k columns of A' are the diagonal form representative of N' (note that the first k columns of A' do not depend on λ). By Proposition 3.7(1), the first k columns of $g(N'|t)$ are also the diagonal form representative of N', so the first k columns of $g(N'|t)$ and A' agree. It remains to consider the last $n-k$ columns.

We claim that for $i \leq n-k$, we have

$$A_{ij} = d_{ij}\frac{\Delta_{[n-k+1,n],\{j\}\cup[i+1,i+k-1]}(A)}{\Delta_{[n-k+1,n],[i+1,i+k]}(A)}, \qquad d_{ij} = \begin{cases} t & \text{if } j \leq i \\ \lambda & \text{if } j > i. \end{cases} \tag{3.25}$$

This is clearly true when $j \in [i+1,i+k]$. Let $A_t = g(N|t)|_{\lambda=(-1)^{k-1}t}$. By Proposition 3.7(2), all size $(k+1)$-minors of A_t vanish. For $j \leq i$, expand the minor $\Delta_{\{i\}\cup[n-k+1,n],\{j\}\cup[i+1,i+k]}(A_t)$ along row i and use the fact that $(A_t)_{ir} = 0$ for $r = i+1,\ldots,i+k-1$, and $(A_t)_{i,i+k} = (-1)^{k-1}t$ to obtain

$$(A_t)_{ij}\Delta_{[n-k+1,n],[i+1,i+k]}(A_t) - t\Delta_{[n-k+1,n],\{j\}\cup[i+1,i+k-1]}(A_t) = 0. \tag{3.26}$$

There are no λ's in the last k rows of A, and $(A_t)_{ij} = A_{ij}$ for $j \leq i$, so we may replace A_t by A in (3.26). By (3.20), the minor $\Delta_{[n-k+1,n],[i+1,i+k]}(A)$ is a ratio of cyclic Plücker coordinates of N', which are nonzero by (3.24). Thus, (3.26) implies the $j \leq i$ case of (3.25).

For $j > i+k$, the same reasoning gives

$$t\Delta_{[n-k+1,n],[i+1,i+k-1]\cup\{j\}}(A_t) + (-1)^k(A_t)_{ij}\Delta_{[n-k+1,n],[i+1,i+k]}(A_t) = 0,$$

and since $(A_t)_{ij} = \dfrac{(-1)^{k-1}t}{\lambda}A_{ij}$ when $j > i+k$, (3.25) holds in this case as well.

Now (3.20) and (3.25) imply that

$$A'_{ij} = A_{n-j+1,n-i+1} = d_{n-j+1,n-i+1}\frac{P_{\{i\}\cup[j-k+1,j-1]}(N')}{P_{[j-k,j-1]}(N')} = g(N'|t)_{ij}$$

for $j \geq k+1$, which completes the proof. □

Note that as an immediate consequence of Proposition 3.15, we have

$$\Delta_{I,J}(g(S(N|t))) = \Delta_{w_0(J),w_0(I)}(g(N|t)). \tag{3.27}$$

Corollary 3.16. *The map $S : \mathbb{X}_{k_1,\ldots,k_d} \to \mathbb{X}_{k_d,\ldots,k_1}$ satisfies the following identities of rational maps:*

$$S^2 = \mathrm{Id} \quad \textit{and} \quad S \circ \mathrm{PR} = \mathrm{PR}^{-1} \circ S.$$

Proof. By Proposition 3.15, (3.18), and (3.10), we have

$$g \circ S^2 = \mathrm{fl}^2 \circ g = g$$

and

$$g \circ S \circ \mathrm{PR} = \mathrm{fl} \circ g \circ \mathrm{PR} = \mathrm{fl} \circ \mathrm{sh} \circ g = \mathrm{sh}^{-1} \circ \mathrm{fl} \circ g = \mathrm{sh}^{-1} \circ g \circ S = g \circ \mathrm{PR}^{-1} \circ S.$$

If $d = 1$, then by Proposition 3.7(1), we may "project" both sides of these equations onto the first k columns to deduce the desired identities. The general case follows from the $d = 1$ case because S and PR act separately on each component of a product. □

Proposition 3.17. *The map S interacts with the geometric crystal structure on $\mathbb{X}_{k_1,\ldots,k_d}$ as follows:*

1. *$\gamma S = w_0\gamma$, where $w_0(z_1, \ldots, z_n) = (z_n, \ldots, z_1)$;*
2. *$\varepsilon_i S = \varphi_{n-i}$ and $\varphi_i S = \varepsilon_{n-i}$ for $i \in \mathbb{Z}/n\mathbb{Z}$;*
3. *$e_i^c S = S e_{n-i}^{c^{-1}}$ for $i \in \mathbb{Z}/n\mathbb{Z}$;*
4. *$fS = f$.*

Proof. The proof is very similar to that of Proposition 3.9. First note that by n-periodicity, fl acts on an unfolded matrix X by $\mathrm{fl}(X)_{ij} = X_{n-j+1,n-i+1}$, so for $X \in B^-$, we have

$$\varepsilon_i(\mathrm{fl}(X)) = \frac{\mathrm{fl}(X)_{i+1,i}}{\mathrm{fl}(X)_{i+1,i+1}} = \frac{X_{n-i+1,n-i}}{X_{n-i,n-i}} = \varphi_{n-i}(X),$$

and similarly $\varphi_i \circ \mathrm{fl} = \varepsilon_{n-i}$, and $\gamma \circ \mathrm{fl} = w_0\gamma$. We also have $\chi \circ \mathrm{fl} = \chi$ because fl preserves the diagonals of an unfolded matrix. Parts (1), (2), and (4) follow from the combination of these identities with (2.16) and Proposition 3.15 (as in the proof of Proposition 3.9).

Suppose $X \in B^-$, and set $X' = \mathrm{fl}(X)$. Using (2.15), part (2), and the fact that fl is an anti-automorphism which maps $\widehat{x}_i(a)$ to $\widehat{x}_{n-i}(a)$, we compute

$$\begin{aligned}\mathrm{fl}(e_i^c(X)) &= \mathrm{fl}\left(\widehat{x}_i\left(\frac{c-1}{\varphi_i(X)}\right) \cdot X \cdot \widehat{x}_i\left(\frac{c^{-1}-1}{\varepsilon_i(X)}\right)\right) \\ &= \widehat{x}_{n-i}\left(\frac{c^{-1}-1}{\varepsilon_i(X)}\right) \cdot X' \cdot \widehat{x}_{n-i}\left(\frac{c-1}{\varphi_i(X)}\right) \\ &= \widehat{x}_{n-i}\left(\frac{c^{-1}-1}{\varphi_{n-i}(X')}\right) \cdot X' \cdot \widehat{x}_{n-i}\left(\frac{c-1}{\varepsilon_{n-i}(X')}\right) = e_{n-i}^{c^{-1}}(\mathrm{fl}(X)).\end{aligned}$$

Now part (3) is proved in the same way as part (3) of Proposition 3.9. In the last step, one computes that if (3) holds for X and Y separately, then for $(x, y) \in X \times Y$,

$$e_i^c S(x,y) = (e_i^{c_1'} S(y), e_i^{c_2'} S(x)) = S(e_{n-i}^{c_2'^{-1}}(x), e_{n-i}^{c_1'^{-1}}(y))$$

where

$$c_1' = \frac{c\varphi_{n-i}(y) + \varepsilon_{n-i}(x)}{\varphi_{n-i}(y) + \varepsilon_{n-i}(x)}, \quad c_2' = \frac{\varphi_{n-i}(y) + \varepsilon_{n-i}(x)}{\varphi_{n-i}(y) + c^{-1}\varepsilon_{n-i}(x)}.$$

By Definition/Proposition 2.32, $e_{n-i}^{c^{-1}}(x,y) = (e_{n-i}^{c_2'^{-1}}(x), e_{n-i}^{c_1'^{-1}}(y))$, so (3) holds for $X \times Y$. □

The Plücker coordinates of $S_t(N)$ will appear frequently enough in later sections that we introduce the following notation for them:

$$Q_t^J(N) := P_{w_0(J)}(S_t(N)). \tag{3.28}$$

By Proposition 3.7(1) (resp., the definition of S), the Plücker coordinates $P_J(N)$ (resp., $Q_t^J(N)$) are the maximal minors of the first k columns (resp., last k rows) of $g(N|t)$. Since the bottom left $k \times k$ submatrix of $g(N|t)$ is upper uni-triangular, we have

$$\frac{P_J(N)}{P_{[n-k+1,n]}(N)} = \Delta_{J,[k]}(g(N|t)) \quad \text{and} \quad \frac{Q_t^J(N)}{Q_t^{[k]}(N)} = \Delta_{[n-k+1,n],J}(g(N|t)). \tag{3.29}$$

Proposition 3.15 allows us to express the entries of the matrix $g(N|t)$ in terms of the Plücker coordinates $Q_t^J(N)$.

Lemma 3.18. *We have*

$$g(N|t)_{ij} = c'_{ij} \frac{Q_t^{[i+1,i+k-1]\cup\{j\}}}{Q_t^{[i+1,i+k]}}, \qquad c'_{ij} = \begin{cases} 1 & \text{if } i > n-k \\ t & \text{if } i \le n-k \text{ and } i \ge j \\ \lambda & \text{if } i \le n-k \text{ and } i < j. \end{cases}$$

Proof. By Proposition 3.15, we have $g(N|t) = \mathrm{fl} \circ g \circ S(N|t)$, so

$$\begin{aligned} g(N|t)_{ij} &= (g \circ S(N|t))_{n-j+1,n-i+1} \\ &= c_{n-j+1,n-i+1} \frac{P_{[n-i-k+2,n-i] \cup \{n-j+1\}}(S_t(N))}{P_{[n-i-k+1,n-i]}(S_t(N))} \\ &= c_{n-j+1,n-i+1} \frac{Q_t^{[i+1,i+k-1] \cup \{j\}}(N)}{Q_t^{[i+1,i+k]}(N)}. \end{aligned}$$

Clearly $c_{n-j+1,n-i+1} = c'_{ij}$, so we are done. □

3.3.3 Duality

In §2.5, we introduced a non-degenerate bilinear form on $\mathbb{C}^n$ such that if $N \in \mathrm{Gr}(k,n)$, and $N^\perp \in \mathrm{Gr}(n-k,n)$ is the orthogonal complement of N with respect to this form, then

$$P_J(N) = P_{\overline{J}}(N^\perp),$$

where $\overline{J}$ is the complement $[n] \setminus J$ (Lemma 2.48). The map studied in this section is the composition of the map $N \mapsto N^\perp$ with the reversal of the standard basis of $\mathbb{C}^n$ and the geometric Schützenberger involution.

Let $T_{w_0} : \mathrm{Gr}(k,n) \to \mathrm{Gr}(k,n)$ be the automorphism induced by reversing the standard basis of $\mathbb{C}^n$. Explicitly, if N' is an $n \times k$ matrix representative for a subspace N, then $T_{w_0}(N)$ is the subspace represented by $Q_{w_0} \cdot N'$, where Q_{w_0} is the permutation matrix corresponding to the longest element of S_n. Note that $\Delta_{J,[k]}(T_{w_0}(N)) = (-1)^{k(k-1)/2} \Delta_{w_0(J),[k]}(N)$, so $P_J(T_{w_0}(N)) = P_{w_0(J)}(N)$ (as projective coordinates), where as above, $w_0(J)$ is the subset obtained by replacing each $i \in J$ with $n-i+1$.

Define $\mu : \mathrm{Gr}(k,n) \to \mathrm{Gr}(n-k,n)$ by

$$\mu(N) = T_{w_0}(N^\perp) = T_{w_0}(N)^\perp.$$

Slightly abusing notation, we also write μ for the map $\mathbb{X}_k \to \mathbb{X}_{n-k}$ which sends $N|t \mapsto T_{w_0}(N^\perp)|t$. For $J \subset [n]$, let $J^* = w_0(\overline{J})$, so that

$$P_J(\mu(N)) = P_{J^*}(N). \tag{3.30}$$

Extend μ to a map $\mathbb{X}_{k_1,\ldots,k_d} \to \mathbb{X}_{n-k_1,\ldots,n-k_d}$ by

$$\mu(x_1, \ldots, x_d) = (\mu(x_1), \ldots, \mu(x_d)).$$

Definition 3.19. Define the *duality map* $D : \mathbb{X}_{k_1,\ldots,k_d} \mapsto \mathbb{X}_{n-k_d,\ldots,n-k_1}$ by

$$D = S \circ \mu.$$

For $N \in \mathrm{Gr}(k,n)$ and $t \in \mathbb{C}^\times$, let $D_t(N) = S_t(\mu(N))$.

We have seen that the unipotent crystal map g intertwines PR with sh, and S with fl. We will now show that $g^{n-k} \circ D(N|t)$ is closely related to the inverse of $g^k(N|t)$ (we use superscripts on g in this section since there are multiple Grassmannians involved). We start by explicitly computing the inverse of $g^k(N|t)$. Define $h^k : \mathbb{X}_k \to B^-$ by $h^k(N|t) = B$, where B is the folded matrix given by

$$(3.31)\quad B_{ij} = (-1)^{i+j} c'_{ij} \frac{P_{[i-k,i]\setminus\{j\}}(N)}{P_{[i-k+1,i]}(N)}, \qquad c'_{ij} = \begin{cases} 1 & \text{if } i > k \\ t & \text{if } i \leq k \text{ and } i \geq j \\ (-1)^n \lambda & \text{if } i \leq k \text{ and } i < j. \end{cases}$$

When k is clear from context, we write h instead of h^k. For example, if $n = 5$ and $k = 3$, then writing $P_J = P_J(N)$, we have

$$h(N|t) = \begin{pmatrix} t\frac{P_{345}}{P_{145}} & 0 & -\lambda & \lambda\frac{P_{135}}{P_{145}} & -\lambda\frac{P_{134}}{P_{145}} \\ -t\frac{P_{245}}{P_{125}} & t\frac{P_{145}}{P_{125}} & 0 & -\lambda & \lambda\frac{P_{124}}{P_{125}} \\ t\frac{P_{235}}{P_{123}} & -t\frac{P_{135}}{P_{123}} & t\frac{P_{125}}{P_{123}} & 0 & -\lambda \\ -1 & \frac{P_{134}}{P_{234}} & -\frac{P_{124}}{P_{234}} & \frac{P_{123}}{P_{234}} & 0 \\ 0 & -1 & \frac{P_{245}}{P_{345}} & -\frac{P_{235}}{P_{345}} & \frac{P_{234}}{P_{345}} \end{pmatrix}.$$

Like g, h is defined for N in the open positroid cell $\mathrm{Gr}^\circ(k,n)$.

Lemma 3.20. *For $t \in \mathbb{C}^\times$ and $N \in \mathrm{Gr}^\circ(k,n)$, we have*

$$h^k(N|t) \cdot g^k(N|t) = (t + (-1)^k \lambda) \cdot Id.$$

Proof. All matrices in this proof are folded. Arguing as in the proof of Lemma 3.3(1), one sees that $h \circ \mathrm{PR} = \mathrm{sh} \circ h$. Thus, since sh is an automorphism, it suffices to prove that

$$(h(N|t) \cdot g(N|t))_{i1} = \delta_{i,1}(t + (-1)^k \lambda).$$

Set $B = h(N|t)$ and $A = g(N|t)$, and write $P_J = P_J(N)$ for the Plücker coordinates of N. By definition,

$$(3.32)\qquad (BA)_{i1} = \sum_\ell B_{i\ell} A_{\ell 1} = \sum_\ell (-1)^{i+\ell} c'_{i\ell} \frac{P_{[i-k,i]\setminus\{\ell\}}}{P_{[i-k+1,i]}} \frac{P_{[n-k+2,n]\cup\{\ell\}}}{P_{[n-k+1,n]}}.$$

If $i = 1$, then $B_{i\ell} A_{\ell 1} = 0$ unless $\ell \in \{1, n-k+1\}$, so we have

$$(BA)_{11} = t + (-1)^k \lambda.$$

If $i > 1$, then $c'_{i\ell}$ has the same value for all nonzero terms appearing in (3.32) (the value is t if $i \in [2,k]$ and 1 if $i > k$), so we have $BA_{i1} = 0$ by the Grassmann–Plücker relations (Proposition 2.41). ☐

In §2.5, we previously defined X^c to be the matrix obtained by replacing X_{ij} with $(-1)^{i+j}X_{ij}$. Extend this definition to unfolded matrices $X \in M_n^\infty(\mathbb{C})$. Note that if A is the folding of X, and we denote the folding of X^c by A^c, then A^c is obtained from A by multiplying the (i,j)-entry by $(-1)^{i+j}$ and replacing λ with $(-1)^n\lambda$, so $A^c_{ij} \neq (-1)^{i+j}A_{ij}$ if n is odd. Define inv on a folded matrix $A \in \mathrm{GL}_n(\mathbb{C}(\lambda))$ by

$$\mathrm{inv}(A) = \mathrm{adj}(A)^c = \mathrm{adj}(A^c),$$

where $\mathrm{adj}(A)$ is the adjoint of A (i.e., $\mathrm{adj}(A)_{ij} = (-1)^{i+j}\Delta_{[n]\setminus\{j\},[n]\setminus\{i\}}(A)$). Note that inv is an anti-automorphism which commutes with sh and fl, and preserves B^- and U.

Remark 3.21. We use the adjoint rather than the inverse in the definition of inv so that the matrix entries remain Laurent polynomials in λ.

Proposition 3.22. *Suppose* $(N_1|t_1,\ldots,N_d|t_d) \in \mathbb{X}_{k_1,\ldots,k_d}$, *with each* $N_j \in \mathrm{Gr}^\circ(k_j,n)$. *Then*

$$\beta \cdot g \circ D(N_1|t_1,\ldots,N_d|t_d) = \mathrm{inv} \circ g(N_1|t_1,\ldots,N_d|t_d),$$

where $\beta = \prod_{j=1}^d (t_j + (-1)^{k_j+n}\lambda)^{n-k_j-1}$.

Proof. Since inv is an anti-automorphism, it suffices to prove the $d = 1$ case. That is, we must show that for $N|t \in \mathbb{X}_k$, with $N \in \mathrm{Gr}^\circ(k,n)$, one has

$$\beta \cdot g^{n-k} \circ D(N|t) = \mathrm{inv} \circ g^k(N|t),$$

where $\beta = (t + (-1)^{k+n}\lambda)^{n-k-1}$.

Let $A = g^k(N|t)$. By Proposition 3.7(3), we have

$$\mathrm{adj}(A) \cdot A = \det(A) \cdot Id = (t + (-1)^k\lambda)^{n-k} \cdot Id.$$

Comparing with Lemma 3.20, we see that $\mathrm{adj}(A) = (t + (-1)^k\lambda)^{n-k-1} \cdot h^k(N|t)$, so

$$\mathrm{inv}(A)_{ij} = \beta \cdot c'_{ij} \frac{P_{[i-k,i]\setminus\{j\}}(N)}{P_{[i-k+1,i]}(N)}, \qquad c'_{ij} = \begin{cases} 1 & \text{if } i > k \\ t & \text{if } i \leq k \text{ and } i \geq j \\ \lambda & \text{if } i \leq k \text{ and } i < j \end{cases}. \tag{3.33}$$

Let $A' = g^{n-k} \circ D(N|t)$. Proposition 3.15 implies that $A' = \mathrm{fl} \circ g^{n-k} \circ \mu(N|t)$. Unraveling the definitions and using Lemma 2.48, we obtain

$$\begin{aligned} A'_{ij} &= c_{n-j+1,n-i+1} \frac{P_{[k-i+2,n-i] \cup \{n-j+1\}}(T_{w_0}(N^\perp))}{P_{[k-i+1,n-i]}(T_{w_0}(N^\perp))} \\ &= c_{n-j+1,n-i+1} \frac{P_{[i+1,i+n-k-1] \cup \{j\}}(N^\perp)}{P_{[i+1,i+n-k]}(N^\perp)} \\ &= c_{n-j+1,n-i+1} \frac{P_{[i-k,i] \setminus \{j\}}(N)}{P_{[i-k+1,i]}(N)} \end{aligned}$$

where

$$c_{ij} = \begin{cases} 1 & \text{if } j \leq n-k \\ t & \text{if } j > n-k \text{ and } i \geq j \\ \lambda & \text{if } j > n-k \text{ and } i < j. \end{cases}$$

Comparing with (3.33), we conclude that $\beta \cdot A' = \mathrm{inv}(A)$. □

Corollary 3.23. *The map $D : \mathbb{X}_{k_1,\ldots,k_d} \to \mathbb{X}_{n-k_d,\ldots,n-k_1}$ satisfies the following identities of rational maps:*

$$D^2 = \mathrm{Id} \qquad D \circ S = S \circ D = \mu \qquad D \circ \mathrm{PR} = \mathrm{PR} \circ D.$$

Proof. The proof that D commutes with S and PR has exactly the same form as the proof that S commutes with PR in Corollary 3.16 (the necessary ingredients are (3.10), Propositions 3.15 and 3.22, and the fact that inv commutes with fl and sh). The identity $S \circ D = \mu$ follows from the definition of D and the fact that S is an involution. This in turn implies $D \circ S = \mu$, which implies that $D^2 = D \circ S \circ \mu = \mu^2 = \mathrm{Id}$. □

Proposition 3.24. *The map D interacts with the geometric crystal structure on $\mathbb{X}_{k_1,\ldots,k_d}$ as follows:*

1. *If $x = (N_1|t_1, \ldots, N_d|t_d)$ and $\gamma(x) = (z_1, \ldots, z_n)$, then $\gamma D(x) = t_1 \cdots t_d \gamma(x)^{-1}$;*
2. *$\varepsilon_i D = \varphi_i$ and $\varphi_i D = \varepsilon_i$ for $i \in \mathbb{Z}/n\mathbb{Z}$;*
3. *$e_i^c D = D e_i^{c^{-1}}$ for $i \in \mathbb{Z}/n\mathbb{Z}$;*
4. *$fD = f$.*

Proof. Suppose $A \in B^-$. We claim that

$$\varepsilon_i(\mathrm{inv}(A)) = \varphi_i(A) \qquad \text{and} \qquad \varphi_i(\mathrm{inv}(A)) = \varepsilon_i(A). \tag{3.34}$$

Since sh commutes with inv, $\varepsilon_{i-1} = \varepsilon_i \circ \mathrm{sh}$, and $\varphi_{i-1} = \varphi_i \circ \mathrm{sh}$, it suffices to prove these identities for $i = 1$. Let $A' = \mathrm{inv}(A)$, and let X, X' be the unfolded matrices corresponding to the folded matrices A, A', respectively. For $i, j \in [n]$, X'_{ij} is the constant coefficient of the polynomial $A'_{ij} = \Delta_{[n]\setminus\{j\},[n]\setminus\{i\}}(A|_{\lambda=(-1)^n\lambda})$. Since there are no negative powers of λ in the entries of A, we have

$$X'_{ij} = \Delta_{[n]\setminus\{j\},[n]\setminus\{i\}}(A|_{\lambda=0}).$$

Since $A|_{\lambda=0}$ is lower triangular and its (i,j)-entry is X_{ij}, we have

$$\varphi_1(X') = \frac{X'_{21}}{X'_{11}} = \frac{\Delta_{[2,n],\{1\}\cup[3,n]}(A|_{\lambda=0})}{\Delta_{[2,n],[2,n]}(A|_{\lambda=0})} = \frac{X_{21}X_{33}\cdots X_{nn}}{X_{22}X_{33}\cdots X_{nn}} = \varepsilon_1(X).$$

The first identity in (3.34) is proved similarly. Using (3.34), (2.15), the fact that inv is an anti-automorphism, and the fact that inv fixes $\widehat{x}_i(a)$ for each i, we obtain

$$\operatorname{inv}(e_i^c(A)) = e_i^{c^{-1}}(\operatorname{inv}(A)). \tag{3.35}$$

Now parts (2) and (3) are proved in the same way as parts (2) and (3) of Proposition 3.17 (the necessary ingredients are (3.34), (3.35), Proposition 3.22, and the observation that if $p(\lambda)$ is any polynomial in λ with nonzero constant term, then $\varepsilon_i(p(\lambda)\cdot A) = \varepsilon_i(A)$, and similarly for φ_i).

To prove (1) and (4), it suffices to consider the $d = 1$ case. Suppose $N|t \in \mathbb{X}_k$. Let $A = g(N|t)$ and $B = gD(N|t)$. and let $B = gD(N|t)$. The proof of Proposition 3.22 shows that the folded matrix B is given explicitly by

$$B_{ij} = c'_{ij}\frac{P_{[i-k,i]\setminus\{j\}}(N)}{P_{[i-k+1,i]}(N)}, \qquad c'_{ij} = \begin{cases} 1 & \text{if } i > k \\ t & \text{if } i \le k \text{ and } i \ge j \\ \lambda & \text{if } i \le k \text{ and } i < j \end{cases}.$$

Comparing with the definition of g, we see that $B_{ii} = t/A_{ii}$, and since $\gamma(N|t)$ is given by the diagonal entries of $g(N|t)$, (1) is proved. Similarly, the decoration f is defined by $f(N|t) = \chi(g(N|t))$. The matrices A and B are $(n-k)$- and k-shifted unipotent, respectively, and the entries along the $(n-k-1)^{th}$ diagonal of the unfolding of A are a reordering of the entries along the $(k-1)^{th}$ diagonal of the unfolding of B. Thus $\chi(A) = \chi(B)$, proving (4). □

We end this section with a result needed for the proof of the positivity of the geometric R-matrix in §5.4.

Lemma 3.25. *Suppose $N|t \in \mathbb{X}_k$. Let $A = g^k(N|t)$ and $A' = g^{n-k}(D(N|t))$, and view these as folded matrices. Then for $I, J \in \binom{[n]}{r}$, we have*

$$\Delta_{I,J}(A') = (t + (-1)^{n-k}\lambda)^{r-(n-k)}\Delta_{\overline{J},\overline{I}}(A|_{\lambda=(-1)^n\lambda}).$$

Proof. Let $C \in \mathrm{GL}_n(\mathbb{C}(\lambda))$ be a folded matrix, and suppose $I, J \in \binom{[n]}{r}$. Since $\mathrm{adj}(C) = \det(C)C^{-1}$ and $\mathrm{inv}(C)$ is obtained from $\mathrm{adj}(C)$ by scaling the i^{th} row and column by $(-1)^i$ and replacing λ with $(-1)^n\lambda$, (2.21) implies that

$$\Delta_{I,J}(\mathrm{inv}(C)) = \det(C|_{\lambda=(-1)^n\lambda})^{r-1}\Delta_{\overline{J},\overline{I}}(C|_{\lambda=(-1)^n\lambda}). \tag{3.36}$$

Set $\alpha = t + (-1)^{n-k}\lambda$. By Proposition 3.22, we have

$$g^{n-k} \circ D(N|t) = \frac{1}{\alpha^{n-k-1}} \,\mathrm{inv} \circ g^k(N|t).$$

Take the (I, J)-minor of both sides of this equation. Proposition 3.7(3) says that $\det(A|_{\lambda=(-1)^n\lambda}) = \alpha^{n-k}$, so by (3.36), we have

$$\Delta_{I,J}(A') = \frac{1}{\alpha^{(n-k-1)r}}\alpha^{(n-k)(r-1)}\Delta_{\overline{J},\overline{I}}(A|_{\lambda=(-1)^n\lambda}) = \alpha^{r-(n-k)}\Delta_{\overline{J},\overline{I}}(A|_{\lambda=(-1)^n\lambda}).$$

$\square$

CHAPTER 4

From geometry to combinatorics

4.1 The Gelfand–Tsetlin parametrization

4.1.1 Definition

Recall from §2.2.4 that a k-rectangle is an array of $k(n-k)+1$ nonnegative integers satisfying certain inequalities; k-rectangles parametrize the set of rectangular SSYTs with k rows. By replacing integers with nonzero complex numbers, we obtain a "rational version" of k-rectangles, as follows. Let

$$\mathbb{T}_k = (\mathbb{C}^\times)^{R_k} \times \mathbb{C}^\times$$

where $R_k = \{(i,j) \mid 1 \leq i \leq k,\ i \leq j \leq i+n-k-1\}$ as in §2.2.4. Denote a point of $\mathbb{T}_k$ by (X_{ij}, t), where (i,j) runs over R_k. We call (X_{ij}, t) a *rational k-rectangle.* Set

$$x_{ij} = X_{ij}/X_{i,j-1} \tag{4.1}$$

for $1 \leq i \leq k$ and $i \leq j \leq i+n-k$, where we set $X_{i,i-1} := 1$ and $X_{i,i+n-k} := t$. The quantity x_{ij} is the rational analogue of the number of j's in the i^{th} row of a tableau (cf. (2.9)). Note that there are no inequality conditions on rational k-rectangles.

We now introduce a parametrization of the variety $\mathbb{X}_k = \mathrm{Gr}(k,n) \times \mathbb{C}^\times$ by the set of rational $(n-k)$-rectangles.

Given $a, b \in [n]$ and $z_a, \ldots, z_b \in \mathbb{C}^\times$, define

$$M_{[a,b]}(z_a, \ldots, z_b) = \sum_{i \in [a,b]} z_i E_{ii} + \sum_{i \in [n] \setminus [a,b]} E_{ii} + \sum_{i \in [a+1,b]} E_{i,i-1} \tag{4.2}$$

where E_{ij} is an $n \times n$ matrix unit. For example, if $n = 5$, then

$$M_{[2,4]}(z_2, z_3, z_4) = \begin{pmatrix} 1 & & & & \\ & z_2 & & & \\ & 1 & z_3 & & \\ & & 1 & z_4 & \\ & & & & 1 \end{pmatrix} \tag{4.3}$$

where only nonzero entries are shown.

Definition 4.1.

1. Define $\Phi_{n-k} : \mathbb{T}_{n-k} \to \mathrm{GL}_n$ by

$$\Phi_{n-k}(X_{ij}, t) = \prod_{i=n-k}^{1} M_{[i,i+k]}(x_{ii}, x_{i,i+1}, \ldots, x_{i,i+k}),$$

   where x_{ij} is defined by (4.1), and the terms in the product are arranged from left to right in decreasing order of i. We call $\Phi_{n-k}(X_{ij}, t)$ a *tableau matrix.*

2. Define $\Theta_k : \mathbb{T}_{n-k} \to \mathbb{X}_k$ by $\Theta_k(X_{ij}, t) = N|t$, where N is the subspace spanned by the first k columns of the tableau matrix $\Phi_{n-k}(X_{ij}, t)$. We call Θ_k the *Gelfand–Tsetlin parametrization* of $\mathbb{X}_k$.

Example 4.2. Suppose $n = 5$ and $k = 2$. For $(X_{ij}, t) \in \mathbb{T}_3$, we have

$$\Phi_3(X_{ij}, t) = \begin{pmatrix} x_{11} & 0 & 0 & 0 & 0 \\ x_{22} & x_{12}x_{22} & 0 & 0 & 0 \\ x_{33} & (x_{12}+x_{23})x_{33} & x_{13}x_{23}x_{33} & 0 & 0 \\ 1 & x_{12}+x_{23}+x_{34} & x_{13}(x_{23}+x_{34}) & x_{24}x_{34} & 0 \\ 0 & 1 & x_{13} & x_{24} & x_{35} \end{pmatrix} \tag{4.4}$$

where x_{ij} is defined by (4.1). We have $\Theta_2(X_{ij}, t) = N|t$, where N is spanned by the first two columns of this tableau matrix.

Remark 4.3. The matrix $M_{[a,b]}\left(X_a, \frac{X_{a+1}}{X_a}, \ldots, \frac{X_{b-1}}{X_{b-2}}, \frac{1}{X_{b-1}}\right)$ has the factorization

$$x_{-a}(X_a)x_{-(a+1)}(X_{a+1}) \cdots x_{-(b-1)}(X_{b-1})$$

where

$$x_{-i}(z) = zE_{ii} + z^{-1}E_{i+1,i+1} + E_{i+1,i} + \sum_{j \neq i,i+1} E_{jj}.$$

Thus, the map Φ_{n-k} is a special case of Berenstein and Kazhdan's parametrization Θ_P^- of the variety $U\overline{w_P}U \cap B^- \times Z(L_P)$, where P is a parabolic subgroup of a reductive group G [BK07a, §3.1]. In our case, the reductive group is $\mathrm{PGL}_n(\mathbb{C})$,[1] P is the maximal parabolic subgroup corresponding to the k^{th} node of the type A_{n-1} Dynkin diagram, w_P is the Grassmannian permutation

$$w_P = \begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ n-k+1 & \cdots & n & 1 & \cdots & n-k \end{pmatrix},$$

and $Z(L_P)$ is the centralizer of the Levi subgroup of P (a one-dimensional sub-torus of the diagonal matrices in $\mathrm{PGL}_n(\mathbb{C})$).

[1] $\mathrm{PGL}_n(\mathbb{C})$ is the Langlands dual of $\mathrm{SL}_n(\mathbb{C})$.

We now state two important results about the maps Φ_{n-k} and Θ_k. The first result gives an explicit formula for the inverse of the map Θ_k. Recall the basic Plücker coordinates $J_{i,j} = [i,j] \cup [n-k+j-i+2, n]$ introduced in §2.5.

Proposition 4.4. *The map Θ_k is an open embedding of $\mathbb{T}_{n-k}$ into $\mathbb{X}_k$. The (rational) inverse is given by $N|t \mapsto (X_{ij}, t)$, where*

$$X_{ij} = \frac{P_{J_{i,j}}(N)}{P_{J_{i+1,j}}(N)} \tag{4.5}$$

for $1 \leq i \leq n-k$ and $i \leq j \leq i+k-1$.

The second result shows that Φ_{n-k} is closely related to the unipotent crystal map $g : \mathbb{X}_k \to B^-$. (In fact, the definition of the map g came out of our desire to "cyclically extend" Φ_{n-k}.)

Proposition 4.5. *Suppose $(X_{ij}, t) \in \mathbb{T}_{n-k}$. If $N|t = \Theta_k(X_{ij}, t)$, then we have*

$$g(N|t)|_{\lambda=0} = \Phi_{n-k}(X_{ij}, t).$$

Propositions 4.4 and 4.5 are proved using planar networks in §4.1.2.

4.1.2 Network representation and formulas for Plücker coordinates

In what follows, we freely use the constructions and results about planar networks from §2.6. Suppose $(X_{ij}, t) \in \mathbb{T}_{n-k}$, and let $x_{ij} = X_{ij}/X_{i,j-1}$ as in (4.1) (so $X_{i,i-1} := 1$ and $X_{i,i+k} := t$). Let $\Gamma_{k,n} = \Gamma_{k,n}(X_{ij}, t)$ be the planar network on the vertex set $\mathbb{Z}^2$ with

- n sinks labeled $1', \ldots, n'$, with the j^{th} sink located at $(0, j)$;
- n sources labeled $1, \ldots, n$, with the j^{th} source located at $(n-k, j-n+k)$;
- a vertical[2] arrow pointing from (i,j) to $(i-1,j)$ for $i = 1, \ldots, n-k$ and $j = 1, \ldots, k$. The weight of this edge is 1;
- a diagonal arrow pointing from $(i, j-i)$ to $(i-1, j-i+1)$ for $i = 1, \ldots, n-k$ and $j = 1, \ldots, n$. The weight of this edge is x_{ij} if $0 \leq j-i \leq k$, and 1 otherwise.

The network $\Gamma_{2,5}$ is shown in Figure 4, and $\Gamma_{3,5}$ appeared previously in Figure 3.

Lemma 4.6. *The matrix associated to $\Gamma_{k,n}(X_{ij}, t)$ is the tableau matrix $\Phi_{n-k}(X_{ij}, t)$.*

[2]We assume the network is drawn using the convention for matrix indices, that is, the first coordinate gives the vertical position, and increases from top to bottom; the second coordinate gives the horizontal position, and increases from left to right.

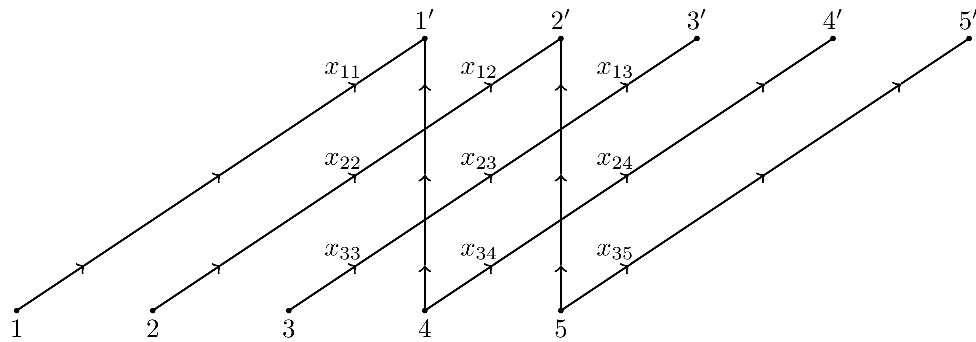


Figure 4: The network $\Gamma_{2,5}$. Unlabeled edges have weight 1.

Proof. By definition,

$$\Phi_{n-k}(X_{ij}, t) = \prod_{i=n-k}^{1} M_{[i,i+k]}(x_{ii}, x_{i,i+1}, \ldots, x_{i,i+k}). \tag{4.6}$$

It's easy to see that the i^{th} "row" from the top of $\Gamma_{k,n}$ (i.e., the part of the network where the first coordinate is between $i-1$ and i) is a network representation of the i^{th} factor in the right-hand side of (4.6). The full network $\Gamma_{k,n}$ is obtained by gluing these "rows" together, and the gluing of networks corresponds to multiplication of the associated matrices. □

Now suppose $N|t = \Theta_k(X_{ij}, t)$. By definition, N is the subspace represented by N', the matrix consisting of the first k columns of $\Phi_{n-k}(X_{ij}, t)$. By erasing everything to the right of the sink k' in $\Gamma_{k,n}$, we obtain a network representation of N'. We also contract the diagonal edges of weight 1 coming out of the first several sources to obtain a slightly more compact network (clearly this does not change the associated matrix), and we call the resulting network $\overline{\Gamma}_{k,n}$. The network $\overline{\Gamma}_{5,9}$ appears in Figure 5.

We will use the networks $\Gamma_{k,n}$ and $\overline{\Gamma}_{k,n}$ to deduce several properties of the maps Φ_{n-k} and Θ_k. We begin with a simple example. Recall that an $n \times k$ matrix is said to have diagonal form if its first k rows are lower triangular with nonzero entries on the main diagonal, and its last k rows are upper uni-triangular.

Lemma 4.7. *Let N' be the first k columns of $\Phi_{n-k}(X_{ij}, t)$. This matrix has diagonal form, and it does not depend on the parameter t.*

Proof. For $i = 1, \ldots, k$, there is a single path in $\overline{\Gamma}_{k,n}$ from source i to sink i', and there are no paths from from source i to source j' for $j > i$. Similarly, for $i = n-k+1, \ldots, n$, there is a single path of weight one from i to $(i-n+k)'$, and no paths from i to j' for $j < i-n+k$. Thus, N' has diagonal form.

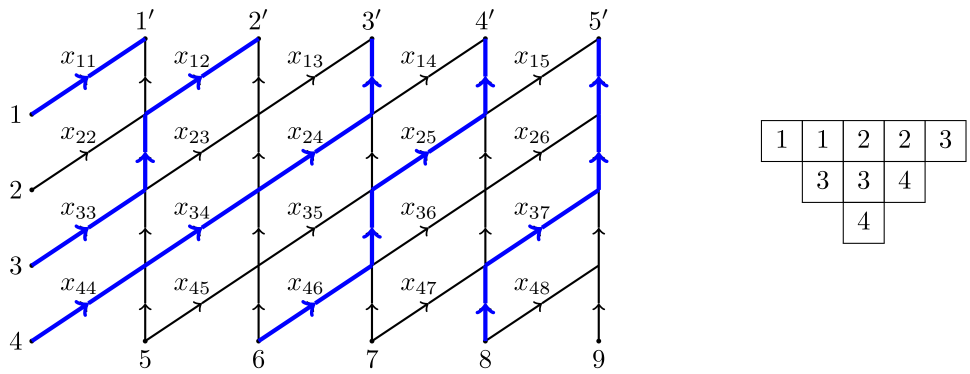


Figure 5: A vertex-disjoint family of paths in $\overline{\Gamma}_{5,9}$ that contributes to the Plücker coordinate P_{13468}, and the corresponding $\{1,3,4,6,8\}$-tableau.

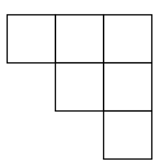
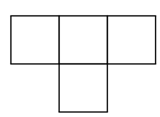
The second assertion follows from the fact that the edge weight x_{ij} only depends on t when $j = i + k$, and the edges of weight $x_{i,i+k}$ are erased in passing from $\Gamma_{k,n}$ to $\overline{\Gamma}_{k,n}$. □

We would like to have formulas for the Plücker coordinates of N (equivalently, for the maximal minors of the diagonal form representative N') in terms of the parameters X_{ij}. The Lindström Lemma expresses these minors as sums of monomials in the edge weights $x_{ij} = X_{ij}/X_{i,j-1}$, where the sum runs over vertex-disjoint families of paths in $\overline{\Gamma}_{k,n}$. We now introduce a combinatorial object that encodes these families of paths.

For $k \in [n-1]$, let

$$D_k = \{(a,b) \in \mathbb{Z}^2 \mid 1 \le a \le b \le k\}$$

be the shifted staircase of size k. We identify D_k with its "Young diagram," so that each point $(a,b) \in D_k$ corresponds to a box in row a and column b of the diagram. Given a subset $J = \{j_1 < j_2 < \cdots < j_k\} \in \binom{[n]}{k}$, let $D_{J,k}$ be the subset of D_k obtained by removing $j_r - n + k$ boxes from the bottom of column r, for each r such that $j_r > n - k$. For example, if $n = 8$, then

$D_3 =$  and $D_{\{4,5,7\},3} =$ .

Definition 4.8. Let $J = \{j_1 < \cdots < j_k\}$. A (J,k)-*tableau* is a map $T : D_{J,k} \to [n-k]$ satisfying the following three properties:

(a) $T(a,b) \le T(a,b+1)$ whenever $(a,b),(a,b+1) \in D_{J,k}$;

(b) $T(a,b) < T(a+1,b)$ whenever $(a,b),(a+1,b) \in D_{J,k}$;

(c) $T(a,a) = j_a$ if $j_a \leq n-k$.

We will often write J-tableau instead of (J,k)-tableau when k is understood.

Let X_{ij} and $x_{ij} = X_{ij}/X_{i,j-1}$ be as above. Define the *weight* of a J-tableau T by

$$\mathrm{wt}(T) = \prod_{(a,b)\in D_{J,k}} x_{T(a,b),T(a,b)+b-a}.$$

If $D_{J,k}$ is empty (i.e., if $J = [n-k+1, n]$), we define the weight of the unique (empty) J-tableau to be 1.

Note that properties (a) and (b) require the rows of T to weakly increase, and the columns to strictly increase.

Example 4.9. Let $n = 8, k = 3$. There are two $\{4,5,7\}$-tableaux, shown here with their weights:

$$\begin{array}{|c|c|c|}\hline 4 & 4 & 4 \\ \hline \multicolumn{1}{c|}{} & 5 & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array} \quad x_{44}x_{45}x_{46}x_{55} \qquad\qquad \begin{array}{|c|c|c|}\hline 4 & 4 & 5 \\ \hline \multicolumn{1}{c|}{} & 5 & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array} \quad x_{44}x_{45}x_{55}x_{56}.$$

Lemma 4.10. *Suppose $(X_{ij}, t) \in \mathbb{T}_{n-k}$, and $N|t = \Theta_k(X_{ij}, t)$. Then*

$$\frac{P_J(N)}{P_{[n-k+1,n]}(N)} = \sum_T \mathrm{wt}(T) \tag{4.7}$$

where the sum runs over all (J,k)-tableaux T.

Proof. The ratio $P_J(N)/P_{[n-k+1,n]}(N)$ is equal to the maximal minor $\Delta_J(N')$, where N' is the diagonal form representative of N. The matrix N' is represented by the network $\overline{\Gamma}_{k,n}$, so by the Lindström Lemma, $\Delta_J(N')$ is equal to the weighted sum over vertex-disjoint families of paths $p_1, \ldots, p_k$ in $\overline{\Gamma}_{k,n}$, where p_r starts at source j_r and ends at sink r'. Let $p_1, \ldots, p_r$ be a (not necessarily vertex-disjoint) family of paths such that p_r goes from source j_r to sink r'. The number of diagonal edges in p_r is, by definition, equal to the number of boxes in the r^{th} column of the diagram $D_{J,k}$. Define $T : D_{J,k} \to [n-k]$ by filling the r^{th} column of $D_{J,k}$ with the heights of the diagonal edges in the r^{th} path (in increasing order), where the height of the edge from (i,j) to $(i-1, j+1)$ is i. See Figure 5 for an example of a family of paths in $\overline{\Gamma}_{k,n}$ and the associated filling of $D_{J,k}$.

It's clear that the association $(p_r)_{1\leq r\leq k} \mapsto T$ is a bijection between (not necessarily vertex-disjoint) families of paths and fillings of $D_{J,k}$ satisfying properties (b) and (c) of Definition 4.8. It's also not hard to see that the rows of T are weakly increasing if and only if the family of paths is vertex-disjoint, and that the association is weight-preserving. This completes the proof. $\square$

Remark 4.11. A similar result, also using an object called J-tableau to record vertex-disjoint families of paths, appeared previously in work of Berenstein–Fomin–Zelevinsky [BFZ96, Proposition 2.6.7]. In that setting, J-tableaux are related to flag minors of an $n \times n$ matrix, rather than maximal minors of an $n \times k$ matrix.

Corollary 4.12. *Let $N|t = \Theta_k(X_{ij}, t)$.*

1. *For all $J \in \binom{[n]}{k}$, $P_J(N)/P_{[n-k+1,n]}(N)$ is a non-zero homogeneous polynomial of degree $|D_{J,k}|$ in the quantities $x_{ij} = X_{ij}/X_{i,j-1}$, with non-negative integer coefficients.*
2. *For a basic subset $J_{i,j} = [i,j] \cup [n-k+j-i+2, n]$, we have*

$$\frac{P_{J_{i,j}}(N)}{P_{[n-k+1,n]}(N)} = \prod_{a \in [i,j] \cap [n-k]} X_{aj}. \tag{4.8}$$

Proof. For any J, there is at least one J-tableau, namely, the tableau with all entries in row r equal to j_r. The weight of every J-tableau is a monomial of degree $|D_{J,k}|$, so (1) is proved.

Next, we claim that for each $J_{i,j}$, there is only one $J_{i,j}$-tableau. Indeed, the first entry in the a^{th} row of a $J_{i,j}$-tableau is required to be $i + a - 1$, and the lengths of the columns of $D_{J_{i,j},k}$ are weakly increasing, so every entry in the a^{th} row must be $i + a - 1$. The weight of this unique tableau is

$$\prod_{a=1}^{m} \prod_{b=a}^{j-i+1} x_{i+a-1,i+b-1} = \prod_{a'=i}^{m+i-1} X_{a'j}$$

where $m = \min(j - i + 1, n - k - i + 1)$. Since $[i, m + i - 1] = [i, j] \cap [n - k]$, we obtain (4.8). ☐

Now we are in position to prove Propositions 4.4 and 4.5.

Proof of Proposition 4.4. Recall that U_k is the subset of $\mathrm{Gr}(k, n)$ where the basic Plücker coordinates are nonzero. Define $\Psi_k : U_k \times \mathbb{C}^\times \to \mathbb{T}_{n-k}$ by $\Psi_k(N|t) = (Y_{ij}, t)$, where

$$Y_{ij} = \frac{P_{J_{i,j}}(N)}{P_{J_{i+1,j}}(N)}.$$

Suppose $(X_{ij}, t) \in \mathbb{T}_{n-k}$, and let $N|t = \Theta_k(X_{ij}, t)$. By part (2) of Corollary 4.12, the basic Plücker coordinates of N are monomials in the X_{ij} (so they are nonzero), and

$$\frac{P_{J_{i,j}}(N)}{P_{J_{i+1,j}}(N)} = \frac{\displaystyle\prod_{a \in [i,j] \cap [n-k]} X_{aj}}{\displaystyle\prod_{a \in [i+1,j] \cap [n-k]} X_{aj}} = X_{ij},$$

so $\Psi_k \circ \Theta_k = \mathrm{Id}$.

Now suppose $N \in U_k$. Set $N|t \overset{\Psi_k}{\mapsto} (X'_{ij}, t) \overset{\Theta_k}{\mapsto} N'|t$. Again by part (2) of Corollary 4.12, we have

$$\frac{P_{J_{i,j}}(N')}{P_{[n-k+1,n]}(N')} = \prod_{a \in [i,j] \cap [n-k]} X'_{aj} = \prod_{a \in [i,j] \cap [n-k]} \frac{P_{J_{a,j}}(N)}{P_{J_{a+1,j}}(N)}$$
$$= \frac{P_{J_{i,j}}(N)}{P_{J_{j+1,j}}(N)} = \frac{P_{J_{i,j}}(N)}{P_{[n-k+1,n]}(N)}.$$

This shows that N and N' have the same nonzero basic Plücker coordinates, so $N = N'$ by Lemma 2.45. Thus, $\Theta_k \circ \Psi_k = \mathrm{Id}$, and we are done. □

Proof of Proposition 4.5. Let $N|t = \Theta_k(X_{ij}, t)$, and let $A = g(N|t)|_{\lambda=0}$. By Lemma 4.7, the first k columns of $\Phi_{n-k}(X_{ij}, t)$ are the diagonal form representative of N. By Proposition 3.7(1) and inspection of the definition of g, the same is true of the first k columns of A. Thus, the first k columns of these matrices agree.

Both A and $\Phi_{n-k}(X_{ij}, t)$ are lower triangular, so it remains to consider the entries in positions (i, j), with $k < j \leq i$. First suppose $j = i$. In the network $\Gamma_{k,n}$, there is a single path from source j to sink j', and this path has weight

$$\frac{t}{X_{j-k,j-1}} \prod_{i \in [j-k+1,j] \cap [n-k]} \frac{X_{ij}}{X_{i,j-1}} = t\frac{P_{J_{j-k+1,j}}(N)}{P_{J_{j-k,j-1}}(N)} = t\frac{P_{[j-k+1,j]}(N)}{P_{[j-k,j-1]}(N)}$$

by Corollary 4.12(2) (note that $X_{j,j-1} = 1$). This shows that

$$\Phi_{n-k}(X_{ij}, t)_{jj} = t\frac{P_{[j-k+1,j]}(N)}{P_{[j-k,j-1]}(N)} = A_{jj}. \tag{4.9}$$

Now suppose $j < i$. We claim that

$$\Delta_{[j-k+1,j] \cup \{i\}, [1,k] \cup \{j\}}(\Phi_{n-k}(X_{ij}, t)) = 0. \tag{4.10}$$

To see this, observe that in $\Gamma_{k,n}$, there is exactly one vertex-disjoint family of paths from $[j-k+1, j]$ to $[1, k]$, and the k^{th} path in this family "blocks off" the only access to the sink j', so for any $i > j$, there is no way to add a path from i to j' which is vertex-disjoint from the other k paths. Thus, the determinant is zero by the Lindström Lemma.

The only nonzero entries in column j and rows $[j-k+1, j] \cup \{i\}$ of $\Phi_{n-k}(X_{ij}, t)$ are in rows j and i. Expand the determinant in (4.10) along the j^{th} column and use (4.9), along with the fact that the first k columns of $\Phi_{n-k}(X_{ij}, t)$ are the diagonal form representative of N, to get

$$0 = \Phi_{n-k}(X_{ij}, t)_{ij} \frac{P_{[j-k+1,j]}(N)}{P_{[n-k+1,n]}(N)} - \Phi_{n-k}(X_{ij}, t)_{jj} \frac{P_{[j-k+1,j-1] \cup \{i\}}(N)}{P_{[n-k+1,n]}(N)}$$
$$= \Phi_{n-k}(X_{ij}, t)_{ij} \frac{P_{[j-k+1,j]}(N)}{P_{[n-k+1,n]}(N)} - t\frac{P_{[j-k+1,j]}(N)}{P_{[j-k,j-1]}(N)} \frac{P_{[j-k+1,j-1] \cup \{i\}}(N)}{P_{[n-k+1,n]}(N)}.$$

This shows that $\Phi_{n-k}(X_{ij},t)_{ij} = t\dfrac{P_{[j-k+1,j-1]\cup\{i\}}(N)}{P_{[j-k,j-1]}(N)} = A_{ij}$, completing the proof. □

We end this section by proving the aforementioned result that the basic (resp., reflected basic) Plücker coordinates "positively generate" all Plücker coordinates.

Proof of Proposition 2.46. Suppose $N \in U_k$. By Proposition 4.4, we have $N|t = \Theta_k(X_{ij},t)$, where $X_{ij} = \dfrac{P_{J_{i,j}}(N)}{P_{J_{i+1,j}}(N)}$. Let N' be the diagonal form representative of N. We saw above that N' is the matrix associated to the network $\overline{\Gamma}_{k,n}(X_{ij},t)$. The edge weights of this network are ratios of the X_{ij}, which are themselves ratios of basic Plücker coordinates of N. Thus, by the Lindström Lemma, every minor of N' is a Laurent polynomial in the basic Plücker coordinates of N with non-negative integer coefficients. Since U_k is a dense subset of $\mathrm{Gr}(k,n)$, the same is true of the Plücker coordinates themselves (i.e., as rational functions on the affine cone over the Grassmannian).

To obtain the result for reflected basic Plücker coordinates, apply the automorphism $T_{w_0} : \mathrm{Gr}(k,n) \to \mathrm{Gr}(k,n)$ that was introduced at the beginning of §3.3.3. □

4.2 Positivity and tropicalization

4.2.1 Positive varieties and positive rational maps

We say that a rational function $h \in \mathbb{C}(z_1,\ldots,z_d)$ is *positive* if it can be expressed as a ratio of two nonzero polynomials in $z_1,\ldots,z_d$ whose coefficients are positive integers. We call such an expression a *positive expression*. We say that h is *non-negative* if it is either positive or zero. For example, $h = z_1^2 - z_1z_2 + z_2^2$ is positive because it has the positive expression $h = \dfrac{z_1^3+z_2^3}{z_1+z_2}$. (We remark that the term "subtraction-free" is often used in place of "positive.")

We say that a rational map $h = (h_1,\ldots,h_{d_2}) : (\mathbb{C}^\times)^{d_1} \to (\mathbb{C}^\times)^{d_2}$ is a *positive map of tori* (or simply *positive*) if each h_i is given by a positive element of $\mathbb{C}(z_1,\ldots,z_{d_1})$.

We now introduce a notion of positivity for rational maps between varieties more complicated than $(\mathbb{C}^\times)^d$. Our definition is a stripped-down version of the definition in [BK07a].

Definition 4.13. A *positive variety* is a pair (X,Θ_X), where X is an irreducible complex algebraic variety, and $\Theta_X : (\mathbb{C}^\times)^d \to X$ is a birational isomorphism. We say that Θ_X is a *parametrization* of X. When there is no danger of confusion, we refer to a positive variety by the name of its underlying variety.

Suppose (X, Θ_X) and (Y, Θ_Y) are positive varieties. A rational map $h : X \to Y$ is a *morphism of positive varieties* (or simply *positive*) if the rational map

$$\Theta h := \Theta_Y^{-1} \circ h \circ \Theta_X : (\mathbb{C}^\times)^{d_1} \to (\mathbb{C}^\times)^{d_2}$$

is a positive map of tori.

Remark 4.14. If $h : X \to Y$ and $g : Y \to Z$ are rational maps, then the composition $g \circ h$ is undefined if the image of h is disjoint from the domain of g. When we say that a composition of rational maps is positive, we implicitly guarantee that it is defined. For example, in the previous definition, h is not positive if the image of h is disjoint from the domain of Θ_Y^{-1}. One nice feature of positive rational maps is that their composition is always defined, by the following result.

Lemma 4.15. *The composition of positive rational maps is positive.*

Proof. Let $(X, \Theta_X), (Y, \Theta_Y), (Z, \Theta_Z)$ be positive varieties, and suppose $h : X \to Y$ and $g : Y \to Z$ are positive rational maps. This means that

$$\Theta h : (\mathbb{C}^\times)^{d_1} \to (\mathbb{C}^\times)^{d_2} \qquad \text{and} \qquad \Theta g : (\mathbb{C}^\times)^{d_2} \to (\mathbb{C}^\times)^{d_3}$$

are positive maps of tori. It's clear that Θh, being positive, is defined on all positive real points $(\mathbb{R}_{>0})^{d_1}$, and it maps these points into $(\mathbb{R}_{>0})^{d_2}$; similarly, Θg is defined on $(\mathbb{R}_{>0})^{d_2}$, so $\Theta g \circ \Theta h = \Theta(g \circ h)$ is defined. Clearly this map is also a positive map of tori, so $g \circ h$ is positive. □

If (X, Θ_X) and (Y, Θ_Y) are positive varieties, then $(X \times Y, \Theta_X \times \Theta_Y)$ is a positive variety, and if Z is another positive variety, it is easy to see that a rational map $(h_1, h_2) : Z \to X \times Y$ is positive if and only if h_1 and h_2 are positive.

The most basic example of a positive variety is the *d-dimensional torus* $((\mathbb{C}^\times)^d, \mathrm{Id})$. A more interesting example comes from the Gelfand–Tsetlin parametrization

$$\Theta_k : (\mathbb{C}^\times)^{k(n-k)} \times \mathbb{C}^\times \to \mathrm{Gr}(k, n) \times \mathbb{C}^\times$$

introduced in the preceding section. By definition, this map sends $(X_{ij}, t) \mapsto N|t$, and by Lemma 4.7, the subspace N only depends on the X_{ij}. We denote the map $(X_{ij}) \mapsto N$ by $\overline{\Theta}_k$, so that $\Theta_k = \overline{\Theta}_k \times \mathrm{Id}$. Proposition 4.4 shows that $\overline{\Theta}_k$ is a birational isomorphism, so the pair $(\mathrm{Gr}(k, n), \overline{\Theta}_k)$ is a positive variety. All positive varieties that we consider below will be products of tori and Grassmannians.

We now prove several necessary and sufficient conditions for rational maps to and from the Grassmannian to be positive.

Say that a rational function $h : \prod_j \mathrm{Gr}(k_j, n) \times (\mathbb{C}^\times)^d \to \mathbb{C}$ is *Plücker-positive* if it can be expressed as a ratio a/b, where a and b are nonzero polynomials with positive integer coefficients in the Plücker coordinates of the various Grassmannians, and the coordinates $z_1, \ldots, z_d$ of $(\mathbb{C}^\times)^d$. We call such an expression a/b a *Plücker-positive expression.* For example, the rational function $h = \dfrac{P_{13}P_{24} - P_{12}P_{34}}{P_{12}P_{34}}$ is Plücker-positive because it can be expressed as $h = \dfrac{P_{14}P_{23}}{P_{12}P_{34}}$ by a three-term Plücker relation.

It's clear that Plücker-positivity is equivalent to positivity for rational functions on $(\mathbb{C}^\times)^d$. In fact, the same is true for rational functions on $\prod_j \mathrm{Gr}(k_j, n) \times (\mathbb{C}^\times)^d$.

Lemma 4.16. *A rational function $h : \prod_j \mathrm{Gr}(k_j, n) \times (\mathbb{C}^\times)^d \to \mathbb{C}$ is positive (i.e., $\Theta h := h \circ (\prod_j \overline{\Theta}_{k_j} \times \mathrm{Id})$ is positive) if and only if it is Plücker-positive.*

Proof. We assume that h is a rational function on $\mathrm{Gr}(k, n)$ to simplify notation (the argument in the general case is the same). Let (X_{ij}) denote the coordinates on $(\mathbb{C}^\times)^{k(n-k)}$.

Suppose h is Plücker-positive. By Corollary 4.12(1), each Plücker coordinate of the subspace $\overline{\Theta}_k(X_{ij})$ is given by a positive rational function in the X_{ij}. By choosing a Plücker-positive expression for h and replacing the Plücker coordinates with these positive expressions in the X_{ij}, we obtain a positive expression for Θh, so h is positive.

Conversely, if h is positive, we may choose a positive expression for Θh in terms of the X_{ij}, and replace each X_{ij} with the ratio of Plücker coordinates in (4.5). This gives a Plücker-positive expression for $h \circ \overline{\Theta}_k \circ \overline{\Theta}_k^{-1} = h$. □

Lemma 4.17. *Let (X, Θ_X) be a positive variety, and let $h : X \to \mathrm{Gr}(k, n)$ be a rational map. The following are equivalent:*

1. *h is positive (i.e., $\overline{\Theta}_k^{-1} \circ h \circ \Theta_X$ is positive);*
2. *The rational functions $(P_J/P_I) \circ h : X \to \mathbb{C}^\times$ are positive for all basic k-subsets I, J;*
3. *$(P_J/P_I) \circ h$ is positive for all reflected basic k-subsets I, J;*
4. *$(P_J/P_I) \circ h$ is positive for all k-subsets I, J.*

We say that a map satisfying these equivalent conditions is Plücker-positive*.*

Proof. Conditions (2)-(4) are equivalent by Proposition 2.46. We now show the equivalence of (1) and (2). Suppose $(X'_{ij}) = \overline{\Theta}_k^{-1}(N)$. Proposition 4.4 shows that

$$\prod_{s=i}^{j} X'_{sj} = \frac{P_{J_{i,j}}(N)}{P_{J_{j+1,j}}(N)} = \frac{P_{J_{i,j}}(N)}{P_{[n-k+1,n]}(N)},$$

so positivity of $\overline{\Theta}_k^{-1} \circ h \circ \Theta_X$ implies positivity of $(P_{J_{i,j}}/P_{[n-k+1,n]}) \circ h \circ \Theta_X$ for all basic subsets $J_{i,j}$. Conversely, each X'_{ij} is a ratio of basic Plücker coordinates of N

(again by Proposition 4.4), so if $(P_J/P_I) \circ h \circ \Theta_X$ is positive for all basic subsets I and J, then $\overline{\Theta}_k^{-1} \circ h \circ \Theta_X$ is positive. □

Lemmas 4.16 and 4.17 show that for the varieties we consider, Plücker-positivity is equivalent to positivity. Thus, we will use the terms "Plücker-positive" and "positive" interchangeably from now on. As an application of this formalism, we show that the geometric crystal maps and symmetries on $\mathbb{X}_{k_1,\ldots,k_d}$ are positive.

Lemma 4.18. *Each of the rational maps $\gamma, \varepsilon_i, \varphi_i, e_i, f, \mathrm{PR}, S, D$ on $\mathbb{X}_{k_1,\ldots,k_d}$ is positive.*

Proof. First consider the $d = 1$ case. From the explicit formulas for $\gamma, \varepsilon_i, \varphi_i$ in §3.1, it's clear that these maps are (Plücker-)positive. The decoration f is positive by the formula in Definition 3.5, and PR is positive by (3.1). By Lemma 3.14, the basic Plücker coordinates of $S_t(N)$ are positive, so S is positive by Lemma 4.17. The map μ is positive by (3.30), so $D = S \circ \mu$ is positive as well by Lemma 4.15.

Now consider $e_i : \mathbb{C}^\times \times \mathbb{X}_k \to \mathbb{X}_k$. By Proposition 3.9(3) and the positivity of PR (and PR^{-1}), it suffices to prove that e_1 is positive. Suppose $e_1^c(N|t) = N'|t$. By the explicit description of e_i in §3.1, N' is obtained from N by adding a scalar multiple of the second row to the first row, so $P_J(N') = P_J(N)$ unless $1 \in J$ and $2 \notin J$. The only basic k-subset which contains 1 but not 2 is $J_{1,1} = \{1\} \cup [n-k+2, n]$, and using the formulas in §3.1, we compute

$$\begin{aligned} P_{\{1\}\cup[n-k+2,n]}(N') &= P_{\{1\}\cup[n-k+2,n]}(N) + \frac{c-1}{\varphi_1(N|t)} \cdot P_{\{2\}\cup[n-k+2,n]}(N) \\ &= P_{\{1\}\cup[n-k+2,n]}(N) + (c-1) \cdot P_{\{1\}\cup[n-k+2,n]}(N) \\ &= cP_{\{1\}\cup[n-k+2,n]}(N). \end{aligned}$$

We conclude that every basic Plücker coordinate of N' is positive, so e_1 is positive by lemma 4.17.

The $d > 1$ case is immediate for PR, S, and D; for $\gamma, \varepsilon_i, \varphi_i, e_i$, and f, it follows from the positivity of the explicit formulas in Definition/Proposition 2.32. □

Let $\Phi_{I,J}^k : \mathbb{T}_k \to \mathbb{C}$ be the rational function $(X_{ij}, t) \mapsto \Delta_{I,J}(\Phi_k(X_{ij}, t))$. The following technical result is needed in Chapter 5.

Lemma 4.19. *Let $I = \{i_1 < \cdots < i_r\}$ and $J = \{j_1 < \cdots < j_r\}$ be two r-subsets of $[n]$, with $r \leq n-k$. Then the rational function $\Phi_{I,J}^k$ is positive if*

$$i_s - k \leq j_s \leq i_s \qquad \text{for } s = 1, \ldots, r, \tag{4.11}$$

and zero otherwise.

Proof. Recall from §4.1 that the matrix $\Phi_k(X_{ij}, t)$ is represented by the planar network $\Gamma_{n-k,n}$ (see Figure 4 for an example of such a network). By the Lindström Lemma, $\Delta_{I,J}(\Phi_k(X_{ij}, t))$ is equal to the sum of the weights of the vertex-disjoint families of paths in $\Gamma_{n-k,n}$ from the sources in I to the sinks in J. Since the edge weights x_{ij} are ratios of the parameters X_{ij} and t, the function $\Phi^k_{I,J}$ is positive if there is at least one vertex-disjoint family of paths from I to J, and zero if there are no such families. Due to the ordering of the sources and sinks, a vertex-disjoint family of paths from I to J must have paths from i_s to j'_s for each s. There is a path from i_s to j'_s if and only if $i_s - k \leq j_s \leq i_s$, so (4.11) is a necessary condition for $\Phi^k_{I,J}$ to be nonzero.

Suppose I and J satisfy (4.11). We show that $\Phi^k_{I,J}$ is positive by constructing an explicit vertex-disjoint family of paths $p_1, \ldots, p_r$ from I to J. If $j_s = i_s$, then p_s is the unique path from i_s to i'_s. If $j_s < i_s$, set $a_s = \max(s, i_s - k)$, and let p_s be the unique path from i_s to j'_s whose vertical steps are on the line containing the sink a'_s. (Note that since there are no vertical steps on the lines containing the sinks $(n-k+1)', \ldots, n'$, the assumption $s \leq r \leq n-k$ is necessary to guarantee the existence of this path.) It is easy to verify that these paths are vertex-disjoint. □

4.2.2 Definition of tropicalization

Tropicalization is a procedure for turning positive[3] rational maps $(\mathbb{C}^\times)^{d_1} \to (\mathbb{C}^\times)^{d_2}$ into piecewise-linear maps $\mathbb{Z}^{d_1} \to \mathbb{Z}^{d_2}$ by replacing the operations $+, \cdot, \div$ with the operations $\min, +, -$, and ignoring constants. More formally, if

$$p = \sum c_{m_1,\ldots,m_d} z_1^{m_1} \cdots z_d^{m_d}$$

is a nonzero polynomial in $z_1, \ldots, z_d$ with positive integer coefficients, set

$$\mathrm{Trop}(p) = \min_{(m_1,\ldots,m_d)} \{m_1 z_1 + \ldots + m_d z_d\}.$$

Given a positive rational function $h \in \mathbb{C}(z_1, \ldots, z_d)$, define its *tropicalization* to be the piecewise-linear function from $\mathbb{Z}^d$ to $\mathbb{Z}$ given by

$$\mathrm{Trop}(h) = \mathrm{Trop}(p) - \mathrm{Trop}(q),$$

where $h = p/q$ is some expression of h as a ratio of polynomials with positive integer coefficients (this definition does not depend on the choice of p and q by, e.g., [BFZ96, Lemma 2.1.6]). For example,

$$\mathrm{Trop}\left(\frac{z_1^2 z_2 + z_3}{z_2^5 + 8z_1z_3 + 4}\right) = \min(2z_1 + z_2, z_3) - \min(5z_2, z_1 + z_3, 0).$$

[3]For a more general notion of tropicalization that removes the positivity assumption, see [BK07a, §4].

Given a positive map of tori $h = (h_1, \ldots, h_{d_2}) : (\mathbb{C}^\times)^{d_1} \to (\mathbb{C}^\times)^{d_2}$, define $\mathrm{Trop}(h)$ to be the piecewise-linear map $(\mathrm{Trop}(h_1), \ldots, \mathrm{Trop}(h_{d_2})) : \mathbb{Z}^{d_1} \to \mathbb{Z}^{d_2}$. If $h, g : (\mathbb{C}^\times)^{d_1} \to (\mathbb{C}^\times)^{d_2}$ are positive, then

$$\mathrm{Trop}(h+g) = \min(\mathrm{Trop}(h), \mathrm{Trop}(g)) \quad \text{and} \quad \mathrm{Trop}(hg^{\pm 1}) = \mathrm{Trop}(h) \pm \mathrm{Trop}(g).$$

Furthermore, tropicalization respects composition of positive maps.

Definition 4.20. Suppose (X, Θ_X) and (Y, Θ_Y) are positive varieties. If $h : X \to Y$ is a positive rational map, define its *tropicalization* $\widehat{h}$ by

$$\widehat{h} = \mathrm{Trop}(\Theta h) := \mathrm{Trop}(\Theta_Y^{-1} \circ h \circ \Theta_X).$$

4.3 Recovering the combinatorial crystals

By tropicalizing the rational maps associated to the geometric crystal $\mathbb{X}_{n-k}$, we obtain piecewise-linear maps on $\widetilde{\mathbb{T}}_k$. We will show that these piecewise-linear maps, when restricted to the set of k-rectangles inside $\widetilde{\mathbb{T}}_k$ (Definition 2.21), give formulas for the affine crystal structure on k-row rectangular tableaux. More generally, the tropicalizations of the maps on $\mathbb{X}_{n-k_1,\ldots,n-k_d}$ describe the crystal structure of the tensor product $\bigotimes_{j=1}^d \bigsqcup_L B^{k_j,L}$.

4.3.1 Tropicalizing the geometric crystal maps and symmetries

The first step is to show that the tropicalization of the decoration f is able to identify the set of k-rectangles inside $\widetilde{\mathbb{T}}_k$. Recall that $f : \mathbb{X}_{n-k} \to \mathbb{C}$ is defined by

$$f(N|t) = \sum_{i \neq n-k} \frac{P_{\{i-n+k\} \cup [i-n+k+2, i]}(N)}{P_{[i-n+k+1, i]}(N)} + t \frac{P_{[2,n-k] \cup \{n\}}(N)}{P_{[1,n-k]}(N)}, \tag{4.12}$$

and the decoration $f : \mathbb{X}_{n-k_1,\ldots,n-k_d} \to \mathbb{C}$ is the sum of the decorations on the individual factors. Using the notation of the previous section, we have the map

$$\Theta f = f \circ (\Theta_{n-k_1} \times \cdots \times \Theta_{n-k_d}) : \mathbb{T}_{k_1,\ldots,k_d} \to \mathbb{C},$$

and since this map is positive, we have its tropicalization

$$\widehat{f} = \mathrm{Trop}(\Theta f) : \widetilde{\mathbb{T}}_{k_1,\ldots,k_d} \to \mathbb{Z}.$$

Proposition 4.21. *Suppose $b_j \in \widetilde{\mathbb{T}}_{k_j}$ for $j = 1, \ldots, d$. Then $\widehat{f}(b_1, \ldots, b_d) \geq 0$ if and only if each b_j is a k_j-rectangle.*

The proof relies on the following technical result, which is proved in §4.3.3.

Lemma 4.22. *The map $\Theta f = f \circ \Theta_{n-k} : \mathbb{T}_k \to \mathbb{C}$ is given by the formula*

$$\Theta f(X_{ij}, t) = X_{kk} + \frac{t}{X_{1,n-k}} + \sum_{\substack{i\in[k] \\ j\in[i+1,i+n-k-1]}} \frac{X_{ij}}{X_{i,j-1}} + \sum_{\substack{i\in[k-1] \\ j\in[i,i+n-k-1]}} \frac{X_{ij}}{X_{i+1,j+1}}. \tag{4.13}$$

Proof of Proposition 4.21. Since $f(x_1, \dots, x_d) = f(x_1) + \dots + f(x_d)$, we have

$$\widehat{f}(b_1, \dots, b_d) = \min(\widehat{f}(b_1), \dots, \widehat{f}(b_d)),$$

so it suffices to consider the $d = 1$ case. Suppose $b = (B_{ij}, L) \in \widetilde{\mathbb{T}}_k$. By inspection of the defining inequalities of a Gelfand–Tsetlin pattern, it's clear that b is a k-rectangle if and only if the following inequalities are satisfied:

1. $B_{kk} \geq 0$
2. $L \geq B_{1,n-k}$
3. $B_{ij} \geq B_{i,j-1}$ for $i \in [k]$ and $j \in [i+1, i+n-k-1]$
4. $B_{ij} \geq B_{i+1,j+1}$ for $i \in [k-1]$ and $j \in [i, i+n-k-1]$.

Tropicalizing the formula (4.13) for Θf, we see that $\widehat{f}(b) \geq 0$ if and only if b satisfies these inequalities. □

By Lemma 4.18, the maps $\gamma, \varepsilon_i, \varphi_i, e_i$ and PR are positive. By tropicalizing, we obtain piecewise-linear maps

$$\widehat{\varepsilon}_i, \widehat{\varphi}_i : \widetilde{\mathbb{T}}_{k_1,\dots,k_d} \to \mathbb{Z} \qquad \widehat{\gamma} : \widetilde{\mathbb{T}}_{k_1,\dots,k_d} \to \mathbb{Z}^n \qquad \widehat{\mathrm{PR}} : \widetilde{\mathbb{T}}_{k_1,\dots,k_d} \to \widetilde{\mathbb{T}}_{k_1,\dots,k_d}$$

and

$$\widehat{e}_i : \mathbb{Z} \times \widetilde{\mathbb{T}}_{k_1,\dots,k_d} \to \widetilde{\mathbb{T}}_{k_1,\dots,k_d}.$$

Since e_i is an action of the multiplicative group $\mathbb{C}^\times$, $\widehat{e}_i$ is an action of the additive group $\mathbb{Z}$.

Theorem 4.23. *Let b_j be a k_j-rectangle for $j = 1, \dots, d$, and let $\mathbf{b} = b_1 \otimes \cdots \otimes b_d$. For $i \in \mathbb{Z}/n\mathbb{Z}$, we have*

1. $\widehat{\gamma}(\mathbf{b}) = \mathrm{wt}(\mathbf{b})$.
2. $\widehat{\varepsilon}_i(\mathbf{b}) = -\widetilde{\varepsilon}_i(\mathbf{b})$ *and* $\widehat{\varphi}_i(\mathbf{b}) = -\widetilde{\varphi}_i(\mathbf{b})$.
3. $\widetilde{e}_i(\mathbf{b})$ *is defined if and only if* $\widehat{f}(\widehat{e}_i(1, \mathbf{b})) \geq 0$*; in this case,* $\widehat{e}_i(1, \mathbf{b}) = \widetilde{e}_i(\mathbf{b})$.
4. $\widetilde{f}_i(\mathbf{b})$ *is defined if and only if* $\widehat{f}(\widehat{e}_i(-1, \mathbf{b})) \geq 0$*; in this case,* $\widehat{e}_i(-1, \mathbf{b}) = \widetilde{f}_i(\mathbf{b})$.

(Note that $\widehat{f}$ is the tropicalization of the decoration, whereas $\widetilde{f}_i$ is a crystal operator!)

The key to the proof of Theorem 4.23 is the following result, which is proved in §4.3.4.

Theorem 4.24. *If b is a k-rectangle, then* $\widehat{\mathrm{PR}}(b) = \widetilde{\mathrm{pr}}(b)$.

Remark 4.25. The map PR clearly has order n, so Theorem 4.24 gives a "birational" proof that $\widetilde{\mathrm{pr}}$ has order n on rectangular tableaux. Grinberg and Roby used a similar birational technique to prove an equivalent result [GR15].

Proof of Theorem 4.23. First assume $d = 1$. We prove each of these statements for $i = 1$, and then Proposition 2.8, Proposition 3.9, and Theorem 4.24 allow us to conjugate by PR at the geometric level and $\widetilde{\mathrm{pr}}$ at the combinatorial level to obtain the statements for all i. (In the case of γ, we show that the first coordinate of $\widehat{\gamma}(b)$ is equal to the first coordinate of $\mathrm{wt}(b)$.)

Let $b = (B_{ij}, L)$ be a k-rectangle, and let $N|t = \Theta_{n-k}(X_{ij}, t)$. By definition, the first coordinate of $\mathrm{wt}(b)$ is the number of 1's in the tableau corresponding to b, which is B_{11} (since 1's can only appear in the first row of a tableau). By the explicit formula for γ in §3.1 and Corollary 4.12(2), we see that the first coordinate of $\gamma(N|t)$ is equal to

$$\frac{P_{\{1\}\cup[k+2,n]}(N)}{P_{[k+1,n]}(N)} = X_{11}.$$

This proves (1).

For (2)-(4), we assume that $k \neq 1, n-1$ to avoid "boundary effects" (the reader may easily check the cases $k = 1, n-1$). By Corollary 4.12(2), we have

$$\varepsilon_1(N|t) = \frac{P_{\{2\}\cup[k+2,n]}(N)P_{\{1\}\cup[k+2,n]}(N)}{P_{[k+1,n]}(N)P_{\{1,2\}\cup[k+3,n]}(N)} = \frac{X_{11}}{X_{12}}$$

and

$$\varphi_1(N|t) = \frac{P_{\{2\}\cup[k+2,n]}(N)}{P_{\{1\}\cup[k+2,n]}(N)} = \frac{X_{22}}{X_{11}}.$$

Thus, we have $-\widehat{\varepsilon}_1(b) = B_{12} - B_{11}$ and $-\widehat{\varphi}_1(b) = B_{11} - B_{22}$, and (2) follows from comparison with Example 2.19.

Now suppose

$$N|t \overset{e_1^c}{\mapsto} N'|t \overset{\Theta_{n-k}^{-1}}{\mapsto} (X'_{ij}, t).$$

By Proposition 4.4, the X'_{ij} depend only on the basic Plücker coordinates of N', and it was shown in the proof of Lemma 4.18 that $P_{\{1\}\cup[k+2,n]}(N') = cP_{\{1\}\cup[k+2,n]}(N)$, and all other basic Plücker coordinates of N and N' are the same. Thus, the effect of $\Theta_{n-k}^{-1} \circ e_i^c \circ \Theta_{n-k}$ on (X_{ij}, t) is to replace X_{11} with cX_{11}, and to leave the other X_{ij} unchanged. This means that $\widehat{e}_1(m, b)$ adds m to B_{11}. Furthermore, (4.13) shows that if $\widehat{f}(b) \geq 0$, then $\widehat{f}(\widehat{e}_1(1,b)) \geq 0$ if and only if $B_{12} > B_{11}$, and $\widehat{f}(\widehat{e}_1(-1,b)) \geq 0$ if and only if $B_{11} > B_{22}$.

We saw in Example 2.19 that $\widetilde{e}_1(b)$ is not defined when $B_{12} = B_{11}$, and otherwise $\widetilde{e}_1(b)$ increases B_{11} by 1; similarly, $\widetilde{f}_1(b)$ is not defined when $B_{11} = B_{22}$, and otherwise $\widetilde{f}_1(b)$ decreases B_{11} by 1. This agrees with the description of $\widehat{e}_1(\pm 1, b)$ in the previous paragraph, so (3) and (4) are proved.

The result for general d follows from the $d = 1$ case by [BK07a, Proposition 6.7]. The point is that the formulas of Definition/Proposition 2.32 tropicalize to the formulas defining the tensor product of crystals. □

Finally, we consider the symmetries S and D. These maps are positive by Lemma 4.18, so we may tropicalize them to get piecewise-linear maps

$$\widehat{S} : \widetilde{\mathbb{T}}_k \to \widetilde{\mathbb{T}}_k \qquad \widehat{D} : \widetilde{\mathbb{T}}_k \to \widetilde{\mathbb{T}}_{n-k}.$$

Recall the symmetries $\widetilde{\mathrm{rot}}$ and $\widetilde{\mathrm{refl}}$ from §2.2.4.

Theorem 4.26. *If b is a k-rectangle, then $\widehat{S}(b) = \widetilde{\mathrm{rot}}(b)$ and $\widehat{D}(b) = \widetilde{\mathrm{refl}}(b)$.*

Proof. Write $J^r_{i,j}$ for the basic subset $[i,j] \cup [n-r+j-i+2, n]$ of size r.

Suppose $(X_{ij}, t) \in \mathbb{T}_k$, and let $N|t = \Theta_{n-k}(X_{ij}, t)$. Set

$$(X'_{ij}, t) = \Theta^{-1}_{n-k} \circ S(N|t) \qquad \text{and} \qquad (X''_{ij}, t) = \Theta^{-1}_k \circ D(N|t).$$

By Proposition 4.4 and Lemma 3.14, we have

$$X'_{ij} = \frac{P_{J^{n-k}_{i,j}}(S_t(N))}{P_{J^{n-k}_{i+1,j}}(S_t(N))} = \frac{t^{\min(j-i+1,k-i+1)} P_{J^{n-k}_{k-i+2,n-j}}(N)}{t^{\min(j-i,k-i)} P_{J^{n-k}_{k-i+1,n-j}}(N)} = \frac{t}{X_{k-i+1,n-j}}.$$

Tropicalizing this equality and comparing with the definition of $\widetilde{\mathrm{rot}}$, we see that $\widehat{S} = \widetilde{\mathrm{rot}}$. Similarly, using Proposition 4.4, Lemma 3.14, (3.30), and the fact that $(J^k_{i,j})^* = J^{n-k}_{k-j+i,n-j}$, we compute

$$X''_{ij} = \frac{P_{J^k_{i,j}}(S_t(\mu(N)))}{P_{J^k_{i+1,j}}(S_t(\mu(N)))} = t\frac{P_{J^k_{n-k-i+2,n-j}}(\mu(N))}{P_{J^k_{n-k-i+1,n-j}}(\mu(N))} = t\frac{P_{J^{n-k}_{j-i+2,j}}(N)}{P_{J^{n-k}_{j-i+1,j}}(N)} = \frac{t}{X_{j-i+1,j}}.$$

Tropicalizing and comparing with the definition of $\widetilde{\mathrm{refl}}$, we conclude that $\widehat{D} = \widetilde{\mathrm{refl}}$. □

Remark 4.27. The compatibility of $\widetilde{\mathrm{rot}}$ and $\widetilde{\mathrm{refl}}$ with the crystal operators on rectangular tableaux (Proposition 2.24) follows from this result, Theorem 4.23, and the compatibility of S and D with the geometric crystal operators (Propositions 3.17 and 3.24).

4.3.2 Examples

One-row tableaux

Let $(X_{1j}, t) = (X_{11}, \ldots, X_{1,n-1}, t)$ be an element of $\mathbb{T}_1$, and set $x_1 = X_{11}, x_j = X_{1j}/X_{1,j-1}$ for $j = 2, \ldots, n-1$, and $x_n = t/X_{1,n-1}$. We have

$$\Phi_1(X_{1j}, t) = \begin{pmatrix} x_1 & & & & & \\ 1 & x_2 & & & & \\ & 1 & x_3 & & & \\ & & & \ddots & & \\ & & & & x_{n-1} & \\ & & & & 1 & x_n \end{pmatrix}.$$

By definition, $\Theta_{n-1}(X_{1j}, t) = N|t$, where N is the $(n-1)$-dimensional subspace spanned by the first $n-1$ columns of $\Phi_1(X_{1j}, t)$. One easily computes

$$\frac{P_{[1,j]\cup[j+2,n]}(N)}{P_{[2,n]}(N)} = x_1 x_2 \cdots x_j = X_{1j}$$

for $j = 1, \ldots, n-1$, in agreement with Proposition 4.4.

Set $N'|t = \mathrm{PR}(N|t)$, and $(X'_{1j}, t) = \Theta_{n-1}^{-1}(N'|t) = \Theta\,\mathrm{PR}(X_{1j}, t)$. We have

$$X'_{1j} = \frac{P_{[1,j]\cup[j+2,n]}(N')}{P_{[2,n]}(N')} = t\frac{P_{[1,j-1]\cup[j+1,n]}(N)}{P_{[1,n-1]}(N)} = t\frac{X_{1,j-1}}{X_{1,n-1}} = x_n x_1 \cdots x_{j-1},$$

and thus in terms of the variables x_i, we have

$$\Theta\,\mathrm{PR}(x_1, \ldots, x_n) = (x_n, x_1, \ldots, x_{n-1}). \tag{4.14}$$

Now we compute Θe_0^c. Set $N''|t = e_0^c(N|t)$, and $(X''_{1j}, t) = \Theta_{n-1}^{-1}(N''|t) = \Theta e_0^c(X_{1j}, t)$. We have $\varphi_0(N|t) = t^{-1}\dfrac{P_{[1,n-1]}(N)}{P_{[2,n]}(N)} = t^{-1}\dfrac{x_1 \cdots x_{n-1}}{1}$, so identifying subspaces with their diagonal form representatives, we have

$$N'' = x_0\left(\frac{(-1)^{n-2}}{t}\frac{c-1}{\varphi_0(N|t)}\right) \cdot N = x_0\left((-1)^n \frac{c-1}{x_1 \cdots x_{n-1}}\right) \cdot N.$$

Left-multiplication by $x_0(a)$ means adding a times row 1 to row n, so we have

$$\Delta_{[2,n],[n-1]}(N'') = \Delta_{[2,n],[n-1]}(N) + \frac{c-1}{x_1 \cdots x_{n-1}}\Delta_{[n-1],[n-1]}(N) = c,$$

and the other maximal minors of N'' are equal to those of N. Thus, $X''_{1j} = c^{-1}X_{1j}$ for all j, so

$$\Theta e_0^c(x_1, \ldots, x_n) = (c^{-1}x_1, x_2, \ldots, x_{n-1}, cx_n). \tag{4.15}$$

Since conjugation by PR sends e_i^c to e_{i-1}^c, (4.14) and (4.15) imply that

$$\Theta e_i^c(x_1, \ldots, x_n) = (x_1, \ldots, cx_i, c^{-1}x_{i+1}, \ldots, x_n) \tag{4.16}$$

for all $i \in \mathbb{Z}/n\mathbb{Z}$ (this can also be computed directly, of course). Thus, we recover the affine geometric crystal structure on $(\mathbb{C}^\times)^n$ described in the introduction of [KOTY03]. Note that the actions of $\widetilde{\mathrm{pr}}$ and $\widetilde{e}_i, \widetilde{f}_i$ on a one-row tableau are indeed given by the tropicalizations of (4.14) and (4.16), where x_i is replaced with the number of i's in the tableau, and c is replaced with ± 1.

The case $n = 4$, $k = 2$

Let $(X_{ij}, t) = (X_{11}, X_{12}, X_{22}, X_{23}, t)$ be a rational 2-rectangle. Set $N|t = \Theta_2(X_{ij}, t)$, $N'|t = \mathrm{PR}(N|t)$, and $(X'_{ij}, t) = \Theta_2^{-1}(N'|t) = \Theta\,\mathrm{PR}(X_{ij}, t)$. We have

$$N = \begin{pmatrix} X_{11} & 0 \\ X_{22} & \dfrac{X_{12}}{X_{11}} X_{22} \\ 1 & \dfrac{X_{12}}{X_{11}} + \dfrac{X_{23}}{X_{22}} \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad N' = \begin{pmatrix} 0 & -t \\ X_{11} & 0 \\ X_{22} & \dfrac{X_{12}}{X_{11}} X_{22} \\ 1 & \dfrac{X_{12}}{X_{11}} + \dfrac{X_{23}}{X_{22}} \end{pmatrix},$$

so Proposition 4.4 gives

$$\begin{aligned} X'_{11} &= \frac{P_{14}(N')}{P_{34}(N')} = \frac{t}{X_{23}} & X'_{12} &= \frac{P_{12}(N')}{P_{24}(N')} = \frac{tX_{11}X_{22}}{X_{11}X_{23} + X_{12}X_{22}} \\ X'_{22} &= \frac{P_{24}(N')}{P_{34}(N')} = \frac{X_{11}X_{23} + X_{12}X_{22}}{X_{22}X_{23}} & X'_{23} &= \frac{P_{23}(N')}{P_{34}(N')} = \frac{X_{12}X_{22}}{X_{23}}. \end{aligned} \tag{4.17}$$

Now suppose $(B_{ij}, L) = (B_{11}, B_{12}, B_{22}, B_{23}, L) \in \widetilde{\mathbb{T}}_2$. Tropicalizing (4.17), we obtain $\widehat{\mathrm{PR}}(B_{ij}, L) = (B'_{ij}, L)$, where

$$\begin{aligned} B'_{11} &= L - B_{23} \\ B'_{12} &= L + B_{11} + B_{22} - \min(B_{11} + B_{23}, B_{12} + B_{22}) \\ B'_{22} &= \min(B_{11} + B_{23}, B_{12} + B_{22}) - B_{22} - B_{23} \\ B'_{23} &= B_{12} + B_{22} - B_{23}. \end{aligned} \tag{4.18}$$

We verify that these piecewise-linear formulas agree with the combinatorial rule for $\widetilde{\mathrm{pr}}$ for a particular tableau. Consider the following 2-row tableau T, and its corresponding 2-rectangle:

$$T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 & 4 & 4 & 4 \\ \hline \end{array} \qquad \longleftrightarrow \qquad \begin{matrix} & & 2 & \\ & 5 & & 1 \\ 6 & & 3 & \end{matrix}\ . \tag{4.19}$$

Using either Bender–Knuth involutions or jeu-de-taquin, one computes

(4.20) $$\widetilde{\mathrm{pr}}(T) = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 3 & 3 \\ \hline 2 & 3 & 3 & 4 & 4 & 4 \\ \hline \end{array} \qquad \longleftrightarrow \qquad \begin{array}{ccccc} & & 3 & & \\ & 4 & & 1 & \\ 6 & & 3 & & \end{array} .$$

The reader may verify that the 2-rectangle corresponding to $\widetilde{\mathrm{pr}}(T)$ agrees with the evaluation of the piecewise-linear formulas (4.18) on the 2-rectangle corresponding to T, in accordance with Theorem 4.24.

4.3.3 Proof of Lemma 4.22

Let $N|t = \Theta_{n-k}(X_{ij}, t)$, and let $P_J = P_J(N)$ denote the Plücker coordinates of N. The formula for Θf follows from (4.12) and the following formulas:

1. $\dfrac{P_{\{k\}\cup[k+2,n]}}{P_{[k+1,n]}} = X_{kk}$
2. $t\dfrac{P_{[2,n-k]\cup\{n\}}}{P_{[1,n-k]}} = \dfrac{t}{X_{1,n-k}}$
3. $\dfrac{P_{[1,r-k]\cup\{r\}\cup[r+2,n]}}{P_{[1,r-k]\cup[r+1,n]}} = \displaystyle\sum_{i=1}^{k} \frac{X_{i,i+r-k}}{X_{i,i+r-k-1}}$ for $r = k+1, \ldots, n-1$
4. $\dfrac{P_{\{r\}\cup[r+2,r+n-k]}}{P_{[r+1,r+n-k]}} = \displaystyle\sum_{j=1}^{n-k} \frac{X_{r,r+j-1}}{X_{r+1,r+j}}$ for $r = 1, \ldots, k-1$.

We now prove these formulas.

By Corollary 4.12(2), we have

$$\frac{P_{\{k\}\cup[k+2,n]}}{P_{[k+1,n]}} = \frac{X_{kk}}{1} \qquad \text{and} \qquad \frac{P_{[2,n-k]\cup\{n\}}}{P_{[1,n-k]}} = \frac{\displaystyle\prod_{a\in[2,n-k]\cap[k]} X_{a,n-k}}{\displaystyle\prod_{a\in[1,n-k]\cap[k]} X_{a,n-k}} = \frac{1}{X_{1,n-k}}$$

which gives (1) and (2).

For (3), let $J = [1, r-k] \cup \{r\} \cup [r+2, n]$, and let T be a J-tableau (see §4.1). The diagram $D_{J,n-k}$ has $r-k+1$ columns and $\min(r-k, k)$ rows, and the lengths of the first $r-k$ columns are weakly increasing. Since the first entry in the a^{th} row of T must be a, the first $r-k$ columns are completely determined. It remains to consider column $r-k+1$, which consists of a single box in the top row. The first $r-k$ boxes in the top row of T are filled with 1, so we may choose any element of $[k]$ for the last column. If we choose i, then the weight of T is $x_{i,i+r-k} \displaystyle\prod_{a\in[r-k]\cap[k]} X_{a,r-k}$.

By Corollary 4.12(2), we have $\dfrac{P_{[1,r-k]\cup[r+1,n]}}{P_{[k+1,n]}} = \displaystyle\prod_{a\in[r-k]\cap[k]} X_{a,r-k}$. Thus, Lemma 4.10 gives

$$\frac{P_{[1,r-k]\cup\{r\}\cup[r+2,n]}}{P_{[1,r-k]\cup[r+1,n]}} = \sum_{i=1}^{k} x_{i,i+r-k} = \sum_{i=1}^{k} \frac{X_{i,i+r-k}}{X_{i,i+r-k-1}}.$$

For (4), let $J = \{r\} \cup [r+2, r+n-k]$, and let T be a J-tableau. The diagram $D_{J,n-k}$ has $n-k$ columns and $\min(n-k, k-r)$ rows, and the column lengths are weakly increasing. For $a \geq 2$, the condition $T(a,a) = j_a = r+a$ implies that every entry in the a^{th} row of T must be $r+a$. There is some choice for the first row. The first entry must be r, but the other $n-k-1$ entries can be any weakly increasing sequence of r's and $r+1$'s. If the first row of T consists of r repeated b times and $r+1$ repeated $n-k-b$ times (for $1 \leq b \leq n-k$), then

$$\begin{aligned}
\mathrm{wt}(T) &= x_{rr}x_{r,r+1}\cdots x_{r,r+b-1}x_{r+1,r+b+1}\cdots x_{r+1,r+n-k} \prod_{a\in[r+2,r+n-k]\cap[k]} X_{a,r+n-k} \\
&= X_{r,r+b-1}\frac{X_{r+1,r+n-k}}{X_{r+1,r+b}} \prod_{a\in[r+2,r+n-k]\cap[k]} X_{a,r+n-k}.
\end{aligned}$$

Thus, using Lemma 4.10 for the numerator and Corollary 4.12(2) for the denominator, we have

$$\frac{P_{\{r\}\cup[r+2,r+n-k]}}{P_{[r+1,r+n-k]}} = \frac{\displaystyle\sum_{b=1}^{n-k} X_{r,r+b-1}\frac{X_{r+1,r+n-k}}{X_{r+1,r+b}} \prod_{a\in[r+2,r+n-k]\cap[k]} X_{a,r+n-k}}{\displaystyle\prod_{a\in[r+1,r+n-k]\cap[k]} X_{a,r+n-k}} = \sum_{b=1}^{n-k} \frac{X_{r,r+b-1}}{X_{r+1,r+b}}.$$

This concludes the proof.

4.3.4 Proof of Theorem 4.24

Recall from §2.2.2 that promotion is defined as the composition

$$\widetilde{\mathrm{pr}} = \widetilde{\sigma}_1\widetilde{\sigma}_2\cdots\widetilde{\sigma}_{n-1},$$

where $\widetilde{\sigma}_r$ is the r^{th} Bender–Knuth involution. Recall also the piecewise-linear formula for the action of a Bender–Knuth involution on a Gelfand–Tsetlin pattern from Lemma 2.20. Our strategy is to "detropicalize" this piecewise-linear formula to obtain "geometric Bender–Knuth involutions," and then to show that applying a sequence of these involutions to an element $(X_{ij}, t) \in \mathbb{T}_k$ has the same effect as applying $\Theta\,\mathrm{PR} = \Theta_{n-k}^{-1} \circ \mathrm{PR} \circ \Theta_{n-k}$ to (X_{ij}, t).

Let $(B_{ij}, L) \in \widetilde{\mathbb{T}}_k$ be a k-rectangle, and let $(B'_{ij}, L) = \widetilde{\sigma}_r(B_{ij}, L)$. By combining Lemma 2.20 with the "embedding" of a k-rectangle into its associated Gelfand–Tsetlin pattern (for an example, see (2.8)), we see that

$$B'_{ij} = \begin{cases} \widetilde{f}_{ir}(B_{ij}, L) + \widetilde{g}_{ir}(B_{ij}, L) - B_{ir} & \text{if } j = r \\ B_{ij} & \text{if } j \neq r \end{cases}$$

$$\text{where} \qquad \widetilde{f}_{ij}(B_{ij}, L) = \begin{cases} \min(B_{i-1,j-1}, B_{i,j+1}) & \text{if } i \neq 1 \text{ and } j \neq n-k-1+i \\ B_{i-1,j-1} & \text{if } i \neq 1 \text{ and } j = n-k-1+i \\ B_{i,j+1} & \text{if } i = 1 \text{ and } j \neq n-k \\ L & \text{if } i = 1 \text{ and } j = n-k \end{cases}$$

$$\text{and} \qquad \widetilde{g}_{ij}(B_{ij}, L) = \begin{cases} \max(B_{i,j-1}, B_{i+1,j+1}) & \text{if } i \neq k \text{ and } j \neq i \\ B_{i+1,j+1} & \text{if } i \neq k \text{ and } j = i \\ B_{i,j-1} & \text{if } i = k \text{ and } j \neq k \\ 0 & \text{if } i = k \text{ and } j = k. \end{cases}$$

Now we naively lift this piecewise-linear formula for $\widetilde{\sigma}_r$ to a rational map $\sigma_r : \mathbb{T}_k \to \mathbb{T}_k$. That is, given $(X_{ij}, t) \in \mathbb{T}_k$, define $\sigma_r(X_{ij}, t) = (X'_{ij}, t)$ by

$$X'_{ij} = \begin{cases} f_{ir}(X_{ij}, t) \cdot g_{ir}(X_{ij}, t) \cdot \dfrac{1}{X_{ir}} & \text{if } j = r \\ X_{ij} & \text{if } j \neq r \end{cases}$$

$$\text{where} \qquad f_{ij}(X_{ij}, t) = \begin{cases} X_{i-1,j-1} + X_{i,j+1} & \text{if } i \neq 1 \text{ and } j \neq n-k-1+i \\ X_{i-1,j-1} & \text{if } i \neq 1 \text{ and } j = n-k-1+i \\ X_{i,j+1} & \text{if } i = 1 \text{ and } j \neq n-k \\ t & \text{if } i = 1 \text{ and } j = n-k \end{cases}$$

$$\text{and} \qquad g_{ij}(X_{ij}, t) = \begin{cases} \dfrac{X_{i,j-1} X_{i+1,j+1}}{X_{i,j-1} + X_{i+1,j+1}} & \text{if } i \neq k \text{ and } j \neq i \\ X_{i+1,j+1} & \text{if } i \neq k \text{ and } j = i \\ X_{i,j-1} & \text{if } i = k \text{ and } j \neq k \\ 1 & \text{if } i = k \text{ and } j = k. \end{cases}$$

Define $\mathrm{pr} : \mathbb{T}_k \to \mathbb{T}_k$ by

$$\mathrm{pr} = \sigma_1 \sigma_2 \cdots \sigma_{n-1}.$$

Clearly $\mathrm{Trop}(\mathrm{pr}) = \widetilde{\mathrm{pr}}$, so to prove Theorem 4.24, it suffices to show that

$$\mathrm{pr} \circ \Theta_{n-k}^{-1} = \Theta_{n-k}^{-1} \circ \mathrm{PR} \tag{4.21}$$

as rational maps from $\mathrm{Gr}(n-k,n)\times\mathbb{C}^\times$ to $\mathbb{T}_k$. Given $N|t \in \mathrm{Gr}(n-k,n)\times\mathbb{C}^\times$, define X_{ij} by $(X_{ij},t) = \Theta_{n-k}^{-1}(N|t)$, and define X'_{ij} by $(X'_{ij},t) = \Theta_{n-k}^{-1}\circ \mathrm{PR}(N|t)$. Write $P_J = P_J(N)$ for the Plücker coordinates of N. By Proposition 4.4 and the definition of PR, we have

$$X_{ij} = \frac{P_{[i,j]\cup[k+j-i+2,n]}}{P_{[i+1,j]\cup[k+j-i+1,n]}} \qquad \text{and} \qquad X'_{ij} = t^{\delta_{i,1}}\frac{P_{[i-1,j-1]\cup[k+j-i+1,n-1]}}{P_{[i,j-1]\cup[k+j-i,n-1]}}.$$

Set $X_{ij}^{(n)} = X_{ij}$, and for $r = 1,\ldots,n-1$, define $X_{ij}^{(r)}$ by

$$(X_{ij}^{(r)},t) = \sigma_r(X_{ij}^{(r+1)},t) = \sigma_r\sigma_{r+1}\cdots\sigma_{n-1}(X_{ij},t).$$

In this notation, (4.21) is the equality $X_{ij}^{(1)} = X'_{ij}$ for all i,j. To prove this, we will show by descending induction on r that

$$X_{ij}^{(r)} = X'_{ij} \qquad \text{for } j = r, r+1, \ldots, n-1. \tag{4.22}$$

If $r = n$, then (4.22) is vacuously true. So suppose $1 \le r \le n-1$. Since σ_a only changes entries in the a^{th} row of the GT pattern, (4.22) holds for $j > r$ by induction, and we need only show that for each i, we have

$$f_{ir}(X_{ij}^{(r+1)},t)\cdot g_{ir}(X_{ij}^{(r+1)},t)\cdot\frac{1}{X_{ir}} = X'_{ir}. \tag{4.23}$$

By the induction hypothesis, the "neighborhood" of X_{ir} in the GT pattern $X_{ij}^{(r+1)}$ looks like

$$\begin{matrix} X_{i-1,r-1} & & X_{i,r-1} \\ & X_{ir} & \\ X'_{i,r+1} & & X'_{i+1,r+1} \end{matrix}.$$

Note that some or all of the NW, NE, and SE neighbors may be "missing," and the SW neighbor may be t. For instance, when $r = n-1$, the SW neighbor is t and the SE neighbor is missing.

We claim that

$$f_{ir}(X_{ij}^{(r+1)},t) = t^{\delta_{i,1}}\frac{P_{[i-1,r-1]\cup[k+r-i+1,n-1]}P_{[i,r]\cup[k+r-i+2,n]}}{P_{[i,r-1]\cup[k+r-i+1,n]}P_{[i,r]\cup[k+r-i+1,n-1]}} \tag{4.24}$$

and

$$g_{ir}(X_{ij}^{(r+1)},t) = \frac{P_{[i,r-1]\cup[k+r-i+1,n]}P_{[i,r]\cup[k+r-i+1,n-1]}}{P_{[i,r-1]\cup[k+r-i,n-1]}P_{[i+1,r]\cup[k+r-i+1,n]}}. \tag{4.25}$$

First we prove (4.24). If $1 < i \le k$ and $i \le r < n-k-1+i$, then the NW and SW neighbors of X_{ir} both exist, and we have

$$\begin{aligned}
f_{ir}(X_{ij}^{(r+1)}, t) &= X_{i-1,r-1} + X'_{i,r+1} \\
&= \frac{P_{[i-1,r-1]\cup[k+r-i+2,n]}}{P_{[i,r-1]\cup[k+r-i+1,n]}} + \frac{P_{[i-1,r]\cup[k+r-i+2,n-1]}}{P_{[i,r]\cup[k+r-i+1,n-1]}} \\
&= \frac{P_{[i-1,r-1]\cup[k+r-i+2,n]}P_{[i,r]\cup[k+r-i+1,n-1]} + P_{[i,r-1]\cup[k+r-i+1,n]}P_{[i-1,r]\cup[k+r-i+2,n-1]}}{P_{[i,r-1]\cup[k+r-i+1,n]}P_{[i,r]\cup[k+r-i+1,n-1]}} \\
&= \frac{P_{[i-1,r-1]\cup[k+r-i+1,n-1]}P_{[i,r]\cup[k+r-i+2,n]}}{P_{[i,r-1]\cup[k+r-i+1,n]}P_{[i,r]\cup[k+r-i+1,n-1]}}
\end{aligned}$$

where in the last step we apply a three-term Plücker relation (Corollary 2.42) to simplify the numerator. We have verified the "general case" of (4.24).

The three "boundary cases" of (4.24) are straightforward to verify: for instance, if $i = 1$ and $r < n-k$, then

$$f_{1r}(X_{ij}^{(r+1)}, t) = X'_{1,r+1} = t\frac{P_{[0,r]\cup[k+r+1,n-1]}}{P_{[1,r]\cup[k+r,n-1]}},$$

which agrees with the right-hand side of (4.24) (recall Convention 2.40). The other two boundary cases are similar, and are left to the reader.

Now we prove (4.25). If $1 \le i < k$ and $i < r \le n-k-1+i$, then the NE and SE neighbors of X_{ir} both exist, and we have

$$\begin{aligned}
g_{ir}(X_{ij}^{(r+1)}, t) &= \frac{X_{i,j-1}X'_{i+1,j+1}}{X_{i,j-1} + X'_{i+1,j+1}} \\
&= \frac{P_{[i,r-1]\cup[k+r-i+1,n]}P_{[i,r]\cup[k+r-i+1,n-1]}}{P_{[i,r-1]\cup[k+r-i+1,n]}P_{[i+1,r]\cup[k+r-i,n-1]} + P_{[i+1,r-1]\cup[k+r-i,n]}P_{[i,r]\cup[k+r-i+1,n-1]}} \\
&= \frac{P_{[i,r-1]\cup[k+r-i+1,n]}P_{[i,r]\cup[k+r-i+1,n-1]}}{P_{[i,r-1]\cup[k+r-i,n-1]}P_{[i+1,r]\cup[k+r-i+1,n]}}
\end{aligned}$$

where in the last step we apply a three-term Plücker relation (Corollary 2.42) to simplify the denominator. This verifies the "general case" of (4.25); we leave the three "boundary cases" to the reader.

Finally, observe that the denominator of (4.24) is equal to the numerator of (4.25), so we have

$$\begin{aligned}
&f_{ir}(X_{ij}^{(r+1)}, t) \cdot g_{ir}(X_{ij}^{(r+1)}, t) \cdot \frac{1}{X_{ir}} \\
&\quad = t^{\delta_{i,1}} \frac{P_{[i-1,r-1]\cup[k+r-i+1,n-1]}P_{[i,r]\cup[k+r-i+2,n]}}{P_{[i,r-1]\cup[k+r-i,n-1]}P_{[i+1,r]\cup[k+r-i+1,n]}} \cdot \frac{P_{[i+1,r]\cup[k+r-i+1,n]}}{P_{[i,r]\cup[k+r-i+2,n]}} \\
&\quad = t^{\delta_{i,1}} \frac{P_{[i-1,r-1]\cup[k+r-i+1,n-1]}}{P_{[i,r-1]\cup[k+r-i,n-1]}} \\
&\quad = X'_{ir}.
\end{aligned}$$

This verifies (4.23) and completes the induction, proving Theorem 4.24.

CHAPTER 5

Lifting the combinatorial R-matrix

5.1 The geometric R-matrix

5.1.1 Definition of R

Fix $\ell, k \in [n-1]$. Consider the unipotent crystals $(\mathbb{X}_\ell, g)$ and $(\mathbb{X}_k, g)$ introduced in §3.1; recall that their product is the unipotent crystal $(\mathbb{X}_\ell \times \mathbb{X}_k, g)$, where $g(u,v) = g(u)g(v)$ for $(u,v) \in \mathbb{X}_\ell \times \mathbb{X}_k$. Recall the geometric Schützenberger involution S and the "evaluation-projection" π_z^k from §3.3.2. Recall also Corollary 3.8, which plays a crucial role in the proofs below.

Define a rational map $\Psi_{k,\ell} : \mathbb{X}_\ell \times \mathbb{X}_k \to \mathbb{X}_k$ by

$$\Psi_{k,\ell}(M|s, N|t) = \pi_t^k \circ g(M|s, N|t).$$

Definition 5.1. The *geometric R-matrix* is the rational map $R : \mathbb{X}_\ell \times \mathbb{X}_k \to \mathbb{X}_k \times \mathbb{X}_\ell$ defined by

$$R = (\Psi_{k,\ell},\ S \circ \Psi_{\ell,k} \circ S).$$

More explicitly, if $R(M|s, N|t) = (N'|t, M'|s)$, then by Corollary 3.8(2) and Proposition 3.15, we have

(5.1)

$$N' = g(M|s)|_{\lambda=(-1)^{k-1}t} \cdot N \qquad \text{and} \qquad S_s(M') = \mathrm{fl}(g(N|t))|_{\lambda=(-1)^{\ell-1}s} \cdot S_s(M).$$

Remark 5.2. The formulas (5.1) show that N' is the image of N under a linear map that depends on M, s, and t, and $S_s(M')$ is the image of $S_s(M)$ under a linear map that depends on N, s, and t. We would very much like to have a geometric interpretation of these linear maps.

The two crucial results about R are the following.

Theorem 5.3. *The geometric R-matrix is positive.*

Theorem 5.4. *We have the identity $g \circ R = g$ of rational maps from $\mathbb{X}_\ell \times \mathbb{X}_k \to B^-$. That is, if $R(u, v) = (v', u')$ and $g(v'), g(u')$ are defined, then*

$$g(u)g(v) = g(v')g(u').$$

Theorems 5.3 and 5.4 are proved in §5.4 and §5.5, respectively.

Remark 5.5.

1. By Lemma 4.15, the positivity of R ensures that compositions such as Rg, RS, Re_i, R^2, etc., are defined (and positive).
2. To prove an equality of rational maps, it suffices to show that the equality holds on a dense subset. We will exploit this in §5.1.2.

Most of the important properties of R are direct consequences of Theorem 5.4. Here is an example.

Lemma 5.6. *We have the identity $R^2 = \mathrm{Id}$ of rational maps from $\mathbb{X}_\ell \times \mathbb{X}_k$ to itself.*

Proof. Suppose $(M|s, N|t) \in \mathbb{X}_\ell \times \mathbb{X}_k$, and

$$(M|s, N|t) \xrightarrow{R} (N'|t, M'|s) \xrightarrow{R} (M''|s, N''|t).$$

By Theorem 5.4, we have

$$M''|s = \pi_s^\ell(g(N'|t)g(M'|s)) = \pi_s^\ell(g(M|s)g(N|t)). \tag{5.2}$$

Corollary 3.8(1) ensures that the first ℓ columns of $g(M|s)g(N|t)|_{\lambda=(-1)^{\ell-1}s}$ are contained in the subspace M. On the other hand, (5.2) shows that these columns span the subspace M'', so we conclude that $M'' = M$.

Let p_1, p_2 be the projections of $\mathbb{X}_\ell \times \mathbb{X}_k$ onto the first and second factors, respectively. We have shown that $p_1 R^2 = p_1$. It's clear that $p_2 = Sp_1S$ and R commutes with S, so we have

$$p_2R^2 = Sp_1SR^2 = Sp_1R^2S = p_2$$

as well. □

Recall the notation $Q_t^J(N) := P_{w_0(J)}(S_t(N))$.

Corollary 5.7. *Suppose $M|s \in \mathbb{X}_\ell, N|t \in \mathbb{X}_k$, and $(N'|t, M'|s) = R(M|s, N|t)$. Let $B = g(M|s)g(N|t), B_s = B|_{\lambda=(-1)^{\ell-1}s}$, and $B_t = B|_{\lambda=(-1)^{k-1}t}$. For ℓ-subsets J and k-subsets I, we have*

$$\frac{P_I(N')}{P_{[n-k+1,n]}(N')} = \frac{\Delta_{I,[k]}(B_t)}{\Delta_{[n-k+1,n],[k]}(B_t)} \qquad \frac{Q_s^J(M')}{Q_s^{[\ell]}(M')} = \frac{\Delta_{[n-\ell+1,n],J}(B_s)}{\Delta_{[n-\ell+1,n],[\ell]}(B_s)} \tag{5.3}$$

$$\frac{P_J(M)}{P_{[n-\ell+1,n]}(M)} = \frac{\Delta_{J,[\ell]}(B_s)}{\Delta_{[n-\ell+1,n],[\ell]}(B_s)} \qquad \frac{Q_t^I(N)}{Q_t^{[k]}(N)} = \frac{\Delta_{[n-k+1,n],I}(B_t)}{\Delta_{[n-k+1,n],[k]}(B_t)}. \tag{5.4}$$

Proof. The equalities (5.3) follow from the definition of R and Proposition 3.15. The equalities (5.4) follow from Lemma 5.6, Theorem 5.4, and (5.3). □

5.1.2 Properties of R

For $\mathbf{k} = (k_1, \ldots, k_d) \in [n-1]^d$, set $\mathbb{X}_{\mathbf{k}} = \mathbb{X}_{k_1,\ldots,k_d} = \mathbb{X}_{k_1} \times \cdots \times \mathbb{X}_{k_d}$. For $i = 1, \ldots, d-1$, let $\sigma_i(\mathbf{k}) = (k_1, \ldots, k_{i+1}, k_i, \ldots, k_d)$, and let

$$R_i : \mathbb{X}_{\mathbf{k}} \to \mathbb{X}_{\sigma_i(\mathbf{k})}$$

be the map which acts as the geometric R-matrix on factors i and $i+1$, and as the identity on the other factors.

Say that a point $N|t \in \mathbb{X}_k$ is *positive* if $t > 0$, and $P_J(N) > 0$ for all J. Let $\mathbb{U}_{\mathbf{k}}$ be the subset of $\mathbb{X}_{\mathbf{k}}$ consisting of $(N_1|t_1, \ldots, N_d|t_d)$ such that each $N_i|t_i$ is positive, and the t_i are distinct. Note that g is defined on $\mathbb{U}_{\mathbf{k}}$, and since the geometric R-matrix is positive and involutive, each R_i is a bijection from $\mathbb{U}_{\mathbf{k}}$ to $\mathbb{U}_{\sigma_i(\mathbf{k})}$.

Lemma 5.8. *If* $(N_1|t_1, \ldots, N_d|t_d) \in \mathbb{U}_{\mathbf{k}}$*, then*

$$\pi_{t_1}^{k_1} \circ g(N_1|t_1, \ldots, N_d|t_d) = N_1|t_1.$$

In other words, the first k_1 *columns of the matrix* $g(N_1|t_1, \ldots, N_d|t_d)|_{\lambda=(-1)^{k_1-1}t_1}$ *span the subspace* N_1*.*

Proof. Let $B = g(N_1|t_1, \ldots, N_d|t_d)$, and $B_{t_1} = B|_{\lambda=(-1)^{k_1-1}t_1}$. By Corollary 3.8(1), the first k_1 columns of B_{t_1} are contained in the subspace N_1. Thus, it suffices to show that the first k_1 columns of B_{t_1} have full rank whenever $(N_1|t_1, \ldots, N_d|t_d) \in \mathbb{U}_{\mathbf{k}}$.

Let

$$(N'_2|t_2, \ldots, N'_d|t_d, N'_1|t_1) = R_{d-1} \circ \cdots \circ R_1(N_1|t_1, \ldots, N_d|t_d).$$

By repeated applications of Theorem 5.4, we have $B = g(N'_2|t_2, \ldots, N'_d|t_d, N'_1|t_1)$. Since the absolute values of the t_i are distinct, $g(N'_2|t_2, \ldots, N'_d|t_d)|_{\lambda=(-1)^{k_1-1}t_1}$ is invertible by Proposition 3.7(3), so the first k_1 columns of B_{t_1} have full rank by Corollary 3.8(2). □

Corollary 5.9. *Suppose* $(M_1|t_1, \ldots, M_d|t_d), (N_1|t_1, \ldots, N_d|t_d) \in \mathbb{U}_{\mathbf{k}}$*. If*

$$g(M_1|t_1, \ldots, M_d|t_d) = g(N_1|t_1, \ldots, N_d|t_d) \tag{5.5}$$

then $M_i = N_i$ *for each* i*.*

Proof. Lemma 5.8 shows that $M_1 = N_1$. By Proposition 3.7(3), the matrix $g(N_1|t_1)$ is invertible (in the ring $M_n(\mathbb{C}(\lambda))$), so we may multiply both sides of (5.5) by $g(N_1|t_1)^{-1}$ and reduce to a smaller value of d. □

Remark 5.10. Corollary 5.9 does not hold for arbitrary points in $\mathbb{X}_{\mathbf{k}}$, even in the case $n = 2, \mathbf{k} = (1, 1)$.

Theorem 5.11.

1. *$R : \mathbb{X}_{k_1} \times \mathbb{X}_{k_2} \to \mathbb{X}_{k_2} \times \mathbb{X}_{k_1}$ is an isomorphism of geometric crystals.*
2. *$R : \mathbb{X}_{k_1} \times \mathbb{X}_{k_2} \to \mathbb{X}_{k_2} \times \mathbb{X}_{k_1}$ commutes with the symmetries* $\mathrm{PR}, S,$ *and* D.
3. *R satisfies the Yang–Baxter relation. That is, we have the equality*

$$R_1 R_2 R_1 = R_2 R_1 R_2 \tag{5.6}$$

of rational maps $\mathbb{X}_{k_1,k_2,k_3} \to \mathbb{X}_{k_3,k_2,k_1}$.

Proof. First we prove (1). By Lemma 5.6, R is invertible, with inverse $R : \mathbb{X}_{k_2} \times \mathbb{X}_{k_1} \to \mathbb{X}_{k_1} \times \mathbb{X}_{k_2}$. Let ρ be one of the maps $\gamma, \varepsilon_i, \varphi_i$. By (2.16) and Theorem 5.4, we have

$$\rho R = \rho g R = \rho g = \rho.$$

It remains to show that R commutes with e_i. Again by (2.16) and Theorem 5.4, we have

$$gRe_i = ge_i = e_i g = e_i g R = g e_i R. \tag{5.7}$$

Suppose $\mathbf{x} = (N_1|t_1, N_2|t_2) \in \mathbb{U}_{k_1,k_2}$, and $c > 0$. Let

$$\mathbf{x}' = (N_2'|t_2, N_1'|t_1) = Re_i^c(\mathbf{x}),$$
$$\mathbf{x}'' = (N_2''|t_2, N_1''|t_1) = e_i^c R(\mathbf{x}).$$

Since R and e_i are positive maps, we have $\mathbf{x}', \mathbf{x}'' \in \mathbb{U}_{k_2,k_1}$, and by (5.7), we have $g(\mathbf{x}') = g(\mathbf{x}'')$. Thus, $Re_i^c(\mathbf{x}) = e_i^c R(\mathbf{x})$ by Corollary 5.9. Since the set of points

$$\{(c, \mathbf{x}) \,|\, c > 0 \text{ and } \mathbf{x} \in \mathbb{U}_{k_1,k_2}\}$$

is dense in $\mathbb{C}^\times \times \mathbb{X}_{k_1,k_2}$, we conclude that $Re_i = e_i R$.

The proof of (2) is formally the same as the proof of $Re_i = e_i R$, with (3.10) and Propositions 3.15, 3.22 playing the role of (2.16) (we also use the positivity of the three symmetries).

The proof of (3) is similar. Suppose $\mathbf{x} = (N_1|t_1, N_2|t_2, N_3|t_3) \in \mathbb{U}_{k_1,k_2,k_3}$, and set

$$\mathbf{x}' = (N_3'|t_3, N_2'|t_2, N_1'|t_1) = R_1 R_2 R_1(\mathbf{x}),$$
$$\mathbf{x}'' = (N_3''|t_3, N_2''|t_2, N_1''|t_1) = R_2 R_1 R_2(\mathbf{x}).$$

Theorem 5.4 implies that $g(\mathbf{x}') = g(\mathbf{x}) = g(\mathbf{x}'')$, and since $\mathbf{x}', \mathbf{x}'' \in \mathbb{U}_{k_3,k_2,k_1}$, we have $\mathbf{x}' = \mathbf{x}''$ by Corollary 5.9. Since $\mathbb{U}_{\mathbf{k}}$ is dense in $\mathbb{X}_{\mathbf{k}}$, this proves (5.6). $\square$

Remark 5.12. Lemma 5.6 and Theorem 5.11(3) show that the maps R_i satisfy the relations of the simple transpositions in the symmetric group. That is, if we repeatedly apply geometric R-matrices to consecutive factors in a product $\mathbb{X}_{k_1} \times \cdots \times \mathbb{X}_{k_d}$, the result depends only on the final permutation of factors; this means that for any permutation $\sigma \in S_d$, there is a well-defined map $R_\sigma : \mathbb{X}_{\mathbf{k}} \to \mathbb{X}_{\sigma(\mathbf{k})}$.

There is an efficient way to pick off the first and last factors of the image of a point in $\mathbb{U}_{\mathbf{k}}$ under R_σ. Indeed, if $\mathbf{x} = (x_1, \ldots, x_d) = (N_1|t_1, \ldots, N_d|t_d) \in \mathbb{U}_{\mathbf{k}}$ and $R_\sigma(\mathbf{x}) = (x'_{\sigma(1)}, \ldots, x'_{\sigma(d)})$, then we have

$$x'_{\sigma(1)} = \pi^{k_{\sigma(1)}}_{t_{\sigma(1)}} \circ g(\mathbf{x}), \qquad x'_{\sigma(d)} = S \circ \pi^{k_{\sigma(d)}}_{t_{\sigma(d)}} \circ \mathrm{fl} \circ g(\mathbf{x})$$

by Theorem 5.4, Lemma 5.8, Proposition 3.15, and the fact that S commutes with R.

5.1.3 Recovering the combinatorial R-matrix

By Theorem 5.4, the map $\Theta R : \mathbb{T}_{k_1} \times \mathbb{T}_{k_2} \to \mathbb{T}_{k_2} \times \mathbb{T}_{k_1}$ is positive, so we may define

$$\widehat{R} = \mathrm{Trop}(\Theta R) : \widetilde{\mathbb{T}}_{k_1} \times \widetilde{\mathbb{T}}_{k_2} \to \widetilde{\mathbb{T}}_{k_2} \times \widetilde{\mathbb{T}}_{k_1}.$$

Theorem 5.13. *If a is a k_1-rectangle and b is a k_2-rectangle, then $\widehat{R}(a \otimes b) = \widetilde{R}(a \otimes b)$, where $\widetilde{R}$ is the combinatorial R-matrix.*

Proof. By (3.8) and Theorem 5.4, we have $fR = f$, where f is the decoration. Thus, Proposition 4.21 and Theorems 4.23 and 5.11(1) imply that for any $L_1, L_2 \geq 0$, $\widehat{R}$ restricts to an affine crystal isomorphism $B^{k_1,L_1} \otimes B^{k_2,L_2} \to B^{k_2,L_2} \otimes B^{k_1,L_1}$. The combinatorial R-matrix is the unique such isomorphism, so we are done. □

Remark 5.14. Theorem 5.13 allows us to deduce the Yang–Baxter relation for the combinatorial R-matrix from the Yang–Baxter relation for the geometric R-matrix, thereby giving a new proof of the former.

Remark 5.15. We used the crystal-theoretic characterization of the combinatorial R-matrix to prove that R tropicalizes to $\widetilde{R}$. Here we outline an alternative proof based on the combinatorial characterization of $\widetilde{R}$ in terms of the tableau product (Proposition 2.12). The idea is that

the product of tableau matrices tropicalizes to the product of tableaux,

where the tableau matrix $\Phi_k(X_{ij}, t)$ is the $n \times n$ matrix from §4.1. To be a bit more precise, let a be a k_1-rectangle corresponding to the tableau T, and b a k_2-rectangle corresponding to the tableau U. Let (C_{ij}) be the Gelfand-Tsetlin pattern corresponding to the tableau $T * U$. Theorem 3.9 in [Fri17] states that the product of tableau matrices $\Phi_{k_1}(x)\Phi_{k_2}(y)$ uniquely determines positive rational functions $Z_{ij}(x,y)$ which tropicalize to formulas for C_{ij} in terms of the entries of a and b. (In fact, the Z_{ij} are ratios of left-justified minors of the product matrix.) If $\Theta R(x,y) = (y',x')$, then by Theorem 5.4 and Proposition 4.5 we have

$$\Phi_{k_1}(x)\Phi_{k_2}(y) = \Phi_{k_2}(y')\Phi_{k_1}(x'),$$

so the rectangular tableaux U' and T' obtained by tropicalizing y' and x' satisfy $U' * T' = T * U$.

The special case of [Fri17, Th. 3.9] where $k_1 = k_2 = 1$ was proved by Noumi and Yamada in their work on a geometric lift of the RSK correspondence [NY04]. The general case is proved by iterating the one-row case. The technical details take up a lot of space, however, and since we do not yet have an application for this result, we have chosen to omit the proof.

At the end of §5.3, we give explicit formulas for ΘR and $\widetilde{R}$ in a small example.

5.2 The geometric coenergy function

Recall that a $\mathbb{Z}$-valued function on a tensor product of two Kirillov–Reshetikhin crystals is a coenergy function if it is invariant under the crystal operators $\widetilde{e}_1, \ldots, \widetilde{e}_{n-1}$, and it interacts with $\widetilde{e}_0$ in a prescribed way (Definition 2.15). In this section, we "lift" the combinatorial definition to define a notion of geometric coenergy function. We show that a certain minor of the product matrix $g(M|s)g(N|t)$ defines a geometric coenergy function on $\mathbb{X}_{k_1} \times \mathbb{X}_{k_2}$, and that this function tropicalizes to the coenergy function $\widetilde{E}$ defined in §2.2.3.

Definition 5.16. A rational function $H : \mathbb{X}_{k_1} \times \mathbb{X}_{k_2} \to \mathbb{C}$ is a *geometric coenergy function* if $H \circ e_i^c = H$ for $i = 1, \ldots, n-1$, and H interacts with e_0^c as follows: if $(u,v) \in \mathbb{X}_{k_1} \times \mathbb{X}_{k_2}$ and $R(u,v) = (v',u')$, then

$$H(e_0^c(u,v)) = H(u,v)\left(\frac{\varepsilon_0(u) + c^{-1}\varphi_0(v)}{\varepsilon_0(u) + \varphi_0(v)}\right)\left(\frac{c\varepsilon_0(v') + \varphi_0(u')}{\varepsilon_0(v') + \varphi_0(u')}\right). \tag{5.8}$$

We now show that this definition "tropicalizes to" the combinatorial definition.

Lemma 5.17. *If H is a positive geometric coenergy function on $\mathbb{X}_{k_1} \times \mathbb{X}_{k_2}$, then the piecewise-linear function $\widehat{H} := \mathrm{Trop}(\Theta H)$, when restricted to $B^{n-k_1,L_1} \otimes B^{n-k_2,L_2} \subset \widetilde{\mathbb{T}}_{n-k_1} \times \widetilde{\mathbb{T}}_{n-k_2}$, is a coenergy function.*

Proof. Clearly $\widehat{H} \circ \widetilde{e}_i = \widehat{H}$ for $i = 1, \ldots, n-1$. If $a \otimes b \in B^{n-k_1,L_1} \otimes B^{n-k_2,L_2}$ and $\widetilde{R}(a \otimes b) = (b' \otimes a')$, then by tropicalizing (5.8) and using Theorems 4.23 and 5.13

(plus the identity $\max(c,d) = -\min(-c,-d)$), we obtain

$$\begin{aligned}\widehat{H}(\widetilde{e}_0(a\otimes b)) &= \widehat{H}(a\otimes b) + \max(\widetilde{\varepsilon}_0(a), \widetilde{\varphi}_0(b)) - \max(\widetilde{\varepsilon}_0(a), \widetilde{\varphi}_0(b)+1) \\ &\qquad + \max(\widetilde{\varepsilon}_0(b'), \widetilde{\varphi}_0(a')) - \max(\widetilde{\varepsilon}_0(b')-1, \widetilde{\varphi}_0(a')) \\ &= \widehat{H}(a\otimes b) + \begin{cases} 0 & \text{if } \widetilde{\varepsilon}_0(a) > \widetilde{\varphi}_0(b) \\ -1 & \text{if } \widetilde{\varepsilon}_0(a) \le \widetilde{\varphi}_0(b) \end{cases} + \begin{cases} 1 & \text{if } \widetilde{\varepsilon}_0(b') > \widetilde{\varphi}_0(a') \\ 0 & \text{if } \widetilde{\varepsilon}_0(b') \le \widetilde{\varphi}_0(a') \end{cases}.\end{aligned}$$

This shows that $\widehat{H}$ satisfies (2.2). □

Definition 5.18. Define $E : \mathbb{X}_{k_1} \times \mathbb{X}_{k_2} \to \mathbb{C}$ by

$$E(u,v) = \Delta_{[n-k+1,n],[k]}(g(u)g(v)),$$

where $k = \min(k_1, k_2)$.

Note that the last k rows of $g(u)$ and the first k columns of $g(v)$ are independent of λ, so by the Cauchy–Binet formula, E is indeed complex-valued. In fact, the Cauchy–Binet formula gives a simple expression for E in terms of Plücker coordinates. Recall the notation $Q_s^J(M) := P_{w_0(J)}(S_s(M))$.

Lemma 5.19. *If $(M|s, N|t) \in \mathbb{X}_{k_1} \times \mathbb{X}_{k_2}$, then*

$$E(M|s,N|t) = \begin{cases} \displaystyle\sum_{I \in \binom{[k_1-k_2+1,n]}{k_2}} \frac{Q_s^{I'}(M)}{Q_s^{[k_1]}(M)} \frac{P_I(N)}{P_{[n-k_2+1,n]}(N)} & \text{if } k_1 \ge k_2 \\ \displaystyle\sum_{I \in \binom{[n-k_2+k_1]}{k_1}} \frac{Q_s^{I}(M)}{Q_s^{[k_1]}(M)} \frac{P_{I''}(N)}{P_{[n-k_2+1,n]}(N)} & \text{if } k_1 \le k_2 \end{cases} \tag{5.9}$$

where $I' = [k_1-k_2] \cup I$, and $I'' = I \cup [n-k_2+k_1+1,n]$.

Proof. We assume $k_1 \ge k_2$ (the case $k_1 \le k_2$ is similar). By Cauchy–Binet,

$$E(M|s,N|t) = \sum_{|I|=k_2} \Delta_{[n-k_2+1,n],I}(g(M|s))\Delta_{I,[k_2]}(g(N|t)).$$

The bottom-left $k_1 \times k_1$ submatrix of $g(M|s)$ is upper uni-triangular, so

$$\Delta_{[n-k_2+1,n],I}(g(M|s)) = \begin{cases} \Delta_{[n-k_1+1,n],[k_1-k_2]\cup I}(g(M|s)) & \text{if } I \subset [k_1-k_2+1,n] \\ 0 & \text{otherwise.} \end{cases}$$

This together with (3.29) proves the $k_1 \ge k_2$ case of (5.9). □

Proposition 5.20. *$E : \mathbb{X}_{k_1} \times \mathbb{X}_{k_2} \to \mathbb{C}$ is a geometric coenergy function.*

Proof. Suppose $u = M|s \in \mathbb{X}_{k_1}$ and $v = N|t \in \mathbb{X}_{k_2}$. Set $B = g(u)g(v)$, $R(u,v) = (v', u')$, and $k = \min(k_1, k_2)$. Since e_i^c commutes with the unipotent crystal map g, we have

$$E(e_i^c(u,v)) = \Delta_{[n-k+1,n],[k]}(e_i^c(B)).$$

By (2.15), the folded matrix $e_i^c(B)$ is obtained from the folded matrix B by adding a multiple of row $i+1$ to row i, and a multiple of column i to column $i+1$ (mod n). If $i \in [n-1]$, then these row and column operations do not change the determinant of a bottom-left justified submatrix, so E is invariant under e_i^c.

Now consider e_0^c. By (2.15), we have

$$E(e_0^c(u,v)) = \Delta_{[n-k+1,n],[k]}\left(x_0\left(\lambda^{-1}\frac{c-1}{\varphi_0(u,v)}\right)\cdot B\cdot x_0\left(\lambda^{-1}\frac{c^{-1}-1}{\varepsilon_0(u,v)}\right)\right) \tag{5.10}$$

where $x_0(z)$ is the $n \times n$ matrix with 1's on the diagonal and z in position $(n,1)$. Suppose $k = k_2$. The left-hand side of (5.10) does not depend on λ, so we may substitute $\lambda = (-1)^{k-1}t$ into the right-hand side and obtain

$$E(e_0^c(u,v)) = \Delta_{[n-k+1,n],[k]}\left(x_0\left(\frac{(-1)^{k-1}}{t}\frac{c-1}{\varphi_0(u,v)}\right)\cdot B_t\cdot x_0\left(\frac{(-1)^{k-1}}{t}\frac{c^{-1}-1}{\varepsilon_0(u,v)}\right)\right),$$

where $B_t = B|_{\lambda=(-1)^{k-1}t}$. By multi-linearity of the determinant (or by Cauchy–Binet), we have

$$\begin{aligned} E(e_0^c(u,v)) = \Delta_{[n-k+1,n],[k]}(B_t) &+ \frac{1}{t}\frac{c-1}{\varphi_0(u,v)}\Delta_{\{1\}\cup[n-k+1,n-1],[k]}(B_t) \\ &+ \frac{1}{t}\frac{c^{-1}-1}{\varepsilon_0(u,v)}\Delta_{[n-k+1,n],[2,k]\cup\{n\}}(B_t) \\ &+ \frac{1}{t^2}\frac{(c-1)(c^{-1}-1)}{\varphi_0(u,v)\varepsilon_0(u,v)}\Delta_{\{1\}\cup[n-k+1,n-1],[2,k]\cup\{n\}}(B_t). \end{aligned} \tag{5.11}$$

Restrict to the open set where $(-1)^{k_1+k_2}s \neq t$, so that $g(M|s)|_{\lambda=(-1)^{k-1}t}$ is invertible by Proposition 3.7(3), and B_t has rank k by Corollary 3.8(2). In a rank k matrix, any set of k columns which are linearly independent span the same subspace, so we have

$$\frac{\Delta_{\{1\}\cup[n-k+1,n-1],[2,k]\cup\{n\}}(B_t)}{\Delta_{[n-k+1,n],[2,k]\cup\{n\}}(B_t)} = \frac{\Delta_{\{1\}\cup[n-k+1,n-1],[k]}(B_t)}{\Delta_{[n-k+1,n],[k]}(B_t)} \tag{5.12}$$

on the open set where both denominators are nonzero. (Since we are trying to prove an identity of rational maps, we may restrict to open subsets.) Using (5.12) and the fact that $\Delta_{[n-k+1,n],[k]}(B_t) = E(u,v)$, we may rewrite (5.11) as

$$E(e_0^c(u,v)) = E(u,v)(1+z_1)(1+z_2) \tag{5.13}$$

where

(5.14)

$$z_1 = \frac{1}{t}\frac{c-1}{\varphi_0(u,v)}\frac{\Delta_{\{1\}\cup[n-k+1,n-1],[k]}(B_t)}{\Delta_{[n-k+1,n],[k]}(B_t)}, \qquad z_2 = \frac{1}{t}\frac{c^{-1}-1}{\varepsilon_0(u,v)}\frac{\Delta_{[n-k+1,n],[2,k]\cup\{n\}}(B_t)}{\Delta_{[n-k+1,n],[k]}(B_t)}.$$

Now we compute

$$\begin{aligned}\varphi_0(u,v) = \varphi_0(v',u') &= \varphi_0(v')\frac{\varepsilon_0(v')+\varphi_0(u')}{\varepsilon_0(v')}\\ &= \frac{1}{t}\frac{\Delta_{[n-k+1,n-1]\cup\{1\},[k]}(B_t)}{\Delta_{[n-k+1,n],[k]}(B_t)}\frac{\varepsilon_0(v')+\varphi_0(u')}{\varepsilon_0(v')},\end{aligned}$$

where the first equality comes from Theorem 5.11(1), the second equality comes from Definition/Proposition 2.32, and the final equality is the formula for φ_0 on $\mathbb{X}_k$, together with (5.3). Similarly, by Proposition 3.17(2), the formula for ε_0 on $\mathbb{X}_k$, and (5.4), we have

$$\begin{aligned}\varepsilon_0(u,v) = \varepsilon_0(v)\frac{\varepsilon_0(u)+\varphi_0(v)}{\varphi_0(v)} &= \varphi_0(S(v))\frac{\varepsilon_0(u)+\varphi_0(v)}{\varphi_0(v)}\\ &= \frac{1}{t}\frac{\Delta_{[n-k+1,n],[2,k]\cup\{n\}}(B_t)}{\Delta_{[n-k+1,n],[k]}(B_t)}\frac{\varepsilon_0(u)+\varphi_0(v)}{\varphi_0(v)}.\end{aligned}$$

Substituting these expressions into (5.14), we get

$$z_1 = \frac{(c-1)\varepsilon_0(v')}{\varepsilon_0(v')+\varphi_0(u')}, \qquad z_2 = \frac{(c^{-1}-1)\varphi_0(v)}{\varepsilon_0(u)+\varphi_0(v)}, \tag{5.15}$$

and then (5.8) is obtained by substituting (5.15) into (5.13).

The $k = k_1$ case is dealt with similarly, using the substitution $\lambda = (-1)^{k_1-1}s$ instead of $\lambda = (-1)^{k_2-1}t$. □

The function E is positive by Lemma 5.19 and the positivity of S, so we may define $\widehat{E} = \mathrm{Trop}(\Theta E) : \widetilde{\mathbb{T}}_{n-k_1} \times \widetilde{\mathbb{T}}_{n-k_2} \to \mathbb{Z}$. Recall the coenergy function $\widetilde{E}$ introduced in §2.2.3.

Theorem 5.21. *The restriction of $\widehat{E}$ to $B^{n-k_1,L_1} \otimes B^{n-k_2,L_2}$ is equal to $\widetilde{E}$.*

The proof of this theorem relies on a technical lemma. At the end of §4.2.1, we defined a rational function $\Phi^k_{I,J} : \mathbb{T}_k \to \mathbb{C}$ by $\Phi^k_{I,J}(X_{ij},t) = \Delta_{I,J}(\Phi_k(X_{ij},t))$ for any subsets I, J of the same cardinality. Let $I = [k+1,n]$, and let J be any $(n-k)$-element subset of $[n]$. By Lemma 4.19, $\Phi^k_{[k+1,n],J}$ is positive (it's clear that the required inequalities hold in this case), so we may tropicalize it to obtain a piecewise-linear function $\widehat{\Phi}^k_{[k+1,n],J} : \widetilde{\mathbb{T}}_k \to \mathbb{Z}$.

Lemma 5.22. *Fix $L \geq 0$, and let b_0 be the classical highest weight element of the KR crystal $B^{k,L}$. For all $J \in \binom{[n]}{n-k}$, we have $\widehat{\Phi}^k_{[k+1,n],J}(b_0) = 0$.*

Proof. Suppose $b = (B_{ij}, L) \in B^{k,L}$, and let $b_{ij} = B_{ij} - B_{i,j-1}$ be the number of j's in the i^{th} row of the corresponding tableau, as in (2.9). Let $\Gamma_{n-k,n}(b)$ be the network $\Gamma_{n-k,n}(X_{ij}, t)$ from §4.1, but with weights b_{ij} instead of x_{ij}, and 0 instead of 1 on unlabeled edges. By the Lindström Lemma, we have

$$\widehat{\Phi}^k_{[k+1,n],J}(b) = \min_{\mathcal{F}:[k+1,n]\to J} \widetilde{\mathrm{wt}}(\mathcal{F}),$$

where $\mathcal{F}$ runs over vertex-disjoint families of paths in $\Gamma_{n-k,n}(b)$ from $[k+1, n]$ to J, and $\widetilde{\mathrm{wt}}(\mathcal{F})$ is the sum of the weights of the edges in the paths.

The classical highest weight element $b_0 \in B^{k,L}$ corresponds to the SSYT whose i^{th} row is filled with the number i, so we have

$$(b_0)_{ij} = \begin{cases} L & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the only edges in $\Gamma_{n-k,n}(b_0)$ with nonzero weights are to the left of source $k+1$, so all edges in the paths that contribute to $\widehat{\Phi}^k_{[k+1,n],J}(b_0)$ have weight zero. $\square$

Proof of Theorem 5.21. By Proposition 5.20 and Lemma 5.17, $\widehat{E}$ is a coenergy function on $B^{n-k_1,L_1} \otimes B^{n-k_2,L_2}$. By Proposition 2.18(1), the coenergy function on such crystals is unique up to a global additive constant, so it suffices to show that $\widehat{E}$ and $\widetilde{E}$ agree on a single element of $B^{n-k_1,L_1} \otimes B^{n-k_2,L_2}$.

Assume $k_1 \leq k_2$ (the other case is basically the same). If $(x, y) \in \mathbb{T}_{n-k_1} \times \mathbb{T}_{n-k_2}$, then by Lemma 5.19, (3.29), and Proposition 4.5, we have

$$\Theta E(x, y) = \sum_{I \in \binom{[n-k_2+k_1]}{k_1}} f_I(x) g_I(y)$$

where

$$f_I(x) = \Delta_{[n-k_1+1,n],I}(\Phi_{n-k_1}(x)), \qquad g_I(y) = \Delta_{I\cup[n-k_2+k_1+1,n],[k_2]}(\Phi_{n-k_2}(y)).$$

Thus,

$$\widehat{E}(a \otimes b) = \min_{I \in \binom{[n-k_2+k_1]}{k_1}} (\widehat{f}_I(a) + \widehat{g}_I(b)), \tag{5.16}$$

where $\widehat{f}_I, \widehat{g}_I$ are the tropicalizations of f_I, g_I (which are positive by Lemma 4.19). Since the bottom $k_2 \times k_2$ submatrix of $\Phi_{n-k_2}(y)$ is upper uni-triangular, we have $g_{[n-k_2+1,n-k_2+k_1]}(y) = 1$ for all y, so

$$\widehat{g}_{[n-k_2+1,n-k_2+k_1]}(b) = 0 \tag{5.17}$$

for all b.

Let a_0 be the classical highest weight element of B^{n-k_1,L_1}. By Lemma 5.22, we have $\widehat{f}_I(a_0) = 0$ for all $I \in \binom{[n]}{k_1}$. Together with (5.16) and (5.17), this implies that $\widehat{E}(a_0 \otimes b) = 0$ for all $b \in B^{n-k_2,L_2}$. By (2.3), $\widetilde{E}(a_0 \otimes b) = 0$ for all $b \in B^{n-k_2,L_2}$, so we are done. □

Corollary 5.25 in the next section gives an explicit formula for $\Theta E : \mathbb{T}_{n-k_1} \times \mathbb{T}_{n-k_2} \to \mathbb{C}$ in the case $k_1 = n - 1$.

5.3 One-row tableaux

Here we give a more explicit description of the geometric R-matrix on $\mathbb{T}_\ell \times \mathbb{T}_k$ in the case $\ell = 1$. When $k = 1$ as well, we recover the one-row geometric (or birational) R-matrix of Yamada and Lam–Pylyavskyy that was discussed in §1.2. At the end of the section we demonstrate our formulas in a small example.

Let $X = (X_{11}, X_{12}, \ldots, X_{1,n-1}, s) \in \mathbb{T}_1$ be a rational 1-rectangle, and define $x_1, \ldots, x_n$ by $x_j = X_{1j}/X_{1,j-1}$ (where $X_{10} := 1$ and $X_{1n} := s$). Let $Y = (Y_{ij}, t) \in \mathbb{T}_k$ be a rational k-rectangle. Suppose

$$\Theta R(X, Y) = ((Y'_{ij}, t), (X'_{1j}, s)),$$

and define x'_j as above. We will work through the various definitions from earlier sections to obtain formulas for Y'_{ij} and x'_j in terms of the inputs x_j, Y_{ij}, and t.

Set $N|t = \Theta_{n-k}(Y), N'|t = \Theta_{n-k}(Y'), A = g(\Theta_{n-1}(X))g(\Theta_{n-k}(Y))$, and $A_t = A|_{\lambda=(-1)^{n-k-1}t}$. For $I \in \binom{[n]}{n-k}$, define

$$\tau_I = \tau_I(X, Y) = \Delta_{I,[n-k]}(A_t)\frac{P_{[k+1,n]}(N)}{P_I(N)}. \tag{5.18}$$

By (5.3) and Proposition 4.4 (applied to both Y_{ij} and Y'_{ij}), we have

$$Y'_{ij} = Y_{ij}\frac{\tau_{[i,j]\cup[k+j-i+2,n]}}{\tau_{[i+1,j]\cup[k+j-i+1,n]}}, \tag{5.19}$$

so we are led to the study of the quantities τ_I. By the Cauchy–Binet formula and Proposition 3.7(1),

$$\tau_I = \sum_J \Delta_{I,J}(C|_{\lambda=(-1)^{n-k-1}t})\frac{P_J(N)}{P_I(N)}, \tag{5.20}$$

where $C = g(\Theta_{n-1}(X))$. Lemma 4.10 expresses the Plücker coordinates of N in terms of the Y_{ij} by summing over J-tableaux, so we regard these Plücker coordinates as well-understood. Now we explicitly compute the minors of the matrix C. Note

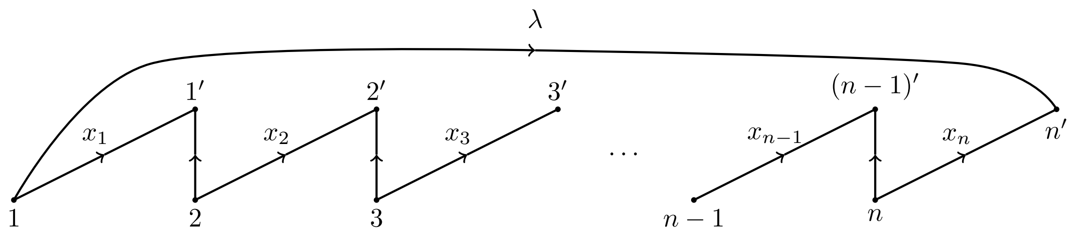


Figure 6: A network representation of the matrix $g(\Theta_{n-1}(X))$. Vertical edges have weight 1.

that C has $x_1, \ldots, x_n$ on the main diagonal, 1's just beneath the main diagonal, λ in the top-right corner, and zeroes elsewhere. For example, if $n = 4$, then

$$C = g(\Theta_3(X)) = \begin{pmatrix} x_1 & 0 & 0 & \lambda \\ 1 & x_2 & 0 & 0 \\ 0 & 1 & x_3 & 0 \\ 0 & 0 & 1 & x_4 \end{pmatrix}.$$

Lemma 5.23. *Let $C = g(\Theta_{n-1}(X))$, and let $I = \{i_1 < \ldots < i_r\}$ be an r-subset of $[n]$, with $r \leq n-1$. For $\epsilon = (\epsilon_1, \ldots, \epsilon_r) \in \{0,1\}^r$, define*

$$I - \epsilon = \{i_1 - \epsilon_1, \ldots, i_r - \epsilon_r\} \subset [n],$$

where if $i_1 = \epsilon_1 = 1$, we take $i_1 - \epsilon_1 = n$. If $I - \epsilon$ has r elements, then we have

$$\Delta_{I,I-\epsilon}(C) = \begin{cases} (-1)^{r-1}\lambda \prod_{s \,|\, \epsilon_s = 0} x_{i_s} & \text{if } i_1 = \epsilon_1 = 1 \\ \prod_{s \,|\, \epsilon_s = 0} x_{i_s} & \text{otherwise.} \end{cases} \tag{5.21}$$

If $J \in \binom{[n]}{r}$ is not of the form $I - \epsilon$ for $\epsilon \in \{0,1\}^r$, then $\Delta_{I,J}(C) = 0$.

Note that by expanding along the last column of C, we have $\det(C) = (-1)^{n+1}\lambda + \prod_{j=1}^n x_j$, so the restriction $r \leq n-1$ is necessary.

Proof. Observe that C is the matrix associated to the planar network in Figure 6. In this network, there are two edges coming out of each source i: an edge to sink i', and an edge to sink $(i-1)'$ (mod n). Thus, if there is a vertex-disjoint family of paths from the sources in I to the sinks in J, then $J = I - \epsilon$ for some $\epsilon \in \{0,1\}^r$; if J is not of this form, then $\Delta_{I,J}(C)$ is zero by the Lindström Lemma.

We claim that for any r-subset J, there is at most one vertex-disjoint family of paths from I to J. To see this, note that the underlying (undirected) graph of the network is a cycle of length $2n$, and a vertex-disjoint family of paths from I to J is a perfect matching in the subgraph induced by the vertices in I and J. Since $r \leq n-1$, the subgraph induced by the vertices in I and J is a forest, and there is at most one perfect matching in any forest.

Suppose $I-\epsilon$ has r elements. In this case, let p_s be the path connecting i_s and $(i_s-\epsilon_s)'$, for $s=1,\ldots,r$. The family of paths (p_s) is clearly vertex-disjoint, and it has weight

$$\begin{cases} \lambda \prod_{s \,|\, \epsilon_s=0} x_{i_s} & \text{if } i_1=\epsilon_1=1 \\ \prod_{s \,|\, \epsilon_s=0} x_{i_s} & \text{otherwise.} \end{cases}$$

The permutation associated to this family has sign $(-1)^{r-1}$ if $i_1=\epsilon_1=1$, and is the identity otherwise, so (5.21) follows from the Lindström Lemma. □

In light of Lemma 5.23, (5.20) becomes

$$\tau_I = \sum_{\epsilon} t^{\delta_{i_1,1}\delta_{\epsilon_1,1}} \cdot \prod_{s \,|\, \epsilon_s=0} x_{i_s} \cdot \frac{P_{I-\epsilon}(N)}{P_I(N)}, \tag{5.22}$$

where the sum is over $\epsilon \in \{0,1\}^{n-k}$ such that $I-\epsilon$ has $n-k$ elements. For example, if $n=7$ and $k=4$, then writing P_J for $P_J(N)$, we have

$$\tau_{145} = \frac{x_1x_4x_5P_{145}+x_1x_5P_{135}+x_1P_{134}+tx_3x_4P_{457}+tx_4P_{357}+tP_{347}}{P_{145}}.$$

Combining (5.19) and (5.22), we have a reasonably explicit formula for the Y'_{ij}. Now we turn to the x'_j. For $j \in \mathbb{Z}/n\mathbb{Z}$, define

$$\kappa_j = \kappa_j(X,Y) = \tau_{[j+k,j+n-1]}(X,Y). \tag{5.23}$$

Proposition 5.24. *We have*

$$x'_j = x_j \frac{\kappa_j}{\kappa_{j+1}}.$$

Furthermore, we have the formula

$$\kappa_j = \sum_{s=0}^{n-k} x_{j+k+s}x_{j+k+s+1}\cdots x_{j+n-1} t^{a_{j,s,k}} \frac{P_{[j+k-1,j+n-1]\setminus\{j+k+s-1\}}(N)}{P_{[j+k,j+n-1]}(N)} \tag{5.24}$$

where

$$a_{j,s,k} = \begin{cases} 1 & \textit{if } n+2-s \le j+k \le n+1 \\ 0 & \textit{otherwise} \end{cases}.$$

Each subscript of κ and x is interpreted mod n.

Proof. By Theorem 5.4, we have the matrix equation

$$g(\Theta_{n-1}(X))g(N|t) = g(N'|t)g(\Theta_{n-1}(X')). \tag{5.25}$$

The diagonal entries of $g(\Theta_{n-1}(X))$ are $x_1, \ldots, x_n$, so by equating the constant coefficients of the diagonal entries of both sides of (5.25) and using the definition of $g(N|t)$ (plus Convention 2.40), we obtain

$$x_j \frac{P_{[j+k+1,j+n]}(N)}{P_{[j+k,j+n-1]}(N)} = x'_j \frac{P_{[j+k+1,j+n]}(N')}{P_{[j+k,j+n-1]}(N')} = x'_j \frac{\Delta_{[j+k+1,j+n],[n-k]}(A_t)}{\Delta_{[j+k,j+n-1],[n-k]}(A_t)}.$$

This shows that $x'_j = x_j \dfrac{\kappa_j}{\kappa_{j+1}}$.

By (5.22), we have

$$\kappa_j = \sum_{\epsilon} t^{\delta_{i_1,1}\delta_{\epsilon_1,1}} \cdot \prod_{s \,|\, \epsilon_s = 0} x_{i_s} \frac{P_{[j+k,j+n-1]-\epsilon}(N)}{P_{[j+k,j+n-1]}(N)}$$

(to compute $[j+k, j+n-1] - \epsilon$, first identify $[j+k, j+n-1]$ with a subset $\{i_1 < \ldots < i_{n-k}\}$ of $[n]$ by reducing mod n, and then subtract ϵ_s from the s^{th} smallest element of this subset). There are $n-k+1$ choices of ϵ such that $[j+k, j+n-1] - \epsilon$ has $n-k$ elements, and one may easily verify that each of these choices gives a term from the right-hand side of (5.24). □

Corollary 5.25. *The geometric coenergy function* $\Theta E : \mathbb{T}_1 \times \mathbb{T}_k \to \mathbb{C}$ *is given by*

$$\Theta E(X, Y) = \kappa_1(X, Y) = \sum_{s=0}^{n-k} x_{k+s+1} x_{k+s+2} \cdots x_n Y_{k,k+s-1},$$

where $Y_{k,k-1} := 1$.

Proof. By definition, $\Theta E(X, Y) = \Delta_{[k+1,n],[n-k]}(A)$, and since this minor is independent of λ, it is equal to $\kappa_1(X, Y)$. The explicit formula for $\kappa_1(X, Y)$ follows from (5.24) and Proposition 4.4. □

Recovering the one-row geometric R-matrix

Now we specialize further to the case $k = 1$. Let $Y = (Y_{11}, \ldots, Y_{1,n-1}, t)$, and define $y_j = Y_{1j}/Y_{1,j-1}$, where $Y_{10} := 1$ and $Y_{1n} := t$. As above, let $N|t = \Theta_{n-1}(Y)$. By Proposition 3.7(1) and Lemma 5.23, we have

$$\frac{P_{[n]\setminus\{a\}}(N)}{P_{[n]\setminus\{b\}}(N)} = \frac{\Delta_{[n]\setminus\{a\},[n-1]}(g(\Theta_{n-1}(Y))}{\Delta_{[n]\setminus\{b\},[n-1]}(g(\Theta_{n-1}(Y))} = \frac{y_1 \cdots y_{a-1}}{y_1 \cdots y_{b-1}} = \begin{cases} y_b \cdots y_{a-1} & \text{if } b \leq a \\ (y_a \cdots y_{b-1})^{-1} & \text{if } a \leq b \end{cases}$$

for $a, b \in [n]$. Setting $k = 1$ in (5.24) and using $t = y_1 \cdots y_n$, we obtain

$$\kappa_j = \sum_{s=0}^{n-j} x_{j+s+1} x_{j+s+2} \cdots x_{j+n-1} \frac{P_{[n]\setminus\{j+s\}}(N)}{P_{[n]\setminus\{j\}}(N)} + t \sum_{s=n-j+1}^{n-1} x_{j+s+1} x_{j+s+2} \cdots x_{j+n-1} \frac{P_{[n]\setminus\{j+s-n\}}(N)}{P_{[n]\setminus\{j\}}(N)}$$
$$= \sum_{s=0}^{n-1} y_j y_{j+1} \cdots y_{j+s-1} x_{j+s+1} x_{j+s+2} \cdots x_{j+n-1}$$

where as above, each subscript of x and y is interpreted mod n.

Proposition 5.26. *The map*

$$\Theta R : ((x_1, \ldots, x_n), (y_1, \ldots, y_n)) \mapsto ((y_1', \ldots, y_n'), (x_1', \ldots, x_n'))$$

is given by

$$y_j' = y_j \frac{\kappa_{j+1}}{\kappa_j}, \qquad x_j' = x_j \frac{\kappa_j}{\kappa_{j+1}}, \qquad \kappa_j = \sum_{s=0}^{n-1} y_j y_{j+1} \cdots y_{j+s-1} x_{j+s+1} x_{j+s+2} \cdots x_{j+n-1}$$

where subscripts are interpreted mod n.

Proof. By the preceding discussion, we have $x_j' = x_j \frac{\kappa_j}{\kappa_{j+1}}$. Arguing as in the proof of Proposition 5.24, we have $x_j y_j = y_j' x_j'$, so $y_j' = y_j \frac{\kappa_{j+1}}{\kappa_j}$. □

Thus, in the one-row by one-row case, our geometric R-matrix agrees with the map found by Yamada [Yam01] and Lam–Pylyavskyy [LP12] (cf. Proposition 1.2).

A small example

Set $n = 4$. Suppose $X = (X_{11}, X_{12}, X_{13}, s) \in \mathbb{T}_1$, $Y = (Y_{11}, Y_{12}, Y_{22}, Y_{23}, t) \in \mathbb{T}_2$, and $(Y', X') = \Theta R(X, Y)$. Define

$$x_1 = X_{11} \qquad x_2 = X_{12}/X_{11} \qquad x_3 = X_{13}/X_{12} \qquad x_4 = s/X_{13},$$

and define x_j' analogously. Define y_{ij} by (4.1), i.e.,

$$\begin{array}{lll} y_{11} = Y_{11} & y_{12} = Y_{12}/Y_{11} & y_{13} = t/Y_{13} \\ y_{22} = Y_{22} & y_{23} = Y_{23}/Y_{22} & y_{14} = t/Y_{23} \end{array}.$$

Note that $t = y_{11} y_{12} y_{13} = y_{22} y_{23} y_{24}$. Let $N|t = \Theta_2(Y) \in \mathrm{Gr}(2, 4) \times \mathbb{C}^\times$. Using the definition of Θ_k, one computes that N is the column span of the matrix

$$\begin{pmatrix} y_{11} & 0 \\ y_{22} & y_{12} y_{22} \\ 1 & y_{12} + y_{23} \\ 0 & 1 \end{pmatrix}.$$

Set $P_J = P_J(N)$. By (5.19), Proposition 5.24, and (5.22), we have

$$(5.26) \qquad Y'_{11} = Y_{11}\frac{\tau_{14}}{\tau_{34}} \qquad Y'_{12} = Y_{12}\frac{\tau_{12}}{\tau_{24}} \qquad Y'_{22} = Y_{22}\frac{\tau_{24}}{\tau_{34}} \qquad Y'_{23} = Y_{23}\frac{\tau_{23}}{\tau_{34}},$$

$$(5.27) \qquad x'_1 = x_1\frac{\kappa_1}{\kappa_2} \qquad x'_2 = x_2\frac{\kappa_2}{\kappa_3} \qquad x'_3 = x_3\frac{\kappa_3}{\kappa_4} \qquad x'_4 = x_4\frac{\kappa_4}{\kappa_1},$$

where

$$\kappa_1 = \tau_{34} = \frac{x_3x_4P_{34} + x_4P_{24} + P_{23}}{P_{34}} = x_3x_4 + x_4y_{22} + y_{22}y_{23}$$
$$\kappa_2 = \tau_{14} = \frac{x_1x_4P_{14} + x_1P_{13} + tP_{34}}{P_{14}} = x_1x_4 + x_1(y_{12} + y_{23}) + y_{12}y_{13}$$
$$\kappa_3 = \tau_{12} = \frac{x_1x_2P_{12} + tx_2P_{24} + tP_{14}}{P_{12}} = x_1x_2 + x_2y_{13} + \frac{y_{11}y_{13}}{y_{22}}$$
$$\kappa_4 = \tau_{23} = \frac{x_2x_3P_{23} + x_3P_{13} + P_{12}}{P_{23}} = x_2x_3 + x_3\frac{y_{11}}{y_{22}y_{23}}(y_{12} + y_{23}) + \frac{y_{11}y_{12}}{y_{23}}$$
$$\tau_{24} = \frac{x_2x_4P_{24} + x_2P_{23} + x_4P_{14} + P_{13}}{P_{24}} = x_2x_4 + x_2y_{23} + \frac{y_{11}}{y_{22}}(x_4 + y_{12} + y_{23}).$$

By tropicalizing these formulas, one obtains piecewise-linear formulas for the combinatorial R-matrix on $B^1 \otimes B^2$. Specifically, let $A = (A_{11}, A_{12}, A_{13}, L_1)$ be a 1-rectangle, let $B = (B_{11}, B_{12}, B_{22}, B_{23}, L_2)$ be a 2-rectangle, and let $B' \otimes A' = \widehat{R}(A \otimes B)$. Define

$$a_1 = A_{11} \qquad a_2 = A_{12} - A_{11} \qquad a_3 = A_{13} - A_{12} \qquad a_4 = L_1 - A_{13},$$

$$b_{11} = B_{11} \qquad b_{12} = B_{12} - B_{11} \qquad b_{13} = L_2 - B_{12}$$
$$b_{22} = B_{22} \qquad b_{23} = B_{23} - B_{22} \qquad b_{24} = L_2 - B_{23},$$

so that a_j is the number of j's in the one-row tableau corresponding to A, and b_{ij} is the number of j's in the i^{th} row of the two-row tableau corresponding to B. Define a'_j, b'_{ij} analogously.

For $I \in \binom{[4]}{2}$, let $\widetilde{\tau}_I$ be the tropicalization of τ_I, where x_j, y_{ij} is replaced with a_j, b_{ij} in the tropicalization. Let $\widetilde{\kappa}_j = \widetilde{\tau}_{\{j+2,j+3\}}$. For example,

$$\widetilde{\kappa}_1 = \widetilde{\tau}_{34} = \min(a_3 + a_4, a_4 + b_{22}, b_{22} + b_{23}),$$
$$\widetilde{\tau}_{24} = \min(a_2 + a_4, a_2 + b_{23}, b_{11} - b_{22} + \min(a_4, b_{12}, b_{23})).$$

By tropicalizing (5.26) and (5.27), we have

$$B'_{11} = B_{11} + \widetilde{\tau}_{14} - \widetilde{\tau}_{34} \qquad B'_{12} = B_{12} + \widetilde{\tau}_{12} - \widetilde{\tau}_{24}$$
$$B'_{22} = B_{22} + \widetilde{\tau}_{24} - \widetilde{\tau}_{34} \qquad B'_{23} = B_{23} + \widetilde{\tau}_{23} - \widetilde{\tau}_{34},$$

$$a'_1 = a_1 + \widetilde{\kappa}_1 - \widetilde{\kappa}_2 \qquad a'_2 = a_2 + \widetilde{\kappa}_2 - \widetilde{\kappa}_3 \qquad a'_3 = a_3 + \widetilde{\kappa}_3 - \widetilde{\kappa}_4 \qquad a'_4 = a_4 + \widetilde{\kappa}_4 - \widetilde{\kappa}_1.$$

Example 5.27. Let A, B correspond to the tableaux T, U from Example 2.13. We have

$$a_1 \quad a_2 \quad a_3 \quad a_4 \;=\; 2 \quad 0 \quad 4 \quad 1 \qquad \text{and} \qquad \begin{matrix} b_{11} & b_{12} & b_{13} \\ b_{22} & b_{23} & b_{24} \end{matrix} = \begin{matrix} 3 & 1 & 1 \\ 2 & 0 & 3 \end{matrix}.$$

We compute

$$\begin{aligned} \widetilde{\kappa}_1 = \widetilde{\tau}_{34} &= \min(5,3,2) = 2 \\ \widetilde{\kappa}_2 = \widetilde{\tau}_{14} &= \min(3, 2 + \min(1,0), 2) = 2 \\ \widetilde{\kappa}_3 = \widetilde{\tau}_{12} &= \min(2,1,2) = 1 \\ \widetilde{\kappa}_4 = \widetilde{\tau}_{23} &= \min(4, 5 + \min(1,0), 4) = 4 \\ \widetilde{\tau}_{24} &= \min(1, 0, 1 + \min(1,1,0)) = 0, \end{aligned}$$

so

$$\begin{aligned} B'_{11} &= 3 + 2 - 2 &&= 3 \\ B'_{12} &= (3+1) + 1 - 0 &&= 5 \\ B'_{22} &= 2 + 0 - 2 &&= 0 \\ B'_{23} &= (2+0) + 4 - 2 &&= 4 \end{aligned} \qquad \text{and} \qquad \begin{aligned} a'_1 &= 2 + 2 - 2 = 2 \\ a'_2 &= 0 + 2 - 1 = 1 \\ a'_3 &= 4 + 1 - 4 = 1 \\ a'_4 &= 1 + 4 - 2 = 3. \end{aligned}$$

The rectangles B' and A' correspond to the tableaux U' and T' from Example 2.13, so we have verified that $\widehat{R} = \widetilde{R}$ in this case. Also, by Corollary 5.25, we have $\widehat{E}(A \otimes B) = \widetilde{\kappa}_1(A \otimes B) = 2$, which agrees with the coenergy of $T \otimes U$ computed in Example 2.17.

5.4 Proof of the positivity of the geometric R-matrix

In this section we prove Theorem 5.3, which states that the geometric R-matrix is positive. We start by reducing this theorem to a statement about the positivity of certain minors of the folded matrix $g(N|t)$ (Proposition 5.29), and then we prove Proposition 5.29 using the Lindström Lemma, the positivity of the symmetries PR, S, and D, and a careful analysis of the structure of $g(N|t)$.

Reduction to Proposition 5.29

Recall the notions of positive varieties and positive rational maps from §4.2.1. Let X be a positive variety, λ an indeterminate, and $f : X \to \mathbb{C}[\lambda]$ a rational map, i.e., a map of the form

$$f = f_0 + f_1\lambda + \ldots + f_d\lambda^d,$$

where $f_i : X \to \mathbb{C}$ are rational functions. For an integer r, we say that f is *r-non-negative* if for each i, the rational function $(-1)^{(r-1)i} f_i$ is non-negative, and we say that f is *r-positive* if f is r-non-negative and nonzero. For example, for any positive variety X, the constant function $f = 1 - \lambda + \lambda^2$ is r-positive for even r, but not for odd r.

We will need the following observation, whose proof is immediate.

Lemma 5.28. *If $f : X \to \mathbb{C}[\lambda]$ is r-non-negative (resp., r-positive), then the rational function $\overline{f} : X \times \mathbb{C}^\times \to \mathbb{C}$ defined by $\overline{f}(x, z) = f(x)|_{\lambda=(-1)^{r-1}z}$ is non-negative (resp., positive).*

For two r-subsets $I, J \subset [n]$, let $\Delta_{I,J} : \mathbb{X}_k \to \mathbb{C}[\lambda]$ denote the rational map which sends $N|t$ to the minor $\Delta_{I,J}(g(N|t))$. Say that a subset of $[n]$ is a *cyclic interval* if its elements are consecutive mod n. Define a *cyclic interval of a subset $I \subset [n]$* to be a maximal collection of elements of I which form a cyclic interval.

Proposition 5.29. *Let $I, J \subset [n]$ be two subsets of size r, at least one of which has no more than two cyclic intervals, and let $\Delta_{I,J} : \mathbb{X}_k \to \mathbb{C}[\lambda]$ be the rational map just defined. Then*

1. *if $r \leq k$, $\Delta_{I,J}$ is r-non-negative;*
2. *if $r > k$, $\Delta_{I,J}$ is equal to $(t + (-1)^k\lambda)^{r-k} f_{I,J}$, where $f_{I,J}$ is an r-non-negative map $\mathbb{X}_k \to \mathbb{C}[\lambda]$.*

Remark 5.30. We expect that Proposition 5.29 holds without the restriction on I and J. We need this restriction in our proof because we do not know the correct generalization of Definition 5.33 and Proposition 5.34 to subsets with more than two cyclic intervals.

Before proving Proposition 5.29, we explain how it implies Theorem 5.3. Since the geometric Schützenberger involution S is positive, it suffices to show that the map $\Psi_{k,\ell} : \mathbb{X}_\ell \times \mathbb{X}_k \to \mathbb{X}_k$ is positive. Suppose $(M|s, N|t) \in \mathbb{X}_\ell \times \mathbb{X}_k$, and let $N'|t = \Psi_{k,\ell}(M|s, N|t), A = g(M|s)$, and $A_t = A_{\lambda=(-1)^{k-1}t}$. Fix a k-subset I. By (5.1) and the Cauchy–Binet formula, we have

$$P_I(N') = \sum_J \Delta_{I,J}(A_t) P_J(N).$$

If I has at most two cyclic intervals, then by Proposition 5.29 and Lemma 5.28, there are non-negative rational functions $f_{I,J} : \mathbb{X}_\ell \times \mathbb{C}^\times \to \mathbb{C}$ such that

$$\Delta_{I,J}(A_t) = (s + (-1)^{\ell+k-1}t)^{\max(0,k-\ell)} f_{I,J}(M|s, t).$$

Furthermore, by Proposition 3.7(3), we have $\det(A_t) = (s + (-1)^{\ell+k-1}t)^{n-\ell}$, so A_t is invertible for $(M|s, t)$ in an open subset of $\mathbb{X}_\ell \times \mathbb{C}^\times$. This means that at least one of the rational functions $f_{I,J}$ is nonzero.

If I and I' are k-subsets with at most two cyclic intervals, then on an open subset of $\mathbb{X}_\ell \times \mathbb{X}_k$, we have

$$\frac{P_I(N')}{P_{I'}(N')} = \frac{\sum f_{I,J}(M|s,t)P_J(N)}{\sum f_{I',J}(M|s,t)P_J(N)},$$

where $f_{I,J}, f_{I',J}$ are non-negative rational functions which are not all zero. In particular, this is true when I and I' are basic k-subsets (Definition 2.44), so $\Psi_{k,\ell}$ is positive by Lemma 4.17.

Proof of Proposition 5.29

I: Exploiting the symmetries

We first use the positivity of the symmetries PR, S, and D to make some further reductions. Suppose I and J are r-subsets, and consider the rational map $\Delta_{I,J} : \mathbb{X}_k \to \mathbb{C}[\lambda]$. By (3.27), we have

$$\Delta_{I,J} \circ S = \Delta_{w_0(J),w_0(I)},$$

so since S is positive, r-non-negativity (resp., r-positivity) of $\Delta_{I,J}$ is equivalent to that of $\Delta_{w_0(J),w_0(I)}$. This allows us to reduce to the case where J has at most two cyclic intervals.

Lemma 3.25 allows us to reduce to the case $r \le k$, as follows. Assume Proposition 5.29 holds for $r \le k$, and fix $r > k$. For $I, J \in \binom{[n]}{r}$, Lemma 3.25 gives the equality

$$\Delta_{I,J} = (t + (-1)^k\lambda)^{r-k}(\Delta_{\overline{J},\overline{I}}|_{\lambda=(-1)^n\lambda} \circ D)$$

of rational maps $\mathbb{X}_k \to \mathbb{C}[\lambda]$. Suppose I or J (equivalently, $\overline{I}$ or $\overline{J}$) has at most two cyclic intervals. By our assumption, the rational map $\Delta_{\overline{J},\overline{I}} : \mathbb{X}_{n-k} \to \mathbb{C}[\lambda]$ is $(n-r)$-non-negative, so $\Delta_{\overline{J},\overline{I}}|_{\lambda=(-1)^n\lambda}$ is r-non-negative. Since D is positive, Proposition 5.29 holds for $\Delta_{I,J}$.

Lemma 3.11 shows that

$$\Delta_{I,J} \circ \mathrm{PR} = \begin{cases} \Delta_{I-1,J-1} & \text{if } 1 \in I \cap J \text{ or } 1 \notin I \cup J \\ (-1)^{r-1}\lambda \cdot \Delta_{I-1,J-1} & \text{if } 1 \in I \setminus J \\ (-1)^{r-1}\lambda^{-1} \cdot \Delta_{I-1,J-1} & \text{if } 1 \in J \setminus I. \end{cases}$$

This, together with the positivity of PR and PR^{-1}, implies the following result.

Lemma 5.31. *$\Delta_{I,J}$ is r-non-negative (resp., r-positive) if and only if $\Delta_{I-1,J-1}$ is r-non-negative (resp., r-positive).*

Recall that a subset is "reflected basic" if it is an interval of $[n]$, or it consists of two disjoint intervals of $[n]$, one of which contains 1 (Definition 2.44). Every subset with at most two cyclic intervals is a cyclic shift of a reflected basic subset, so combining the observations above, we see that it suffices to prove Proposition 5.29 in the case where $r \leq k$, and J is a reflected basic subset.

II: Non-negativity of minors that do not depend on λ

Let A be the folded matrix $g(N|t)$, where $N|t \in \mathbb{X}_k$. Here we view A as an array of n^2 rational maps $A_{ij} : \mathbb{X}_k \to \mathbb{C}[\lambda]$. By the definition of g, the maps A_{ij} split up into three categories:

$$A_{ij} \text{ is } \begin{cases} \text{a nonzero map to } \mathbb{C} & \text{if } i-n+k \leq j \leq i \\ \text{a nonzero map to } \mathbb{C}\cdot\lambda & \text{if } j \geq i+k \\ 0 & \text{if } j < i-n+k \text{ or } i < j < i+k. \end{cases} \tag{5.28}$$

In the second case, we say that A_{ij} *depends on* λ; otherwise we say that A_{ij} is *independent of* λ. Given subsets $I, J \subset [n]$, say that the submatrix $A_{I,J}$ is *independent of* λ if A_{ij} is independent of λ for all $i \in I, j \in J$. If $A_{I,J}$ is independent of λ, then $\Delta_{I,J}$ is a rational function $\mathbb{X}_k \to \mathbb{C}$, so r-positivity of $\Delta_{I,J}$ is the same thing as (ordinary) positivity of $\Delta_{I,J}$.

Lemma 5.32. *Let $I = \{i_1 < \cdots < i_r\}$ and $J = \{j_1 < \cdots < j_r\}$ be two r-subsets of $[n]$, with $r \leq k$. If the submatrix $A_{I,J}$ is independent of λ, then the rational map $\Delta_{I,J}$ is positive (equivalently, r-positive) if*

$$i_s - n + k \leq j_s \leq i_s \qquad \textit{for } \; s = 1, \ldots, r, \tag{5.29}$$

and zero otherwise.

Proof. Recall that $\Phi_{I,J}^{n-k} : \mathbb{T}_{n-k} \to \mathbb{C}$ is the rational map $(X_{ij}, t) \mapsto \Delta_{I,J}(\Phi_{n-k}(X_{ij}, t))$. Since $A_{I,J}$ is independent of λ, Proposition 4.5 implies that

$$\Phi_{I,J}^{n-k} = \Delta_{I,J} \circ \Theta_k.$$

By Lemma 4.19, $\Phi_{I,J}^{n-k}$ is positive if (5.29) holds, and zero otherwise, so the same is true of $\Delta_{I,J}$ (since by definition, $\Delta_{I,J}$ is positive if and only if $\Delta_{I,J} \circ \Theta_k$ is positive). □

III: Reflected basic subsets and zero rows

Following Convention 2.40, we interpret an interval $[c, d] \subset \mathbb{Z}$ as a cyclic interval of $[n]$ by reducing each element of $[c, d]$ mod n. As usual, $[c, d]$ is the empty set if

$c > d$. For example, if $n \geq 6$, then $[-2, 3]$ and $[n-2, n+3]$ both represent the cyclic interval $[1, 3] \cup [n-2, n]$, but $[n-2, 3]$ is the empty set.

Given a subset $J \subset [n]$, let $Z(J)$ be the rows of the submatrix $A_{[n],J}$ which are identically zero. We call $Z(J)$ the *zero rows* of the columns J. (By convention, we set $Z(\emptyset) = \emptyset$.) It follows from (5.28) that the j^{th} column of A has zeroes in rows $[j-k+1, j-1]$. This implies that if $s \geq 1$ and $c \in \mathbb{Z}$, then

$$Z([c, c+s-1]) = [c-k+s, c-1]. \tag{5.30}$$

Definition 5.33. Fix $r \leq k, a \in [0, r]$, and $b \in [0, n-r]$, and consider the reflected basic r-subset

$$J^{a,b} = [1, a] \cup [a+b+1, r+b].$$

(Note that every reflected basic r-subset is of this form.) Let Z_1 be the zero rows of columns $[1, a]$, and let Z_2 be the zero rows of columns $[a+b+1, r+b]$. We say that a subset $I \in \binom{[n]}{r}$ *satisfies condition* $C^r_{a,b}$ (or $C_{a,b}$ if r is understood) if

$$I \cap Z(J^{a,b}) = \emptyset, \qquad |I \cap Z_1| \leq r - a, \qquad |I \cap Z_2| \leq a.$$

Note that if $r = k$, then $|Z_1| = k - a$ and $|Z_2| = a$ by (5.30), and $Z(J^{a,b})$ is empty because each row of A has only $k-1$ zeroes. Thus, condition $C^k_{a,b}$ always holds. Note also that by (5.30), we have

$$Z_1 = \begin{cases} \emptyset & \text{if } a = 0 \\ [n+a-k+1, n] & \text{if } a > 0 \end{cases}, \qquad Z_2 = \begin{cases} [r+b-k+1, a+b] & \text{if } a < r \\ \emptyset & \text{if } a = r \end{cases}, \tag{5.31}$$

and thus

$$Z(J^{a,b}) = \begin{cases} [r+b-k+1, b] & \text{if } a = 0 \\ [n+a-k+1, a+b] \cup [n+r+b-k+1, n] & \text{if } a \in [1, r-1] \\ [n+r-k+1, n] & \text{if } a = r. \end{cases} \tag{5.32}$$

Proposition 5.34. *Fix $r \leq k$. Let $J^{a,b}$ be a reflected basic r-subset, and let $Z_1 = Z([1, a]), Z_2 = Z([a+b+1, r+b])$ be the zero rows of the two intervals of $J^{a,b}$. Then for $I \in \binom{[n]}{r}$, the rational map $\Delta_{I,J^{a,b}}$ is r-positive if I satisfies condition $C_{a,b}$, and zero otherwise.*

Thanks to the reductions based on PR, S, and D, Proposition 5.34 implies Proposition 5.29. The proof of Proposition 5.34 is rather technical. The idea is to use Lemma 5.32 and the cyclic shifting map to show that a large class of the minors $\Delta_{I,J^{a,b}}$ are r-positive, and then to show that all other minors of the form $\Delta_{I,J^{a,b}}$ are either zero, or can be expressed as positive Laurent polynomials in the minors that are known to be r-positive. We carry out the first step with the following lemma.

Lemma 5.35. *Fix $r \leq k$.*

1. *The submatrix $A_{I,J^{a,b}}$ is independent of λ if and only if $I \subset [r+b-k+1, n]$, or $a = r$.*
2. *If $A_{I,J^{a,b}}$ is independent of λ, then $\Delta_{I,J^{a,b}}$ is positive (equivalently, r-positive) if I satisfies condition $C_{a,b}$, and zero otherwise.*
3. *If there is some c such that $J^{a,b} - c = J^{a',b'}$ and the submatrix $A_{I-c,J^{a',b'}}$ is independent of λ, then Proposition 5.34 holds for $\Delta_{I,J^{a,b}}$. Here $S - c$ is the subset obtained by subtracting c from each element of S, and interpreting the result mod n.*

Proof. Let $I = \{i_1 < \ldots < i_r\}$ and $J^{a,b} = \{j_1 < \ldots < j_r\}$. By (5.28), A_{ij} is independent of λ if and only if $j - i < k$, so $A_{I,J^{a,b}}$ is independent of λ if and only if

$$j_r - i_1 < k. \tag{5.33}$$

If $a = r$, then $J^{a,b} = [r]$, so (5.33) holds for every I. If $a \neq r$, then $j_r = r + b$, so (5.33) holds if and only if $i_1 > r + b - k$. This proves (1).

Now suppose $A_{I,J^{a,b}}$ is independent of λ. Specializing Lemma 5.32 to the case $J = J^{a,b}$, we see that $\Delta_{I,J^{a,b}}$ is positive if I satisfies

$$i_s \in \begin{cases} [s, s+n-k] & \text{if } s = 1, \ldots, a \\ [s+b, s+b+n-k] & \text{if } s = a+1, \ldots, r, \end{cases} \tag{5.34}$$

and zero otherwise. So to prove (2), we must show that:

Given the assumption $I \subset [r+b-k+1, n]$ or $a = r$, I satisfies condition $C_{a,b}$ if and only if I satisfies (5.34).

To see this, first suppose $a \in [r-1]$. In this case, (5.34) is equivalent to the three inequalities

$$i_a \leq n + a - k, \qquad i_{a+1} \geq a + b + 1, \qquad i_r \leq r + b + n - k. \tag{5.35}$$

Using (5.31), (5.32) and considering separately the cases $r + b \geq k$ and $r + b < k$, it is straightforward to check that for $I \subset [r+b-k+1, n]$, (5.35) is equivalent to condition $C_{a,b}$. If $a = r$, (5.34) is equivalent to the first inequality of (5.35); if $a = 0$, (5.34) is equivalent to the last two inequalities of (5.35). The verification of the claim in these cases is similar.

For (3), suppose $A_{I',J^{a',b'}}$ is independent of λ, where $I' = I - c$ and $J^{a',b'} = J^{a,b} - c$. The cyclic symmetry of the locations of zeroes in the matrix A implies that I' satisfies condition $C_{a',b'}$ if and only if I satisfies condition $C_{a,b}$. Thus, Proposition 5.34 holds for $\Delta_{I,J^{a,b}}$ by Lemma 5.31 and part (2). □

IV: Conclusion of the proof

We now complete the proof of Proposition 5.29 (and thus the proof of Theorem 5.3) by proving Proposition 5.34.

Suppose I does not satisfy condition $C_{a,b}$. If $I \cap Z(J^{a,b}) \neq \emptyset$, then the submatrix $A_{I,J^{a,b}}$ has a row of zeroes, so its determinant vanishes. If $|I \cap Z_1| > r - a$, then the first a columns of $A_{I,J^{a,b}}$ have at least $r - a + 1$ zero rows, so again the determinant vanishes. The case $|I \cap Z_2| > a$ is similar.

Now suppose I satisfies condition $C_{a,b}$. If $J^{a,b}$ is contained in a cyclic interval of size k, then there is some c so that $J^{a,b} - c = J^{a',b'} \subset [k]$. The first k columns of A are independent of λ, so $\Delta_{I,J^{a,b}}$ is r-positive by Lemma 5.35(3).

Assume that $J^{a,b}$ is not contained in a cyclic interval of size k. The zeroes in each row of A are located in $k-1$ cyclically consecutive columns, so in this case, the submatrix $A_{[n],J^{a,b}}$ does not have a row of zeroes. In other words, $Z(J^{a,b}) = \emptyset$, so the first part of condition $C_{a,b}$ is automatically satisfied. Also, Z_1 and Z_2 are both non-empty, and $|Z_1| = k - a$, $|Z_2| = k - r + a$, so I satisfies condition $C_{a,b}$ if and only if

$$|Z_i \setminus I| \geq k - r \quad \text{for } i = 1, 2.$$

Thus, we need to show that $\Delta_{I,J^{a,b}}$ is r-positive whenever

$$Y_1 \in \binom{Z_1}{k-r}, \qquad Y_2 \in \binom{Z_2}{k-r}, \qquad \text{and} \qquad I \in \binom{[n] \setminus (Y_1 \cup Y_2)}{r}. \tag{5.36}$$

Let Y_1, Y_2, and I be as in (5.36). Suppose first that $I \supset Z_1 \setminus Y_1$. In this case, since $Z_1 = [n + a - k + 1, n]$, the lower left $(r - a) \times a$ submatrix of $A_{I,J^{a,b}}$ consists entirely of zeroes, so we have

$$\Delta_{I,J^{a,b}}(A) = \Delta_{I',[1,a]}(A)\Delta_{I'',[a+b+1,r+b]}(A),$$

where $I' = I \setminus (Z_1 \setminus Y_1)$ and $I'' = Z_1 \setminus Y_1$. The first k columns and the last k rows of A are independent of λ, so the submatrices $A_{I',[1,a]}$ and $A_{I'',[a+b+1,r+b]}$ are independent of λ. The a-subset I' is disjoint from Z_1, so I' satisfies condition $C^a_{a,0}$; similarly, since $Z_1 \cap Z_2 = Z(J_{a,b}) = \emptyset$, the $r-a$ subset I'' is disjoint from Z_2, so I'' satisfies condition $C^{r-a}_{0,a+b}$. Thus, $\Delta_{I',[1,a]}$ and $\Delta_{I'',[a+b+1,r+b]}$ are positive rational functions by Lemma 5.35(2), so $\Delta_{I,J^{a,b}}$ is a positive (hence r-positive) rational function.

If $I \supset Z_2 \setminus Y_2$, set $c = a + b$, so that $J^{a,b} - c = J^{r-a,n-r-b}$. Let $I' = I - c$, $Z_1' = Z_2 - c$, and $Y_1' = Y_2 - c$. Clearly Z_1' consists of the zero rows of columns $[1, r - a]$, and $I' \supset Z_1' \setminus Y_1'$, so $\Delta_{I',J^{r-a,n-r-b}}$ is r-positive by the previous paragraph, and $\Delta_{I,J^{a,b}}$ is r-positive by Lemma 5.31.

It remains to consider the case where $I \not\supset Z_i \setminus Y_i$ for $i = 1, 2$. This case is subtler, and we proceed indirectly. Set $S = [n] \setminus (Y_1 \cup Y_2)$. Call a subset of S an *S-interval* if it is of the form $S \cap [c,d]$, with $1 \leq c \leq d \leq n$. Let I be a basic r-subset of S; this means that I consists of one or two S-intervals, and if there are two S-intervals, one of them contains the largest element of S. If $I \subset [r+b-k+1,n]$, then $\Delta_{I,J^{a,b}}$ is r-positive by Lemma 5.35, so suppose $I \not\subset [r+b-k+1,n]$. We claim that

$$I \subset [1,a+b] \cup [n+a-k+1,n]. \tag{5.37}$$

To prove (5.37), recall that $Z_1 = [n+a-k+1,n]$, and $Z_2 = [r+b-k+1,a+b]$. Since $I \not\subset [r+b-k+1,n]$, we must have $r+b-k+1 \geq 2$, so $[r+b-k+1,a+b]$ is an "honest" (i.e., non-cyclic) interval of $[n]$, and we have

$$S = [1,r+b-k] \cup (Z_2 \setminus Y_2) \cup [a+b+1,n+a-k] \cup (Z_1 \setminus Y_1).$$

By assumption, I is a basic r-subset of S which intersects $[1,r+b-k]$ and does not contain all of $Z_2 \setminus Y_2$ or $Z_1 \setminus Y_1$. There are two possibilities: either I is a single S-interval contained in $[1,r+b-k] \cup Z_2 \setminus Y_2$, or $I = I_1 \cup I_2$, where I_1 is an S-interval contained in $[1,r+b-k] \cup Z_2 \setminus Y_2$, and I_2 is an S-interval contained in $Z_1 \setminus Y_1$. In either case, $I \cap [a+b+1,n+a-k] = \emptyset$, so (5.37) holds.

Now let $I' = I - (a+b)$, and note that $J^{a,b} - (a+b) = J^{r-a,n-r-b}$. By (5.37), we have

$$I' \subset [n-b-k+1,n] = [r+(n-r-b)-k+1,n],$$

so $\Delta_{I,J^{a,b}}$ is r-positive by Lemma 5.35.

Let J be an r-subset of S. Proposition 2.46 says that $\Delta_{J,J^{a,b}}$ can be expressed as a positive Laurent polynomial in the $\Delta_{I,J^{a,b}}$ with I basic. We have shown that these $\Delta_{I,J^{a,b}}$ are r-positive, and furthermore, it's clear from the proof that each of these $\Delta_{I,J^{a,b}}$ is a monomial with respect to λ. It follows that $\Delta_{J,J^{a,b}}$ is r-positive (although not necessarily a monomial). We have shown that $\Delta_{I,J^{a,b}}$ is r-positive whenever I satisfies (5.36), so we are done.

5.5 Proof of the identity $g \circ R = g$

In this section we prove Theorem 5.4. Suppose $u = M|s \in \mathbb{X}_\ell$ and $v = N|t \in \mathbb{X}_k$. Let $A = g(u)g(v)$ (viewed as a folded matrix), and let $A_t = A|_{\lambda=(-1)^{k-1}t}$, and $A_s = A|_{\lambda=(-1)^{\ell-1}s}$. Define $v' = N'|t \in \mathbb{X}_k$ and $u' = M'|s \in \mathbb{X}_\ell$ by $R(u,v) = (v',u')$. We must show that

$$g(u)g(v) = g(v')g(u'). \tag{5.38}$$

For $I \in \binom{[n]}{k}$, let $P'_I = P_I(N')$, and for $J \in \binom{[n]}{\ell}$, let $Q'_J = Q_s^J(M') = P_{w_0(J)}(S_s(M'))$. By (5.3), we have

$$\frac{P'_I}{P'_{[n-k+1,n]}} = \frac{\Delta_{I,[k]}(A_t)}{\Delta_{[n-k+1,n],[k]}(A_t)} \qquad \text{and} \qquad \frac{Q'_J}{Q'_{[\ell]}} = \frac{\Delta_{[n-\ell+1,n],J}(A_s)}{\Delta_{[n-\ell+1,n],[\ell]}(A_s)}. \tag{5.39}$$

The key to the proof of Theorem 5.4 is the following identity.

Proposition 5.36. *For $r = 1, \ldots, n$, we have*

$$(5.40) \qquad (t + (-1)^k \lambda) \frac{Q'_{[\ell-1]\cup\{r\}}}{Q'_{[\ell]}} = \sum_{a=1}^{n} (-1)^{n+a} \frac{P'_{[n-k,n]\setminus\{a\}}}{P'_{[n-k+1,n]}} A_{ar}.$$

(Note that by Convention 2.40, the terms on the right-hand side of (5.40) with $a < n-k$ are zero, and the left-hand side is zero when $r \le \ell - 1$.)

Before proving this identity, we use it to deduce Theorem 5.4. Consider the folded matrices

$$B = (t + (-1)^k \lambda) \cdot g(u') \qquad \text{and} \qquad C = h(v')g(u)g(v)$$

where $h : \mathbb{X}_k \to B^-$ is defined by (3.31). The left-hand side of (5.40) is equal to B_{nr} by Lemma 3.18, and the right-hand side is equal to C_{nr} by the definition of h, so Proposition 5.36 says that $B_{nr} = C_{nr}$ for all r. Recall the cyclic shift map PR and the shift automorphism sh from §3.1. Since $\mathrm{sh} \circ g = g \circ \mathrm{PR}$ by Lemma 3.3(1), $\mathrm{sh} \circ h = h \circ \mathrm{PR}$ by a similar argument, and PR commutes with R by Theorem 5.11(2), the equality of the last rows of B and C implies that $B = C$.

By Lemma 3.20, the matrix $h(v')$ satisfies

$$g(v')h(v') = (t + (-1)^k \lambda) \cdot Id,$$

so left-multiplying B and C by $g(v')$ gives the desired equality (5.38).

Proof of Proposition 5.36. Let

$$p_r(\lambda) = \sum_{a=n-k}^{n} (-1)^{n+a} \frac{P'_{[n-k,n]\setminus\{a\}}}{P'_{[n-k+1,n]}} A_{ar}$$

be the right-hand side of (5.40). Let X be the unfolding of A. The entries of the matrices $g(u)$ and $g(v)$ are at most linear in λ, so the entries of their product A are at most quadratic in λ, and

$$A_{ij} = X_{ij} + \lambda X_{n+i,j} + \lambda^2 X_{2n+i,j}.$$

Recall from §3.1 that an unfolded matrix X is m-shifted unipotent if $X_{ij} = 0$ when $i - j > m$, and $X_{ij} = 1$ when $i - j = m$. The matrices $g(u)$ and $g(v)$ are $(n-\ell)$- and $(n-k)$-shifted unipotent, respectively, so their product is $(2n - \ell - k)$-shifted unipotent. This implies, in particular, that if $a \ge n - k$, then A_{ar} is either constant or linear in λ, so $p_r(\lambda)$ is a polynomial of degree at most one. Proposition 5.36 is therefore an immediate consequence of the following two claims:

1. $(-1)^{k-1} t$ is a root of $p_r(\lambda)$;

2. the coefficient of λ in $p_r(\lambda)$ is $(-1)^k \dfrac{Q'_{[\ell-1]\cup\{r\}}}{Q'_{[\ell]}}$.

To prove the first claim, let $D = (A_t)_{[n-k,n],<1,2,\ldots,k,r>}$ denote the (generalized) submatrix of A_t consisting of the last $k+1$ rows, and columns $1,\ldots,k,r$, in that order. If $r \in [k]$, then clearly $\det(D) = 0$; if $r \notin [k]$, then $\det(D)$ is still zero because $g(v)|_{\lambda=(-1)^{k-1}t}$ has rank k by Proposition 3.7(2). On the other hand, expanding the determinant along column r gives $\det(D) = p_r((-1)^{k-1}t)$, so (1) follows.

It remains to prove the second claim. Since the coefficient of λ in A_{ar} is $X_{n+a,r}$, claim (2) can be rephrased as the identity

$$\frac{Q'_{[\ell-1]\cup\{r\}}}{Q'_{[\ell]}} = \sum_{a=n-k}^{n} (-1)^{k+n+a} \frac{P'_{[n-k,n]\setminus\{a\}}}{P'_{[n-k+1,n]}} X_{n+a,r}. \tag{5.41}$$

If $r \leq \ell-1$, then $X_{n+a,r} = 0$ for $a \geq n-k$ (since X is $(2n-\ell-k)$-shifted unipotent), so (5.41) holds trivially in this case.

To prove (5.41) for $r \geq \ell$, we start by massaging the (folded) matrices $g(u)$ and $g(v)$ into a simpler form. By Proposition 3.7(2), the matrix $g(u)|_{\lambda=(-1)^{\ell-1}s}$ has rank ℓ. This means that we may add linear combinations of the last ℓ rows of $g(u)$ (which are linearly independent, and do not depend on λ) to the first $n-\ell$ rows to obtain the matrix $g(u)^*$, where

$$g(u)^*_{ij} = \begin{cases} g(u)_{ij} & \text{if } i \geq n-\ell+1 \\ (\lambda + (-1)^\ell s) g(u)_{ij} & \text{if } j-i \geq \ell \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we may add linear combinations of the first k columns of $g(v)$ to the last $n-k$ columns to obtain the matrix $g(v)^*$, where

$$g(v)^*_{ij} = \begin{cases} g(v)_{ij} & \text{if } j \leq k \\ (\lambda + (-1)^k t) g(v)_{ij} & \text{if } j-i \geq k \\ 0 & \text{otherwise.} \end{cases}$$

Define $A^* = g(u)^* g(v)^*$. See Figure 7 for an example of the matrices $g(u)^*, g(v)^*$, and A^*.

Given two subsets $I, J \subset [n]$ of the same cardinality, say that (I,J) is a *good pair* if I contains or is contained in the interval $[n-\ell+1, n]$, and J contains or is contained in the interval $[1,k]$. The construction of $g(u)^*$ and $g(v)^*$, together with the Cauchy–Binet formula, implies that

$$\Delta_{I,J}(A^*) = \Delta_{I,J}(A) \quad \text{if } (I,J) \text{ is a good pair.} \tag{5.42}$$

$$\begin{pmatrix} * & 0 & \lambda & \lambda * \\ * & * & 0 & \lambda \\ 1 & * & * & 0 \\ 0 & 1 & * & * \end{pmatrix} \begin{pmatrix} * & 0 & \lambda & \lambda * \\ * & * & 0 & \lambda \\ 1 & * & * & 0 \\ 0 & 1 & * & * \end{pmatrix} = \begin{pmatrix} X_{11}+\lambda & \lambda X_{52} & \lambda X_{53} & \lambda X_{54} \\ X_{21} & X_{22}+\lambda & \lambda X_{63} & \lambda X_{64} \\ X_{31} & X_{32} & X_{33}+\lambda & \lambda X_{74} \\ X_{41} & X_{42} & X_{43} & X_{44}+\lambda \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \lambda+s & (\lambda+s)* \\ 0 & 0 & 0 & \lambda+s \\ 1 & * & * & 0 \\ 0 & 1 & * & * \end{pmatrix} \begin{pmatrix} * & 0 & \lambda+t & (\lambda+t)* \\ * & * & 0 & \lambda+t \\ 1 & * & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= \left(\begin{array}{cc|cc} \lambda+s & (\lambda+s)X_{52} & 0 & 0 \\ 0 & \lambda+s & 0 & 0 \\ \hline X_{31} & X_{32} & \lambda+t & (\lambda+t)X_{74} \\ X_{41} & X_{42} & 0 & \lambda+t \end{array}\right)$$

Figure 7: Suppose $n=4, \ell=k=2$, and $u=M|s, v=N|t \in \mathbb{X}_2$. The first line shows the product $g(u)g(v)=A$, where the $*$'s are ratios of Plücker coordinates of M or N, possibly scaled by s or t, and X is the unfolding of A. The next lines show the product $g(u)^*g(v)^*=A^*$, with the blocks of A^* indicated.

Set

$$\alpha_s = \lambda + (-1)^\ell s, \qquad \alpha_t = \lambda + (-1)^k t.$$

The reader may easily verify that the entries of A^* are given by

$$(A^*)_{ij} = \begin{cases} X_{ij} & \text{if } i \geq n-\ell+1 \text{ and } j \leq k \\ \alpha_s X_{n+i,j} & \text{if } i \leq n-\ell \text{ and } j \leq k \\ \alpha_t X_{n+i,j} & \text{if } i \geq n-\ell+1 \text{ and } j \geq k+1 \\ \alpha_s\alpha_t X_{2n+i,j} & \text{if } i \leq n-\ell \text{ and } j \geq k+1. \end{cases} \tag{5.43}$$

In other words, A^* has the block form

$$\begin{array}{r|c|c|} \hline n-\ell & \alpha_s E & \alpha_s\alpha_t F \\ \hline \ell & G & \alpha_t H \\ \hline \multicolumn{1}{r}{} & \multicolumn{1}{c}{k} & \multicolumn{1}{c}{n-k} \end{array}$$

$$\left(\begin{array}{cc|ccccc} \alpha_s X_{81} & \alpha_s X_{82} & 0 & 0 & 0 & 0 & \alpha_s\alpha_t \\ \alpha_s & \alpha_s X_{92} & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_s & 0 & 0 & 0 & 0 & 0 \\ \hline X_{41} & X_{42} & \alpha_t & \alpha_t X_{11,4} & \alpha_t X_{11,5} & \alpha_t X_{11,6} & \alpha_t X_{11,7} \\ X_{51} & X_{52} & 0 & \alpha_t & \alpha_t X_{12,5} & \alpha_t X_{12,6} & \alpha_t X_{12,7} \\ X_{61} & X_{62} & 0 & 0 & \alpha_t & \alpha_t X_{13,6} & \alpha_t X_{13,7} \\ X_{71} & X_{72} & 0 & 0 & 0 & \alpha_t & \alpha_t X_{14,7} \end{array}\right)$$

Figure 8: The matrix A^* in the case $n = 7, \ell = 4, k = 2$, with blocks indicated.

where G is filled with the constant terms of the corresponding entries of A, E and H with the coefficients of λ in the corresponding entries of A, and F with the coefficients of λ^2 in the corresponding entries of A (examples are shown in Figures 7 and 8). Furthermore, the matrices E, F, G, H satisfy:

- H and $\mathrm{fl}(E)$ (see §3.3.2) are upper uni-triangular;
- all entries in the first column and last row of F are zero.

Taken together, these properties imply that row $n - \ell$ (resp., column $k + 1$) of A^* has only one nonzero entry, namely, $(A^*)_{n-\ell,k} = \alpha_s$ (resp., $(A^*)_{n-\ell+1,k+1} = \alpha_t$).

The argument now splits into two cases.

Case 1: $\ell \neq k$.

First note that the argument used to deduce Theorem 5.4 from Proposition 5.36 can be reversed to deduce the latter from the former, so these two results are in fact equivalent. Thanks to Proposition 3.15 and the fact that the geometric Schützenberger involution commutes with the geometric R-matrix, (5.38) holds for $(u, v) \in \mathbb{X}_\ell \times \mathbb{X}_k$ if and only if it holds for $(S(v), S(u)) \in \mathbb{X}_k \times \mathbb{X}_\ell$. Thus, the ℓ, k case of Theorem 5.4 is equivalent to the k, ℓ case, so the same is true of Proposition 5.36, and we may assume $\ell > k$ here.

Using (5.42), (5.43), and the fact that the matrix H defined above is upper uni-triangular (it may be helpful to refer to Figure 8), we compute

$$\begin{aligned} \Delta_{[n-\ell+1,n],[\ell-1]\cup\{r\}}(A) &= \Delta_{[n-\ell+1,n],[\ell-1]\cup\{r\}}(A^*) \\ &= (-1)^{k(\ell-k)}\alpha_t^{\ell-k-1} \sum_{a=n-k}^{n} (-1)^{a-n+k} A^*_{ar} \Delta_{[n-k,n]\setminus\{a\},[k]}(A^*) \\ &= (-1)^{k(\ell-k)}\alpha_t^{\ell-k} \sum_{a=n-k}^{n} (-1)^{a-n+k} X_{n+a,r} \Delta_{[n-k,n]\setminus\{a\},[k]}(A). \end{aligned}$$

In particular,

$$\Delta_{[n-\ell+1,n],[\ell]}(A) = (-1)^{k(\ell-k)}\alpha_t^{\ell-k}\Delta_{[n-k+1,n],[k]}(A),$$

and thus

$$\frac{\Delta_{[n-\ell+1,n],[\ell-1]\cup\{r\}}(A)}{\Delta_{[n-\ell+1,n],[\ell]}(A)} = \sum_{a=n-k}^{n} (-1)^{a-n+k} X_{n+a,r} \frac{\Delta_{[n-k,n]\setminus\{a\},[k]}(A)}{\Delta_{[n-k+1,n],[k]}(A)}. \tag{5.44}$$

By (5.39), the two sides of (5.41) are obtained by evaluating the two sides of (5.44) at $\lambda = (-1)^{\ell-1}s$ and $\lambda = (-1)^{k-1}t$, respectively. Note, however, that the entries of the submatrix $A_{[n-\ell+1,n],[k]}$ do not depend on the value of λ, so since $k < \ell$, the right-hand side of (5.44) is independent of λ. This means the left-hand side is also independent of λ, and (5.41) follows.

Case 2: $\ell = k$.

Since $X_{n+a,k}$ is 1 when $a = n-k$ and 0 when $a \geq n-k$, (5.41) clearly holds when $r = k$. Fix $r \geq k+1$. Let z denote the coefficient of λ in $\Delta_{[n-k,n],[k]\cup\{r\}}(A)$. We will deduce (5.41) by computing z in two ways. On the one hand, we use (5.42), (5.43), and the fact that the only nonzero entry in the $(n-k)$th row of A^* is $A^*_{n-k,k} = \alpha_s$ to compute

$$\begin{aligned}\Delta_{[n-k,n],[k]\cup\{r\}}(A) &= \Delta_{[n-k,n],[k]\cup\{r\}}(A^*)\\ &= \alpha_s \sum_{a=n-k+1}^{n} (-1)^{a-n+k-1} A^*_{ar} \Delta_{[n-k+1,n]\setminus\{a\},[k-1]}(A^*)\\ &= \alpha_s\alpha_t \sum_{a=n-k+1}^{n} (-1)^{a-n+k-1} X_{n+a,r} \Delta_{[n-k+1,n]\setminus\{a\},[k-1]}(A),\end{aligned}$$

which shows that

$$z = -(s+t) \sum_{a=n-k+1}^{n} (-1)^{a-n} X_{n+a,r} \Delta_{[n-k+1,n]\setminus\{a\},[k-1]}(A). \tag{5.45}$$

On the other hand, the coefficient of λ in the determinant $\Delta_{[n-k,n],[k]\cup\{r\}}(A)$ can be computed from the unfolded matrix X by taking the alternating sum of determinants of submatrices of columns $[1,k]\cup\{r\}$ of X, where k of the rows come from $[n-k,n]$ and one row comes from $[2n-k,2n]$, i.e.,

$$z = \sum_{a=n-k}^{n} (-1)^{n-a} \Delta_{([n-k,n]\setminus\{a\})\cup\{n+a\},[k]\cup\{r\}}(X). \tag{5.46}$$

Consider the term in this sum with $a = n-k$. Expanding the determinant along row $2n-k$ (it may be helpful to refer to the first line of Figure 7), we obtain

$$\Delta_{[n-k+1,n]\cup\{2n-k\},[k]\cup\{r\}}(X) = X_{2n-k,r}\Delta_{[n-k+1,n],[k]}(X) - \Delta_{[n-k+1,n],[k-1]\cup\{r\}}(X).$$

Observe that the entries of the bottom-left $k \times k$ submatrix of A do not depend on λ, and the entries A_{ar} for $a \geq n-k+1$ are polynomials in λ of degree at most one. This means that

$$\Delta_{[n-k+1,n],[k]}(X) = \Delta_{[n-k+1,n],[k]}(A) = \Delta_{[n-k+1,n],[k]}(A_s) = \Delta_{[n-k+1,n],[k]}(A_t) \tag{5.47}$$

and

$$\begin{aligned}\Delta_{[n-k+1,n],[k-1]\cup\{r\}}(X) &= \Delta_{[n-k+1,n],[k-1]\cup\{r\}}(A) \\ &\quad - \lambda \sum_{a=n-k+1}^{n} (-1)^{n-a} X_{n+a,r} \Delta_{[n-k+1,n]\setminus\{a\},[k-1]}(A).\end{aligned}$$

Since the left-hand side of this equation is independent of λ, we may substitute $\lambda = (-1)^{k-1}s$ into the right-hand side to obtain

$$\begin{aligned}\Delta_{[n-k+1,n],[k-1]\cup\{r\}}(X) &= \Delta_{[n-k+1],[k-1]\cup\{r\}}(A_s) \\ &\quad - (-1)^{k-1} s \sum_{a=n-k+1}^{n} (-1)^{n-a} X_{n+a,r} \Delta_{[n-k+1,n]\setminus\{a\},[k-1]}(A).\end{aligned}$$

By similar reasoning,

$$\Delta_{([n-k,n]\setminus\{a\})\cup\{n+a\},[k]\cup\{r\}}(X) = X_{n+a,r}\left(\Delta_{[n-k,n]\setminus\{a\},[k]}(A_t) - t\Delta_{[n-k+1,n]\setminus\{a\},[k-1]}(A)\right)$$

for $a = n-k+1, \ldots, n$.

Putting all of this together, we may rewrite (5.46) as

$$\begin{aligned}z &= (-1)^k X_{2n-k,r} \Delta_{[n-k+1,n],[k]}(A) - (-1)^k \Delta_{[n-k+1,n],[k-1]\cup\{r\}}(A_s) \\ &\quad + \sum_{a=n-k+1}^{n} (-1)^{n-a} X_{n+a,r} (\Delta_{[n-k,n]\setminus\{a\},[k]}(A_t) - (s+t)\Delta_{[n-k+1,n]\setminus\{a\},[k-1]}(A)).\end{aligned} \tag{5.48}$$

Equating the expressions for z in (5.45) and (5.48), we obtain

$$\begin{aligned}\Delta_{[n-k+1,n],[k-1]\cup\{r\}}(A_s) &= X_{2n-k,r} \Delta_{[n-k+1,n],[k]}(A) \\ &\quad + \sum_{a=n-k+1}^{n} (-1)^{n-a+k} X_{n+a,r} \Delta_{[n-k,n]\setminus\{a\},[k]}(A_t).\end{aligned}$$

Divide both sides of this equation by $\Delta_{[n-k+1,n],[k]}(A)$ and use (5.47) and (5.39) to obtain (5.41). This completes the proof. □

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