Applications of Canonical Metrics on Berkovich Spaces

by

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To my family and teachers

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ABSTRACT

This thesis examines the nature of Temkin's canonical metrics on the sheaves of differentials of Berkovich spaces, and discusses 3 applications thereof. First, we show a comparison theorem between Temkin's metric on the $\beth$-analytification of a smooth variety over a trivially-valued field of characteristic zero, and a weight metric defined in terms of log discrepancies. This result is the trivially-valued counterpart to a comparison theorem of Temkin between his metric and the weight metric of Mustață–Nicaise in the discretely-valued setting.

These weight metrics are used to define an essential skeleton of a pair over a trivially-valued field; this is done following the approach of Brown–Mazzon in the discretely-valued case, and we show a compatibility result between the essential skeletons of pairs in the two settings. Furthermore, a careful study of the closures of these skeletons enables us to realize the toric skeleton of a toric variety as an essential skeleton.

On the Berkovich unit disc, Temkin's metric acts a substitute for the Lebesgue measure. Adopting this philosophy, we show a non-Archimedean version of the Ohsawa–Takegoshi extension theorem. As a corollary, we deduce a non-Archimedean analogue of Demailly's regularization theorem for quasisubharmonic functions on the Berkovich disc.

Finally, we employ Temkin's metric and essential skeletons to compute the dual boundary complexes of two classes of character varieties that arise in non-abelian

Hodge theory. These two results provide the first non-trivial evidence for the geometric $P = W$ conjecture of Katzarkov–Noll–Pandit–Simpson in the compact case. For each result, we give two proofs: one using non-Archimedean geometry over a trivially-valued field, and another in the discretely-valued setting. The latter produces degenerations of compact hyper-Kähler manifolds, which are of independent interest.

CHAPTER I

Introduction

Analytic methods in algebraic geometry are powerful tools. From Hodge theory and Kodaira's vanishing theorem to the deformation invariance of plurigenera, there are many theorems in algebraic geometry whose proofs either necessitate or are facilitated by input from complex-analytic geometry. However, when working over a non-Archimedean field, the story is more mysterious. The theory of non-Archimedean analytic geometry was born in the early 1960's when Tate initiated in [Tat71] the study of rigid-analytic spaces over a (nontrivially-valued) non-Archimedean field in order to generalize the uniformization of elliptic curves to the p-adic setting. In the decades since Tate's foundational work, the field of non-Archimedean geometry has grown rapidly and several approaches have appeared in the literature. We will be interested in the theory of Berkovich spaces.

In [Ber90, Ber93], Berkovich introduced his theory of analytic spaces over a non-Archimedean field. Berkovich builds on Tate's rigid spaces and it generalizes it in several directions; in particular, Berkovich's theory allows the base field to be trivially-valued. Berkovich spaces provide a more honest geometric alternative to rigid spaces, which enables the use of methods similar to those in complex geometry. Furthermore, Berkovich spaces are close cousins of Huber's adic spaces (see [Hub93, Hub94]) and of Raynaud's formal models (see [Bos14]), and they are intimately related to the objects of tropical geometry (see [Pay09]).

The theory of Berkovich spaces enjoys a deep connection with algebraic geometry: to a variety X over a complete non-Archimedean field k, we can associate a Berkovich space X^{an}, called its Berkovich analytification. The analytification X^{an} is a non-Archimedean analogue of the classical analytification of a complex variety, as in [Ser56]. The underlying topological space of X^{an} consists of valuations on the residue fields of X that extend the given valuation on the base field k. When k is trivially-valued, the Berkovich analytification contains a compact subspace $X^{\beth} \subseteq X^{\mathrm{an}}$ of particular interest, called the $\beth$-analytification of X. First introduced by Thuillier in [Thu07], $X^{\beth}$ is a compactification of the space of rank-1 valuations on the function field of X that restrict to the trivial valuation on the base field k and admit a centre on X.

In recent years, the theory of Berkovich spaces has found many applications across algebraic geometry and other branches of mathematics. Several significant examples are given below:

- Harris and Taylor's proof in [HT01] of the local Langlands conjecture (for GL_n over a discretely-valued field of characteristic zero) crucially uses the étale cohomology theory developed for Berkovich spaces in [Ber93].

- In [Ber94, Ber96a], Berkovich proves a conjecture of Deligne, which states that the vanishing cycles sheaves of a scheme over a discrete valuation ring depend only on the formal completion along the special fibre.

- Berkovich spaces serve as ideal topological spaces on which to study complex and p-adic dynamics, e.g. as in [BR10].

- Following the work of Kontsevich–Soibelman in [KS06], Berkovich's theory has enabled progress towards the construction of the SYZ fibration in mirror symmetry, and the construction of the non-Archimedean SYZ fibration as in [Yu16, NXY18].

- In [Thu07], Thuillier used $\beth$-analytifications to prove that the homotopy type of the dual complex of the boundary divisor of a compactification of a smooth variety over a perfect field is independent of the choice of compactification.

- There are non-Archimedean analogues of complex-geometric theorems that have been developed for Berkovich spaces, most notably the solution by Boucksom–Favre–Jonsson of a non-Archimedean Calabi–Yau problem in [BFJ15].

- In [ACP15], Abramovich–Caporaso–Payne realize the tropical moduli space of curves as a combinatorial subspace of the Berkovich analytification of the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of $\mathcal{M}_{g,n}$, which has revolutionized the study of tropical moduli.

- Berkovich's framework can be used to construct a non-Archimedean limit of a degeneration of complex-analytic spaces, which can be then used to understand the limiting behaviour of the degeneration, as in [Ber09, Jon16, Fav16, BJ17].

- Berkovich spaces are used to give criteria for K-stability and the existence of Kähler–Einstein metrics on Fano manifolds, as in the proof of Boucksom–Berman–Jonsson of the Yau–Tian–Donaldson conjecture in [BBJ15].

In [Tem16], Temkin introduces canonical metrics on the sheaves of differentials of a Berkovich space, building on the work of [KS06, MN15]. In complex geometry, canonical metrics and canonical volume forms are ubiquitous and their utility is abundant; one can ask whether the same holds true in the non-Archimedean setting. To this end, we examine in this thesis the nature of Temkin's metric and discuss 3 applications thereof: one to the study of skeletons of Berkovich analytifications, another to potential theory on Berkovich curves, and finally one to the geometric $P = W$ conjecture from non-abelian Hodge theory. This work appears in the papers [Ste18, MMS18].

1.1 Temkin's canonical metric and weight functions

For a smooth variety X over a discretely-valued field k of residue characteristic zero, Mustață and Nicaise introduce in [MN15] the notion of a weight function on X^{an} associated to a rational pluricanonical form on X. More precisely, for a section η of $\omega_{X/k}^{\otimes m}$ defined on a Zariski-open subset $U \subseteq X$, they construct a lower-semicontinuous function

$$\mathrm{wt}_\eta \colon U^{\mathrm{an}} \to \mathbf{R} \cup \{\pm\infty\}$$

on the Berkovich analytification U^{an} of U. When η is a global section of $\omega_{X/k}^{\otimes m}$, the weight function wt_η is defined in terms of the geometry of models of X over the valuation ring k° of k, and hence wt_η enjoys many connections with the relative minimal model program (see [NX16, BN16]). The weight functions were originally introduced by Kontsevich and Soibelman in [KS06] in the case when $k = \mathbf{C}((t))$ and X is a maximally-degenerate K3 surface in order to build the mirror family of X using the SYZ fibration. Moreover, the definition was recently extended to the case of pairs in [BM17] in order to study degenerations of hyper-Kähler varieties.

The weight function wt_η can be thought of as a way to measure the "length" of the form η at each point of X^{an}. In this way, the collection of all weight functions on X^{an} gives rise to a weight metric on the analytification $(\omega_{X/k}^{\otimes m})^{\mathrm{an}}$ of the pluricanonical bundle $\omega_{X/k}^{\otimes m}$ on X. In [Tem16], Temkin describes how the weight metric can be realized as a special case of a general construction in Berkovich geometry: for any non-Archimedean field k and any Berkovich space Z over k, Temkin constructs a canonical metric on the pluricanonical sheaves of Z, which we call Temkin's metric. Temkin shows in [Tem16, Theorem 8.3.3] that the canonical metric and the weight metric coincide when k is a discretely-valued field of residue characteristic zero and $Z = X^{\mathrm{an}}$ for a smooth variety X over k. A thorough discussion of Temkin's work appears in §3.3.

Now, let k be a trivially-valued field of characteristic zero, X a normal variety over k, K_X a canonical divisor on X, D a Weil divisor on X, and $m \in \mathbf{Z}_{>0}$ be such that $m(K_X + D)$ is Cartier. In §3.2.2, we construct a weight function on the $\beth$-analytification $X^{\beth}$ associated to a rational pluricanonical form on X. As in the discretely-valued setting, these weight functions can be bundled into a weight metric on the $\beth$-analytification $\mathcal{O}_X(m(K_X + D))^{\beth}$ of the logarithmic pluricanonical bundle.

The weight functions on $X^{\beth}$ are built in terms of log discrepancy functions. The log discrepancy of a prime divisor over X is a ubiquitous notion in birational geometry, and it can be used to define a log discrepancy function on the set of divisorial valuations on X. Further, it extends to a lower-semicontinuous function on spaces of (semi)valuations by [JM12, BdFFU15, BJ18a]. Now, for each rational pluricanonical form η with poles along D, the weight function wt_η is, up to scaling, defined to be the log discrepancy of the pair $(X, D_{\mathrm{red}} - \mathrm{div}(\eta))$; see §3.2.1 for a more precise definition.

The definition of the weight metric in the trivially-valued setting is justified by the following comparison theorem in the case when X is smooth and the boundary D is empty.

Theorem A. *Let k be a trivially-valued field of characteristic zero and let X be a* *smooth variety over k. Then, Temkin's metric on $(\omega_{X/k}^{\otimes m})^{\beth}$ coincides with the weight metric.*

Theorem A is the trivially-valued analogue of Temkin's comparison theorem in the discretely-valued setting, and the proof of Theorem A proceeds by reducing to the discretely-valued case. The strategy of proof can be summarized in 3 steps:

- We show that both metrics in Theorem A are determined on the subspace X^{div} of divisorial points in $X^{\beth}$ (see Theorem 3.3.3.3).

- For $x \in X^{\mathrm{div}}$ and a suitable discretely-valued extension k' of k, the fibre of the ground field extension map $X_{k'}^{\mathrm{an}} \to X^{\mathrm{an}}$ above x contains a distinguished point x'. We give an explicit description of x' in terms of models of $X_{k'}$ over the valuation ring $(k')^{\circ}$ of k' (see §3.4.2).

- Finally, we relate Temkin's metric at x and x', and the weight metric at x and x', and appeal to Temkin's comparison theorem in the discretely-valued setting (see Lemma 3.3.2.1 and Proposition 3.4.3.1).

The major use of Theorem A is that it provide a computable expression for Temkin's metric on $X^{\beth}$, i.e. one in terms of the birational geometry of X. This enables the use of Temkin's metric in several applications. Furthermore, Theorem A shows that the log discrepancy function is the "correct" extension of its values on divisorial valuations, since it coincides with Temkin's metric, which is a canonically-defined object.

1.2 Dual complexes and essential skeletons

Degeneration and compactification of algebraic varieties are powerful tools in algebraic geometry: they recast the study of non-proper varieties into that of proper varieties and their invariants. General theorems on resolutions of singularities ensure the existence of simple normal crossing (snc) degenerations and compactifications in characteristic zero: this means that the central fibre of the degeneration or the boundary of the compactification are divisors with smooth irreducible components, which intersect one another transversally.

To such a divisor D on a variety X, we associate a regular Δ-complex $\mathcal{D}(D)$, namely the dual intersection complex of D. The dual complex encodes the combinatorial structure of the connected components of intersections of irreducible components of D, called the strata of D, and it captures aspects of the geometry of $X \setminus D$. For instance, it follows from Deligne [Del71] that there exists a correspondence between the reduced rational homology of $\mathcal{D}(D)$ and the top dimensional pieces of the weight filtration on the cohomology of $X \setminus D$. See also [Ber00, Theorem 1.1.(c)] and [Pay13, Theorem 4.4].

Many conjectures in the theory of singularities, tropical geometry, mirror symmetry, or even non-abelian Hodge theory involve understanding the homotopy or homeomorphism type of particular dual complexes. This can often be done by reframing the study of dual complexes in terms of non-Archimedean geometry. This is done by constructing a Berkovich space Z over a non-Archimedean field k and a polyhedral subspace $S \subseteq Z$, called a skeleton of Z, such that the following two conditions hold:

- S is homeomorphic to the dual complex in question;

- Z admits a strong deformation retraction onto S.

The homotopy or homeomorphism type of the skeleton S, and hence of the dual complex, can then be computed using non-Archimedean techniques.

In general, there is not a unique skeleton of a Berkovich space; rather, skeletons usually arise from the choice of some auxiliary data. Nonetheless, in certain settings one can construct a canonical skeleton of a Berkovich space, called an essential skeleton. This originates in the work of Mustaţă–Nicaise [MN15] and Kontsevich–Soibelman [KS06]. In the next sections, we discuss further contributions to this theory.

1.2.1 Skeletons over a discretely-valued field

For a smooth, proper variety X over a discretely-valued field k, a common source of skeletons of X^{an} are the dual complexes of suitable pairs on models of X over the valuation ring k° of k. More precisely, let $\mathscr{X}$ be a degeneration of X over k° such that the special fibre $\mathscr{X}_0$ is an snc divisor on $\mathscr{X}$; in this case, we say $\mathscr{X}$ is an snc model of X. Such models always exist when the residue characteristic of k is zero. The order of vanishing along a component of $\mathscr{X}_0$ defines a valuation on the function field of X, and hence a point of X^{an}. In this way, the vertices of the dual complex of $\mathscr{X}_0$ are embedded in X^{an}, and the results of [Ber99, MN15] show that this extends to an embedding of the entire dual complex into X^{an}; the image, denoted $\mathrm{Sk}(\mathscr{X})$, is called the skeleton of $\mathscr{X}$. More generally, we can associate skeletons to log-regular models of X over k°, or enhance the special fibre of $\mathscr{X}$ with horizontal components, as in [BM17, GRW16]. In both cases, the skeleton is a polyhedral subspace of the Berkovich analytification.

The aforementioned skeletons of X^{an} all depend on some additional input, and they are not intrinsic to the variety X in question. Motivated to construct a canonical skeleton of X^{an}, Mustaţă and Nicaise introduce in [MN15] the notion of the essential skeleton of X when k has residue characteristic zero. For each global pluricanonical form $\eta \in H^0(X, \omega_{X/k}^{\otimes m})$, they consider the minimality locus $\mathrm{Sk}(X, \eta) \subseteq X^{\mathrm{an}}$ of the weight function wt_η, called the Kontsevich–Soibelman skeleton of (X, η). The union

$$\mathrm{Sk}^{\mathrm{ess}}(X) := \bigcup_{\eta} \mathrm{Sk}(X, \eta)$$

is the essential skeleton of X. The essential skeleton $\mathrm{Sk}^{\mathrm{ess}}(X)$ is a union of faces in the skeleton $\mathrm{Sk}(\mathscr{X})$ of any snc model $\mathscr{X}$ of X, and, when X is a curve of genus greater than zero, $\mathrm{Sk}^{\mathrm{ess}}(X)$ coincides with the skeleton of the minimal snc model of X. In general, X may not admit a minimal snc model; however, if X is projective and $\omega_{X/k}$ is semiample, then X admits a good minimal dlt model $\mathscr{X}_{\mathrm{dlt}}$, and in fact

$$\mathrm{Sk}^{\mathrm{ess}}(X) = \mathrm{Sk}(\mathscr{X}_{\mathrm{dlt}}).$$

This is a result of Nicaise–Xu (see [NX16, Theorem 3.3.3]), and the existence of such a model depends on deep results from the minimal model program.

Furthermore, under these stronger hypotheses, the analytification X^{an} admits a strong deformation retraction onto $\mathrm{Sk}^{\mathrm{ess}}(X)$; this is deduced by Nicaise–Xu from the work of de Fernex–Kollár–Xu [dFKX17]. It is worth noting that, when X is Calabi–Yau, this retraction is a non-Archimedean SYZ fibration; see [Yu16, NXY18].

1.2.2 Skeletons over a trivially-valued field

For a variety X over a trivially-valued field k, skeletons arise not from a choice of model over the valuation ring but from a choice of divisor on X. To that end, let D be an effective Weil divisor on X such that (X, D) is a log regular pair; this is the case e.g. when X is smooth and D is snc, or when X is a toric variety and D is the toric boundary. Following [BM17], we construct in §4.2 a skeleton $\mathrm{Sk}(X, D) \subseteq X^{\beth}$ associated to D, which has the structure of a cone complex with the vertex corresponding to the trivial valuation. In fact, $\mathrm{Sk}(X, D)$ coincides with the skeleton produced by Ulirsch in [Uli17, §6] and, when k is perfect, with Thuiller's skeleton associated to a toroidal embedding without self-intersection, as in [Thu07, §3]. Furthermore, if the boundary D is snc, then $\mathrm{Sk}(X, D)$ coincides with the cone complex of quasi-monomial valuations in D of [JM12], and it is homeomorphic to the cone over the dual complex $\mathcal{D}(D)$. Thus, the language of skeletons in the trivially-valued setting can provide a good formalism with which to study dual complexes.

Suppose now that the characteristic of k is zero. For a global pluricanonical form η on X with poles along D, write $\mathrm{wt}_\eta\colon X^\beth \to \mathbf{R} \cup \{\pm\infty\}$ for the associated weight function, as in §1.1. The minimality locus $\mathrm{Sk}(X, D, \eta)$ of wt_η is called the Kontsevich–Soibelman skeleton of η, and the essential skeleton of the pair (X, D) is the union

$$\mathrm{Sk}^{\mathrm{ess}}(X, D) := \bigcup_\eta \mathrm{Sk}(X, D, \eta)$$

of the Kontsevich–Soibelman skeletons for $\eta \in H^0(X, \mathcal{O}_X(m(K_X + D)))$ and for $m \in \mathbf{Z}_{>0}$ sufficiently divisible; such a form η is said to be a (global) D-logarithmic pluricanonical form. As in [BM17], we prove that the Kontsevich–Soibelman skeletons are each contained in $\mathrm{Sk}(X, D)$, and hence $\mathrm{Sk}^{\mathrm{ess}}(X, D)$ is as well. In other words, the weight functions of (X, D) cut out essential faces from $\mathrm{Sk}(X, D)$, and the union of these faces defines the essential skeleton $\mathrm{Sk}^{\mathrm{ess}}(X, D)$.

In §4.2.8, we establish a compatibility result between the weight functions in the trivially-valued and discretely-valued settings, similar to one used in the proof of Theorem A. As a consequence, we obtain that the essential skeleton in the former setting is a cone over the essential skeleton in the latter, as stated below. This result is also illustrated in Fig. 4.1 in the special case of a semistable degeneration of a Tate elliptic curve.

Theorem B. *Let k be a trivially-valued field of characteristic zero. Let $\mathscr{X}$ be a* *degeneration over the ring of formal power series* $k[[\varpi]]$ *that arises as the base change of* $X \to C$ *along the map* $\mathrm{Spec}(\widehat{\mathcal{O}}_{C,0}) \to C$*, where* C *is the germ of a smooth* k*-curve,* $0 \in C(k)$*, and* $\widehat{\mathcal{O}}_{C,0} \simeq k[[\varpi]]$*. Suppose that the generic fibre* $\mathscr{X}_{k((\varpi))}$ *of* $\mathscr{X}$ *is smooth, and* X *is a normal, flat, projective* C*-scheme such that the special fibre* X_0 *is reduced. If* (X, X_0) *is log canonical and* $K_X + X_0$ *is semiample, then*

$$\mathrm{Sk}^{\mathrm{ess}}(\mathscr{X}_{k((\varpi))}) = \mathrm{Sk}^{\mathrm{ess}}(X, X_0) \cap \mathscr{X}^{\mathrm{disc}},$$

where $\mathscr{X}^{\mathrm{disc}} \subseteq X^{\beth}$ *is the* $k((\varpi))$*-analytic generic fibre of* $\mathscr{X}$*.*

The assumption in Theorem B that the degeneration $\mathscr{X}$ is defined over a k-curve C is a technical condition, and we expect the result to hold without it (on the other hand, the log canonicity assumption is necessary; see Remark 4.2.8.4). This assumption is needed in the proof of Theorem B in order to apply the results of [NX16]; there, it is used to apply the results of the minimal model program to schemes over rings of formal power series.

More generally, Temkin's metric can be used to define the essential skeleton of a (quasi-smooth) Berkovich space over any non-Archimedean field. This approach is adopted in [HN17, Proposition 4.3.2] and [KY19] to realize intrinsically-defined skeletons as essential skeletons. For example, Halle and Nicaise show that the intrinsic skeleton of an abelian variety (constructed in [Ber90, §6.5] by Berkovich in terms of the non-Archimedean uniformization) coincides with the essential skeleton over any field. By Theorem A, this definition of the essential skeleton in terms of Temkin's metric coincides with the one above, when both are defined. Moreover, Theorem A provides a concrete and computable description of the essential skeleton over a trivially-valued field of characteristic zero.

As a result of these constructions, there is a unified framework of studying canonical skeletons of Berkovich analytifications in both the trivially-valued and discretely-valued settings. In §6.1, we will describe how these tools are applied to compute two classes of dual complexes arising in non-abelian Hodge theory.

1.2.3 The closure of Kontsevich–Soibelman skeletons

The methods discussed in §1.2.1 and §1.2.2 allow us to discuss the weight functions, the Kontsevich–Soibelman skeletons, and the essential skeletons simultaneously in the trivially-valued and discretely-valued settings. In this section, this uniform approach is adopted to describe the closures of these skeletons, and to realize an intrinsically-defined skeleton of a toric variety as an essential skeleton.

Let k be a non-Archimedean field that is either trivially or discretely-valued, and let (X, D) be a log-regular pair over k. For a log-regular model $(\mathscr{X}, D_{\mathscr{X}})$ of (X, D) over the valuation ring of k (which we understand as $(\mathscr{X}, D_{\mathscr{X}}) = (X, D)$ when k is trivially-valued), we can explicitly describe the closure of the associated skeleton $\mathrm{Sk}(\mathscr{X}, D_{\mathscr{X}})$ in X^{an}. More precisely, we show in Proposition 4.3.1.5 that the closure of $\mathrm{Sk}(\mathscr{X}, D_{\mathscr{X}})$ is a disjoint union of skeletons associated to the strata of $D_{\mathscr{X}}$; this extends [Thu07, Proposition 3.17].

Suppose that k has residue characteristic zero. If D is snc with irreducible components $\{D_i\}$ and η is a regular D-logarithmic pluricanonical form on X, then the above decomposition induces one on the closure of the Kontsevich–Soibelman skeleton $\mathrm{Sk}(X, D, \eta)$. For each stratum W of D, we write $\mathrm{Res}_W(\eta)$ for the residue form of η along W, and $(W, \sum_{j\colon W \not\subseteq D_j} D_j|_W)$ for the induced log-regular structure on W (see Proposition 4.3.1.1 for a precise definition). In the result below, we describe the closure of $\mathrm{Sk}(X, D, \eta)$ in terms of the Kontsevich–Soibelman skeletons of the residue forms $\mathrm{Res}_W(\eta)$ of η along the various strata W of D.

Theorem C. *Let D be an snc divisor on X and η be a non-zero regular D-logarithmic pluricanonical form on X. Then, the closure of the Kontsevich–Soibelman skeleton* $\mathrm{Sk}(X, D, \eta)$ *in* X^{an} *lies in the disjoint union of the Kontsevich–Soibelman skeletons*

$$\bigsqcup_{W} \mathrm{Sk}\Big(W, \sum_{j\colon W \not\subseteq D_j} D_j|_W, \mathrm{Res}_W(\eta)\Big),$$

where the index runs over all strata W of D.

In addition, we show that the inclusion in Theorem C is an equality when k is trivially-valued (see Proposition 4.3.3.2), while it is false in the discretely-valued setting (see Example 4.3.3.3).

There are instances of similar decompositions that occur in the literature. For example, if X is the toric variety over k associated to a rational polyhedral fan Σ, then Σ admits a natural compactification, which is endowed with a decomposition indexed by the strata of the toric boundary divisor; see [Pay09, §3] and [Rab12, Proposition 3.4]. In [Thu07, §2], this compactification of the support of Σ is embedded into X^{an} and the image is called the toric skeleton of X. Building on the ideas of Berkovich in [Ber90, §5.1], Thuillier shows that X^{an} admits a strong deformation retraction onto the toric skeleton. The toric skeleton is the principal connection between non-Archimedean and tropical geometry; for example, Thuillier's strong deformation retraction is precisely the tropicalization map of X.

The toric skeleton only depends on the choice of open torus T in X, which determines a toric boundary divisor $D = X \backslash T$ on X. We show that the toric skeleton can be realized in terms of the essential skeleton of the pair (X, D).

Theorem D. *Let X be a normal toric variety over k, and D be the toric boundary divisor on X. Then, the closure of* $\mathrm{Sk}^{\mathrm{ess}}(X, D)$ *in* X^{an} *coincides with the toric skeleton.*

The proof of Theorem D proceeds by decomposing the skeletons into disjoint unions of subspaces indexed by the strata of D, and then explicitly comparing these pieces. See §4.3.2 for further details.

1.3 A non-Archimedean Ohsawa–Takegoshi extension theorem

The Ohsawa–Takegoshi theorem is one of the fundamental extensions theorems in complex geometry. Originating in the foundational paper [OT87] of Ohsawa–Takegoshi, many generalizations and improvements have since been shown; see [Man93, Ber96b, Siu96, Dem00, MV07]. Its many applications include Siu's much-celebrated proof [Siu98] of the deformation invariance of plurigenera. In its simplest form, the classical Ohsawa–Takegoshi theorem asserts the following: given a plurisubharmonic function φ on the complex unit disc $\mathbf{D}$, a point $z \in \mathbf{D}\backslash\{\varphi = -\infty\}$, and a value $a \in \mathbf{C}$, there is a holomorphic function f on $\mathbf{D}$ such that $f(z) = a$ and

$$\int_{\mathbf{D}} |f(x)|^2 e^{-2\varphi(x)} d\lambda \leq \pi |f(z)|^2 e^{-2\varphi(z)}.$$

The constant π is optimal, as shown in [Bo13, Theorem 1]. There is also an adjoint formulation of the result, which concerns the extension of a holomorphic 1-form rather than of a holomorphic function.

Let k be a non-Archimedean field, $k\{T\}$ be the Tate algebra in one variable over k, $X = E_k(1)$ be the Berkovich closed unit disc over k (see Fig. 5.1 for a picture of X when k is algebraically closed). In order to state a non-Archimedean version of the Ohsawa–Takegoshi extension theorem, we must discuss the non-Archimedean analogues of (pluri)subharmonic functions and volume forms. The former is well-understood: there is a class of quasisubharmonic functions on the Berkovich closed unit disc, which are the non-Archimedean analogue of (pluri)subharmonic functions on the complex unit disc. These are briefly discussed in §5.1.4, and a comprehensive treatment is given in [BR10, Jon15].

On the complex unit disc, the Lebesgue measure is a canonical volume form. In general, volume forms on a complex manifold naturally correspond to smooth metrics on the canonical bundle. Thus, a non-Archimedean analogue of the Lebesgue measure on the disc is a canonical metric on the canonical bundle of X. This role is played by Temkin's metric: let $\|dT\|_{\mathrm{geom}}: X \to \mathbf{R}_+$ denote (the geometric version of) Temkin's metric applied to the global section dT of $\omega_{X/k}$. In fact, $\|dT\|$ coincides with the radius function on X; see §5.1.2 for more details. Set $A_X = -\log \|dT\|$. As suggested by the notation, A_X coincides with the log discrepancy function on $X = \mathbf{A}_k^{1,\beth}$ when k is a trivially-valued field of characteristic zero.

For any quasisubharmonic function φ on X and analytic function $f \in k\{T\}$, consider the norm

$$\|f\|_\varphi \coloneqq \sup_{X \setminus Z(\varphi)} |f| e^{-\varphi - A_X}$$

where $Z(\varphi) \coloneqq \{\varphi = -\infty\}$. The function $\varphi + A_X$ can be thought of as a metric on the canonical bundle $\omega_{X/k}$, and the norm $\|f\|_\varphi$ measures the length of the section $f dT$ in the metric $\varphi + A_X$.

Now, we can state a non-Archimedean version of the Ohsawa–Takegoshi extension theorem on the Berkovich closed unit disc $X = E_k(1)$.

Theorem E. *Assume k is algebraically closed, trivially-valued, or is spherically complete of residue characteristic zero. Let φ be a quasisubharmonic function on $X = E_k(1)$. For any $z \in X$, there exists a nonzero polynomial $f \in k[T]$ such that*

$$\lim_{\epsilon \to 0^+} \|f\|_{(1+\epsilon)\varphi} \leq |f(z)| e^{-\varphi(z)}.$$

If $\varphi(z) = -\infty$, then we may find f such that $\lim_{\epsilon\to 0^+} \|f\|_{(1+\epsilon)\varphi} < +\infty$. Moreover, if k is algebraically closed and z is a rigid point of X, then for any value $a \in \mathcal{H}(z)^* = k^*$, *we may find f such that $f(z) = a$.*

The techniques involved in the proof of Theorem E rely crucially on the tree structure of the Berkovich unit disc; in particular, the proof does not generalize to higher dimensions. More precisely, the proof proceeds by first constructing a finite subtree Γ_φ of X that captures the worst of the singularities of φ, and we reduce to proving Theorem E on the convex hull of $\Gamma_\varphi \cup \{z\}$. For each end of the tree Γ_φ, we construct a new quasisubharmonic function ϕ such that $\Gamma_\phi \subsetneq \Gamma_\varphi$ and reduce to proving Theorem E for ϕ. This inductively reduces Theorem E to a simple case, which is solved directly. The proof of Theorem E appears in §5.2.

In Theorem E, an analytic function is measured using the sequence of norms $\|\cdot\|_{(1+\epsilon)\varphi}$ as $\epsilon \to 0^+$, instead of with the single norm $\|\cdot\|_\varphi$ (which is what one might expect from the classical Ohsawa–Takegoshi theorem). Nonetheless, the former proves to give the correct analogy with the complex setting, as the following example demonstrates. Consider the (pluri)subharmonic function $\varphi = \alpha \log|z|$, with $\alpha > 0$, on the complex unit disc $\mathbf{D}$. It is elementary to verify that $\int_{\mathbf{D}} e^{-2\varphi} d\lambda < +\infty$ if and only if $\alpha < 1$. Similarly, consider the quasisubharmonic function $\varphi = \alpha \log|T|$, with $\alpha > 0$, on the Berkovich unit disc $X = E_k(1)$; then, $\lim_{\epsilon\to 0^+} \|1\|_{(1+\epsilon)\varphi} < +\infty$ if and only if $\alpha < 1$.

1.3.1 A regularization theorem

As an application of Theorem E, we prove a regularization theorem for quasisubharmonic functions on the Berkovich unit disc. Certain results in this direction already exist in the literature: in [Jon15, Theorem 2.10], it was shown that any quasisubharmonic function on X is the decreasing limit of bounded quasisubharmonic functions. A similar argument appears in [FRL06a, §4.6]. However, these constructions use only the tree structure on the Berkovich unit disc (in particular, they do not incorporate the analytic structure). It is therefore unlikely that these proofs can be generalized to higher dimensions.

As inspiration, we use the much-celebrated regularization theorem of Demailly for a plurisubharmonic function ϕ on a bounded pseudoconvex domain $\Omega \subset \mathbf{C}^n$. For such a plurisubharmonic function ϕ on Ω and a positive integer m, we associate the Hilbert space $\mathscr{H}_{m\phi}$ of holomorphic functions on Ω satisfying the integrability condition

$$\int_\Omega |f|^2 e^{-2m\phi} d\lambda \; < +\infty.$$

The Demailly approximation associated to $\mathscr{H}_{m\phi}$ is a plurisubharmonic function ϕ_m on Ω with analytic singularities, and it is given by

$$\phi_m(z) = \sup_{f \in \mathscr{H}^\circ_{m\phi}} \frac{1}{m} \log |f(z)|$$

for $z \in \Omega$, where $\mathscr{H}^\circ_{m\phi} \subseteq \mathscr{H}_{m\phi}$ denotes the unit ball. Demailly uses the Ohsawa–Takegoshi theorem to show that the sequence $(\phi_m)_{m=1}^\infty$ converges pointwise and in L^1_{loc} to ϕ. See [Dem92] for further details.

We adopt the same philosophy in the non-Archimedean setting: to a quasisubharmonic function φ on X, we associate the ideal $\mathcal{H}_\varphi$ of the Tate algebra $k\{T\}$ consisting of those analytic functions f satisfying the finiteness condition

$$\|f\|^+_\varphi := \lim_{\epsilon \to 0^+} \sup_{X \setminus Z(\varphi)} |f| e^{-(1+\epsilon)\varphi - A_X} < +\infty$$

For each positive integer m, we define the non-Archimedean Demailly approximation φ_m by the formula

$$\varphi_m := \frac{1}{m} \left(\sup_{f \in \mathcal{H}_{m\varphi} \setminus \{0\}} \log \frac{|f|}{\|f\|^+_{m\varphi}} \right)^*,$$

where $(-)^*$ denotes the upper-semicontinuous regularization. We show that φ_m is a quasisubharmonic function on X with analytic singularities.

The ideal sheaf on X associated to $\mathcal{H}_\varphi$ behaves like a non-Archimedean multiplier ideal associated to φ, and may be of independent interest. This idea is briefly explored in §5.3.3, where we show that $\mathcal{H}_\varphi$ generates the stalks of a locally-defined multiplier ideal sheaf associated to φ. These multiplier ideals are used to show that the ideals $\mathcal{H}_\varphi$ satisfy a subadditivity property.

We prove the following non-Archimedean analogue of Demailly's regularization theorem.

Theorem F. *Assume k is algebraically closed, trivially-valued, or is spherically complete with residue characteristic zero. For a quasisubharmonic function φ on X with $\varphi \leq 0$, we have*

$$\varphi \leq \varphi_m \leq \varphi + \frac{A_X}{m}.$$

In particular, the sequence $(\varphi_m)_{m=1}^{\infty}$ converges pointwise to φ on $\{A_X < +\infty\} \subseteq X$.

In Theorem F, the crucial inequality $\varphi_m \geq \varphi$ is a consequence of Theorem E; see §5.3 for the proof. In principle, a statement similar to Theorem F ought to be possible in higher dimensions, but this requires a higher-dimensional version of the Ohsawa–Takegoshi theorem for polydiscs.

There has been much work done on the development of pluripotential theory on Berkovich spaces. Quite generally, Chambert-Loir and Ducros have introduced in [CD12] the notion of *continuous* plurisubharmonic functions. In addition, semipositive metrics on line bundles were studied in detail by [Zha95, Gub98], among others. On analytic curves, potential theory is well-established, due to the work of Thuillier [Thu05]; in the continuous case, this coincides with the potential theory of Chambert-Loir and Ducros by a theorem of Wanner in [Wan18]. A (global) regularization theorem similar to Theorem F is proven in [BFJ16, Theorem B] for analytifications of smooth, projective varieties over a discretely-valued field of residue characteristic zero. A related discussion appears in [BFJ08, §5].

1.4 New evidence for the geometric $P = W$ conjecture.

By studying the behaviour of the weight functions over $\mathbf{C}$ (equipped with the trivial valuation), we determine the homeomorphism type of the dual complex of pairs that arise from compactifications of character varieties. In particular, our computation provides new evidence for the geometric $P = W$ conjecture, formulated by Katzarkov, Noll, Pandit, and Simpson in [KNPS15, Conjecture 1.1]; see alternatively [Sim16, Conjecture 11.1]. We give a brief overview of the content of this conjecture in §6.1.

For a reductive algebraic group G, consider the G-character variety

$$M_G \coloneqq \mathrm{Hom}(\pi_1(X), G) /\!\!/ G$$

of G-representations of the topological fundamental group $\pi_1(X)$ of a proper, smooth curve X over $\mathbf{C}$; M_G is also known as the *Betti moduli space* for the group G. When X is a compact Riemann surface of genus one, M_G can be realized as the GIT quotient

$$\{(A, B) \in G \times G \colon AB = BA\} /\!\!/ G,$$

where G acts by conjugation on each factor of $G \times G$. For example, when $G = \mathrm{GL}_n$, M_{GL_n} is isomorphic to the n-fold symmetric product $(\mathbf{C}^* \times \mathbf{C}^*)^{(n)}$ of the torus $\mathbf{C}^* \times \mathbf{C}^*$.

In this setting, the geometric $P = W$ conjecture asserts that the dual boundary complex $\mathcal{D}(\partial M_G)$ of M_G has the homotopy type of a sphere (of a particular dimension, depending on G). It is not a priori clear how one can make sense of $\mathcal{D}(\partial M_G)$, since M_G can be a singular affine variety, hence it may not admit an snc compactification. Thus, the task is to find mildly singular compactifications to which a dual complex may still be attached. Our solution is to consider dlt compactifications (see §6.2.1 for a precise definition). Such compactifications have a well-defined dual complex, whose homotopy type is independent of the choice of a specific dlt compactification by [dFKX17]. Further, when the group G is either GL_n or SL_n, the existence of dlt compactifications follows from the existence of dlt blow-ups by Hacon (see [Fuj11, Theorem 10.4] and [KK10, Theorem 3.1]), and the fact that M_G has canonical and $\mathbf{Q}$-factorial singularities (as shown in [BS16, Theorems 1.20 and 1.21]).

Among all possible dlt compactifications of M_G, it is convenient to restrict to special ones, namely the dlt log Calabi–Yau compactifications. This is an algebraic condition which rigidifies the configuration of divisors at infinity, and in practice it simplifies the description of the dual complex. The dual complex of any dlt log Calabi–Yau compactification identifies a distinguished homeomorphism class in the homotopy equivalence class of the dual complex of any dlt compactification. Moreover, it is expected that M_G actually admits a log Calabi–Yau compactification; see [Sim16]. These observations suggest the following strengthening of the homotopy equivalence in the geometric $P = W$ conjecture.

Conjecture 1.4.0.1. *The Betti moduli space M_G admits a dlt log Calabi–Yau compactification $(\overline{M}_G, \partial M_G)$ and the associated dual complex $\mathcal{D}(\partial M_G)$ is homeomorphic to a sphere.*

In our final main results, we prove Conjecture 1.4.0.1 when G is GL_n or SL_n. These results provide the first non-trivial evidence for the geometric $P = W$ conjecture in the compact case.

Theorem G. *The dual boundary complex $\mathcal{D}(\partial M_{\mathrm{GL}_n})$ of a dlt log Calabi–Yau com-*

pactification of M_{GL_n} *has the homeomorphism type of* $\mathbb{S}^{2n-1}$.

Theorem H. *The dual boundary complex* $\mathcal{D}(\partial M_{\mathrm{SL}_n})$ *of a dlt log Calabi–Yau compactification of* M_{SL_n} *has the homeomorphism type of* $\mathbb{S}^{2n-3}$.

We are not aware of an explicit dlt compactification of M_G. To overcome this issue, we recast the problem in terms of non-Archimedean geometry. In this approach, strata of the boundary divisor are thought as centres of suitable monomial valuations (in the sense of Proposition 4.2.1.1). These valuations can be studied abstractly, independent of a choice of compactification of M_G and without concern for the singularities that may arise in a compactification. In particular, this new viewpoint allows us to reinterpret the dual complex $\mathcal{D}(\partial M_G)$ as the level set of a suitable function inside a space of valuations, namely as the minimality locus of a log discrepancy function. More precisely, we show that $\mathcal{D}(\partial M_G)$ is homeomorphic to the link of the essential skeleton of a log Calabi–Yau pair (see Definitions 6.2.4.2 and 6.2.4.3).

In fact, one could determine the homotopy type of $\mathcal{D}(\partial M_G)$ by using Thuillier's skeleton and the arguments in the trivially-valued proofs of Theorems G and H. However, our new definition of essential skeleton allows us to establish the actual homeomorphism class of the dual boundary complex and it adapts well to the more singular case of dlt compactifications.

We give two proofs for each of Theorems G and H, one in the trivially-valued setting and the other in the discretely-valued one. The latter is technically more demanding, since it requires the construction of a degeneration of compact hyper-Kähler manifolds; see §6.2.6 and §6.3.1. However, this construction is of independent interest, as it suggests a relationship between the geometric $P = W$ conjecture and the conjecture below.

Conjecture 1.4.0.4. *Let $\mathscr{X}$ be a maximally unipotent good minimal dlt degeneration of compact hyper-Kähler manifolds over $\mathbf{C}((t))$. Then, the dual complex of the special fibre of $\mathscr{X}$ is homeomorphic to $\mathbf{P}^n(\mathbf{C})$.*

CHAPTER II

Preliminaries

2.1 Conventions

A *non-Archimedean field* is a field k equipped with a multiplicative norm

$$|\cdot|\colon k \to \mathbf{R}_+ := [0, +\infty)$$

that satisfies the ultrametric inequality and with respect to which k is complete. The associated valuation on k is $v = -\log|\cdot|$. Write $k^\circ := \{|\cdot| \leqslant 1\}$ for the valuation ring, $k^{\circ\circ} := \{|\cdot| < 1\}$ for the maximal ideal, $\widetilde{k} := k^\circ/k^{\circ\circ}$ for the residue field, and

$$\sqrt{|k^\times|} := \{c \in \mathbf{R}_+^*\colon \exists\, \ell \in \mathbf{Z} \text{ such that } c^\ell \in |k^\times|\},$$

for the divisible value group, where $\mathbf{R}_+^* := (0, +\infty)$. Further, we say k is *trivially-valued* if $|k^\times| = \{1\}$, and *discretely-valued* if there is $r \in (0,1)$ such that $|k^\times| = r^{\mathbf{Z}}$. In the former case, we will always denote the trivial norm by $|\cdot|_0$ and the trivial valuation by v_0. A *valued extension* k'/k is a field extension k' of k that is a non-Archimedean field whose norm restricts to the norm on k. We freely use the language of k-analytic spaces from [Ber90, Ber93].

A *variety* is an integral separated scheme of finite type over a field. The terms line bundle and invertible sheaf are used interchangeably. A *pair* (resp. a *sub-pair*) (X, D) is the datum of a normal scheme X and a Weil $\mathbf{Q}$-divisor D with coefficients in $(0, 1]$

(resp. in $(-\infty, 1]$); the divisor D is called a *boundary* (resp. *sub-boundary*). Write $D^{=1} = \sum_i D_i$ for the union of all irreducible components of D whose coefficient equals 1. The irreducible components of the intersection $D_{i_1} \cap \ldots \cap D_{i_r}$ of r components of $D^{=1}$ are called *strata* of codimension r. A pair (X, D) is *simple normal crossing* (snc) if X is a regular scheme and the support of D is an effective Cartier divisor on X such that for any $x \in \operatorname{Supp}(D)$, there are local equations $f_1, \ldots, f_r \in \mathfrak{m}_x$ of the components of $\operatorname{Supp}(D)$ containing x that form a regular system of parameters in the local ring $\mathcal{O}_{X,x}$.

2.2 Berkovich analytifications

Let k be a non-Archimedean field and let X be a scheme locally of finite type over k. Consider the functor $F_X \colon \mathbf{An}_k \to \mathbf{Sets}$ from the category $\mathbf{An}_k$ of good analytic spaces over k to the category **Sets** of sets given by

$$Z \mapsto \operatorname{Hom}(Z, X),$$

where $\operatorname{Hom}(Z, X)$ denotes the set of morphisms $Z \to X$ as locally k-ringed spaces.

Theorem 2.2.0.1. *The functor F_X is representable by a boundaryless, strictly k-analytic space X^{an} and a surjective morphism* $\ker \colon X^{\mathrm{an}} \to X$ *of locally k-ringed spaces.*

Proof. See [Ber90, Theorem 3.4.1] in the nontrivially-valued case and [Ber90, Theorem 3.5.1] in the trivially-valued case. □

The space X^{an} is called the *Berkovich analytification* of X, and ker is called the *kernel map*. The assignment $X \mapsto X^{\mathrm{an}}$ is functorial, and it commutes with fibre products and ground field extension. For a morphism of schemes $f \colon X \to Y$, write

$f^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ for the induced map on Berkovich analytifications. The functor $X \mapsto X^{\mathrm{an}}$ has many permanence properties (see [Ber90, §3.4-3.5]) that we use freely.

The underlying topological space of X^{an} admits a concrete description: a point $x \in X^{\mathrm{an}}$ is a pair $(\xi_x, |\cdot|_x)$, where $\xi_x \in X$ is a scheme-theoretic point of X, and $|\cdot|_x$ is an absolute value on the residue field $\kappa(x)$ of X at ξ_x that extends $|\cdot|$ on k; write $v_x = -\log|\cdot|_x$ for the associated valuation on $\kappa(x)$. It is often convenient to think of $|\cdot|_x$ as a seminorm on the stalk $\mathcal{O}_{X,\xi_x}$ that restricts to the given norm on k. This seminorm is defined by the composition

$$\mathcal{O}_{X,\xi_x} \to \kappa(x) \xrightarrow{|\cdot|_x} \mathbf{R}_+,$$

the kernel of which is precisely the maximal ideal $\mathfrak{m}_{\xi_x}$ of $\mathcal{O}_{X,\xi_x}$. The *completed residue field* $\mathcal{H}(x)$ of X^{an} at x is the completion of $\kappa(x)$ with respect to $|\cdot|_x$. The kernel map is given by $\ker(x) := \xi_x$. The analytification X^{an} is equipped with the weakest topology such that ker is continuous, and, for any Zariski-open $U \subseteq X$ and any $f \in \mathcal{O}_X(U)$, the map $\ker^{-1}(U) \to \mathbf{R}_+$ given by

$$x \mapsto |f(x)| := |f(\xi_x)|_x$$

is continuous.

Equivalently, if $X = \mathrm{Spec}(A)$ is affine, then X^{an} is the set of multiplicative seminorms $|\cdot|_x\colon A \to \mathbf{R}_+$ that restrict to the given norm on k, equipped with the topology of pointwise convergence. The kernel map is given by $\ker(x) = \{f \in A\colon |f(x)| = 0\}$.

When X is a variety, the subset of *birational points* $X^{\mathrm{bir}} \subseteq X^{\mathrm{an}}$ of X^{an} is the ker-preimage of the generic point of X. Alternatively, X^{bir} is the space of valuations on the function field of X that extend the given valuation on k.

The Berkovich analytification X^{an} has good topological properties: it is locally compact and locally Hausdorff. Moreover, it satisfies the following topological GAGA results.

Proposition 2.2.0.2. *1. X is separated if and only if X^{an} is Hausdorff.*

2. X is proper if and only if X^{an} is compact and Hausdorff.

3. X is connected if and only if X^{an} is arcwise connected.

In fact, if X is connected, then the analytification X^{an} is locally arcwise connected by [Ber90, Theorem 3.2.1].

Proof. See [Ber90, Theorem 3.4.8] in the nontrivially-valued case and [Ber90, Theorem 3.5.3] in the trivially-valued case. □

Write $\mathcal{O}_{X^{\mathrm{an}}}$ for the structure sheaf on X^{an}, called the *sheaf of analytic functions* on X^{an}. Given a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on X, the pullback

$$\mathcal{F}^{\mathrm{an}} := \ker^*(\mathcal{F}) = \ker^{-1}(\mathcal{F}) \otimes_{\ker^{-1}(\mathcal{O}_X)} \mathcal{O}_{X^{\mathrm{an}}}$$

is a coherent sheaf of $\mathcal{O}_{X^{\mathrm{an}}}$-modules on X^{an}, called the *analytification* of $\mathcal{F}$. Further, if L is a line bundle on X, then $L^{\mathrm{an}} := \ker^*(L)$ is a line bundle on X^{an} . If X is proper or if X is separated and k is trivially-valued, then the functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$ on coherent sheaves satisfies various cohomologyical GAGA theorems; see [Ber90, §3.4-3.5].

2.3 Analytic generic fibres

Let k be a non-Archimedean field. A formal k°-scheme $\mathfrak{X}$ is *locally finitely presented* if it is locally of the form $\mathrm{Spf}(A)$, where A is the quotient of the ring $k^\circ\{T_1, \ldots, T_n\}$ of restricted power series by a finitely-generated ideal. Write $\mathfrak{X}_0$ for the special fibre of $\mathfrak{X}$, which is a $\widetilde{k}$-scheme locally of finite type. Berkovich constructs in [Ber94, Ber96a] a k-analytic space associated to $\mathfrak{X}$, which we describe below. This work mirrors Raynaud's construction of the generic fibre functor from rigid-analytic geometry.

For a k-analytic space Z, write $G_{\mathfrak{X}}(Z)$ for the set of morphisms $\varphi\colon Z \to \mathfrak{X}$ of locally k°-ringed spaces satisfying the following two conditions:

1. for any open $\mathfrak{U} \subseteq \mathfrak{X}$ and any affinoid domain $V \subseteq Z$ such that $\varphi(V) \subseteq U$, the induced morphism $\varphi^*\colon \mathcal{O}_{\mathfrak{X}}(\mathfrak{U}) \to \mathcal{O}_Z(V)$ satisfies $\|\varphi^*(f)\|_V \leq 1$ for all $f \in \mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$, where $\|\cdot\|_V$ is the norm on the k-affinoid algebra $\mathcal{O}_Z(V)$;

2. for any $z \in Z$, the composition $\mathcal{O}_{\mathfrak{X},\varphi(z)} \to \mathcal{O}_{Z,z} \to \mathcal{H}(z)$ takes values in $\mathcal{H}(z)^\circ$, and $\mathcal{O}_{\mathfrak{X},\varphi(z)} \to \mathcal{H}(z)^\circ$ is a local homomorphism of local rings.

This gives rise to a functor $G_{\mathfrak{X}}\colon k\text{-}\mathbf{An} \to \mathbf{Sets}$ given by $Z \mapsto G_{\mathfrak{X}}(Z)$.

Theorem 2.3.0.1. *The functor $G_{\mathfrak{X}}$ is representable by a strictly k-analytic space $\mathfrak{X}_\eta$ and a surjective morphism* $\mathrm{red}_{\mathfrak{X}}\colon \mathfrak{X}_\eta \to \mathfrak{X}_0$ *from the G-topological site of $\mathfrak{X}_\eta$ to the Zariski site of $\mathfrak{X}_0$.*

Proof. See [Thu07, Proposition 1.3] for a proof, as well as [Ber94, §1] and [Ber96a, §1] for the basic properties of $G_{\mathfrak{X}}$. □

The space $\mathfrak{X}_\eta$ is called the *analytic generic fibre* of $\mathfrak{X}$, and $\mathrm{red}_{\mathfrak{X}}$ is the *reduction map*. The rule $\mathfrak{X} \mapsto \mathfrak{X}_\eta$ is functorial. As a map on topological spaces, $\mathrm{red}_{\mathfrak{X}}$ is anti-continuous, and it is a morphism of sites in the following sense: $\mathrm{red}_{\mathfrak{X}}^{-1}(U)$ is an affinoid domain for any Zariski-open $U \subseteq \mathfrak{X}_0$, and the $\mathrm{red}_{\mathfrak{X}}$-preimage of a Zariski-open cover of $\mathfrak{X}_0$ is an affinoid cover of $\mathfrak{X}_\eta$.

Suppose now that $\mathscr{X}$ is a locally finitely presented k°-scheme, and $\widehat{\mathscr{X}}$ is the ϖ-adic formal completion of $\mathscr{X}$ for some choice of pseudouniformizer $\varpi \in k^{\circ\circ}$ (i.e. $\varpi \in k^{\circ\circ}$ is nonzero if k is nontrivially valued, and $\varpi = 0$ otherwise). Write $\mathscr{X}_k := \mathscr{X} \times_{k^\circ} k$

for the generic fibre of $\mathscr{X}$, which is a scheme locally of finite type over k. The special fibre $\mathscr{X}_0 := \mathscr{X} \times_{k^\circ} \widetilde{k}$ coincides with the special fibre $\widehat{\mathfrak{X}}_0$ of $\widehat{\mathfrak{X}}$.

In the special case when $\mathscr{X} = \mathrm{Spec}(\mathcal{A})$ is affine, $\widehat{\mathscr{X}_\eta}$ is the affinoid domain in $\mathscr{X}_k^{\mathrm{an}}$ given by

$$\widehat{\mathscr{X}_\eta} = \mathcal{M}(\widehat{\mathcal{A}} \otimes_{k^\circ} k) = \{x \in \mathscr{X}_k^{\mathrm{an}} \colon |f(x)| \leq 1 \text{ for all } f \in \mathcal{A}\},$$

where $\widehat{\mathcal{A}}$ denotes the ϖ-adic completion of $\mathcal{A}$. More generally, $\widehat{\mathscr{X}_\eta}$ and $\mathscr{X}_k^{\mathrm{an}}$ are related as follows.

Proposition 2.3.0.2. *If $\mathscr{X}$ is separated and finitely presented over k°, then there is a closed embedding $\iota_{\mathscr{X}} \colon \widehat{\mathscr{X}_\eta} \hookrightarrow \mathscr{X}_k^{\mathrm{an}}$ of $\widehat{\mathscr{X}_\eta}$ as a compact strictly analytic domain in $\mathscr{X}_k^{\mathrm{an}}$. Furthermore,*

1. *a point $x \in \mathscr{X}_k^{\mathrm{an}}$ lies in $\widehat{\mathscr{X}_\eta}$ if and only if the morphism $\mathrm{Spec}(\mathcal{H}(x)) \to \mathscr{X}_k$ of k-schemes extends to a morphism $\mathrm{Spec}(\mathcal{H}(x)^\circ) \to \mathscr{X}$ of k°-schemes.*

2. *$\iota_{\mathscr{X}}$ is an isomorphism if and only if $\mathscr{X}$ is proper over k°.*

If $x \in \mathscr{X}_k^{\mathrm{an}}$ lies in $\widehat{\mathscr{X}_\eta}$, then the image of the closed point via $\mathrm{Spec}(\mathcal{H}(x)^\circ) \to \mathscr{X}$ coincides with $\mathrm{red}_{\mathscr{X}}(x) \in \mathscr{X}_0$, and we say that x *admits a centre on* $\mathscr{X}$.

Proof. See [Ber94, §5] for a proof. See also [Con99, Theorem A.3.1] for a proof of the corresponding statement in rigid-analytic geometry. □

For $x \in \widehat{\mathscr{X}_\eta}$, the reduction $\mathrm{red}_{\mathscr{X}}(x) \in \mathscr{X}_0$ is a specialization of $\ker(x) \in \mathscr{X}_k \subseteq \mathscr{X}$, and so $\mathcal{O}_{\mathscr{X},\ker(x)}$ is a localization of $\mathcal{O}_{\mathscr{X},\mathrm{red}_{\mathscr{X}}(x)}$; in particular, the seminorm x on $\mathcal{O}_{\mathscr{X},\ker(x)}$ restricts to a seminorm on $\mathcal{O}_{\mathscr{X},\mathrm{red}_{\mathscr{X}}(x)}$. For a coherent sheaf $\mathfrak{a}$ of fractional ideals on $\mathscr{X}$, set

$$v_x(\mathfrak{a}) := \min_{f \in \mathfrak{a}_{\mathrm{red}_{\mathscr{X}}(x)}} v_x(f).$$

Further, for a $\mathbf{Q}$-Cartier divisor D on $\mathscr{X}$, set $v_x(D) := m^{-1}v_x(\mathcal{O}_{\mathscr{X}}(-mD))$, where $m \in \mathbf{Z}_{>0}$ is such that mD is Cartier.

2.4 $\beth$-analytifications

If k is trivially valued, then a locally finitely presented formal k°-schemes is the same as a k-scheme locally of finite type. Let X be such a k-scheme, and write $X^\beth := \widehat{X}_\eta$ for the analytic generic fibre. The space $X^\beth$ is called the $\beth$-*analytification* of X, and it was introduced and studied in [Thu07]. In this setting, the reduction map is called the *centre map* and it is written $c_X\colon X^\beth \to X$. The reason for the name is as follows: when X is integral and separated, $X^{\mathrm{bir}} \cap X^\beth$ is the space of valuations on the function field of X that admit a centre on X (in the classical sense), and c_X is the map that sends any valuation to its centre.

As with any analytic space over a trivially-valued field, there is a $\mathbf{R}_+$-action on $X^\beth$: for $a \in \mathbf{R}_+$ and $x \in X^\beth$, the point $a \cdot x \in X^\beth$ is given by

$$|f(a \cdot x)| := |f(x)|^a$$

for $f \in \kappa(x)$. In terms of valuations, the action is $v_{a\cdot x} = a \cdot v_x$. Moreover, if $a > 0$, then $c_X(a \cdot x) = c_X(x)$.

2.5 Models

Suppose k is discretely-valued. For a scheme $\mathscr{X}$ of finite type over k°, write $\mathscr{X}_k := \mathscr{X} \times_{k^\circ} k$ for the generic fibre and $\mathscr{X}_0 := \mathscr{X} \times_{k^\circ} \widetilde{k}$ for the special fibre. If X is a variety over k, a *model* for X over k° (or more classically, a *degeneration* of X) is a normal, flat, separated scheme $\mathscr{X}$ of finite type over k° endowed with an isomorphism of k-schemes $\mathscr{X}_k \xrightarrow{\simeq} X$. A *morphism of models* is a morphism $\mathscr{X}' \to \mathscr{X}$ of k°-schemes between two models of X such that the induced map $\mathscr{X}'_k \xrightarrow{\simeq} \mathscr{X}_k$ on generic fibres is an isomorphism such that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{X}'_k & \xrightarrow{\simeq} & \mathscr{X}_k \\ & \underset{\simeq}{\searrow} & \downarrow\simeq \\ & & X \end{array}$$

By Proposition 2.3.0.2, the analytic generic fibre $\widehat{\mathscr{X}_\eta}$ is a compact strictly analytic domain in X^{an} equipped with the reduction map $\mathrm{red}_{\mathscr{X}} : \widehat{\mathscr{X}_\eta} \to \widehat{\mathscr{X}_0} = \mathscr{X}_0$. Note that if $\mathscr{X}' \to \mathscr{X}$ is a proper morphism of models of X, then it induces an isomorphism $\widehat{\mathscr{X}'}_\eta \xrightarrow{\simeq} \widehat{\mathscr{X}_\eta}$ on the analytic generic fibres.

A model $\mathscr{X}$ is *semistable* if the special fibre $\mathscr{X}_0$ is reduced, and $\mathscr{X}$ is an *snc model* if $\mathscr{X}$ is a regular scheme and the special fibre $\mathscr{X}_0$ is an snc divisor on $\mathscr{X}$. Note that the special fibre $\mathscr{X}_0$ is always a principal Cartier divisor: it is the divisor $\mathrm{div}_{\mathscr{X}}(\varpi)$ of a uniformizer $\varpi \in k^{\circ\circ}\backslash\{0\}$. When k is of residue characteristic zero, snc models always exist by Hironaka's theorem on resolution of singularities; see [Hir64].

Let (X, D) be a pair. A *model* for (X, D) is a pair $(\mathscr{X}, \mathscr{D})$, where $\mathscr{X}$ is a model of X over k° and $\mathscr{D} = \overline{D} + \mathscr{X}_{0,\mathrm{red}}$; here, $\overline{D}$ denotes the closure of D in $\mathscr{X}$ and $\mathscr{X}_{0,\mathrm{red}}$ is the reduced special fibre of $\mathscr{X}$. Furthermore, we require that $K_{\mathscr{X}} + \mathscr{D}_{\mathrm{red}}$ is $\mathbf{Q}$-Cartier when $K_X + D_{\mathrm{red}}$ is so.

2.6 Monomial and quasi-monomial points

Let k be a non-Archimedean field. For a valued extension $k \subseteq k'$, consider

$$\begin{cases} s(k'/k) \coloneqq \mathrm{tr.deg}(\widetilde{k'}/\widetilde{k}), \\ t(k'/k) \coloneqq \dim_{\mathbf{Q}}(\sqrt{|k'^\times|}/\sqrt{|k^\times|}), \\ d(k'/k) \coloneqq s(k'/k) + t(k'/k). \end{cases}$$

The invariant $s(k'/k)$ is called the *transcendence degree* of k'/k, and $t(k'/k)$ is called the *rational rank*. The basic properties of these invariants are collected in the lemma below.

Lemma 2.6.0.1. *1. If $k \subseteq k'$ is a valued extension, then $d(k'/k) \leq \operatorname{tr.deg}(k'/k)$.*

2. If $k \subseteq k' \subseteq k''$ is a tower of valued extensions, then

$$d(k''/k) = d(k''/k') + d(k'/k),$$

and the analogous equalities hold for s and t.

3. If $k \subseteq k'$ is an immediate valued extension, then

$$d(k'/k) = s(k'/k) = t(k'/k) = 0.$$

In particular, if $k \subseteq k' \subseteq \ell \subseteq \ell'$ is a tower of valued extensions such that both k'/k and ℓ'/ℓ are immediate, then

$$d(\ell'/k') = d(\ell/k).$$

Proof. The inequality in (1) is the Abhyankar inequality; see e.g. [Bou89, VI, §10.3, Corollary 1]. The equalities in (2) and (3) are elementary. □

Let X be a variety over k. For $x \in X$, set $s(x) = s(\mathcal{H}(x)/k)$, $t(x) = t(\mathcal{H}(x)/k)$, and $d(x) = d(\mathcal{H}(x)/k)$. By Lemma 2.6.0.1, we have $d(x) \leq \dim(X)$ for all $x \in X^{\mathrm{an}}$. A point $x \in X$ is *Abhyankar* if $d(x) = \dim(X)$; such points are dense in X^{an} by [Poi13, Corollaire 4.8]. Among the Abhyankar points of X^{an}, we will be particular interested in the subclass of divisorial points. A point $x \in X^{\mathrm{an}}$ is *divisorial* if the following condition holds:

- if k is trivially-valued, then $s(x) = \dim(X) - 1$ and $t(x) = 1$;

- if k is nontrivially-valued, then $s(x) = \dim(X)$ and $t(x) = 0$.

Write $X^{\mathrm{div}} \subseteq X^{\mathrm{an}}$ for the subset of divisorial points. In the literature, the points x satisfying $s(x) = \dim(X)$ and $t(x) = 0$ are known as the *Shilov points* of X^{an}. While

Shilov and divisorial points coincide when k is nontrivially-valued, there are many more divisorial points than Shilov points in the trivially-valued setting. Indeed, when k is trivially-valued, the only Shilov point of X^{an} is the trivial norm.

In both the trivially-valued and discretely-valued settings, divisorial points admit a geometric criterion, provided that the variety X is normal.

Proposition 2.6.0.2. *Let X be a normal variety over k and $x \in X^{\mathrm{an}}$.*

1. *If k is trivially-valued and $x \in X^{\beth}$, then $x \in X^{\mathrm{div}}$ if and only if there exists a constant $c > 0$, a proper birational morphism $h\colon Y \to X$ from a normal k-variety Y, and a prime divisor $E \subseteq Y$ such that*

$$|f(x)| = e^{-c\,\mathrm{ord}_E(h^* f)}$$

for $f \in k(X)$; in this case, we say x is determined by the triple $(c, Y \xrightarrow{h} X, E)$.

2. *If k is discretely-valued and x admits a centre on some model of X, then $x \in X^{\mathrm{div}}$ if and only if there exists a model $\mathscr{X}$ of X and an irreducible component $E \subseteq \mathscr{X}_0$ such that*

$$|f(x)| = |\varpi|^{\mathrm{ord}_E(f)/\mathrm{ord}_E(\varpi)}$$

for $f \in k(X)$, where ϖ is a uniformizer of k; in this case, we say x is determined by the pair $(\mathscr{X}, E)$.

Proof. See [ZS60, VI,§14, Theorem 31] for a proof of (1), and [MN15, Proposition 2.4.8] for a proof of (2). □

Proposition 2.6.0.3. *If* $\mathrm{char}(\widetilde{k}) = 0$*, then X^{div} is dense.*

Proof. This is [JM12, Remark 4.11] in the trivially-valued setting, and [MN15, Proposition 2.4.9] in the discretely-valued case. □

Remark 2.6.0.4. For a k-analytic space X, a point $x \in X$ is called a *Shilov point* if $s(x) = \dim_x(X)$ and $t(x) = 0$, where $\dim_x(X)$ denotes the local dimension of X at x (see [Duc07] for a discussion of dimension theory on k-analytic spaces). If X is a good strictly k-analytic space of equidimension d (i.e. $\dim_x(X) = d$ for all $x \in X$), then [Poi13, Corollaire 4.5] shows that the locus

$$\{x \in X \colon s(x) = d\}$$

is dense in X. As Shilov points and divisorial points coincide on a good, equidimensional strictly k-analytic space over a nontrivially-valued field, this result can be seen as a generalization of Proposition 2.6.0.3.

When $\mathrm{char}(\widetilde{k}) = 0$ and k is either discretely-valued or trivially-valued, the set of Abhyankar points in X^{an} admits a geometric description that generalizes the one for divisorial points in Proposition 2.6.0.2.

Consider first the case when k is trivially-valued. A point $x \in X^{\beth}$ is *quasi-monomial* if there exists

- a proper birational morphism $h \colon Y \to X$ from a normal k-variety Y,
- a regular system of parameters $(y_1, \dots, y_r)$ at a regular point ξ of Y,
- an r-tuple $(\alpha_1, \dots, \alpha_r) \in \mathbf{R}_+^r$

such that x is the unique minimal real valuation with $v_x(y_i) = \alpha_i$; see [JM12, Proposition 3.1] for the construction of v_x. If, in addition, Y is smooth, ξ is a stratum of a reduced, snc divisor D on Y, and the y_i's local equations for the components of D containing ξ, then we say x is *determined by* the pair (Y, D). The *skeleton* of the pair (Y, D) is the subset $\mathrm{Sk}(Y, D) \subseteq X^{\beth}$ of quasi-monomial points determined by the pair (Y, D). The $\mathbf{R}_+$-scaling action on $X^{\beth}$ gives $\mathrm{Sk}(Y, D)$ the structure of a cone complex, with each face of $\mathrm{Sk}(Y, D)$ determined by a stratum of D.

Suppose now that k is discretely-valued and let ϖ be a uniformizer of k. A point $x \in X^{\mathrm{an}}$ is *monomial* if there exists

- an snc model $\mathscr{X}$ of X,
- an r-tuple $(E_1, \ldots, E_r)$ of distinct irreducible components of $\mathscr{X}_0$,
- local equations y_i for E_i at the generic point ξ of a connected component of $E_1 \cap \ldots \cap E_r$,
- an r-tuple $(\alpha_1, \ldots, \alpha_r) \in \mathbf{R}_+^r$

such that $\sum_{i=1}^r \alpha_i \operatorname{ord}_{E_i}(\varpi) = 1$ and x is the unique minimal real valuation with $v_x(y_i) = \alpha_i$; see [MN15, Proposition 2.4.4] for the construction of v_x. We say x is *determined on* the model $\mathscr{X}$. The *skeleton* of $\mathscr{X}$ is the subset $\mathrm{Sk}(\mathscr{X}) \subseteq \widehat{\mathscr{X}_\eta} \subseteq X^{\mathrm{an}}$ of monomial points determined on $\mathscr{X}$. The skeleton $\mathrm{Sk}(\mathscr{X})$ carries the structure of a cell complex, with cells of dimension r corresponding to connected components of intersections of r irreducible components of $\mathscr{X}_0$.

We denote by X^{mon} the set of quasi-monomial or monomial points in X^{an}, and we note that $X^{\mathrm{div}} \subseteq X^{\mathrm{mon}} \subseteq X^{\mathrm{bir}} \subseteq X^{\mathrm{an}}$.

Proposition 2.6.0.5. *Assume* $\mathrm{char}(\widetilde{k}) = 0$ *and let* $x \in X^{\mathrm{an}}$.

1. *If k is trivially-valued and $x \in X^{\beth}$, then $x \in X^{\mathrm{mon}}$ if and only if x is Abhyankar.*
2. *If k is discretely-valued and x admits a centre on some model of X, then $x \in X^{\mathrm{mon}}$ if and only if x is Abhyankar.*

The condition that $x \in X^{\mathrm{an}}$ admits a centre on some model of X is necessary because the models we consider in this section are *algebraic* (however, if X is proper, then this condition is always satisfied). If one instead allows *formal* models and defines the class of monomial points analogously, then Proposition 2.6.0.5(ii) no longer requires this condition by [GM16, Proposition A.5]. (When X is proper, one does not need to make the distinction between algebraic and formal models by [GM16, Lemma 2.4].)

Proof. The equivalence in (1) is [ELS03, Proposition 2.8], and (2) follows from [BFJ16, Remark 3.8]. □

2.7 Gauss extensions

Let k be a non-Archimedean field and pick $r \in (0,1)\backslash\sqrt{|k^*|}$. Consider the k-subalgebra k_r of $k((\varpi))$ that consists of those bi-infinite series $\sum_{j\in\mathbf{Z}} a_j\varpi^j$ with $a_j \in k$ such that $|a_j|r^j \to 0$ as $|j| \to +\infty$. The k-algebra k_r is in fact a field and it is complete with respect to the norm

$$|\sum_{j\in\mathbf{Z}} a_j\varpi^j|_r := \max_{j\in\mathbf{Z}} |a_j|_{\mathcal{K}} r^j.$$

Introduced in [Ber90, §2.1], the extension k_r/k of non-Archimedean fields is often referred to as a *Gauss extension* in the literature. If k is trivially-valued, then k_r is simply a Laurent series field $k((\varpi))$ over k equipped with the ϖ-adic norm satisfying $|\varpi|_r = r$.

Let Z be a k-analytic space, and write $p_r\colon Z_r := Z \times_k k_r \to Z$ for the ground field extension. For any $z \in Z$, the fibre $p_r^{-1}(z) \subseteq Z_r$ is naturally identified with the spectrum $\mathcal{M}(\mathcal{H}(z)\hat{\otimes}_k k_r)$. If the tensor product seminorm on $\mathcal{H}(z)\widehat{\otimes}_k k_r$ is multiplicative, then it defines the unique Shilov point $\sigma_r(z)$ of $\mathcal{M}(\mathcal{H}(z)\hat{\otimes}_k k_r)$.

Proposition 2.7.0.1. *1. For any $z \in Z$, $\sigma_r(z)$ is well-defined and the natural map $\mathcal{H}(z)\widehat{\otimes}_k k_r \to \mathcal{H}(\sigma_r(z))$ is an isometric isomorphism.*

2. The map $\sigma_r\colon Z \to Z_r$ is a continuous section of p_r.

3. If X is a k-variety and $Z = X^{\mathrm{an}}$, then $x \in X^{\mathrm{bir}}$ if and only if $\sigma_r(x) \in X_{k_r}^{\mathrm{bir}}$.

Proof. The assertion (1) is clear from the definition, (2) is [Ber93, Lemma 2.2.5], and (3) follows by working affinoid-locally and applying [Ber93, Lemma 2.2.5]. □

The invariants s, t, and d introduced in §2.6 behave well under Gauss extension.

Proposition 2.7.0.2. *Let $z \in Z$.*

1. *If $r \in \sqrt{|\mathcal{H}(z)^\times|}$, then $s(\sigma_r(z)) = s(z) + 1$ and $t(\sigma_r(z)) = t(z) - 1$.*

2. *If $r \notin \sqrt{|\mathcal{H}(z)^\times|}$, then $s(\sigma_r(z)) = s(z)$ and $t(\sigma_r(z)) = t(z)$.*

In particular, $d(\sigma_r(z)) = d(z)$. Moreover, if X is a k-variety and $Z = X^{\mathrm{an}}$, then

3 z is Abhyankar if and only if $\sigma_r(z)$ is so;

4 z is divisorial or trivial if and only if $\sigma_r(x)$ is divisorial and $r \in \sqrt{|\mathcal{H}(z)^\times|}$.

Proof. This is [Poi13, Lemme 4.6]. See also [BJ18b, Corollary 1.4]. □

2.8 Logarithmic geometry

In this section, we will briefly review the terminology of log schemes and we refer to [Kat89, Kat94] for a general exposition. See also [BM17, BM19].

Denote a log scheme by $X^+ = (X, \mathcal{M}_{X^+})$, where X is a scheme and $\mathcal{M}_{X^+} \subseteq \mathcal{O}_X$ is the structural sheaf of monoids. All log schemes are assumed to be fine and saturated (*fs*) log schemes defined with respect to the Zariski topology. That is, $\mathcal{M}_{X^+}$ is a sheaf in the Zariski topology on X, and for every $x \in X$, the stalk $\mathcal{M}_{X^+,x}$ contains $\mathcal{O}_{X,x}^\times$ and it has the following three properties:

- $\mathcal{M}_{X^+,x}$ is a finitely-generated monoid;
- the groupification morphism $\mathcal{M}_{X^+,x} \to \mathcal{M}_{X^+,x}^{\mathrm{gp}}$ is injective;
- if $f^n \in \mathcal{M}_{X^+,x}$ for some $f \in \mathcal{M}_{X^+,x}^{\mathrm{gp}}$ and $n \in \mathbf{Z}_{>0}$, then $f \in \mathcal{M}_{X^+,x}$.

The *characteristic sheaf* of X^+ is the quotient sheaf

$$\mathcal{C}_{X^+} := \mathcal{M}_{X^+}/\mathcal{O}_X^\times.$$

For $x \in X$, write $\mathcal{I}_{X^+,x}$ for the ideal of $\mathcal{O}_{X,x}$ generated by $\mathcal{M}_{X^+,x} \setminus \mathcal{O}_{X,x}^\times$.

The prototypical example of a log scheme that we will consider is the following: let D be an effective Weil divisor on a scheme X, and consider the sheaf

$$U \mapsto \mathcal{M}_D(U) := \{f \in \mathcal{O}_X(U) \colon f|_{X \setminus D} \text{ is invertible}\}$$

of monoids on X. The log scheme $X^+ = (X, \mathcal{M}_D)$ is called the *divisorial log structure* on X associated to D, and we write it as $X^+ = (X, D)$. Note that X^+ depends only on $\mathrm{Supp}(D)$. If D is an snc divisor and x is a stratum of D, then $\mathcal{C}_{X^+,x}$ is a free monoid on the local equations of the components of D passing through x, and $\mathcal{I}_{X^+,x}$ is the maximal ideal of $\mathcal{O}_{X,x}$.

A log scheme X^+ is *log-regular* at $x \in X$ if the following two conditions hold:

1. $\mathcal{O}_{X,x}/\mathcal{I}_{X^+,x}$ is a regular local ring;
2. $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{X,x}/\mathcal{I}_{X^+,x}) + \mathrm{rank}(\mathcal{C}^{\mathrm{gp}}_{X^+,x})$, where $\mathcal{C}^{\mathrm{gp}}_{X^+,x}$ is the groupification of the monoid $\mathcal{C}_{X^+,x}$.

The log scheme X^+ is *log-regular* if it is log-regular at all points. Two common examples of log-regular log schemes are toric varieties with the divisorial log structure associated to the torus-invariant boundary divisor, or smooth varieties equipped with the divisorial log structure associated to an snc divisor.

For a log-regular log scheme X^+, the locus $\{x \in X \colon \mathcal{M}_{X^+,x} \neq \mathcal{O}_{X,x}^\times\}$ where the log structure is nontrivial is a reduced divisor on X, which we will denote by D_{X^+}. In fact, the log scheme X^+ is isomorphic to X equipped wtih the divisorial log structure on X induced by D_{X^+} by [Kat94, Theorem 11.6].

For a log-regular log scheme $X^+ = (X, D_{X^+})$, the *Kato fan* $F_{X^+} = (F, \mathcal{M}_F)$ is the log scheme consisting of the subscheme $F \subseteq X$ of generic points x of the intersections of irreducible components of D_{X^+}, and $\mathcal{M}_{F,x} = \mathcal{C}_{X^+,x}$. The Kato fan first appears in [Kat94] and the above description is detailed in [BM17, Lemma 2.2.3].

2.9 Polyhedral complexes

The *dual (intersection) complex* of a (pure-dimensional) snc divisor D is the cell complex $\mathcal{D}(D)$ whose vertices are in correspondence with the irreducible components $\{D_i\}_{i\in I}$ of D, and whose r-dimensional cells correspond to strata of codimension $r+1$. The attaching maps are prescribed by the inclusion relation. The complex $\mathcal{D}(D)$ is simplicial precisely when the intersection $\bigcap_{i\in J} D_i$ is irreducible for any subset $J \subseteq I$.

Given a topological space A, the *cone* over A, denoted $\mathrm{Cone}(A)$, is the quotient of $A \times \mathbf{R}_+$ under the identification $(a_1, 0) \sim (a_2, 0)$ for any a_1, $a_2 \in A$. The vertex of $\mathrm{Cone}(A)$ is the image of $A \times \{0\}$ under the quotient map. The group $\mathbf{R}_+^*$ acts by rescaling on the second factor and descends to an action on $\mathrm{Cone}(A)$. If C is a topological space homeomorphic to $\mathrm{Cone}(A)$, then A is homeomorphic to the quotient of the punctured cone $C^* \coloneqq C \setminus \{\text{vertex}\}$ by the $\mathbf{R}_+^*$-action.

Given two topological spaces A and B, the *join* of A and B, denoted $A * B$, is the quotient of the space $A \times B \times I$, where $I = [0, 1]$, under the identifications

$$(a, b_1, 0) \sim (a, b_2, 0) \quad \forall a \in A, b_1, b_2 \in B,$$

$$(a_1, b, 1) \sim (a_2, b, 1) \quad \forall a_1, a_2 \in A, b \in B.$$

In other words, the join is the space of all segments joining points in A and B, with two segments meeting only at common endpoints.

The homeomorphism $A \times B \times I \times \mathbf{R}_+ \to A \times \mathbf{R}_+ \times B \times \mathbf{R}_+$ given by

$$(a, b, t, r) \mapsto (a, r(1-t), b, rt)$$

descends to a $\mathbf{R}_+^*$-equivariant homeomorphism

$$\mathrm{Cone}(A * B) \simeq \mathrm{Cone}(A) \times \mathrm{Cone}(B), \tag{2.1}$$

where the cones are endowed with the $\mathbf{R}_+^*$-action defined above, and the product has the diagonal action.

CHAPTER III

Canonical metrics on sheaves of differentials

3.1 Metrics on non-Archimedean line bundles

Let k be a non-Archimedean field. In this section, we will discuss the formalism of metrics on line bundles on k-analytic spaces.

3.1.1 Definition of a metric on a line bundle

Definition 3.1.1.1. Let X be a k-analytic space. Given a line bundle L on X, a *metric* ϕ on L is the data of a function $\phi(\cdot, x)\colon L_x \to \overline{\mathbf{R}}$ for each $x \in X$, with the following transformation property: for any $s \in L_x$ and $f \in \mathcal{O}_{X,x}$, we have

$$\phi(fs, x) = v_x(f) + \phi(s, x). \tag{3.1}$$

A metric ϕ is *continuous* if for any open subset $U \subseteq X$ and any section $s \in \Gamma(U, L)$, the function

$$U \ni x \mapsto \phi(s, x) \in \overline{\mathbf{R}}$$

is continuous. This definition coincides with the continuous metrics of [CL11], provided that $\phi(\cdot, x) \not\equiv +\infty$ for all $x \in X$. Similarly, one defines *upper-semicontinuous* (usc) and *lower-semicontinuous* (lsc) metrics.

For line bundles L_1 and L_2 on X with metrics ϕ_1 and ϕ_2, the tensor product

$L_1 \otimes L_2$ is equipped with the metric $\phi_1 \otimes \phi_1$ given by

$$(\phi_1 \otimes \phi_2)(s_1 \otimes s_2, x) = \phi_1(s_1, x) + \phi_2(s_2, x)$$

for $x \in X$, $s_1 \in L_{1,x}$, and $s_2 \in L_{2,x}$. Similarly, a metric on a line bundle induces metrics on the inverse and on the powers of the line bundle.

Remark 3.1.1.2. In the literature, it is common to write a metric ϕ on line bundles in the 'multiplicative' notation $\|\cdot\| = r^{\phi}$ for some $r \in (0,1)$, as opposed to the 'additive' notation introduced in Definition 3.1.1.1. That is, a metric on L^{an} can also be defined as a collection of functions $\|\cdot\|_x \colon L^{\mathrm{an}}_x \to \mathbf{R}_+$ such that

$$\|f \cdot s\|_x = |f(x)| \cdot \|s\|_x$$

for $s \in L^{\mathrm{an}}_x$ and $f \in \mathcal{O}_{X^{\mathrm{an}},x}$. See [CL11] for further details. The multiplicative notation is adopted especially in §3.3-3.4, whereas the additive notation is more convenient elsewhere.

3.1.2 Metrics on analytifications of line bundles

Suppose now that X is a variety over k and L is a line bundle on X. In this section, we will discuss metrics on the analytification L^{an} of L.

For $x \in X$, the stalk L^{an}_x can be realized as

$$L^{\mathrm{an}}_x \simeq L_{\ker(x)} \otimes_{\mathcal{O}_{X,\ker(x)}} \mathcal{O}_{X^{\mathrm{an}},x}. \tag{3.2}$$

The (semi)continuity of a metric on L^{an} can be verified on algebraic sections of L, in the following sense.

Lemma 3.1.2.1. *A metric ϕ on L^{an} is (semi)continuous if and only if for any $x \in X$, the function $\phi(\cdot, x) \colon L_{\ker(x)} \to \overline{\mathbf{R}}$ is (semi)continuous.*

This is often a more convenient condition to check for metrics that are defined in terms of algebro-geometric data. The proof of Lemma 3.1.2.1 is immediate from (3.2) and the transformation property for a metric.

A metric ϕ is determined by its values on the algebraic stalks $L_{\ker(x)}$ of L; indeed, given a function $\phi(\cdot, x)$ on $L_{\ker(x)}$ satisfying the transformation property (3.1), it extends to a function on L_x^{an} by using the isomorphism (3.2). More precisely, if $s \in L_{\ker(x)}$ is an $\mathcal{O}_{X,\ker(x)}$-module generator, then $s \otimes 1 \in L_x^{\mathrm{an}}$ is an $\mathcal{O}_{X^{\mathrm{an}},x}$-module generator, so the value of $\phi(\cdot, x)$ on L_x^{an} is completely determined by its value on $s \otimes 1$ and the formula (3.1).

Furthermore, if k is trivially-valued, then we may consider metrics on the $\beth$-analytification

$$L^{\beth} := L^{\mathrm{an}}|_{X^{\beth}}$$

of L. In this setting, a metric on $L^{\beth}$ is in fact determined at a point x by its values on the stalks $L_{c_X(x)}$; indeed, the localization map $\mathcal{O}_{X,c_X(x)} \hookrightarrow \mathcal{O}_{X,\ker(x)}$ gives rise to an isomorphism

$$L_{\ker(x)} \simeq L_{c_X(x)} \otimes_{\mathcal{O}_{X,c_X(x)}} \mathcal{O}_{X,\ker(x)}$$

and we can argue as before.

Definition 3.1.2.2. Assume k is trivially-valued. Given a line bundle L on X, the *trivial metric* $\phi_{\mathrm{triv},L}$ on $L^{\beth}$ assigns to a point $x \in X^{\beth}$ and a local section $s \in L_{c_X(x)}$ the number

$$\phi_{\mathrm{triv},L}(s, x) = v_x(f), \tag{3.3}$$

where s is given by the function $f \in \mathcal{O}_{X,c_X(x)}$ locally at $c_X(x)$. Said differently, pick any $\mathcal{O}_{X,c_X(x)}$-module generator $\delta \in L_{c_X(x)}$, and write $s = f\delta$ in $L_{c_X(x)}$. The expression (3.3) is independent of the choice of generator δ, since any two generators differ by a unit $u \in \mathcal{O}_{X,c_X(x)}^{\times}$, and $v_x(u) = 0$.

The trivial metric $\phi_{\mathrm{triv},L}$ allows us to identify a function $\varphi\colon X^{\beth} \to \overline{\mathbf{R}}$ with a metric $\varphi + \phi_{\mathrm{triv},L}$ on $L^{\beth}$; that is, to a point $x \in X^{\beth}$ and a local section $s \in L_{c_X(x)}$, the metric $\varphi + \phi_{\mathrm{triv},L}$ assigns the number

$$(\varphi + \phi_{\mathrm{triv},L})(s, x) \coloneqq \varphi(x) + v_x(f),$$

where, locally at $c_X(x)$, s is given by the function $f \in \mathcal{O}_{X,c_X(x)}$. In fact, every metric on $L^{\beth}$ arises in this manner. See [BJ18b, §2.8] for further details.

Remark 3.1.2.3. If X is proper over a trivially-valued field k, the trivial metric $\phi_{\mathrm{triv},L}$ is the non-Archimedean metric on $L^{\beth}$ associated to the trivial test configuration of (X, L), in the sense of [BHJ17, Remark 3.3]. The relationship between test configurations and non-Archimedean metrics yields new insights in the study of K-stability; see [BJ18a] for an overview.

3.2 Weight metrics

In this section, we introduce the notion of a weight function associated to a rational pluricanonical form on a variety defined over a trivially-valued field of characteristic zero. The weight functions are crucial to define and compute the essential skeleton of a pair in the trivially-valued setting.

To this end, we briefly recall the formalism of metrics on the analytification of a line bundle over an arbitrary non-Archimedean field k. We introduce the weight metric on the analytification of the pluricanonical bundle; in the discretely-valued case, the weight metric originates in [MN15] and it is studied further in [NX16, BN16, Tem16, BM17]. To do so, we assume that k has residue characteristic zero: this guarantees the divisorial points are dense in the Berkovich analytification (see Proposition 2.6.0.3), a property that we employ in the construction of weight functions and weight metrics.

Throughout the section, let X be a normal variety over a non-Archimedean field k, K_X a canonical divisor on X, and D a Weil $\mathbf{Q}$-divisor on X such that $K_X + D_{\mathrm{red}}$ is $\mathbf{Q}$-Cartier. For $m \in \mathbf{Z}_{>0}$ sufficiently divisible, the sections of the line bundle

$$\omega_{(X,D_{\mathrm{red}})}^{\otimes m} \coloneqq \mathcal{O}_X(m(K_X + D_{\mathrm{red}}))$$

are called *logarithmic m-pluricanonical forms of* (X, D), while the sections of the rank-1 reflexive sheaf

$$\omega_{(X,D)}^{\otimes m} \coloneqq \mathcal{O}_X(m(K_X + D))$$

are called *D-logarithmic m-pluricanonical forms on X*.

3.2.1 The weight metric over a discretely-valued field

Suppose that k is a discretely-valued field with residue characteristic zero, and let $\varpi \in k^{\circ\circ}$ be a uniformizer. Generalizing the ideas of Kontsevich and Soibelman in [KS06], Mustață and Nicaise in [MN15] construct a $\overline{\mathbf{R}}$-valued function on the analytification X^{an} associated to a rational pluricanonical forms η of X, called the *weight function associated to* η and denoted by wt_η. We briefly recall the definition of the weight function and prove a maximality property; see [MN15, NX16, BM17] for further details.

Let η be a rational section of $\omega_{(X,D)}^{\otimes m}$. The definition of the weight function associated to η on divisorial points is as follows. If $x \in X^{\mathrm{div}}$ has a divisorial representation on a model $\mathscr{X}$ of X, then we may assume that $(\mathscr{X}, D_{\mathscr{X}})$ is a log-regular model of (X, D_{red}), where $D_{\mathscr{X}} = \overline{D}_{\mathrm{red}} + (\mathscr{X}_0)_{\mathrm{red}}$. Then, we set

$$\mathrm{wt}_\eta(x) \coloneqq v_x(\mathrm{div}_{(\mathscr{X}, D_{\mathscr{X}} - \mathrm{div}_{\mathscr{X}}(\varpi))}(\eta)) + m, \tag{3.4}$$

where $\operatorname{div}_{(\mathscr{X}, D_{\mathscr{X}} - \operatorname{div}_{\mathscr{X}}(\varpi))}(\eta)$ denotes the divisor on $\mathscr{X}$ determined by η, which is thought of as a rational section of the line bundle

$$\mathcal{O}_{\mathscr{X}}(m(K_{\mathscr{X}/k^\circ} + D_{\mathscr{X}} - \operatorname{div}_{\mathscr{X}}(\varpi))).$$

By [BM17, Lemma 4.1.4], the formula for the weight function in (3.4) is equivalent to the original definition of the weight function introduced in [MN15, §4.3].

Theorem 3.2.1.1. *Suppose X is smooth. For any rational section η of $\omega_{(X,D)}^{\otimes m}$, there is a unique maximal lower-semicontinuous extension* $\mathrm{wt}_\eta\colon X^{\mathrm{an}} \to \overline{\mathbf{R}}$ *of the weight function* $\mathrm{wt}_\eta\colon X^{\mathrm{div}} \to \mathbf{R}$.

The extension was produced by Mustață and Nicaise in [MN15, §4.4], and the maximality property is demonstrated below. This property is presumably well-known to experts, but we are not aware of a proof appearing in the literature.

Proof. Pick a smooth compactification $X \subseteq \overline{X}$ of X, so $X^{\mathrm{bir}} = \overline{X}^{\mathrm{bir}}$. Such a compactification exists by Nagata's compactification theorem and resolution of singularities. The construction of a lower-semicontinuous extension $\mathrm{wt}_{X,\eta}\colon X^{\mathrm{an}} \to \overline{\mathbf{R}}$ is made in [MN15, §4.4], and similarly we have an extension $\mathrm{wt}_{\overline{X},\eta}\colon \overline{X}^{\mathrm{an}} \to \overline{\mathbf{R}}$. By [MN15, Proposition 4.5.5], we have

$$\mathrm{wt}_{X,\eta} = \mathrm{wt}_{\overline{X},\eta}$$

on X^{an}. We now prove that $\mathrm{wt}_{X,\eta}$ is maximal: if $W\colon X^{\mathrm{an}} \to \overline{\mathbf{R}}$ is another lower-semicontinuous extension of wt_η rom X^{div}, then we must show the inequality

$$W(x) \leqslant \mathrm{wt}_{X,\eta}(x) \tag{3.5}$$

for all $x \in X^{\mathrm{an}}$. To this end, we first prove (3.5) for $x \in X^{\mathrm{mon}}$, and then for any $x \in X^{\mathrm{an}}$ by approximating x by monomial points. If $x \in X^{\mathrm{mon}}$, pick a sequence (x_j)

of divisorial points that converges to x, all of whom lie in the skeleton of a fixed snc model of X. By the lower-semicontinuity of W, we have

$$W(x) \leqslant \liminf_j W(x_j) = \liminf_j \mathrm{wt}_{X,\eta}(x_j) = \mathrm{wt}_{X,\eta}(x),$$

where the final equality $\liminf_j \mathrm{wt}_{X,\eta}(x_j) = \mathrm{wt}_{X,\eta}(x)$ follows from the continuity of the weight function on a fixed skeleton, as in [MN15, Proposition 4.4.3]. If $x \in X^{\mathrm{an}}$, then [BFJ16, Corollary 3.2] implies that $x = \lim_{\overline{\mathscr{X}}} \rho_{\overline{\mathscr{X}}}(x)$, where the limit runs over all snc models $\overline{\mathscr{X}}$ of $\overline{X}$ and $\rho_{\overline{\mathscr{X}}}\colon \overline{X}^{\mathrm{an}} \to \mathrm{Sk}(\overline{\mathscr{X}})$ denotes the retraction onto the skeleton from [MN15, §3.1]. As $\rho_{\overline{\mathscr{X}}}(x) \in \overline{X}^{\mathrm{mon}} = X^{\mathrm{mon}}$ for all snc models $\overline{\mathscr{X}}$, the lower-semicontinuity of W shows that

$$\begin{aligned} W(x) \leqslant \liminf_{\overline{\mathscr{X}}} W(\rho_{\overline{\mathscr{X}}}(x)) &\leqslant \liminf_{\overline{\mathscr{X}}} \mathrm{wt}_{X,\eta}(\rho_{\overline{\mathscr{X}}}(x)) \\ &= \liminf_{\overline{\mathscr{X}}} \mathrm{wt}_{\overline{X},\eta}(\rho_{\overline{\mathscr{X}}}(x)) \\ &\leqslant \sup_{\overline{\mathscr{X}}} \mathrm{wt}_{\overline{X},\eta}(\rho_{\overline{\mathscr{X}}}(x)) \\ &= \mathrm{wt}_{\overline{X},\eta}(x) = \mathrm{wt}_{X,\eta}(x). \end{aligned}$$

The uniqueness of the extension follows from the maximality, and we write it simply as $\mathrm{wt}_\eta = \mathrm{wt}_{X,\eta}$. □

Definition 3.2.1.2. The *weight metric* $\mathrm{wt}_{\mathrm{disc}}$ is the metric on $(\omega_{(X,D_{\mathrm{red}})}^{\otimes m})^{\mathrm{an}}$ satisfying

$$\mathrm{wt}_{\mathrm{disc}}(\eta, x) = \mathrm{wt}_\eta(x) \tag{3.6}$$

for any $x \in X^{\mathrm{an}}$ and rational section η of $\omega_{(X,D_{\mathrm{red}})}^{\otimes m}$ that is regular at $\ker(x)$. By Theorem 3.2.1.1, $\mathrm{wt}_{\mathrm{disc}}$ is the maximal lower-semicontinuous metric on $(\omega_{(X,D_{\mathrm{red}})}^{\otimes m})^{\mathrm{an}}$ such that (3.6) holds on X^{div}. Write $\|\cdot\|_{\mathrm{wt}_{\mathrm{disc}}}$ for the weight metric in multiplicative notation, as in Remark 3.1.1.2.

3.2.2 The weight metric over a trivially-valued field

Suppose that k is a trivially-valued field of characteristic zero, and assume that $K_X + D$ is $\mathbf{Q}$-Cartier.

Definition 3.2.2.1. Let $x \in X^{\mathrm{div}} \cap X^{\beth}$ be the divisorial point determined by the triple $(c, Y \xrightarrow{h} X, E)$. Pick canonical divisors K_Y on Y and K_X on X such that $h_*(K_Y) = K_X$. The *log discrepancy* $A_{(X,D)}(x)$ of x is the value

$$A_{(X,D)}(x) \coloneqq c\left(1 + \mathrm{ord}_E\left(K_Y - \frac{1}{m}h^*(m(K_X + D))\right)\right) \tag{3.7}$$

for $m \in \mathbf{Z}_{>0}$ sufficiently divisible. The pair (X, D) is *log canonical* if $A_{(X,D)}(x) \geqslant 0$ for all $x \in X^{\mathrm{div}} \cap X^{\beth}$.

It is easy to verify that the log discrepancy $A_{(X,D)}(x)$ depends only on x, and not on the choice of m, nor on the choice of the birational model Y of X.

There is a maximal lower-semicontinuous extension $A_{(X,D)} \colon X^{\beth} \to \overline{\mathbf{R}}$ of the log discrepancy on the divisorial points $X^{\mathrm{div}} \cap X^{\beth}$; explicitly, it is given by

$$A_{(X,D)}(x) = \sup_{U \ni x} \inf_{y \in U \cap X^{\mathrm{div}}} A_{(X,D)}(y), \tag{3.8}$$

where the supremum runs over all open neighbourhoods U of x in $X^{\beth}$. The extension $A_{(X,D)}$, which we also refer to as the log discrepancy function, is $\mathbf{R}_+$-homogeneous and it is non-negative when (X, D) is log canonical. The restriction to $X^{\mathrm{bir}} \cap X^{\beth}$ admits an alternative characterization; see [Blu18, §3.2].

The log discrepancy function is well studied in the literature: when X is smooth and $D = \emptyset$, it is introduced in [JM12, §5] as a function $A_X \colon X^{\mathrm{bir}} \cap X^{\beth} \to \overline{\mathbf{R}}_+$. The same holds for normal varieties by [BdFFU15]. The function A_X is extended to all of $X^{\beth}$ when X is smooth in [BJ18a, Appendix A], and it is constructed in positive characteristic in [Can17, §3].

Definition 3.2.2.2. For a rational section η of $\omega_{(X,D_{\mathrm{red}})}^{\otimes m}$ that is regular on the Zariski open $U \subseteq X$, the *weight function* $\mathrm{wt}_\eta \colon U^{\beth} \to \overline{\mathbf{R}}$ of η is given by

$$\mathrm{wt}_\eta(x) = mA_{(X,D_{\mathrm{red}})}(x) + \phi_{\mathrm{triv},\omega_{(X,D_{\mathrm{red}})}^{\otimes m}}(\eta, x).$$

The *weight metric* $\mathrm{wt}_{\mathrm{triv}}$ is the metric on $(\omega_{(X,D_{\mathrm{red}})}^{\otimes m})^{\beth}$ satisfying

$$\mathrm{wt}_{\mathrm{triv}}(\eta, x) = \mathrm{wt}_\eta(x) \tag{3.9}$$

for any $x \in X^{\beth}$ and rational section η of $\omega_{(X,D_{\mathrm{red}})}^{\otimes m}$ that is regular at $\ker(x)$. It follows that $\mathrm{wt}_{\mathrm{triv}}$ is the the maximal lower-semicontinuous metric on $(\omega_{(X,D_{\mathrm{red}})}^{\otimes m})^{\beth}$ such that (3.9) holds on $X^{\mathrm{div}} \cap X^{\beth}$. Write $\| \cdot \|_{\mathrm{wt}_{\mathrm{triv}}}$ for the weight metric in multiplicative notation.

Remark 3.2.2.3. There is another construction in [MN15, §6.1] of a weight function in the trivially-valued setting, which is distinct from the weight function of Definition 3.2.2.2 (indeed, it does not take a pluricanonical section as an argument).

3.2.3 Alternative expressions for the weight function

Assume that k is a trivially-valued field of characteristic zero. For a rational section η of $\omega_{(X,D_{\mathrm{red}})}^{\otimes m}$, set

$$D_\eta \coloneqq D_{\mathrm{red}} - \mathrm{div}_{(X,D_{\mathrm{red}})}(\eta),$$

where $\mathrm{div}_{(X,D_{\mathrm{red}})}(\eta)$ denotes the divisor of η, thought of as a rational section of the line bundle $\omega_{(X,D_{\mathrm{red}})}^{\otimes m}$. In the following proposition, we provide an alternative expression for the weight function associated to η, which is purely in terms of a log discrepancy function.

Proposition 3.2.3.1. *For any $x \in X^{\beth}$, we have* $\mathrm{wt}_\eta(x) = mA_{(X,D_\eta)}(x)$.

Proof. Using the maximality properties of the weight function and of the log discrepancy function, it suffices to check the equality on divisorial points. If $x \in X^{\mathrm{div}} \cap X^{\beth}$ is determined by the triple $(c, Y \xrightarrow{h} X, E)$, then we have that

$$\begin{aligned} \mathrm{wt}_\eta(x) &= mA_{(X,D_{\mathrm{red}})}(x) + \phi_{\mathrm{triv},\omega^{\otimes m}_{(X,D_{\mathrm{red}})}}(\eta, x) \\ &= mc\left(1 + \mathrm{ord}_E\left(K_Y - \frac{1}{m}h^*(m(K_X + D_{\mathrm{red}}))\right)\right) + c\,\mathrm{ord}_E(h^*\mathrm{div}_{(X,D_{\mathrm{red}})}(\eta)) \\ &= mA_{(X,D_{\mathrm{red}}-\mathrm{div}_{(X,D_{\mathrm{red}})}(\eta))}(x) \\ &= mA_{(X,D_\eta)}(x), \end{aligned}$$

as required. □

Corollary 3.2.3.2. *If $x \in X^{\mathrm{div}} \cap X^{\beth}$ is the divisorial point determined by the triple $(c, Y \xrightarrow{h} X, E)$, then*

$$\mathrm{wt}_\eta(x) = v_x(\mathrm{div}_{(Y,D_Y)}(h^*\eta)),$$

where $D_Y = \widetilde{D}_{\mathrm{red}} + \sum_i E_i$, $\widetilde{D}_{\mathrm{red}}$ denotes the strict transform of D_{red} via h, and the E_i's are the irreducible h-exceptional divisors on Y.

Corollary 3.2.3.2 shows that the weight function wt_η on $X^{\mathrm{div}} \cap X^{\beth}$ can be computed much as in the discretely-valued setting; indeed, this result is the analogue of [BM17, Lemma 4.1.4]. Moreover, Corollary 3.2.3.2 can be deduced from Proposition 3.2.3.1, but we find enlightening to provide a different proof of the statement using a local calculation.

Proof. By definition of weight function and of the log discrepancy function, we have that

$$\begin{aligned}\mathrm{wt}_\eta(x) &= mA_{(X,D_{\mathrm{red}})}(x) + \phi_{\mathrm{triv},\omega^{\otimes m}_{(X,D_{\mathrm{red}})}}(\eta,x)\\ &= mc\left(1+\mathrm{ord}_E\left(K_Y - \frac{1}{m}h^*(m(K_X+D_{\mathrm{red}}))\right)\right) + \phi_{\mathrm{triv},\omega^{\otimes m}_{(X,D_{\mathrm{red}})}}(\eta,x)\\ &= c\,\mathrm{ord}_E\left(\left(\omega^{\otimes m}_{(Y,D_Y)}\right)^{-1}\otimes h^*\omega^{\otimes m}_{(X,D_{\mathrm{red}})}\right) + \phi_{\mathrm{triv},\omega^{\otimes m}_{(X,D_{\mathrm{red}})}}(\eta,x).\end{aligned}$$

Let $\xi = c_X(x)$ be the centre of x on X, and let ξ' be the generic point of E in Y. Consider a $\mathcal{O}_{X,\xi}$-module generator δ of $\omega^{\otimes m}_{(X,D_{\mathrm{red}}),\xi}$. Then, locally at ξ, we write the section η as $\eta = f\delta$ for some $f \in \mathrm{Frac}(\mathcal{O}_{X,\xi})$, so that

$$\phi_{\mathrm{triv},\omega^{\otimes m}_{(X,D_{\mathrm{red}})}}(\eta,x) = v_x(f) = c\,\mathrm{ord}_E(h^*f).$$

Consider now a $\mathcal{O}_{Y,\xi'}$-module generator α of the stalk

$$\left(\omega^{\otimes m}_{(Y,D_Y)}\otimes\left(h^*\omega^{\otimes m}_{(X,D_{\mathrm{red}})}\right)^{-1}\right)_{\xi'}.$$

It follows that $\alpha\otimes h^*\delta$ is a $\mathcal{O}_{Y,\xi'}$-module generator of $\left(\omega^{\otimes m}_{(Y,D_Y)}\right)_{\xi'}$, and we write

$$h^*\eta = (\alpha^{-1}h^*f)\cdot\alpha\otimes h^*\delta$$

locally at ξ'. It follows that

$$\begin{aligned}v_x(\mathrm{div}_{(Y,D_Y)}(h^*\eta)) &= v_x(\alpha^{-1}h^*f)\\ &= v_x(\alpha^{-1}) + v_x(h^*f)\\ &= c\,\mathrm{ord}_E\left(\left(\omega^{\otimes m}_{(Y,D_Y)}\right)^{-1}\otimes h^*\omega^{\otimes m}_{(X,D_{\mathrm{red}})}\right) + c\,\mathrm{ord}_E(h^*f)\\ &= \mathrm{wt}_\eta(x),\end{aligned}$$

which concludes the proof. □

3.3 Temkin's metrization of pluricanonical sheaves

In this section, we review Temkin's construction from [Tem16] of an intrinsic metric on the sheaves of differentials of an analytic space. The metrics in this section are written multiplicatively as in Remark 3.1.1.2, following the conventions of [Tem16]. By doing so, one avoids changing the base of logarithms when passing between the trivially-valued and discretely-valued settings.

3.3.1 Seminorms on modules of Kähler differentials

Let $(k, |\cdot|_k)$ denote a non-Archimedean field. Let $(A, |\cdot|_A)$ be a seminormed k-algebra, and let $\widehat{A}$ denote the separated completion of $(A, |\cdot|_A)$. Let $\Omega^1_{A/k}$ be the (algebraic) module of Kähler differentials, which we equip with the seminorm

$$\|\eta\|_{A/\mathcal{K}} := \inf \max_i |a_i|_A \cdot |b_i|_A, \qquad \text{for } \eta \in \Omega^1_{A/k},$$

where the infimum ranges over all finite expressions of the form $\eta = \sum_i a_i db_i$ with $a_i, b_i \in A$. By [Tem16, Lemma 4.1.3], $\|\cdot\|_{A/k}$ is the maximal A-module seminorm such that the differential $d\colon A \to \Omega^1_{A/k}$ is a contractive k-module morphism.

The *completed module of Kähler differentials* $\widehat{\Omega}^1_{A/k}$ of A is the separated completion of $(\Omega^1_{A/k}, \|\cdot\|_{A/k})$, and we write the resulting norm on $\widehat{\Omega}^1_{A/k}$ also as $\|\cdot\|_{A/k}$. In [Tem16], the seminorm $\|\cdot\|_{A/k}$ on $\Omega^1_{A/k}$ is referred to as the *Kähler seminorm*, and the norm $\|\cdot\|_{A/k}$ on $\widehat{\Omega}^1_{A/k}$ is known as the *Kähler norm*.

There is an alternate, intrinsic description of the completed module of Kähler differentials.

Proposition 3.3.1.1. *The composition $\widehat{d}\colon A \xrightarrow{d} \Omega^1_{A/k} \to \widehat{\Omega}^1_{A/k}$ is the universal contractive k-derivation with values in a Banach $\widehat{A}$-module, where $\widehat{A}$ denotes the separated completion of A.*

Proof. This is [Tem16, Lemma 4.3.3]. □

Suppose A is a k-affinoid algebra, and let $\mathcal{J}$ be the kernel of the multiplication map $A\widehat{\otimes}_k A \to A$, which we can view as a finite Banach A-module. Consider the contractive morphism $q\colon A \to \mathcal{J}$ of Banach A-modules given by

$$a \mapsto 1\widehat{\otimes}a - a\widehat{\otimes}1.$$

The map q can be used to realize $\widehat{\Omega}^1_{A/k}$ in a manner analogous to the construction of the module of Kähler differentials for rings.

Proposition 3.3.1.2. *There is a natural isomorphism*

$$\widehat{\Omega}^1_{A/k} \simeq \mathcal{J}/\mathcal{J}^2$$

of Banach A-modules that identifies $\widehat{d}$ with the composition $A \xrightarrow{q} \mathcal{J} \to \mathcal{J}/\mathcal{J}^2$.

Proof. See [Tem16, Remark 4.3.4] and [Ber93, §3.3]. □

For a good k-analytic space Z, one can construct a coherent sheaf $\Omega^1_{Z/k}$ of Kähler differentials on Z such that for any affinoid domain $V = \mathcal{M}(A)$ in Z, we have

$$\Gamma(V, \Omega^1_{Z/k}) = \widehat{\Omega}^1_{A/k}.$$

Strictly speaking, $\Omega^1_{Z/k}$ is defined as a sheaf in the G-topology on Z, but there is no distinction by [Ber93, Proposition 1.3.4]. The sheaf of Kähler differentials satisfies the following GAGA theorem.

Proposition 3.3.1.3. *For a finite type k-scheme X, there is a natural isomorphism*

$$\Omega^1_{X^{\mathrm{an}}/k} \simeq (\Omega^1_{X/k})^{\mathrm{an}} \tag{3.10}$$

of coherent sheaves of $\mathcal{O}_{X^{\mathrm{an}}}$-modules. Furthermore, if k is trivially valued, then (3.10) *restricts to a natural isomorphism $\Omega^1_{X^{\beth}/k} \simeq (\Omega^1_{X/k})^{\beth}$ of $\mathcal{O}_{X^{\beth}}$-modules.*

Proof. The isomorphism (3.10) is [Duc11, §5.1.4], and the corresponding statement for $\beth$-analytifications holds because $X^{\beth}$ is an analytic domain of X^{an}. □

For more details on the construction of the sheaf of differentials, see [Ber93, §3.3] and [Duc11, §5.1].

3.3.2 Temkin's metric

Let Z be a good k-analytic space. For each $z \in Z$, the stalk $(\Omega^1_{Z/k})_z$ is the filtered colimit

$$\Omega^1_{Z/k,z} = \varinjlim_{\mathcal{M}(A)\ni z} \widehat{\Omega}^1_{A/k}$$

over all affinoid neighbourhoods $\mathcal{M}(A) \subseteq Z$ of z. In particular, the Kähler norms on each $\widehat{\Omega}^1_{A/k}$ induce a colimit seminorm the stalk $\Omega^1_{Z/k,z}$, which we denote by $\|\cdot\|_z$. The pair $(\Omega^1_{Z/k,z}, \|\cdot\|_z)$ is a seminormed $\mathcal{O}_{Z,z}$-module, which is not complete in general. This collection $\{\|\cdot\|_z\}_{z\in Z}$ of seminorms on each stalk of $\Omega^1_{Z/k}$ is known as *Temkin's metric on the sheaf* $\Omega^1_{Z/k}$; it gives $\Omega^1_{Z/k}$ the structure of a seminormed sheaf of $\mathcal{O}_Z$-modules in the sense of [Tem16, §3.1].

The stalks $\Omega^1_{Z/k,z}$ can be difficult to describe; for example, $\Omega^1_{Z/k,z}$ is not isomorphic to $\Omega^1_{\mathcal{O}_{Z,z}/k}$ as normed $\mathcal{O}_{Z,z}$-algebras. Nevertheless, the completed fibres admit a much nicer description: for any affinoid neighbourhood $\mathcal{M}(A) \subseteq Z$ of z, the universal property of $\widehat{\Omega}^1_{Z/k}$ yields a contractive morphism $\widehat{\Omega}^1_{A/k} \to \widehat{\Omega}^1_{\mathcal{H}(z)/k}$ of Banach A-modules, and the universal property of the colimit gives rise to a morphism ψ_z and a commutative diagram

$$\begin{array}{ccc} \widehat{\Omega}^1_{A/k} & & \\ \downarrow & \searrow & \\ \Omega^1_{Z/k,z} & \overset{\psi_z}{\dashrightarrow} & \widehat{\Omega}^1_{\mathcal{H}(z)/k}. \end{array}$$

By [Tem16, Theorem 6.1.8], the morphism ψ_z identifies $\widehat{\Omega}^1_{\mathcal{H}(z)/k}$ with the separated completion of the module $(\Omega^1_{Z/k,z}, \|\cdot\|_z)$. In fact, ψ_z factors through the fibre $\Omega^1_{Z/k}(z) \coloneqq \Omega^1_{Z/k,z} \otimes_{\mathcal{O}_{Z,z}} \mathcal{H}(z)$, which is equipped with the tensor product seminorm. This factorization identifies $\widehat{\Omega}^1_{\mathcal{H}(z)/k}$ with the separated completion of $\Omega^1_{Z/k}(z)$; see [Tem16, Corollary 6.1.9].

Now, let Z be a quasi-smooth k-analytic space in the sense of [Duc11, Definition

5.2.4]. For example, take Z to be the analytification of a smooth k-scheme, or the $\beth$-analytification of a smooth k-scheme when k is trivially-valued. By [Duc11, Corollary 5.3.2], the sheaf $\Omega^1_{Z/k}$ is locally free.

Let $\ell, m \in \mathbf{Z}_{>0}$. The stalk of the exterior power $\Omega^{\ell}_{Z/k} := \bigwedge_{i=1}^{\ell} \Omega^1_{Z/k}$ at a point $z \in Z$ acquires an $\mathcal{O}_{Z,z}$-module seminorm as follows: let $\|\cdot\|_{z,\ell}$ be the largest $\mathcal{O}_{Z,z}$-module seminorm on $\Omega^{\ell}_{Z/k,z}$ such that for any local sections $s_1, \ldots, s_\ell \in \Omega^1_{Z/k,z}$, we have

$$\|s_1 \wedge \ldots \wedge s_\ell\|_{z,\ell} \leq \prod_{i=1}^{\ell} \|s_i\|_z.$$

Similarly, the stalks of the tensor power $(\Omega^{\ell}_{Z/k})^{\otimes m}$ are also equipped with seminorms; see [Tem16, §3.2] for a more complete discussion. In particular, if Z is of equidimension n, then the *m-pluricanonical sheaf* $\omega^{\otimes m}_{Z/k} := (\Omega^n_{Z/k})^{\otimes m}$ is a line bundle on Z, and it carries a metric

$$\|\cdot\|_{\mathrm{Tem}} = \{\|\cdot\|_{\mathrm{Tem},z}\}_{z\in Z},$$

which we will also refer to as *Temkin's metric*. Moreover, for a fixed local section s of $\omega^{\otimes m}_{Z/k}$, the function $\|s\|_{\mathrm{Tem}}$ is upper-semicontinuous on the locus where s is defined. Thus, in the terminology of §3.1, Temkin's metric is lower-semicontinuous.

In the lemma below, we describe the behaviour of Temkin's metric on $\omega^{\otimes m}_{Z/k}$ under Gauss extensions (as defined in §2.7). For $r \in \mathbf{R}^*_+$, write $Z_r := Z \times_k k_r$, $p_r \colon Z_r \to Z$ for the ground field extension map, and $\sigma_r \colon Z \to Z_r$ for the Gauss extension.

Lemma 3.3.2.1. *Let Z be a good k-analytic space, $r \in (0,1) \backslash \sqrt{|k^*|}$, and $\ell, m \in \mathbf{Z}_{>0}$. Then, for any $z \in Z$,*

$$\|\cdot\|_{\mathrm{Tem},z} = \|(p_r)^*_z(\cdot)\|_{\mathrm{Tem},\sigma_r(z)}$$

as $\mathcal{O}_{Z,z}$-module seminorms on $(\Omega^{\ell}_{Z/k})^{\otimes m}_z$, where $(p_r)^*_z \colon (\Omega^{\ell}_{Z/k})^{\otimes m}_z \to (\Omega^{\ell}_{Z_r/k_r})^{\otimes m}_{\sigma_r(z)}$ denotes the pullback map at z.

Proof. We may assume that $m = \ell = 1$. Consider the commutative diagram

$$\begin{array}{ccc} \widehat{\Omega}^1_{\mathcal{H}(z)/k}\widehat{\otimes}_{\mathcal{H}(z)}\left(\mathcal{H}(z)\widehat{\otimes}_k k_r\right) & \longrightarrow & \widehat{\Omega}^1_{(\mathcal{H}(z)\widehat{\otimes}_k k_r)/k_r} \\ \downarrow & & \downarrow \\ \widehat{\Omega}^1_{\mathcal{H}(z)/k}\widehat{\otimes}_{\mathcal{H}(z)}\mathcal{H}(\sigma_r(z)) & \longrightarrow & \widehat{\Omega}^1_{\mathcal{H}(\sigma_r(z))/k_r} \end{array}$$

Arguing as in [Tem16, Theorem 6.3.11], it suffices to show that the bottom horizontal map is an isometry. Indeed, the vertical maps are isometric isomorphisms because the natural map $\mathcal{H}(z)\widehat{\otimes}_k k_r \to \mathcal{H}(\sigma_r(z))$ is so, and the top horizontal map is an isometric isomorphism by [Tem16, Lemma 4.2.6]. □

3.3.3 Temkin's metric on divisorial points

When k is a nontrivially-valued field of residue characteristic zero, Temkin's metric $\|\cdot\|_{\mathrm{Tem}}$ on $\omega_{Z/k}^{\otimes m}$ is the maximal lower-semicontinuous extension of its values on the divisorial points $Z^{\mathrm{div}} \subseteq Z$ (in the sense of [Tem16, §3.2.7]). This is shown in [Tem16, Corollary 8.2.10]. When k is trivially-valued of characteristic zero, one can show that Temkin's metric is determined by the set of divisorial points and by the trivial norm; this is done by reducing to the nontrivially-valued setting by means of the Gauss extensions (as in §2.7).

Definition 3.3.3.1. Let Z be a good k-analytic space, $W \subseteq Z$ a subset, L a line bundle on Z, and $\|\cdot\| = \{\|\cdot\|_z\}_{z\in Z}$ a metric on L. We say that $\|\cdot\|$ is *determined on* W if for any $z \in Z$ and any section $s \in L_z$, we have

$$\|s\|_z = \inf_{U\ni z} \sup_{y\in U\cap W} \|(s_U)_y\|_y, \tag{3.11}$$

where the infimum ranges over all open neighbourhoods U of z and local sections $s_U \in \Gamma(U, L)$ that restrict to s_z, and $(s_U)_y$ denotes the image of s_U in the stalk L_y at y.

Note that the right-hand side of (3.11) defines the minimal upper-semicontinuous metric on L that extends the collection of seminorms $\{\|\cdot\|_z\}_{z\in W}$. In particular, if $\|\cdot\|$ is upper-semicontinuous, then $\|\cdot\|_z$ dominates this seminorm for any $z \in Z$.

Now, let Z be a good, quasi-smooth, equidimensional k-analytic space, and let $Z^{\mathrm{Shv}} \subseteq Z$ be the subset of Shilov points of Z. In [Tem16], Temkin shows the following result.

Theorem 3.3.3.2. *Suppose* $\mathrm{char}(\widetilde{k}) = 0$ *and* $m \in \mathbf{Z}_{>0}$. *If* Z *is a compact, strictly* k*-analytic space and* $s \in \Gamma(Z, \omega_{Z/k}^{\otimes m})$, *then*

$$\max_{z\in Z} \|s_z\|_{\mathrm{Tem},z} = \max_{z\in Z^{\mathrm{Shv}}} \|s_z\|_{\mathrm{Tem},z}, \tag{3.12}$$

where s_z *denotes the image of* s *in the stalk* $\omega_{Z/k,z}^{\otimes m}$. *In particular, if* k *is nontrivially-valued, then Temkin's metric on* $\omega_{Z/k}^{\otimes m}$ *is determined on* Z^{div}.

Proof. The equality (3.12) is [Tem16, Corollary 8.2.10]. The second statement follows from the first by observing that any open neighbourhood of a point contains a strictly k-affinoid neighbourhood by [Ber90, Proposition 2.2.3(iii)], Shilov points are dense by Remark 2.6.0.4, and Shilov and divisorial points coincide in this setting. □

The goal of this section is to prove the following trivially-valued analogue of Theorem 3.3.3.2.

Theorem 3.3.3.3. *Let* k *be a trivially-valued field of characteristic zero. For any* $z \in Z$ *and* $s \in \Gamma(Z, \omega_{Z/k}^{\otimes m})$, *there is an affinoid neighbourhood* $V \subseteq Z$ *of* z *such that*

$$\max_{z\in V} \|s_z\|_{\mathrm{Tem},z} = \max_{z\in V\cap(Z^{\mathrm{div}}\cup\{z_0\})} \|s_z\|_{\mathrm{Tem},z}, \tag{3.13}$$

where s_z *denotes the image of* s *in the stalk* $\omega_{Z/k,z}^{\otimes m}$, *and* z_0 *denotes the trivial norm. In particular, Temkin's metric on* $\omega_{Z/k}^{\otimes m}$ *is determined on* $Z^{\mathrm{div}} \cup \{z_0\}$.

The proof of Theorem 3.3.3.3 proceeds by reduction to Theorem 3.3.3.2 using Gauss extensions.

Lemma 3.3.3.4. *For any $z \in Z$ and any affinoid neighbourhood $V \subseteq Z$ of z, either V is strictly k-affinoid or there exists a smaller affinoid neighbourhood $W \subseteq V$ of z and $r \notin \sqrt{|k^*|}$ such that $W_{k_r} := W \times_k k_r$ is strictly k_r-affinoid.*

It is a standard trick in the theory of analytic spaces to pick $r_1, \ldots, r_n \notin \sqrt{|k^*|}$ such that $V_{k_{\vec{r}}} := V \times_k k_{\vec{r}}$ is strictly $k_{\vec{r}}$-affinoid, where $k_{\vec{r}} := k_{r_1} \hat{\otimes}_k \ldots \hat{\otimes}_k k_{r_n}$. The point of Lemma 3.3.3.4, however, is that this can be done with $n = 1$, after possibly passing to a smaller affinoid neighbourhood.

Proof of Lemma 3.3.3.4. Suppose V is not strictly k-affinoid; in particular, $\sqrt{|k^*|}$ is not all of $\mathbf{R}_+^*$. Let A_V be the k-affinoid algebra corresponding to V. An admissible epimorphism from a generalized Tate algebra to A_V induces a closed immersion of V into a polydisc, with z landing in the interior of the image. As Laurent domains form a basis of closed neighbourhoods of x, there is an affinoid neighbourhood $\mathcal{M}(B) \subseteq V$ of z, where B is of the form

$$B = \frac{k\{s_i^{-1}S_i, t_j^{-1}T_j\}}{(S - f_i, g_jT_j - 1)},$$

for some $s_i, t_j \in \mathbf{R}_+^*$, $f_i, g_j \in k\{s^{-1}S, t^{-1}T\}$, and $i = 1, \ldots, \ell$, $j = 1, \ldots, m$. By assumption, $|f_i(z)| < s_i$ and $|g_j(z)| > t_j$ for all i, j. For any $r \in \mathbf{R}_+^*$, the subgroup $r^{\mathbf{Q}}$ is dense in $\mathbf{R}_+^*$, so there are exponents $p_i, q_j \in \mathbf{Q}$ such that $|f_i(z)| \leq r^{p_i} < s_i$ and $|g_j(z)| \geq r^{q_j} > t_j$ for all i, j. Pick r so that $r \notin \sqrt{|k^*|}$. Set

$$B' := \frac{k\{(r^{p_i})^{-1}S_i, (r^{q_j})'^{-1}T_j\}}{(S_i - f_i, g_jT_j - 1)},$$

then $W := \mathcal{M}(B') \subseteq \mathcal{M}(B) \subseteq V$ is an affinoid neighbourhood of z and, by construction, W_{k_r} is strictly k_r-affinoid. □

Proof of Theorem 3.3.3.3. Let V be an affinoid neighbourhood of x, and assume that V is not strictly k-affinoid. By Lemma 3.3.3.4, we may assume that there exists $r \in \sqrt{|k^*|}$ such that V_{k_r} is strictly k_r-affinoid. Set

$$M = \max_{z \in V} \|s\|_{\mathrm{Tem},z} \quad \text{and} \quad \widetilde{M} = \max_{z' \in V_{k_r}} \|p_r^* s\|_{\mathrm{Tem},z'},$$

where $p_r \colon V_{k_r} \to V$ denotes the ground field extension map. Write $\sigma_r \colon V \to V_{k_r}$ for the Gauss extension.

We claim that $M = \widetilde{M}$. Since p_r is surjective, [Tem16, Lemma 6.3.2] implies that $M \geq \widetilde{M}$. Further, Lemma 3.3.2.1 asserts that $\|s\|_{\mathrm{Tem},z} = \|p_r^* s\|_{\mathrm{Tem},\sigma_r(z)} \leq \widetilde{M}$ for all $z \in Z$. It follows that $M = \widetilde{M}$.

By [Tem16, Corollary 8.2.10] (and the assumption that V_{k_r} is strictly k_r-affinoid!), there exists $y \in V_{k_r} \cap Z_{k_r}^{\mathrm{div}}$ such that $\|p_r^* s\|_{\omega,y} = \widetilde{M}$. Set $z = p_r(y)$. Applying [Tem16, Lemma 6.3.2], we have

$$\widetilde{M} = \|p_r^* s\|_{\mathrm{Tem},y} \leq \|s\|_{\mathrm{Tem},z} \leq M,$$

and hence all of the above inequalities are equalities; in particular, z also achieves the maximum M. Thus, it suffices to show that z is either a divisorial point or the trivial norm.

It suffices to show that z is an Abhyankar point with $t(\mathcal{H}(z)/k) \leq 1$. By [Ber90, Corollary 9.3.2], $d(\mathcal{H}(y)/k_r) \leq d(\mathcal{H}(\sigma_r(z))/k_r)$; in particular, $\sigma_r(z)$ is Abhyankar since y is so. By Proposition 2.7.0.2(3), this occurs if and only if z is Abhyankar (while Proposition 2.7.0.2 is written only for analytifications of varieties, the same result holds more generally when Abhyankar points are defined using the local dimension as in [Duc07]). It remains to show that $t(\mathcal{H}(z)/k) \leq 1$. As z is obtained via restriction from y, it follows that

$$t(\mathcal{H}(z)/k) = \dim_{\mathbf{Q}}(\sqrt{|\mathcal{H}(z)^*|}) \leq \dim_{\mathbf{Q}}(\sqrt{|\mathcal{H}(y)^*|/}) = 1,$$

which completes the proof. □

3.4 Comparison theorems with Temkin's metric

The goal of this section is to first review Temkin's comparison theorem [Tem16, Theorem 8.3.3] with the weight metric in the discretely-valued setting, and then to prove Theorem A by passing to a discretely-valued extension and applying Temkin's comparison result.

3.4.1 Temkin's comparison theorem with the weight metric

One of the main results of [Tem16] is a comparison theorem between Temkin's metric and the weight metric over a discretely-valued field of residue characteristic zero. Let k be such a field, and let ϖ be an uniformizer of k.

To state Temkin's comparison theorem, we write the weight metric multiplicatively as in Remark 3.1.1.2. For a normal k-variety X such that $\omega_{X/k}^{\otimes m}$ is invertible for $m \in \mathbf{Z}_{>0}$, recall that the weight metric $\|\cdot\|_{\mathrm{wt}_{\mathrm{disc}}}$ on the canonical bundle $(\omega_{X/k}^{\otimes m})^{\mathrm{an}} \simeq \omega_{X^{\mathrm{an}}/k}^{\otimes m}$ is defined as follows: for any $x \in X^{\mathrm{an}}$ and local section $s \in \omega_{X/k,\ker(x)}^{\otimes m}$, set

$$\|s\|_{\mathrm{wt}_{\mathrm{disc},x}} := |\varpi|^{\mathrm{wt}_s(x)}.$$

This formula determines the seminorm $\|\cdot\|_{\mathrm{wt}_{\mathrm{disc},x}}$ on all of the stalks $\omega_{X^{\mathrm{an}}/k,x}^{\otimes m}$ as in §3.1. For a divisorial point $x \in X^{\mathrm{div}}$ corresponding to a k°-model $\mathscr{X}$ of X and an irreducible component $E \subseteq \mathscr{X}_0$, the weight metric admits a simple description: pick a $\mathcal{O}_{\mathscr{X},E}$-module generator δ of the stalk $\omega_{\mathscr{X}/k^\circ,E}^{\otimes m}$ and write $s = f\delta$ for some $f \in k(X)$, then

$$\|s\|_{\mathrm{wt}_{\mathrm{disc},x}} = |f(x)| \cdot |g_E(x)|^m,$$

where g_E is a local equation of E at its generic point on $\mathscr{X}$. This expression is independent of the choice of δ since any two generators differ by a multiplicative factor $u \in \mathcal{O}_{\mathscr{X},E}^{\times}$ and $|u(x)| = 1$.

Theorem 3.4.1.1. [Tem16, Theorem 8.3.3] *For a smooth k-variety X and $m \in \mathbf{Z}_{>0}$, we have*

$$\| \cdot \|_{\mathrm{wt}_{\mathrm{disc}}} = |\varpi|^m \| \cdot \|_{\mathrm{Tem}}$$

as metrics on $(\omega_{X/k}^{\otimes m})^{\mathrm{an}} \simeq \omega_{X^{\mathrm{an}}/k}^{\otimes m}$.

Proof. The proof of Theorem 3.4.1.1, as outlined in [Tem16, Remark 8.3.4(i)], very much requires the description of the weight function as the maximal lower-semicontinuous extension of its values on divisorial points as in Theorem 3.2.1.1; combining this with Theorem 3.3.3.2, it suffices to check equality on divisorial points. This is done by appealing to results from log geometry and almost mathematics. Further, the proof of Theorem 3.4.1.1 uses that X is smooth in order to reduce to the case $m = 1$. It is not clear whether the assumptions in Theorem 3.4.1.1 can be weakened to assume only that X is $\mathbf{Q}$-Gorenstein. □

When proving Theorem 3.4.1.1, Temkin uses a description of the weight metric that does not involve references to a pair $(\mathscr{X}, E)$. For the sake of completeness, we review Temkin's construction (as in [Tem16, §8.3.1]) and prove that it coincides with our definition of the weight metric.

Temkin's construction requires the following commutative algebra lemma, which does not seem to appear explicitly in the literature. Using the machinery of cotangent complexes, it can be deduced from [GR03, Theorem 6.5.12]. We thank Rankeya Datta for his help in formulating a more elementary proof.

Lemma 3.4.1.2. *If $\ell \subseteq K$ is a finite separable extension of discretely-valued fields,* *then* K° *is a finite-type* ℓ°*-algebra of the form*

$$\frac{\ell^\circ[s_1, \ldots, s_m]}{(f_1, \ldots, f_m)}$$

such that $\Delta = \det\left(\frac{\partial f_i}{\partial s_j}\right)$ *is nonzero in* K°.

Note that in Lemma 3.4.1.2 neither K nor ℓ are required to be complete, and we have made no assumptions on the characteristic of the fields.

Remark 3.4.1.3. The proof of Lemma 3.4.1.2 uses a selection of commutative algebra definitions and results, which we recall below.

1. Let k be a field and A be a k-algebra. We say that A is a *local complete intersection* (lci) if there exists a cover $\{D(g_i)\}_{i \in I}$ of $\operatorname{Spec}(A)$ by distinguished opens such that for each $i \in I$, there is a presentation

$$A[1/g_i] = \frac{k[x_1, \ldots, x_n]}{(f_1, \ldots, f_c)}$$

   with $\dim(A[1/g_i]) = n - c$. If no localizations are necessary, then we say that A is a *(global) complete intersection*. See [Sta19, Tag 00S9] for more details.

2. A ring map $A \to B$ is *syntomic* if it is flat, of finite presentation, and for every $\mathfrak{p} \in \operatorname{Spec}(A)$, $\kappa(\mathfrak{p}) \to B \otimes_A \kappa(\mathfrak{p})$ is a local complete intersection. See [Sta19, Tag 00SL] for more details.

3. By [Sta19, Tag 00SY], for any ring map $A \to B$ is a ring map, the following two conditions are equivalent:

   (a) there exists $g \in B$ such that $A \to B[1/g]$ is syntomic;

   (b) there exists $g \in B$ such that $A \to B[1/g]$ is a global complete intersection.

Proof of Lemma 3.4.1.2. Let A be the integral closure of ℓ° in K. By [Mat89, Corollary on p.85], then A is a Dedekind domain, and $\ell^\circ \hookrightarrow A$ is finite by [Sta19, Tag

032L]. Moreover, $A \subseteq K^\circ$ because K° is a valuation ring containing ℓ°. It follows that $\mathfrak{n} := K^{\circ\circ} \cap A$ is a nonzero prime ideal of A (indeed, it contains the nonzero maximal ideal of ℓ°), hence $\mathfrak{n}$ is maximal in A. Thus, $A_\mathfrak{n} \subseteq K^\circ$. Since discrete valuation rings are maximal with respect to dominance, we must have $K^\circ = A_\mathfrak{n}$.

Furthermore, the ring A is semilocal. Indeed, any maximal ideal of A must contract to $\ell^{\circ\circ}$ under the local homomorphism $\ell^\circ \hookrightarrow A$, hence the contractions contains a uniformizer ϖ_ℓ of ℓ°. However, the extension $\varpi_\ell A$ has a finite presentation

$$\varpi_\ell A = \mathfrak{m}_1^{\alpha_1} \cdots \mathfrak{m}_r^{\alpha_r},$$

and the maximal ideals of A that appear in the presentation are precisely those that contain ϖ_ℓ. Thus, A is semilocal.

Now, assume without loss of generality that $\mathfrak{n} = \mathfrak{m}_1$. For each $i = 2, \ldots, r$, take $g_i \in \mathfrak{m}_i \backslash \mathfrak{n}$, and set $g := g_2 \cdots g_r$. Then, $A[1/g]$ is local with maximal ideal $\mathfrak{n}A[1/g]$, so

$$A[1/g] = (A[1/g])_{\mathfrak{n}A[1/g]} = A_\mathfrak{n}[1/g] = K^\circ.$$

In particular, $\ell^\circ \hookrightarrow K^\circ$ is of finite type, since it is the composition of the finite map $\ell^\circ \hookrightarrow A$ and the finite type map $A \hookrightarrow A[1/g] = K^\circ$.

Applying Remark 3.4.1.3(3) with $g = 1$ (and using the fact that both ℓ° and K° are of dimension 1), it suffices to show that $\ell^\circ \hookrightarrow K^\circ$ is syntomic. Observe that K° is flat over ℓ°, since flatness over a dvr is equivalent to torsion-freeness. Moreover, we have seen that $\ell^\circ \hookrightarrow K^\circ$ is of finite type, and hence of finite presentation because ℓ° is noetherian. Therefore, to construct the desired presentation, it remains to show that each fibre of $\ell^\circ \hookrightarrow K^\circ$ is lci. In fact, we will show that each fibre is a global complete intersection.

The fibre above the generic point of ℓ° is $\ell \hookrightarrow K$, which is finite and separable by assumption. By the primitive element theorem, we can write $K = \ell[T]/(f)$ for some irreducible polynomial $f \in \ell[T]$; in particular, K is a global complete intersection over ℓ. Furthermore, the derivative $f'(T)$ is nonzero because the extension $\ell \hookrightarrow K$ is nontrivial, so $L \hookrightarrow K$ is smooth.

The fibre above the closed point of ℓ° is $\widetilde{\ell} \hookrightarrow K^\circ/(\varpi_K^e)$, where ϖ_K is a uniformizer of K° and $e \in \mathbf{Z}_{\geq 1}$ is the ramification index of the extension K/ℓ. The quotient $K^\circ/(\varpi_K^e)$ is a noetherian local ring of dimension zero, hence it is complete. Since it contains the field $\widetilde{\ell}$, it has a coefficient field, i.e. there is an injective homomorphism $\widetilde{K} \hookrightarrow K^\circ/(\varpi_K^e)$.

Consider the surjective $\widetilde{K}$-algebra homomorphism $\widetilde{K}[[T]] \to K^\circ/(\varpi_K^e)$ given by $T \mapsto \varpi_K$ (this is well-defined precisely because $K^\circ/(\varpi_K^e)$ is complete). The power series ring $\widetilde{K}[[T]]$ is a dvr, so every ideal must be of the form (T^β) for some $\beta \in \mathbf{Z}_{\geq 0}$; in particular, the kernel must be (T^e). Therefore, $\widetilde{\ell} \hookrightarrow K^\circ/(\varpi_K^e)$ decomposes as

$$\widetilde{\ell} \hookrightarrow \widetilde{K} \hookrightarrow \widetilde{K}[T]/(T^e) \simeq \widetilde{K}[[T]]/(T^e) \simeq K^\circ/(\varpi_K^e).$$

The finite field extension $\widetilde{\ell} \hookrightarrow \widetilde{K}$ decomposes into a sequence

$$\widetilde{\ell} = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_d = \widetilde{K}$$

of nontrivial finite simple extensions, i.e. for all $i = 1, \ldots, d$, we have $F_i = F_{i-1}[S_i]/(h_i)$ and $h_i \in F_{i-1}[S_i]$ is irreducible. By induction, we can write

$$\widetilde{K} = \frac{\widetilde{\ell}[S_1, \ldots, S_d]}{(h_1, \ldots, h_d)}.$$

It follows that

$$K^\circ/(\varpi_K^e) \simeq \frac{\widetilde{\ell}[S_1, \ldots, S_d, T]}{(h_1, \ldots, h_d, T^e)}.$$

Note that the special fibre need not be smooth (indeed, it is not even reduced if $e > 1$). This completes the proof that there is a presentation of the desired form.

Finally, it remains to show that the Jacobian determinant Δ (as in the statement) is nonzero in K°, but this is clear: we have seen that the generic fibre $\ell \hookrightarrow K$ is smooth, so Δ is invertible in K by the Jacobian criterion, hence nonzero. □

Remark 3.4.1.4. There are purely inseparable, finite extensions $\ell \subseteq K$ of discretely-valued fields such that $\ell^\circ \hookrightarrow K^\circ$ is not finite. For example, there are non-excellent non-F-finite dvrs, from which one can construct an extension $\ell^\circ \hookrightarrow K^\circ$ of dvrs that is not finite; see [DS18, §4.1].

Construction 3.4.1.5. As before, k is discretely-valued field of residue characteristic zero, X is a normal k-variety, $n = \dim(X)$, and $x \in X^{\mathrm{div}}$. Let $K(x)$ denote the function field $k(X)$ equipped with the norm x; $K(x)$ is a discretely-valued field and the separated completion of $K(x)$ is the completed residue field $\mathcal{H}(x)$ of X^{an} at x. Let ϖ_x be a uniformizer of $K(x)$. We will construct a metric $\|\cdot\|_{\alpha,x}$ on the fibre $\omega_{X^{\mathrm{an}}/k,x}(x)$ by specifying its value on a nonzero element of the $K(x)$-suspace $\omega_{K(x)/k}$.

Let $u_1, \ldots, u_n \in K(x)^\circ$ be such that $u_1, \ldots, u_n$ form a transcendence basis of $K(x)/k$ and $\widetilde{u}_1, \ldots, \widetilde{u}_n$ form a transcendence basis of $\widetilde{K(x)}/\widetilde{k}$; such a collection exists by the assumption that x is divisorial and [Tem16, 8.3.1]. Set $\ell(x) = k(u_1, \ldots, u_n)$, so $\ell(x) \subseteq K(x)$ is a finite separable extension, since $K(x)$ is a finitely-generated extension of k. By Lemma 3.4.1.2, there is a presentation of the form

$$K(x)^\circ = \frac{\ell(x)^\circ[s_1, \ldots, s_m]}{(f_1, \ldots, f_m)}$$

such that $\Delta_x \coloneqq \det\left(\frac{\partial f_i}{\partial s_j}\right)$ is nonzero in $K(x)^\circ$. By [Liu02, Corollary 4.14], the relative canonical module $\omega_{K(x)^\circ/\ell(x)^\circ}$ is generated by Δ_x^{-1} as a $K(x)^\circ$-submodule of $K(x)$. Hence, the relative canonical module $\omega_{K(x)^\circ/k^\circ}$ is generated by

$$\eta_x \coloneqq \Delta_x^{-1} du_1 \wedge \ldots \wedge du_n \tag{3.14}$$

as a $K(x)^\circ$-submodule of $\Omega^n_{K(x)/k}$. The metric $\|\cdot\|_{\alpha,x}$ on $\omega_{X^{\mathrm{an}}/k,x}(x)$ is uniquely determined by declaring

$$\|\eta_x\|_{\alpha,x} := |\varpi_x(x)|.$$

That is, there is a unique way to write any element of $\omega_{X^{\mathrm{an}}/k,x}(x)$ as $f\eta_x$ for some $f \in \mathcal{H}(x)$, and we set

$$\|f\eta_x\|_{\alpha,x} = |f(x)| \cdot \|\eta_x\|_{\alpha,x} = |f\varpi_x\Delta_x(x)|.$$

It is easy to check that the function $\|\cdot\|_{\alpha,x}$ is independent of the choices involved.

The following shows that the metric $\|\cdot\|_{\alpha,x}$ from Construction 3.4.1.5 coincides with the weight metric $\|\cdot\|_{\mathrm{wt}_{\mathrm{disc}},x}$ as defined at the start of the section.

Proposition 3.4.1.6. *For $x \in X^{\mathrm{div}}$ and $s \in \omega_{X^{\mathrm{an}}/k,x}$, we have $\|s\|_{\alpha,x} = \|s\|_{\mathrm{wt}_{\mathrm{disc}},x}$.*

Proof. It suffices to show $\|\eta_x\|_{\alpha,x} = \|\eta_x\|_{\mathrm{wt}_{\mathrm{disc}},x}$, where $\eta_x \in \omega_{K(x)^\circ/k^\circ}$ is as in (3.14). Suppose x is determined by the pair $(\mathscr{X}, E)$, in which case there are isomorphisms $K(x)^\circ \simeq \mathcal{O}_{\mathscr{X},E}$ and $\omega_{K(x)^\circ/k^\circ} \simeq \omega_{\mathscr{X}/k^\circ,E}$. The form η_x gives a $K(x)^\circ$-module isomorphism

$$\sigma_x \colon \omega_{K(x)^\circ/k^\circ} \simeq \mathcal{O}_{\mathscr{X},E}$$

given by $f\eta_x \mapsto f$; in particular, $\sigma_x(\eta_x) = 1$. Thus,

$$\mathrm{ord}_E(\mathrm{div}_{\mathscr{X}}(\eta_x) + \mathscr{X}_{0,\mathrm{red}}) = \mathrm{ord}_E(\sigma_x(\eta_x)) + 1 = 1,$$

and hence

$$\|\eta_x\|_{\mathrm{wt}_{\mathrm{disc}},x} = |\varpi|^{b_E^{-1}\,\mathrm{ord}_E(\mathrm{div}_{\mathscr{X}}(\eta_x)+\mathscr{X}_{0,\mathrm{red}})} = |\varpi|^{b_E^{-1}} = |\varpi_x| = \|\eta_x\|_{\alpha,x},$$

where $b_E = \mathrm{ord}_E(\mathscr{X}_0)$. □

3.4.2 Divisorial points under Gauss extensions

Let k be a trivially-valued field of characteristic zero. Let X be a normal k-variety, and let $x \in X^{\mathrm{div}} \cap X^{\beth}$ be the divisorial point determined by the triple $(c, Y \xrightarrow{h} X, E)$. Assume that X is quasi-projective, and Y and E are smooth. For $r \in (0,1)$, recall that $k_r = k((\varpi))$ is the Gauss extension of k with $|\varpi|_r = r$; write $p_r \colon X^{\mathrm{an}}_{k_r} \to X^{\mathrm{an}}$ for the ground field extension map and $\sigma_r \colon X^{\mathrm{an}} \to X^{\mathrm{an}}_{k_r}$ for the Gauss extension map. For any such r, the point $\sigma_r(x) \in X^{\mathrm{an}}_{k_r}$ is divisorial by Proposition 2.7.0.2.

The goal of this section is to pick an $r \in (0,1)$ such that we can construct an explicit divisorial representation of $\sigma_r(x) \in X^{\mathrm{div}}_{k_r}$. This is done in three steps.

1. Construct an explicit $k[[\varpi]]$-model of $X_{k((\varpi))}$, together with an irreducible component F of its special fibre, with the property that for any element $a \in k(X)$, we have
$$\mathrm{ord}_E(a) = \mathrm{ord}_F(a).$$
2. Endow $k((\varpi))$ with the ϖ-adic norm $|\varpi|_r = r$ for a suitable choice of $r \in (0,1)$ so that the divisorial valuation $y_F \in X^{\mathrm{an}}_{k_r}$ determined by F satisfies $p_r(y_F) = x$.
3. Show that $\sigma_r(x) = y_F$.

The construction we will present is inspired by a similar phenomenon involving test configurations, as in [BHJ17, BJ18b]; this relationship is described further in Remark 3.4.2.2.

Step 1. We may assume that h is projective by [KM08, Lemma 2.45], and so [Har77, II, Theorem 7.17] implies that there exists a coherent ideal $I \subseteq \mathcal{O}_X$ such that $Y = \mathrm{Bl}_I X$ and h is identified with the blow-up morphism. Let $\alpha \in \mathbf{Z}_{\geqslant 0}$ be the multiplicity of E in the exceptional locus of h.

Consider the fibre product $\mathcal{X} \coloneqq X \times_k k[[\varpi]]$: this is the trivial $k[[\varpi]]$-model of $X_{k((\varpi))}$, and its special fibre $\mathcal{X}_0$ is naturally identified with X. We set $\mathcal{I} \coloneqq (I, \varpi) \subseteq \mathcal{O}_{\mathcal{X}}$, which is a coherent ideal sheaf on $\mathcal{X}$ that is cosupported on $\mathcal{X}_0$. Let $\nu \colon \mathcal{Y} \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ along $\mathcal{I}$. As the vanishing locus of $\mathcal{I}$ lies in $\mathcal{X}_0$, it follows that $\mathcal{Y}$ is again a model of $X_{k((\varpi))}$. The strict transform of $\mathcal{X}_0$ via ν can be identified with Y by [Har77, II, Corollary 7.15]. Under this identification, $\mathcal{Y}_0$ contains a copy of the divisor E, which we write as $\widetilde{E}$. Further, let $\rho \colon \mathcal{Y} \dashrightarrow Y \times_k k[[\varpi]]$ be the birational map given by the composition of $\nu \colon \mathcal{Y} \to \mathcal{X}$ with the inverse of $Y \times_k k[[\varpi]] \to \mathcal{X}$. These objects are collected in Section 3.4.2.

Write η (resp. $\widetilde{\eta}$) for the generic point of E (resp. $\widetilde{E}$) in Y (resp. $\mathcal{Y}$). We claim that the composition of ρ with the projection $Y \times_k k[[\varpi]] \to Y$ onto the special fibre sends $\widetilde{\eta}$ to η. Indeed, observe that the diagram

$$\begin{array}{ccc} Y \times_k k[[\varpi]] & \overset{\rho}{\dashleftarrow} & \mathcal{Y} \\ \downarrow & & \uparrow \\ Y \supset E & \longrightarrow & \widetilde{E} \subset \mathcal{Y}_0 \end{array}$$

is commutative, and that the bottom arrow restricts to an isomorphism from $\widetilde{E}$ to E. Hence, it suffices to show that $\widetilde{E}$ is not contained in the indeterminacy locus of ρ, and we show this with the following local computation. Suppose $X = \operatorname{Spec}(A)$ and $I = (f_1, \ldots, f_\ell)$, in which case an affine chart of Y is given by $U = \operatorname{Spec}(B)$, where

$$B = \frac{A[S_2, \ldots, S_\ell]}{(f_1 S_i - f_i \colon i = 2, \ldots, \ell)}.$$

There is a corresponding affine chart of $\mathcal{Y}$ given by $\mathcal{U} = \operatorname{Spec}(\mathcal{B})$, where

$$\mathcal{B} = \frac{\mathcal{A}[S_2, \ldots, S_\ell, \widetilde{S}]}{(f_1 \widetilde{S} - \varpi, f_1 S_i - f_i \colon i = 2, \ldots, \ell)}$$

and $\mathcal{A} = A \otimes_k k[[\varpi]]$. The birational map $\rho : \mathcal{Y} \dashrightarrow Y \times_k k[[\varpi]]$ is given on these charts by the composition of the two top arrows in the diagram below:

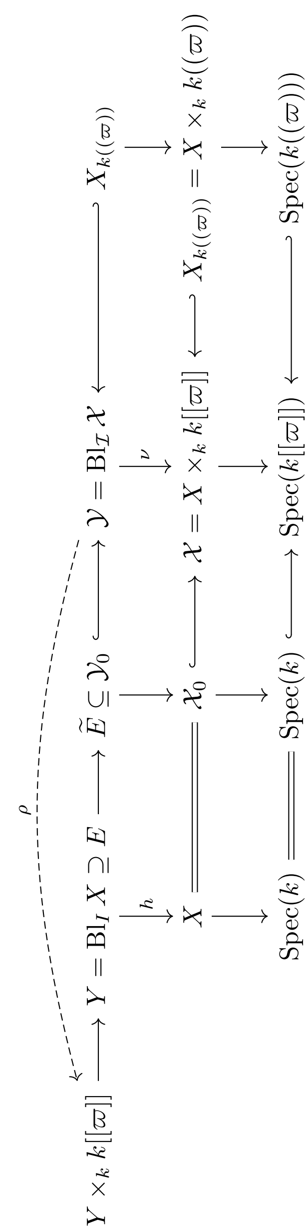

Figure 3.1: A commutative diagram describing the model associated to the Gauss extension.

$$\begin{array}{ccc} B \otimes_k k[[\varpi]] \longrightarrow (B \otimes_k k[[\varpi]])[\widetilde{S}] & \longrightarrow & \mathcal{B} = \frac{(B\otimes_k k[[\varpi]])[\widetilde{S}]}{(f_1\widetilde{S}-\varpi)} \\ \uparrow & & \downarrow \\ B & \twoheadleftarrow & \mathcal{B}/(\varpi) \end{array}$$

Thus, the construction of the map $\mathcal{Y} \dashrightarrow Y \times_k k[[\varpi]] \to Y$ yields a ring morphism $\mathcal{O}_{Y,\eta} \to \mathcal{O}_{\mathcal{Y},\widetilde{\eta}}$, which sends $\widetilde{\eta}$ to η, as required.

The irreducible subscheme $\widetilde{E}$ of $\mathcal{Y}$ is not a divisor (indeed, it has codimension 2 in $\mathcal{Y}$), so consider the blow-up $\mu\colon \mathcal{Z} \to \mathcal{Y}$ of $\widetilde{E}$. Note that $\mathcal{Z}$ is again a model of $X_{k((\varpi))}$. Write $F \subseteq \mathcal{Z}_0$ for the exceptional divisor of μ, which is irreducible since $\widetilde{E}$ is so.

We claim that $\operatorname{ord}_F(a) = \operatorname{ord}_E(a)$ for all $a \in k(X)$. It suffices to show the equality for $a \in \mathcal{O}_{Y,\eta}$. With notation as above, the exceptional divisors of h in the affine chart $U = \operatorname{Spec}(B)$ of Y is defined by f_1. Let g be a local equation of E at η. In the model $\mathcal{Y}$, $\widetilde{E}$ is locally cut out by g and the equations defining the strict transform of $\mathcal{X}_0$. Therefore, in $\mathcal{Z}$, g is a local equation of F at its generic point.

Write $a = ug^\lambda$ for $u \in \mathcal{O}_{Y,\eta}^\times$ and $\lambda \in \mathbf{Z}_{\geqslant 0}$ (so that $\operatorname{ord}_E(a) = \lambda$). The image of this expression for a via the map

$$\mathcal{O}_{Y,\eta} \to \mathcal{O}_{\mathcal{Y},\widetilde{\eta}} \to \mathcal{O}_{\mathcal{Z},F}$$

gives an expression for a in $\mathcal{O}_{\mathcal{Z},F}$. As u remains a unit in $\mathcal{O}_{\mathcal{Z},F}$, we deduce that

$$\operatorname{ord}_F(a) = \lambda \operatorname{ord}_F(g) = \lambda = \operatorname{ord}_E(a),$$

as required.

Step 2. We will find $r \in (0,1)$ such that the divisorial valuation $y_F \in X_{k_r}^{\mathrm{an}}$ determined by $(\mathcal{Z}, F)$ satisfies $p_r(y_F) = x$. To that end, we first compute the multiplicity of F in the special fibre of $\mathcal{Z}$. Working in an affine chart of the blowup as before, the composition $\mathcal{Z} \to \mathcal{Y} \to \mathcal{X} \to \operatorname{Spec}(k[[\varpi]])$ can locally be written as

$$k[[\varpi]] \longrightarrow \mathcal{A} = A \otimes_k k[[\varpi]] \longrightarrow \mathcal{B} = \frac{\mathcal{A}[S_2,\ldots,S_\ell,\widetilde{S}]}{(f_1\widetilde{S}-\varpi, f_1S_i - f_i\colon i=2,\ldots,\ell)}$$
$$\big\downarrow$$
$$\frac{\mathcal{A}[S_2,\ldots,S_\ell,\widetilde{S},Q]}{(\widetilde{S}-gQ, f_1\widetilde{S}-\varpi, f_1S_i-f_i\colon i=2,\ldots,\ell)}.$$

In particular, we can write $\varpi = f_1 g Q$ at the generic point of F. As Q is a unit in $\mathcal{O}_{\mathcal{Z},F}$, we conclude that

$$\mathrm{ord}_F(\varpi) = \mathrm{ord}_F(f_1) + \mathrm{ord}_F(g) = \mathrm{ord}_E(f_1) + \mathrm{ord}_F(g) = \alpha + 1.$$

Set $r = e^{-c(\alpha+1)}$, where recall that x is determined by the triple $(c, Y \to X, E)$. For any $a \in k(X)$, we have that

$$|a(x)| = e^{-c\,\mathrm{ord}_E(a)} = r^{\frac{\mathrm{ord}_E(a)}{\alpha+1}} = r^{\frac{\mathrm{ord}_F(a)}{\mathrm{ord}_F(\varpi)}} = r^{v_{y_F}(a)} = |a(y_F)|.$$

That is, $p_r(y_F) = x$.

Step 3. It remains to show that $y_F = \sigma_r(x)$. This is done by appealing to the following construction from [BJ18b, §1.6]. The group k^* acts on $k[[\varpi]]$ as follows: for $c \in k^*$ and $f = \sum_{j\in\mathbf{Z}} a_j \varpi^j$ with $a_j \in k$ and $a_j = 0$ for $j \ll 0$, set

$$c \cdot f := \sum_{j\in\mathbf{Z}} c^{-j} a_j \varpi^j.$$

For a k-scheme X, there is an induced k^*-action on the product $X \times_k k[[\varpi]]$, and we say that an ideal sheaf on $X \times_k k[[\varpi]]$ is $k^\times$*-invariant* if the corresponding closed subscheme of $X \times_k k[[\varpi]]$ is fixed pointwise by the k^*-action. For example, if $\mathcal{J} \subseteq \mathcal{O}_X$ is an ideal sheaf, then $(\mathcal{J}, \varpi) \subseteq \mathcal{O}_{X\times k[[\varpi]]}$ is k^*-invariant.

Lemma 3.4.2.1. *Let $\mathcal{X}^{(\ell)} \to \mathcal{X}^{(\ell-1)} \to \ldots \to \mathcal{X}^{(1)} \to \mathcal{X}$ be a sequence of models of $X_{k((\varpi))}$, where each morphism $\mathcal{X}^{(i+1)} \to \mathcal{X}^{(i)}$ is the blow-up of a k^*-invariant ideal, and $\mathcal{X} = X \times_k k[[\varpi]]$ is the trivial model of $X_{k((\varpi))}$. If k is an infinite field and $r \in (0,1)$, then any divisorial point of $X^{\mathrm{an}}_{k_r}$ determined by an irreducible component of $\mathcal{X}^{(\ell)}_0$ lies in the image of σ_r.*

In [BJ18b, §1.7], Lemma 3.4.2.1 is used to interpret the k^*-invariant divisorial points of $X^{\mathrm{an}}_{k((\varpi))}$ in terms of test configurations.

Proof. This follows from [BJ18b, Proposition 1.6]; see also [BHJ17, Lemma 4.5]. □

The point $y_F \in X^{\mathrm{an}}_{k_r}$ satisfies the hypotheses of Lemma 3.4.2.1 by construction, so it lies in the image of σ_r. As σ_r is a section of the projection p_r and $p_r(y_F) = x$ by Step 2, we conclude that $\sigma_r(x) = y_F$.

Remark 3.4.2.2. The construction of the point y_F is inspired by one in the proof of [BHJ17, Proposition 4.11]. There, for any $r \in (0, 1)$, the authors view the Gauss extension as a continuous map $\sigma_r \colon X^{\beth} \to (X \times_k \mathbf{A}^1_k)^{\beth}$, and one can show the following: if $x \in X^{\beth}$ is the divisorial point given by the triple $(-\log(r), Y \to X, E)$, then $\sigma_r(x)$ is a monomial valuation on the birational model $Y \times_k \mathbf{A}^1_k \to X \times_k \mathbf{A}^1_k$ in the snc divisor $E \times_k \mathbf{A}^1_k + Y \times_k \{0\}$.

The construction of $y_F = \sigma_r(x)$ can be rephrased in the above language. We first consider the blow-up ν of $X \times_k \mathbf{A}^1_k$ at $\overline{\{c_X(x)\}} \times \{0\}$, and then the blow-up μ of the intersection of $\mathrm{Exc}(\nu)$ and the strict transform of $X \times_k \{0\}$ via ν. The valuation $\sigma_r(x)$ is realized as an order of vanishing along $\mathrm{Exc}(\mu)$. The advantage of realizing $\sigma_r(x)$ in this manner is that the blow-ups occur only above the origin of $\mathbf{A}^1_k$.

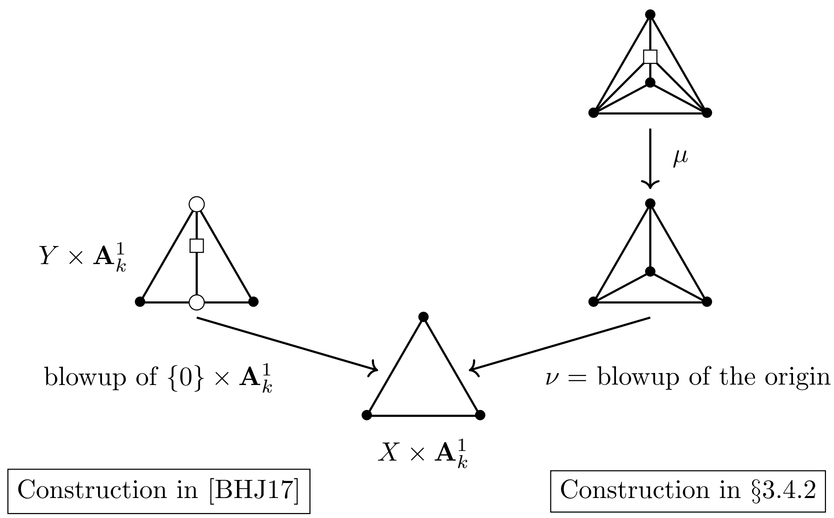

Figure 3.2: A comparison between the constructions of [BHJ17] and §3.4.2. We illustrate the two approaches to the construction of $\sigma_r(x)$ in Remark 3.4.2.2 with a toric example. Consider $X = \mathbf{A}^2_k$ and the blow-up $Y \to X$ at the origin with exceptional divisor $E \subseteq Y$. Let $x \in X^{\beth}$ be the divisorial point determined by the triple $(-\log(r), Y \to X, E)$. In the above figure, the triangles represent a slice of the fans of the various toric blow-ups that occur in the two constructions. Following [BHJ17], $\sigma_r(x)$ is a monomial valuation in the divisors corresponding to the white nodes, which we picture as a square on the segment joining them. On the other side, according to §3.4.2, we extract a divisor corresponding to $\sigma_r(x)$ with a sequence of two blow-ups, and we mark this divisor with a square.

3.4.3 Proof of Theorem A

The goal of this section is to prove Theorem A, which is the trivially-valued analogue of Theorem 3.4.1.1. This justifies the definition of the weight metric in the trivially-setting from (3.2.2). The proof of Theorem A proceeds by reduction to Theorem 3.4.1.1.

Throughout this section, let k be a trivially-valued field of characteristic zero, and let X be a normal, $\mathbf{Q}$-Gorenstein k-variety. Fix $m \in \mathbf{Z}_{>0}$ such that $\omega_{X/k}^{\otimes m}$ is a line bundle. For $x \in X^{\beth}$, recall that we write

$$\| \cdot \|_{\mathrm{wt}_{\mathrm{triv}},x} = e^{-\mathrm{wt}_{\mathrm{triv}}(\cdot,x)}$$

for the multiplicative form of the weight metric on the stalk $(\omega_{X/k}^{\otimes m})^{\beth}_{x}$.

Proposition 3.4.3.1. *Let $x \in X^{\beth}$ be the divisorial point determined by the triple $(c, Y \xrightarrow{h} X, E)$. With notation as in §3.4.2, for any rational section s of $\omega_{X/k}^{\otimes m}$, we have*

$$\|s\|_{\mathrm{wt}_{\mathrm{triv}},x} = r^{-m}\|q_r^* s\|_{\mathrm{wt}_{\mathrm{disc}},\sigma_r(x)}, \tag{3.15}$$

where $q_r \colon X_{k_r} \to X$ denotes the (algebraic) ground field extension.

Proof. Set $\xi = c_X(x)$. Let s be a $\mathcal{O}_{X,\xi}$-module generator of the stalk $\omega_{X/k,\xi}^{\otimes m}$. It suffices to show (3.15) for s; indeed, any local section at ξ can be written as fs for some $f \in \mathcal{O}_{X,\xi}$, in which case both sides of (3.15) are multiplied by $|f(x)|$. By working locally around ξ, we may assume that $X = \mathrm{Spec}(A)$ is affine and s is globally-defined. With the same notation as in Step 2 of Section 3.4.2, we have

$$\|s\|_{\mathrm{wt}_{\mathrm{triv}},x} = e^{-cm(1+(\ell-1)\alpha)}, \tag{3.16}$$

since $\mathrm{ord}_E(K_{Y/X}) = \mathrm{ord}_E(f_1^{\ell-1}) = (\ell-1)\alpha$. By [Liu02, Corollary 6.4.14], the stalk of the relative canonical sheaf $\omega_{\mathcal{Z}/\mathcal{X}}$ at the generic point of F can be viewed as the $\mathcal{O}_{\mathcal{Z},F}$-submodule of the function field of $X_{k((\varpi))}$; further, it is generated by $(gf_1^{\ell})^{-1}$. The m-th power of these generators multiplied by $q_r^* s$ thus gives a $\mathcal{O}_{\mathcal{Z},F}$-module generator of the stalk $\omega_{\mathcal{Z}/k[[\varpi]],F}^{\otimes m}$. It follows that

$$\begin{aligned}
\|q_r^* s\|_{\mathrm{wt}_{\mathrm{disc}},\sigma_r(x)} &= \|q_r^* s\|_{\mathrm{wt}_{\mathrm{disc}},y_F} \\
&= |(gf_1^{\ell})^m(y_F)| \cdot |g(y_F)|^m \\
&= |g(y_F)|^{m(2+l\alpha)} \\
&= e^{-cm(2+\ell\alpha)}.
\end{aligned}$$

Thus, combining the above with (3.16), it follows that

$$r^{-m}\|q_r^* s\|_{\mathrm{wt}_{\mathrm{disc}},\sigma_r(x)} = e^{cm(\alpha+1)}e^{-cm(2+\ell\alpha)} = e^{-cm(1+(\ell-1)\alpha)} = \|s\|_{\mathrm{wt}_{\mathrm{triv}},x},$$

as required. □

Now, Proposition 3.4.3.1 is the key tool to prove the trivially-valued analogue of Theorem 3.4.1.1, which is stated as Theorem A in the introduction.

Theorem 3.4.3.2. *If X is a smooth k-variety, then $\|\cdot\|_{\mathrm{wt}_{\mathrm{triv}}} = \|\cdot\|_{\mathrm{Tem}}$ as metrics on $(\omega_{X/k}^{\otimes m})^{\beth} \simeq \omega_{X^{\beth}/k}^{\otimes m}$.*

Proof. By Theorem 3.3.3.3 and using that $A_{(X,\emptyset)}$ is the maximal lower-semicontinuous extension of its values on divisorial points, it suffices to show the equality on the points in $X^{\mathrm{div}} \cap X^{\beth}$. Fix $x \in X^{\mathrm{div}} \cap X^{\beth}$ and let $r' \in (0,1)$ be chosen as in Step 2 of Section 3.4.2. It suffices to check equality on elements of the stalk $(\omega_{X/k}^{\otimes m})_{\ker(x)}$, i.e. on a rational section s of $\omega_{X/k}^{\otimes m}$. Now, applying Proposition 3.4.3.1, Theorem 3.4.1.1, and Lemma 3.3.2.1 we find that

$$\|s\|_{\mathrm{wt}_{\mathrm{triv}},x} = (r')^{-m}\|q_{r'}^{*}s\|_{\mathrm{wt}_{\mathrm{disc}},\sigma_{r'}(x)} = (r')^{-m}\left((r')^{m}\|q_{r'}^{*}s\|_{\mathrm{Tem},\sigma_{r'}(x)}\right) = \|s\|_{\mathrm{Tem},x},$$

which completes the proof. □

Remark 3.4.3.3. Let k, X, m be as above. To any Cartier divisor D on X, we can associate a canonical *singular metric* $\|\cdot\|_D$ on the line bundle $\mathcal{O}_X(D)^{\mathrm{an}}$ in the following manner: the divisor D induces an embedding ι_D of $\mathcal{O}_X(D)$ into the constant sheaf $k(X)$, and for any $x \in X^{\mathrm{an}}$ and $f \in \mathcal{O}_X(D)_{\ker(x)}$, set

$$\|f\|_{D,x} \coloneqq |\iota_D(f)(x)|.$$

Now, Temkin's metric $\|\cdot\|_{\mathrm{Tem}}$ and $\|\cdot\|_{D_{\mathrm{red}}}$ induce a tensor product metric on $(\omega_{(X,D_{\mathrm{red}})}^{\otimes m})^{\beth}$. By Theorem 3.4.3.2, this tensor product metric coincides with the weight metric $\mathrm{wt}_{\mathrm{triv}}$. It would be interesting if this metric $(\omega_{(X,D_{\mathrm{red}})}^{\otimes m})^{\beth}$ arises in a similar fashion as in [Tem16], or if similar results could be obtained in the singular setting.

CHAPTER IV

Essential skeletons of pairs

4.1 Skeletons over a discretely-valued field

In this section, we briefly review the construction of the skeleton associated to a log regular model from [BM17], and of the Kontsevich–Soibelman and essential skeletons from [MN15]. These are constructions whose trivially-valued analogues are discussed in §4.2, where more detail is given. Throughout, let k be a discretely-valued field, $\varpi \in k^{\circ\circ}$ a uniformizer, $S = \mathrm{Spec}(k^\circ)$, and let S^+ be the divisorial log structure on S defined by the closed point $\mathrm{Spec}(\widetilde{k})$.

Let $\mathscr{X}^+$ be a log-regular log scheme over S^+, $x \in F_{\mathscr{X}^+}$, $\overline{g_1}, \dots, \overline{g_m} \in \mathcal{C}_{\mathscr{X}^+,x}$ monoid generators, and $g_1, \dots, g_m \in \mathcal{M}_{\mathscr{X}^+,x}$ lifts of the $\overline{g_1}, \dots, \overline{g_m}$. Note that $g_1, \dots, g_m$ is a generating system for the maximal ideal $\mathfrak{m}_x \subseteq \mathcal{O}_{\mathscr{X},x}$. In particular, any $f \in \mathcal{O}_{\mathscr{X},x}$ admits a decomposition of the form

$$\sum_{\beta \in \mathbf{Z}_+^m} c_\beta g^\beta \tag{4.1}$$

in the $\mathfrak{m}_x$-adic completion $\widehat{\mathcal{O}}_{\mathscr{X},x}$ of $\mathcal{O}_{\mathscr{X},x}$, where $c_\beta \in \widehat{\mathcal{O}}_{\mathscr{X},x}^\times \cup \{0\}$. A decomposition of f as in (4.1) is called an *admissible expansion* of f.

Proposition 4.1.0.1. *For any monoid homomorphism* $\alpha \in \mathrm{Hom}(\mathcal{C}_{\mathscr{X}^+,x}, \overline{\mathbf{R}}_+)$ *such* *that* $\alpha(\varpi) = 1$, *there exists a unique minimal semivaluation*

$$v_\alpha \colon \mathcal{O}_{\mathscr{X},x} \backslash \{0\} \to \overline{\mathbf{R}}_+$$

such that

1. v_α *extends the discrete valuation* v_k *on* $k \hookrightarrow \mathcal{O}_{\mathscr{X},x}$*;*

2. *for any* $f \in \mathcal{M}_{X^+,x}$*, we have* $v_\alpha(f) = \alpha(\overline{f})$*;*

3. *for any* $f \in \mathcal{O}_{\mathscr{X},x}$ *and any admissible expansion* $f = \sum_{\beta \in \mathbf{Z}_+^m} c_\beta g^\beta$*, we have*

$$v_\alpha(f) = \min_\beta v_k(c_\beta) + \alpha(\overline{g}^\beta).$$

Proof. This is [BM17, Proposition 3.2.10]. In the special case when $\mathscr{X}_0$ is an snc divisor on $\mathscr{X}$ and $\mathscr{X}^+ = (\mathscr{X}, \mathscr{X}_0)$, this is [MN15, Proposition 2.4.4]. □

Any semivaluation as in Proposition 4.1.0.1 defines a point of the analytic generic fibre $\widehat{\mathscr{X}_\eta}$. The collection of all such semivaluations is a piecewise-linear subspace $\mathrm{Sk}(\mathscr{X}^+) \subseteq \widehat{\mathscr{X}_\eta}$ called the *skeleton* of $\mathscr{X}^+$, which is a polyhedral complex in X^{bir} with (possibly) unbounded faces. In the special case when $\mathscr{X}_0$ is an snc divisor on $\mathscr{X}$ and $\mathscr{X}^+ = (\mathscr{X}, \mathscr{X}_0)$, then $\mathrm{Sk}(\mathscr{X}^+) = \mathrm{Sk}(\mathscr{X})$ is the skeleton from [MN15, §3].

Assume now that $\mathrm{char}(\widetilde{k}) = 0$. Let X be a smooth, proper variety over k, D an snc boundary on X, and $X^+ = (X, D)$ for the associated divisorial log structure on X. For any $\eta \in H^0(X, m(K_X + D))$, let $\mathrm{wt}_\eta(X, D)$ denote the minimal value of the weight function wt_η on X^{an}. The *Kontsevich–Soibelman skeleton* $\mathrm{Sk}(X, D, \eta)$ of η is the locus of points $x \in X^{\mathrm{bir}}$ such that $\mathrm{wt}_\eta(x) = \mathrm{wt}_\eta(X, D)$. By [BM17, Proposition 4.1.6], $\mathrm{Sk}(X, D, \eta) \subseteq \mathrm{Sk}(\mathscr{X}^+)$ for any log regular model $\mathscr{X}^+$ of X^+ over S^+. The *essential skeleton* $\mathrm{Sk}^{\mathrm{ess}}(X, D)$ of the pair is the union of all Kontsevich–Soibelman skeletons; in particular, it is also contained in the skeleton of any log regular model of

X^+. In the special case when $D = 0$, $\mathrm{Sk}^{\mathrm{ess}}(X, D) = \mathrm{Sk}^{\mathrm{ess}}(X)$ is the essential skeleton of Mustață–Nicaise. See [MN15, BM17, BM19] for further details.

4.2 Skeletons over a trivially-valued field

In this section, we construct a skeleton associated to a log-regular log scheme over a trivially valued field k. This generalizes the construction of the simplicial cones of quasi-monomial valuations in [JM12, §3], and it is a trivially-valued analogue of the skeletons of §4.1. Moreover, our skeleton coincides with that of [Uli17, §6], but the explicit descriptions of the points that arises in the two constructions is slightly different. Our realization of the skeleton, inspired by [MN15], enables us to describe the minimality loci of the weight functions of §3.2.2, and ultimately to define the essential skeleton of a pair over k, when k has characteristic zero.

Throughout the section, let k be a trivially-valued field, X a normal variety over k, and D an effective $\mathbf{Q}$-divisor on X such that $K_X + D_{\mathrm{red}}$ is $\mathbf{Q}$-Cartier, and assume that the log scheme $X^+ = (X, D_{\mathrm{red}})$ is log-regular; in particular $D_{X^+} = D_{\mathrm{red}}$. Note that, under these assumptions, the pair (X, D_{red}) is log canonical.

4.2.1 The faces of the skeleton of a log-regular scheme.

In the following proposition, we construct the valuations that will form the skeleton of X^+. Over a perfect field, the log scheme X^+ has toroidal singularities, and the valuations of its skeleton are the toric or monomial valuations of the local toric model, parametrized by the realification of the cocharacter lattice, as in [Thu07]. For an arbitrary log-regular log scheme X^+, the valuations are expressed in terms of the log-geometric data.

Proposition 4.2.1.1. *For any $x \in F_{X^+}$ and $\alpha \in \mathrm{Hom}(\mathcal{C}_{X^+,x}, \overline{\mathbf{R}}_+)$, there exists a* *unique minimal semivaluation*

$$v_\alpha\colon \mathcal{O}_{X,x}\backslash\{0\} \to \overline{\mathbf{R}}_+$$

such that

1. *v_α extends the trivial valuation v_0 on $k \hookrightarrow \mathcal{O}_{X,x}$;*

2. *for any $f \in \mathcal{M}_{X^+,x}$, we have $v_\alpha(f) = \alpha(\overline{f})$.*

Moreover, v_α is a valuation if and only if $\alpha \in \mathrm{Hom}(\mathcal{C}_{X^+,x}, \mathbf{R}_+)$.

Proof. The proof is analogous to that of Proposition 4.1.0.1 (see [BM17, Proposition 3.2.10]). We briefly outline the construction in this setting. Pick a multiplicative section $\sigma\colon \mathcal{C}_{X^+,x} \to \mathcal{M}_{X^+,x}$ of the quotient map $\mathcal{M}_{X^+,x} \to \mathcal{C}_{X^+,x}$. By [BM17, Lemma 3.2.3], any $f \in \mathcal{O}_{X,x}$ can be expressed as

$$f = \sum_{\gamma\in\mathcal{C}_{X^+,x}} a_\gamma \cdot \sigma(\gamma)$$

as an element of the $\mathfrak{m}_x$-adic completion $\widehat{\mathcal{O}}_{X,x}$, where $a_\gamma \in \mathcal{O}_{X,x}^\times \cup \{0\}$. Such an expression will be referred to as an *admissible expansion* of f. Now, set

$$v_\alpha(f) \coloneqq \inf_{\gamma\in\mathcal{C}_{X,x}} v_0(a_\gamma) + \alpha(\gamma). \tag{4.2}$$

Following [BM17, Proposition 3.2.10], one can show that $v_\alpha(f)$ is independent of the choice of admissible expansion of f or of the choice of section σ, the infimum is in fact a minimum, and v_α defines a semivaluation on $\mathcal{O}_{X,x}$ that satisfies the desired properties. □

For any $x \in F_{X^+}$, consider the subset

$$\mathrm{Sk}_x(X^+) \coloneqq \{v_\alpha\colon \alpha \in \mathrm{Hom}(\mathcal{C}_{X^+,x}, \mathbf{R}_+)\}$$

of $X^{\beth}$, equipped with the subspace topology inherited from $X^{\beth}$. Alternatively, $\mathrm{Sk}_x(X^+)$ can be equipped with the topology of pointwise convergence inherited from the identification with the space $\mathrm{Hom}(\mathcal{C}_{X^+,x}, \mathbf{R}_+)$; that is, for a sequence $(\alpha_n)_{n=1}^{\infty}$ and α in $\mathrm{Hom}(\mathcal{C}_{X^+,x}, \mathbf{R}_+)$, we have $v_{\alpha_n} \to v_\alpha$ in $\mathrm{Sk}_x(X^+)$ if and only if $\alpha_n(\gamma) \to \alpha(\gamma)$ for all $\gamma \in \mathcal{C}_{X^+,x}$. These two topologies are compared below.

Lemma 4.2.1.2. *The topology of pointwise convergence on* $\mathrm{Sk}_x(X^+)$ *coincides with the subspace topology inherited from* $X^{\beth}$.

Proof. Given a sequence $(\alpha_n)_{n=1}^{\infty}$ and α in $\mathrm{Hom}(\mathcal{C}_{X^+,x}, \mathbf{R}_+)$, it suffices to show that $v_{\alpha_n}(f) \to v_\alpha(f)$ for all $f \in \mathcal{O}_{X,x}$ if and only if $\alpha_n(\gamma) \to \alpha(\gamma)$ for all $\gamma \in \mathcal{C}_{X^+,x}$. Granted the latter assumption, the convergence $v_{\alpha_n}(f) \to v_\alpha(f)$ follows by (4.2). Conversely, for any lift $\widetilde{\gamma} \in \mathcal{M}_{X^+,x}$ of γ, we have that

$$\alpha_n(\gamma) = v_{\alpha_n}(\widetilde{\gamma}) \to v_\alpha(\widetilde{\gamma}) = \alpha(\gamma).$$

□

Lemma 4.2.1.3. *For any* $x \in F_{X^+}$*, the closure* $\overline{\mathrm{Sk}}_x(X^+)$ *of* $\mathrm{Sk}_x(X^+)$ *in* $X^{\beth}$ *coincides with the subset*

$$\{v_\alpha \colon \alpha \in \mathrm{Hom}(\mathcal{C}_{X^+,x}, \overline{\mathbf{R}}_+)\}. \tag{4.3}$$

In addition, $\overline{\mathrm{Sk}}_x(X^+) \cap X^{\mathrm{bir}} = \mathrm{Sk}_x(X^+)$

Proof. Denote by Z_x the subset of $X^{\beth}$ defined in (4.3). It is clear that $\mathrm{Sk}_x(X^+) \subseteq Z_x$; thus, we need to show that Z_x is contained in $\overline{\mathrm{Sk}}_x(X^+)$ and Z_x is closed in $X^{\beth}$.

Consider a net $(v_{\alpha_\epsilon})_\epsilon$ in Z_x such that $v_{\alpha_\epsilon} \to v$ for some $v \in X^{\beth}$. For any $f \in \mathcal{O}_{X,x}^{\times}$, $v(f) = \lim_\epsilon v_{\alpha_\epsilon}(f) = 0$, so the restriction of v to $\mathcal{M}_{X^+,x}$ descends to a monoid morphism $\alpha \colon \mathcal{C}_{X^+,x} \to \overline{\mathbf{R}}_+$. Arguing as in Lemma 4.2.1.2, one sees that $\alpha_\epsilon \to \alpha$ and hence $v_{\alpha_\epsilon} \to v_\alpha$ in $X^{\beth}$; thus, $v = v_\alpha$ lies in Z_x, so Z_x is contained in $\overline{\mathrm{Sk}}_x(X^+)$.

Now, following the proof of Lemma 4.2.1.2, we observe that the map

$$\mathrm{Hom}(\mathcal{C}_{X^+,x}, \overline{\mathbf{R}}_+) \twoheadrightarrow Z_x \subseteq X^{\beth}$$

$$\alpha \mapsto v_\alpha$$

is continuous, so Z_x is the image of a compact space into a Hausdorff space via a continuous map, and hence Z_x is closed. It follows that $Z_x = \overline{\mathrm{Sk}}_x(X^+)$. Finally, $Z_x \cap X^{\mathrm{bir}} = \mathrm{Sk}_x(X^+)$ by Proposition 4.2.1.1. □

4.2.2 The skeleton of a log-regular scheme.

The subsets $\mathrm{Sk}_x(X^+)$ of $X^{\beth}$, for $x \in F_{X^+}$, can be glued together in a manner that is compatible with the relation of specialization in the Kato fan F_{X^+}. Indeed, consider $x, y \in F_{X^+}$ where x is a specialization of y, i.e. $x \in \overline{\{y\}}$. The localization map $\mathcal{O}_{X,x} \hookrightarrow \mathcal{O}_{X,y}$ descends to a surjective monoid morphism $\tau_{x,y} \colon \mathcal{C}_{X^+,x} \twoheadrightarrow \mathcal{C}_{X^+,y}$. In this case, the two subsets of $\mathrm{Sk}_x(X^+)$ and $\mathrm{Sk}_y(X^+)$ of $X^{\beth}$ are related as follows:

Lemma 4.2.2.1. *The map* $\mathrm{Sk}_y(X^+) \to \mathrm{Sk}_x(X^+)$*, given by* $v_\alpha \mapsto v_{\alpha\circ\tau_{x,y}}$*, is continuous and injective. Furthermore, this map identifies* $\mathrm{Sk}_y(X^+)$ *as a subspace of* $\mathrm{Sk}_x(X^+)$ *in* $X^{\beth}$*.*

Proof. The continuity is immediate from Lemma 4.2.1.2, and the injectivity follows from the surjectivity of $\tau_{x,y}$. Finally, note that v_α and $v_{\alpha\circ\tau_{x,y}}$ coincide as points of $X^{\beth}$ by the uniqueness in Proposition 4.2.1.1. □

Definition 4.2.2.2. The *skeleton* of X^+ is the subspace

$$\mathrm{Sk}(X^+) \coloneqq \bigcup_{x\in F_{X^+}} \mathrm{Sk}_x(X^+) \subseteq X^{\beth},$$

where $\mathrm{Sk}_y(X^+)$ is identified as a subset of $\mathrm{Sk}_x(X^+)$ whenever $x \in \overline{\{y\}}$ via Lemma 4.2.2.1.

By construction, $\mathrm{Sk}(X^+)$ has the structure of a polyhedral cone complex with vertex v_0 where

$$\{v_0\} = \mathrm{Hom}(\{0\}, \mathbf{R}_+) = \mathrm{Sk}_{\eta_X}(X^+)$$

and $\eta_X \in F_{X^+}$ is the generic point of X. The faces of $\mathrm{Sk}(X^+)$ are precisely the subsets $\mathrm{Sk}_x(X^+)$ for $x \in F_{X^+}$. Write $\overline{\mathrm{Sk}}(X^+)$ for the closure of $\mathrm{Sk}(X^+)$ in $X^\beth$. Lemma 4.2.1.3 shows that $\overline{\mathrm{Sk}}(X^+)$ is the union of the subsets $\overline{\mathrm{Sk}}_x(X^+)$ for $x \in F_{X^+}$ with the suitable identifications as in Lemma 4.2.2.1.

For any log-regular log scheme X^+ over k, [Kat94, Proposition 9.8] shows that there is a regular k-scheme X', a reduced snc divisor D' on X', and a morphism $X'^+ = (X', D') \to X^+$ of log schemes such that $F_{X'^+}$ is obtained from F_{X^+} via subdivisions. As subdivisions of the Kato fan do not change the associated skeleton, it follows that $X'^\beth \to X^\beth$ restricts to a homeomorphism $\mathrm{Sk}(X'^+) \simeq \mathrm{Sk}(X^+)$. Two consequences of this fact are detailed below:

- The skeleton $\mathrm{Sk}(X^+)$ coincides with the subspace $\mathrm{QM}(X', D_{X'}) \subseteq X^\beth$ of quasi-monomials valuations in $(X', D_{X'})$ constructed in [JM12, §3]. It follows that $\mathrm{Sk}(X^+)$ lies in the locus of quasi-monomial points of $X^\beth$.

- Under the identification $\mathrm{Sk}(X'^+) \simeq \mathrm{Sk}(X^+)$, the skeleton $\mathrm{Sk}(X^+)$ is endowed with the structure of a simplicial cone complex, and moreover with an integral piecewise affine structure (analogous to [JM12, §4.2]).

4.2.3 The retraction to the skeleton.

Proposition 4.2.3.1. *There is a continuous retraction map*

$$\rho_{X^+} \colon X^\beth \to \overline{\mathrm{Sk}}(X^+)$$

such that $c_X(v) \in \overline{\{c_X(\rho_{X^+}(v))\}}$ *for all* $v \in X^{\beth}$. *Moreover,* ρ_{X^+} *restricts to a continuous retraction map* $X^{\mathrm{bir}} \cap X^{\beth} \to \mathrm{Sk}(X^+)$.

Proof. Given $v \in X^{\beth}$, the construction of the retraction $\rho_{X^+}(v)$ is described below. Write $x = c_X(v)$, and let y be a Kato point in F_{X^+} to which x specializes; that is, y is the generic point of a stratum of D_{X^+} to which x specializes. Let $\mathcal{I}_{X^+,x}$ denotee the ideal of $\mathcal{O}_{X,x}$ generated by $\mathcal{M}_{X^+,x} \backslash \mathcal{O}^{\times}_{X^+,x}$, so the natural map $\mathcal{O}_{X,x} \hookrightarrow \mathcal{O}_{X,y}$ is the localization at $\mathcal{I}_{X^+,x}$, and hence $\mathfrak{m}_y = \mathcal{I}_{X^+,x}\mathcal{O}_{X,y}$. For any multiplicative section $\sigma\colon \mathcal{C}_{X^+,x} \to \mathcal{M}_{X^+,x}$, [BM17, Lemma 3.2.3] shows that any $f \in \mathcal{O}_{X,x}$ can be expressed as

$$f = \sum_{\gamma \in \mathcal{C}_{X^+,x}} a_\gamma \cdot \sigma(\gamma) \tag{4.4}$$

as an element of $\widehat{\mathcal{O}}_{X,y}$, where $a_\gamma \in (\mathcal{O}_{X,x} \backslash \mathcal{I}_{X^+,x}) \cup \{0\}$. For any expansion of f as in (4.4), set

$$\widetilde{v}(f) \coloneqq \min_{\gamma \in \mathcal{C}_{X^+,x}} v_0(a_\gamma) + v(\sigma(\gamma)). \tag{4.5}$$

Following the proof of [BM17, Proposition 3.2.10], one can show that $\widetilde{v}$ is well-defined and is a semivaluation $\widetilde{v}\colon \mathcal{O}_{X,x} \to \overline{\mathbf{R}}_+$. Further, it is clear that x specializes to $c_X(\widetilde{v})$ since $\widetilde{v}(f) \geqslant 0$ for all $f \in \mathcal{O}_{X,x}$.

We claim that $\widetilde{v} \in \overline{\mathrm{Sk}}_y(X^+)$. To see this, we construct a monoid morphism $\widetilde{\alpha} \in \mathrm{Hom}(\mathcal{C}_{X^+,y}, \overline{\mathbf{R}}_+)$ such that $\widetilde{v} = v_{\widetilde{\alpha}}$ as semivaluations, where $v_{\widetilde{\alpha}}$ is the semivaluation constructed in Proposition 4.2.1.1. Observe that any $f \in \mathcal{O}_{X,y}$ can be written as $f = g/h$ with $g \in \mathcal{O}_{X,x}$ and $h \in \mathcal{O}_{X,x} \backslash \mathcal{I}_{X^+,x}$, so $\widetilde{v}(h) = 0$ and hence $\widetilde{v}(f) = \widetilde{v}(g) \geqslant 0$. In addition, f is invertible in $\mathcal{O}_{X,y}$ if and only if g is, which is equivalent to $g \in \mathcal{O}_{X,x} \backslash \mathcal{I}_{X^+,x}$; in this case, $\widetilde{v}(f) = \widetilde{v}(g) = 0$ by construction. Thus, the restriction of $\widetilde{v}$ to $\mathcal{M}_{X^+,y}$ descends to a monoid morphism $\widetilde{\alpha}\colon \mathcal{C}_{X^+,y} \to \overline{\mathbf{R}}_+$. The uniqueness in Proposition 4.2.1.1 guarantees that $\widetilde{v} = v_{\widetilde{\alpha}}$; thus, set $\rho_{X^+}(v) \coloneqq \widetilde{v} \in \overline{\mathrm{Sk}}_y(X^+)$.

Note that if $v \in \overline{\mathrm{Sk}}(X^+)$, then we have $\widetilde{v} = v$. Indeed, if $x = c_X(v) \in F_{X^+}$, then the formula (4.5) defining $\widetilde{v}$ on elements of $\mathcal{O}_{X,x}$ coincides with (4.2). That is, ρ_{X^+} is a retraction of $X^\beth$ onto $\overline{\mathrm{Sk}}(X^+)$ for the inclusion $\overline{\mathrm{Sk}}(X^+) \to X^\beth$.

It remains to show that ρ_{X^+} is continuous. For each $w \in X^\beth$, consider the subset $U_w = c_X^{-1}(\overline{\{c_X(\rho_{X^+}(w))\}})$ of $X^\beth$, which is an open neighbourhood of w since the centre map is anticontinuous. As $\{U_w\}_{w \in X^\beth}$ is an open cover of $X^\beth$, it suffices to show that the restriction $\rho_{X^+}|_{U_w}$ is continuous for each $w \in X^\beth$. Note that the image of $\rho_{X^+}|_{U_w}$ lies in $\overline{\mathrm{Sk}}_{c_X(\rho_{X^+}(w))}(X^+)$ because $c_X(\rho_{X^+}(w))$ is a Kato point to which $c_X(w')$ specializes for all $w' \in U_w$. The continuity of $\rho_{X^+}|_{U_w}$ is then a consequence of the following: for any $f \in \mathcal{O}_{X,c_X(\rho_{X^+}(w))}$, the map

$$\begin{aligned} U_w &\to \overline{\mathbf{R}}_+ \\ w' &\mapsto v_{\rho_{X^+}(w')}(f) \end{aligned}$$

is continuous. Indeed, if $f = \sum_\gamma a_\gamma \cdot \sigma(\gamma)$ is an admissible expansion in $\widehat{\mathcal{O}}_{X,c_X(\rho_{X^+}(w))}$, then

$$v_{\rho_{X^+}(w')}(f) = \min_\gamma v_0(a_\gamma) + w'(\sigma(\gamma))$$

is continuous in w'. Hence, $\rho_{X^+}|_{U_w}$ is continuous, which concludes the proof. $\square$

The retraction of Proposition 4.2.3.1 is related to other constructions in the literature.

- If $X^+ = (X, D_{X^+})$ is an snc pair, the retraction ρ_{X^+} of Proposition 4.2.3.1 restricts to the retraction $X^{\mathrm{bir}} \cap X^\beth \to \mathrm{Sk}(X^+)$ of [JM12, §4.3]. Note that [JM12] denotes the space $X^{\mathrm{bir}} \cap X^\beth$ by Val_X.

- After identifying $\overline{\mathrm{Sk}}(X^+)$ with the extended cone complex $\overline{\Sigma}_{X^+}$ following [Uli17, §6.1], ρ_{X^+} coincides with the tropicalization map $X^\beth \to \overline{\Sigma}_{X^+}$. In particular, [Uli17, Theorem 1.2] implies that ρ_{X^+} recovers Thuillier's (strong deformation) retraction map from [Thu07, §3.2].

4.2.4 Functoriality of the skeleton.

Given log-regular log schemes X^+ and Y^+ over k and a morphism $\varphi\colon X \to Y$ of k-schemes, write $\varphi^\beth\colon X^\beth \to Y^\beth$ for the $\beth$-analytification. The retraction map of Proposition 4.2.3.1 shows that $\varphi^\beth$ restricts to a continuous map

$$\overline{\mathrm{Sk}}(X^+) \hookrightarrow X^\beth \xrightarrow{\varphi^\beth} Y^\beth \xrightarrow{\rho_{Y^+}} \overline{\mathrm{Sk}}(Y^+) \tag{4.6}$$

between the closures of the skeletons. If φ is a dominant map, then (4.6) restricts to a continuous map

$$\mathrm{Sk}(X^+) \hookrightarrow X^{\mathrm{bir}} \cap X^\beth \xrightarrow{\varphi^\beth} Y^{\mathrm{bir}} \cap Y^\beth \xrightarrow{\rho_{Y^+}} \mathrm{Sk}(Y^+). \tag{4.7}$$

That is, the formation of the skeleton is functorial with respect to dominant morphisms.

4.2.5 Comparison with the dual complex

In [MN15, Proposition 3.1.4], Mustaţă and Nicaise remark that, given a variety X over a discretely valued field, the skeleton associated to an snc model $\mathscr{X}$ of X over the valuation ring is homeomorphic to the dual intersection complex of the special fibre $\mathscr{X}_0$. In the trivially-valued field case, consider a log-regular pair $X^+ = (X, D_{X^+})$ such that D_{X^+} is an snc divisor, and let $\mathcal{D}(D_{X^+})$ denote the dual intersection complex of D_{X^+}. In the following proposition, we compare $\mathcal{D}(D_{X^+})$ to the skeleton $\mathrm{Sk}(X^+)$; this result is well-known to experts, but we include a proof for the sake of completeness.

Proposition 4.2.5.1. *There is a homeomorphism between* $\mathrm{Sk}(X^+)$ *and the cone over* $\mathcal{D}(D_{X^+})$*.*

In §6.2, we extend Proposition 4.2.5.1 to more singular pairs; see e.g. Lemma 6.2.4.4.

Proof. A point $x \in F_{X^+}$ is the generic point of a stratum of D_{X^+} of codimension r, for some r; since D_{X^+} is snc, a choice of local equations for D_{X^+} at x yields an isomorphism $\mathcal{C}_{X^+,x} \simeq \mathbf{N}^r$. This induces an isomorphism

$$\mathrm{Sk}_x(X^+) \simeq \mathrm{Hom}(\mathcal{C}_{X^+,x}, \mathbf{R}_+) \simeq (\mathbf{R}_+)^r$$

of topological monoids.

A face of $\mathcal{D}(D_{X^+})$ correspond to a stratum Z of D_{X^+} of codimension r for some r, and is isomorphic to the standard simplex Δ^{r-1}. Thus, the cone over this face is homeomorphic to $(\mathbf{R}_+)^r$, i.e. to $\mathrm{Sk}_x(X^+)$ where x is the generic point of Z.

As the gluing maps on the dual complex are compatible with the identifications on $\mathrm{Sk}(X^+)$, we conclude that the cone over the dual complex is homeomorphic to the skeleton of X^+. □

Definition 4.2.5.2. The *link* of the skeleton $\mathrm{Sk}(X^+)$ is the (topological) quotient $\mathrm{Sk}(X^+)^*/\mathbf{R}_+^*$ by the $\mathbf{R}_+^*$-rescaling action.

Proposition 4.2.5.3. *The spaces* $\mathrm{Sk}(X^+)^*/\mathbf{R}_+^*$ *and* $\mathcal{D}(D_{X^+})$ *are homeomorphic.*

Proof. The proof is identical to that of [Thu07, Proposition 4.7], and we sketch it below. The rescaling action on the punctured cone over $\mathcal{D}(D_{X^+})$ makes the homeomorphism of Proposition 4.2.5.1 into an $\mathbf{R}_+^*$-equivariant one. The assertion follows by taking quotients by the $\mathbf{R}_+^*$-actions. □

It follows from Proposition 4.2.5.3 that $\mathrm{Sk}(X^+)^*/\mathbf{R}_+^*$ has the structure of a (compact) cell complex induced by the homeomorphism with $\mathcal{D}(D_{X^+})$.

4.2.6 The skeleton of a product

Let k be a trivially-valued field and let $X^+ = (X, D_{X^+})$ and $Y^+ = (Y, D_{Y^+})$ be log-regular pairs over k. We denote by $Z^+ = (Z, D_{Z^+})$ the product in the category of fine and saturated log schemes. In particular, Z^+ is log-regular and

$$D_{Z^+} = D_{X^+} \times Y + X \times D_{Y^+}.$$

The goal of this section is to compare the skeleton associated to Z^+ with the product of skeletons of X^+ and Y^+ in the category of topological spaces.

Lemma 4.2.6.1. *The projection maps* $(\mathrm{pr}_X, \mathrm{pr}_Y)\colon Z \to X \times_k Y$ *induces an isomorphism* $F_{Z^+} \xrightarrow{\simeq} F_{X^+} \times F_{Y^+}$.

Proof. As any stratum of D_{Z^+} is of the form $D_x \times D_y$ for some $x \in F_{X^+}$ and $y \in F_{Y^+}$, we have a bijective correspondence between F_{Z^+} and $F_{X^+} \times F_{Y^+}$ that is compatible with the projections to the factors. Moreover, this bijection is actually an isomorphism of Kato fans, observing that

$$\mathcal{C}_{Z^+,z} \simeq \mathcal{C}_{X^+,x} \oplus \mathcal{C}_{Y^+,y}$$

when the Kato point $z \in F_{Z^+}$ maps to $(x, y) \in F_{X^+} \times F_{Y^+}$. $\square$

The projections $\mathrm{pr}_X : Z^+ \to X^+$ and $\mathrm{pr}_Y : Z^+ \to Y^+$ are dominant morphisms of log-regular log schemes, hence they induce a continuous map of skeletons

$$\left(\mathrm{pr}_{\mathrm{Sk}(X^+)}, \mathrm{pr}_{\mathrm{Sk}(Y^+)}\right) : \mathrm{Sk}(Z^+) \to \mathrm{Sk}(X^+) \times \mathrm{Sk}(Y^+)$$

that is constructed as in 4.2.4; that is, $\left(\mathrm{pr}_{\mathrm{Sk}(X^+)}, \mathrm{pr}_{\mathrm{Sk}(Y^+)}\right)$ is the composition

$$\mathrm{Sk}(Z^+) \hookrightarrow Z^{\beth} \cap Z^{\mathrm{bir}} \xrightarrow{(\mathrm{pr}^{\beth}_{X^+}, \mathrm{pr}^{\beth}_{Y^+})} (X^{\beth} \cap X^{\mathrm{bir}}) \times (Y^{\beth} \cap Y^{\mathrm{bir}}) \xrightarrow{(\rho_{X^+}, \rho_{Y^+})} \mathrm{Sk}(X^+) \times \mathrm{Sk}(Y^+).$$

It follows that there is a commutative diagram

$$\begin{array}{ccc} Z^{\mathrm{bir}} \cap Z^{\beth} & \xrightarrow{(\mathrm{pr}^{\beth}_{X^+}, \mathrm{pr}^{\beth}_{Y^+})} & (X^{\mathrm{bir}} \cap X^{\beth}) \times (Y^{\mathrm{bir}} \cap Y^{\beth}) \\ {\scriptstyle \rho_{Z^+}} \downarrow & & \downarrow {\scriptstyle (\rho_{X^+}, \rho_{Y^+})} \\ \mathrm{Sk}(Z^+) & \xrightarrow{(\mathrm{pr}_{\mathrm{Sk}(X^+)}, \mathrm{pr}_{\mathrm{Sk}(Y^+)})} & \mathrm{Sk}(X^+) \times \mathrm{Sk}(Y^+). \end{array} \tag{4.8}$$

In the following lemma, we show that the map $\mathrm{pr}_{\mathrm{Sk}(X^+)} \colon \mathrm{Sk}(Z^+) \to \mathrm{Sk}(X^+)$ is in fact induced by the restriction of morphisms of monoids.

Lemma 4.2.6.2. *Let $z = (x, y) \in F_Z^+$ be a Kato point, $\varepsilon \in \mathrm{Hom}(\mathcal{C}_{Z^+,z}, \mathbf{R}_+)$, and let $i_{x,z} \colon \mathcal{C}_{X^+,x} \hookrightarrow \mathcal{C}_{Z^+,z}$ and $i_{y,z} \colon \mathcal{C}_{Y^+,y} \hookrightarrow \mathcal{C}_{Z^+,z}$ denote the inclusions of characteristic sheaves. Then,*

$$\begin{cases} \mathrm{pr}_{\mathrm{Sk}(X^+)}(v_\varepsilon) = v_{\varepsilon \circ i_{x,z}}, \\ \mathrm{pr}_{\mathrm{Sk}(Y^+)}(v_\varepsilon) = v_{\varepsilon \circ i_{y,z}}. \end{cases}$$

Proof. It suffices to show the first equality. By the definition of the projection map to the skeleton, we have that

$$\mathrm{pr}_{\mathrm{Sk}(X^+)}(v_\varepsilon) = \rho_{X^+}(\mathrm{pr}^{\beth}_{X^+}(v_\varepsilon)).$$

Since $\mathrm{pr}_{\mathrm{Sk}(X^+)}(v_\varepsilon)$ is a point of $\mathrm{Sk}_x(X^+)$, there exists $\alpha \in \mathrm{Hom}(\mathcal{C}_{X^+,x}, \mathbf{R}_+)$ such that

$$\rho_{X^+}(\mathrm{pr}^{\beth}_{X^+}(v_\varepsilon)) = v_\alpha.$$

Hence, it suffices to show $\alpha = \varepsilon \circ i_{x,z}$. By Proposition 4.2.1.1, for any $m \in \mathcal{M}_{X,x}$ we have

$$\alpha(\overline{m}) = \mathrm{pr}^{\beth}_{X^+}(v_\varepsilon)(m)$$

and, since pr_X induces the inclusion of fraction fields $i : k(X) \hookrightarrow k(Z)$, we obtain that

$$\mathrm{pr}^{\beth}_{X^+}(v_\varepsilon)(m) = (v_\varepsilon \circ i)(m) = v_\varepsilon(m) = \varepsilon(\overline{m}).$$

On the other hand, for any $m \in \mathcal{M}_{X,x}$ we also have

$$v_{\varepsilon \circ i_{x,z}}(m) = (\varepsilon \circ i_{x,z})(\overline{m}) = \varepsilon(\overline{m}),$$

which concludes the proof. □

Similar to [BM17, Proposition 3.4.3], we prove that log-regular skeletons are well-behaved under products.

Proposition 4.2.6.3. *The map* $\left(\mathrm{pr}_{\mathrm{Sk}(X^+)}, \mathrm{pr}_{\mathrm{Sk}(Y^+)}\right)$ *restricts to a homeomorphism*

$$\mathrm{Sk}(Z^+) \simeq \mathrm{Sk}(X^+) \times \mathrm{Sk}(Y^+).$$

Proof. It suffices to show that $\left(\mathrm{pr}_{\mathrm{Sk}(X^+)}, \mathrm{pr}_{\mathrm{Sk}(Y^+)}\right)$ restricts to a homeomorphism $\mathrm{Sk}_z(Z^+) \simeq \mathrm{Sk}_x(X^+) \times \mathrm{Sk}_y(Y^+)$ for each $z = (x, y) \in F_{Z^+}$. By Lemma 4.2.6.2, this is equivalent to showing that the map

$$\mathrm{Hom}(\mathcal{C}_{Z^+,z}, \mathbf{R}_+) \to \mathrm{Hom}(\mathcal{C}_{X^+,x}, \mathbf{R}_+) \times \mathrm{Hom}(\mathcal{C}_{Y^+,y}, \mathbf{R}_+),$$

given by $\varepsilon \mapsto (\varepsilon \circ i_{x,z}, \varepsilon \circ i_{y,z})$, is a homeomorphism. It is clearly continuous and, if $q_{z,x} \colon \mathcal{C}_{Z^+,z} \to \mathcal{C}_{X^+,x}$ and $q_{z,y} \colon \mathcal{C}_{Z^+,z} \to \mathcal{C}_{Y^+,y}$ denote the projections, then

$$(\varepsilon_1, \varepsilon_2) \mapsto \varepsilon_1 \circ q_{z,x} + \varepsilon_2 \circ q_{z,y}$$

is a continuous inverse. □

Proposition 4.2.6.4. *There is a homeomorphism*

$$\mathrm{Sk}(Z^+)^*/\mathbf{R}_+^* \simeq (\mathrm{Sk}(X^+)^*/\mathbf{R}_+^*) * (\mathrm{Sk}(Y^+)^*/\mathbf{R}_+^*),$$

where $(\mathrm{Sk}(X^+)^*/\mathbf{R}_+^*) * (\mathrm{Sk}(Y^+)^*/\mathbf{R}_+^*)$ *denotes the join of* $\mathrm{Sk}(X^+)^*/\mathbf{R}_+^*$ *and* $\mathrm{Sk}(Y^+)^*/\mathbf{R}_+^*$.

Proof. Observe that the proof of Proposition 4.2.6.3 yields a $\mathbf{R}_+^*$-equivariant homeomorphism

$$\mathrm{Sk}(Z^+) \simeq \mathrm{Sk}(X^+) * \mathrm{Sk}(Y^+),$$

where the product is endowed with the diagonal action. As explained in §2.9, there exists a $\mathbf{R}_+^*$-equivariant homeomorphism

$$\mathrm{Sk}(Z^+) \simeq \mathrm{Cone}((\mathrm{Sk}(X^+)^*/\mathbf{R}_+^*) * (\mathrm{Sk}(Y^+)^*/\mathbf{R}_+^*)).$$

The statement now follows from the results of §2.9. □

4.2.7 The Kontsevich–Soibelman and essential skeletons

Assume now that k has characteristic zero and D is a Weil $\mathbf{Q}$-divisor on X such that $K_X + D_{\mathrm{red}}$ is $\mathbf{Q}$-Cartier. Following the approach of Kontsevich and Soibelman, for any rational D-logarithmic pluricanonical form η on X, we can construct a subset $\mathrm{Sk}(X, D, \eta)$ of $X^\beth$ as the set of birational points satisfying a minimality condition with respect to η. More precisely, we define

$$\mathrm{wt}_\eta(X, D) \coloneqq \inf\{\mathrm{wt}_\eta(x) \colon x \in X^\beth\} \in \overline{\mathbf{R}}.$$

Definition 4.2.7.1. The *Kontsevich–Soibelman skeleton* of the triple (X, D, η) is

$$\mathrm{Sk}(X, D, \eta) = \{x \in X^{\mathrm{bir}} \cap X^\beth \colon \ \mathrm{wt}_\eta(x) = \mathrm{wt}_\eta(X, D)\}.$$

In fact, as in [MN15, Theorem 4.7.5], $\mathrm{Sk}(X, D, \eta)$ is the closure in $X^{\mathrm{bir}} \cap X^\beth$ of the points $x \in X^{\mathrm{div}} \cup \{v_0\}$ such that $\mathrm{wt}_\eta(x) = \mathrm{wt}_\eta(X, D)$.

Assume in addition that $X^+ = (X, D_{\mathrm{red}})$ is log-regular, hence log canonical. In this case, the function $A_{(X,D_{\mathrm{red}})}$ is non-negative on $X^\beth$, and it has value exactly 0 at any divisorial point in $\mathrm{Sk}(X^+)$, thus on $\mathrm{Sk}(X^+)$. In fact, the only $x \in X^{\mathrm{bir}} \cap X^\beth$ with $A_{(X,D_{\mathrm{red}})}(x) = 0$ are those in the skeleton by [Blu18, Proposition 3.2.5].

Proposition 4.2.7.2. *For a non-zero regular D-logarithmic pluricanonical form η on X and $x \in X^\beth$, we have*

$$\mathrm{wt}_\eta(x) \geqslant \mathrm{wt}_\eta(\rho_{X^+}(x)),$$

and if $x \in X^{\mathrm{bir}} \cap X^{\beth}$, then equality holds if and only if $x \in \mathrm{Sk}(X^+)$.

Proof. By the maximal lower-semicontinuity of the weight function, it suffices to show the inequality on $X^{\mathrm{bir}} \cap X^{\beth}$ (or even on $X^{\mathrm{div}} \cap X^{\beth}$). Let $x \in X^{\mathrm{bir}} \cap X^{\beth}$. Denote by ξ and ξ' the centres of x and $\rho_{X^+}(x)$, respectively. By construction of the retraction ρ_{X^+}, we have that $\xi \in \overline{\{\xi'\}}$, and hence there exists a trivializing open $U \subseteq X$ for the logarithmic pluricanonical bundle $\omega_{(X,D_{\mathrm{red}})}^{\otimes m}$ that contains both ξ and ξ'. On such an open set U, the form $\eta|_U$ corresponds to a regular function f on U, and the weight functions can be computed as

$$\begin{cases} \mathrm{wt}_\eta(x) = A_{(X,D_{\mathrm{red}})}(x) + v_x(f), \\ \mathrm{wt}_\eta(\rho_{X^+}(x)) = A_{(X,D_{\mathrm{red}})}(\rho_{X^+}(x)) + v_{\rho_{X^+}(x)}(f). \end{cases}$$

Locally at ξ', f has an admissible expansion of the form $f = \sum_{\gamma \in \mathcal{C}_{X^+,\xi'}} c_\gamma \gamma$. The ultrametric inequality gives

$$v_x(f) \geqslant \min_\gamma \{v_0(c_\gamma) + v_x(\gamma)\} = v_{\rho_{X^+}(x)}(f), \tag{4.9}$$

and $A_{(X,D_{\mathrm{red}})}(x) \geqslant 0 = A_{(X,D_{\mathrm{red}})}(\rho_{X^+}(x))$ by [Blu18, Proposition 3.2.5]; adding this to (4.9), we get that $\mathrm{wt}_\eta(x) \geqslant \mathrm{wt}_\eta(\rho_{X^+}(x))$.

Assume, in addition, that the equality $A_{(X,D_{\mathrm{red}})}(x) + v_x(f) = v_{\rho_{X^+}(x)}(f)$ holds. As $v_x(f) \geqslant v_{\rho_{X^+}(x)}(f)$ and $A_{(X,D_{\mathrm{red}})}(v_x) \geqslant 0$, this assumption implies that $A_{(X,D_{\mathrm{red}})}(x) = 0$. It follows that x lies in the skeleton $\mathrm{Sk}(X^+)$. □

Definition 4.2.7.3. The *essential skeleton* $\mathrm{Sk}^{\mathrm{ess}}(X,D)$ of (X,D) is the union of all Kontsevich–Soibelman skeletons $\mathrm{Sk}(X,D,\eta)$, where η runs over all non-zero regular D-logarithmic pluricanonical forms on X. In symbols,

$$\mathrm{Sk}^{\mathrm{ess}}(X,D) \coloneqq \bigcup_\eta \mathrm{Sk}(X,D,\eta).$$

For any regular D-logarithmic pluricanonical form η, the function $\phi_{\mathrm{triv},\omega^{\otimes m}_{(X,D_{\mathrm{red}})}}(\eta,\cdot)$ is non-negative, and hence wt_η is as well. Further, if η is non-zero, $\mathrm{wt}_\eta(v_0) = 0$, where v_0 is the trivial valuation. It follows that $\mathrm{wt}_\eta(X,D) = 0$ and $v_0 \in \mathrm{Sk}(X,D,\eta)$ for every such form η. In particular, the essential skeleton of (X,D) is nonempty whenever there exists a non-zero regular D-logarithmic form on X.

By arguing as in [MN15, Proposition 4.5.5(v)], one can show that

$$\mathrm{Sk}(X,D,\eta^{\otimes m}) = \mathrm{Sk}(X,D,\eta)$$

for any $m \in \mathbf{Z}_{>0}$. In particular, $\mathrm{Sk}^{\mathrm{ess}}(X,D)$ can be computed as the union of Kontsevich–Soibelman skeletons of sections of $m(K_X+D)$ with $m \in \mathbf{Z}_{>0}$ sufficiently divisible.

Remark 4.2.7.4. There are two fundamental reasons why the essential skeleton is defined in terms of non-zero regular D-logarithmic pluricanonical forms. They are the following:

- If $\xi \in X$ and δ is a generating section of $\omega^{\otimes m}_{(X,D_{\mathrm{red}}),\xi}$, then any regular section η of $\omega^{\otimes m}_{(X,D)}$ can be written, locally at ξ, as $\eta = f\delta$ for some $f \in \mathcal{O}_{X,\xi}$. For any $x \in X^{\beth}$ such that f is regular at $c_X(x)$ and $c_X(\rho_{X^+}(x))$, we have $v_x(f) \geqslant v_{\rho_{X^+}(x)}(f)$, as in Proposition 4.2.7.2. In particular, the minimality locus of wt_η on $X^{\mathrm{bir}} \cap X^{\beth}$ (and hence the essential skeleton) lies in the log-regular skeleton $\mathrm{Sk}(X^+)$.

- The definition of the essential skeleton is in terms of D-logarithmic pluricanonical forms, as opposed to logarithmic pluricanonical forms. This is done so that the faces of $\mathrm{Sk}(X^+)$ corresponding to components of D with coefficient strictly less than 1 do not lie in the essential skeleton. This choice is compatible with the correspondence between the dual complex of a dlt boundary divisor and the essential skeleton in the discretely-valued setting, as explored in [NX16, Theorem 3.3.3] and [BM17, Proposition 5.1.7].

Furthermore, when (X, D) is a logCY pair, we will show in Proposition 6.2.4.1 that the essential skeleton $\mathrm{Sk}^{\mathrm{ess}}(X, D)$ in fact coincides with the skeleton $\mathrm{Sk}(X, D^{=1})$. This plays a crucial role in the proof of Theorem G.

4.2.8 Comparison of the trivially-valued and discretely-valued settings

This section explores a relationship between the weight functions in the trivially-valued and in the discretely-valued cases. To this end, we work in a setting where both the weight functions are defined and interact, namely on the total space of a degeneration. Proposition 4.2.8.3 shows that we can regard an essential skeleton, defined in the trivially-valued setting, as a cone over the essential skeleton in the discretely-valued setting.

Let k be a trivially-valued field of characteristic zero. Let $\mathscr{X}$ be a degeneration over $k[[\varpi]]$, i.e. a normal, flat, separated scheme of finite type over $k[[\varpi]]$. The formal completion $\widehat{\mathscr{X}}$ of $\mathscr{X}$ along the special fibre $\mathscr{X}_0$ is a formal scheme topologically of finite type over $k[[\varpi]]$, and the structure morphism

$$\widehat{\mathscr{X}} \to \mathrm{Spf}(k[[\varpi]]) \tag{4.10}$$

is a morphism of special formal k-schemes in the sense of [Ber96a, §1]. The morphism (4.10) induces a morphism

$$\mathscr{X}^{\mathrm{triv}} \to D^1_k(0,1)$$

on the analytic generic fibres, where $D^1_k(0,1)$ denotes the open unit disc over k. We can identify $D^1_k(0,1)$ with the interval $[0,1)$ by sending $r \in [0,1)$ to the ϖ-adic seminorm $|\cdot|_r$ on $k[[\varpi]]$ normalized so that $|\varpi|_r = r$. Under this identification, the fibre of $\mathscr{X}^{\mathrm{triv}} \to D^1_k(0,1)$ above $1/e$ is the generic fibre of $\mathscr{X}$, denoted $\mathscr{X}^{\mathrm{disc}}$, as an analytic space over the field $K \coloneqq (k((\varpi)), |\cdot|_{1/e})$; see [Nic11, Lemma 4.2] for details.

Definition 4.2.8.1. We say that $\mathscr{X}$ is *defined over a curve* if there exists a germ of a smooth curve C over k, a closed point $0 \in C(k)$, an isomorphism $\widehat{\mathcal{O}}_{C,0} \simeq k[[\varpi]]$ (which we write as an equality from now on), and a normal, flat, separated scheme X over C such that

$$\mathscr{X} = X \times_C \operatorname{Spec}(\widehat{\mathcal{O}}_{C,0}).$$

For the rest of the section, fix a morphism $X \to C$ and $0 \in C(k)$ as in Definition 4.2.8.1. There is a cartesian square of analytic spaces over k given by

$$\begin{array}{ccccc} \mathscr{X}^{\mathrm{disc}} & \hookrightarrow & \mathscr{X}^{\mathrm{triv}} & \hookrightarrow & X^{\beth} \\ \downarrow & & \downarrow & & \downarrow \\ \{1/e\} \simeq \mathcal{M}(K) & \hookrightarrow & [0,1) \simeq D^1_k(0,1) & \hookrightarrow & C^{\beth}. \end{array}$$

Let $X_0 \subseteq X$ denote the fibre above 0. Suppose that X_0 is reduced, $\mathscr{X}_{k((\varpi))}$ is smooth, and $K_X + X_0$ is $\mathbf{Q}$-Cartier. For any regular section η of $\omega^{\otimes m}_{(X,X_0)}$, write η_K for the *Gelfand–Leray form* associated to η: this is the regular section of $\omega^{\otimes m}_{\mathscr{X}_K}$ characterized by the property that $\eta_K \wedge d\varpi$ coincides with the pullback of η along $\mathscr{X} \to X$, or equivalently it is the contraction of η with the vector field $\partial/\partial\varpi$. See [NS07, Definition 9.5] for more details. We can define weight functions on $\mathscr{X}^{\mathrm{triv}}$ and $\mathscr{X}^{\mathrm{disc}}$ as follows:

- the weight function $\mathrm{wt}^{\mathrm{disc}}_{\eta_K} \colon \mathscr{X}^{\mathrm{disc}} \to \overline{\mathbf{R}}$ is defined as in Theorem 3.2.1.1, where we consider η_K as a regular section of $\omega^{\otimes m}_{(\mathscr{X},\mathscr{X}_0)}$;
- the weight function $\mathrm{wt}^{\mathrm{triv}}_{\eta} \colon \mathscr{X}^{\mathrm{triv}} \to \overline{\mathbf{R}}$ is the restriction of the weight function $\mathrm{wt}_\eta \colon X^{\beth} \to \overline{\mathbf{R}}$ defined as in Definition 3.2.2.2.

Note that the reason we assume that $\mathscr{X}$ is defined over a curve is that our definition of $\mathrm{wt}_\eta^{\mathrm{triv}}$ only holds on the $\beth$-analytification of a k-variety, but not on a general k-analytic space.

Proposition 4.2.8.2. *Let $m \in \mathbf{Z}_{>0}$ be such that $m(K_X + X_0)$ is Cartier. For $\eta \in H^0(X, m(K_X + X_0))$ and $x \in \mathscr{X}^{\mathrm{disc}}$, we have*

$$\mathrm{wt}_{\eta_K}^{\mathrm{disc}}(x) = \mathrm{wt}_\eta^{\mathrm{triv}}(x). \tag{4.11}$$

If in addition $X \to C$ is proper (and hence $\mathscr{X}^{\mathrm{disc}} = \mathscr{X}_K^{\mathrm{an}}$), then there is an inclusion of Kontsevich–Soibelman skeletons

$$\mathrm{Sk}(\mathscr{X}_K, \eta_K) \supseteq \mathrm{Sk}(X, X_0, \eta) \cap \mathscr{X}^{\mathrm{disc}}, \tag{4.12}$$

which is an equality provided that (X, X_0) is log canonical and that there is a component of X_0 along which η does not vanish identically.

Proof. We prove (4.11) in two steps. For the first step, assume that $x \in \mathscr{X}^{\mathrm{disc}} \cap X^{\mathrm{div}}$ and is determined by a prime divisor on a proper birational model $h\colon Y \to X$ of X, where h is an isomorphism away from X_0. Let $\mathscr{Y} := Y \times_X \mathscr{X}$; it is equipped with a proper birational morphism $\mathscr{Y} \to \mathscr{X}$, also denoted by h, that is an isomorphism outside of $\mathscr{X}_0$. In particular, $\mathscr{Y}$ is a model of $\mathscr{X}_K$.

Set $\xi = \mathrm{red}_{\mathscr{X}}(x)$ and take a $\mathcal{O}_{\mathscr{X},\xi}$-module generator δ of $\omega_{(\mathscr{X},\mathscr{X}_0),\xi}^{\otimes m}$. Locally at ξ, write the section η_K as $\eta_K = f\delta$ for some $f \in \mathcal{O}_{\mathscr{X},\xi}$. Consider the identity

$$\begin{aligned} m(K_{\mathscr{Y}/k[[\varpi]]} + \mathscr{Y}_{0,\mathrm{red}} - \mathrm{div}_{\mathscr{Y}}(\varpi)) - \left(\sum_i ma(E_i)E_i + \mathrm{div}_{\mathscr{Y}}(h^*f) - m\mathrm{div}_{\mathscr{Y}}(\varpi) \right) \\ = h^*(m(K_{\mathscr{X}/k[[\varpi]]} + \mathscr{X}_0) - \mathrm{div}_{\mathscr{X}}(f)), \end{aligned} \tag{4.13}$$

where E_i are the exceptional prime divisors of h, and $a(\cdot)$ is the log discrepancy function with respect to $(\mathscr{X}, \mathscr{X}_0)$. Note that $a(E_i) = A_{(X,X_0)}(\mathrm{ord}_{E_i})$.

Now, (3.4) and (4.13) yield the equalities

$$\begin{aligned}\mathrm{wt}^{\mathrm{disc}}_{\eta_K}(x) &= v_x(\mathrm{div}_{(\mathscr{Y},\mathscr{Y}_{0,\mathrm{red}}-\mathrm{div}_{\mathscr{Y}}(\varpi))}(\eta_K)) + m \\ &= m\,a(x) + v_x(h^* f) - m v_x(\varpi) + m \\ &= m A_{(X,X_0)}(x) + v_x(f) \\ &= m A_{(X,X_0-\mathrm{div}_{(X,X_0)}(\eta))}(x) \\ &= \mathrm{wt}^{\mathrm{triv}}_{\eta}(x),\end{aligned}$$

where the second-to-last equality follows from [Kol13, Lemma 2.5] and the last equality from Proposition 3.2.3.1. Thus, (4.11) holds for any $x \in \mathscr{X}^{\mathrm{disc}} \cap X^{\mathrm{div}}$. Note that $(\mathscr{X}_K)^{\mathrm{div}} = \mathscr{X}^{\mathrm{disc}} \cap X^{\mathrm{div}}$, since the blow-up of a formal ideal on $\mathscr{X}$ that is cosupported on $\mathscr{X}_0$ can be realized as the completion of an algebraic blow-up of X.

Now, we proceed to the second step: to prove the equality (4.11) on all of $\mathscr{X}^{\mathrm{disc}}$, it suffices to check that both $\mathrm{wt}^{\mathrm{disc}}_{\eta_K}$ and $\mathrm{wt}^{\mathrm{triv}}_{\eta}$ are maximal lower-semicontinuous extensions on $\mathscr{X}^{\mathrm{disc}}$ of

$$\mathrm{wt}^{\mathrm{disc}}_{\eta_K}|_{(\mathscr{X}_K)^{\mathrm{div}}} = \mathrm{wt}^{\mathrm{triv}}_{\eta}|_{\mathscr{X}^{\mathrm{disc}} \cap X^{\mathrm{div}}}.$$

This follows immediately for $\mathrm{wt}^{\mathrm{disc}}_{\eta_K}$ from Theorem 3.2.1.1.

By Definition 3.2.2.2 and since the inclusion $\mathscr{X}^{\mathrm{triv}} \hookrightarrow X^{\beth}$ is an open immersion, the weight function $\mathrm{wt}^{\mathrm{triv}}_{\eta}$ is the maximal lower-semicontinuous extension of $\mathrm{wt}^{\mathrm{triv}}_{\eta}|_{\mathscr{X}^{\mathrm{triv}} \cap X^{\mathrm{div}}}$. By construction, $\mathrm{wt}^{\mathrm{triv}}_{\eta}$ is $\mathbf{R}^*_+$-homogeneous, i.e. we have $\mathrm{wt}^{\mathrm{triv}}_{\eta}(a \cdot x) = a \cdot \mathrm{wt}^{\mathrm{triv}}_{\eta}(x)$ for $a \in \mathbf{R}^*_+$. By homogeneity, the restriction of $\mathrm{wt}^{\mathrm{triv}}_{\eta}$ to $\mathscr{X}^{\mathrm{disc}}$ is the maximal lower-semicontinuous extension of $\mathrm{wt}^{\mathrm{triv}}_{\eta}|_{\mathscr{X}^{\mathrm{disc}} \cap X^{\mathrm{div}}}$. This completes the proof of (4.11).

The inclusion (4.12) can be deduced from (4.11) as follows: it implies that

$$\mathrm{wt}^{\mathrm{triv}}_{\eta}(X, X_0) \leqslant \mathrm{wt}^{\mathrm{disc}}_{\eta_K}(\mathscr{X}_K).$$

By Definition 4.2.7.1, $\mathrm{Sk}(X, X_0, \eta) \cap \mathscr{X}^{\mathrm{disc}}$ consists of those $x \in \mathscr{X}_K^{\mathrm{bir}}$ such that $\mathrm{wt}_\eta^{\mathrm{triv}}(x) = \mathrm{wt}_\eta^{\mathrm{triv}}(X, X_0)$. Thus, for such an x, we have

$$\mathrm{wt}_{\eta_K}^{\mathrm{disc}}(x) = \mathrm{wt}_\eta^{\mathrm{triv}}(x) = \mathrm{wt}_\eta^{\mathrm{triv}}(X, X_0) \leqslant \mathrm{wt}_{\eta_K}^{\mathrm{disc}}(\mathscr{X}_K) \leqslant \mathrm{wt}_{\eta_K}^{\mathrm{disc}}(x),$$

and hence these are equalities. It follows that $x \in \mathrm{Sk}(\mathscr{X}_K, \eta_K)$ by [MN15, Theorem 4.7.5].

We show equality in (4.12) under the additional hypotheses that (X, X_0) is log canonical and there is a component $E \subseteq X_0$ such that $\mathrm{ord}_E(\mathrm{div}_{(X,X_0)}(\eta))$ is zero. The former assumption guarantees that $\mathrm{wt}_\eta^{\mathrm{triv}}(X, X_0) = 0$, which in turn implies that $\mathrm{wt}_\eta^{\mathrm{triv}}(\mathrm{ord}_E) = mA_{(X,X_0)}(\mathrm{ord}_E) = 0$. After rescaling ord_E, we find that there is a point $x \in \mathscr{X}^{\mathrm{disc}}$ such that $\mathrm{wt}_{\eta_K}^{\mathrm{disc}}(x) = 0$; in particular,

$$0 = \mathrm{wt}_\eta^{\mathrm{triv}}(X, X_0) \leqslant \mathrm{wt}_{\eta_K}^{\mathrm{disc}}(\mathscr{X}_K) \leqslant \mathrm{wt}_{\eta_K}^{\mathrm{disc}}(x) = 0.$$

Thus, both sides of the inclusion (4.12) consist of those $x \in (\mathscr{X}^{\mathrm{disc}})^{\mathrm{bir}}$ such that $\mathrm{wt}_{\eta_K}^{\mathrm{disc}}(x) = 0$, hence they coincide. □

Now, we have developed the tools to prove Theorem B from the introduction.

Proposition 4.2.8.3. *If $X \to C$ is projective, then there is an inclusion of essential skeletons*

$$\mathrm{Sk}^{\mathrm{ess}}(\mathscr{X}_K) \supseteq \mathrm{Sk}^{\mathrm{ess}}(X, X_0) \cap \mathscr{X}^{\mathrm{disc}}, \tag{4.14}$$

which is an equality when (X, X_0) is log canonical and $K_X + X_0$ is semiample.

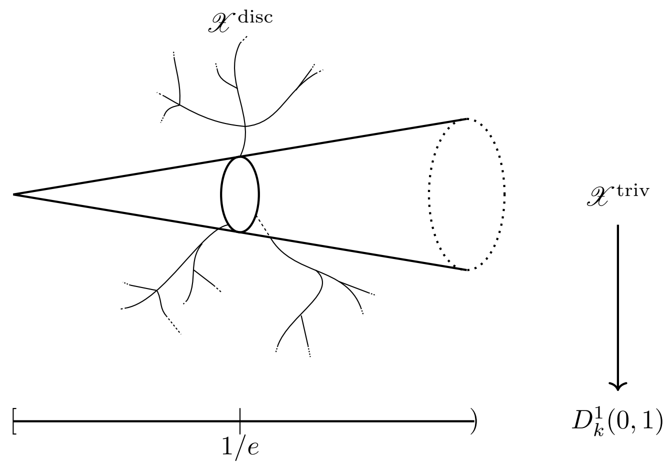

Figure 4.1: An illustration of Proposition 4.2.8.3 for a Tate elliptic curve. Consider the degeneration $\mathscr{X} := \{xyz + \varpi(x^3 + y^3 + z^3) = 0\} \subseteq \mathbf{P}^2_{\mathbf{C}[[\varpi]]}$, where $\mathbf{P}^2_{\mathbf{C}[[\varpi]]}$ has homogeneous coordinates $[x\colon y\colon z]$. The equality of (4.14) can be illustrated for $\mathscr{X}$ as above: the cone is $\mathrm{Sk}^{\mathrm{ess}}(X, X_0)$, and its intersection with the fibre $\mathscr{X}^{\mathrm{disc}}$ is a circle (that is, the essential skeleton of the Tate elliptic curve $\mathscr{X}_K^{\mathrm{an}}$).

Proof. The inclusion (4.14) is immediate from (4.12) and the fact that Kontsevich–Soibelman skeleton can be computed in terms of tensor powers of the given form. For the equality, assume now that (X, X_0) is log canonical and $K_X + X_0$ is semiample. Pick $m \in \mathbf{Z}_{>0}$ such that $m(K_X + X_0)$ is Cartier and globally generated, and pick global generators $\eta_1, \dots, \eta_N \in H^0(X, m(K_X + X_0))$ that do not vanish along all of X_0. As (X, X_0) is log canonical, [Kol13, Corollary 1.36] shows that there is a dlt pair $(X^{\mathrm{dlt}}, X_0^{\mathrm{dlt}})$, equipped with a crepant birational morphism $(X^{\mathrm{dlt}}, X_0^{\mathrm{dlt}}) \to (X, X_0)$ that is an isomorphism on the snc-locus of (X, X_0). In particular,

$$\mathscr{X}^{\mathrm{dlt}} := X^{\mathrm{dlt}} \times_C \mathrm{Spec}(\widehat{\mathcal{O}}_{C,0})$$

is a good minimal dlt model of $\mathscr{X}_K$ that dominates the model $\mathscr{X}$; this is a technical condition needed to apply the results of [NX16], and it is discussed further in Definition 6.2.6.2.

Write δ_i for the pullback of η_i to X^{dlt}, and $\delta_{i,K}$ for the restriction to the generic fibre $\mathscr{X}_K^{\mathrm{dlt}} = \mathscr{X}_K$. Since the map $(X^{\mathrm{dlt}}, X_0^{\mathrm{dlt}}) \to (X, X_0)$ is crepant, the sections

$\delta_1, \ldots, \delta_N$ of $H^0(X^{\text{dlt}}, m(K_{X^{\text{dlt}}} + X_0^{\text{dlt}}))$ are global generators for $m(K_{X^{\text{dlt}}} + X_0^{\text{dlt}})$. Then,

$$\mathrm{Sk}^{\mathrm{ess}}(\mathscr{X}_K) = \bigcup_{i=1}^{N} \mathrm{Sk}(\mathscr{X}_K^{\text{dlt}}, \delta_{i,K}) = \bigcup_{i=1}^{N} \mathrm{Sk}(\mathscr{X}_K, \eta_{i,K}), \tag{4.15}$$

where the first equality follows from [NX16, Theorem 3.3.3], and the second equality follows from [MN15, Proposition 4.7.2]. Now, observe that (4.14) and (4.15) yield inclusions

$$\begin{aligned} \mathrm{Sk}^{\mathrm{ess}}(X, X_0) \cap \mathscr{X}^{\mathrm{disc}} &\subseteq \mathrm{Sk}^{\mathrm{ess}}(\mathscr{X}_K) \\ &= \textstyle\bigcup_{i=1}^{N} \mathrm{Sk}(\mathscr{X}_K, \eta_{i,K}) \\ &= \textstyle\bigcup_{i=1}^{N} \mathrm{Sk}(X, X_0, \eta_i) \cap \mathscr{X}^{\mathrm{disc}} \\ &\subseteq \mathrm{Sk}^{\mathrm{ess}}(X, X_0) \cap \mathscr{X}^{\mathrm{disc}}, \end{aligned}$$

where the final equality follows from the case of equality in (4.12). This completes the proof. □

Remark 4.2.8.4. If (X, X_0) is not log canonical, then the equality in (4.14) does not necessarily hold. For example, take a semistable degeneration $X \to C$ of an elliptic curve to a cusp with $K_X + X_0$ trivial, such as

$$X = \{zy^2 = x^3 + \varpi z^3\} \subseteq \mathbf{P}_k^2 \times_k \mathrm{Spec}\,(k[\varpi])\,.$$

The pair (X, X_0) is not log canonical e.g. by [Kol13, Theorem 2.31], and hence $\mathrm{Sk}^{\mathrm{ess}}(X, X_0)$ is empty. However, $\mathrm{Sk}^{\mathrm{ess}}(\mathscr{X}_K)$ is the skeleton of the minimal regular model of $\mathscr{X}_K$, which is non-empty.

4.3 Closure of the skeleton of a log-regular pair

The skeleton of a log-regular model of X is a polyhedral complex in X^{bir} with (possibly) unbounded faces. The closure of the skeleton in the Berkovich analytification X^{an} has itself a decomposition into skeletons associated to the strata of the log-regular structure of X. This decomposition is treated in details in [Thu07, Proposition 3.17] in the trivially-valued setting, and the case of a toroidal embedding is mentioned in [ACP15, Example 2.4.2 and Proposition 2.6.2]. In this section, we review and extend their description for a log-regular log scheme, in order to prove analogous results for the closure of the Kontsevich–Soibelman skeletons when the residue characteristic is zero.

Let k be a field that is either trivially or discretely-valued. In the latter case, let ϖ be a uniformizer of k°. Write S^+ for the log structure on $S = \operatorname{Spec}(k^\circ)$ that is trivial when k is trivially-valued, and the divisorial log structure determined by the special fibre $\operatorname{Spec}(\widetilde{k})$ when k is discretely-valued. Let $X^+ = (X, D_{X^+})$ be a log-regular log scheme over k, and let $\mathcal{X}^+ = (\mathcal{X}, D_{\mathcal{X}^+})$ be a log-regular model of X^+ over S^+, which we understand as $\mathcal{X}^+ = X^+$ when k is trivially-valued. Let $\mathcal{X}^{\mathrm{an}}$ denote $X^\beth$ when k is trivially-valued, and $\widehat{\mathscr{X}_\eta}$ when k is discretely-valued.

Moreover, we denote by

$$D_{\mathcal{X}^+} = \textstyle\sum_{i=1}^N D_{\mathcal{X}^+,i}$$

the sum of the irreducible components of $D_{\mathcal{X}^+}$, $I_x \subseteq \{1, \ldots, N\}$ the (possibly empty) index set such that $\overline{\{x\}}$ is the irreducible component of

$$\bigcap_{i \in I_x} D_{\mathcal{X}^+,i}$$

with generic point $x \in F_{\mathcal{X}^+}$, $\mathcal{D}_x = \overline{\{x\}}$ for each $x \in F_{\mathcal{X}^+}$, and $D_x = (\mathcal{D}_x)_k$ for the generic fibre of $\mathcal{D}_x$ for each $x \in F_{X^+}$. For $x \in F_{X^+}$, let $\mathcal{D}_x^{\mathrm{an}}$ denote $D_x^\beth$ if k is trivially-valued, and the analytic generic fibre of $\mathcal{D}_x$ if k is discretely-valued. In both cases, the closed immersion $\mathcal{D}_x \hookrightarrow \mathcal{X}$ induces a closed immersion $\mathcal{D}_x^{\mathrm{an}} \hookrightarrow \mathcal{X}^{\mathrm{an}}$ on analytifications.

4.3.1 The decomposition of the closure of the skeleton

In order to decompose the closure $\overline{\mathrm{Sk}}(\mathcal{X}^+)$ into a disjoint union of skeletons associated to the strata $\mathcal{D}_y^+$ for any $y \in F_{\mathcal{X}^+}$, we endow the subscheme $\mathcal{D}_y^+$ with the log-regular structure prescribed by the following proposition.

Proposition 4.3.1.1. *Let $x, y \in F_{\mathcal{X}^+}$ be such that $x \in \overline{\{y\}}$ and consider the submonoid*

$$\mathcal{I}_y = \{f \in \mathcal{M}_{\mathcal{X}^+,x} \colon f(y) = 0\}$$

of $\mathcal{M}_{\mathcal{X}^+,x}$. Then, the log structure associated to $\mathcal{M}_{\mathcal{X}^+,x} \setminus \mathcal{I}_y \to \mathcal{O}_{\mathcal{X},x}/\mathcal{I}_y\mathcal{O}_{\mathcal{X},x}$ on the scheme $\mathrm{Spec}(\mathcal{O}_{\mathcal{X},x}/\mathcal{I}_y\mathcal{O}_{\mathcal{X},x})$ is log-regular.

Proof. This follows immediately from [Kat94, Proposition 7.2]. □

For each $y \in F_{\mathcal{X}^+}$, the log-regular structure obtained in Proposition 4.3.1.1 is called the *trace* of $\mathcal{X}^+$ on $\mathcal{D}_y$. More geometrically, the trace log structure on $\mathcal{D}_y$ is

$$(\mathcal{D}_y, \sum_{j \notin I_y} D_{\mathcal{X}^+,j}|_{\mathcal{D}_y}).$$

In particular, the Kato fan of $\mathcal{D}_y^+$ is given by

$$F_{\mathcal{D}_y^+} = F_{\mathcal{X}^+} \cap \mathcal{D}_y = \{x \in F_{\mathcal{X}^+} \colon x \in \overline{\{y\}}\},$$

and the characteristic sheaf of $\mathcal{D}_y^+$ at $x \in F_{\mathcal{D}_y^+}$ is $\mathcal{C}_{\mathcal{D}_y^+,x} = \{\gamma \in \mathcal{C}_{\mathcal{X}^+,x} \colon \gamma(y) \neq 0\}$. Thus, for any $x \in F_{\mathcal{D}_y^+}$, there is an injective monoid morphism

$$\mathcal{C}_{\mathcal{D}_y^+,x} \hookrightarrow \mathcal{C}_{\mathcal{X}^+,x}.$$

For any $y \in F_{\mathcal{X}^+}$ such that $\mathcal{D}_y^+$ dominates the base log scheme S^+, we can construct the skeleton $\mathrm{Sk}(\mathcal{D}_y^+)$. In the case when k is discretely-valued field and the scheme $\mathcal{D}_y^+$ is vertical (i.e. $\mathcal{D}_y^+$ is supported on the closed fibre of $\mathcal{X}^+$), we set $\mathrm{Sk}(\mathcal{D}_y^+) = \emptyset$.

Lemma 4.3.1.2. *For any $x \in F_{\mathcal{X}^+}$, the closure $\overline{\mathrm{Sk}}_x(\mathcal{X}^+)$ of $\mathrm{Sk}_x(\mathcal{X}^+)$ in $\mathcal{X}^{\mathrm{an}}$ coincides with the subset*

$$\mathcal{Z}_x := \begin{cases} \{v_\alpha \colon \alpha \in \mathrm{Hom}(\mathcal{C}_{\mathcal{X}^+,x}, \overline{\mathbf{R}}_+)\}, & \text{if } k \text{ is trivially-valued,} \\ \{v_\alpha \colon \alpha \in \mathrm{Hom}(\mathcal{C}_{\mathcal{X}^+,x}, \overline{\mathbf{R}}_+), \alpha(\varpi) = 1\}, & \text{if } k \text{ is discretely-valued,} \end{cases} \tag{4.16}$$

of $\mathcal{X}^{\mathrm{an}}$. In particular,

$$\overline{\mathrm{Sk}}(\mathcal{X}^+) = \bigcup_{x \in F_{\mathcal{X}^+}} \mathcal{Z}_x.$$

Proof. In the trivially-valued case, the claim coincides with Lemma 4.2.1.3. Similarly, in the discretely-valued case, it is clear that $\mathrm{Sk}_x(\mathcal{X}^+) \subseteq \mathcal{Z}_x$ and that $\mathcal{Z}_x$ is closed. By [BM17, Proposition 3.2.15], it follows that $\mathcal{Z}_x \subseteq \overline{\mathrm{Sk}}_x(\mathcal{X}^+)$, hence we conclude that $\mathcal{Z}_x$ is the closure of $\mathrm{Sk}_x(\mathcal{X}^+)$. □

Proposition 4.3.1.3. *For any $x \in F_{\mathcal{X}^+}$, the closure $\overline{\mathrm{Sk}}_x(\mathcal{X}^+)$ of $\mathrm{Sk}_x(\mathcal{X}^+)$ in $\mathcal{X}^{\mathrm{an}}$ coincides with the disjoint union*

$$\overline{\mathrm{Sk}}_x(\mathcal{X}^+) = \bigsqcup_{\substack{y \in F_{\mathcal{X}^+} \\ x \in \overline{\{y\}}}} \mathrm{Sk}_x(\mathcal{D}_y^+).$$

Proof. A valuation in the closure of $\mathrm{Sk}_x(\mathcal{X}^+)$ is of the form v_α for some morphism $\alpha \in \mathrm{Hom}(\mathcal{C}_{\mathcal{X}^+,x}, \overline{\mathbf{R}}_+)$ by Lemma 4.3.1.2. If $\mathrm{Im}(\alpha) \subseteq \mathbf{R}_+$, then $v_\alpha \in \mathrm{Sk}_x(\mathcal{X}^+)$; otherwise, the subset

$$\mathcal{I}_\alpha = \{f \in \mathcal{O}_{\mathcal{X},x} \colon v_\alpha(f) = +\infty\}.$$

is non-empty, and it forms an ideal of $\mathcal{O}_{\mathcal{X},x}$.

Claim 4.3.1.4. *There exists $y \in F_{\mathcal{X}^+}$ such that $x \in \overline{\{y\}}$ and the vanishing locus $V(\mathcal{I}_\alpha) \subseteq \mathrm{Spec}(\mathcal{O}_{\mathcal{X},x})$ of $\mathcal{I}_\alpha$ is equal to $\overline{\{y\}}$.*

Proof of Claim 4.3.1.4. First, observe that

$$V(\mathcal{I}_\alpha) = \bigcap_{f \in \mathcal{I}_\alpha} V(f) = \bigcap_{f \in \mathcal{I}_\alpha \cap \mathcal{M}_{\mathcal{X}^+,x}} V(f).$$

Thus, it suffices to prove that

$$\bigcap_{f\in\mathcal{I}_\alpha} V(f) \supseteq \bigcap_{f\in\mathcal{I}_\alpha\cap\mathcal{M}_{\mathcal{X}^+,x}} V(f).$$

Let $f \in \mathcal{I}_\alpha$, then any admissible expansion $\sum_{\gamma\in\mathcal{C}_{\mathcal{X}^+,x}} c_\gamma\gamma$ is such that, if $c_\gamma \neq 0$, then $\gamma \in \mathcal{I}_\alpha$. Therefore, for any $f \in \mathcal{I}_\alpha$

$$\bigcap_{\gamma\in\mathcal{I}_\alpha\cap\mathcal{M}_{\mathcal{X}^+,x}} V(\gamma) \subseteq \bigcap_{\gamma:\, c_\gamma\neq 0 \text{ in } f=\sum c_\gamma\gamma} V(\gamma) \subseteq V(f)$$

and we obtain the required inclusion. Therefore, $V(\mathcal{I}_\alpha)$ is a stratum of $D_{\mathcal{X}^+}$, since $\mathcal{I}_\alpha$ is a prime ideal and $V(f)$ is the union of irreducible components of $D_{\mathcal{X}^+}$ for any $f \in \mathcal{I}_\alpha \cap \mathcal{M}_{\mathcal{X}^+,x}$. By definition of a Kato point, we have that

$$\overline{\{x\}} = \bigcap_{f\in\mathcal{M}_{\mathcal{X}^+,x}\setminus\mathcal{O}^\times_{\mathcal{X},x}} V(f) \subseteq \bigcap_{\gamma\in\mathcal{I}_\alpha\cap\mathcal{M}_{\mathcal{X}^+,x}} V(\gamma) = \overline{\{y\}},$$

hence we conclude that $x \in \overline{\{y\}}$. This completes the proof of the claim. □

Now, let $y \in F_{\mathcal{X}^+}$ be such that $x \in \overline{\{y\}}$ and $V(\mathcal{I}_\alpha) = \overline{\{y\}}$. Any element γ of $\mathcal{C}_{\mathcal{X}^+,x}$ satisfies $\gamma \in \mathcal{C}_{\mathcal{D}^+_y,x}$ if and only if $\gamma(y) \neq 0$, or equivalently $\gamma \notin \mathcal{I}_\alpha$. Thus, the restriction α_y of the morphism α to $\mathcal{C}_{\mathcal{D}^+_y,x}$ does not attain the value $+\infty$. To such a morphism we associate a valuation v_{α_y} that, by construction, lies in the skeleton $\mathrm{Sk}_x(\mathcal{D}^+_y)$. Therefore, by restriction of morphisms, we obtain an injective map

$$\overline{\mathrm{Sk}}_x(\mathcal{X}^+) \hookrightarrow \bigsqcup_{y\in F_{\mathcal{X}^+}\,:\, x\in\overline{\{y\}}} \mathrm{Sk}_x(\mathcal{D}^+_y).$$

It remains to show the surjectivity of this map. Given a valuation $v_\beta \in \mathrm{Sk}_x(\mathcal{D}^+_y)$ for some $y \in F_{\mathcal{X}^+}$ with $x \in \overline{\{y\}}$ and $\beta \in \mathrm{Hom}(\mathcal{C}_{\mathcal{D}^+_y,x}, \mathbf{R}_+)$, we can extend β to a morphism $\widetilde{\beta}$ on $\mathcal{C}_{\mathcal{X}^+,x}$ by

$$\widetilde{\beta}(\gamma) := \begin{cases} \beta(\gamma), & \gamma \in \mathcal{C}_{\mathcal{D}^+_y,x}, \\ +\infty, & \text{otherwise.} \end{cases}$$

The associated valuation $v_{\widetilde{\beta}}$ lies in the closure $\overline{\mathrm{Sk}}_x(\mathcal{X}^+)$ in $\mathcal{X}^{\mathrm{an}}$, therefore we get

$$\bigsqcup_{y \in F_{\mathcal{X}^+}\,:\, x \in \overline{\{y\}}} \mathrm{Sk}_x(\mathcal{D}_y^+) \hookrightarrow \overline{\mathrm{Sk}}_x(\mathcal{X}^+).$$

The two maps are inverse of each other by construction, which completes the proof.

□

Proposition 4.3.1.5. *The closure of the skeleton* $\mathrm{Sk}(\mathcal{X}^+)$ *in* $\mathcal{X}^{\mathrm{an}}$ *admits the decomposition*

$$\overline{\mathrm{Sk}}(\mathcal{X}^+) = \bigsqcup_{y \in F_{\mathcal{X}^+}} \mathrm{Sk}(\mathcal{D}_y^+),$$

where $\mathrm{Sk}(\mathcal{D}_y^+)$ *is viewed as a subset of* $\mathcal{X}^{\mathrm{an}}$ *by the inclusion* $\mathcal{D}_y^{\mathrm{an}} \hookrightarrow \mathcal{X}^{\mathrm{an}}$*. Further,*

$$\mathrm{Sk}(\mathcal{D}_y^+) = \overline{\mathrm{Sk}}(\mathcal{X}^+) \cap \ker^{-1}(y).$$

Proof. From Proposition 4.3.1.3, it follows that

$$\begin{aligned}
\overline{\mathrm{Sk}}(\mathcal{X}^+) = \bigcup_{x \in F_{\mathcal{X}^+}} \overline{\mathrm{Sk}}_x(\mathcal{X}^+) &= \bigcup_{x \in F_{\mathcal{X}^+}} \bigsqcup_{y \in F_{\mathcal{X}^+}\,:\, x \in \overline{\{y\}}} \mathrm{Sk}_x(\mathcal{D}_y^+) \\
&= \bigsqcup_{y \in F_{\mathcal{X}^+}} \bigcup_{x \in F_{\mathcal{D}_y^+}} \mathrm{Sk}_x(\mathcal{D}_y^+) \\
&= \bigsqcup_{y \in F_{\mathcal{X}^+}} \mathrm{Sk}(\mathcal{D}_y^+).
\end{aligned}$$

For any $y \in F_{\mathcal{X}^+}$, if the skeleton $\mathrm{Sk}(\mathcal{D}_y^+)$ is non-empty, then it consists of birational points of $\mathcal{D}_y^{\mathrm{an}}$, hence of valuations whose image via the kernel map is the generic point of $\mathcal{D}_y$, thus y. Therefore, the kernel map distinguishes the different part of the disjoint union in $\overline{\mathrm{Sk}}(\mathcal{X}^+)$. □

4.3.2 The case of the toric varieties

Let M be a finitely-generated free abelian group, $N = \mathrm{Hom}(M, \mathbf{Z})$ be the cocharacter lattice, and $\langle \cdot, \cdot \rangle \colon M \times N \to \mathbf{Z}$ be the evaluation pairing. Set $M_{\mathbf{R}} \coloneqq M \otimes_{\mathbf{Z}} \mathbf{R}$

and $N_{\mathbf{R}} \coloneqq N \otimes_{\mathbf{Z}} \mathbf{R}$. Let Σ be a rational polyhedral fan in $N_{\mathbf{R}}$. Given a cone $\sigma \in \Sigma$, consider the monoid $S_\sigma \coloneqq \sigma^\vee \cap M$.

Let X_Σ (resp. $\mathcal{X}_\Sigma$) be the normal toric variety over k (resp. model over S) associated to the fan Σ. Let D_Σ denote the (reduced) toric boundary divisor of X_Σ. Write $D_{\mathcal{X}^+}$ for D_Σ when k is trivially-valued, or for $\overline{D_\Sigma} + (\mathcal{X}_\Sigma)_{0,\mathrm{red}}$ when k is discretely-valued. The log scheme $\mathcal{X}^+ = (\mathcal{X}_\Sigma, D_{\mathcal{X}^+})$ is log-regular, and the goal of this section is to describe the closure in $\mathcal{X}_\Sigma^{\mathrm{an}}$ of the essential skeleton of (X_Σ, D_Σ).

The support of the fan Σ admits a natural compactification $\overline{\Sigma}$, as in [Pay09, §3] and [Rab12, Proposition 3.4]; see also [ACM+15, §7.2]. The construction is reviewed below. Given a cone $\sigma \in \Sigma$, we denote by $\overline{\sigma} \coloneqq \mathrm{Hom}(S_\sigma, \overline{\mathbf{R}}_+)$, equipped with the topology of pointwise convergence. The space $\overline{\sigma}$ admits a locally closed stratification by the quotient monoids σ/σ', for each face $\sigma' \preccurlyeq \sigma$, where the embedding $\sigma/\sigma' \hookrightarrow \overline{\sigma}$ is given by

$$u + \sigma' \mapsto \left[m \mapsto \begin{cases} \langle m, u \rangle, & m \in \sigma'^{\perp}, \\ +\infty, & \text{otherwise}, \end{cases} \right] \tag{4.17}$$

for $u \in \sigma$. For example, the natural inclusion $\sigma \hookrightarrow \overline{\sigma}$ coincides with the embedding $\sigma/\sigma' \hookrightarrow \overline{\sigma}$ associated to the face $\sigma' = 0$. If $\tau \preccurlyeq \sigma$, then the natural map $S_\sigma \to S_\tau$ induces an open embedding $\overline{\tau} \hookrightarrow \overline{\sigma}$; moreover, if $\sigma' \preccurlyeq \tau \preccurlyeq \sigma$, then the embedding $\overline{\tau} \hookrightarrow \overline{\sigma}$ restricts to the natural inclusion $\tau/\sigma' \hookrightarrow \sigma/\sigma'$. Consequently, the cones $\{\overline{\sigma} \colon \sigma \in \Sigma\}$ glue to give an extended cone

$$\overline{\Sigma} \coloneqq \bigcup_{\sigma \in \Sigma} \overline{\sigma} = \bigsqcup_{\sigma \in \Sigma} \bigcup_{\sigma \preccurlyeq \tau} \tau/\sigma, \tag{4.18}$$

which is a compactification of the support of Σ in $N \otimes_{\mathbf{Z}} \overline{\mathbf{R}}$. In [Thu07, §2], Thuillier constructs an embedding J_Σ of $\overline{\Sigma}$ into $\mathcal{X}_\Sigma^{\mathrm{an}}$, as well as a strong deformation retraction of $\mathcal{X}_\Sigma^{\mathrm{an}}$ onto the image of the embedding. The work of [Thu07] is over a trivially-valued field, but these constructions in fact hold over any field, as pointed out in [ACM$^+$15, Proposition 7.6]. The image of J_Σ in $\mathcal{X}_\Sigma^{\mathrm{an}}$ is called the *toric skeleton* of $\mathcal{X}_\Sigma$. Note that both the embedding J_Σ and the toric boundary D_Σ are completely determined by the choice of dense torus in X_Σ.

For any cone $\sigma \in \Sigma$, the cones τ that contain σ as a face form a fan in $N/\langle\sigma\rangle \otimes_{\mathbf{Z}} \mathbf{R}$, whose associated toric S-scheme is the orbit closure $\mathcal{V}(\sigma)$ corresponding to the cone σ. Further, the subscheme $\mathcal{V}(\sigma)$ is a stratum of $D_{\mathcal{X}^+}$, so it can be endowed with the trace log structure $\mathcal{V}(\sigma)^+ = (\mathcal{V}(\sigma), D_{\mathcal{V}(\sigma)^+})$ as in §4.3.1. The stratification of the skeleton of $\mathcal{V}(\sigma)^+$ is related to the decomposition (4.18) by the embedding J_Σ, as demonstrated below.

Proposition 4.3.2.1. *For any cone σ of Σ, J_Σ restricts to a homeomorphism*

$$\bigcup_{\sigma \preccurlyeq \tau} \tau/\sigma \simeq \mathrm{Sk}(\mathcal{V}(\sigma)^+).$$

Proof. This follows from [Thu07, Proposition 2.13], [Uli17, Theorem 1.2 and §3.4], and Proposition 4.2.1.1. □

Now, we show that the toric skeleton can be realized as the closure of an essential skeleton, which was stated as Theorem D in the introduction.

Corollary 4.3.2.2. *Assume* $\mathrm{char}(\widetilde{k}) = 0$. *The closure of the essential skeleton of* (X_Σ, D_Σ) *in* $\mathcal{X}_\Sigma^{\mathrm{an}}$ *coincides with the toric skeleton; that is,*

$$J_\Sigma \colon \overline{\Sigma} = \bigsqcup_{\sigma\in\Sigma} \bigcup_{\sigma \preccurlyeq \tau} \tau/\sigma \xrightarrow{\simeq} \bigsqcup_{\sigma\in\Sigma} \mathrm{Sk}(\mathcal{V}(\sigma)^+) = \overline{\mathrm{Sk}}(\mathcal{X}^+) = \overline{\mathrm{Sk}}^{\mathrm{ess}}(X_\Sigma, D_\Sigma).$$

Proof. The homeomorphism between $\overline{\Sigma}$ and $\overline{\mathrm{Sk}}(\mathcal{X}^+)$ follows immediately from Corollary 4.3.1.5 and Corollary 4.3.2.1. As a toric variety with its toric boundary defines a logCY pair, the last homeomorphism will follow from Proposition 6.2.4.1 in the trivially-valued field case, and applying [BM17, Lemma 5.1.2] in the discretely-valued field case. □

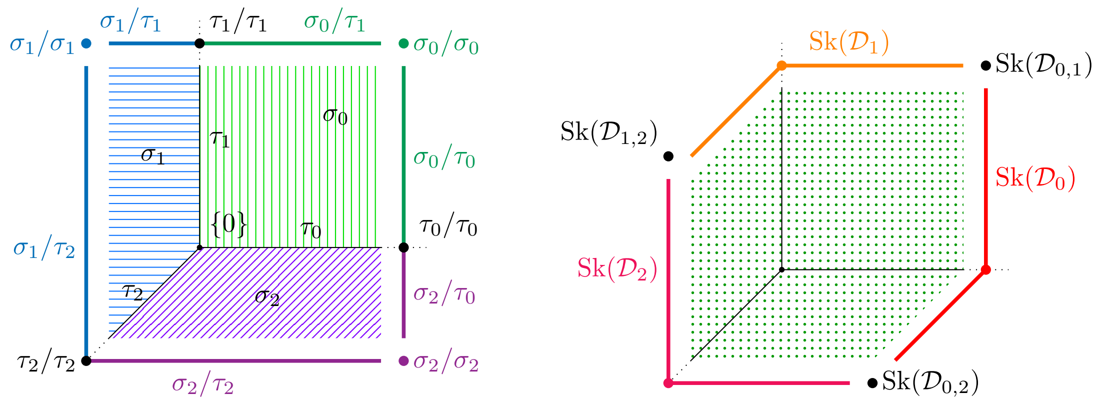

Figure 4.2: A comparison of the decomposition of the extended fan and the closure of the skeleton for a model of $\mathbf{P}^2$. Let Σ be the usual fan in $\mathbf{R}^2$ associated to the k°-toric variety $\mathcal{X}_\Sigma = \mathbf{P}^2_{k^\circ}$. In the picture on the left, we have the compactification $\overline{\Sigma}$ and its decomposition as in (4.18). In the picture on the right, there is the stratification of $\overline{\mathrm{Sk}}(\mathcal{X}^+)$ from Proposition 4.3.1.5.

4.3.3 The closure of a Kontsevich–Soibelman skeleton

Assume $\mathrm{char}(\widetilde{k}) = 0$. Let (X, D) be a pair over k such that $D = \sum_i a_i D_i$ is a $\mathbf{Q}$-boundary divisor with snc support, so the log scheme $X^+ = (X, D_{\mathrm{red}})$ is log-regular. Assume $K_X + D_{\mathrm{red}}$ is $\mathbf{Q}$-Cartier. Let $\mathcal{X}$ be an snc model of X over S. We set $D_{\mathcal{X}^+} = D_{\mathrm{red}}$ if k is trivially-valued, and $D_{\mathcal{X}^+} = \overline{D}_{\mathrm{red}} + \mathcal{X}_{0,\mathrm{red}}$ if k is discretely-valued. Then, $\mathcal{X}^+ = (\mathcal{X}, D_{\mathcal{X}^+} = \sum_i D_{\mathcal{X}^+,i})$ is a log-regular model of (X, D_{red}) over S^+.

Taking advantage of the decomposition of the closure of the skeleton $\mathrm{Sk}(\mathcal{X}^+)$ described in Proposition 4.3.1.5, we study the closure of Kontsevich–Soibelman skeletons. More precisely, in Proposition 4.3.3.1, we show that for any non-zero D-logarithmic pluricanonical form η, the valuations in the complement

$$\overline{\mathrm{Sk}}(X, D, \eta) \setminus \mathrm{Sk}(X, D, \eta)$$

are minima of weight functions associated to suitable forms on the strata of $D_{X^+}(= D_{\mathrm{red}} = \sum_i D_i)$. In the trivially-valued setting, this characterization can be made even more precise (Proposition 4.3.3.2).

As we assume that the divisor $D_{\mathcal{X}^+}$ is snc, the characteristic monoid $\mathcal{C}_{\mathcal{X}^+,x}$ at any Kato point x of $\mathcal{X}^+$ is a free monoid isomorphic to $\mathbf{N}^{|I_x|}$, where the isomorphism is determined by choosing local equations z_i of the components $D_{\mathcal{X}^+,i}$ at x. In this case, any $f \in \mathcal{O}_{\mathcal{X},x}$ at x has an admissible expansion of the form

$$f = \sum_{\gamma \in \mathbf{Z}_{\geqslant 0}^{|I_x|}} c_\gamma z^\gamma,$$

in the completion $\widehat{\mathcal{O}}_{\mathcal{X},x}$, where $c_\gamma \in \{0\} \cup \mathcal{O}_{\mathcal{X},x}^\times$.

Let η be a non-zero regular D-logarithmic m-pluricanonical form on X, and let x be the generic point of an irreducible component D_i of D. The *residue* $\mathrm{Res}_{D_i}(\eta)$ of the form η along D_i is a (possibly zero) regular $\left(\sum_{j\neq i} a_j D_j\right)|_{D_i}$-logarithmic m-pluricanonical form on D_i, whose local description we review below. If the divisor D_i is locally defined at x by the equation $z_i = 0$, then η can locally be written at x as

$$\eta = \left(\frac{dz_i}{z_i}\right)^{\otimes m} \wedge \mu$$

for some local section μ of $\bigwedge^{n-1}(\Omega^1_{X/\mathcal{K}}(\log D))^{\otimes m}$, where n is the relative dimension of X over $\mathcal{K}$. The form $\mathrm{Res}_{D_i}(\eta)$ is a global section in $H^0\Big(D_i, \omega^{\otimes m}_{(D_i, \sum_{j\neq i} a_j D_j|_{D_i})}\Big)$ that is locally given by the restriction $\mu|_{D_i}$.

For a general Kato point $x \in F_{X^+}$, D_x is a stratum of D, i.e. a component of an intersection of $\{D_i \colon i \in I_x\}$, so we can iterate the above construction; that is, if z_i is a local equation of the component D_i at x for each $i \in I_x$, then write

$$\eta = \bigwedge_{i \in I_x} \left(\frac{dz_i}{z_i}\right)^{\otimes m} \wedge \mu$$

for some local section μ of $\bigwedge^{n-|I_x|}(\Omega^1_{X/\mathcal{K}}(\log D))^{\otimes m}$. The form $\mathrm{Res}_{D_x}(\eta)$ on D_x is locally given by $\eta\big|_{D_x}$. See [EV92, §2] for further details.

This leads us to the following result, which was stated as Theorem C in the introduction.

Proposition 4.3.3.1. *Under the same assumptions on (X, D), if η is a non-zero regular D-logarithmic pluricanonical form on X and $x \in F_{X^+}$, then there is an inclusion*

$$\overline{\mathrm{Sk}}(X, D, \eta) \cap \ker^{-1}(x) \subseteq \mathrm{Sk}(D_x, \textstyle\sum_{j\notin I_x} a_j D_j|_{D_x}, \mathrm{Res}_{D_x}(\eta)).$$

Proof. By Proposition 4.2.7.2 and Proposition 4.3.1.5,

$$\overline{\mathrm{Sk}}(X, D, \eta) \cap \ker^{-1}(x) \subseteq \overline{\mathrm{Sk}}(\mathcal{X}^+) \cap \ker^{-1}(x) = \mathrm{Sk}(\mathcal{D}_x^+) = \bigcup_{y\in F_{\mathcal{D}_x^+}} \mathrm{Sk}_y(\mathcal{D}_x^+),$$

$$\mathrm{Sk}(D_x, \textstyle\sum_{j\notin I_x} a_j D_j|_{D_x}, \mathrm{Res}_{D_x}(\eta)) \subseteq \mathrm{Sk}(\mathcal{D}_x^+) = \bigcup_{y\in F_{\mathcal{D}_x^+}} \mathrm{Sk}_y(\mathcal{D}_x^+).$$

Therefore, we may prove the desired inclusion for a valuation that lies in $\mathrm{Sk}_y(\mathcal{D}_x^+)$, for some $y \in F_{\mathcal{D}_x^+}$. In order to relate the closure of the Kontsevich–Soibelman skeleton $\mathrm{Sk}(X, D, \eta)$ to the Kontsevich–Soibelman skeleton $\mathrm{Sk}(D_x, \sum_{j\notin I_x} a_j D_j|_{D_x}, \mathrm{Res}_{D_x}(\eta))$, we will compute explicitely the associated weight functions on the faces $\mathrm{Sk}_y(\mathcal{X}^+)$ and $\mathrm{Sk}_y(\mathcal{D}_x^+)$. To that end, recall that the forms η and $\mathrm{Res}_{D_x}(\eta)$ respectively induce divisors $\mathrm{div}_{(\mathcal{X}, D_{\mathcal{X}^+})}(\eta)$ and $\mathrm{div}_{(\mathcal{D}_x, D_{\mathcal{D}_x^+})}(\mathrm{Res}_{D_x}(\eta))$ on $\mathcal{X}$ and $\mathcal{D}_x$ when k is trivially-valued, and $\mathrm{div}_{(\mathcal{X}, D_{\mathcal{X}^+}-\mathrm{div}(\pi))}(\eta)$ and $\mathrm{div}_{(\mathcal{D}_x, D_{\mathcal{D}_x^+}-\mathrm{div}(\pi))}(\mathrm{Res}_{D_x}(\eta))$ in the discretely-valued case. To compute the weight functions wt_η and $\mathrm{wt}_{\mathrm{Res}_{D_x}(\eta)}$ on $\mathrm{Sk}_y(\mathcal{X}^+)$ and $\mathrm{Sk}_y(\mathcal{D}_x^+)$, we first determine local equations for these divisors at y.

The Kato point $y \in F_{\mathcal{D}_x^+}$ satisfies $y \in \overline{\{x\}}$ and $I_x \subseteq I_y$. Let $z_1, \ldots, z_n$ be local parameters at y such that z_i is a local equation of $D_{\mathcal{X}^+,i}$ for each $i \in I_y$. Then, the forms η and $\operatorname{Res}_{D_x}(\eta)$ can be written locally at y as

$$\begin{aligned}
\eta &= fg^{-1} \cdot g \bigwedge_{i \in I_x} \left(\frac{dz_i}{z_i}\right)^{\otimes m} \bigwedge_{j \in I_y \setminus I_x} \left(\frac{dz_j}{z_j}\right)^{\otimes m} \bigwedge_{h \notin I_y} dz_h^{\otimes m}, \\
\operatorname{Res}_{D_x}(\eta) &= \left(fg^{-1} \cdot g \bigwedge_{j \in I_y \setminus I_x} \left(\frac{dz_j}{z_j}\right)^{\otimes m} \bigwedge_{h \notin I_y} dz_h^{\otimes m}\right)\Big|_{D_x} \\
&= fg^{-1}\big|_{D_x} \cdot g \bigwedge_{j \in I_y \setminus I_x} \left(\frac{dz_j}{z_j}\right)^{\otimes m} \bigwedge_{h \notin I_y} dz_h^{\otimes m}
\end{aligned}$$

for some $f \in \mathcal{O}_{\mathcal{X},y}$, with $g = \varpi^m$ if k is discretely-valued and $g \in \mathcal{O}_{\mathcal{X},y}^\times$ if k is trivially-valued. Thus, fg^{-1} and $fg^{-1}\big|_{D_x}$ are the required local equations at y. An admissible expansion of f in $\widehat{\mathcal{O}}_{\mathcal{X},y}$ can be decomposed as

$$f = \sum_{\gamma \in \mathbf{Z}_{\geqslant 0}^{|I_y|}} c_\gamma z^\gamma = \sum_{\gamma :\, \exists i \in I_x, \gamma_i \neq 0} c_\gamma z^\gamma + \sum_{\gamma :\, \forall i \in I_x, \gamma_i = 0} c_\gamma z^\gamma, \tag{4.19}$$

with $c_\gamma \in \{0\} \cup \mathcal{O}_{\mathcal{X},y}^\times$. It follows that

$$f\big|_{D_x} = \sum_{\gamma :\, \forall i \in I_x, \gamma_i = 0} c_\gamma\big|_{D_x} z^\gamma \tag{4.20}$$

in $\widehat{\mathcal{O}}_{D_x,y}$ and $c_\gamma\big|_{D_x} \in \{0\} \cup \mathcal{O}_{D_x,y}^\times$, so (4.20) is an admissible expansion of $f\big|_{D_x}$ at y. Thus, for valuations $v_\alpha \in \operatorname{Sk}_y(\mathcal{X}^+)$ and $v_\beta \in \operatorname{Sk}_y(\mathcal{D}_x^+)$, we have

$$\begin{aligned}
\operatorname{wt}_\eta(v_\alpha) = v_\alpha(f) &= \min\left\{ \min_{\gamma :\, \exists i \in I_x, \gamma_i \neq 0} \{v_\alpha(c_\gamma) + \alpha(z^\gamma)\}, \min_{\gamma :\, \forall i \in I_x, \gamma_i = 0} \{v_\alpha(c_\gamma) + \alpha(z^\gamma)\} \right\}, \\
\operatorname{wt}_{\operatorname{Res}_{D_x}(\eta)}(v_\beta) = v_\beta(f\big|_{D_x}) &= \min_{\gamma :\, \forall i \in I_x, \gamma_i = 0} \{v_\beta(c_\gamma) + \beta(z^\gamma)\}.
\end{aligned} \tag{4.21}$$

Due to 4.21 the weight function wt_η extends to a continuous function on $\overline{\operatorname{Sk}}_y(\mathcal{X}^+)$ and restricts to $\operatorname{wt}_{\operatorname{Res}_{D_x}(\eta)}$ on $\operatorname{Sk}_y(\mathcal{D}_x^+)$. Indeed, if $v_\beta \in \operatorname{Sk}_y(\mathcal{D}_x^+)$, then $\beta(z^\gamma) = +\infty$

for any γ such that $\gamma_i \neq 0$ for some $i \in I_x$. As a result we have that

$$\begin{aligned}
\mathrm{wt}_\eta(v_\beta) &= v_\beta(f) \\
&= \min\left\{ \min_{\gamma:\ \exists i \in I_x, \gamma_i \neq 0} \{v_\beta(c_\gamma) + \beta(z^\gamma)\}, \min_{\gamma:\ \forall i \in I_x, \gamma_i = 0} \{v_\beta(c_\gamma) + \beta(z^\gamma)\} \right\} \\
&= \min_{\gamma:\ \forall i \in I_x, \gamma_i = 0} \{v_\beta(c_\gamma) + \beta(z^\gamma)\} \\
&= v_\beta(f\big|_{D_x}) = \mathrm{wt}_{\mathrm{Res}_{D_x}(\eta)}(v_\beta).
\end{aligned}$$

The minimality locus of wt_η along $\mathrm{Sk}_y(\mathcal{D}_x^+)$ are contained in the minimality locus of $\mathrm{wt}_{\mathrm{Res}_{D_x}(\eta)}$. Hence, we conclude that

$$\overline{\mathrm{Sk}}(X, D, \eta) \cap \ker^{-1}(x) \subseteq \mathrm{Sk}(D_x, \textstyle\sum_{j \notin I_x} a_j D_j|_{D_x}, \mathrm{Res}_{D_x}(\eta)).$$

□

When k is a trivially-valued field, the inclusion of Proposition 4.3.3.1 is in fact an equality.

Proposition 4.3.3.2. *Under the same assumptions on (X, D), suppose that k is trivially-valued. If η is a non-zero regular D-logarithmic pluricanonical form on X and $x \in F_{X^+}$, then*

$$\overline{\mathrm{Sk}}(X, D, \eta) \cap \ker^{-1}(x) = \mathrm{Sk}(D_x, \textstyle\sum_{j \notin I_x} a_j D_j|_{D_x}, \mathrm{Res}_{D_x}(\eta)).$$

Proof. Under the same assumption and notation of the proof of Proposition 4.3.3.1, consider $v_\beta \in \mathrm{Sk}_y(\mathcal{D}_x^+)$, where β is a morphism on $\mathcal{C}_{\mathcal{D}_x^+, y} \simeq \mathbf{N}^{|I_y \setminus I_x|}$. We construct a sequence of valuations $(v_{\alpha_n})_{n=1}^\infty$ in $\mathrm{Sk}_y(\mathcal{X}^+)$ that converge to v_β as follows:

$$\begin{cases} \alpha_n(z_i) = \beta(z_i), & i \in I_y \setminus I_x, \\ \alpha_n(z_i) = n, & i \in I_x. \end{cases}$$

We have $\alpha_n(z_i) \to +\infty$ as $n \to +\infty$ for any $i \in I_x$; moreover, for sufficiently large n, $v_{\alpha_n}(f)$ can be written as

$$\begin{aligned} v_{\alpha_n}(f) &= \min_{\gamma:\,\forall i \in I_x, \gamma_i = 0} \{v_{\alpha_n}(c_\gamma) + \alpha_n(z^\gamma)\} \\ &= \min_\gamma \alpha_n(z^\gamma) \\ &= \min_\gamma \beta(z^\gamma) = v_\beta(f\big|_{D_x}), \end{aligned} \tag{4.22}$$

where the two right-hand minima range over all $\gamma \in \mathbf{Z}_{\geqslant 0}^{|I_y|}$ such that $c_\gamma \neq 0$ and such that for all $i \in I_x$, $\gamma_i = 0$. Thus, given any valuation v_β in $\mathrm{Sk}_y(\mathcal{D}_x^+)$, we can construct a sequence of valuations in $\mathrm{Sk}_y(\mathcal{X}^+)$ that converge to v_β, and moreover by (4.21) attaining the same weight with respect to $\mathrm{Res}_{D_x}(\eta)$ and η.

Assume now that $v_\beta \in \mathrm{Sk}(D_x, \sum_{j \notin I_x} a_j D_j|_{D_x}, \mathrm{Res}_{D_x}(\eta)) \cap \mathrm{Sk}_y(\mathcal{D}_x^+)$ and consider a sequence $(v_{\alpha_n})_{n=1}^\infty$ in $\mathrm{Sk}_y(\mathcal{X}^+)$ converging to v_β, as above. The minimal weight with respect to either form is zero and we know that

$$\mathrm{wt}_\eta(v_{\alpha_n}) = v_{\alpha_n}(f) = v_\beta(f|_{D_x}) = \mathrm{wt}_{\mathrm{Res}_{D_x}(\eta)}(v_\beta) = 0$$

by (4.21) and (4.22). It follows that $v_{\alpha_n} \in \mathrm{Sk}(X, D, \eta)$ for all n sufficiently large. In other words, any valuation in the Kontsevich–Soibelman skeleton

$$\mathrm{Sk}(D_x, \textstyle\sum_{j \notin I_x} a_j D_j|_{D_x}, \mathrm{Res}_{D_x}(\eta))$$

is the accumulation point of a sequence of valuations in $\mathrm{Sk}(X, D, \eta)$, hence we have the inclusion of $\mathrm{Sk}(D_x, \sum_{j \notin I_x} a_j D_j|_{D_x}, \mathrm{Res}_{D_x}(\eta))$ in $\overline{\mathrm{Sk}}(X, D, \eta) \cap \ker^{-1}(x)$, as required. $\square$

Example 4.3.3.3. Over a discretely-valued field k, the inclusion of Proposition 4.3.3.1 may be strict. Indeed, consider the k°-scheme

$$\mathscr{X} = \mathrm{Spec}\left(\frac{k^\circ[T_1, T_2, T_3]}{(\varpi - T_1^2 T_2 T_3)}\right),$$

and let D_i be the reduced vertical divisor on $\mathscr{X}$ given by the equation $T_i = 0$, for $i = 1, 2, 3$. Let D_4 be the horizontal divisor on $\mathscr{X}$ given by the equation $T_1 - a$, for some $a \in k^\circ \backslash \{0\}$. Consider the log scheme $\mathscr{X}^+ = (\mathscr{X}, D_{\mathscr{X}^+})$ with the divisorial log structure given by $D_{\mathscr{X}^+} = \sum_{i=1}^4 D_i$. In Figure 4.3 below, we describe the closure of $\mathrm{Sk}(\mathscr{X}^+)$, as well as the decomposition $\overline{\mathrm{Sk}}(\mathscr{X}^+) = \mathrm{Sk}(\mathscr{X}^+) \bigsqcup \mathrm{Sk}(D_4^+)$ from Proposition 4.3.1.5. There, the face of $\overline{\mathrm{Sk}}(\mathscr{X}^+)$ corresponding to the generic point of the intersection $\bigcap_{i \in J} D_i$, for $J \subseteq \{1, \ldots, 4\}$, is denoted by $x_{\cap_{i \in J} D_i}$.

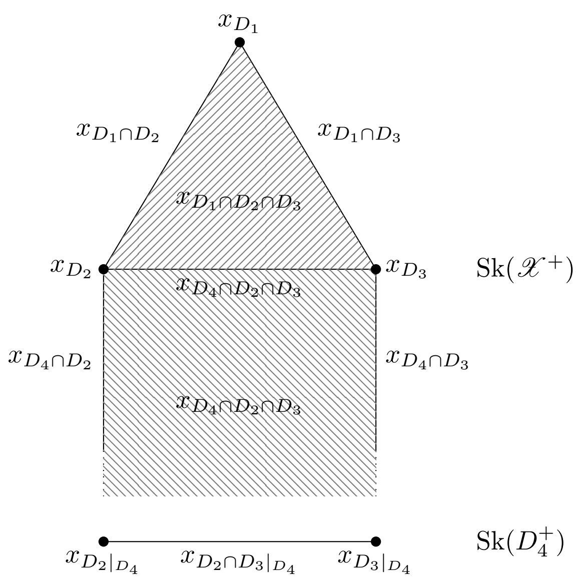

Figure 4.3: An example illustrating that the inclusion of Proposition 4.3.3.1 may be strict. The figure shows the closure $\overline{\mathrm{Sk}}(\mathscr{X}^+)$ in $\widehat{\mathscr{X}_\eta}$ of the skeleton of $\mathscr{X}^+$.

Consider the logarithmic canonical forms given by

$$\begin{cases} \eta = \frac{T_1^2 T_2^2 T_3^2}{T_1 - a} \frac{dT_2}{T_2} \wedge \frac{dT_3}{T_3} = 2 \frac{T_1^2 T_2^2 T_3^2}{T_1 - a} \frac{dT_1}{T_1} \wedge \frac{dT_3}{T_3} = -2 \frac{T_1^2 T_2^2 T_3^2}{T_1 - a} \frac{dT_1}{T_1} \wedge \frac{dT_2}{T_2}, \\ \mathrm{Res}_{D_4}(\eta) = 2a T_2^2 T_3^2 \frac{dT_3}{T_3} = -2a T_2^2 T_3^2 \frac{dT_2}{T_2}. \end{cases}$$

The Kontsevich–Soibelman skeleton of η is $\mathrm{Sk}(\mathscr{X}_k, (D_4)_k, \eta) = \{v_{D_1}\}$, as

$$\mathrm{wt}_\eta(v_{D_i}) = v_{D_i}\left(\frac{T_1^2 T_2^2 T_3^2}{T_1 - a}\right) + 1 = \begin{cases} 2, & \text{for } i = 1, \\ 3, & \text{for } i = 2, 3. \end{cases}$$

where $\mathrm{wt}_\eta(v_{D_1}) = 2$ since the multiplicity of D_1 is 2. However, the Kontsevich–Soibelman skeleton of $\mathrm{Res}_{D_4}(\eta)$ is the skeleton $\mathrm{Sk}(D_4^+)$. It follows that

$$\overline{\mathrm{Sk}}(\mathscr{X}_k, (D_4)_k, \eta) \cap \ker^{-1}(x_{D_4}) = \emptyset \quad \subsetneq \quad \mathrm{Sk}(D_4^+) = \mathrm{Sk}((D_4)_k, \emptyset, \mathrm{Res}_{D_4}(\eta)).$$

Thus, the inclusion of Proposition 4.3.3.1 may be strict.

CHAPTER V

A non-Archimedean Ohsawa–Takegoshi extension theorem

Let k be a non-Archimedean field and let $K := \widehat{k^a}$ denote its completed algebraic closure.

5.1 The structure of the Berkovich unit disc

In this section, we review a variety of features of the Berkovich unit disc over k: the classification of points, the metric structure, Temkin's metric on the disc, and the theory of quasisubharmonic functions on the disc.

5.1.1 The Berkovich unit disc

The *Tate algebra* $k\{T\}$ in the variable T is the k-subalgebra of $k[[T]]$ consisting of those power series $f = \sum_{i=0}^{\infty} a_i T^i$, with $a_i \in k$, such that $|a_i| \to 0$ as $i \to \infty$. The Tate algebra is a Banach k-algebra when equipped with the *Gauss norm*

$$|f(x_G)| := \max_{i \geq 0} |a_i|.$$

See [BGR84, §5.1] for further details.

The *Berkovich unit disc* is the set $X := \mathcal{M}(k\{T\})$ of multiplicative seminorms on the Tate algebra $k\{T\}$ that extend the given absolute value on k and are bounded above by the Gauss norm x_G. When equipped with the topology of pointwise convergence, X is a compact Hausdorff path-connected space.

Define a partial order $\leq$ on X by declaring that $x \leq y$ if and only if $|f(x)| \leq |f(y)|$ for all $f \in k\{T\}$; in this way, the pair $(X, \leq)$ becomes a rooted tree with root at the Gauss point x_G. The tree structure on X is discussed in more detail in Section 5.1.3.

Let $\mathrm{Gal}(k^a/k)$ denote the group of automorphisms of k^a fixing k (though k^a/k is not Galois when k is not perfect), and let $X_K := \mathcal{M}(K\{T\})$ denote the ground field extension of X to K, which comes equipped with a continuous surjective map $p_K \colon X_K \to X$. The ground field extension X_K carries a $\mathrm{Gal}(k^a/k)$-action, extending the natural action on K by isometries, such that p_K induces a homeomorphism

$$X_K/\,\mathrm{Gal}(k^a/k) \xrightarrow{\sim} X,$$

where $X_K/\,\mathrm{Gal}(k^a/k)$ has the quotient topology. See [Ber90, p.18] and [BR10, §1] for further details.

Given $z \in K^\circ$ and $r \in [0,1]$, we may construct a point $x_{z,r} \in X_K$ as the sup-norm over the closed disc $\overline{D}(z,r) := \{a \in K^\circ \colon |z-a| \leq r\} \subset K^\circ$ of radius r about z; that is,

$$|f(x_{z,r})| := \sup_{z' \in \overline{D}(z,r)} |f(z')|, \qquad f \in K\{T\}.$$

The point $p_K(x_{z,r}) \in X$ will again be denoted by $x_{z,r}$. When $K \neq k$, it is possible that two pairs (z,r) and (z',r') may define the same point of X; for example, if $z' \in \mathrm{Gal}(k^a/k) \cdot \{z\}$ and $r > 0$, then $x_{z',r} = x_{z,r}$.

When $z \in (k^a)^\circ$ and $r = 0$, the associated point $x_{z,0}$ is called a *rigid point*; the seminorm $x_{z,0}$ coincides with the seminorm induced by the maximal ideal of $k\{T\}$ generated by the minimal polynomial of z over k. Let X^{rig} denote the subset of rigid points. The points of X can be classified into 4 types:

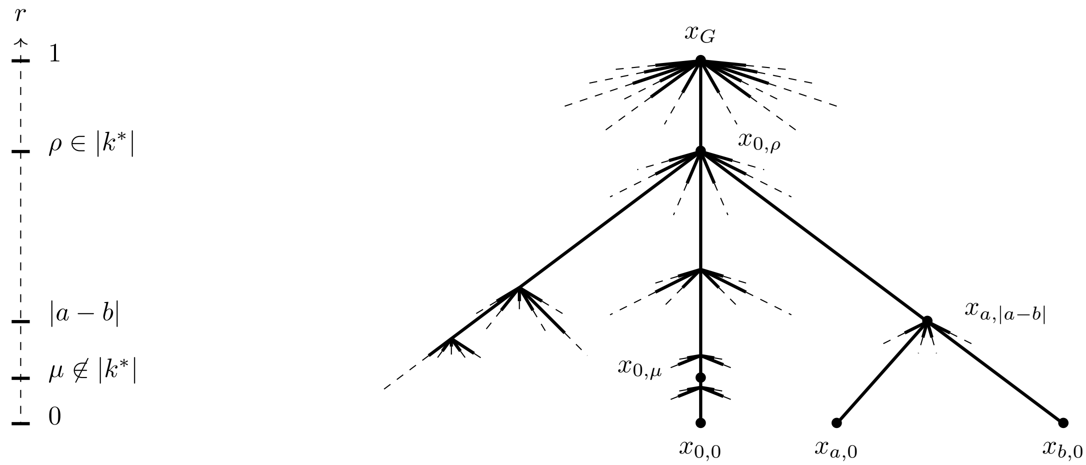

Figure 5.1: The Berkovich unit disc X over an algebraically closed, spherically complete field, with the radius function shown on the vertical axis. It has the structure of a metric **R**-tree, with the rigid points X^{rig} lying at the leaves.

1. a type 1 point is of the form $x_{z,0} \in X$ for $z \in K^\circ$;
2. a type 2 point is of the form $x_{z,r} \in X$ for $z \in K^\circ$ and $r \in (0,1] \cap |K^*|$;
3. a type 3 point is of the form $x_{z,r} \in X$ for $z \in K^\circ$ and $r \in (0,1] \backslash |K^*|$;
4. a type 4 point is the pointwise limit of x_{z_i,r_i} such that the corresponding discs $\overline{D}(z_i, r_i)$ form a decreasing sequence with empty intersection.

See [Ber90, p.18] for further details; when $K \neq k$, see also [Ked11, Proposition 2.2.7]. The set of type 1 points is precisely the set of rigid points and, when k is nontrivially-valued, the set of rigid points and the set of type 2 points are both dense in X. Points of type 4 exist only when k is not spherically complete.

The *radius function* $r\colon X \to [0,1]$ is an upper-semicontinuous function on X that sends a point $x_{z,r}$ of type 1, 2, or 3 to r, and a point $x = \lim_i x_{z_i,r_i}$ of type 4 to $\lim_i r_i$. Note that this function is not continuous in general: if k is nontrivially-valued, then there is a net $(x_i) \subseteq X^{\mathrm{rig}}$ such that $x_i \to x_G$, but $r(x_i) = 0$ for all i and $r(x_G) = 1$.

For any $x \in X$, the *multiplicity* of x is $m(x) := \#p_K^{-1}(x) \in \mathbf{Z}_{>0} \cup \{\infty\}$. The points of type 1 with finite multiplicity are precisely the rigid points. All points of type 2 and type 3 have finite multiplicity.

Lemma 5.1.1.1. *Assume k is trivially-valued or has residue characteristic zero. For any $x \in X$ of type 2 or 3, there exists $x' \in X^{\mathrm{rig}}$ such that $x' \leq x$ and $m(x') = m(x)$.*

Proof. Let $d = m(x)$, let $x_1, \ldots, x_d$ be the p_K-preimages of x, and let $G := \mathrm{Gal}(k^a/k)$. Each x_i is the sup-norm over a closed disc $\overline{D}_i \subseteq K^\circ$ of radius $r(x) > 0$. It suffices to show that there is $z_1 \in \overline{D}_1 \cap (k^a)^\circ$ such that $G \cdot \{z_1\} = \{z_1, \ldots, z_d\}$ with $z_i \in \overline{D}_i$. Fix any $z_1 \in \overline{D}_1 \cap (k^a)^\circ$, then it is easy to see that any such z_1 has $\#G \cdot \{z_1\} \geq d$.

Assume k is trivially-valued. If $r(x) = 1$, then the problem is trivial. If $r(x) < 1$, then each $\overline{D}_i$ contains a unique rigid point; in particular, $G \cdot \{z_1\}$ has size precisely equal to d.

Assume now that k has residue characteristic zero. The case when $d = 1$ is [Ax70, Proposition 2'] and we deduce the general case using a similar strategy (the $d = 1$ case also follows from the main result of [Sch15]).

Let $z_1, \ldots, z_\ell \in G \cdot \{z_1\}$ be those conjugates of z_1 that lie in $\overline{D}_1$. We claim each $\overline{D}_i$ contains precisely ℓ elements of the orbit $G \cdot \{z_1\}$. Indeed, by assumption there exists $\sigma_i \in G$ such that $\sigma_i(z_1) \in \overline{D}_i$, from which it follows that $\sigma_i(z_j) \in \overline{D}_i$ for all $j = 1, \ldots, \ell$ since G acts by isometries. As σ_i is an automorphism of k^a, it must give a bijection $G \cdot \{z_1\} \cap \overline{D}_1 \xrightarrow{\sim} G \cdot \{z_1\} \cap \overline{D}_i$, which gives the claim. Now, for all $i = 1, \ldots, d$, set

$$w_i := \ell^{-1} \sum_{u \in G \cdot \{z_1\} \cap \overline{D}_i} u.$$

As the residue characteristic of k is zero, $|\ell| = 1$ and hence

$$|\sigma_i(z_1) - w_i| = \left| \ell \sigma_i(z_1) - \sum_{u \in G \cdot \{z_1\} \cap \overline{D}_i} u \right| \leq \max_{u \in G \cdot \{z_1\} \cap \overline{D}_i} |\sigma_i(z_1) - u| \leq r(x).$$

In particular, $w_i \in \overline{D}_i$. Moreover, we have $G \cdot \{w_1\} = \{w_1, \ldots, w_d\}$ by construction.

□

Example 5.1.1.2. The statement of Lemma 5.1.1.1 is not necessarily true for a nontrivially-valued field of positive residue characteristic. Consider $k = \mathbf{Q}_p$ with the p-adic norm, normalized so that $|p| = p^{-1}$. If x is the type 2 point of $X_{\mathbf{C}_p}$ corresponding to the closed disc

$$E := \overline{D}(p^{1/p}, |p|^{\frac{2p-1}{p(p-1)}}) \subseteq (\mathbf{C}_p)^{\circ},$$

then $p_K(x) \in X$ is a type 2 point with multiplicity $m(\pi(x)) = 1$, but we claim that there is no $\mathbf{Q}_p$-rational point lying below it, i.e. $E \cap \mathbf{Q}_p = \emptyset$. We thank Kiran Kedlaya for pointing out this example.

Let ζ_p denote a primitive p-th root of unity in k^a, and set

$$c_i := \zeta_p^i p^{1/p} \in K.$$

Then, $c_1, \ldots, c_p$ is the $\mathrm{Gal}(k^a/k)$-orbit of p-th roots of p in K and we claim that

$$|c_i - c_j| = p^{-\frac{1}{p-1}}$$

for $i \neq j$. Indeed, plugging 1 into the p-th cyclotomic polynomial over $\mathbf{Q}_p$ yield the identity

$$p = \prod_{1 \leq i \leq p-1} (1 - \zeta_p^i),$$

and taking norms yields $|p| = |1 - \zeta_p|^{p-1}$. As $|\zeta_p^i - \zeta_p^j| = |1 - \zeta_p|$ for $i \neq j$, we have

$$|c_i - c_j| = |1 - \zeta_p| \cdot |p|^{1/p} = |p|^{\frac{1}{p-1} + \frac{1}{p}} = p^{-\frac{2p-1}{p(p-1)}}.$$

Set $r := p^{-\frac{2p-1}{p(p-1)}}$. Now, suppose there exists $a \in \mathbf{Q}_p \cap E$. Write $|a| = p^{-n}$ for some $n \in \mathbf{Z}$. Note that $|c_1| \notin p^{\mathbf{Z}}$, so either $|a| > |c_1|$ or $|c_1| > |a|$.

- If $|a| > |c_1|$, then $n < 1/p$ and $|a - c_1| = |a| \leq r$ implies that $n \geq \frac{2p-1}{p(p-1)} > 0$, but there are no integers satisfying both inequalities.

- If $|c_1| > |a|$, then $p^{-1/p} = |c_1| = |c_1 - a| \leq r$, so $-\frac{1}{p} \leq -\frac{2p-1}{p(p-1)}$, which is equivalent to demanding $p \leq 0$.

Thus, $E \cap \mathbf{Q}_p = \emptyset$, as required.

5.1.2 Temkin's metric on the Berkovich unit disc

The sheaf $\Omega^1_{X/k}$ of Kähler differentials on X is the free $\mathcal{O}_X$-module on the differential dT. Following the notation of §3.3, write $\|\cdot\|_{\mathrm{Tem},x}$ for Temkin's metric on the stalk $\Omega^1_{X/k,x}$ at $x \in X$.

Proposition 5.1.2.1. *Suppose k is algebraically closed. For any $x \in X$, we have*

$$\|dT\|_{\mathrm{Tem},x} = r(x).$$

Proof. This is [Tem16, §6.3.1]. □

When working over fields that are not algebraically closed or of nonzero residue characteristic, it is often better to work with what Temkin calls the *geometric Kähler seminorm*; this is done because Temkin's metric may be poorly behaved when k or $\widetilde{k}$ has wild extensions, as in [Tem16, §6.2]. For $x \in X$ and $s \in \Omega^1_{X/k,x}$, the geometric Kähler seminorm $\|s\|_{\mathrm{geom},x}$ is defined as

$$\|s\|_{\mathrm{geom},x} := \|p_K^* s\|_{\mathrm{Tem},x'}$$

for $x' \in p_K^{-1}(x)$. This is independent of the choice of x'. See [Tem16, §6.3.15] for more details. In the context of the unit disc, the geometric Kähler seminorm can be described as follows.

Proposition 5.1.2.2. *For any $x \in X$, we have $\|dT\|_{\mathrm{geom},x} = r(x)$.*

Proof. This is immediate from Proposition 5.1.2.1 and the definition of $\|\cdot\|_{\mathrm{geom},x}$. $\square$

By analogy with the log discrepancy function, write $A := -\log r = -\log \|dT\|_{\mathrm{geom}}$; this a lower-semicontinuous function $X \to \overline{\mathbf{R}}_+$. Thus, any global section of $\Omega^1_{X/k}$ can be written as $f dT$ for some $f \in k\{T\}$, and

$$\|f dT\|_{\mathrm{geom},x} = |f(x)| e^{-A(x)}$$

for any $x \in X$.

5.1.3 Metric structure on the Berkovich unit disc

In this section, we discuss the metric tree structure on the Berkovich unit disc. For a comprehensive treatment, see [BR10] and [Jon15].

For each $x \in X$, define an equivalence relation $\sim$ on the set $X \backslash \{x\}$ by declaring $y \sim z$ if the paths $(x, y]$ and $(x, z]$ intersect. The *tangent space* $T_{X,x}$ at x is the set of equivalences classes of $X \backslash \{x\}$ modulo $\sim$. For each $\vec{v} \in T_{X,x}$, let $U(\vec{v})$ be the set of points of X representing $\vec{v}$, and set

$$m(\vec{v}) := \inf_{y \in U(\vec{v})} m(y).$$

The subsets of the form $U(\vec{v})$, for some tangent direction $\vec{v}$, form a subbasis of open sets for the topology on X. See [Jon15, §2.3] for further details.

The closed unit disc X may be equipped with a generalized metric, in the sense that the distance between two points may be infinite; in particular, X gains the structure of a metric tree. This generalized metric is described as follows. Let $r\colon X \to [0, 1]$ denote the radius function, which can be thought of as a $\mathrm{Gal}(k^a/k)$-equivariant function on X_K. Define a function $\alpha\colon X \to \overline{\mathbf{R}}_+$ by specifying that

$\alpha(x_G) = 0$ and

$$\alpha(x) - \alpha(y) = -\int_y^x \frac{1}{m(z)} d\left(\log r(z)\right), \tag{5.1}$$

for any two distinct points $x, y \in X$, where the integral is taken over the unique path in X joining the points x and y. These constraints completely determine a function $\alpha\colon X \to [-\infty, +\infty]$ whose restriction to any segment is monotone decreasing. Observe that, when $k = k^a$, $\alpha = -\log r = A$. This, in turn, induces a generalized metric d on X by setting

$$d(x, y) \coloneqq |\alpha(x) - \alpha(x \vee y)| + |\alpha(y) - \alpha(x \vee y)|$$

for $x, y \in X$; here, $x \vee y$ is the least upper bound of x and y. The rooted tree $(X, \leq)$ acquires the structure of a metric tree when equipped with the generalized metric d. It is important to note that the topology on X induced by d is strictly finer than the native topology.

5.1.4 Quasisubharmonic functions on the Berkovich unit disc

One can discuss potential theory and the notion of quasisubharmonic functions on any metric tree, as developed in [Jon15, §2.5]. We briefly recall this theory in the special case of the Berkovich disc. This is also discussed in [BR10, §5] (though our conventions differ slightly).

Fix a finite atomic measure ρ_0 supported at x_G, i.e. ρ_0 is a positive real multiple of the Dirac mass δ_{x_G} at the Gauss point x_G. A function $\varphi\colon X \to [-\infty, \infty)$ is called ρ_0-*subharmonic* if it satisfies:

1. for every finite subtree $Y \subset X \backslash X^{\mathrm{rig}}$ containing x_G, $\varphi|_Y$ is a continuous function on Y such that:

   (a) $\varphi|_Y$ is convex on any segment in Y that does not contain x_G;

(b) for any $y \in Y$,

$$\rho_0\{y\} + \sum_{\vec{v} \in T_y Y} d_{\vec{v}}\,(\varphi|_Y) \geq 0,$$

where $d_{\vec{v}}\,(\varphi|_Y)$ denotes the directional derivative of $\varphi|_Y$ in the direction $\vec{v}$.

2. φ is the limit of its retractions to finite subtrees containing x_G; more precisely, if $\{Y_i\}_{i \in I}$ denotes the net of finite subtrees of $X \backslash X^{\mathrm{rig}}$ containing x_G and if $\mathfrak{r}_i \colon X \to Y_i$ is the (continuous) retraction map of X onto Y_i, then $\varphi = \lim_i \mathfrak{r}_i^* \varphi$.

The condition (b) is equivalent to the subaverage property: for any $y \in Y$, there exists $r > 0$ such that

$$\varphi(y) \leq \frac{1}{|B_Y(y,r)|} \sum_{z \in B_Y(y,r)} \varphi(z) + \mathrm{mass}(\rho_0) r \cdot \mathbf{1}_{\{x_G\}}(y),$$

where $B_Y(y,r) \coloneqq \{z \in Y \colon d(y,z) = r\}$ is the ball of radius r about y in Y, and $\mathbf{1}_{\{x_G\}}$ is the indicator function of the point x_G. This is reminiscent of the classical definition of subharmonic functions on $\mathbf{R}^n$.

A function is said to be *quasisubharmonic* if it is ρ_0-subharmonic for some measure ρ_0 as above. The class of ρ_0-subharmonic functions on X is a convex set, which is closed under taking finite maxima and decreasing (pointwise) limits. A quasisubharmonic function is upper-semicontinuous, but it may take the value $-\infty$; this can only occur at those points $x \in X$ such that $\alpha(x) = +\infty$.

Given a topological space Z and a function $\phi \colon Z \to [-\infty, \infty)$ that is locally bounded above, the *upper-semicontinuous (usc) regularization* ϕ^* of ϕ is defined by the formula

$$\phi^*(z) \coloneqq \limsup_{y \to z} \phi(y), \qquad z \in Z.$$

The usc regularization ϕ^* is the smallest upper-semicontinuous function such that $\phi^* \geq \phi$.

Lemma 5.1.4.1. *If* $\{\varphi_i\}_{i\in I}$ *is a net of* ρ_0*-subharmonic functions on* X *which is locally bounded above, then the usc regularization* ψ^* *of* $\psi := \sup_{i\in I}\varphi_i$ *is* ρ_0*-subharmonic. Furthermore,* $\psi^* = \psi$ *on* $X\setminus\{\alpha = +\infty\}$*.*

Proof. This is [BR10, Proposition 8.23(E)]. □

The rest of this section is devoted to a brief discussion of the Laplacian of a quasisubharmonic function. Given a ρ_0-subharmonic function φ and any finite subtree $Y \subset X$ containing x_G, let $\Delta(\varphi|_Y)$ be the unique signed Borel measure on Y determined by the following rule: if $\vec{v}_1, \ldots, \vec{v}_n$ are tangent directions in Y such that $U(\vec{v}_i) \cap U(\vec{v}_j) \neq \emptyset$ and $U(\vec{v}_i) \not\subseteq U(\vec{v}_j)$ for all $i \neq j$, then

$$\Delta(\varphi|_Y)\left(\bigcap_{i=1}^{n} U(\vec{v}_i)\right) = -\sum_{i=1}^{n} d_{\vec{v}_i}\left(\varphi|_Y\right).$$

The *Laplacian* is the signed Borel measure $\Delta\varphi$ uniquely characterized by the following property: for any finite subtree $Y \subset X$ containing x_G,

$$(\mathfrak{r}_Y)_*(\rho_0 + \Delta\varphi) = \rho_0 + \Delta\left(\varphi|_Y\right),$$

where $\mathfrak{r}_Y\colon X \to Y$ is the (continuous) retraction map of X onto Y. There are several Laplacians defined in the literature, and the above definition follows the conventions of [FJ04, Jon15] and it differs from that of [BR10] by a negative sign; see [BR10, §5.8] for a discussion of the various Laplacian operators.

Quasisubharmonic functions and their Laplacians behave well with respect to the ground field extension map $p_K\colon X_K \to X$, in the following sense: given a ρ_0-subharmonic function φ on X, $p_K^*\varphi$ is a ${}^*_K\rho_0$-subharmonic function on X_K and $\Delta\varphi = (p_K)_*\Delta(p_K^*\varphi)$. Indeed, this follows from the fact that a finite subtree of X_K containing the Gauss point is mapped onto a finite subtree of X containing the Gauss point.

Example 5.1.4.2. For any irreducible $f \in k\{T\}$, the function $\varphi := \log|f|$ is ρ_0-subharmonic, where $\rho_0 = m(x)\cdot\delta_{x_G}$ and $x \in X^{\mathrm{rig}}$ is the rigid point of X corresponding to the maximal ideal (f) of $k\{T\}$. It is easy to check that

$$\Delta\varphi = m(x)\,(\delta_x - \delta_{x_G})\,.$$

More generally, for any $f \in k\{T\}$, the function $\log|f|$ is quasisubharmonic and its Laplacian can be identified, up to scaling and adding a multiple of δ_{x_G}, with the divisor of zeros of f via the Poincaré–Lelong formula. See [BR10, Example 5.20].

In fact, for any $x \in X$ with $m(x) < +\infty$, the function $\varphi = -\alpha(x \vee \cdot)$ is $m(x)\cdot\delta_{x_G}$-subharmonic with Laplacian given by $\Delta\varphi = m(x)\,(\delta_x - \delta_{x_G})$.

For a quasisubharmonic function φ on X, we may construct a metric on $\mathcal{O}_X$: to a local section f of $\mathcal{O}_X$ over an analytic domain $V \subset X$, we assign the function $x \mapsto |f(x)|e^{-\varphi(x)}$, for $x \in V$. This convention mirrors how (semipositive) metrics on line bundles on complex manifolds are locally given by plurisubharmonic functions.

5.2 An Ohsawa–Takegoshi-type extension theorem

Assume that k is algebraically closed, trivially-valued, or is spherically complete of residue characteristic zero. Let $X = \mathcal{M}(k\{T\})$ be the Berkovich closed unit disc over k with Gauss point x_G, let $r\colon X \to [0,1]$ be the radius function, and let $A := -\log r\colon X \to \overline{\mathbf{R}}_+$.

In this section, we prove the following variant of Theorem E and deduce Theorem E as an easy consequence.

Theorem 5.2.0.1. *Let φ be a quasisubharmonic function on X with $\varphi(x_G) = 0$, and let $Z(\varphi) = \{\varphi = -\infty\}$ denote the polar locus of φ. For any $z \in X$, there exists* *a constant $\epsilon_0 > 0$ and a nonzero polynomial $f \in k[T]$ such that*

$$\|f\|_{(1+\epsilon)\varphi} := \sup_{x \in X \setminus Z(\varphi)} |f(x)| e^{-(1+\epsilon)\varphi(x) - A(x)} \leq |f(z)| e^{-\varphi(z)} \text{ for all } \epsilon \in [0, \epsilon_0].$$

If $\varphi(z) = -\infty$, then we may find f such that $\|f\|_{(1+\epsilon)\varphi} < +\infty$ for all $\epsilon \in [0, \epsilon_0]$. Moreover, if k is algebraically closed and $z \in X^{\mathrm{rig}}$, then for any value $a \in \mathcal{H}(z)^* = k^*$, we may find f such that $f(z) = a$.

We will prove Theorem 5.2.0.1 in §5.2.1. The hypotheses of Theorem 5.2.0.1 may be weakened to allow $\varphi(x_G) \geq 0$, but it is false if $\varphi(x_G) < 0$ (e.g. if $\varphi = -1$, then the only f that could satisfy the inequality in Theorem 5.2.0.1 at the Gauss point is $f = 0$). Nonetheless, Theorem E, which has no hypothesis on the value of $\varphi(x_G)$, may be easily deduced from Theorem 5.2.0.1. Furthermore, the presence of the hypotheses on the field k is discussed immediately after the proof of Lemma 5.2.1.4

Proof of Theorem E. Given a quasisubharmonic function φ on X and a point $z \in X$, set $\phi := \varphi - \varphi(x_G)$. Theorem 5.2.0.1 asserts that there is a nonzero polynomial $f \in k[T]$ such that $\|f\|_{(1+\epsilon)\phi} \leq |f(z)| e^{-\phi(z)}$ for all $\epsilon > 0$ sufficiently small. Thus,

$$\lim_{\epsilon \to 0^+} \|f\|_{(1+\epsilon)\varphi} = e^{\varphi(x_G)} \lim_{\epsilon \to 0^+} e^{\epsilon\varphi(x_G)} \|f\|_{(1+\epsilon)\phi} \leq |f(z)| e^{-\phi(z) + \varphi(x_G)} = |f(z)| e^{-\varphi(z)}.$$

This completes the proof of Theorem E. □

It is a result of Błocki (see [Bo13, Theorem 1]) that the optimal constant appearing in the classical Ohsawa–Takegoshi theorem for the complex unit disc is π. In the non-Archimedean setting, however, the optimal constant is 1. This is demonstrated by the result below.

Corollary 5.2.0.2. *For any $z \in X$, let $c(z)$ be the smallest positive number such that for any quasisubharmonic function φ on X, there exists a nonzero $f \in k\{T\}$*

satisfying

$$\lim_{\epsilon\to 0^+} \|f\|_{(1+\epsilon)\varphi} \leq c(z)|f(z)|e^{-\varphi(z)}.$$

Then, $c(z) = 1$.

Proof. If $\varphi \equiv 0$, then

$$|f(x_G)| = \lim_{\epsilon\to 0^+} \|f\|_{(1+\epsilon)\varphi} \leq c(z)|f(z)| \leq c(z)|f(x_G)|.$$

As $|f(x_G)| \neq 0$, it follows that $c(z) \geq 1$. Theorem E asserts that the lower bound $c(z) = 1$ is achieved for any z. □

5.2.1 Proof of Theorem 5.2.0.1

Fix $z \in X$. Suppose φ is ρ_0-subharmonic for some finite (atomic) measure ρ_0 supported at the Gauss point x_G, and $\varphi(x_G) = 0$. To simplify the exposition, we assume $\varphi(z) > -\infty$. When $\varphi(z) = -\infty$, the proof is similar: one proves analogues of the sequence of lemmas below, each of which is made easier because one is only concerned with ensuring that a quantity is finite, as opposed to ensuring that the same quantity is less than some fixed value.

The strategy of the proof is to use strong induction on $\lfloor \mathrm{mass}(\rho_0) \rfloor$. For this reason, the following terminology will be quite helpful.

Definition 5.2.1.1. Let $\epsilon_0 > 0$. A nonzero polynomial $f \in k[T]$ is a (φ, ϵ_0)-*extension at* z if the inequality

$$\|f\|_{(1+\epsilon)\varphi} \leq |f(z)|e^{-\varphi(z)}$$

holds for all $\epsilon \in [0, \epsilon_0]$. Equivalently, f is a (φ, ϵ_0)-extension if for all $x \in X\backslash Z(\varphi)$ and all $\epsilon \in [0, \epsilon_0]$, we have

$$\log|f(x)| - \log|f(z)| \leq A(x) + (1+\epsilon)\varphi(x) - \varphi(z). \tag{5.2}$$

As $\varphi \leq 0$, in order to verify that f is an (φ, ϵ_0)-extension at z, it suffices to check $\|f\|_{(1+\epsilon_0)\varphi} \leq |f(z)|e^{-\varphi(z)}$, as opposed to verifying it for all $\epsilon \in [0, \epsilon_0]$.

The key players in the proof of Theorem 5.2.0.1 are the finite subtrees

$$\Gamma_{\varphi,n} := \left\{x \in X \colon (\rho_0 + \Delta\varphi)\{y \leq x\} \geq \frac{n}{n+1} m(x)\right\}.$$

for $n \geq 1$. Note that $\Gamma_{\varphi,n}$ is indeed a finite subtree of X, since any end x of $\Gamma_{\varphi,n}$ satisfies

$$(\rho_0 + \Delta\varphi)\{x\} \geq \frac{n}{n+1} m(x)$$

and $\rho_0 + \Delta\varphi$ is a positive measure of finite mass equal to $\operatorname{mass}(\rho_0)$. It is clear that $\Gamma_{\varphi,n} \supseteq \Gamma_{\varphi,n+1}$ for any $n \geq 1$; in particular,

$$\Gamma_\varphi := \bigcap_{n\geq 1} \Gamma_{\varphi,n} = \{x \in X \colon (\rho_0 + \Delta\varphi)\{y \leq x\} \geq m(x)\}$$

is a finite subtree of X that is contained in $\Gamma_{\varphi,n}$ for any $n \geq 1$.

The subtrees $\Gamma_{\varphi,n}$ are crucial in reducing Theorem 5.2.0.1 to a "finite" problem. Observe that any point of $\Gamma_{\varphi,n}$ must have a finite multiplicity because $\operatorname{mass}(\rho_0)$ is finite and, moreover, $\Gamma_{\varphi,n} \cap X^{\mathrm{rig}} \subseteq Z(\varphi)$. Variants of this tree appear in [Jon15, Prop 2.8] and [FJ04, Lem 7.7].

The following is the base case for our induction.

Lemma 5.2.1.2. *If* $\operatorname{mass}(\rho_0) < 1$, *there exists a constant* $\epsilon_0 > 0$ *such that any nonzero constant function is a* (φ, ϵ_0)*-extension at* z.

Proof. Set $\epsilon_0 := \frac{1-\operatorname{mass}(\rho_0)}{\operatorname{mass}(\rho_0)}$. Since $\operatorname{mass}(\rho_0) < 1$, $\Gamma_\varphi = \emptyset$ and

$$|d_{\vec{v}}(1+\epsilon)\varphi| \leq (1+\epsilon)\operatorname{mass}(\rho_0) \leq 1$$

for all tangent directions $\vec{v}$ in X, provided $\epsilon \in [0, \epsilon_0]$. To check that any constant function is a (φ, ϵ_0)-extension at z, it suffices to check that

$$0 \leq A + (1+\epsilon)\varphi - \varphi(z)$$

on $X\backslash Z(\varphi)$. This inequality is satisfied at the Gauss point:

$$A(x_G) + (1+\epsilon)\varphi(x_G) - \varphi(z) = -\varphi(z) \geq 0.$$

Moreover, the convexity of φ and of A ensures that it is enough to check that, in any tangent direction at x_G, the function $(1+\epsilon)\varphi + A$ only increases in that direction. However, in any such direction $\vec{v}$,

$$d_{\vec{v}}\left((1+\epsilon)\varphi + A\right) \geq -(1+\epsilon)\,\mathrm{mass}(\rho_0) + 1 \geq 0,$$

which completes the proof. □

Now, if $\mathrm{mass}(\rho_0) \geq 1$, then $x_G \in \Gamma_{\varphi,n}$ for all $n \geq 1$, because $m(x_G) = 1$. In particular, $\Gamma_{\varphi,n} \neq \emptyset$ and the set of ends $\mathrm{Ends}(\Gamma_{\varphi,n})$ is nonempty. If Γ is a subtree of X that contains x_G, we adopt the following conventions: if $\Gamma \supsetneq \{x_G\}$, then $\mathrm{Ends}(\Gamma)$ consists of those points in Γ with a unique tangent direction in Γ; if $\Gamma = \{x_G\}$, then $\mathrm{Ends}(\Gamma) = \Gamma$.

Let $\Gamma'_{\varphi,n}$ denote the convex hull of $\Gamma_{\varphi,n} \cup \{z\}$. Let $\mathfrak{r}_{\Gamma'_{\varphi,n}} : X \to \Gamma'_{\varphi,n}$ denote the retraction of X onto $\Gamma'_{\varphi,n}$. For any $n \geq 1$, observe that a $(\mathfrak{r}^*_{\Gamma'_{\varphi,n}}\varphi, \epsilon_0)$-extension f at z is also a (φ, ϵ_1)-extension at z, where $\epsilon_1 = \min\{\epsilon_0, \frac{1}{n}\}$. Indeed, for any $x \in \Gamma'_{\varphi,n}$ and any direction $\vec{v} \in T_{X,x}$ such that $U(\vec{v}) \cap \Gamma'_{\varphi,n} = \emptyset$, we have

$$d_{\vec{v}}((1+\epsilon)\varphi + A) > \left(-(1+\epsilon)\frac{n}{n+1} + 1\right) m(x) \geq 0$$

for all $\epsilon \in [0, \epsilon_1]$. Thus, for fixed $n \geq 1$, we may replace φ with $\mathfrak{r}^*_{\Gamma'_{\varphi,n}}\varphi$ to assume that φ is locally constant off of $\Gamma'_{\varphi,n}$.

Fix $n \geq 1$ such that

$$\Gamma_{\varphi,n}\backslash\Gamma_{\varphi} = \bigsqcup_{j \in J} [x_j, \tilde{x}_j), \tag{5.3}$$

where J is a finite index set, and for all $j \in J$, $\tilde{x}_j \in \Gamma_\varphi$ and the point x_j is of type 2 or 3 such that the multiplicity function is constant on $[x_j, \tilde{x}_j)$. Moreover, for each

$j \in J$, if $\vec{v}_j \in T_{X,\tilde{x}_j}$ denotes the unique tangent direction at $\tilde{x}_j$ with $x_j \in U(\vec{v}_j)$, then $d_{\vec{v}_j}\varphi = -m(x_j)$.

As mentioned before, the strategy of the proof of Theorem 5.2.0.1 is by strong induction on $\lfloor \mathrm{mass}(\rho_0) \rfloor$. In the following sequence of lemmas, we explain how to reduce the problem of the existence of a (φ, ϵ_0)-extension at z to the existence of a (ϕ, ϵ_1)-extension at z, where ϕ is a ρ-subharmonic for some finite measure ρ with $\mathrm{mass}(\rho) \leq \mathrm{mass}(\rho_0) - 1$ and $0 < \epsilon_1 \leq \epsilon_0$. As $\mathrm{mass}(\rho_0)$ is finite, after finitely-many such reductions, we must find ourselves in the setting of Lemma 5.2.1.2. The hypotheses of each lemma are concerned with which types of points may arise in the finite set $\mathrm{Ends}(\Gamma_{\varphi,n})$; in particular, if one assumes that k is spherically complete, one can ignore Lemma 5.2.1.5.

Lemma 5.2.1.3. *Suppose that* $\mathrm{mass}(\rho_0) \geq 1$ *and that there exists* $x \in \mathrm{Ends}(\Gamma_{\varphi,n})$ *of type 1. Then, there exists* $\epsilon_1 > 0$ *and a* ρ*-subharmonic function* ϕ*, with* $\mathrm{mass}(\rho) \leq \mathrm{mass}(\rho_0) - 1$*, such that a* (φ, ϵ_0)*-extension* f *at* z *may be constructed from a* (ϕ, ϵ_1)*-extension* $\tilde{f}$ *at* z*.*

Proof. By assumption, $x \in \Gamma_\varphi$ and so it satisfies $m(x) < +\infty$; in particular, $x \in X^{\mathrm{rig}}$, because x is of type 1. In addition, as $\Gamma_\varphi \cap X^{\mathrm{rig}} \subseteq Z(\varphi)$ and $z \notin Z(\varphi)$, we have $x \neq z$. Let $\mathfrak{m}_x$ denote the maximal ideal of $k\{T\}$ corresponding to x, and let g be a polynomial generator of $\mathfrak{m}_x$ with $|g(x_G)| = 1$. As type-1 points are minimal with respect to the partial order $\leq$, we have

$$1 \leq m(x) \leq (\rho_0 + \Delta\varphi)\{y \in X \colon y \leq x\} = \Delta\varphi\{x\}.$$

Set $c \coloneqq \frac{\Delta\varphi\{x\}}{m(x)} \geq 1$, $\gamma \coloneqq \lfloor c \rfloor$, and $\rho \coloneqq \rho_0 - \gamma\delta_{x_G}$, then the function $\phi \coloneqq \varphi - \gamma \log|g|$ is ρ-subharmonic and $\phi(x_G) = 0$. Suppose there exists $\epsilon_1 > 0$ and a (ϕ, ϵ_1)-extension $\tilde{f} \in k[T]$ at z and set $f \coloneqq g^\gamma \tilde{f}$. We claim that there exists $\epsilon_0 \in (0, \epsilon_1]$ such that f is a (φ, ϵ_0)-extension at z.

Case 1. Consider the case when c is an integer. If $z' := \mathfrak{r}_{\Gamma_{\varphi,n}}(z)$, then $\Gamma'_{\varphi,n} = \Gamma_{\varphi,n} \cup [z, z']$. If $z' \neq z$, let $\vec{v}_z \in T_{X,z'}$ be the unique direction at z' with $z \in U(\vec{v}_z)$. The function $y \mapsto |f(y)|e^{-(1+\epsilon_0)\varphi(y)-A(y)}$ is decreasing in the direction $\vec{v}_z$ provided $d_{\vec{v}_z}(-(1+\epsilon_0)\varphi - A) \leq 0$. This occurs if $\epsilon_0 \leq \frac{m(\vec{v}_z)}{nm(z')}$. Thus, it suffices to find $\epsilon_0 > 0$ such that

$$|f(y)|e^{-(1+\epsilon_0)\varphi(y)-A(y)} \leq |f(z)|e^{-\varphi(z)} = |\tilde{f}(z)|e^{-\phi(z)}$$

for all $y \in \Gamma_{\varphi,n}$. Set $\widetilde{\Gamma}_{\phi,n} = \Gamma_{\phi,n} \cup \{x_G\}$ (where we must include x_G in case $\Gamma_{\phi,n}$ is empty). If $y \in \widetilde{\Gamma}_{\phi,n}$, then $\phi(y) \leq -\frac{n}{n+1}\alpha(y)$, and hence

$$|g(y)|^{-\epsilon_0 c} e^{(\epsilon_1-\epsilon_0)\phi(y)} \leq \left(e^{-\alpha(y)}\right)^{-\epsilon_0 cm(x)+\frac{n}{n+1}(\epsilon_1-\epsilon_0)}$$

and this is less than or equal to 1 provided $\epsilon_0 \leq \frac{\epsilon_1 n}{m(x)c(n+1)+n}$. Granted this, observe that

$$\begin{aligned}
|f(y)|e^{-(1+\epsilon_0)\varphi(y)-A(y)} &= |g(y)|^{-\epsilon_0 c} e^{(\epsilon_1-\epsilon_0)\phi(y)} |\tilde{f}(y)|e^{-(1+\epsilon_1)\phi(y)-A(y)} \\
&\leq |\tilde{f}(y)|e^{-(1+\epsilon_1)\phi(y)-A(y)} \\
&\leq |\tilde{f}(z)|e^{-\phi(z)},
\end{aligned}$$

as required. Let $\tilde{x}$ denote the minimal element of $\widetilde{\Gamma}_{\phi,n} \cap [x, x_G]$ and let $\vec{v}_x \in T_{X,\tilde{x}}$ be the unique direction at $\tilde{x}$ such that $x \in U(\vec{v}_x)$. As $\Gamma_{\varphi,n} = \widetilde{\Gamma}_{\phi,n} \cup [x, x_G]$, it suffices to find an $\epsilon_0 > 0$ such that the function $y \mapsto |f(y)|e^{-(1+\epsilon_0)\varphi(y)-A(y)}$ is decreasing in the direction $\vec{v}_x$, or equivalently

$$d_{\vec{v}_x}\left(\epsilon_0 m(x)c\alpha - (1+\epsilon_0)\phi - A\right) \leq 0.$$

This occurs if $\epsilon_0 \leq \frac{m(\vec{v}_x)}{(c(n+1)+n)m(x)}$.

Case 2. Now, suppose c is not an integer, i.e. $c > \gamma$. Write $\varphi = \psi + c\log|g|$ for some quasisubharmonic function ψ on X with $\psi \leq 0$. Set $\epsilon_0 = \frac{\epsilon_1(c-\gamma)}{c}$. It is straightforward to check that $\epsilon_1\phi \leq \epsilon_0\varphi$ on all of X. Thus, for any $y \in X\backslash Z(\varphi)$, we have

$$\begin{aligned}|f(y)|e^{-(1+\epsilon_0)\varphi(y)-A(y)} &= e^{\epsilon_1\phi(y)-\epsilon_0\varphi(y)}|\tilde{f}(y)|e^{-(1+\epsilon_1)\phi(y)-A(y)} \\ &\leq |\tilde{f}(z)|e^{-\phi(z)} \\ &= |f(z)|e^{-\varphi(z)},\end{aligned}$$

which completes the proof.

□

Lemma 5.2.1.4. *Suppose that* $\mathrm{mass}(\rho_0) \geq 1$ *and that there exists* $x \in \mathrm{Ends}(\Gamma_{\varphi,n})$ *of type 2 or of type 3 such that* $\mathfrak{r}_{\Gamma_{\varphi,n}}(z) \neq x$. *Then, there exists* $\epsilon_1 > 0$ *and a* ρ*-subharmonic function* ϕ*, with* $\mathrm{mass}(\rho) \leq \mathrm{mass}(\rho_0) - 1$*, such that a* (φ, ϵ_0)*-extension* f *at* z *may be constructed from a* (ϕ, ϵ_1)*-extension* $\tilde{f}$ *at* z*.*

Proof. By Lemma 5.1.1.1, there exists $x' \in X^{\mathrm{rig}}$ such that $\mathfrak{r}_{\Gamma_{\varphi,n}}(x') = x$ and $m(x') = m(x)$. If $x \in \Gamma_\varphi$, then set $c := \frac{\Delta\varphi\{x\}}{m(x)} \geq 1$. We construct a new ρ_0-subharmonic function φ' by extending φ linearly with slope $cm(x)$ from the end x to the rigid point x'; more precisely, φ' is given by the formula

$$\varphi'(y) := \varphi(y) + cm(x)\left(\alpha(y \vee x) - \alpha(y \vee x')\right).$$

It is clear that $\varphi'(x_G) = 0$, $\varphi'(z) = \varphi(z)$, and $\varphi' \leq \varphi$. In particular, any (φ', ϵ_0)-extension at z is also a (φ, ϵ_0)-extension at z.

If $x \in \Gamma_{\varphi,n}\backslash\Gamma_\varphi$, then by the assumption (5.3), we can replace φ with a function that is linear on $[x, \mathfrak{r}_{\Gamma_\varphi}(x)]$ with slope $m(\mathfrak{r}_{\Gamma_\varphi}(x))$ and then repeat the same argument as above. □

The proof of Lemma 5.2.1.4 is one point in the proof of Theorem 5.2.0.1 where the additional assumptions on the field k are needed (in order to be able to apply Lemma 5.1.1.1). If one were to pick $x' \in X^{\mathrm{rig}}$ such that $m(x') > m(x)$, then the function φ' need not be quasisubharmonic.

In addition, the conditions on the field k are such that the hypotheses of the following lemma can only occur when k is algebraically closed and nontrivially-valued.

Lemma 5.2.1.5. *Suppose that* $\mathrm{mass}(\rho_0) \geq 1$ *and that there exists* $x \in \mathrm{Ends}(\Gamma_{\varphi,n})$ *of type 4 and* $x \neq z$*. Then, there exists* $\epsilon_1 > 0$ *and a* ρ*-subharmonic function* ϕ*, with* $\mathrm{mass}(\rho) \leq \mathrm{mass}(\rho_0) - 1$*, such that a* (φ, ϵ_0)*-extension* f *at* z *may be constructed from a* (ϕ, ϵ_1)*-extension* $\tilde{f}$ *at* z*.*

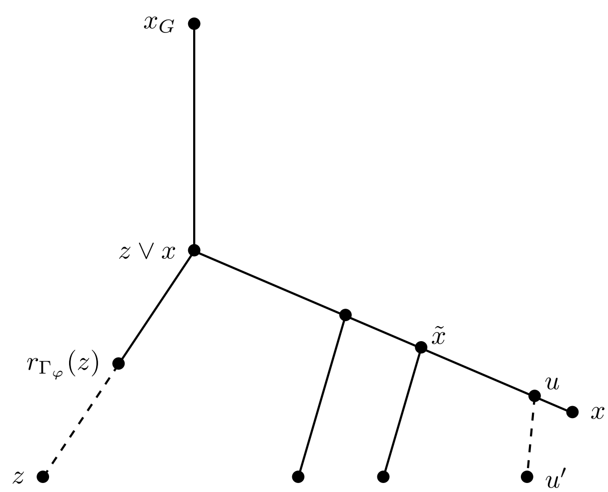

Figure 5.2: A possible configuration for $\Gamma_{\varphi,n}$ in Lemma 5.2.1.5.

Proof. By assumption, k is algebraically closed and nontrivially-valued (so the type 2 points of X are dense). Moreover, all multiplicities are equal to 1 and $\alpha = A$.

Note that, since x is of type 4, $A(x) \in (0, +\infty)$. Recall that $\Gamma'_{\varphi,n}$ denotes the convex hull of $\Gamma_{\varphi,n} \cup \{z\}$ (we allow the possibility that $z \in \Gamma_{\varphi,n}$, so it is possible that $\Gamma'_{\varphi,n} = \Gamma_{\varphi,n}$). Let $\tilde{x} \in (x, x_G]$ be the minimal point with the property that $\{y \in \Gamma'_{\varphi,n} \colon y \leq \tilde{x}\} \neq [x, \tilde{x}]$. That is, $\tilde{x}$ is the minimal point on $(x, x_G]$ such that $(\tilde{x}, x]$ does not intersect any branches of $\Gamma'_{\varphi,n}$ other than the one containing x. It follows that $\mathfrak{r}_{\Gamma_{\varphi,n}}(z) \notin (\tilde{x}, x]$. An example is in Fig. 5.2.

Set $c := -d_{\vec{v}}\varphi \geq 1$, where $\vec{v} \in T_{X,\tilde{x}}$ is the unique tangent direction at $\tilde{x}$ with $x \in U(\vec{v})$; then, after possibly replacing φ with a smaller quasisubharmonic function, we may assume that φ is linear of slope $-c$ on $[x, \tilde{x}]$, i.e.

$$\varphi(y) = \varphi(\tilde{x}) - c\,(\alpha(y) - \alpha(\tilde{x}))$$

for $y \in [x, \tilde{x}]$. If $\gamma := \lfloor c \rfloor$, and $\rho := \rho_0 - \gamma\delta_{x_G}$, then the function $\phi(y) := \varphi(y) + \gamma\alpha(y \vee x)$ is ρ-subharmonic and $\phi(x_G) = 0$. Suppose there exists $\epsilon_1 > 0$ and a (ϕ, ϵ_1)-extension $\tilde{f} \in k[T]$ at z.

Pick $u \in (x, \tilde{x})$ of type 2 such that $\alpha(x) - \alpha(u) \in (0, \eta)$, where η is a positive number chosen to be less than $\frac{A(x)}{c}$ and, if $\phi(x) < 0$, then we also require that $\eta < \frac{-\epsilon_1\phi(x)}{c}$.

Pick $u' \in X^{\mathrm{rig}}$ such that $\mathfrak{r}_{\Gamma_{\varphi,n}}(u') = u$, and a polynomial generator g of $\mathfrak{m}_{u'}$ with $|g(x_G)| = 1$. Set $f := \tilde{f}g^{\gamma}$. By construction, $z \vee u = z \vee x$, and hence we have

$$|f(z)|e^{-\varphi(z)} = |g(z)|^{\gamma}|\tilde{f}(z)|e^{\gamma\alpha(z\vee u)}e^{-\phi(z)} = |\tilde{f}(z)|e^{-\phi(z)}.$$

Case 1. Suppose that c is an integer and $\phi(x) = 0$. Then, $\phi|_{[x,x_G]} = 0$, and hence $\varphi(y) = -c\alpha(y \vee x)$ for $y \in [x, x_G]$. Arguing as in Case 1 of Lemma 5.2.1.3, it suffices to find an $\epsilon_0 > 0$ such that

$$|f(y)|e^{-(1+\epsilon_0)\varphi(y)-A(y)} \leq |\tilde{f}(z)|e^{-\phi(z)} \tag{5.4}$$

for $y \in \Gamma_{\varphi,n}$. By assumption, $\Gamma_{\phi,n} \cap [x, x_G] \subseteq \{x_G\}$, so the left-hand side of (5.4) is equal to $|\tilde{f}(y)|e^{-(1+\epsilon_0)\phi(y)-A(y)}$ for $y \in \Gamma_{\phi,n}$, in which case (5.4) holds. Thus, it suffices to verify (5.4) for $y \in [x, x_G]$. It holds when $y = x_G$ by assumption, and the function $y \mapsto |f(y)|e^{-(1+\epsilon_0)\varphi(y)-A(y)}$ is decreasing on $[u, x_G]$, but increasing on $[u, x]$. Therefore, it is enough to check (5.4) at $y = x$. Let $\beta := \log|\tilde{f}(x)| - \log|\tilde{f}(x_G)|$, so $\beta \leq 0$. Suppose there exists an $\epsilon_0 > 0$ such that

$$e^{\beta}|g(x)|^{c}e^{-(1+\epsilon_0)\varphi(x)-A(x)} \leq 1. \tag{5.5}$$

Then, $|f(x)|e^{-(1+\epsilon_0)\varphi(x)-A(x)} \leq |\tilde{f}(x_G)|$, and $|\tilde{f}(x_G)|$ is less than or equal to $|\tilde{f}(z)|e^{-\phi(z)}$ by assumption. Thus, it suffices to show (5.5). Observe that

$$\begin{aligned} &e^{\beta}|g(x)|^c e^{-(1+\epsilon_0)\varphi(x)-A(x)} \\ &= \exp\left(\beta - c\alpha(u) - (1+\epsilon_0)\varphi(\tilde{x}) + (1+\epsilon_0)c\left(\alpha(x) - \alpha(\tilde{x})\right) - A(x)\right) \\ &\leq \exp\left(-c\left(\alpha(u) - \alpha(x)\right) - A(x) + \epsilon_0 c\alpha(x)\right), \end{aligned}$$

so (5.5) holds at $y = x$ if $\epsilon_0 \leq \frac{A(x)-c\eta}{c\alpha(x)}$.

Case 2. Suppose that c is an integer and $\phi(x) < 0$. As in Case 1 of Lemma 5.2.1.3, it suffices to find $\epsilon_0 > 0$ such that (5.4) holds for all $y \in \Gamma_{\varphi,n}$. Set $\widetilde{\Gamma}_{\phi,n} = \Gamma_{\phi,n} \cup \{x_G\}$. Again arguing as in Case 1 of Lemma 5.2.1.3, (5.4) holds for $y \in \widetilde{\Gamma}_{\phi,n}$ provided

$$\epsilon_0 \leq \frac{\epsilon_1 n}{n + c(n+1)}.$$

Finally, as in Case 1, the function $y \mapsto |f(y)|e^{-(1+\epsilon_0)\varphi(y)-A(y)}$ on $\Gamma_{\varphi,n}\backslash\widetilde{\Gamma}_{\phi,n}$ is maximized at $y = x$, so it suffices to find an $\epsilon_0 > 0$ so that holds (5.4) there. Observe that

$$|g(u)|^c e^{(\epsilon_1-\epsilon_0)\phi(x)+(1+\epsilon_0)c\alpha(x)} \leq 1$$

provided

$$\epsilon_0 \leq \frac{-\epsilon_1\phi(x) - c\eta}{-\phi(x) + c\alpha(x)},$$

where $-\epsilon_1\phi(x) - c\eta > 0$ by the choice of u. It follows that

$$\begin{aligned} |f(x)|e^{-(1+\epsilon_0)\varphi(x)-A(x)} &= |g(u)|^c e^{(\epsilon_1-\epsilon_0)\phi(x)} e^{(1+\epsilon_0)c\alpha(x)} |\tilde{f}(x)| e^{-(1+\epsilon_1)\phi(x)-A(x)} \\ &\leq |\tilde{f}(z)|e^{-\phi(z)}, \end{aligned}$$

which completes the proof in Case 2.

Case 3. Suppose $c > \gamma$. Following the previous two cases, it suffices to find $\epsilon_0 > 0$ such that (5.4) holds for all $y \in \Gamma_{\varphi,n}$. Set $\widetilde{\Gamma}_{\phi,n} = \Gamma_{\phi,n} \cup \{x_G\}$. If $y \in \widetilde{\Gamma}_{\phi,n}$, then

$\phi(y) \leq -\frac{n}{n+1}\alpha(y)$, and hence

$$e^{(\epsilon_1-\epsilon_0)\phi(y)+\gamma\epsilon_0\alpha(y\vee u')} \leq \left(e^{-\alpha(y)}\right)^{\frac{n}{n+1}(\epsilon_1-\epsilon_0)-\gamma\epsilon_0}.$$

This last quantity is less than or equal to 1 provided

$$\epsilon_0 \leq \frac{\epsilon_1 n}{\gamma(n+1)+n}.$$

In this case, we have

$$\begin{aligned}|f(y)|e^{-(1+\epsilon_0)\varphi(y)-A(y)} &\leq e^{\gamma\epsilon_0\alpha(y\vee u')+(\epsilon_1-\epsilon_0)\phi(y)}|\tilde{f}(y)|e^{-(1+\epsilon_1)\phi(y)-A(y)} \\ &\leq |\tilde{f}(z)|e^{-\phi(z)}.\end{aligned}$$

Let $x'' \in [u, x_G]$ be the minimal point of $\widetilde{\Gamma}_{\phi,n} \cap [u, x_G]$. If $\vec{v}'' \in T_{X,x''}$ is the unique direction at x'' with $x \in U(\vec{v}'')$, then the function $y \mapsto |f(y)|e^{-(1+\epsilon_0)\varphi(y)-A(y)}$ is decreasing on $[u, x'']$ provided

$$d_{\vec{v}''}(\epsilon_0\gamma\alpha - (1+\epsilon_0)\phi - A) \leq 0.$$

Using the fact that $d_{\vec{v}''}\phi \geq -\frac{n}{n+1}$, this holds if $\epsilon_0 \leq \frac{1}{(n+1)\gamma+n}$.

Now, the function $y \mapsto |f(y)|e^{-(1+\epsilon_0)\varphi(y)-A(y)}$ on $[x, u)$ achieves its maximum at $y = x$, so it suffices to find an $\epsilon_0 > 0$ such that (5.4) holds at $y = x$. If $\phi(x) = 0$, one can argue as in Case 1. If $\phi(x) < 0$, then it follows that

$$|g(u)|^{\gamma}e^{(\epsilon_1-\epsilon_0)\phi(x)+(1+\epsilon_0)\gamma\alpha(x)} \leq e^{\gamma\eta+\epsilon_1\phi(x)-\epsilon_0\varphi(x)},$$

and this is bounded above by 1 provided

$$\epsilon_0 \leq \frac{-\phi(x)\epsilon_1 - \gamma\eta}{-\varphi(x)}.$$

Note that $-\phi(x)\epsilon_1 - \gamma\eta > 0$ by the choice of u. Therefore, we have

$$\begin{aligned}|f(x)|e^{-(1+\epsilon_0)\varphi(x)-A(x)} &\leq |g(u)|^{\gamma}e^{(\epsilon_1-\epsilon_0)\phi(x)+\gamma(1+\epsilon_0)\alpha(x)}|\tilde{f}(x)|e^{-(1+\epsilon_1)\phi(x)-A(x)} \\ &\leq |\tilde{f}(z)|e^{-\phi(z)},\end{aligned}$$

which completes the proof. ∎

If none of the hypotheses of the previous 4 lemmas hold, then $\Gamma'_{\varphi,n}$ must be the interval $[z, x_G]$. This special case is addressed below.

Lemma 5.2.1.6. *Suppose* $\mathrm{mass}(\rho_0) \geq 1$ *and* $\Gamma'_{\varphi,n} = [z, x_G]$*, then there exists* $\epsilon_0 > 0$ *such that any nonzero constant function is a* (φ, ϵ_0)*-extension at* z*.*

Proof. It suffices to find $\epsilon_0 > 0$ such that $-(1+\epsilon_0)\varphi(y) - A(y) \leq -\varphi(z)$ for any $y \in [z, x_G]$. For such y, $\varphi(z) - \varphi(y) \leq 0$ and hence it suffices to find $\epsilon_0 > 0$ such that $-\epsilon_0\varphi(y) - A(y) \leq 0$ for all such y. This holds for any $\epsilon_0 > 0$ at $y = x_G$. Thus, if $\vec{v} \in T_{X,x_G}$ is the unique direction at x_G with $z \in U(\vec{v})$, then it suffices to find $\epsilon_0 > 0$ such that $d_{\vec{v}}(-\epsilon_0\varphi - A) \leq 0$. This holds if $\epsilon_0 \leq \frac{m(\vec{v})}{-d_{\vec{v}}\varphi}$. □

Let us now summarize the proof of Theorem 5.2.0.1: if $\mathrm{mass}(\rho_0) < 1$, then a (φ, ϵ_0)-extension at z exists by Lemma 5.2.1.2. Suppose now that $\mathrm{mass}(\rho_0) \geq 1$; in particular, $\Gamma_{\varphi,n} \neq \emptyset$ and $\mathrm{Ends}(\Gamma_{\varphi,n}) \neq \emptyset$. As discussed at the start of the section, we assume that φ is locally constant off of the convex hull of $\Gamma_\varphi \cup \{z\}$. By repeatedly applying Lemma 5.2.1.3, Lemma 5.2.1.4, and Lemma 5.2.1.5, we may assume that $\mathrm{Ends}(\Gamma_{\varphi,n})$ consists of a single type-2 or type-3 point onto which z retracts. This is precisely the setting of Lemma 5.2.1.6, which then asserts that a (φ, ϵ_0)-extension at z exists. Now, if $z \in X^{\mathrm{rig}}$ and $k = k^a$, then $\mathcal{H}(z) = k$, so the second assertion is immediate from the first. This concludes the proof of Theorem 5.2.0.1.

5.3 A non-Archimedean Demailly approximation

Let X be the Berkovich closed unit disc over k. Given a quasisubharmonic φ on X, one may wish to approximate it by a sequence of quasisubharmonic functions whose singularities are controlled. One such result is already well known: [Jon15, Theorem 2.10] shows that there is a decreasing sequence of bounded quasisubharmonic functions $(\varphi_n)_{n=1}^{\infty}$ on X which decrease pointwise to φ. Let us briefly recall the construction: if φ is ρ_0-subharmonic, for each $n \geq 1$, consider the finite subtree

$$\Gamma_n := \left\{ x \in X \colon (\rho_0 + \Delta\varphi)\{y \geq x\} \geq 2^{-n} \text{ and } d(x_G, x) \leq 2^n \right\},$$

where d is the generalized metric on X from §5.1.3. If $\mathfrak{r}_n \colon X \to \Gamma_n$ denotes the retraction map of X onto Γ_n, then φ_n is, up to translation by a constant, equal to $(\mathfrak{r}_n)^*\varphi$. A similar argument appears in [FRL06b, §4.6].

Notice that the construction of the sequence $(\varphi_n)_{n=1}^{\infty}$ heavily uses the tree structure on X (indeed, the same proof yields an analogous result for any metric tree) and moreover it depends on the choice of exhausting sequence of subtrees Γ_n. Without involving the analytic structure, it is unlikely that such a regularization result will generalize to higher-dimensional analytic spaces. The goal of this section is to construct a canonical regularization of a quasisubharmonic function on the Berkovich unit disc. The inspiration is the much-celebrated regularization theorem of Demailly [Dem92, Proposition 3.1].

Let us briefly recall the construction of the Demailly approximation of a plurisubharmonic function on the complex unit disc. Let $\mathbf{D}$ be the open unit disc in $\mathbf{C}$, and let φ be a plurisubharmonic function on $\mathbf{D}$. Consider the Hilbert space $\mathscr{H}_\varphi$ of holomorphic functions f on $\mathbf{D}$ satisfying the integrability condition

$$\|f\|_\varphi^2 := \int_{\mathbf{D}} |f|^2 e^{-2\varphi} d\lambda < +\infty,$$

where $d\lambda$ is the Lebesgue measure. For each $m \geq 1$, let $(f_{m,n})_{n=1}^{\infty}$ be an orthonormal basis of $\mathscr{H}_{m\varphi}$ and define a plurisubharmonic function φ_m on $\mathbf{D}$ by the formula

$$\varphi_m = \frac{1}{2m} \log \left(\sum_{n=1}^{\infty} |f_{m,n}|^2 \right).$$

This function φ_m is called the *Demailly approximation* associated to $\mathscr{H}_{m\varphi}$; φ_m has analytic singularities, in the sense that it can be locally written as

$$c \log \left(\sum_{i=1}^{N} |g_i|^2 \right) + \beta$$

for some constant $\alpha > 0$, local holomorphic functions g_i, and a locally bounded function β. The Demailly approximation may also be expressed as

$$\varphi_m = \sup_f \frac{1}{m} \log |f|,$$

where the supremum ranges over all holomorphic functions f in the unit ball of $\mathscr{H}_{m\varphi}$. In [Dem92, Proposition 3.1], it is shown that the sequence $(\varphi_m)_{m=1}^{\infty}$ converges pointwise (and in L^1_{loc}) to φ and, after passing to a subsequence, is decreasing in m.

In §5.3.1, given a quasisubharmonic function φ on the Berkovich unit disc X, we construct an ideal $\mathcal{H}_\varphi$ of the Tate algebra, and for each $m \geq 1$ we construct the non-Archimedean Demailly approximation φ_m associated to $\mathcal{H}_{m\varphi}$. Using Theorem 5.2.0.1, we show that the sequence $(\varphi_m)_{m=1}^{\infty}$ converges to φ. In §5.3.3, we briefly discuss the connection between the ideals $\mathcal{H}_\varphi$ and a locally-defined non-Archimedean multiplier ideal associated to φ.

5.3.1 Construction of the non-Archimedean Demailly approximation

Let φ be a quasisubharmonic function on X with $\varphi \leq 0$. The case when φ may be positive will be addressed later. As in §5.2, consider the function

$$\| \cdot \|_\varphi \colon k\{T\} \to \overline{\mathbf{R}}_+$$

defined by

$$\|f\|_\varphi := \sup_{x \in X \setminus Z(\varphi)} |f(x)| e^{-\varphi(x) - A(x)}$$

for $f \in k\{T\}$. The function $\| \cdot \|_\varphi$ is a non-Archimedean norm on the locus $\{\| \cdot \|_\varphi < +\infty\}$ in $k\{T\}$. The norms of this form are not submultiplicative in general, but they do satisfy a monotonicity property: if φ, ϕ are quasisubharmonic and such that $\varphi \le \phi \le 0$, then $\|f\|_\phi \le \|f\|_\varphi$ for any $f \in k\{T\}$. In particular, the limit

$$\|f\|_\varphi^+ := \lim_{\epsilon \to 0^+} \|f\|_{(1+\epsilon)\varphi}$$

exists (though it may be infinite). Consider the subset $\mathcal{H}_\varphi$ of the Tate algebra consisting of those series $f \in k\{T\}$ such that $\|f\|_\varphi^+ < +\infty$. The function $\|\cdot\|_\varphi^+$ defines a non-Archimedean norm on $\mathcal{H}_\varphi$, which is not submultiplicative in general. With this norm, $\mathcal{H}_\varphi$ is a normed $k\{T\}$-module.

Lemma 5.3.1.1. *The subset $\mathcal{H}_\varphi$ is a principal ideal of $k\{T\}$, which is complete for the norm $\|\cdot\|_\varphi^+$. In particular, $\mathcal{H}_\varphi$ is a Banach $k\{T\}$-module.*

Proof. It is clear that $\mathcal{H}_\varphi$ is closed under addition. Given $g \in k\{T\}$, the image of the function $x \mapsto |g(x)|$ is contained in the interval $[0, |g(x_G)|]$. For any $f \in \mathcal{H}_\varphi$ and for any $\epsilon > 0$,

$$\|fg\|_{(1+\epsilon)\varphi} \le |g(x_G)| \cdot \|f\|_{(1+\epsilon)\varphi}.$$

In particular, $fg \in \mathcal{H}_\varphi$. Furthermore, the Tate algebra $k\{T\}$ is a principal ideal domain (when k is nontrivially-valued, see [Bos14, Corollary 2.2.10]; otherwise, $k\{T\} = k[T]$) and hence $\mathcal{H}_{m\varphi}$ is principal.

Let $(f_j)_{j=1}^\infty \subset \mathcal{H}_\varphi$ be a Cauchy sequence in the norm $\|\cdot\|_\varphi^+$. It follows that $(f_j)_{j=1}^\infty$ is also a Cauchy sequence for the Gauss norm, since $|\cdot|_{x_G} \le \|\cdot\|_\varphi^+$. By the completeness of $k\{T\}$, the sequence $(f_j)_{j=1}^\infty$ admits a limit $f \in k\{T\}$. All ideals of $k\{T\}$ are closed, so $f \in \mathcal{H}_\varphi$. If $\mathcal{H}_\varphi = (h)$, then we can write $f_j = g_j h$ and $f = gh$ for some $g_j, g \in k\{T\}$ such that $g_j \to g$ in $|\cdot|_{x_G}$. For any $\delta > 0$, take $\epsilon > 0$ sufficiently small so that $\|h\|_{(1+\epsilon)\varphi} < +\infty$ and take $j \gg 0$ so that $|(g_j - g)(x_G)| < \frac{\delta}{\|h\|_\varphi^+}$. It follows that

$$\begin{aligned}\|f_j - f\|_\varphi^+ &\leq \|f_j - f\|_{(1+\epsilon)\varphi} \\ &\leq \left(\sup_{x \in X \setminus Z(\varphi)} |(g_j - g)(x)| \right) \|h\|_{(1+\epsilon)\varphi} \\ &= |(g_j - g)(x_G)| \cdot \|h\|_{(1+\epsilon)\varphi} < \frac{\delta}{\|h\|_\varphi^+} \|h\|_{(1+\epsilon)\varphi} \\ &\leq \delta.\end{aligned}$$

Therefore, $f_j \to f$ in the norm $\|\cdot\|_\varphi^+$, and so $\mathcal{H}_\varphi$ is complete with respect to $\|\cdot\|_\varphi^+$. □

Proposition 5.3.1.2. *Let φ, ϕ be quasisubharmonic functions on X with $\varphi, \phi \leq 0$. If there exist $C_1, C_2 > 0$ such that $\phi - C_2 \leq \varphi \leq \phi + C_1$ on X, then*

$$e^{-C_1} \| \cdot \|_\phi^+ \leq \| \cdot \|_\varphi^+ \leq e^{C_2} \| \cdot \|_\phi^+$$

as functions on $k\{T\}$. In particular, $\mathcal{H}_\varphi$ and $\mathcal{H}_\phi$ coincide as ideals of $k\{T\}$ and the identity map between them is an isomorphism of Banach $k\{T\}$-modules. Moreover, if φ is bounded, then $\mathcal{H}_\varphi = k\{T\}$.

The proof of Proposition 5.3.1.2 is elementary. However, Proposition 5.3.1.2 may be used to define $\mathcal{H}_\varphi$ for a quasisubharmonic function φ such that $\sup_X \varphi > 0$, in which case it is not clear that the limit defining the norm $\| \cdot \|_\varphi^+$ exists.

The ideals $\mathcal{H}_\varphi$ satisfy the subadditivity-type property below.

Proposition 5.3.1.3. *If φ, ϕ are quasisubharmonic functions on X, then*

$$\mathcal{H}_{\varphi+\phi} \subseteq \mathcal{H}_\varphi \mathcal{H}_\phi.$$

The proof of Proposition 5.3.1.3 requires studying the extension of the ideal $\mathcal{H}_\varphi$ to each local ring $\mathcal{O}_{X,x}$ of X. This is discussed in §5.3.3. This proof is simpler than in the complex setting, where the proof of the subadditivity theorem for multiplier ideals relies crucially on the Ohsawa–Takegoshi theorem on the unit bidisc.

We will now define the non-Archimedean analogue of a plurisubharmonic function with analytic singularities.

Definition 5.3.1.4. A quasisubharmonic function φ on X is said to have *analytic singularities* if there exists a cover $\{V_i\}_{i\in I}$ of X by affinoid domains such that for each $i \in I$, there are analytic functions $f_{i,1}, \dots, f_{i,n_i} \in \mathcal{O}_X(V_i)$, positive numbers $\alpha_{i,1}, \dots, \alpha_{i,n_i} > 0$, and a bounded function $\beta_i \colon V_i \to \mathbf{R}$ such that

$$\varphi|_{V_i} = \beta_i + \sum_{j=1}^{n_i} \alpha_{i,j} \log |f_{i,j}| \quad \text{on } V_i.$$

Quasisubharmonic functions with analytic singularities, like their complex counterparts, are quite well-behaved. One instance of this is illustrated below.

Example 5.3.1.5. Let φ be a quasisubharmonic function on X with analytic singularities and suppose it admits a decomposition $\varphi = \beta + \alpha \log |f|$, with $f \in k\{T\}$ irreducible, $\alpha > 0$, and $\beta \colon X \to \mathbf{R}$ bounded. Then, $\mathcal{H}_\varphi = (f^{\lfloor \alpha \rfloor})$.

Given a quasisubharmonic function φ on X and a positive integer $m \geq 1$, the *non-Archimedean Demailly approximation* associated to $\mathcal{H}_{m\varphi}$ is the function

$$\varphi_m := \frac{1}{m} \left(\sup_{f \in \mathcal{H}_{m\varphi} \setminus \{0\}} \log \frac{|f|}{\|f\|^+_{m\varphi}} \right)^*,$$

where $(-)^*$ denotes the upper-semicontinuous regularization. When k is nontrivially-valued, φ_m may equivalently be defined in terms of a supremum over

$$f \in B(1)_{m\varphi} := \{f \in \mathcal{H}_{m\varphi} \colon \|f\|^+_{m\varphi} \leq 1\},$$

the unit ball in $\mathcal{H}_{m\varphi}$. This mirrors the definition of the Demailly approximation in the complex setting.

Proposition 5.3.1.6. *Let φ be a quasisubharmonic function on X and let $m \geq 1$. The non-Archimedean Demailly approximation φ_m is quasisubharmonic with analytic singularities.*

Proof. For each $f \in \mathcal{H}_{m\varphi}$, $\frac{1}{m} \log \frac{|f|}{\|f\|^+_{m\varphi}} \leq \varphi(x_G)$ and hence by Lemma 5.1.4.1, the function φ_m is quasisubharmonic. Let $h \in \mathcal{H}_{m\varphi}$ be any generator. To show φ_m has analytic singularities, it suffices to show that the function

$$\beta(x) \coloneqq \frac{1}{m}\left(\sup_{f \in \mathcal{H}_{m\varphi} \setminus \{0\}} \log \frac{|f(x)|}{\|f\|^+_{m\varphi}}\right)^* - \frac{1}{m} \log \frac{|h(x)|}{\|h\|^+_{m\varphi}}$$

is bounded. As $\|gh\|^+_{m\varphi} \geq |g(x_G)||h(x_G)|e^{-m\varphi(x_G)} \neq 0$ for any nonzero $g \in k\{T\}$, we have

$$\sup_{g \in k\{T\} \setminus \{0\}} \log \frac{|h(x)||g(x)|}{\|gh\|^+_{m\varphi}} \leq \log \frac{|h(x)|}{|h(x_G)|} + m\varphi(x_G).$$

This upper bound is upper semicontinuous in x, so it follows that

$$\beta(x) \leq \frac{1}{m} \log \frac{\|h\|^+_{m\varphi}}{|h(x_G)|} + \varphi(x_G).$$

For a lower bound, taking $g = 1$ gives $\beta \geq 0$. Note that the decomposition $\varphi_m = \beta + \frac{1}{m} \log \frac{|h(x)|}{\|h\|^+_{m\varphi}}$ is not unique – it depends on the choice of generator h of the ideal $\mathcal{H}_{m\varphi}$. □

Remark 5.3.1.7. It is not true in general that, if φ has analytic singularities, then $\varphi_m = \varphi$ for all $m \geq 1$. Rather, φ_m is an "algebraic approximation" to φ, as the following example demonstrates. Let $\varphi = \alpha \cdot \log |f|$ for some $\alpha > 0$ and $f \in k\{T\}$ irreducible. It is easy to verify that

$$\varphi_m = \frac{\lfloor m\alpha \rfloor}{m} \log |f|.$$

If $x \in X^{\mathrm{rig}}$ is the rigid point corresponding to the maximal ideal (f) of $k\{T\}$, then φ_m has rational slope along the branch $[x, x_G]$ of X; however, this will not be the case for φ if $\alpha \in \mathbf{R} \setminus \mathbf{Q}$.

Example 5.3.1.8. The sequence $(\varphi_m)_{m=1}^{\infty}$ does not, in general, decrease monotonically in m. If $\varphi = \frac{3}{2}\log|T|$, then

$$\varphi_m = \begin{cases} \varphi, & m = 2n \\ \frac{3n+1}{2n+1}\log|T|, & m = 2n+1. \end{cases}$$

In particular, $\varphi_{2n} = \varphi$ for all $n \geq 0$, but the subsequence $(\varphi_{2n+1})_{n\geq 0}$ decreases monotonically to φ. It is not clear whether or not the sequence φ_m admits a decreasing subsequence in general. For related results in the complex case, see [Kim14].

5.3.2 A regularization theorem

The following is a non-Archimedean analogue of Demailly's regularization theorem [Dem92, Proposition 3.1].

Theorem 5.3.2.1. *Assume k is algebraically closed, trivially-valued, or is spherically complete of residue characteristic zero. Let φ be a quasisubharmonic function on X with $\varphi \leq 0$, and let $m \geq 1$. For any $x \in X$,*

$$\varphi(x) \leq \varphi_m(x) \leq \varphi(x) + \frac{1}{m}A(x).$$

In particular, φ_m converges pointwise to φ on $\{A < +\infty\} \subseteq X$.

The estimate in Theorem 5.3.2.1 in fact yields a stronger assertion than just the pointwise convergence of the non-Archimedean Demailly approximation φ_m to φ. Indeed, for any compact subset $K \subseteq X$ such that $\alpha|_K$ is bounded above, φ_m converges uniformly to φ; this is reminiscent of the L^1_{loc} convergence of the Demailly approximation in the complex case. The estimate in Theorem 5.3.2.1 can thus be thought of as asserting that φ_m converges to φ in a non-Archimedean version of the Hartog's sense, as introduced in [DS09, Definition 3.2.3].

The key ingredient in the proof of Theorem 5.3.2.1 is Theorem 5.2.0.1. Moreover, the proof shows that the hypothesis that $\varphi \leq 0$ is only needed for the upper bound, whereas the lower bound always holds.

Proof. Consider first the upper bound on φ_m. If $A(x) = +\infty$, then the upper bound is trivial. Suppose that $A(x) < +\infty$, in which case $\alpha(x) < +\infty$ as well, so $\varphi_m(x)$ may be calculated without the upper-semicontinuous regularization by Lemma 5.1.4.1. The upper bound then follows from the observation that

$$m(\varphi_m - \varphi) - A = \sup_{f \in \mathcal{H}_{m\varphi}} \log \frac{|f|e^{-m\varphi - A}}{\|f\|^+_{m\varphi}} \leq 0,$$

where we have used that $\varphi \leq 0$ to conclude that $|f|e^{-m\varphi-A} \leq \|f\|^+_{m\varphi}$.

To get the lower bound, let $\phi := \varphi - \varphi(x_G)$ and observe that $\mathcal{H}_{m\varphi} = \mathcal{H}_{m\phi}$ and $\|f\|^+_{m\phi} = e^{m\varphi(x_G)}\|f\|^+_{m\varphi}$ for any $f \in \mathcal{H}_{m\varphi}$. For any $x \in X$, Theorem 5.2.0.1 asserts that there exists an $\epsilon_0 > 0$ and a $(m\phi, \epsilon_0)$-extension $f \in k[T]$ at x. In particular, $f \in \mathcal{H}_{m\phi}$ and

$$1 \leq \frac{|f(x)|}{\|f\|^+_{m\phi}} e^{-m\phi(x)} = \frac{|f(x)|}{\|f\|^+_{m\varphi}} e^{-m\varphi(x)},$$

or equivalently $\varphi(x) \leq \frac{1}{m} \log \frac{|f(x)|}{\|f\|^+_{m\varphi}} \leq \varphi_m(x)$. □

Remark 5.3.2.2. A quasisubharmonic function cannot, in general, be bounded pointwise below by a quasisubharmonic function with analytic singularities. Indeed, a general quasisubharmonic function can have a dense set of poles in X: for example, if $\{a_j \colon j \geq 0\}$ forms a dense subset of k°, set

$$\varphi := \sum_{j=0}^{\infty} c_j \log |T - a_j|,$$

for some $c_j > 0$ satisfying $\sum_{j=0}^{\infty} c_j = 1$. However, a quasisubharmonic function with analytic singularities on X must have a finite set of poles. This is analogous to the fact that a subharmonic function with analytic singularities on the complex unit disc has a discrete set of poles.

5.3.3 Non-Archimedean multiplier ideals

Let $\mathcal{A} = k\{T\}$ and $X = \mathcal{M}(\mathcal{A})$. The stalk $\mathcal{O}_{X,x}$ of the structure sheaf $\mathcal{O}_X$ at $x \in X$ is a noetherian local ring, which may be computed as the direct limit of the affinoid algebras $\mathcal{A}_V$ over all affinoid neighborhoods V of x. The unique maximal ideal $\underline{\mathfrak{m}}_x$ of $\mathcal{O}_{X,x}$ consists of those germs $f \in \mathcal{O}_{X,x}$ such that $|f(x)| = 0$. We say that a germ $f \in \mathcal{O}_{X,x}$ is *defined on* an affinoid domain $V \subset X$ if it lies in the image of the natural map $\mathcal{A}_V \to \mathcal{O}_{X,x}$.

For $x \in X^{\mathrm{rig}}$, let $\mathfrak{m}_x$ denote the corresponding maximal ideal of $\mathcal{A}$. As in [BGR84, 7.3.2/1], $\underline{\mathfrak{m}}_x$ is the extension of $\mathfrak{m}_x$ along the natural map $\mathcal{A} \to \mathcal{O}_{X,x}$ (though this is false for arbitrary points of an affinoid space, as demonstrated by [Ber93, Remark 2.2.9]). Moreover, for $x \in X^{\mathrm{rig}}$, the natural map $\mathcal{A} \to \mathcal{O}_{X,x}$ factors through the localization $\mathcal{A}_{\mathfrak{m}_x}$, which induces an isomorphism $\widehat{\mathcal{A}}_{\mathfrak{m}_x} \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}$ on completions.

In complex geometry, we associate to each plurisubharmonic function (or more generally, to a semipositive metric on a line bundle) a *multiplier ideal sheaf*, which measures the singularities of the plurisubharmonic function. We propose a non-Archimedean analogue of this notion.

Definition 5.3.3.1. Let φ be a quasisubharmonic function on X with $\varphi \leq 0$. For any $x \in X$, the *local multiplier ideal* of φ at x, denoted $\mathcal{J}(\varphi)_x$, consists of those germs $f \in \mathcal{O}_{X,x}$ such that for all $\epsilon > 0$, there is an affinoid neighborhood V_ϵ of x on which f is defined such that

$$\sup_{V_\epsilon \setminus Z(\varphi)} |f| e^{-(1+\epsilon)\varphi - A} < +\infty.$$

It is easy to check that the local multiplier ideal $\mathcal{J}(\varphi)_x$ is indeed an ideal of the local ring $\mathcal{O}_{X,x}$. Furthermore, it is immediate from the definition that the local multiplier ideals satisfy $\mathcal{J}(\varphi+\beta)_x = \mathcal{J}(\varphi)_x$ where β is a bounded upper-semicontinuous function on X. In particular, if $\sup_X \varphi > 0$, we may define the local multiplier ideal of φ at x to be $\mathcal{J}(\varphi - \varphi(x_G))_x$. For simplicity, we assume from now on that all quasisubharmonic functions are nonpositive.

The name of a multiplier ideal is justified by examples of the following form: if $f \in \mathcal{A}$ is irreducible, $c > 0$, and $\varphi := c \cdot \log|f|$, then $\mathcal{J}(\varphi)_x = \underline{\mathfrak{m}}_x^{\lfloor c \rfloor}$, where $x \in X^{\mathrm{rig}}$ is the rigid point corresponding to the maximal ideal (f) of $\mathcal{A}$. More generally, the local multiplier ideals admit a similar description for a general quasisubharmonic function, as is made precise in the following lemma.

Lemma 5.3.3.2. *Let $x \in X$.*

1. *For $x \in X^{\mathrm{rig}}$, $\mathcal{J}(\varphi)_x = \underline{\mathfrak{m}}_x^{\lfloor c_x \rfloor}$, where $c_x := \Delta\varphi\{x\}/m(x)$.*

2. *For $x \in X \backslash X^{\mathrm{rig}}$, $\mathcal{J}(\varphi)_x = \mathcal{O}_{X,x}$.*

The quantity c_x may be thought of as the "non-Archimedean Lelong number" of φ at the point x.

Proof. Fix $x \in X^{\mathrm{rig}}$. For a germ $f \in \mathcal{O}_{X,x}$, let $\mathrm{ord}_x(f)$ denote the maximal power of $\underline{\mathfrak{m}}_x$ to which f belongs. To say $f \in \mathcal{J}(\varphi)_x$ is equivalent the existence of an $\epsilon > 0$ such that $\mathrm{ord}_x(f) - (1+\epsilon)c_x + 1 \geq 0$, which occurs if and only if $\mathrm{ord}_x(f) > c_x - 1$, i.e. $\mathrm{ord}_x(f) \geq \lfloor c_x \rfloor$. Thus, $\mathcal{J}(\varphi)_x = \underline{\mathfrak{m}}_x^{\lfloor c_x \rfloor}$.

Fix $x \in X \backslash X^{\mathrm{rig}}$ and any $\epsilon_0 > 0$. There are only finitely-many rigid points $z_1, \ldots, z_n \in X^{\mathrm{rig}}$ satisfying $(1+\epsilon_0)c_{z_i} \geq 1$, so we may find an affinoid neighborhood V of x which avoids $z_1, \ldots, z_n$ and hence

$$\sup_{V \backslash Z(\varphi)} |1| e^{-(1+\epsilon)\varphi - A} < +\infty$$

for all $\epsilon \in [0, \epsilon_0]$. Thus, $1 \in \mathcal{J}(\varphi)_x$. ∎

One can show that the local multiplier ideals $\mathcal{J}(\varphi)_x$ arise as the stalks of a coherent sheaf of ideals on X. More precisely, the local multiplier ideals satisfy a "coherence" property similar to [Dem12, Proposition 5.7], originally due to Nadel [Nad90].

Lemma 5.3.3.3. *For $x \in X^{\mathrm{rig}}$, $\mathcal{J}(\varphi)_x = \mathcal{H}_\varphi \cdot \mathcal{O}_{X,x}$.*

Proof. The inclusion $\mathcal{H}_\varphi \cdot \mathcal{O}_{X,x} \subseteq \mathcal{J}(\varphi)_x$ is clear. The local ring $\mathcal{O}_{X,x}$ is a dvr, so the ideal $\mathcal{H}_\varphi \cdot \mathcal{O}_{X,x}$ can be written as $\underline{\mathfrak{m}}_x^d$ for some $d \in \mathbf{Z}_{\geq 0}$; the power d is the largest integer such that $\mathcal{H}_\varphi \subseteq \mathfrak{m}_x^d$. We first show that $d \geq \lfloor c_x \rfloor$: pick a net $(x_j)_{j\in J}$ in $\{\alpha < +\infty\}$ that converges to x such that $x_j \geq x$ for all $j \in J$. After shifting φ by a constant, assume that $\varphi(x_G) = 0$, and arguing as in [Jon15, Lemma 2.9] we have

$$\begin{aligned}\varphi(x_j) &= -\int_{x_G}^{x_j} (\Delta\varphi)\{y \in X \colon y \leq x_j\} d\alpha(y) \\ &\leq -\int_{x_G}^{x_j} (\Delta\varphi)\{x\} d\alpha(y) \\ &= -\Delta\varphi\{x\}\alpha(x_j).\end{aligned}$$

Fix a generator g_x of $\mathfrak{m}_x$. For any $f \in \mathcal{H}_\varphi$, there is a unique factorization $f = g_x^a h$ (up to units) for some $a \in \mathbf{Z}_{\geq 0}$ and $h \in \mathcal{A}\backslash\mathfrak{m}_x$. For sufficiently small $\epsilon > 0$, the above upper bound on $\varphi(x_j)$ gives

$$\begin{aligned}+\infty > \sup_{X\backslash Z(\varphi)} |f|e^{-(1+\epsilon)\varphi - A} &\geq \sup_{j\in J} |f(x_j)|e^{-(1+\epsilon)\varphi(x_j) - A(x_j)} \\ &\geq \sup_{j\in J} |h(x_j)|e^{-am(x_j)\alpha(x_j) + (1+\epsilon)(\Delta\varphi\{x\})\alpha(x_j) - m(x_j)\alpha(x_j)} \\ &\geq \sup_{j\in J} |h(x_j)|e^{\alpha(x_j)(-am(x) + (1+\epsilon)(\Delta\varphi\{x\}) - m(x))}\end{aligned}$$

where we have used that $m(x) \geq m(x_j)$ and $m(x_j)\alpha(x_j) \geq A(x_j)$, which follows from (5.1). This last supremum is finite only if

$$-am(x) + (1+\epsilon)(\Delta\varphi\{x\}) - m(x) \leq 0,$$

and this must hold for all $\epsilon > 0$ sufficiently small; said differently, $a \geq \lfloor c_x \rfloor$. Thus, $d \geq \lfloor c_x \rfloor$.

Now, any generator f of $\mathcal{H}_\varphi$, there is a unique way to write $f = g_x^d h$ (up to units) for some $h \in \mathcal{A} \backslash \mathfrak{m}_x$. If $d > \lfloor c_x \rfloor$, then it is easy to check that $g_x^{\lfloor c_x \rfloor} h$ lies in $\mathcal{H}_\varphi$ (it is the same calculation as above near x, and hence $g_x^{\lfloor c_x \rfloor} h$ must be a generator of $\mathcal{H}_\varphi$; this is a contradiction, so $d = \lfloor c_x \rfloor$. Thus, $\mathcal{H}_\varphi \cdot \mathcal{O}_{X,x} = \underline{\mathfrak{m}}_x^{\lfloor c_x \rfloor}$ and we apply Lemma 5.3.3.2 to conclude. □

It is not hard to check that the local multiplier ideals satisfy many of the usual properties of multiplier ideals on complex algebraic varieties, e.g. invariance under small perturbations, the behavior under adding integral divisors, and the subadditivity property (in the subsequent lemma). This further justifies the terminology.

Lemma 5.3.3.4. *If φ, ϕ are quasisubharmonic functions on X and $x \in X$, then*

$$\mathcal{J}(\varphi + \phi)_x \subseteq \mathcal{J}(\varphi)_x \mathcal{J}(\phi)_x.$$

Proof. The assertion follows from Lemma 5.3.3.2 and the observation that

$$\lfloor \Delta(\varphi + \phi)\{x\} \rfloor \geq \lfloor \Delta\varphi\{x\} \rfloor + \lfloor \Delta\phi\{x\} \rfloor.$$

□

Proof of Proposition 5.3.1.3. After translating φ and ϕ by constants, we may assume that they are nonpositive. The inclusion $\mathcal{H}_{\varphi+\phi} \subseteq \mathcal{H}_\varphi \mathcal{H}_\phi$ holds if and only for every maximal ideal $\mathfrak{m}_x$ of $\mathcal{A}$, there is an inclusion $\mathcal{H}_{\varphi+\phi} \widehat{\mathcal{A}}_{\mathfrak{m}_x} \subseteq (\mathcal{H}_\varphi \mathcal{H}_\phi) \widehat{\mathcal{A}}_{\mathfrak{m}_x}$ of the extensions along $\mathcal{A} \to \widehat{\mathcal{A}}_{\mathfrak{m}_x}$. The claim then follows from Lemma 5.3.3.3 and Lemma 5.3.3.4. □

As demonstrated by Lemma 5.3.3.2, the local multiplier ideals on the Berkovich unit disc are somewhat degenerate, principally because the types of singularities of quasisubharmonic functions that can occur in one dimension are limited. In this paper, they serve only to prove Proposition 5.3.1.3. However, the definition naturally extends to higher dimensions and, in that more complicated setting, could prove quite useful.

Remark 5.3.3.5. Given a quasisubharmonic function φ on X and $x \in X$, it is easy to see that ideals $\mathcal{H}_\varphi$ and $\mathcal{J}(\varphi)_x$ satisfy an analogue of the openness conjecture. That is, $\mathcal{H}_\varphi = \bigcup_{\epsilon > 0} \mathcal{H}_{(1+\epsilon)\varphi}$ and similarly for $\mathcal{J}(\varphi)_x$. The openness conjecture, for a plurisubharmonic function defined on a neighborhood of the origin in $\mathbf{C}^n$, has been solved in [Ber13].

CHAPTER VI

On the geometric P=W conjecture

The theory of weight functions and of essential skeletons developed in the previous chapters can be employed to describe certain dual complexes that are of particular interest in non-abelian Hodge theory. The goal of this section is the study of the dual boundary complex $\mathcal{D}(\partial M_{\mathrm{GL}_n})$ of the GL_n-character variety M_{GL_n} associated to a Riemann surface of genus one, and to prove Theorems G and H from the introduction. Throughout, all varieties are defined over $\mathbf{C}$, which is thought of as a non-Archimedean field equipped with the trivial norm.

6.1 The geometric $P = W$ conjecture.

In this section, we give a brief overview of the geometric $P = W$ conjecture, formulated by Katzarkov, Noll, Pandit, and Simpson in [KNPS15, Conjecture 1.1]; see alternatively [Sim16, Conjecture 11.1]. To do so, we begin with the cornerstone of non-abelian Hodge theory, namely the Corlette–Simpson correspondence. For a reductive algebraic group G, this correspondence is a homeomorphism between the G-character variety, or *Betti moduli space*,

$$M_B \coloneqq \mathrm{Hom}(\pi_1(X), G) /\!\!/ G$$

of G-representations of the topological fundamental group of a smooth curve X over $\mathbf{C}$, and Hitchin's moduli space M_{Dol} of semistable principal Higgs G-bundles on X with vanishing Chern classes, also known as the *Dolbeault moduli space*. See e.g. [Sim94] for further details and generalizations.

The spaces M_B and M_{Dol} are non-proper varieties, and the "behaviour at infinity" of the Corlette–Simpson correspondence is a topic of great interest in the literature; see [KNPS15]. To that end, consider the following extra data:

- a compactification $\overline{M}_B$ (resp. $\overline{M}_{\text{Dol}}$) of M_B (resp. of M_{Dol});
- the boundary $\partial M_B \coloneqq \overline{M}_B \setminus M_B$ (resp. $\partial M_{\text{Dol}} \coloneqq \overline{M}_{\text{Dol}} \setminus M_{\text{Dol}}$);
- a neighbourhood at infinity N_B (resp. N_{Dol}) of M_B (resp. of M_{Dol}), i.e. a tubular neighbourhood of ∂M_B (resp. of ∂M_{Dol});
- a punctured neighbourhood $N_B^* \coloneqq N_B \setminus \partial M_B$ (resp. $N_{\text{Dol}}^* \coloneqq N_{\text{Dol}} \setminus \partial M_{\text{Dol}}$) of ∂M_B (resp. of ∂M_{Dol}).

Note that the Corlette–Simpson correspondence induces a homotopy equivalence $N_{\text{Dol}}^* \sim N_B^*$. Hitchin's moduli space M_{Dol} comes equipped with the Hitchin map

$$H : M_{\text{Dol}} \to \mathbf{C}^N,$$

with $2N = \dim_{\mathbf{C}}(M_{\text{Dol}})$ (see [Hit87, Equation 4.4]), which induces a map from N_{Dol}^* to a neighbourhood at infinity of $\mathbf{C}^N$. Composing with the radial projection to the sphere $\mathbb{S}^{2N-1}$, we obtain a map

$$h : N_{\text{Dol}}^* \xrightarrow{H} \mathbf{C}^N \setminus \{0\} \xrightarrow{\sim} \mathbb{S}^{2N-1}.$$

Now, assume that the dual boundary complex $\mathcal{D}(\partial M_B)$ is well-defined. By means of a partition of unity, one can define a map from N_B^* to the dual boundary complex

$\mathcal{D}(\partial M_B)$, written

$$\alpha : N_B^* \to \mathcal{D}(\partial M_B).$$

If ∂M_B is an snc divisor, then the homotopy type of $\mathcal{D}(\partial M_B)$ is independent of the choice of the snc compactification by the works of many authors, e.g. [Dan75, Ste06, KS06, Thu07, ABW13, Pay13].

The geometric $P = W$ conjecture proposes a correspondence between the dual boundary complex of M_B and the sphere at infinity of the Hitchin base for M_{Dol}.

Conjecture 6.1.0.1 (Geometric $P = W$ conjecture)**.** *There exists a homotopy equivalence*

$$\mathcal{D}(\partial M_B) \sim \mathbb{S}^{2N-1} \tag{6.1}$$

such that the following diagram is homotopy commutative

$$\begin{array}{ccc} N_{\text{Dol}}^* & \xrightarrow{\sim} & N_B^* \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathbb{S}^{2N-1} & \xrightarrow{\sim} & \mathcal{D}(\partial M_B). \end{array} \tag{6.2}$$

The results in [Sim16] provide evidence for the conjecture: when M_B is the SL_2-character variety of local systems on a punctured sphere (such that conjugacy classes of the monodromies around the punctures are fixed), Simpson proves in [Sim16, Theorem 1.1] that the dual boundary complex $\mathcal{D}(\partial M_B)$ has the homotopy type of a sphere; see also [Kom15, Theorem 1.4]. However, there is no known proof of the commutativity of the diagram 6.2. In the same paper, Simpson suggests to study the case of character varieties associated to compact Riemann surfaces. In the sequel, we explain our contribution in the genus one case.

6.2 Dual boundary complex of GL_n-character varieties of a genus one surface

The goal of this section is to prove the following result, stated as Theorem G in the introduction.

Theorem 6.2.0.1. *The dual boundary complex $\mathcal{D}(\partial M_{\mathrm{GL}_n})$ of a dlt log Calabi–Yau compactification of M_{GL_n} has the homeomorphism type of $\mathbb{S}^{2n-1}$.*

This character variety admits a concrete description: one can show that M_{GL_n} is the n-fold symmetric product of the two-dimensional algebraic torus $\mathbf{C}^* \times \mathbf{C}^*$; see e.g. [FT16, Corollary 5.6]. Symmetric products of toric surfaces are natural candidates for compactifications of M_{GL_n}. However, these compactifications are not dlt, although log canonical and log Calabi–Yau (see §6.2.2). In §6.2.3 we adapt the strategy of [KX16] to prove Theorem G for $n = 2$. For higher n, this approach is not sufficient and we instead employ techniques from Berkovich geometry (see §6.2.4 and §6.2.6).

It is worth remarking that a related conjecture, known as cohomological $P = W$ conjecture, holds for a crepant resolution of M_{GL_n} thanks to [dCHM13]. For more details about this cohomological version, we refer the interested reader to [dCHM12] and to the excellent survey [Mig17].

6.2.1 Dlt modifications and dual complexes

Given a log canonical (lc) pair (X, Δ), a *lc centre* of the pair is the centre of a divisorial valuation $x \in X^{\beth}$ with $A_{(X,\Delta)}(x) = 0$. The *snc locus* X^{snc} is the largest open subset in X such that the pair (X, Δ) restricts to an snc pair. The pair (X, Δ) is said to be *divisorial log terminal* (dlt) if none of its lc centres are contained in $X \setminus X^{\mathrm{snc}}$; see [KM08, Definition 2.37] for more details. Recall that two pairs (X, Δ_X)

and (Y,Δ_Y) are *crepant birational* if X and Y are birational and $A_{(X,\Delta_X)} = A_{(Y,\Delta_Y)}$ as functions on $X^{\mathrm{bir}} = Y^{\mathrm{bir}}$; see [Kol13, Definition 2.23] for further details.

There are several advantages to working with dlt pairs over snc pairs. Most notably, we use the fact that any lc pair (X,Δ) is crepant birational to a (non-unique) dlt pair $(X^{\mathrm{dlt}},\Delta^{\mathrm{dlt}})$, while the corresponding statement fails in general for snc pairs. This fact is a consequence of the existence of *dlt modifications*, as in [Kol13, Corollary 1.36], which asserts that there exists a proper birational morphism $g : X^{\mathrm{dlt}} \to X$ with exceptional divisors $\{E_i\}_{i\in I}$ such that

1. (dlt) the pair $(X^{\mathrm{dlt}},\Delta^{\mathrm{dlt}} := g_*^{-1}\Delta + \sum_{i\in I} E_i)$ is dlt, where $g_*^{-1}\Delta$ is the strict transform of Δ via g;
2. (crepant) $K_{X^{\mathrm{dlt}}} + \Delta^{\mathrm{dlt}} \sim_{\mathbf{Q}} g^*(K_X + \Delta)$.

It is always possible to construct a dual intersection complex for a dlt pair (X,Δ) by following the same prescriptions as for snc pairs (while this is not in general possible for lc pairs). In fact, this coincides with the dual complex of the snc pair

$$(X^{\mathrm{snc}}, \Delta^{=1}|_{X^{\mathrm{snc}}})$$

by [dFKX17, §2]. The dual complex of a lc pair (X,Δ) can be defined as the homeomorphism class of the dual complex of any dlt modification of (X,Δ), and it is denoted by $\mathcal{DMR}(X,\Delta)$; the notation is an abbreviation for Dual complex of a Minimal dlt partial Resolution. The homeomorphism class $\mathcal{DMR}(X,\Delta)$ is well-defined, as it is independent of the choice of a dlt modification by [dFKX17, Definition 15].

6.2.2 Hilbert scheme of n points of a toric surface

Let Z be a smooth, projective toric surface, and let Δ be its toric boundary. Let Σ_Z be a toric fan for Z, write $|\Sigma_Z|$ for its support, and $|\Sigma_Z|^* := |\Sigma_Z| \backslash \{0\}$. Note that:

1. $Z^+ := (Z, \Delta)$ is an snc logCY pair;

2. $Z \setminus \Delta \simeq \mathbf{C}^* \times \mathbf{C}^*$;

3. $\mathcal{D}(Z^+) \simeq \mathrm{Sk}^{\mathrm{ess}}(Z, \Delta)^*/\mathbf{R}_+^* = \mathrm{Sk}(Z^+)^*/\mathbf{R}_+^* = |\Sigma_Z|^*/\mathbf{R}_+^* \simeq (\mathbf{R}^2\backslash\{0\})/\mathbf{R}_+^* \simeq \mathbb{S}^1$.

Denote by $Z^{[n]}$ the Hilbert scheme of n points of Z; see [Bea83, §6] for an overview of the construction. Recall that the Hilbert scheme of n points of a projective surface is a crepant resolution of its n-fold symmetric product. In a diagram, we have

$$\begin{array}{ccc} & & Z^n := \underbrace{Z \times \ldots \times Z}_{n-\text{times}} \\ & & \big\downarrow q \\ Z^{[n]} & \xrightarrow{\rho_{\mathrm{HC}}} & Z^{(n)} := Z^n/\mathfrak{S}_n, \end{array}$$

where the crepant birational map ρ_{HC} is the Hilbert–Chow morphism, and q is the quotient of the product Z^n by the action of the symmetric group $\mathfrak{S}_n$ of degree n, which acts by permuting the factors of Z^n. This gives rise to the following diagram of lc logCY pairs:

$$\begin{array}{ccc} & & (Z^n, \Delta^n := \mathrm{pr}_1^* \Delta + \ldots + \mathrm{pr}_n^* \Delta)\,. \\ & & \big\downarrow q \\ (Z^{[n]}, \Delta^{[n]} := {\rho_{\mathrm{HC}}}^* \Delta^{(n)}) & \xrightarrow{\rho_{\mathrm{HC}}} & (Z^{(n)}, \Delta^{(n)} := q_* \Delta^n) \end{array}$$

The variety $Z^{(n)}$ is a compactification of $M_{\mathrm{GL}_n} \simeq (\mathbf{C}^* \times \mathbf{C}^*)^{(n)}$, since we can identify $\mathbf{C}^* \times \mathbf{C}^* \simeq Z \setminus \Delta \subseteq Z$. Further, since the lc pairs $(Z^{[n]}, \Delta^{[n]})$ and $(Z^{(n)}, \Delta^{(n)})$ are crepant birational, it follows from [dFKX17, Proposition 11] that

$$\mathcal{D}(\partial M_{\mathrm{GL}_n}) \simeq \mathcal{DMR}(Z^{(n)}, \Delta^{(n)}) \simeq \mathcal{DMR}(Z^{[n]}, \Delta^{[n]}) \simeq \mathcal{D}(\partial(\mathbf{C}^* \times \mathbf{C}^*)^{[n]}). \quad (6.3)$$

Remark 6.2.2.1. Unfortunately, the pair $(Z^{(n)}, \Delta^{(n)} = Z^{(n)} \setminus (\mathbf{C}^* \times \mathbf{C}^*)^{(n)})$ fails to be dlt. In light of (6.3), one could eventually consider the Hilbert scheme $Z^{[n]}$, but even in that case the compactification is not dlt, as we show in the following. For simplicity, in this section we will focus our attention on the case $n = 2$.

Let $(\mathbf{C}^2_{x_1,x_2}, (x_1x_2 = 0))$ be a local toric chart for (Z, Δ). As above, consider the product pair

$$(\mathbf{C}^2_{x_1,x_2} \times \mathbf{C}^2_{y_1,y_2}, (x_1x_2y_1y_2 = 0)).$$

There is an involution which swaps x_1 and x_2 with y_1 and y_2 respectively. Via the change of coordinates $(u, v, r, s) = (x_1 + y_1, x_2 + y_2, x_1 - y_1, x_2 - y_2)$, the involution sends (u, v, r, s) to $(u, v, -r, -s)$. Hence, the previous diagram has the following form:

$$\begin{array}{ccc} & & \mathbf{C}^2_{u,v} \times \mathbf{C}^2_{r,s} \\ & & \Big\downarrow q \\ \mathbf{C}^2_{u,v} \times \mathrm{Bl}_0(\mathbf{C}^2_{r,s}/(\mathbf{Z}/2\mathbf{Z})) & \xrightarrow{\rho_{HC}} & \mathbf{C}^2_{u,v} \times \mathbf{C}^2_{r,s}/(\mathbf{Z}/2\mathbf{Z}) \simeq \mathbf{C}^2_{u,v} \times \mathrm{Spec}\left(\frac{\mathbf{C}[x,y,z]}{(xz-y^2)}\right), \end{array}$$

where the maps q is given in coordinates by

$$q : (u, v, r, s) \mapsto (u, v, r^2, rs, s^2).$$

Consider the chart of the blowup $\mathrm{Bl}_0(\mathbf{C}^2_{r,s}/(\mathbf{Z}/2\mathbf{Z})) \subseteq \mathbf{C}^3_{x,y,z} \times \mathbf{P}^2_{[X:Y:Z]}$ given by

$$\begin{aligned} \mathbf{C}^2_{x',y'} &\hookrightarrow \mathrm{Bl}_0(\mathbf{C}^2_{r,s}/(\mathbf{Z}/2\mathbf{Z})) \subseteq \mathbf{C}^3_{x,y,z} \times \mathbf{P}^2_{[X:Y:Z]} \\ (x', y') &\mapsto ((x', x'y', x'y'^2), [1 : y' : y'^2]). \end{aligned}$$

In these local coordinates, the boundaries are given by the following equations:

1. $\Delta^2 = (x_1x_2y_1y_2 = 0) = ((u^2 - r^2)(v^2 - s^2) = 0)$;

2. $\Delta^{(2)} = ((u^2 - x)(v^2 - z) = 0)$;

3. $\Delta^{[2]} \overset{\text{loc}}{=} ((u^2 - x')(v^2 - x'y'^2) = 0)$.

One of the components of $\Delta^{(2)}$ and $\Delta^{[2]}$ is non-normal, and so none of the pairs $(X^{(2)}, \Delta^{(2)})$ and $(X^{[2]}, \Delta^{[2]})$ can be dlt by [KM08, Corollary 5.52].

6.2.3 A proof of Theorem 6.2.0.1 for $n = 2$.

The $n = 2$ case of Theorem 6.2.0.1 can be deduced from results in [KX16] and the Poincaré conjecture, as it is explained below. In the following lemma, the fundamental group of a variety refers to the topological fundamental group of the associated complex-analytic variety.

Lemma 6.2.3.1. *For $n \geq 2$, the dual boundary complex $\mathcal{D}(\partial(\mathbf{C}^* \times \mathbf{C}^*)^{[n]})$ is simply connected, i.e.*

$$\pi_1(\mathcal{D}(\partial(\mathbf{C}^* \times \mathbf{C}^*)^{[n]})) = 1.$$

Proof. Consider a dlt modification $h : (Z^{[n],\text{dlt}}, \Delta^{[n],\text{dlt}}) \to (Z^{[n]}, \Delta^{[n]})$. By [KX16, Theorem 36], there is a surjection of fundamental groups

$$\pi_1((Z^{[n],\text{dlt}})^{\text{sm}}) \longrightarrow \pi_1(\mathcal{DMR}(Z^{[n]}, \Delta^{[n]})),$$

where the superscript 'sm' denotes the restriction to the smooth locus. Since h is a birational contraction (that is, the exceptional locus of the inverse map h^{-1} has complex codimension $\geqslant 2$), there exists a surjection

$$\pi_1((Z^{[n]})^{\text{sm}}) \longrightarrow \pi_1((Z^{[n],\text{dlt}})^{\text{sm}})$$

by [KX16, Lemma 41]. However, $Z^{[n]}$ is smooth and rationally connected, and hence $\pi_1(Z^{[n]}) = 1$; see e.g. [Deb01, Corollary 4.18.(c)]. It follows that

$$\mathcal{DMR}(Z^{[n]}, \Delta^{[n]}) \simeq \mathcal{D}(\partial(\mathbf{C}^* \times \mathbf{C}^*)^{[n]})$$

is simply connected. □

Proof of Theorem 6.2.0.1 for $n = 2$. By [KX16], $\mathcal{DMR}(Z^{[2]}, \Delta^{[2]})$ is a real 3-manifold with the rational homology of the 3-sphere $\mathbb{S}^3$. By Lemma 6.2.3.1, it is also simply connected, and hence the Poincaré conjecture implies that it is homeomorphic to the 3-sphere $\mathbb{S}^3$. By (6.3), the same holds for $\mathcal{D}(\partial M_{\mathrm{GL}_2})$. □

The methods of the proof of Theorem 6.2.0.1 for the $n = 2$ case are not sufficient to prove the theorem in the general case. The problem is that they do not provide a control on the torsion of $H_i(\mathcal{D}(\partial(\mathbf{C}^* \times \mathbf{C}^*)^{(n)}), \mathbf{Z})$. In the sequel, we avoid this issue by constructing an explicit homeomorphism between $\mathcal{DMR}(Z^{(n)}, \Delta^{(n)})$ and the sphere $\mathbb{S}^{2n-1}$, which is a non-Archimedean avatar of the geometric construction of the Hilbert scheme via products and finite quotients.

6.2.4 The essential skeleton of a logCY pair

The construction of the dual complex of a lc pair relies on the intersection poset of the strata of a dlt modification. It is convenient to think of these strata as the associated monomial valuations, suitably normalized, as defined in Proposition 4.2.1.1. The advantage of this viewpoint is that these valuations are independent of the choice of dlt modification, and they embed in a common ambient space, namely the $\beth$-analytification, with image equal to the essential skeleton of the pair (see Lemma 6.2.4.4).

As defined in §4.2.7, the essential skeleton of a pair (X, Δ_X) is given by the union of the minimality loci of a collection of weight functions in the $\beth$-analytification. In the proper logCY case, the weight functions associated to regular Δ_X-pluricanonical forms coincide with the log discrepancy $A_{(X,\Delta_X)}$, as we show in Proposition 6.2.4.1.

Proposition 6.2.4.1. *Let (X, Δ_X) be a proper log-regular logCY pair. If η is a regular section of $\mathcal{O}_X(m(K_X + \Delta_X))$, then $\mathrm{wt}_\eta = mA_{(X,\Delta_X)}$ as functions on $X^\beth$.*

Moreover, if $(X, \Delta_X^{=1})$ *is log-regular, then*

$$\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X) = \mathrm{Sk}(X, \Delta_X^{=1}). \tag{6.4}$$

Proof. By properness, there exists a unique regular section η of $\mathcal{O}_X(m(K_X + \Delta_X))$ up to scaling, for $m \in \mathbf{Z}_{>0}$ sufficiently divisible. As a result, the weight functions are independent of the choice of a Δ_X-logarithmic m-pluricanonical section, and so $\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X) = \mathrm{Sk}(X, \Delta_X, \eta)$.

We now proceed as in the proof of Proposition 3.2.3.1 and Corollary 3.2.3.2. Let δ be a local generator of $\mathcal{O}_X(m(K_X + \Delta_{X,\mathrm{red}}))$ and f be a local regular function such that $\eta = f\delta$. As η is a global generator of $\mathcal{O}_X(m(K_X + \Delta_X))$, then f provides a local equation for $m(\Delta_{X,\mathrm{red}} - \Delta_X)$. Hence, from Proposition 3.2.3.1, we get that

$$\mathrm{wt}_\eta(x) = A_{(X, \Delta_{X,\mathrm{red}} - (\Delta_{X,\mathrm{red}} - \Delta_X))} = A_{(X,\Delta_X)}(x).$$

We conclude that

$$\begin{aligned} \mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X) &= \mathrm{Sk}(X, \Delta_X, \eta) \\ &= \{x \in X^{\mathrm{bir}} : A_{(X,\Delta_X)}(x) = 0\} \\ &= \{x \in X^{\mathrm{bir}} : A_{(X,\Delta_X^{=1})}(x) = 0\} \\ &= \mathrm{Sk}(X, \Delta_X^{=1}), \end{aligned}$$

where the intermediate equality follows from the fact that the log centres of the pairs $(X, \Delta_X^{=1})$ and (X, Δ_X) coincide. Indeed, one can first assume that (X, Δ_X) is an snc pair by passing to a log resolution that, locally at the generic point of the strata of $\Delta_X^{=1}$, is given by a sequence of blow-ups induced by subdivisions of the corresponding Kato fan. One then applies [Kol13, Proposition 2.7]. □

In fact, if there exists a boundary $\Delta \leqslant \Delta_X$ such that (X, Δ_X) is a log-regular pair, then one can define a skeleton of (X, Δ) as in §4.2 by throwing away suitable faces of $\mathrm{Sk}(X, \Delta_X)$. With this definition, the equality (6.4) holds without the additional hypothesis that $(X, \Delta_X^{=1})$ is log-regular. Nonetheless, Proposition 6.2.4.1 suggests the following generalization of the definition of the essential skeleton to a proper lc logCY sub-pair, which agrees with the skeleton of [BJ17, Proposition 5.6] in the dlt case.

Definition 6.2.4.2. Let (X, Δ_X) be a proper lc logCY sub-pair. The *essential skeleton* of (X, Δ_X) is

$$\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X) \coloneqq \{x \in X^{\mathrm{bir}} \cap X^{\beth} \colon A_{(X,\Delta_X)}(x) = 0\}.$$

As in Definition 4.2.5.2, we can consider the link of the skeleton.

Definition 6.2.4.3. The *link* of the essential skeleton $\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)$ is the quotient of the punctured skeleton $\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)^* := \mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X) \backslash \{v_0\}$ via rescaling, denoted by

$$\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)^* / \mathbf{R}_+^*.$$

Lemma 6.2.4.4. *If the proper lc logCY sub-pairs (X, Δ_X) and (Y, Δ_Y) are crepant birational, then*

$$\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X) = \mathrm{Sk}^{\mathrm{ess}}(Y, \Delta_Y), \tag{6.5}$$

$$\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)^* / \mathbf{R}_+^* = \mathrm{Sk}^{\mathrm{ess}}(Y, \Delta_Y)^* / \mathbf{R}_+^*, \tag{6.6}$$

$$\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)^* / \mathbf{R}_+^* \simeq \mathcal{DMR}(X, \Delta_X). \tag{6.7}$$

Proof. The equalities (6.5) and (6.6) follow from the fact that $A_{(X,\Delta_X)} = A_{(Y,\Delta_Y)}$ on $X^{\mathrm{bir}} = Y^{\mathrm{bir}}$. The equality (6.7) is a consequence of the existence of a (crepant) dlt modification, and Proposition 4.2.5.3, once we restrict to the snc locus of the dlt modification. □

Remark 6.2.4.5. Given a lc logCY pair (X, Δ_X), $X^{\beth}$ admits a strong deformation retraction onto the closure of $\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)$. Indeed, $X^{\beth}$ retracts onto the closure of the skeleton of a pair $(Y, f_*^{-1}(\Delta_X) + \sum_i E_i)$ by [Thu07, Theorem 3.26], where $f : Y \to X$ is an snc modification of (X, Δ_X) and E_i are the exceptional divisors of f. Then $\mathrm{Sk}(Y, f_*^{-1}(\Delta_X) + \sum_i E_i)$ retracts onto $\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)$ by [dFKX17, Theorem 28.(2)]. While this is not needed in the sequel, the existence of this retraction affirms the use of the terminology 'skeleton' used in Definition 6.2.4.2.

Lemma 6.2.4.6. *Let (X, Δ_X) be a proper lc logCY pair. Let G be a finite group acting on X so that the quotient map $q : X \to X/G$ is quasi-étale, i.e. étale away from a subscheme of codimension $\geqslant 2$. Then,*

$$\mathrm{Sk}^{\mathrm{ess}}(X/G, \Delta_{X/G} := q_*\Delta_X) = q^{\beth}(\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)) \simeq \mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)/G. \tag{6.8}$$

In particular,

$$\mathrm{Sk}^{\mathrm{ess}}(X/G, \Delta_{X/G})^*/\mathbf{R}_+^* \simeq \mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)^*/(\mathbf{R}_+^* \times G). \tag{6.9}$$

Proof. Observe that the skeleton $\mathrm{Sk}^{\mathrm{ess}}(X/G, \Delta_{X/G})$ is well-defined since the pair $(X/G, \Delta_{X/G})$ is lc logCY. Indeed, $q^*(K_{X/G} + \Delta_{X/G}) = K_X + \Delta \sim_{\mathbf{Q}} 0$, because q is quasi-étale. In particular, [KM08, Proposition 5.20] implies that the pair $(X/G, \Delta_{X/G})$ is lc as (X, Δ_X) is so.

In order to show the first equality of (6.8), it is enough to show that the surjective map $q^{\beth} : X^{\beth} \to (X/G)^{\beth}$ restricts to a surjective map

$$q^{\beth}|_{\mathrm{Sk}^{\mathrm{ess}}} : \mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X) \to \mathrm{Sk}^{\mathrm{ess}}(X/G, \Delta_{X/G})$$

on essential skeletons. To this end, we first prove that the image of $\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)$ via $q^{\beth}$ lies in $\mathrm{Sk}^{\mathrm{ess}}(X/G, \Delta_{X/G})$ and that $q^{\beth}|_{\mathrm{Sk}^{\mathrm{ess}}}$ is surjective on divisorial points.

Let $x \in X^{\mathrm{div}} \cap X^{\beth}$ be the divisorial point determined by the triple $(c, Y \xrightarrow{h} X, E)$. By [Kol13, Lemma 2.22], there exists a commutative diagram

$$\begin{array}{ccc} E \subset Y & \overset{q'}{\dashrightarrow} & F \subset Y' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ X & \xrightarrow{\quad q \quad} & X/G \end{array}$$

where Y' is a normal variety and F is a divisor on Y' satisfying

- the morphism h and h' are birational;

- the map q' is rational and dominant;

- the image of the divisor E via q' is the divisor F.

Note that the image $q^{\beth}(x)$ is determined by the triple $(c \cdot r(E), Y' \xrightarrow{h'} X/G, F)$, where $r(E)$ is the ramification index of q' along E. Indeed, we have that

$$c \cdot \mathrm{ord}_E(f \circ q \circ h) = c \cdot \mathrm{ord}_E(f \circ h' \circ q') = c \cdot r(E)\, \mathrm{ord}_F(f \circ h')$$

for any rational function $f \in K(X/G)$. By [KM08, Proposition 5.20], we have that $A_{(X/G, \Delta_{X/G})}(q^{\beth}(x))$ is zero if $A_{(X,\Delta_X)}(x)$ is zero, hence $q^{\beth}(x) \in \mathrm{Sk}^{\mathrm{ess}}(X/G, \Delta_{X/G})$ for any divisorial point $x \in \mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)$. Similarly, the proof of [KM08, Proposition 5.20] shows that $q^{\beth}|_{\mathrm{Sk}^{\mathrm{ess}}}$ is surjective on divisorial points.

In fact, $q^{\beth}|_{\mathrm{Sk}^{\mathrm{ess}}}$ is surjective onto the whole skeleton $\mathrm{Sk}^{\mathrm{ess}}(X/G, \Delta_{X/G})$. Indeed, since $q^{\beth}$ is equivariant with respect to the $\mathbf{R}_+^*$-action, it is enough to check that the induced map

$$q^{\beth}|_{\mathrm{Sk}^{\mathrm{ess}*}/\mathbf{R}_+^*} : \mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)^*/\mathbf{R}_+^* \to \mathrm{Sk}^{\mathrm{ess}}(X/G, \Delta_{X/G})^*/\mathbf{R}_+^*$$

is surjective. If $q^{\beth}|_{\mathrm{Sk}^{\mathrm{ess}}}$ is surjective on divisorial points, then $q^{\beth}|_{\mathrm{Sk}^{\mathrm{ess}*}/\mathbf{R}_+^*}$ is a continuous map from a compact topological space to a Hausdorff space with dense image. Hence, $q^{\beth}|_{\mathrm{Sk}^{\mathrm{ess}}}$ is surjective.

Finally, the second equality of (6.8) follows from [Ber95, Corollary 5]. Since the actions of G and $\mathbf{R}_+^*$ commute and the homeomorphism

$$\mathrm{Sk}^{\mathrm{ess}}(X/G, \Delta_{X/G}) \simeq \mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)/G$$

is $\mathbf{R}_+^*$-equivariant, we conclude that also (6.9) holds. □

6.2.5 Proof of Theorem 6.2.0.1

Proof of Theorem 6.2.0.1. The desired homeomorphism is obtained by applying the preceding sequence of lemmas as follows:

$$\begin{aligned}
\mathcal{D}(\partial M_{\mathrm{GL}_n}) &\simeq \mathcal{DMR}(Z^{(n)}, \Delta^{(n)}) \\
&\simeq \mathrm{Sk}^{\mathrm{ess}}(Z^{(n)}, \Delta^{(n)})^*/\mathbf{R}_+^* \\
&\simeq \mathrm{Sk}^{\mathrm{ess}}(Z^n, \Delta^n)^*/(\mathbf{R}_+^* \times \mathfrak{S}_n) \\
&\simeq \Big((\mathrm{Sk}^{\mathrm{ess}}(Z, \Delta)^*/\mathbf{R}_+^*) * \ldots * (\mathrm{Sk}^{\mathrm{ess}}(Z, \Delta)^*/\mathbf{R}_+^*) \Big)/\mathfrak{S}_n \\
&\simeq (\mathbb{S}^1 * \ldots * \mathbb{S}^1)/\mathfrak{S}_n \\
&\simeq \mathbb{S}^{2n-1}/\mathfrak{S}_n \\
&\simeq \mathbb{S}^{2n-1},
\end{aligned}$$

where the final homeomorphism follows from Lemma 6.2.5.1, which is stated immediately after this proof. □

We conclude the section by proving the topological lemma mentioned at the end of the proof of Theorem 6.2.0.1. It is presumably well-known, but the authors are not aware of a reference.

Lemma 6.2.5.1. *Consider the linear action of the symmetric group $\mathfrak{S}_n$ that permutes the standard coordinates of $\mathbf{C}^n$. The quotient of the unit sphere $\mathbb{S}^{2n-1} \subset \mathbf{C}^n$ by this action is homeomorphic to the sphere $\mathbb{S}^{2n-1}$.*

Proof. Consider the finite morphism $q\colon \mathbf{C}\times\ldots\times\mathbf{C}\to\mathbf{C}^{(n)}\simeq\mathbf{C}[z]_{n,1}$ given by

$$(z_1,\ldots,z_n)\mapsto\prod_{i=1}^{n}(z-z_i),$$

where we identify the symmetric product $\mathbf{C}^{(n)}$ with the space $\mathbf{C}[z]_{n,1}$ of monic polynomials of degree n in one variable with complex coefficients. The restriction of q to the boundary of the closed unit polydisc $\mathbb{D}^{2n}$

$$q\colon \mathbb{S}^{2n-1}\simeq\partial\mathbb{D}^{2n}=\partial\mathbb{D}^1*\ldots*\partial\mathbb{D}^1\to q(\mathbb{S}^{2n-1})\simeq\mathbb{S}^{2n-1}/\mathfrak{S}_n$$

is the given quotient map. The space $\mathbf{C}[z]_{n,1}$ is isomorphic to $\mathbf{C}^n$ through the identification $\psi:\mathbf{C}[z]_{n,1}\to\mathbf{C}^n$ of a monic polynomial with the n-uples of its coefficients; more explicitly, ψ is by

$$\psi\left(\prod_{i=1}^{n}(z-z_i)\right)=\psi(z^n+r_1e^{i\theta_1}z^{n-1}+\ldots+r_ne^{i\theta_n})=(r_1e^{i\theta_1},\ldots,r_ne^{i\theta_n}),$$

where $(r_j,\theta_j)_{1\leqslant j\leqslant n}$ are polar coordinates on $\mathbf{C}^n\simeq\mathbf{R}^{2n}$. Further, let $\varphi\colon\mathbf{C}^n\to\mathbf{C}^n$ be the homeomorphism given by

$$\varphi(r_1e^{i\theta_1},\ldots,r_ne^{i\theta_n})=(r_1e^{i\theta_1},\sqrt[2]{r_2}e^{i\theta_1},\ldots,\sqrt[n]{r_n}e^{i\theta_n}).$$

We can restrict the composition $\frac{\varphi}{|\varphi|}\circ\psi\circ q:\mathbf{C}^n\setminus\{0\}\to\mathbf{C}^n\setminus\{0\}$ to a morphism of spheres which factors through the symmetric quotient by construction, as shown in the following diagram

$$\begin{array}{ccc} \mathbb{S}^{2n-1} & & \\ \downarrow{\scriptstyle q} & \searrow^{\frac{\varphi}{|\varphi|}\circ\psi\circ q} & \\ \mathbb{S}^{2n-1}/\mathfrak{S}_n & \xrightarrow[\frac{\varphi}{|\varphi|}\circ\psi]{} & \mathbb{S}^{2n-1}. \end{array}$$

We claim that the map

$$\frac{\varphi}{|\varphi|}\circ\psi:\mathbb{S}^{2n-1}/\mathfrak{S}_n\to\mathbb{S}^{2n-1}$$

is a homeomorphism. Indeed, since it is a continuous map from a compact topological space to a Hausdorff space, it is enough to check that it is bijective. This is equivalent to show that the preimage of any point in $\mathbb{S}^{2n-1}$ via the map $\frac{\varphi}{|\varphi|} \circ \psi \circ q$ is a $\mathfrak{S}_n$-orbit. Alternatively, we need to prove that the preimage of any real half-line $\{(re^{i\theta_1}, \dots, re^{i\theta_n}) \colon r \in \mathbf{R}_+\} \subseteq \mathbf{C}^n$ via the map $\varphi \circ \psi \circ q$ is the orbit of a half-line $\{(rz_1, \dots, rz_n) \colon r \in \mathbf{R}_+\} \subseteq \mathbf{C}^n$. This follows from the fact that

$$
\begin{aligned}
(\varphi \circ \psi \circ q)^{-1}(re^{i\theta_1}, \dots, re^{i\theta_n}) &= (\psi \circ q)^{-1}(re^{i\theta_1}, r^2 e^{i\theta_2}, \dots, r^n e^{i\theta_n}) \\
&= \bigcup_{\sigma \in \mathfrak{S}_n} (rz_{\sigma(1)}, \dots, rz_{\sigma(n)})
\end{aligned}
$$

for any $r \in \mathbf{R}_+$, where the values z_j are chosen in such a way that

$$
q(z_1, \dots, z_n) = (e^{i\theta_1}, \dots, e^{i\theta_n}).
$$

□

6.2.6 An alternative proof of Theorem 6.2.0.1.

The proof of Theorem 6.2.0.1 is inspired by the result in [BM17, Proposition 6.2.4]. There, Brown and the second author show that the dual complex of a degeneration of the Hilbert scheme of n points of K3 surfaces induced by a maximal unipotent semistable degeneration of K3 surfaces is homeomorphic to the complex projective space $\mathbf{P}^n(\mathbf{C})$. Both proofs crucially rely on the compatibility of the construction of the essential skeleton with products and finite quotients.

In this section, we exhibit a direct connection between the two results: we show how Theorem 6.2.0.1 can be deduced from [BM17, Proposition 6.2.4]. This alternate proof relies on the construction of an explicit degeneration of Calabi–Yau varieties (see Proposition 6.2.6.3), and a global-to-local argument (Lemma 6.2.6.1) that relates the dual complex of the degeneration to that a logCY pair. While the proof of Theorem 6.2.0.1 presented in §6.2.5 is technically more elementary, we expect both strategies to prove useful for future calculations of dual complexes. Furthermore, the existence of a degeneration as in Proposition 6.2.6.3 is of independent interest: loosely speaking, it realizes a character variety as a "limit" of compact hyper-Kähler manifolds.

Let (X, Δ_X) be a dlt pair with $\Delta_X^{=1} := \sum_{i=1}^m \Delta_i$. For every stratum W of (X, Δ_X), there exists a $\mathbf{Q}$-divisor $\mathrm{Diff}^*_W(\Delta_X)$ such that

$$(K_X + \Delta_X)|_W \sim_{\mathbf{Q}} K_W + \mathrm{Diff}^*_W(\Delta_X).$$

See [Kol13, §4.18] for more details. By adjunction, we have that $\mathrm{Diff}^*_W(\Delta_X)^{=1}$ coincides with the trace of Δ_X on W, i.e.

$$\mathrm{Diff}^*_W(\Delta_X)^{=1} = \sum_{W \not\subseteq \Delta_i} \Delta_i|_W.$$

In particular, any stratum W of a dlt (logCY) pair has an induced structure of (logCY) pair $(W, \mathrm{Diff}^*_W(\Delta_X))$ such that

$$\mathcal{D}(\mathrm{Diff}^*_W(\Delta_X)^{=1}) \simeq \mathcal{D}\bigg(\sum_{W \not\subseteq \Delta_i} \Delta_i|_W \bigg). \tag{6.10}$$

Lemma 6.2.6.1 (Global-to-local argument)**.** *Let (X, Δ_X) be a dlt pair such that the dual complex of $\mathcal{D}(\Delta_X)$ is a topological manifold. Then, $\mathcal{D}(\mathrm{Diff}^*_W(\Delta_X))$ is homeomorphic to a sphere for any stratum W of $\Delta_X^{=1}$.*

Proof. Up to baricentrical subdivisions, the link of a neighbourhood of a cell associated to W in $\mathcal{D}(\Delta_X)$ is isomorphic to $\mathcal{D}(\mathrm{Diff}^*_W(\Delta_X))$. Since $\mathcal{D}(\Delta_X)$ is a topological manifold, this link is homeomorphic to a sphere. □

We will construct a degeneration of Hilbert schemes of a K3 surface with a component of the special fibre that, paired with the different of the special fibre, is crepant birational to a dlt compactification of M_{GL_n}. This is then combined with the global-to-local argument to compute the dual complex in Theorem 6.2.0.1. The properties of the required degeneration are collected below.

Definition 6.2.6.2. A model $\mathscr{X}$ over $\mathbf{C}[[t]]$ is *good minimal dlt* if $\mathscr{X}$ is $\mathbf{Q}$-factorial, the pair $(\mathscr{X}, \mathscr{X}_{0,\mathrm{red}})$ is dlt, and $K_{\mathscr{X}} + \mathscr{X}_{0,\mathrm{red}}$ is semiample.

Proposition 6.2.6.3. *Let (X, Δ_X) be a lc logCY pair. Assume there exist*

(a) a maximal unipotent semistable good minimal dlt model $\mathscr{S}$ of a K3 surface S over $\mathbf{C}((t))$,

(b) a good minimal dlt model $\mathscr{S}^{[n],\mathrm{dlt}}$ of the Hilbert scheme of n points of S,

such that (X, Δ_X) is crepant birational to $(D, \mathrm{Diff}^*_D(\mathscr{S}^{[n],\mathrm{dlt}}))$ for some irreducible component D of the special fibre $\mathscr{S}_0^{[n],\mathrm{dlt}}$. Then, $\mathcal{D}(\Delta_X)$ is homeomorphic to a sphere.

Proof. It follows from the combination of Lemma 6.2.6.1, [dFKX17, Proposition 11], [BM17, Proposition 6.2.4], and [NX16, Propositon 3.3.3]. Note that we only use [BM17, Proposition 6.2.4] to grant that the dual complex of the degeneration is a manifold, and not the fact that it is actually homeomorphic to a complex projective space. □

Proof of Theorem 6.2.0.1. Let $\mathscr{S}$ be a semistable good minimal (although not $\mathbf{Q}$-factorial) snc model over $\mathbf{C}[[t]]$ of a quartic surface S in $\mathbf{P}^3_{\mathbf{C}((t))}$, degenerating to the union of four hyperplanes $\mathscr{S}_0 = \sum_{i=0}^3 D_i$. For example, take the Dwork pencil

$$\mathscr{S} := \left\{ x_0x_1x_2x_3 + t\sum_{i=0}^{3} x_i^4 = 0 \right\} \subseteq \mathbf{P}^3_{[x_0:x_1:x_2:x_3]} \times \mathrm{Spec}(\mathbf{C}[[t]]).$$

The degeneration $\mathscr{S}$ is a model of the K3 surface S as in Proposition 6.2.6.3(a), and the proof proceeds in two steps: we construct a model $\mathscr{S}^{[n],\mathrm{dlt}}$ of $S^{[n]}$ as in Proposition 6.2.6.3(b), and then we identify a component of the special fibre $\mathscr{S}_0^{[n],\mathrm{dlt}}$ which, paired with the different of $\mathscr{S}_0^{[n],\mathrm{dlt}}$, is crepant birational to a dlt compactification of M_{GL_n}.

For the first step, let $(\mathscr{S}^{(n)}, \mathscr{S}_0^{(n)})$ and $(\mathscr{S}^{[n]}, \mathscr{S}_0^{[n]})$ be the pairs given by the relative n-fold symmetric product and the relative Hilbert scheme of n points on $\mathscr{S}$ respectively, together with their special fibres. Consider a log resolution of $(\mathscr{S}^{[n]}, \mathscr{S}_0^{[n]})$, written

$$g\colon (\mathscr{Y}, \Delta_{\mathscr{Y}} \coloneqq g_*^{-1}\mathscr{S}_0^{[n]} + E) \to (\mathscr{S}^{[n]}, \mathscr{S}_0^{[n]}),$$

which is an isomorphism on the snc locus of $(\mathscr{S}^{[n]}, \mathscr{S}_0^{[n]})$, where E is the sum of the g-exceptional divisors. Note that the composition $g \circ \rho_{\mathrm{HC}}$ of g with the Hilbert–Chow morphism ρ_{HC} gives a log resolution of the pair $(\mathscr{S}^{(n)}, \mathscr{S}_0^{(n)})$ as well. The $(K_{\mathscr{Y}/\mathbf{C}[[t]]} + \Delta_{\mathscr{Y}})$-MMP with scaling terminates with a $\mathbf{Q}$-factorial, dlt, minimal model of $S^{[n]}$

$$h : (\mathscr{S}^{[n],\mathrm{dlt}}, \mathscr{S}_{0,\mathrm{red}}^{[n],\mathrm{dlt}} = h_*^{-1}\mathscr{S}_0^{[n]} + E') \to \mathscr{S}^{[n]},$$

where E' is the sum of the $(g \circ \rho_{\mathrm{HC}})$-exceptional divisors that lie in the special fibre, and $\mathscr{S}_{0,\mathrm{red}}^{[n],\mathrm{dlt}}$ is the reduced special fibre of $\mathscr{S}^{[n],\mathrm{dlt}}$. The existence of such a h follows from [Kol13, Corollary 1.36]; note that the degeneration $\mathscr{S}$ is defined over a curve (see Definition 4.2.8.1), so we can run a relative MMP as usual. Note also that the pair $(\mathscr{S}^{(n)}, \mathscr{S}_0^{(n)})$ is (reduced) lc logCY, since $(\mathscr{S}, \mathscr{S}_0)$ is so. The pair $(\mathscr{S}^{[n],\mathrm{dlt}}, \mathscr{S}_{0,\mathrm{red}}^{[n],\mathrm{dlt}})$ is logCY as well, as h is a crepant morphism of pairs (see [Kol13, §1.35]). Hence, $\mathscr{S}^{[n],\mathrm{dlt}}$ is a good minimal dlt model of $S^{[n]}$, as required in order to apply Proposition 6.2.6.3.

Now, we show that there exist irreducible components Δ_i^{dlt} of $\mathscr{S}_{0,\mathrm{red}}^{[n],\mathrm{dlt}}$ such that the pairs

$$(\Delta_i^{\mathrm{dlt}}, \mathrm{Diff}^*_{\Delta_i^{\mathrm{dlt}}}(\mathscr{S}_{0,\mathrm{red}}^{[n],\mathrm{dlt}}))$$

are crepant birational to a dlt compactification of $M_{\mathrm{GL}_n} \simeq (\mathbf{C}^* \times \mathbf{C}^*)^{(n)}$, equivalently of $(\mathbf{C}^* \times \mathbf{C}^*)^{[n]}$. To this end, note that the special fibre $\mathscr{S}_0^{(n)}$ contains irreducible components $\Delta_i \simeq (\mathbf{P}^2)^{(n)}$, $i \in I \simeq \{0, \dots, 3\}$, which are the n-fold symmetric products of the hyperplanes D_i. Denote by Δ_i' and Δ_i^{dlt} the strict transform of Δ_i under ρ_{HC} and h, respectively. By [Kol13, Proposition 4.6], the following pairs are crepant birational:

$$(\Delta_i', \mathrm{Diff}_{\Delta_i'}(\mathscr{S}_{0,\mathrm{red}}^{[n]})) \sim (\Delta_i, \mathrm{Diff}_{\Delta_i}(\mathscr{S}_0^{(n)})) \sim (\Delta_i^{\mathrm{dlt}}, \mathrm{Diff}_{\Delta_i^{\mathrm{dlt}}}(\mathscr{S}_{0,\mathrm{red}}^{[n],\mathrm{dlt}})).$$

Further, the inclusion of $D_i \setminus \cup_{j \neq i} D_j \simeq \mathbf{C}^* \times \mathbf{C}^*$ into D_i induces the embedding of $\Delta_i^\circ := (D_i \setminus \cup_{j \neq i} D_j)^{[n]}$ into Δ_i', which is isomorphic to $(\mathbf{C}^* \times \mathbf{C}^*)^{[n]}$. We need the following technical lemma.

Lemma 6.2.6.4. $\mathrm{Diff}_{\Delta_i'}(\mathscr{S}_{0,\mathrm{red}}^{[n]}) = \Delta_i' \setminus \Delta_i^\circ$.

Proof. It is clear that $\mathrm{Diff}_{\Delta_i'}(\mathscr{S}_{0,\mathrm{red}}^{[n]}) \geqslant (\mathscr{S}_0^{[n]} \setminus \Delta_i')|_{\Delta_i'}$. For the equality, it is enough to prove that no divisor whose generic point is contained in Δ_i° belongs to the support of $\mathrm{Diff}_{\Delta_i'}(\mathscr{S}_{0,\mathrm{red}}^{[n]})$. By [Kol13, Proposition 4.5 (1)], it is sufficient to prove that $\mathscr{S}^{[n]}$ is regular along Δ_i°. To this aim, let $\xi \in \Delta_i^\circ$ be a scheme of length n in $\mathscr{S}_0$. Since the immersion of a (formal) neighbourhood of ξ in $\mathscr{S}_0^{[n]}$ factors through Δ_i, the subscheme ξ is unobstructed by [Fog68, Theorem 2.4]; it follows from [Kol96, Theorem 2.10] that $\mathscr{S}^{[n]}$ is regular at ξ. $\square$

Finally, we conclude that

$$\begin{aligned}\mathcal{D}(\partial(M_{\mathrm{GL}_n}) &\simeq \mathcal{DMR}(\Delta_i', \mathrm{Diff}_{\Delta_i'}(\mathscr{S}_{0,\mathrm{red}}^{[n]})) \\ &\simeq \mathcal{DMR}(\Delta_i^{\mathrm{dlt}}, \mathrm{Diff}_{\Delta_i^{\mathrm{dlt}}}(\mathscr{S}_{0,\mathrm{red}}^{[n],\mathrm{dlt}})) \\ &\simeq \mathbb{S}^{2n-1},\end{aligned}$$

where the first homeomorphism holds by Lemma 6.2.6.4 and (6.3), the second is [dFKX17, Proposition 11], and the final homeomorphism is Proposition 6.2.6.3. □

6.3 Dual boundary complex of $\mathbf{SL}_n$-character varieties of a genus one surface

In this section, we determine the homeomorphism class of the dual boundary complex of the SL_n-character variety M_{SL_n} associated to a Riemann surface of genus one. The following result is stated as Theorem H in the introduction.

Theorem 6.3.0.1. *The dual boundary complex $\mathcal{D}(\partial M_{\mathrm{SL}_n})$ of a dlt log Calabi–Yau compactification of M_{SL_n} has the homeomorphism type of $\mathbb{S}^{2n-3}$.*

Proof. Observe that M_{SL_n} is the fibre of the determinant morphism

$$(\mathbf{C}^* \times \mathbf{C}^*)^{(n)} \simeq M_{\mathrm{GL}_n} \to \mathbf{C}^* \times \mathbf{C}^* \tag{6.11}$$

$$((a_i, b_i))_{i=1}^n \simeq [(A, B)] \mapsto (\det A, \det B) = \left(\textstyle\prod_{i=1}^n a_i, \prod_{i=1}^n b_i\right), \tag{6.12}$$

where the pair (A, B) of matrices in GL_n represents a point in M_{GL_n}, and $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ are their eigenvalues; see for instance [BS16, Lemma 8.17]. We proceed in several steps.

First, we will show that the character variety M_{SL_n} admits a lc logCY compactification $\overline{M}_{\mathrm{SL}_n}$. Indeed, consider the diagram

$$\begin{array}{ccccc} L := m_r^{-1}(\mathbf{1}) \simeq (\mathbf{C}^* \times \mathbf{C}^*)^{n-1} & \longrightarrow & (\mathbf{C}^* \times \mathbf{C}^*)^n & \xrightarrow{m_n} & \mathbf{C}^* \times \mathbf{C}^* \ni \mathbf{1} = (1,1) \\ \downarrow & & \downarrow q & \nearrow & \\ M_{\mathrm{SL}_n} & \longrightarrow & (\mathbf{C}^* \times \mathbf{C}^*)^{(n)}, & & \end{array}$$

where m_n is the multiplication map, and q the quotient by the action of the symmetric group $\mathfrak{S}_n$, permuting the factors. The projective closure $\overline{L}$ of L in $(\mathbf{P}^2)^n$ is invariant with respect to the action

$$(\mathbf{C}^* \times \mathbf{C}^*)^{n-1} \times (\mathbf{P}^2)^n \to (\mathbf{P}^2)^n$$

given as follows: for $((a_i, b_i))_{i=1}^{n-1} \in (\mathbf{C}^* \times \mathbf{C}^*)^{n-1}$ and $([x_j : y_j : z_j])_{j=1}^n \in (\mathbf{P}^2)^n$, the point $((a_i, b_i))_{i=1}^{n-1} \cdot ([x_j : y_j : z_j])_{j=1}^n$ of $(\mathbf{P}^2)^n$ has j-th factor given by

$$\begin{cases} [a_j x_j \colon b_j y_j \colon z_j], & j = 1, \dots, n-1, \\ \left[\prod_{i=1}^{n-1} a_i^{-1} \cdot x_n : \prod_{i=1}^{n-1} b_i^{-1} \cdot y_n : z_n\right], & j = n. \end{cases}$$

for $j = 1, \dots, n$.

As $L \simeq (\mathbf{C}^* \times \mathbf{C}^*)^{n-1}$ is a dense orbit of this algebraic action, it follows that $\overline{L}$ is a toric compactification of L. In particular, the pair $(\overline{L}, \partial L \coloneqq \overline{L} \setminus L)$ is a (normal) lc logCY pair. Since $\overline{L}$ is $\mathfrak{S}_n$-invariant and the restriction of the quotient map $q : (\mathbf{P}^2)^n \to (\mathbf{P}^2)^{(n)}$ to $\overline{L}$ is quasi-étale, then the projective closure $\overline{M}_{\mathrm{SL}_n}$ of M_{SL_n} in $(\mathbf{P}^2)^{(n)}$ is a lc logCY compactification of M_{SL_n}. Thus, we can construct the essential skeleton $\mathrm{Sk}^{\mathrm{ess}}(\overline{M}_{\mathrm{SL}_n}, \partial \overline{M}_{\mathrm{SL}_n})$ as in Definition 6.2.4.2. Although $\overline{L}$ is a toric variety, it is worth pointing out that the embedding $\overline{L} \hookrightarrow (\mathbf{P}^2)^n$ is not toric.

Next, let Δ be the toric boundary of $\mathbf{P}^2$ and N be the cocharacter lattice of the torus $\mathbf{C}^* \times \mathbf{C}^* \subseteq \mathbf{P}^2$. By Proposition 6.2.4.1, $\mathrm{Sk}^{\mathrm{ess}}(\mathbf{P}^2, \Delta)$ is the skeleton of the log-regular pair $(\mathbf{P}^2, \Delta)$, and hence the multiplication m_n induces a map $\alpha_n \colon \mathrm{Sk}^{\mathrm{ess}}(\mathbf{P}^2, \Delta)^n \to \mathrm{Sk}^{\mathrm{ess}}(\mathbf{P}^2, \Delta)$ by functoriality, as in §4.7. In particular, in this toric case, the essential skeleton $\mathrm{Sk}^{\mathrm{ess}}(\mathbf{P}^2, \Delta)$ can be identified with $N_{\mathbf{R}} \simeq \mathbf{R}^2$ (see

§4.3.2), and α_n is given by the linear map

$$(N_{\mathbf{R}})^n \simeq \mathbf{R}^{2n} \to N_{\mathbf{R}} \simeq \mathbf{R}^2,$$
$$(x_i, y_i)_{i=1}^n \mapsto \Big(\textstyle\sum_{i=1}^n x_i, \sum_{i=1}^n y_i \Big).$$

Finally, observe that the symmetric quotient of the kernel of α_n is isomorphic to the additive group $\mathbf{C}^{n-1}$, i.e.

$$\alpha_n^{-1}(\mathbf{0})/\mathfrak{S}_n \simeq \mathbf{C}^{n-1}.$$

This follows from the diagram below:

$$\begin{array}{ccccc}
\alpha_n^{-1}(\mathbf{0}) & \longrightarrow & \mathrm{Sk}^{\mathrm{ess}}((\mathbf{P}^2)^n, \Delta^n) \simeq \mathbf{C}^n \simeq \mathbf{R}^{2n} & \xrightarrow{\alpha_n} & \mathrm{Sk}^{\mathrm{ess}}(\mathbf{P}^2, \Delta) \simeq \mathbf{C} \simeq \mathbf{R}^2 \\
\downarrow & & {\scriptstyle q^{\beth}}\downarrow & \nearrow_{\mathrm{pr}} & \\
\alpha_n^{-1}(\mathbf{0})/\mathfrak{S}_n & \longrightarrow & \mathrm{Sk}^{\mathrm{ess}}((\mathbf{P}^2)^{(n)}, \Delta^{(n)}) \simeq \mathbf{C}^n, & &
\end{array}$$

where the map pr is the linear projection to the $\mathfrak{S}_n$-invariant coordinate α_n.

Now, the essential skeleton of the pair $(\overline{M}_{\mathrm{SL}_n}, \partial\overline{M}_{\mathrm{SL}_n})$ is homeomorphic to the symmetric quotient of the fibre of α_n, namely

$$\mathrm{Sk}^{\mathrm{ess}}(\overline{M}_{\mathrm{SL}_n}, \partial\overline{M}_{\mathrm{SL}_n}) \simeq \alpha_n^{-1}(\mathbf{0})/\mathfrak{S}_n.$$

This statement can be shown following the same strategy of [BM17, Proposition 6.3.3]. Indeed, the latter relies on the functoriality of skeletons via finite quotients and products, which we have reproved in the trivially-valued setting in Proposition 4.2.6.3 and Lemma 6.2.4.6.

Finally, in the same fashion as in §6.2.5, we conclude that

$$\mathcal{D}(\partial M_{\mathrm{SL}_n}) \simeq \mathcal{DMR}(\overline{M}_{\mathrm{SL}_n}, \partial\overline{M}_{\mathrm{SL}_n}) \simeq \mathbb{S}^{2n-3}.$$

□

6.3.1 An alternative proof of Theorem 6.3.0.1

Following the same strategy as in §6.2.6, one can invoke a global-to-local argument to reduce the proof of Theorem 6.3.0.1 to the construction of a degeneration as in the following proposition. Observe that the role of the Hilbert scheme of a K3 surface in Proposition 6.2.6.3 is replaced by the generalised Kummer variety of an abelian surface.

Proposition 6.3.1.1. *There exists a good minimal dlt model $\mathscr{K}_{n-1}^{\mathrm{dlt}}$ of a generalised Kummer variety and an irreducible component Δ^{dlt} of the special fibre $\mathscr{K}_{n-1,0}^{\mathrm{dlt}}$ such that the pair $(\Delta^{\mathrm{dlt}}, \mathrm{Diff}_{\Delta^{\mathrm{dlt}}}(\mathscr{K}_{n-1,0,\mathrm{red}}^{\mathrm{dlt}}))$ is crepant birational to a lc logCY compactification of M_{SL_n}.*

The proof of Proposition 6.3.1.1 relies on some local computations on the *Tate curve*. Following [DR73, VII], we recall that the Tate curve $\overline{\mathscr{G}}_m$ is a model over $\mathbf{C}[[t]]$ of the multiplicative group $\mathbf{G}_m$ with special fibre given by an infinite chain of $\mathbf{P}^1$'s; see Section 6.4 for the construction. In fact, $\overline{\mathscr{G}}_m$ is the universal cover of the minimal model of a Tate elliptic over $\mathbf{C}((t))$, as in [Tat95] (see also [Sil09, C §14]). Mind that $\overline{\mathscr{G}}_m$ is the completion of a $\mathbf{C}$-scheme that is only *locally* of finite type over $\mathbf{C}$.

The model $\mathscr{G}_m$, obtained from $\overline{\mathscr{G}}_m$ by removing the nodes of the special fibre, is the Néron model of $\mathbf{G}_m$; see [DR73, VII, Example 1.2.c)]. In particular, $\mathscr{G}_m$ is endowed with a multiplication morphism

$$\mathscr{G}_m^n := \mathscr{G}_m \times \ldots \times \mathscr{G}_m \to \mathscr{G}_m,$$

which extends the multiplication $\mathbf{G}_m^n \to \mathbf{G}_m$ on the generic fibre. Let $\mathscr{V}_{n-1}$ denote the fibre of the identity section via the multiplication map $\mathscr{G}_m^n \to \mathscr{G}_m$. By a local computation in the coordinates of [DR73, VII], one can show that the pair $(\overline{\mathscr{V}}_{n-1}, \overline{\mathscr{V}}_{n-1,0})$, given by the closure of $\mathscr{V}_{n-1}$ in the fibre product $\overline{\mathscr{G}}_m^n$ together with its special fibre, is normal, reduced, and toric. Further, the intersection $\overline{\mathscr{V}}_{n-1} \cap (\overline{\mathscr{G}}_m^n \setminus \mathscr{G}_m^n)$ has codimension two in $\overline{\mathscr{V}}_{n-1}$. The proof of these facts appear in Section 6.4.

Proof of Proposition 6.3.1.1. Let E be an elliptic curve over $\mathbf{C}((t))$ with multiplicative reduction (c.f. [Liu02, Definition 10.2.2]), and $\mathscr{E}$ be a semistable good minimal snc model of E over $\mathbf{C}[[t]]$. In order to later run a MMP, assume further that $\mathscr{E}$ is defined over a curve in the sense of Definition 4.2.8.1. For example, take $\mathscr{E}$ to be the Dwork pencil of cubic curves that appears in Fig. 4.1. The Néron model $\mathscr{N}$ of $\mathscr{E}$ is the group scheme obtained from $\mathscr{E}$ by removing the nodes of the special fibre; see [Liu02, Theorem 10.2.14].

We first perform the classical construction of a singular generalised Kummer variety, as in [Bea83, §7], but in the relative setting. Let $\mathscr{X}_{n-1}$ be the fibre of the identity section of the multiplication morphism

$$m_n : (\mathscr{N} \times \mathscr{N})^n := (\mathscr{N} \times_{\mathbf{C}[[t]]} \mathscr{N}) \times_{\mathbf{C}[[t]]} \dots \times_{\mathbf{C}[[t]]} (\mathscr{N} \times_{\mathbf{C}[[t]]} \mathscr{N}) \to (\mathscr{N} \times_{\mathbf{C}[[t]]} \mathscr{N}).$$

The closure $\overline{\mathscr{X}}_{n-1}$ of $\mathscr{X}_{n-1}$ in $(\mathscr{E} \times \mathscr{E})^n$ is invariant under the action of the symmetric group $\mathfrak{S}_n$, which acts by permuting the factors of $(\mathscr{E} \times \mathscr{E})^n$. As a result, the quotient

$$\mathscr{K}_{n-1}^{\text{sing}} := \overline{\mathscr{X}}_{n-1}/\mathfrak{S}_n$$

is a model of the singular generalised Kummer variety K_{n-1}^{sing} associated to the abelian surface $E \times E$. Let $\mathscr{K}_{n-1,0}^{\text{sing}}$ be the special fibre of $\mathscr{K}_{n-1}^{\text{sing}}$.

Lemma 6.3.1.2. *The pair $(\mathscr{K}_{n-1}^{\text{sing}}, \mathscr{K}_{n-1,0}^{\text{sing}})$ is reduced lc logCY.*

Proof. We omit the subscript $n-1$ for brevity. Since the quotient map $\overline{\mathscr{X}} \to \mathscr{K}^{\text{sing}}$ is quasi-étale, it is equivalent to check that the pair $(\overline{\mathscr{X}}, \overline{\mathscr{X}}_0)$ is reduced lc logCY. To this end, observe that the universal cover of $(\mathscr{E} \times \mathscr{E})^n$ is the fibre product $(\overline{\mathscr{G}}_m \times \overline{\mathscr{G}}_m)^n$ of Tate curves. Therefore, the pair $(\overline{\mathscr{X}}, \overline{\mathscr{X}}_0)$ is reduced lc, since it is étale-locally isomorphic to the pair $(\overline{\mathscr{V}} \times \overline{\mathscr{V}}, (\overline{\mathscr{V}} \times \overline{\mathscr{V}})_0)$, which is the fibre product of the reduced toric pair $(\overline{\mathscr{V}}, \overline{\mathscr{V}}_0)$ by Proposition 6.4.0.1.

In order to verify that $K_{\overline{\mathscr{X}}/\mathbf{C}[[t]]} + \overline{\mathscr{X}}_0$ is trivial, it suffices to check that its restriction

$$\left(K_{\overline{\mathscr{X}}/\mathbf{C}[[t]]} + \overline{\mathscr{X}}_0\right)|_{\mathscr{X}} = K_{\mathscr{X}/\mathbf{C}[[t]]} + \mathscr{X}_0 \sim K_{\mathscr{X}/\mathbf{C}[[t]]}$$

to $\mathscr{X}$ is trivial; indeed, $\overline{\mathscr{X}} \setminus \mathscr{X}$ has codimension two in $\overline{\mathscr{X}}$ by Proposition 6.4.0.1. Let $N_{\mathscr{X}}((\mathscr{N} \times \mathscr{N})^n)$ denote the normal bundle of $\mathscr{X}$ in $(\mathscr{N} \times \mathscr{N})^n$. As $\mathscr{X}$ is a fibre of the locally trivial fibration m_n, it follows that $\det(N_{\mathscr{X}}((\mathscr{N} \times \mathscr{N})^n))$ is trivial; in particular, we have

$$K_{\mathscr{X}/\mathbf{C}[[t]]} \sim K_{(\mathscr{N} \times \mathscr{N})^n/\mathbf{C}[[t]]}|_{\mathscr{X}} \otimes \det\left(N_{\mathscr{X}}((\mathscr{N} \times \mathscr{N})^n)\right) \sim 0,$$

since $(\mathscr{N} \times \mathscr{N})^n$ is Calabi–Yau. Thus, the pair $(\overline{\mathscr{X}}, \overline{\mathscr{X}}_0)$ is logCY, as required. □

Restrict now the construction of the relative generalised Kummer variety to the identity component of the special fibre of $\mathscr{N}$, which is isomorphic to $\mathbf{C}^*$: this gives the construction of M_{SL_n} in 6.11. As a consequence, an irreducible component of the special fibre $\mathscr{K}^{\mathrm{sing}}_{n-1,0}$ is a lc logCY compatification of M_{SL_n} in $(\mathbf{P}^1 \times \mathbf{P}^1)^{(n)}$. This is shown by arguing as in the proof of Theorem 6.3.0.1.

Finally, the good minimal dlt model $\mathscr{K}^{\mathrm{dlt}}_{n-1}$ of the generalised Kummer variety K_{n-1} associated to $E \times E$ can be obtained, following [Kol13, Corollary 1.38], by extracting the exceptional divisors of the Hilbert–Chow morphism $\rho_{\mathrm{HC}} \colon K_{n-1} \to K^{\mathrm{sing}}_{n-1}$. □

Proof of Theorem 6.3.0.1. The result follows from Lemma 6.2.6.1, Proposition 6.3.1.1, and [BM17, Proposition 6.3.4]. □

6.4 Local computations on the Tate curve

The goal of this section is to prove Proposition 6.4.0.1, which is a technical ingredient needed in the proof of Proposition 6.3.1.1. The former result involves the Tate curve of [DR73, VII], whose existence and basic properties were discussed in §6.3.1. We begin by recalling its construction.

Let $(x_i)_{i\in\mathbf{Z}}$ be a collection of indeterminates. The Tate curve $\overline{\mathscr{G}}_m$ over the base ring $R \coloneqq \mathbf{C}[[t]]$ is the union of the affine charts $(U_{i+1/2})_{i\in\mathbf{Z}}$ given by

$$U_{i+1/2} \coloneqq \operatorname{Spec}\left(\frac{R[x_i, y_{i+1}]}{(x_i y_{i+1} - t)}\right).$$

For each $i \in \mathbf{Z}$, the charts $U_{i-1/2}$ and $U_{i+1/2}$ are glued along the open subscheme

$$\begin{aligned} T_i &\coloneqq U_{i-1/2} \cap U_{i+1/2} \\ &= \operatorname{Spec}\left(\mathcal{O}(U_{i+1/2})[x_i^{-1}]\right) = \operatorname{Spec}\left(R[x_i, x_i^{-1}]\right) \qquad (y_{i+1} = t/x_i) \\ &= \operatorname{Spec}\left(\mathcal{O}(U_{i-1/2})[y_i^{-1}]\right) = \operatorname{Spec}\left(R[y_i, y_i^{-1}]\right) \qquad (x_{i-1} = t/y_i) \end{aligned}$$

via the identification $x_i y_i = 1$.

The R-group scheme $\mathscr{G}_m \coloneqq \bigcup_{i\in\mathbf{Z}} T_i$, obtained from $\overline{\mathscr{G}}_m$ by removing the nodes in the special fibre, is the Néron model of the multiplicative group $\mathbf{G}_m$, as explained in [DR73, Example 1.2.c]. In particular, the n-th multiplication map

$$\mathbf{G}_m \times \ldots \times \mathbf{G}_m \to \mathbf{G}_m$$

extends to a homomorphism

$$\mu_n \colon \mathscr{G}_m^n \coloneqq \mathscr{G}_m \times_R \ldots \times_R \mathscr{G}_m \to \mathscr{G}_m$$

of R-group schemes, which (when $n = 2$) is given in local charts by

$$\begin{aligned} T_i \times_R T_j &\longrightarrow T_{i+j} \\ R[x_i, x_i^{-1}] \otimes_R R[x_j, x_j^{-1}] &\longleftarrow R[x_{i+j}, x_{i+j}^{-1}] \\ x_i \otimes x_j &\hookleftarrow x_{i+j}. \end{aligned}$$

As $x_{i-1}y_i = t$ and $x_iy_i = 1$, it follows that $x_i = t^{-i}x_0$. In particular, the identity section $\mathcal{I}d$ of $\mathscr{G}_m$ is cut out in the chart T_i by the equation $x_i = t^{-i}$.

Let $\mathscr{V}_{n-1} \coloneqq \mu_n^{-1}(\mathcal{I}d)$ be the fibre of the identity section $\mathcal{I}d$ via the n-th multiplication map μ_n, and let $\overline{\mathscr{V}}_{n-1}$ denote the closure of $\mathscr{V}_{n-1}$ in $\overline{\mathscr{G}}_m$. The proposition below describes the singularities of the pair $(\overline{\mathscr{V}}_{n-1}, \overline{\mathscr{V}}_{n-1,0})$, where $\overline{\mathscr{V}}_{n-1,0}$ is the special fibre of $\overline{\mathscr{V}}_{n-1}$.

Proposition 6.4.0.1. *The pair $(\overline{\mathscr{V}}_{n-1}, \overline{\mathscr{V}}_{n-1,0})$ is normal, reduced, and toric (i.e. it is Zariski-locally isomorphic to a normal toric scheme with its reduced toric boundary). Furthermore, the intersection $\overline{\mathscr{V}}_{n-1} \cap (\overline{\mathscr{G}}_m^n \setminus \mathscr{G}_m^n)$ has codimension two in $\overline{\mathscr{V}}_{n-1}$.*

The Proposition 6.4.0.1 is an immediate corollary of Lemma 6.4.0.2 below. Indeed, the assertions in Proposition 6.4.0.1 are local: we may work on the the open subsets

$$U_{\alpha+1/2} \coloneqq U_{\alpha_1+1/2} \times_R \ldots \times_R U_{\alpha_n+1/2},$$

for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}^n$, since the $U_{\alpha+1/2}$'s cover $\overline{\mathscr{G}}_m^n$.

For brevity, we omit the subscript $n-1$ from now on; let $\mathscr{V}_\alpha$ be the restriction of $\mathscr{V}$ to $T_\alpha \coloneqq T_{\alpha_1} \times_R \ldots \times_R T_{\alpha_n}$, $\overline{\mathscr{V}}_\alpha$ be its closure in $U_{\alpha+1/2}$, and $\overline{\mathscr{V}}_{\alpha,0}$ be the special fibre of $\overline{\mathscr{V}}_\alpha$. In local coordinates, we have

$$U_{\alpha+1/2} = \operatorname{Spec}\left(\frac{R[x_{\alpha_1}, y_{\alpha_1+1}, \dots, x_{\alpha_n}, y_{\alpha_n+1}]}{(x_{\alpha_i}y_{\alpha_i+1} - t)}\right),$$

$$T_\alpha = \operatorname{Spec}\left(R[x_{\alpha_1}^{\pm 1}, \dots, x_{\alpha_n}^{\pm 1}]\right) \subseteq U_{\alpha+1/2},$$

$$\mathscr{V}_\alpha = \left\{\prod_{i=1}^n x_{\alpha_i} = t^{-\sum \alpha_i}\right\} \subseteq T_\alpha.$$

Lemma 6.4.0.2. *For any $\alpha \in \mathbf{Z}^n$, the pair $(\overline{\mathscr{V}}_\alpha, \overline{\mathscr{V}}_{\alpha,0})$ is normal, reduced, and toric. Furthermore, the intersection $\overline{\mathscr{V}}_\alpha \cap (U_{\alpha+1/2} \setminus \mathscr{G}_m^n)$ has codimension two in $\overline{\mathscr{V}}_\alpha$.*

Proof. The proof is divided into cases depending on the sign of $|\alpha| \coloneqq \sum_{i=1}^n \alpha_i$. For the first case, assume that $|\alpha| > 0$. In this case, $\overline{\mathscr{V}}_\alpha$ is cut out of $U_{\alpha+1/2}$ by the equation $t^{|\alpha|}\prod_{i=1}^n x_{\alpha_i} = 1$, so t is invertible on $\overline{\mathscr{V}}_\alpha$. In particular, $\overline{\mathscr{V}}_\alpha$ and $\mathscr{V}_\alpha$ both coincide with the generic fibre of $U_{\alpha+1/2}$, which is isomorphic to the Frac(R)-scheme $\mathbf{G}_m^n$. Thus, there is nothing to prove.

In the second case, assume $|\alpha| = 0$. As $\prod_{i=1}^n x_{\alpha_i} = 1$ on $\overline{\mathscr{V}}_\alpha$, it follows that the x_{α_i}'s are invertible there, and hence the variables $y_{\alpha_i+1} = x_{\alpha_i}^{-1}x_{\alpha_1}y_{\alpha_1+1}$ can be eliminated. Thus, we have

$$\overline{\mathscr{V}}_\alpha = \operatorname{Spec}\left(R[x_{\alpha_1}^{\pm 1}, \dots, x_{\alpha_{n-1}}^{\pm 1}, y_{\alpha_1+1}]\right) \simeq \mathbf{G}_{m,R}^{n-1} \times_R \mathbf{A}_R^1,$$

$$\overline{\mathscr{V}}_{\alpha,0} = \{x_{\alpha_1}y_{\alpha_1+1} = 0\} = \{y_{\alpha_1+1} = 0\},$$

$$\overline{\mathscr{V}}_\alpha \cap (U_{\alpha+1/2} \setminus \mathscr{G}_m^n) \subseteq \overline{\mathscr{V}}_\alpha \cap \left\{\prod_{i=1}^n x_{\alpha_i} = 0\right\} = \emptyset.$$

It is clear from the above equations that $(\overline{\mathscr{V}}_\alpha, \overline{\mathscr{V}}_{\alpha,0})$ satisfies the required properties.

For the third and final case, assume $|\alpha| < 0$. We will show that $\overline{\mathscr{V}}_\alpha$ is normal by showing the conditions S_2 and R_1, and in the process we deduce that $(\overline{\mathscr{V}}_\alpha, \overline{\mathscr{V}}_{\alpha,0})$ is toric and $\overline{\mathscr{V}}_{\alpha,0}$ is reduced. This will be done in two steps.

First, observe that $\overline{\mathscr{V}}_\alpha$ is contained in the closed, toric subscheme $\mathscr{Z}_\alpha$ of $U_{\alpha+1/2}$

given by the equations

$$\begin{cases} t \cdot \prod_{i=1}^{n} x_{\alpha_i} = t^{-|\alpha|+1}, \\ x_{\alpha_1} y_{\alpha_1+1} = \ldots = x_{\alpha_n} y_{\alpha_n+1} = t \end{cases}$$

in $\operatorname{Spec}(R[x_{\alpha_1}, y_{\alpha_1+1}, \ldots, x_{\alpha_n}, y_{\alpha_n+1}])$. The fibres of $\mathscr{Z}_\alpha$ over R are easily described: over the generic fibre, $\mathscr{Z}_\alpha$ coincides with $\overline{\mathscr{V}}_\alpha$; over the special fibre, it is $(U_{\alpha+1/2})_0$, hence given by the equations

$$x_{\alpha_1} y_{\alpha_1+1} = \ldots = x_{\alpha_n} y_{\alpha_n+1} = t = 0.$$

Recall that if a Gorenstein scheme of pure dimension d is a union of two closed subschemes of pure dimension n and one is Cohen-Macaulay, then the other is Cohen-Macaulay; see [Kol11, Lemma 7]. Thus, since $\mathscr{Z}_\alpha = \overline{\mathscr{V}}_\alpha \cup (U_{\alpha+1/2})_0$ and both $\mathscr{Z}_\alpha$ and $(U_{\alpha+1/2})_0$ are complete intersections, it follows that $\overline{\mathscr{V}}_\alpha$ is Cohen-Macaulay, hence S_2. In particular, the pair $(\overline{\mathscr{V}}_\alpha, \overline{\mathscr{V}}_{\alpha,0})$ is toric, as both $\overline{\mathscr{V}}_\alpha$ and $\overline{\mathscr{V}}_{\alpha,0}$ are torus-invariant subschemes of $\mathscr{Z}_\alpha$.

Now, it is enough to check the condition R_1 at the generic point of each irreducible component of $\overline{\mathscr{V}}_{\alpha,0}$. As $\overline{\mathscr{V}}_\alpha$ is a toric R-scheme, such components are toric strata of $(U_{\alpha+1/2})_0$ of dimension $n-1$. Let (J, j) be the datum of a non-empty subset $J \subseteq I \coloneqq \{1, \ldots, n\}$, along with a distinguished element $j \in J$. Consider the $(n-1)$-dimensional stratum $Z_{(J,j)}$ of $\overline{\mathscr{V}}_{\alpha,0}$ given by the equations

$$Z_{(J,j)} \coloneqq \begin{cases} x_{\alpha_i} = 0 & i \in J, \\ y_{\alpha_i+1} = 0 & i \in (I \setminus J) \cup \{j\}, \\ t = 0. \end{cases}$$

Up to relabeling of the indices, we can assume that $1 \in J$ and $j = 1$, in which case we write $Z_{(J,j)}$ simply as Z_J. After localizing at the generic point of Z_J, the functions

$\{x_{\alpha_i} \colon i \in I \setminus J\}$, and $\{y_{\alpha_i+1} \colon i \in J \setminus \{1\}\}$ become invertible, and hence the variables

$$y_{\alpha_i+1} = x_{\alpha_i}^{-1} x_{\alpha_1} y_{\alpha_1+1} \qquad i \in I \setminus J,$$
$$x_{\alpha_i} = y_{\alpha_i+1}^{-1} x_{\alpha_1} y_{\alpha_1+1} \qquad i \in J \setminus \{1\},$$

can be eliminated. Thus, locally at the generic point of Z_J, we have

$$\begin{aligned}\overline{\mathscr{V}}_\alpha &\overset{\text{loc}}{=} \left\{ x_{\alpha_1} \cdot \left(\prod_{i \in J\setminus\{1\}} y_{\alpha_i+1}^{-1} \right) \cdot (x_{\alpha_1} y_{\alpha_1+1})^{|J|-1} \cdot \left(\prod_{i \in I \setminus J} x_{\alpha_i} \right) = (x_{\alpha_1} y_{\alpha_1+1})^{-|\alpha|} \right\} \\ &= \big\{ (\text{invertible}) \cdot (x_{\alpha_1})^{|J|+|\alpha|} (y_{\alpha_1+1})^{|J|+|\alpha|-1} = 1 \big\}\end{aligned}$$

in $\operatorname{Spec}\big(R[x_{\alpha_i}^{\pm 1}, y_{\alpha_l+1}^{\pm 1} : i \in I \setminus J, l \in J \setminus \{1\}][x_{\alpha_1}, y_{\alpha_1+1}]\big)$.

If $|J| + |\alpha| > 1$ or $|J| + |\alpha| < 0$, then $\overline{\mathscr{V}}_{\alpha,0}$ does not contain Z_J, and there is nothing to prove.

If $|J| + |\alpha| = 0$, then y_{α_1+1} is invertible and it is a function of $x_{\alpha_i}^{\pm 1}$ and $y_{\alpha_l+1}^{\pm 1}$ with $i \in I \setminus J$ and $l \in J \setminus \{1\}$, so that

$$\overline{\mathscr{V}}_\alpha \overset{\text{loc}}{=} \operatorname{Spec}\big(R[x_{\alpha_i}^{\pm 1}, y_{\alpha_l+1}^{\pm 1} : i \in I \setminus J, l \in J \setminus \{1\}][x_{\alpha_1}]\big) \simeq \mathbf{G}_{m,R}^{n-1} \times_R \mathbf{A}_R^1.$$

In particular, $\overline{\mathscr{V}}_{\alpha,0} = \{x_{\alpha_1} = 0\}$.

Finally, if $|J| + |\alpha| = 1$, then x_{α_1} is invertible and it is a function of $x_{\alpha_i}^{\pm 1}$ and $y_{\alpha_l+1}^{\pm 1}$ with $i \in I \setminus J$ and $l \in J \setminus \{1\}$, so that

$$\overline{\mathscr{V}}_\alpha \overset{\text{loc}}{=} \operatorname{Spec}\big(R[x_{\alpha_i}^{\pm 1}, y_{\alpha_l+1}^{\pm 1} : i \in I \setminus J, l \in J \setminus \{1\}][y_{\alpha_1+1}]\big) \simeq \mathbf{G}_{m,R}^{n-1} \times_R \mathbf{A}_R^1.$$

In particular, $\overline{\mathscr{V}}_{\alpha,0} = \{y_{\alpha_1+1} = 0\}$. We conclude that $\overline{\mathscr{V}}_\alpha$ is a normal toric irreducible scheme locally of finite type. The local computation above shows also that the divisor $\overline{\mathscr{V}}_{\alpha,0}$ is reduced. Further, in order to prove that $\overline{\mathscr{V}}_\alpha \cap (U_{\alpha+1/2} \setminus \mathscr{G}_m^n)$ has codimension two in $\overline{\mathscr{V}}_{n-1}$, it is enough to check that this intersection does not contain the generic point of any stratum Z_J. A point in $(U_{\alpha+1/2} \setminus \mathscr{G}_m^n)$ is characterized by the property that a pair of coordinates $(x_{\alpha_i}, y_{\alpha_i+1})$ for $i \in I$ vanishes simultaneously. However, this cannot happen at the generic point of Z_J, as the local equations above show. This concludes the proof of Lemma 6.4.0.2. $\square$

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