

SOME APPLICATIONS OF A THEOREM OF RADO

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It has been shown by Mirsky and Perfect [1] that the theorem of Rado [2], linking matroid theory and transversal theory, has important applications in combinatorial theory. In this note I use it to obtain necessary and sufficient conditions for two families of sets to have a common transversal containing a given set, and then I show how it may be used to obtain a variant of a well-known theorem that was obtained by Hoffman and Kuhn [3] using linear programming methods.

I use the notation of Mirsky and Perfect [4]. If $\mathbf{A} = \{A_i; i \in I\}$ is any finite family of sets, the set of distinct elements $\{x_i; i \in I\}$ is a *transversal* of \mathbf{A} if $x_i \in A_{\sigma(i)}$, ($i \in I$), for some permutation σ of I . If $\mathbf{A} = \{A_i; i \in I\}$ and $\mathbf{B} = \{B_i; i \in I\}$ are two finite families with the same index set, the set of distinct elements $\{x_i; i \in I\}$ is a *common transversal* of \mathbf{A} and \mathbf{B} if it is both a transversal of \mathbf{A} and a transversal of \mathbf{B} . (Although the individual sets A_i, B_i are not assumed to be finite, in practice this assumption may usually be made without loss of generality. This is because the various conditions considered will clearly hold for infinite sets if and only if they hold for suitably chosen finite subsets.)

Hoffman and Kuhn [5], completing a result of Mann and Ryser [6], gave necessary and sufficient conditions for $\mathbf{A} = \{A_i; i \in I\}$ to have a transversal which contains a given finite set C , namely that for all subsets J of I

$$\min(A(J), |I| - |C - A(J)|) \geq |J|,$$

where we use $A(J)$ to denote the union of those A_i whose indices are members of J . These conditions are clearly necessary; it would be easy to prove their sufficiency by applying Rado's theorem to the matroid $(A(I), \mathbf{M})$ where $X \in \mathbf{M}$ if $|X \cup C| \leq |I|$. Mirsky and Perfect [1] have given a proof based on the notion of deltoid independence.

Ford and Fulkerson [7] gave necessary and sufficient conditions for $\mathbf{A} = \{A_i; i \in I\}$ and $\mathbf{B} = \{B_i; i \in I\}$ to have a common transversal, namely that

$$|A(J) \cap B(K)| \geq |J| + |K| - |I|$$

for all subsets J, K , of I . Mirsky and Perfect [1] give a proof based on Rado's theorem.

I first prove a common generalisation of the above two results.

THEOREM 1. *Two families \mathbf{A} and \mathbf{B} , with common finite index set I , have a common transversal which contains the set C , if, and only if,*

$$|\{A(J) \cap B(K)\} - C| + |A(J) \cap C| + |B(K) \cap C| \geq |J| + |K| + |C| - |I|$$

for all subsets J, K of I .

We recover the result of Ford and Fulkerson by taking $C = \emptyset$ and that of Hoffman and Kuhn [5] on taking $B_k = A(I)$ for all $k \in I$ and considering the two cases $K = \emptyset, K \neq \emptyset$.

Before proving Theorem 1, I review the matroid terminology used. A *matroid* (S, \mathbf{M}) is a family \mathbf{M} of *independent* subsets of a finite set S such that:

- (1) $\emptyset \in \mathbf{M}$;
- (2) $X \in \mathbf{M}$ and $Y \subset X$ implies $Y \in \mathbf{M}$;
- (3) if Z is any subset of S , the maximal independent subsets of Z have the same cardinality. This cardinality is called the *rank* of Z in (S, \mathbf{M}) and will be denoted by $r(Z)$.

A *base* of (S, \mathbf{M}) is a maximal independent subset of S , and alternative axioms defining a matroid in terms of its bases are given by Whitney [8]:

- (4) no proper subset of a base is a base;
- (5) if B, B' , are bases and $e \in B$, then there exists $f \in B'$ such that $B - e + f$ is a base.

For further details of matroid theory see [8] or Tutte [9]. We state now the result of Rado [2] which is the central theorem on which this paper is based.

RADO'S THEOREM. *If (S, \mathbf{M}) is a matroid, and $\mathbf{A} = \{A_i; i \in I\}$ is any family of subsets of S , then \mathbf{A} has a transversal which is independent in the matroid \mathbf{M} , if, and only if, for all $J \subset I$*

$$r(A(J)) \geq |J|.$$

Proof of Theorem 1. We may suppose that $A(I)$ is finite. Let \mathbf{T} denote the set of transversals of the family \mathbf{A} . Then by a theorem of Edmonds and Fulkerson [10] (also obtained independently by Mirsky and Perfect [11]), \mathbf{T} is the set of bases of a matroid on S , which we call the *transversal matroid* determined by \mathbf{A} and denote by $[S, \mathbf{T}(\mathbf{A})]$. (We allow \mathbf{T} to be the null set in which case $[S, \mathbf{T}(\mathbf{A})]$ is the trivial matroid having the null set as its only base.)

Define a new matroid (S, \mathbf{T}) on S by the rule that a subset X is a base of \mathbf{T} if and only if X is a base of $[S, \mathbf{T}(\mathbf{A})]$ and in addition X contains the set C . It is trivial to show that this does in fact define a matroid on S by axioms (4) and (5). It is clear that \mathbf{A} and \mathbf{B} have a common transversal containing the set C if and only if \mathbf{B} has a transversal which is a base of the matroid (S, \mathbf{T}) . Hence by the theorem of Rado, \mathbf{A}, \mathbf{B} have a common transversal containing C if, and only if,

$$(6) \quad r(\mathbf{B}(K)) \geq |K|,$$

where $r(\)$ is the rank function of the matroid (S, \mathbf{T}) , and K is any subset of I . Since any independent set in a matroid can be augmented to a base, it is not difficult to see that for any subset X of S

$$r(X) \geq k,$$

if and only if $X \cup C$ contains a partial transversal of \mathbf{B} of cardinality not less than

$$k + |C| - |C \cap X|.$$

This means that

$$\{(X \cup C) \cap B_i; i \in I\}$$

must have a partial transversal with defect

$$|I| - k - |C| + |C \cap X|.$$

By a theorem of Ore [11] the condition for this is that

$$|(X \cup C) \cap B(K)| \geq |K| - |I| + k + |C| - |C \cap X|$$

for any subset K of I .

Hence (6) is satisfied if, and only if,

$$(7) \quad |(A(J) \cup C) \cap B(K)| \geq |J| + |K| + |C| - |I| - |C \cap A(J)|$$

for any subsets J, K , of I .

It is trivial to put (7) into the symmetric form used in the statement of the theorem.

Another application of Rado's theorem is to prove

THEOREM 2. *If $\{E_i; i \in J\}$ is any partition of a finite set S into disjoint subsets, and $A = \{A_i; i \in I\}$ is any finite family of subsets of S , then A has a transversal X such that*

$$|X \cap E_j| \leq a_j, \quad j \in J,$$

where the $a_j, j \in J$, are prescribed integers, if, and only if,

$$|A(K)| - \sum_j (|A(K) \cap E_j| - a_j)^+ \geq |K|$$

for all subsets K of I (where, as in Mirsky and Perfect [4], we write z^+ for $\max(z, 0)$).

This theorem, although only a weaker form of a theorem of Hoffman and Kuhn [3], seems worth mentioning because of the very simple proof.

Proof. Let (S, M) be the matroid which has as bases those subsets Z of S which satisfy

$$|Z \cap E_j| = a_j, \quad (j \in J).$$

It is trivial, using axioms (4) and (5), to verify that this is in fact a matroid. Now if $r(\cdot)$ denotes the rank function in (S, M) then for any $X \subset S$

$$(8) \quad r(X) = |X| - \sum_j (|X \cap E_j| - a_j)^+.$$

But the family A has a transversal X which satisfies

$$|X \cap E_j| \leq a_j, \quad (j \in J),$$

if, and only if, A has a transversal X which is an independent set in (S, M) . By Rado's theorem this is so if and only if

$$r(A(K)) \geq |K|$$

for all $K \subset I$. The theorem follows from (8).

Using the same technique we now establish an alternative version of the main theorem of Hoffman and Kuhn [3].

Let $E = \{E_j; j \in J\}$ be any partition of the finite set S into disjoint subsets. Let $a_j, b_j, (j \in J)$ be integers with

$$0 \leq a_j \leq b_j, \quad (j \in J).$$

For any subset X of S we define a *lower deficiency* $L(X)$ by

$$L(X) = \sum_j (a_j - |X \cap E_j|)^+,$$

and an *upper deficiency* $H(X)$ by

$$H(X) = \sum_j (|X \cap E_j| - b_j)^+.$$

We prove

THEOREM 3. Let $A = \{A_i; i \in I\}$ be a finite family of subsets of a finite set S . Let $\{E_j; j \in J\}$ be any partition of S into pairwise disjoint subsets. Let $a_j, b_j, (j \in J)$ be integers such that

$$0 \leq a_j \leq b_j, \quad (j \in J).$$

Then \mathbf{A} has a transversal X with the property

$$(9) \quad a_j \leq |X \cap E_j| \leq b_j, \quad (j \in J),$$

if and only if for all $K \subset I$

$$\left(|I| - |A(K)| - L(A(K)) + H(A(K))\right)^+ \leq |I| - |K| - L(A(K)).$$

The conditions given are at first sight unattractive but from a computational point of view they are certainly not more formidable than those given by Hoffman and Kuhn.

Proof. Let (S, \mathbf{M}) be the matroid having as bases only those subsets B of S such that

$$(10) \quad |B| = |I|,$$

$$(11) \quad a_j \leq |B \cap E_j| \leq b_j.$$

We show that the family \mathbf{B} of subsets of S satisfying (10) and (11) also satisfy the base axioms of a matroid. Clearly no member of \mathbf{B} can properly contain another. Let B_1, B_2 be distinct members of \mathbf{B} and let $e \in B_1$. Since $\{E_j; j \in J\}$ partition S , e is a member of exactly one E_j , say E_1 . Hence

$$a_1 - 1 \leq |(B_1 - e) \cap E_1| \leq b_1 - 1.$$

If $B_2 \cap E \not\subset (B_1 - e) \cap E_1$, let $f \in (B_2 \cap E_1) - (B_1 \cap E_1)$, and $B_3 = B_2 - e + f$ satisfies (10) and (11).

If $(B_2 \cap E) \subset (B_1 - e) \cap E_1$, then since B_2 satisfies (11),

$$a_1 \leq |(B_1 - e) \cap E_1| \leq b_1 - 1.$$

Also, since $|B_2| = |B_1 - e| + 1$, there must exist some E_j ($j \neq 1$) such that

$$|(B_1 - e) \cap E_j| < |B_2 \cap E_j| \leq b_1.$$

Let $f \in (B_2 \cap E_j) - (B_1 - e)$; then clearly $B_1 - e + f$ satisfies (10) and (11) and thus (S, \mathbf{M}) is a matroid.

By the definition of (S, \mathbf{M}) , the family \mathbf{A} has a transversal which satisfies the intersection properties (9) if and only if \mathbf{A} has a transversal which is a base of (S, \mathbf{M}) . Again using Rado's theorem this will be so if, and only if, for any $K \subset I$,

$$r(A(K)) \geq |K|,$$

where $r(\)$ is the rank function of (S, \mathbf{M}) .

Now if X is any subset of S , $r(X) \geq u$ if, and only if, there exists a subset Y of S such that $X \cup Y$ contains a base of (S, \mathbf{M}) , and

$$|Y| \leq r(S) - u = |I| - u.$$

But this subset Y must contain

$$L(X) + (|I| - |X| - L(X) + H(X))^+$$

elements.

Hence $r(X) \geq u$ if and only if

$$L(X) + (|I| - |X| - L(X) + H(X))^+ \leq |I| - u.$$

This proves the theorem.

Conclusion. The purpose of this paper is to illustrate the power of the theorem of Rado, proved as early as 1942. Many other theorems on transversal theory proved by methods of linear programming or *ad hoc* combinatorial methods may be deduced from it, and it certainly appears to have a strong claim to being one of the fundamental theorems of combinatorial theory.

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