TREES WITH HAMILTONIAN SQUARE

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Plummer (see [**2**; p. 69]) conjectured that the square of every block is hamiltonian, and this has just been proved by Fleischner [**1**]. It was shown by Karaganis [**3**] that the cube of every connected graph, and hence the cube of every tree, is hamiltonian. Our present object is to characterize those trees whose *square* is hamiltonian in three equivalent ways.

We follow the terminology and notation of the book [**2**]. In particular, the following concepts are used in stating our main result. A graph is *hamiltonian* if it has a cycle containing all its points. The graph with the same points as G, in which two points are adjacent if their distance in G is at most 2, is denoted by G^2 and is called the *square* of G. The *subdivision graph* $S(G)$ is formed (Figure 1) by inserting a point of degree two on each line of G.

Fig. 1.—The subdivision of the graph $K_{1,3}$.

The *crossing number* of G, denoted $\nu(G)$, is the smallest number of crossings of pairs of edges when G is drawn in the plane. We now develop a variation on $\nu(G)$ for bipartite graphs. To do this, we draw a bigraph B with points of one color on the abscissa and those of the other color at unit distance above the abscissa. We require each line of B to be drawn as a straight line segment. Then the *bipartite crossing number* $\nu_2(B)$ is the smallest possible number of crossings occurring in all such embeddings of B. Figure 2 shows two bigraphs drawn in this way. The first is $S(K_{1,3})$ from Figure 1 which has $\nu_2 = 1$, the second is a tree T with $\nu_2 = 0$.

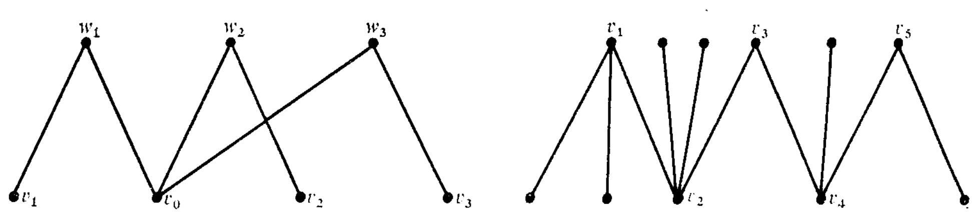

Fig. 2a. Fig. 2b.

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We can now state the main theorem which gives several characterizations of trees with hamiltonian square.

THEOREM 1. *Let T be a tree at least 3 points. The following statements are equivalent*:

(1) *T^2 is hamiltonian.*
(2) *T does not contain $S(K_{1,3})$ as a subgraph.*
(3) *T minus its endpoints is a path, P.*
(4) $\nu_2(T) = 0$.

Proof. (1) *implies* (2). Let C be a hamiltonian cycle of T^2, and assume T has a subgraph $S(K_{1,3})$ labelled as in Fig. 1. Let V_i be the set of points in the connected component of w_i in $T - v_0$, for $i = 1, 2, 3$. Certainly C must enter V_i and leave it at least once. Since each $|V_i| \geqslant 2$ by the assumption, there must be two distinct points which are the first and last to appear as C passes through V_i, and one of these, say u_i, is *not* w_i since w_i cannot occur twice in a cycle. Now every path between T_i and $T - T_i$ must contain w_i and v_0, and so, the only point of $T - T_i$ whose distance from u_i is at most two is v_0. But C contains a line between u_i and $T - T_i$, so this line must be $u_i v_0$. Since this holds for $i = 1, 2, 3$, it follows that v_0 is incident with three lines in C, a contradiction. Thus, T contains no subgraph $S(K_{1,3})$.

(2) *implies* (3). By hypothesis, T contains no $S(K_{1,3})$. Let T' be the tree obtained by deleting all the endpoints of T. If T' is not a path, it has a point of degree at least three. It is then evident that v is the central point of some $S(K_{1,3})$ subgraph in T. Since this violates the hypothesis, we conclude that T' is a path.

(3) *implies* (4). By hypothesis, T' is a path P which can be labelled $v_1, v_2, \ldots, v_n$. Embed P on two parallel lines in a zigzag fashion with no crossing as illustrated in Fig. 2b for $n = 5$. Then, for each endpoint of T adjacent to v_i, insert a point on the herizental line opposite v_i between v_{i-1} and v_{i+1}. The resulting embedding of T has no crossings, so $\nu_2(T) = 0$.

(4) *implies* (1). Suppose T is embedded with no crossings (as in Fig. 2b). It is convenient to label the upper points from left to right by $u_1, u_2, \ldots, u_k$ and the lower points from right to left by $w_1, w_2, \ldots, w_{p-k}$. Observe that

$$d(u_i, u_{i+1}) = d(w_j, w_{j+1}) = 2,$$

for $1 \leqslant i \leqslant k - 1$ and $1 \leqslant j \leqslant p - k - 1$, since any longer path would have to cross itself. Similarly, $d(u_1, w_{p-k}) = d(w_1, u_k) = 1$. Consequently,

$$u_1, u_2, \ldots, u_k\, w_1, w_2, \ldots, w_{p-k}\, u_1,$$

is a hamiltonian cycle in T^2.

We notice that the hamiltonian cycle found above uses just two lines of T, and these are the endlines of a longest path. This happens to be always true.

THEOREM 2. *If T^2 is hamiltonian, then any hamiltonian cycle contains exactly two lines of T, and these are the endlines of a longest path. Moreover, the endlines of every longest path lie on some hamiltonian cycle.*

Proof. The result is trivial when T has 3 points. We assume the theorem has been proven for trees on fewer than p points, and proceed to prove it for a p-point tree T, whose square is hamiltonian.

If T is itself a path, it can be labelled $v_1, v_2, \ldots, v_p$. It is easily seen that T^2 has a unique hamiltonian cycle, and this cycle contains only $v_1 v_2$ and $v_{p-1} v_p$ and no other lines of T.

If T is not a path, consider any longest path P, and an endpoint v_1 not lying on P. Let v_2 and v_3 be the points which precede and follow v_1 on a hamiltonian cycle C of T^2. Now either v_1, v_2 and v_3 are all adjacent in T to a fourth point v_4 or else $v_1 v_2 v_3$ is a path in T. In either case $d(v_2, v_3) \leqslant 2$. Consequently, the graph $(T - v_1)^2$ has a hamiltonian cycle C' which is formed by deleting v_1 from C. Clearly P is also a longest path of $T - v_1$, so by the induction hypothesis, only the endlines of P can appear in the cycle C'. Consequently, $d(v_2, v_3) \neq 1$, so $d(v_2, v_3) = 2$. Thus, the only lines of T used in C are the end-lines of P, completing the proof.

As a consequence of Theorem 2, we notice that if T^2 is hamiltonian, all its hamiltonian cycles are identical (up to the order of traversing the endpoints adjacent to a common point).

In [4], Neumann developed necessary and sufficient conditions for a given line of a tree T to lie on a hamiltonian cycle, if any, of T^2. His rather involved result is subsumed by our Theorem 2.

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