

MEAN-VALUES OF THE RIEMANN ZETA-FUNCTION

K. SOUNDARARAJAN

§1. *Introduction.* Let

$$I_k(T) = \int_T^{2T} |\zeta(\frac{1}{2} + it)|^{2k} dt.$$

Asymptotic formulae for $I_k(T)$ have been established for the cases $k = 1$ (Hardy-Littlewood, see [13]) and $k = 2$ (Ingham, see [13]). However, the asymptotic behaviour of $I_k(T)$ remains unknown for any other value of k (except the trivial $k = 0$, of course). Heath-Brown, [6], and Ramachandra, [10], [11], independently established that, assuming the Riemann Hypothesis, when $0 \leq k \leq 2$, $I_k(T)$ is of the order $T(\log T)^{k^2}$. One believes that this is the right order of magnitude for $I_k(T)$ even when $k = 2$ and indeed expects an asymptotic formula of the form

$$I_k(T) = (C_k + o(1))T(\log T)^{k^2},$$

where C_k is a suitable positive constant. It is not clear what the value of C_k should be.

In [3], Conrey and Ghosh showed that the Riemann Hypothesis implies

$$I_k(T) \geq (1 + o(1))T \sum_{n \leq T} \frac{d_k^2(n)}{n} = (c_k + o(1))T(\log T)^{k^2}$$

where

$$c_k = \frac{1}{\Gamma(k^2 + 1)} \prod_p \left\{ \left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{k(k+1) \dots (k+m-1)}{m!} \right)^2 p^{-m} \right\}.$$

The dependency on the Riemann Hypothesis was removed by Balasubramanian and Ramachandra, [2], in the special case where k is an integer. In a later paper, [4], Conrey and Ghosh used results of Balasubramanian, Conrey and Heath-Brown, [1], to improve this lower bound. In this paper we obtain further improvements on the results of Conrey and Ghosh. In the case when $k \geq 3$ is an integer our results double the previous bounds and a comparable (although smaller) improvement is obtained in the non-integral cases as well. This work is motivated and inspired by Ramachandra's proof of the fourth power moment, [9].

Let $k \geq 3$ be an integer. Let $N = T^\theta$ where $\theta \in (0, 1)$ will be fixed later. Let $r = k - 1$ and

$$A_r(s, P) = \sum_{n \leq N} \frac{d_r(n)P(\log n/\log N)}{n^s}$$

where $s = \sigma + it$ is a complex variable and P is a polynomial which will be chosen appropriately. The lower bounds of [4] were obtained by analysing the right-hand side of the inequality

$$0 \leq \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} |\zeta(s)|^2 |\zeta^r(s) - A_r(s, P)|^2 ds.$$

We 'twist' the right-hand side and consider

$$0 \leq \int_{1/2+iT}^{1/2+2iT} |\zeta(s)|^2 |\zeta^r(s) - A_r(s, P) - \chi^r(s)A_r(1-s, P)|^2 ds$$

where $\chi(s)$ is the factor arising from the asymmetric functional equation

$$\zeta(s) = \chi(s)\zeta(1-s).$$

THEOREM 1.1. *Suppose $\theta < \frac{1}{2}$ when $k = 3$. Put*

$$J_4 = \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} |\zeta(s)|^2 \chi^r(s) A_r(1-s, P)^2 ds.$$

With the above notations the bound

$$J_4 = O(T^{1-\epsilon})$$

holds, unconditionally when $k = 3$ and on the assumption of the Lindelöf Hypothesis when $k \geq 4$, for arbitrary fixed $\epsilon > 0$.

COROLLARY 1.2. *The lower bound*

$$I_3 \geq (20 \cdot 26 + o(1))c_3 T(\log T)^9$$

holds unconditionally. If the Lindelöf Hypothesis is assumed then the following asymptotic lower bounds for $F_k = I_k / (c_k T(\log T)^{k^2})$, ($4 \leq k \leq 6$) hold:

$$F_4 \geq 410, \quad F_5 \geq 6484, \quad F_6 \geq 56260.$$

Observe that Corollary 1.2 improves Corollary 1 of [4] by a factor of 2. Conrey and Ghosh also obtain results assuming the validity of their Theorems 1 and 2 in a wider range of θ . This assumption is roughly of the same strength as the conjecture in [1]. Upon making this assumption we obtain corresponding improvements since our Theorem 1.1 is valid for all $\theta < 1$ when $k \geq 4$.

COROLLARY 1.3. *Assuming the Lindelöf Hypothesis and that Theorems 1 and 2 of Conrey and Ghosh, [4], remain valid for any $\theta < 1$ the improved asymptotic lower bounds*

$$I_4 \geq 43056c_4 T(\log T)^{16}; \quad I_5 \geq 96877600c_5 T(\log T)^{25}$$

hold. Also as $k \rightarrow \infty$

$$I_k \geq 2c(ek/2)^{2k-3/2} c_k T(\log T)^{k^2}$$

where $c = 1/(e\sqrt{2\pi e})$.

If $k \geq 7$ then the assumption that $\theta < 1$ is permissible is necessary to obtain improvements over the bound of Conrey and Ghosh, [3]. This is a consequence of the θ^{k^2} , present in our Lemma 2.3, which rapidly goes to 0 as k becomes large. However the results of Conrey and Ghosh, [3], can be improved unconditionally by considering

$$\int_{1/2+iT}^{1/2+2iT} |\zeta^k(s) - A_k(s, P) - \chi^k(s)A_k(1-s, P)|^2 ds.$$

Let

$$K_4 = \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} \chi^k(s)A_k(1-s, P)^2 ds.$$

THEOREM 1.4. *If k is an integer and $\theta < 1$ then, unconditionally,*

$$|K_4| = O(T^{1-\varepsilon}).$$

Theorem 1.4 may be proved along the exact same lines as Theorem 1.1. As a consequence we obtain the aforementioned improvement.

COROLLARY 1.5. *If $k \geq 2$ is an integer then*

$$I_k \geq (2 + o(1))c_k T(\log T)^{k^2}.$$

Note that Corollary 1.5 is independent of any hypothesis. This is a consequence of the results of [2] which remove the dependency upon the Riemann Hypothesis in [3].

We now turn to the case when k is not an integer. The difficulties in this case arise from the failure of the equality

$$\frac{\zeta^k(s)}{\zeta^k(1-s)} = \chi^k(s)$$

when k is not an integer.

THEOREM 1.6. *Suppose $k > 2$ and assume the truth of the Riemann Hypothesis. Put*

$$H(x, y, z) = \log \left(\frac{(r+1) \log T - 2 \log (xy/z)}{(r-1) \log T - 2 \log z} \right) + r \left(\theta - \frac{\log y}{\log T} \right) \log \left(\frac{(r-\theta) \log T + 2 \log z - \log x}{(r-1-\theta) \log T - \log z} \right).$$

Then, if $\theta < \min (1/2, (r-1)/2)$,

$$|J_4| = \left| \int_{1/2+iT}^{1/2+2iT} |\zeta(s)|^2 \frac{\zeta^r(s)}{\zeta^r(1-s)} A_r(1-s, P)^2 ds \right| \leq \frac{T \log T}{\pi} |e^{2\pi ik} - 1| \sum_{u, v \leq T^\theta} \frac{b_u b_v}{[u, v]} H(u, v, (u, v)),$$

where $b_u = d_r(u)P(\log u / (\theta \log T))$.

COROLLARY 1.7. *Put*

$$Q(x) = P(x) \left(\log \left(\frac{r+1-2\theta}{r-1-2\theta} \right) + r(\theta - \theta x) \log \left(\frac{r}{r-1-2\theta} \right) \right)$$

and

$$G(x) = \left(\int_x^1 r(z-x)^{r-1} P(z) dz \right) \left(\int_x^1 r(z-x)^{r-1} Q(z) dz \right).$$

Then, if $\theta < \min (1/2, (r-1)/2)$,

$$|J_4| \leq \frac{\Gamma(k^2+1)}{\Gamma^2(r+1)\Gamma(r^2+1)} c_k \theta^{k^2-1} T(\log T)^{k^2} \left(\int_0^1 y^{r^2-1} G(y) dy \right).$$

COROLLARY 1.8. *If $\theta < 1$ is permissible in Theorems 1 and 2 of Conrey and Ghosh, [4], and in our Theorem 1.6, then, as $k \rightarrow \infty$,*

$$I_k \geq \left(2 + O\left(\frac{1}{k}\right) \right) c(ek/2)^{2k-3/2} c_k T(\log T)^{k^2}$$

where, as in Corollary 1.3, $c = 1/(e\sqrt{2\pi e})$.

As in the integral case, when k becomes large it is necessary to assume the $\theta = 1$ conjectures to obtain improvements upon the bound of [3]. Again, as in

the integral case, considering

$$0 \leq \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} \left| \zeta^k(s) - A_k(s, P) - \frac{\zeta^k(s)}{\zeta^k(1-s)} A_k(1-s, P) \right|^2 ds$$

leads to improvements that are independent of the $\theta = 1$ conjectures.

THEOREM 1.9. *Suppose k is not an integer and assume the validity of the Riemann Hypothesis. Then, if $\theta < \min(1, k/2)$,*

$$\begin{aligned} |K_4| &= \left| \int_{1/2+iT}^{1/2+2iT} \frac{\zeta^k(s)}{\zeta^k(1-s)} A_k(1-s, P)^2 ds \right| \\ &\leq |e^{2\pi ik} - 1| \frac{T}{\pi} \left(\sum_{n \leq T^\theta} \frac{d_k(n)^2 P(\log n / (\theta \log T))^2 \log T}{n(k \log T - 2 \log n)} \right. \\ &\quad \left. + 2k \sum_{n \leq T^\theta} \frac{d_k(n)^2 P(\log n / (\theta \log T))^2}{n} \log \left(\frac{k \log T - 2 \log n}{(k - \theta) \log T - \log n} \right) \right). \end{aligned}$$

COROLLARY 1.10. *Put $\theta = \frac{1}{2}k - \varepsilon$ if $k \leq 2$ and $\theta = 1 - \varepsilon$ if $k \geq 2$. Take*

$$P(x) = \left(1 + |e^{2\pi ik} - 1| \left(\frac{1}{2\pi(k - 2\theta x)} + \frac{k}{\pi} \log \left(\frac{k - 2\theta x}{k - \theta - \theta x} \right) \right) \right)^{-1}.$$

Then

$$I_k(x) \geq (1 + o(1)) 2c_k \theta^{k^2} T (\log T)^{k^2} \left(\int_0^1 k^2 x^{k^2-1} P(x) dx \right).$$

As $k \rightarrow \infty$,

$$I_k \geq \left(2 + O\left(\frac{1}{k}\right) \right) c_k T (\log T)^{k^2}.$$

Theorem 1.9 may be proved using exactly the same techniques as the proof of Theorem 1.6. If $1 < k < 2$, Conrey and Ghosh, [4], have explicit lower bounds for $I_k(T)$. If the $\theta = 1$ conjectures are not assumed then these bounds are never better than

$$I_k(T) \geq 1.7c_k T (\log T)^{k^2}.$$

It is evident that our Corollary 1.10 gives

$$I_k(T) \geq (2 - \varepsilon)c_k T (\log T)^{k^2}$$

provided k is sufficiently close to 2. In practice, this improvement is noticeable only when $k \geq 1.95$. It is possible to refine the bounds of Theorems 1.6 and 1.9 by moving the line of integration to $\frac{1}{2} + c/\log T$ (instead of just $\mu > \frac{1}{2}$) with an appropriately chosen c . This would enable us to take longer Dirichlet

polynomials A , when $k < 2$ (in Theorem 1.9) or $r < 2$ (in Theorem 1.6). However, the improvements arising from these refinements do not justify the extra efforts required.

We often write $A \leq B$ when we mean $A \leq (1 + o(1))B$. This should not be cause for confusion.

I am grateful to Professors H. L. Montgomery and T. D. Wooley for encouragement.

§2. *Some Lemmata.* Let

$$J_2 = \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} |\zeta(s)|^2 \zeta^r(s) A_r(1-s, P) ds$$

and

$$J_3 = \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} |\zeta(s)|^2 |A_r(s, P)|^2 ds.$$

LEMMA 2.1. *With the above notation*

$$I_k \geq 4\Re J_2 - 2J_3 - 2\Re J_4.$$

Proof. Observe that

$$\begin{aligned} 0 &\leq \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} |\zeta(s)|^2 |\zeta^r(s) - A_r(s, P) - \chi^r(s) A_r(1-s, P)|^2 ds \\ &= \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} |\zeta(s)|^{2k} ds - 4\Re \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} |\zeta(s)|^2 \zeta^r(s) A_r(1-s, P) ds \\ &\quad + \frac{2}{i} \int_{1/2+iT}^{1/2+2iT} |\zeta(s) A_r(s, P)|^2 ds + 2\Re \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} |\zeta(s)|^2 \chi^r(s) A_r(1-s, P)^2 ds. \end{aligned}$$

Rearranging we obtain the lemma.

Let

$$\begin{aligned} K_2 &= \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} \zeta^k(s) A_k(1-s, P) ds, \\ K_3 &= \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} |A_k(s, P)|^2 ds, \end{aligned}$$

and

$$K_4 = \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} \frac{\zeta^k(s)}{\zeta^k(1-s)} A_k(1-s, P)^2 ds.$$

LEMMA 2.2. *With the above notation*

$$I^k(T) \geq 4\Re K_2 - 2K_3 - 2\Re K_4.$$

Proof. The proof is identical to the proof of Lemma 2.1.

LEMMA 2.3. *Let*

$$h(\alpha) = \int_0^1 (\beta - \alpha)^r P(\beta) d\beta$$

and let ${}_2F_1$ denote the usual hypergeometric function. Then, if $\theta < \frac{1}{2}$,

$$J_3 = (1 + o(1)) T (\log N)^{k^2} \frac{c_k \Gamma(k^2 + 1)}{\Gamma(r + 1)^2 \Gamma(r^2)} \\ \times \int_0^1 \alpha^{r^2-1} \left(\frac{h'(\alpha)^2}{\theta} + 2rh(\alpha)h'(\alpha) \right) d\alpha$$

and

$$J_2 = (1 + o(1)) T (\log N)^{k^2} \frac{c_k \Gamma(k^2 + 1) \theta^{-1-r}}{\Gamma(r + 2) \Gamma(r^2 + r)} \\ \times \int_0^1 P(\alpha) \alpha^{k^2-r-2} {}_2F_1(-r, -r-1, k^2-r-1, -\alpha\theta) d\alpha.$$

Proof. See Theorems 1 and 2 of Conrey and Ghosh, [4].

LEMMA 2.4. *The following asymptotic formulae hold.*

$$K_3 = (1 + o(1)) T \sum_{n \leq T^\theta} \frac{d_k^2(n) P(\log n / (\theta \log T))^2}{n}$$

and

$$K_2 = (1 + o(1)) T \sum_{n \leq T^\theta} \frac{d_k^2(n) P(\log n / (\theta \log T))}{n}.$$

Proof. This is easily obtained by slightly modifying the results in Conrey and Ghosh, [3].

LEMMA 2.5. Let $x, T > 1$ and $\rho = \beta + iy$ run over the complex zeros of $\zeta(s)$. Then

$$\sum_{T \leq \gamma \leq 2T} x^\rho = -\frac{T}{2\pi} \Lambda(x) + O(x \log(2x) \log \log(3x) + x \log 2T) + O\left(\log x \min\left(T, \frac{x}{\langle x \rangle}\right) + \min\left(\frac{\log T}{\log x}, T \log T\right)\right)$$

where $\langle x \rangle$ is the distance from x to the nearest prime power other than x .

Proof. This is Theorem 1 of Gonek, [5].

LEMMA 2.6. Let $r \geq 0$ and $h = \prod_p p^{h_p}$ be a positive integer. Define

$$D_r(h, s) = \prod_{p|h} \left((1 - p^{-s})^r \sum_{m=0}^{\infty} d_r(p^{m+h_p}) p^{-ms} \right)$$

so that when $\Re s > 1$,

$$D_r(h, s) \zeta^r(s) = \sum_{n=1}^{\infty} \frac{d_r(hn)}{n^s}.$$

Then

$$\sum_{n \leq x} \frac{d_r(hn)}{n} = \frac{D_r(h, 1)(\log x)^r}{\Gamma(r+1)} + O(E(h, r)),$$

say, where

$$\sum_{h \leq H} |E(r, h)| = O(H(\log H)^{r-1}).$$

Also

$$\sum_{n \leq x} \frac{D_r(n, 1)^2 \varphi(n)}{n} = (1 + o(1)) \frac{c_{r+1} \Gamma(1 + (r+1)^2)}{\Gamma(r^2)} x(\log x)^{r^2-1}.$$

Proof. These results are easily obtainable *via* the results of Selberg, [12]. They may also be found as Lemmata 4 and 5, and equations (36) and (37) of Conrey and Ghosh, [4].

§3. *Proof of Theorem 1.1 and its Corollaries.* We first treat the case $k \geq 4$ and when the Lindelöf Hypothesis is assumed. By Cauchy's theorem

$$J_4 = \frac{1}{i} \left(\int_{\mu+iT}^{\mu+2iT} + \int_{1/2+iT}^{\mu+iT} - \int_{1/2+2iT}^{\mu+2iT} \right) \zeta^2(s) \chi(s)^{r-1} A_r(1-s, P)^2 ds$$

where $\mu > \frac{1}{2}$ is some constant. Since $|A_r(1-s, P)| = O(T^{\theta(\sigma+\epsilon)})$, $|\chi(\frac{1}{2} + it)| = 1$ and $|\zeta(s)| = O(T^\epsilon)$ by the Lindelöf Hypothesis, the horizontal integrals are

bounded by $O(T^{\theta+\epsilon}) = O(T^{1-\epsilon})$. Since $|\chi(s)| = O(T^{(1/2-\sigma)})$ it follows that the integral on the μ -line is bounded by

$$O\left(T^{(r-1)(1/2-\mu)+\epsilon} \int_{\mu+iT}^{\mu+2iT} |A_r(1-s, P)|^2 ds\right).$$

By the mean-value theorem for Dirichlet polynomials, this is

$$O\left(T^{1+(r-1)(1/2-\mu)+\epsilon} \sum_{n \leq N} \frac{d_r^2(n) P^2(\log n / \log N)}{n^{(2-2\mu)}}\right) = O(T^{1+(r-1-2\theta)(1/2-\mu)+\epsilon}) = O(T^{1-\epsilon}).$$

It remains now to treat the case $k=3$ and $\theta < \frac{1}{2}$. Again, by Cauchy's theorem,

$$J_4 = \frac{1}{i} \left(\int_{\mu+it_0}^{\mu+it_1} + \int_{1/2+it_0}^{\mu+it_0} - \int_{1/2+it_1}^{\mu+it_1} \right) \zeta^2(s) \chi^{r-1}(s) A_r(1-s, P)^2 ds + O\left(T^{\theta+\epsilon} \left(\int_T^{t_0} + \int_{t_1}^{2T} \right) |\zeta(\frac{1}{2}+it)|^2 dt\right)$$

where $t_0 \in (T+T^{1/3+\epsilon}, T+T^{1/2-\epsilon})$ and $t_1 \in (2T-T^{1/2-\epsilon}, 2T-T^{1/3+\epsilon})$ will be fixed later. Since the mean-square of the ζ -function is known with an error $O(T^{1/3+\epsilon})$ (see [13]) the error term in the above relation is

$$O(T^{\theta+\epsilon}(t_0 - T + 2T - t_1)^{1+\epsilon}) = O(T^{1-\epsilon}).$$

Also the integral on the μ -line is easily seen, as before, to be small.

It suffices to estimate the horizontal integrals. We choose t_0 so that

$$\begin{aligned} & \int_{1/2+it_0}^{\mu+it_0} \zeta^2(s) \chi^{r-1}(s) A_r(1-s, P)^2 ds \\ &= O\left(\frac{T^\epsilon}{T^{1/2}} \int_{T+T^{1/3+\epsilon}}^{T+T^{1/2-\epsilon}} \int_{1/2}^{\mu} |\zeta(\sigma+it)^2 \chi(\sigma+it)^{r-1} A_r(1-\sigma-it, P)^2| d\sigma dt\right) \\ &= O\left(\frac{1}{T^{1/2-\epsilon}} \int_{T+T^{1/3+\epsilon}}^{T+T^{1/2-\epsilon}} \int_{1/2}^{\mu} T^{2\theta\sigma+(r-1)(1/2-\sigma)} |\zeta(\sigma+it)|^2 d\sigma dt\right) \\ &= O\left(T^\epsilon \int_{1/2}^{\mu} T^{2\theta\sigma+(r-1)(1/2-\sigma)} d\sigma\right) = O(T^{1/2}). \end{aligned}$$

A similar bound holds for the integral on it_1 . This completes the proof.

To prove Corollaries 1.2 and 1.3, we use Theorem 1.1 with Lemma 2.1 to see that

$$I_k(T) \geq 4\Re \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} |\zeta(s)|^2 \zeta^r(s) A_r(1-s, P) ds - \frac{2}{i} \int_{1/2+iT}^{1/2+2iT} |\zeta(s) A_r(s, P)|^2 ds.$$

The bounds of Conrey and Ghosh, [4], were based on the inequality

$$I_k(T) \geq 2\Re \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} |\zeta(s)|^2 \zeta^r(s) A_r(1-s, P) ds - \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} |\zeta(s) A_r(s, P)|^2 ds.$$

Thus making the same choice of P which leads to their Corollaries 1 and 2 we obtain our improved Corollaries 1.2 and 1.3.

§4. *Proof of Theorem 1.6 and its Corollaries.* Let $S(t) = 1/\pi \arg \zeta(\frac{1}{2} + it)$ and $N(t)$ denote the number of zeros of $\zeta(s)$ with ordinates between 0 and t . Then

$$\frac{\zeta^r(\frac{1}{2} + it)}{\zeta^r(\frac{1}{2} - it)} = e^{2\pi i r S(t)} = e^{2\pi i r N(t)} \chi^r(\frac{1}{2} + it).$$

It follows that

$$|J_4| \leq \left| \sum_{2T \geq \gamma_l \geq T} e^{2\pi i r l} \int_{\gamma_l}^{\gamma_{l+1}} |\zeta(\frac{1}{2} + it)|^2 \chi^r(\frac{1}{2} + it) A_r(\frac{1}{2} - it, P)^2 dt \right|$$

where $0 < \gamma_1 \leq \gamma_2 \leq \dots$ are the ordinates of the zeros of the ζ -function in the upper half-plane.

Let

$$J_4(\gamma_l, \gamma_{l+1}) = \int_{\gamma_l}^{\gamma_{l+1}} |\zeta(\frac{1}{2} + it)|^2 \chi^r(\frac{1}{2} + it) A_r(\frac{1}{2} - it, P)^2 dt.$$

Let $\mu > \frac{1}{2}$ be some fixed constant and put

$$J_{41}(x, y) = \int_x^y \zeta(\mu + it) \zeta(1 - \mu - it) \chi^r(\mu + it) A(1 - \mu - it, P)^2 dt$$

and

$$J_{42}(x) = \int_{1/2}^{\mu} \zeta(\sigma + ix) \zeta(1 - \sigma - ix) \chi^r(\sigma + ix) A_r(1 - \sigma - ix, P)^2 d\sigma.$$

Using Cauchy's theorem it is easily seen that

$$iJ_4(\gamma_l, \gamma_{l+1}) = iJ_{41}(\gamma_l, \gamma_{l+1}) + J_{41}(\gamma_l) - J_{41}(\gamma_{l+1}).$$

Thus

$$\begin{aligned} |J_4| &\leq \left| \sum_{2T \geq \gamma_l \geq T} e^{2\pi i r l} J_4(\gamma_l, \gamma_{l+1}) \right| \\ &\leq \left| \sum_{2T \geq \gamma_l \geq T} e^{2\pi i r l} J_{41}(\gamma_l, \gamma_{l+1}) \right| + |e^{2\pi i r} - 1| \left| \sum_{2T \geq \gamma_l \geq T} e^{2\pi i r l} J_{42}(\gamma_l) \right| \\ &\leq \int_{\mu+iT}^{\mu+2iT} |\chi^r(s) A_r(1-s, P)^2 \zeta(s) \zeta(1-s) ds| + |e^{2\pi i r} - 1| \\ &\quad \times \int_{1/2}^{\mu} \sum_{2T \geq \gamma_l \geq T} |\zeta(\sigma + i\gamma_l) \zeta(1-\sigma - i\gamma_l) \chi^r(\sigma + i\gamma_l) A_r(1-\sigma - i\gamma_l, P)^2| d\sigma \\ &= L_1 + |e^{2\pi i r} - 1| L_2, \end{aligned}$$

say.

Clearly

$$\begin{aligned} L_1 &\ll T^{(1/2-\mu)(r-1)+\varepsilon} \int_T^{2T} |A_r(1-\mu-it, P)|^2 dt \\ &\ll T^{(1/2-\mu)(r-1)+\varepsilon} \times T \times T^{(2\mu-1)\theta} \ll T^{1-\varepsilon}, \end{aligned}$$

if $\theta \leq \frac{1}{2}(r-1)$. Next,

$$\begin{aligned} L_2 &\leq \int_{1/2}^{\mu} T^{(r-1)(1/2-\sigma)} \left(\sum_{2T \geq \gamma_l \geq T} |A_r(1-\sigma - i\gamma_l, P) \zeta(\sigma + i\gamma_l)|^2 \right) d\sigma \\ &= \int_{1/2}^{\mu} T^{(r-1)(1/2-\sigma)} X(\sigma) d\sigma, \end{aligned}$$

say. By the approximate functional equation (see [13])

$$\zeta(\sigma + it) = \sum_{n \leq \sqrt{t/2\pi}} \frac{1}{n^{\sigma+it}} + \chi(\sigma + it) \sum_{n \leq \sqrt{t/2\pi}} \frac{1}{n^{1-\sigma-it}} + O(t^{-\sigma/2}).$$

Hence

$$\begin{aligned} X(\sigma) &= \sum_{\gamma_l} |A_r(1-\sigma - i\gamma_l, P) \zeta(\sigma + i\gamma_l)|^2 \\ &= \sum_{\gamma_l} \left| A_r(1-\sigma - i\gamma_l, P) \left(\sum_{n \leq \sqrt{T/2\pi}} \left(\frac{1}{n^{\sigma+i\gamma_l}} + \frac{\chi(\sigma + i\gamma_l)}{n^{1-\sigma-i\gamma_l}} \right) \right) \right|^2 \\ &\leq 2(X_1(\sigma) + X_2(\sigma)), \end{aligned}$$

where

$$X_1(\sigma) = \sum_{\gamma_l} \left| A_r(1 - \sigma - i\gamma_l, P) \sum_{n \leq \sqrt{T/2\pi}} \frac{1}{n^{\sigma + i\gamma_l}} \right|^2$$

and

$$X_2(\sigma) = \sum_{\gamma_l} \left| A_r(1 - \sigma - i\gamma_l, P) \chi(\sigma + i\gamma_l) \sum_{n \leq \sqrt{T/2\pi}} \frac{1}{n^{1 - \sigma - i\gamma_l}} \right|^2.$$

For brevity, write

$$A_r(s, P) = \sum_{n \leq T^\theta} b_n n^{-s}$$

and

$$A_r(1 - \sigma - it, P) \sum_{n \leq \sqrt{T/2\pi}} n^{-\sigma + it} = \sum_{n \leq T^{\theta+1/2}/\sqrt{2\pi}} c_n(\sigma) n^{it}$$

where

$$c_n(\sigma) = \sum_{\substack{uv=n \\ u \leq T^\theta, v \leq \sqrt{T/2\pi}}} b_u u^{\sigma-1} v^{-\sigma} = n^{-\sigma} \sum_{\substack{u|n \\ n/\sqrt{2\pi}/T \leq u \leq T^\theta}} b_u u^{2\sigma-1}.$$

Thus,

$$\begin{aligned} \frac{2\pi X_1(\sigma)}{T \log T} &= \frac{2\pi}{T \log T} \sum_{m, n \leq T^{\theta+1/2}/\sqrt{2\pi}} c_m(\sigma) c_n(\sigma) \sum_{T \leq \gamma \leq 2T} \left(\frac{m}{n}\right)^{i\gamma} \\ &\leq (1 + o(1)) \sum_{n \leq T^{\theta+1/2}/\sqrt{2\pi}} c_n(\sigma)^2 + 2 \left| \sum_{m > n} c_m(\sigma) c_n(\sigma) \sum_{T \leq \gamma \leq 2T} \left(\frac{m}{n}\right)^{i\gamma} \right|. \end{aligned}$$

From Lemma 2.5 and the Riemann Hypothesis we see that

$$\begin{aligned} \sum_{T \leq \gamma \leq 2T} \left(\frac{m}{n}\right)^{i\gamma} &= -\frac{T}{2\pi} \sqrt{\frac{n}{m}} \Lambda\left(\frac{m}{n}\right) \\ &\quad + O\left(\sqrt{\frac{m}{n}} \log^2 T + \frac{\sqrt{m} \log T}{\sqrt{n} \langle m/n \rangle} + \frac{\sqrt{n} \log T}{\sqrt{m} \log(m/n)}\right). \end{aligned}$$

Observe that

$$\begin{aligned} &\int_{1/2}^{\mu} c_m(\sigma) c_n(\sigma) T^{(r-1)(1/2-\sigma)} d\sigma \\ &= \sum_{\substack{u|n \\ n\sqrt{2\pi}/T \leq u \leq T^\theta}} \sum_{\substack{v|m \\ m\sqrt{2\pi}/T \leq v \leq T^\theta}} \frac{b_u b_v}{\sqrt{nm}} \int_{1/2}^{\mu} \left(\frac{T^{(r-1)} nm}{(uv)^2}\right)^{1/2-\sigma} d\sigma \\ &= \sum_u \sum_v \frac{(1 + o(1)) b_u b_v}{\sqrt{nm}((r-1) \log T + \log(nm) - 2 \log(uv))}, \end{aligned}$$

since $\theta < \frac{1}{2}(r-1)$. In particular

$$\int_{1/2}^{\mu} c_m(\sigma)c_n(\sigma)T^{(r-1)(1/2-\sigma)}d\sigma = O\left(\frac{d_k(n)d_k(m)}{\sqrt{nm}}\right).$$

From this estimate and our earlier estimate for $\sum_{T \leq \gamma \leq 2T} (m/n)^{\gamma}$, it easily follows that

$$\begin{aligned} & \int_{1/2}^{\mu} T^{(r-1)(1/2-\sigma)}X_1(\sigma)d\sigma \\ & \leq \frac{T}{2\pi} \int_{1/2}^{\mu} T^{(r-1)(1/2-\sigma)} \left(\log T \sum_{n \leq T^{\theta+1/2}/\sqrt{2\pi}} c_n(\sigma)^2 \right. \\ & \qquad \qquad \qquad \left. + 2 \sum_{T^{\theta+1/2}/\sqrt{2\pi} > m > n} c_m(\sigma)c_n(\sigma) \sqrt{\frac{n}{m}} \Lambda\left(\frac{m}{n}\right) \right) d\sigma. \end{aligned}$$

Now

$$\begin{aligned} & \int_{1/2}^{\mu} T^{(r-1)(1/2-\sigma)} \sum_{n \leq T^{\theta+1/2}/\sqrt{2\pi}} c_n(\sigma)^2 d\sigma \\ & \leq \sum_{\substack{u,v \leq T^{\theta} \\ n \leq \sqrt{Tuv/2\pi}, [u,v]|n}} \frac{(1+o(1))b_u b_v}{(r-1) \log T + 2 \log n - 2 \log(uv)} \\ & \leq \sum_{u,v \leq T^{\theta}} \frac{b_u b_v(u,v)}{2uv} \log \left(\frac{r \log T - \log(uv)}{(r-1) \log T - 2 \log(u,v)} \right). \end{aligned}$$

Next,

$$\begin{aligned} & \int_{1/2}^{\mu} T^{(r-1)(1/2-\sigma)} \sum_{n \leq m} c_n(\sigma)c_m(\sigma) \left(\frac{n}{m}\right)^{1/2} \Lambda\left(\frac{m}{n}\right) d\sigma \\ & = \sum_p \frac{\log p}{p} \sum_{u,v \leq T^{\theta}} b_u b_v \sum_{\substack{u|n, v|np \\ n \leq \sqrt{T} \min(u,v/p)^{1/2}}} \int_{1/2}^{\mu} T^{(r-1)(1/2-\sigma)} \frac{(uv)^{2\sigma-1}}{(n^2 p)^{\sigma}} d\sigma \\ & = \sum_p \frac{\log p}{p} \sum_{u,v \leq T^{\theta}} b_u b_v \sum_{\substack{u|n, v|np \\ n \leq \sqrt{T} \min(u,v/p)^{1/2}}} \int_{1/2}^{\mu} T^{(r-1)(1/2-\sigma)} \frac{(uv)^{2\sigma-1}}{(n^2 p)^{\sigma}} d\sigma \\ & = \sum_p \frac{\log p}{p} \sum_{u,v \leq T^{\theta}} b_u b_v \sum_n \frac{(1+o(1))}{n((r-1) \log T + \log(n^2 p) - 2 \log(uv))} \\ & = Y_1 + Y_2 \end{aligned}$$

where Y_1 corresponds to those terms for which $p \nmid v$ and Y_2 to those for which $p \mid v$.

Putting $h = h(u, v, p) = \min(u, v/p) \leq \sqrt{uv/p}$, we see that,

$$\begin{aligned} Y_1 &\leq \sum_p \frac{\log p}{p} \sum_{u,v \leq T^\theta} b_u b_v \sum_{\substack{[u,v] \mid n \\ n \leq h\sqrt{T}}} \frac{1 + o(1)}{n((r-1) \log T + \log(n^2 p) - 2 \log(uv))} \\ &\leq \sum_p \frac{\log p}{p} \sum_{u,v} \frac{b_u b_v}{[u,v]} \sum_{n \leq \sqrt{Th/[u,v]}} \frac{1 + o(1)}{n((r-1) \log T - 2 \log(u, v) + \log(pn^2))} \\ &\leq (1 + o(1)) \sum_{p \leq T^\theta} \frac{\log p}{p} \sum_{u,v} \frac{b_u b_v}{2[u,v]} \log \left(\frac{r \log T + \log(h^2 p) - 2 \log(uv)}{(r-1) \log T - 2 \log(u, v) + \log p} \right) \\ &\leq (\theta + o(1)) \log T \sum_{u,v} \frac{b_u b_v}{2[u,v]} \log \left(\frac{r \log T - \log(uv)}{(r-1) \log T - 2 \log(u, v)} \right). \end{aligned}$$

Further

$$\begin{aligned} Y_2 &\leq \sum_p \frac{\log p}{p} \sum_{u \leq T^\theta, v \leq T^\theta/p} \sum_{\substack{[u,v] \mid n \\ n \leq \min(u,v)\sqrt{T}}} \frac{(1 + o(1))b_u b_{vp}}{n((r-1) \log T + \log(n^2 p) - 2 \log(uvp))} \\ &\leq \sum_p \frac{\log p}{p} \sum_{u,v} \frac{b_u b_{pv}}{[u,v]} \sum_{n \leq v\sqrt{T/[u,v]}} \frac{1 + o(1)}{n((r-1) \log T + \log(n^2) - 2 \log(u, v) - \log p)} \\ &\leq \sum_p \frac{\log p}{p} \sum_{u,v} \frac{b_u b_{pv}}{2[u,v]} \log \left(\frac{r \log T - 2 \log v - \log p}{(r-1) \log T - 2 \log(u, v) - \log p} \right) \\ &\leq r \sum_{u,v} \frac{b_u b_v}{2[u,v]} (\theta \log T - \log v) \log \left(\frac{(r-\theta) \log T - \log v}{(r-1-\theta) \log T - \log(u, v)} \right). \end{aligned}$$

Piecing these results together and using $\theta < \min(\frac{1}{2}, \frac{1}{2}(r-1))$ we see that

$$\begin{aligned} \int_{1/2}^\mu T^{(r-1)(1/2-\sigma)} X_1(\sigma) d\sigma &\leq \frac{T \log T}{2\pi} \sum_{u,v} \frac{b_u b_v}{[u,v]} \left(\log \left(\frac{r \log T - \log(uv)}{(r-1) \log T - 2 \log(u, v)} \right) \right. \\ &\quad \left. + r \left(\theta - \frac{\log v}{\log T} \right) \log \left(\frac{(r-\theta) \log T - \log v}{(r-1-\theta) \log T - \log(u, v)} \right) \right). \end{aligned}$$

Similarly, we see that,

$$\begin{aligned} \int_{1/2}^\mu T^{(r-1)(1/2-\sigma)} X_2(\sigma) d\sigma &\leq \frac{T \log T}{2\pi} \sum_{u,v} \frac{b_u b_v}{[u,v]} \left(\log \left(\frac{(r+1) \log T - 2 \log [uv]}{r \log T - \log(u, v)} \right) \right. \\ &\quad \left. + r \left(\theta - \frac{\log v}{\log T} \right) \log \left(\frac{(r-\theta) \log T + 2 \log(u, v) - \log u}{(r-\theta) \log T - \log v} \right) \right). \end{aligned}$$

Thus

$$\begin{aligned} \frac{\pi L_2}{T \log T} &\leq \frac{2\pi}{T \log T} \int_{1/2}^{\mu} T^{(r-1)(1/2-\sigma)} (X_1(\sigma) + X_2(\sigma)) d\sigma \\ &\leq \sum_{u,v} \frac{b_u b_v}{[u, v]} \left(\log \left(\frac{(r+1) \log T - 2 \log [u, v]}{(r-1) \log T - 2 \log (u, v)} \right) \right. \\ &\quad \left. + r \left(\theta - \frac{\log v}{\log T} \right) \log \left(\frac{(r-\theta) \log T + 2 \log (u, v) - \log u}{(r-1-\theta) \log T - \log (u, v)} \right) \right) \\ &\leq \sum_{u,v} \frac{b_u b_v}{[u, v]} H(u, v, (u, v)). \end{aligned}$$

This proves Theorem 1.6. To obtain Corollary 1.7, note that if $u, v \leq T^\theta$, then

$$H(u, v, (u, v)) \leq \log \left(\frac{r+1-2\theta}{r-1-2\theta} \right) + r \left(\theta - \frac{\log v}{\log T} \right) \log \left(\frac{r}{r-1-2\theta} \right).$$

Hence

$$\begin{aligned} \frac{\pi L_2}{T \log T} &\leq \sum_{u,v \leq T^\theta} \frac{b_u b_v}{[u, v]} \left(\log \left(\frac{r+1-2\theta}{r-1-2\theta} \right) + r \left(\theta - \frac{\log v}{\log T} \right) \log \left(\frac{r}{r-1-2\theta} \right) \right) \\ &= \sum_{w \leq T^\theta} \sum_{\substack{u,v \leq T^\theta/w \\ (u,v)=1}} \frac{b_{uw} b_{vw}}{uvw} \\ &\quad \times \left(\log \left(\frac{r+1-2\theta}{r-1-2\theta} \right) + r \left(\theta - \frac{\log wv}{\log T} \right) \log \left(\frac{r}{r-1-2\theta} \right) \right) \\ &= \sum_{w \leq T^\theta} \sum_{u,v \leq T^\theta/w} \left(\sum_{d|u, d|v} \mu(d) \right) \frac{b_{uw} b_{vw}}{uvw} \\ &\quad \times \left(\log \left(\frac{r+1-2\theta}{r-1-2\theta} \right) + r \left(\theta - \frac{\log wv}{\log T} \right) \log \left(\frac{r}{r-1-2\theta} \right) \right) \\ &= \sum_{w \leq T^\theta} \sum_{d \leq T^\theta/w} \frac{\mu(d)}{d^2 w} \sum_{u,v \leq T^\theta/wd} \frac{b_{udw} b_{vdw}}{uv} \\ &\quad \times \left(\log \left(\frac{r+1-2\theta}{r-1-2\theta} \right) + r \left(\theta - \frac{\log wdv}{\log T} \right) \log \left(\frac{r}{r-1-2\theta} \right) \right). \end{aligned}$$

Using partial summation in conjunction with Lemma 2.6, we see

$$\begin{aligned} & \sum_{u,v \leq T^\theta/wd} \frac{b_{udw} b_{vdw}}{uv} \left(\log \left(\frac{r+1-2\theta}{r-1-2\theta} \right) + r \left(\theta - \frac{\log wdv}{\log T} \right) \log \left(\frac{r}{r-1-2\theta} \right) \right) \\ & \leq \frac{D_r(wd, 1)^2}{\Gamma^2(r+1)} \left(\int_1^{T^\theta/wd} \frac{r(\log y)^{r-1}}{y} P \left(\frac{\log(ydw)}{\theta \log T} \right) dy \right) \\ & \quad \times \left(\int_1^{T^\theta/wd} \frac{r(\log y)^{r-1}}{y} Q \left(\frac{\log(ydw)}{\theta \log T} \right) dy \right). \end{aligned}$$

By an obvious change of variables this is

$$\begin{aligned} & = \frac{D_r(wd, 1)^2}{\Gamma^2(r+1)} (\theta \log T)^{2r} \left(\int_{\log(dw)/(\theta \log T)}^1 r \left(z - \frac{\log(dw)}{\theta \log T} \right)^{r-1} P(z) dz \right) \\ & \quad \times \left(\int_{\log(dw)/(\theta \log T)}^1 r \left(z - \frac{\log(dw)}{\theta \log T} \right)^{r-1} Q(z) dz \right) \\ & = G \left(\frac{\log(dw)}{\theta \log T} \right) \frac{D_r(wd, 1)^2}{\Gamma^2(r+1)} (\theta \log T)^{2r}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{\pi L_2}{T \log T} & \leq \frac{(\theta \log T)^{2r}}{\Gamma^2(r+1)} \sum_{w \leq T^\theta} \sum_{d \leq T^\theta/w} \frac{\mu(d) D_r(wd, 1)^2}{d^2 w} G \left(\frac{\log(dw)}{\theta \log T} \right) \\ & = \frac{(\theta \log T)^{2r}}{\Gamma^2(r+1)} \sum_{n \leq T^\theta} \frac{D_r(n, 1)^2 \varphi(n)}{n^2} G \left(\frac{\log n}{\theta \log T} \right). \end{aligned}$$

Corollary 1.7 follows immediately by partial summation and Lemma 2.6. Corollary 1.8 is a simple consequence of Corollary 1.7 and Corollary 2 of Conrey and Ghosh, [4].

References

1. R. Balasubramanian, J. B. Conrey and D. R. Heath-Brown. Asymptotic mean square of the Riemann Zeta-function and a Dirichlet polynomial. *J. Reine Angew. Math.*, 357 (1985), 161-181.
2. R. Balasubramanian and K. Ramachandra. Proof of some conjectures on the mean value of Titchmarsh series, I. *Hardy Ramanujan J.*, 13 (1990), 1-20.
3. J. B. Conrey and A. Ghosh. Mean Values of the Riemann Zeta-function. *Mathematika*, 31 (1984), 159-164.
4. J. B. Conrey and A. Ghosh. Mean Values of the Riemann Zeta-function, III. *Proc. of the Amalfi Conference on Analytic Number Theory* (1992), 35-59.
5. S. M. Gonek. A formula of Landau and Mean Values of $\zeta(s)$. In *Topics in Analytic Number Theory* (edited by S. W. Graham and J. D. Vaaler) 92-97.
6. D. R. Heath-Brown. Fractional Moments of the Riemann Zeta-function. *J. Lond. Math. Soc.* (2), 24 (1981), 65-78.

7. D. R. Heath-Brown. Fractional Moments of the Riemann Zeta-function, II. *Quart. J. of Math. Oxford* (2), 44 (1993), 185–197.
8. A. Ivic. Mean Values of the Riemann Zeta function. *T.I.F.R. Lectures in Mathematics and Physics*, 82 (1991).
9. K. Ramachandra. Application of a Theorem of Montgomery and Vaughan to the Zeta-function. *J. Lond. Math. Soc.* (2), 10 (1975), 482–486.
10. K. Ramachandra. Some Remarks on the mean value of the Riemann Zeta-function and other Dirichlet series, II. *Hardy Ramanujan J.* 3 (1980), 1–24.
11. K. Ramachandra. Some Remarks on the mean value of the Riemann Zeta-function and other Dirichlet series, III. *Ann. Acad. Sci. Fenn. Ser. A. I.* 5 (1980), 145–158.
12. A. Selberg. Note on a paper of L. G. Sathe. *J. of the Indian Math. Soc. B.* 18 (1954), 83–87.
13. E. C. Titchmarsh. The theory of the Riemann Zeta-function. Clarendon Press (second edition), Oxford (1986).

K. Soundararajan,
Department of Mathematics,
University of Michigan,
Ann Arbor, MI 48109,
U.S.A.

11M06: *NUMBER THEORY; Zeta and L-functions, analytic theory; $\zeta(s)$.*

Received on the 11th of April, 1994.