Stability of Nonlinear Systems with Parameter Uncertainty

by

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DEDICATION

To my family.

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LIST OF ABBREVIATIONS

IC initial condition

SEP stable equilibrium point

recovery value recovery value of the parameters

recovery boundary recovery boundary in parameter space

RoA region of attraction

UEP unstable equilibrium point

LIST OF SYMBOLS

Ω nonwandering set

M smooth manifold

C^1 continuously differentiable

C^∞ smooth

$\mathbb{R}^n$ Euclidean space

V vector field

x_e equilibrium point

$W^s(\gamma)$ stable manifold of γ

$W^u(\gamma)$ unstable manifold of γ

$\hat{A}$ differentiable submanifold

$\hat{B}$ differentiable submanifold

$T_x(\hat{A})$ tangent space of $\hat{A}$ at x

$\hat{\tau}$ Poincaré first return map

$\overline{E}$ topological closure of E

∂E topological boundary of E

$\mathbf{int}\ E$ topological interior of E

$\tilde{f}|_E$ restriction of the function $\tilde{f}$ to E

D_ϵ ϵ-neighborhood of D

$\mathcal{C}(K)$ the set of nonempty closed subsets of K

d_h Hausdorff distance

d_S set distance

ϕ flow

ϕ_t time-t flow

$\tilde{M}$ smooth manifold

d_{C^1} C^1 distance

$d\tilde{f}_x$ differential of $\tilde{f}$ at x

V_p vector field corresponding to parameter value p

S Poincaré cross section

X critical element

$W^s_{\mathbf{loc}}(X)$ local stable manifold of X

$W^u_{\mathbf{loc}}(X)$ local unstable manifold of X

$W^s(X)$ stable manifold of X

$W^u(X)$ unstable manifold of X

$\Omega(V)$ nonwandering set under V

$\omega(y, V)$ ω-limit set of y under V

J connected smooth manifold which represents parameter values

X_p critical element corresponding to parameter value p

X_Q $\bigcup_{p \in Q} X_p$

$W^s(X_Q)$ $\sqcup_{p \in Q} W^s(X_p)$

$W^u(X_Q)$ $\sqcup_{p \in Q} W^u(X_p)$

$X^s(p)$ perturbation of $X^s(p_0)$ corresponding to parameter value p

$X^i(p)$ perturbation of $X^i(p_0)$ corresponding to parameter value p

$W^s(X^s(p))$ stable manifold of $X^s(p)$

$W^s(X^i(p))$ stable manifold of $X^i(p)$

$W^u(X^i(p))$ unstable manifold of $X^i(p)$

X^s_J $\sqcup_{p\in J} X^s(p)$

$W^s(X^s_J)$ $\sqcup_{p\in J} W^s(X^s(p))$

X^i_J $\sqcup_{p\in J} X^i(p)$

$W^s(X^i_J)$ $\sqcup_{p\in J} W^s(X^i(p))$

$W^u(X^i_J)$ $\sqcup_{p\in J} W^u(X^i(p))$

S^s $W^s_{\text{loc}}(X^s(p_0))$

S^i $X^i(p_0)$

S^i_s $W^s_{\text{loc}}(X^i(p_0))$

S^i_u $W^u_{\text{loc}}(X^i(p_0))$

F^i maps p onto $X^i(p)$

F^i_s maps p onto $W^s_{\text{loc}}(X^i(p))$

F^i_u maps p onto $W^u_{\text{loc}}(X^i(p))$

F^s maps p onto $W^s_{\text{loc}}(X^s(p))$

s event or switching function

σ function that gives the switching state status

s^j_p event or switching function j corresponding to parameter value p

Z^p_j switching surface j corresponding to parameter value p

Z^p $\bigcup_{j=1}^m Z^p_j$

Z_j $\sqcup_{p\in J} Z^p_j$

Z^J $\sqcup_{p\in J} Z^p$

C^J $(M \times J)\backslash Z^J$

$S^i_s(p)$ $W^s_{\text{loc}}(X^i(p))$

$S^j_u(p)$ $W^u_{\text{loc}}(X^j(p))$

$\hat{y}_p$ initial condition corresponding to parameter value p

R set of recovery parameter values

∂R recovery parameter boundary

C set of critical parameter values

ABSTRACT

A novel approach is introduced to assess stability of nonlinear systems in the presence of parameter uncertainty. The idea is to consider the deterministic dynamics of the system as a function of parameter values, where the parameter-dependent initial condition may for example be the output of a particular finite-time disturbance. The goal is to numerically determine the boundary in parameter space, referred to as the *recovery boundary*, between parameter values which lead to recovery and those which lead to a failure to recover to an initial stable equilibrium point. *Critical parameter values*, which are defined to be those parameter values whose corresponding initial conditions lie on the boundary of the region of attraction of their corresponding stable equilibrium points, have the potential to provide an explicit connection between the recovery boundary in parameter space and the region of attraction boundary in state space that can be exploited for algorithm design.

However, examples are provided to illustrate that the recovery boundary may not contain critical parameter values when the boundary of the region of attraction of the stable equilibrium point varies discontinuously with parameter. Fortunately, it is shown that, for a large class of vector fields possessing stable equilibrium points, the boundaries of the regions of attraction of these equilibrium points vary continuously with respect to small variations in parameter values. This region of attraction boundary continuity ensures that the recovery boundary consists entirely of critical parameter values and that the nearest critical parameter value to any non-critical parameter value lies on the recovery boundary.

Two classes of theoretically motivated algorithms are developed to compute critical parameter values by exploiting the structure and behavior of the region of attraction boundary under parameter variation. The system trajectory corresponding to a critical parameter value converges to an invariant set and, therefore, spends an infinite amount of time in any neighborhood of that invariant set. A first class of algorithms proceed by varying parameter values so as to maximize the time in a neighborhood of the invariant set. Under reasonable assumptions, the system trajectory corresponding to a critical parameter value becomes infinitely sensitive to small changes in parameter value. A second class of algorithms proceed by varying parameter values so as to maximize the trajectory sensitivities to parameters.

Theoretical motivation is provided for both classes of algorithms, and shows that under reasonable assumptions they will drive parameter values to their critical values. Both of these approaches transform the abstract problem of finding critical parameter values into concrete numerical optimization problems. Based on these approaches, algorithms are developed to find the closest parameter value on the recovery boundary in the case of one-dimensional parameter space, trace the recovery boundary in two-dimensional parameter space, and find the nearest point on the recovery boundary in parameter space of arbitrary dimension.

The algorithms are applied to assess fault vulnerability in power systems. Results from the test cases of a modified IEEE 37-bus feeder and a modified IEEE 39-bus system illustrate the algorithm performance. The emphases in these test cases are, respectively, to explore the onset of induction motor stalling during Fault Induced Delayed Voltage Recovery, and to analyse stability of a system of synchronous machines under high levels of load uncertainty.

CHAPTER I

Introduction

Engineered systems experience disturbances which have the potential to disrupt desired operation, such as a lightning strike on a particular transmission line in a power system. From a systems perspective, the disturbance can be thought of as a parameter dependent initial condition (IC) to the post-disturbance dynamical system (which itself is parameter dependent). The desired operating point is typically modeled as a stable equilibrium point (SEP) of the post-disturbance system. Ability of the system to recover from a particular finite-time disturbance to the desired operating point depends on the system parameters. Define a recovery value of the parameters (recovery value) to be a value for which the system is able to recover from the disturbance to the desired SEP. It is an important and challenging problem to determine the set of recovery values for a particular disturbance and parameter space. Solving this problem is of value for many applications, for example assessing fault vulnerability in power systems (a fault is a short circuit that causes instability in power systems and in some instances may lead to blackout conditions).

As parameter values are generally time-varying and uncertain in practice, knowledge of the recovery values provides system operators and engineers with a quantitative measure of the margins for safe operation. In order to determine the set of recovery values, the approach taken here focuses on determining their (topological) boundary in parameter space, which we term the recovery boundary in parameter space (recovery boundary). Variants of this problem have existed for a long time, but it remains of major importance to many application areas today. For example, it has existed since the first large scale power systems were developed about a century ago, but is as important as ever in power systems as increased penetration of renewable energy generation increases parameter uncertainty, and increased demand coupled with aging infrastructure cause the grid to operate closer to its limits. We note that the discussion here applies equally well to systems with uncertain, parameter dependent initial conditions; that the initial condition is the output of another dynamical system (such as a finite-time disturbance) is not required.

In power systems, current industry practice involves the construction of inner approximations of the recovery boundary known as nomograms [MWTP97]. These nomograms are typically piecewise linear, consisting of a polytope defined by bounding hyperplanes. Nomograms are constructed by a variety of heuristic methods. One such technique involves initially fixing the values of all parameters other than one, varying that last parameter value and running simulations until the recovery boundary is encountered, changing the other parameter values, and repeating. Another method is to start from an initial set of parameter values, choose a "stress" direction in parameter space, vary the distance traveled along this stress direction and run simulations until the recovery boundary is encountered, change the stress direction, and repeat. Every simulation is run for a time interval determined to be sufficient to assess whether or not the system has recovered. The recovery boundary is encountered numerically when parameter values are sampled which lead to recovery and failure to recover (ie. parameter values both within and outside of the recovery boundary) that are sufficiently close in parameter space.

Each of these approaches requires a large number of computationally intensive time domain simulations and, in particular, the number of required simulations grows rapidly with the dimension of parameter space. Hence, to avoid computational intractability, in practice power system operators typically limit parameter space to two or three important parameters of interest - often selected based on operator experience and intuition - and perform the construction of nomograms only during offline planning studies. Due to the low dimensionsionality, the nomograms can be visualized as a piecewise linear curve in the case of two parameter dimensions. In three dimensional parameter space, they can be viewed as a contour plot consisting of a piecewise linear curve for each value of a third parameter. Recent work has extended the techniques to parameter space with higher dimension than three, such as in [MDL+12], but nomogram construction remains a method for low dimensional parameter space. Overall, major limitations of current industry practice include:

1. The small number of parameters of interest are chosen based on operator intuition and experience, so many parameters which could be critical to system recovery may be overlooked.

2. The methods are inherently approximate and tend to be overly conservative or just inaccurate (especially as parameter values vary during real-time operation).

3. Nomogram construction is limited to offline planning studies, and they are not updated during real-time operation.

These drawbacks of standard industry practice have long motivated a search for alternative approaches to accurately and efficiently determine or approximate the recovery boundary.

Historical work focused on parameter independent post-disturbance dynamics, and the main parameter of interest was the duration of the finite-time disturbance. In this simpler setting, the post-disturbance system has a unique (parameter independent) region of attraction (RoA) of its unique SEP. Assuming (at least local) continuity of the IC with respect to disturbance duration, a disturbance duration value lies on the recovery boundary precisely when its corresponding IC lies on the boundary of the RoA of the post-disturbance SEP. More generally, we coin the term *critical parameter value* to refer to a parameter value whose corresponding IC lies on the boundary of the RoA of the corresponding SEP.

Classical algorithms relied primarily on the use of energy functions to estimate the critical disturbance duration, known in the power systems literature as critical fault clearing time. These methods were initially introduced by [Mag47, Ayl58, Gle66, EAN66] and subsequently extended in several directions. The common steps behind the algorithms were as follows:

1. Identify a real-valued function on state space, called the energy function, for the system of interest such that the SEP is a local minimum of the energy function, near the SEP this function is positive definite, and near the SEP its time derivative along system trajectories is negative definite.

2. For a particular disturbance, identify a critical energy such that if the energy of the IC is not larger than this value, then the IC lies inside the RoA of the SEP.

3. Numerically integrate the disturbance dynamics (no longer enforcing the finite-time limit) until the critical energy is encountered. The time at which this occurs is used to approximate the critical disturbance duration.

The intuition behind this technique is that since the SEP is a local minimum of the energy function, it is positive definite near the SEP, and its time derivative along system trajectories is negative definite near the SEP, there exists a sublevel set of the energy function in which all trajectories converge to the SEP in forwards time. Hence, this sublevel set is contained in the RoA of the SEP. Therefore, the disturbance duration is guaranteed to be at a recovery value if its corresponding IC lies in this sublevel set or, equivalently, if the energy of its corresponding IC lies beneath some critical level. Numerically integrating the disturbance dynamics until this critical energy is encountered therefore finds the disturbance time which puts its corresponding IC at the boundary of this sublevel set, thereby providing a conservative estimate of the critical disturbance duration.

The main differences in the various energy function approaches consist primarily of the choice of energy function and of the method to determine the critical energy. Many modified energy functions were proposed in early attempts to apply the methods to more accurate power systems models [BH81, BHD86, NM84, TAV85], but energy functions for more realistic, industry standard power system models have yet to be developed. To determine the critical energy, the initial focus was on the so-called closest unstable equilibrium point (UEP) method. The idea was to set the critical energy to be the minimum of the energy function over the boundary of the RoA of the SEP. Then, whenever the IC has energy less than this critical energy, it must lie inside the RoA. There was an implicit assumption in the closest UEP approach that every trajectory in the boundary of the RoA would converge to an equilibrium point in forwards time (it should be noted that this assumption is not true in general for more realistic power system models).

Define the *stable manifold* (resp. *unstable manifold*) of an equilibrium point to be the set of initial conditions that converge to it in forwards (resp. backwards) time. Then the assumption can be restated as: the RoA boundary is equal to the union of the stable manifolds of the equilibria it contains. As any energy function decreases in time along system trajectories, it follows that it achieves a minimum over the stable manifold of an equilibrium point at that equilibrium point. Therefore, the minimum of the energy function over the RoA boundary would be equal to the minimum of the energy function over the equilibria in the RoA boundary. Hence, methods were devised to determine the equilibrium point on the RoA boundary of minimum energy [TS72, PEA75], called the closest UEP.

It was quickly realized that the closest UEP method for determining critical energy yielded highly conservative estimates. One alternative approach, known as the potential energy boundary surface (PEBS) method [KOH78, APV79], was developed to estimate the critical energy by estimating the energy at the point when the disturbance system exits the post-disturbance RoA boundary (this exit point is difficult to estimate as well). If the IC has energy less than this for the given disturbance, then the system cannot escape the post-disturbance RoA. The PEBS technique estimated the exit point energy by first estimating the exit point energy of the simpler system given by the gradient of the potential energy (which was typically found by integrating the disturbance trajectory until a maximum in potential energy was encountered).

Another approach, which came to be known as the controlling UEP method, was initially alluded to by [KOH78]. The key idea is that the trajectory of the disturbance system must cross the post-disturbance RoA boundary. So, by the decomposition above, it must cross the stable manifold of some UEP, which is called the controlling UEP. The controlling UEP method sets the critical energy to be the energy at the controlling UEP. So long as the IC has energy less than at the controlling UEP, the disturbance trajectory could not have crossed the stable manifold of the controlling UEP (since this stable manifold has higher energy than the controlling UEP). So, the system could not have escaped the post-disturbance RoA, which implies that the IC lies inside this RoA.

Several methods for identifying the controlling UEP were developed, including the mode of disturbance (MOD) method [FVO84], the exit point method (also known as the BCU method) [CWV88], and the shadowing method [TVK96], which ultimately rely on using numerical integration of the system dynamics to attempt to find a point near the controlling UEP which can be used as an initial guess to solve for the actual controlling UEP using Newton's method. Power system models contain many equilibria, and these methods are not guaranteed to converge to the single desired controlling UEP among the large number of possible equilibria.

Limitations for practical application of these energy methods were identified early on [BPRP+84]. These include:

1. Lack of suitable energy functions for more realistic generator, load, and network power system models.

2. Inherent conservativeness of the energy function approach.

3. Inability to handle hybrid dynamics including switching behavior and controller limits that are characteristic of modern power systems.

4. Methods are only able to determine the critical clearing time (as opposed to other critical parameter values), which does not adequately measure system stability.

To this list, we add:

5. The energy function methods are not able to satisfactorily handle the case when the exit point of the disturbance trajectory converges to an invariant set other than an equilibrium point.

6. Less conservative energy function methods require knowledge of the controlling UEP, which is challenging to find even for relatively small power system models.

Despite decades of research since their initial introduction in the 1960's, these fundamental challenges to energy function methods have yet to be overcome. However, the development of the energy function methods stimulated research into the topological structure of the RoA boundary of nonlinear systems, and its application to estimation of critical disturbance time. The key papers in this area were [CHW88, ZHZL88], which showed that for a large class of continuously differentiable (C^1) vector fields possessing a SEP, the boundary of its RoA is equal to the union of the stable manifolds of the equilibrium points and periodic orbits it contains. These results were of limited practical value because the assumptions under whch they were proved include that the RoA boundary is contained in the union of stable manifolds. Establishing that the RoA boundary is not just contained in but is actually equal to this union has not provided additional motivation or intuition for the development of algorithms for computing critical disturbance times. The foundation for this theoretical work came from developments in the theory of hyperbolic dynamical systems.

One such area involved the study of *generic* properties of differentiable dynamical systems. These were properties or behaviors that one might expect of a typical vector field, such as if a vector field could be chosen at random, but could still exclude certain pathological cases. The notion of generic in a topological space is similar to the concept of probability one in a probability space in that both are meant to capture the idea of being true almost always. Before discussing some generic properties, a few definitions are required. An equilibrium point (resp. periodic orbit) is *hyperbolic* if the linearization of the vector field at any point in it possesses exactly zero (resp. one) eigenvalues with zero real part. Two differentiable manifolds $\hat{A}$ and $\hat{B}$ inside some smooth manifold (M) are *transverse* if at every point x in their intersection, the tangent spaces to $\hat{A}$ and $\hat{B}$ at x together span the tangent space of M at x. A point x in a differentiable manifold M is said to be nonwandering for a vector field, which possesses an associated flow, if for every open neighborhood U of x and every $T > 0$, there exists $t > T$ such that the flow of U forward or backward by time t has nonempty intersection with U. Let nonwandering set (Ω) be the set of all nonwandering points in M. Then Ω includes all the equilibrium points, periodic orbits, and nontrivial recurrence (including chaotic behavior) of the flow. Furthermore, every trajectory that does not diverge to infinity must converge to a subset of Ω.

The study of generic properties made major progress when it was shown that for a generic vector field, all its equilibrium points and periodic orbits are hyperbolic, and their stable and unstable manifolds are pairwise transverse [Sma62, Kup63, Pei66]. Around the same time, a class of vector fields, which came to be known as Morse-Smale vector fields, was introduced [Sma60] which satisfied the following assumptions:

1. The nonwandering set consists of a finite union of equilibrium points and periodic orbits.

2. All equilibrium points and periodic orbits are hyperbolic.

3. The stable and unstable manifolds of the equilibrium points and periodic orbits are pairwise transverse.

It was initially conjectured that Morse-Smale vector fields could be generic [Sma60], although the same author later showed that this is not the case. It was also conjectured [Sma60] that a C^1 vector field was structurally stable, meaning informally that its orbit structure would be preserved under perturbations, if and only if it was a Morse-Smale vector field. Soon afterwards it was shown that Morse-Smale vector fields are open in the topological space of C^1 vector fields [Pal69], that they are structurally stable [PS68], and that there exist structurally stable C^1 vector fields which are not Morse-Smale [Sma63].

In [CHW88, ZHZL88], the authors make assumptions that are slightly more general than that the vector field is Morse-Smale along the RoA boundary of the SEP in order to arrive at their decomposition of the RoA boundary into a union of stable manifolds. However, these more general assumptions lead them to require a lemma [CHW88, Lemma 3-5], which is disproven via a counterexample in Section 2.1, to complete the proof of their main theorem. Therefore, to avoid the use of this flawed lemma, we will assume here that the vector field is Morse-Smale along the RoA boundary. More generally, the techniques used by [CHW88, ZHZL88] in their proofs were inherited from the study of Morse-Smale systems, especially [Pal69]. Since that initial work in 1988, there has not been significant progress towards obtaining new results regarding the structure and behavior of the RoA boundary.

Chiang's work related to the study of systems exhibiting Morse-Smale-like properties along their RoA boundaries has received criticism in the past. In particular, in [CC95, Theorem 5-2] Chiang makes an assumption that the stable and unstable manifolds of the equilibria in the RoA boundary are transversal over an entire one-parameter family (as a parameter is varied from zero to one). This assumption was challenged, both from the construction of simple counterexamples [LLM+95] and an illustration that the assumption is not generic [PL99]. Hence, from both practical and theoretical perspectives, this one-parameter family transversality assumption was not a good assumption. Fortunately, the transversality assumption used in [CHW88] and employed in this work requires only transversality of stable and unstable manifolds at one particular parameter value, which is a generic assumption, rather than over an entire one-parameter family. Hence, the transversality assumption used here was not the subject of this earlier criticism, and is generically true for C^1 vector fields.

To address the problem of determining the recovery boundary, it is beneficial to extend the historical work to incorporate behavior of the RoA boundary under parameter variation. Recall that a critical parameter value refers to a parameter value whose corresponding IC lies on the boundary of the RoA of the corresponding SEP. The recovery boundary is defined to be the (topological) boundary in parameter space of the set of recovery values. The definition of recovery boundary does not a priori provide a link between the parameter values it contains and their corresponding state space behavior. On the other hand, critical parameter values provide an explicit connection to the RoA boundary, which introduces an opportunity to develop algorithms that can exploit the structure and behavior of the RoA boundary. Therefore, from the perspective of recovery boundary estimation or computation, it would be highly desirable if the recovery boundary consisted entirely of critical parameter values.

By definition, every recovery value has a corresponding IC which lies inside the RoA of the corresponding SEP. Therefore, it seems natural to expect that every parameter value in the recovery boundary would have a corresponding IC which lies in the boundary of the RoA of the corresponding SEP. This would imply that every recover parameter value would be a critical parameter value. If the IC is (at least locally) continuous with respect to parameter, and if the post-disturbance vector field is parameter independent, then this is always true. This motivates the classical approach of designing algorithms to estimate the critical disturbance time in order to estimate the disturbance time recovery boundary (in fact, no distinction is made between these terms in the literature).

Unfortunately, the situation becomes more complex in the case where the post-disturbance vector field is parameter dependent. We will see in Example 2.2.1 that even if the IC is continuous with respect to parameter and the vector field is perturbed smoothly in a compact region, it is possible that the recovery boundary contains no critical parameter values at all. The reason for this is that, even though the IC is continuous and the vector field varies smoothly, the RoA boundary varies discontinuously and "jumps" over the IC so that they do not intersect. Furthermore, this behavior does not just hold for contrived, pathological cases: Example 2.3.3 shows such behavior for a (globally) Morse-Smale vector field over Euclidean space. Therefore, it is important to study conditions under which continuity of the RoA boundary can be guaranteed. This will help ensure both that the recovery boundary consists entirely of critical parameter values and that the closest critical parameter value to any recovery value lies on the recovery boundary. In turn, these will facilitate the design of practical algorithms for computing the recovery boundary,

The main results of this effort are that for C^1 vector fields which are Morse-Smale along the RoA boundary of a SEP in a compact Riemannian manifold, or in Euclidean space with some additional generic assumptions and control near infinity, under sufficiently small variations in parameter values the RoA boundary will vary continuously. The proof proceeds in two main steps:

1. Under the additional assumption that the nonwandering set in the RoA boundary does not grow under parameter perturbations, prove continuity of the RoA boundary.

2. Show that the above assumptions imply that the nonwandering set in the RoA boundary does not grow under parameter perturbations.

As part of the continuity proof, it is also shown that the decomposition of the RoA boundary into the union of the stable manifolds of the equilibrium points and periodic orbits it contains persists under parameter perturbations. Afterwards, the RoA boundary continuity results are extended to a class of hybrid dynamical systems which exhibit restricted switching and limit behavior. The RoA boundary continuity results are then used to establish that the recovery boundary consists entirely of critical parameter values. Finally, it is shown that the closest critical parameter value to any recovery value lies on the recovery boundary.

The next step is to exploit the structure of the RoA boundary to motivate algorithms for numerically computing critical parameter values, thereby obtaining the desired recovery boundary. Theoretical motivation is provided for two such algorithms. The first algorithm builds upon the historical notion of the controlling UEP. It exploits the property that, at a critical parameter value, the corresponding system trajectory converges to an invariant set (such as an equilibrium point or periodic orbit) in the RoA boundary, and therefore spends an infinite amount of time in any neighborhood of this so-called controlling invariant set. It is shown that, assuming the system trajectory intersects a neighborhood of the controlling invariant set transversely, the time the system trajectory spends inside this neighborhood is continuous over the set of recovery values, and diverges to infinity as any critical parameter value is approached from within the set of recovery values. This motivates an algorithm to compute critical parameter values by varying parameter values so as to maximize the time the system trajectory spends inside a neighborhood (typically chosen to be a ball for convenience) of the controlling invariant set, thereby driving the parameter value to its critical value.

To motivate the second algorithm, consider first a hyperbolic linear system with a saddle equilibrium point. For any trajectory in the stable eigenspace of the equilibrium point, a slight perturbation would push it off the stable eigenspace and cause it to diverge to infinity along the unstable direction in forwards time. In particular, the system trajectory along the stable manifold of the equilibrium point (which is equal to its stable eigenspace) becomes infinitely sensitive to small perturbations as time goes to infinity. More generally, at a critical parameter value assume the corresponding system trajectory is in the stable manifold of a hyperbolic (or normally hyperbolic) invariant set. Then the sensitivity of the system trajectory to small changes in parameter value (ie. small perturbations) diverges to infinity as time goes to infinity. It is shown that the supremum over time of the sensitivity of the trajectory with respect to parameter is continuous over the set of recovery values, and diverges to infinity as any critical parameter value is approached from within the set of recovery values. This motivates an algorithm to numerically compute critical parameter values by varying parameter values so as to maximize the supremum over time of the sensitivity of the trajectory with respect to parameter, thereby driving the parameter value to its critical value. It is shown that, as recovery values approach the recovery boundary, the infimum over time of the inverse trajectory sensitivities approaches a dot product. This motivates the design of algorithms for computing critical parameter values by exploiting this linearity. In both cases discussed above, properties of the RoA boundary are exploited to convert the abstract problem of computing critical parameter values into a concrete optimization or algebraic problem which can be solved numerically.

Motivated by the theory discussed above, several algorithm variations are developed to numerically compute critical parameter values. In the case of the choice of a one-dimensional parameter space, methods are proposed to find the unique parameter value on the recovery boundary. If a two-dimensional parameter space is chosen, techniques are developed to numerically trace the locally one-dimensional manifold recovery boundary using continuation methods. However, it is important to note that the recovery boundary can be nonsmooth in general. Nevertheless, empirical observations indicate that, at least on the test cases considered so far, the recovery boundary is smooth for practical power system models. For the case of arbitrary parameter space dimension, multiple approaches are devised to find the nearest set of parameter values on the recovery boundary from some initial set of parameter values. These techniques address many of the limitations of current industry practice:

1. The parameter space of interest can be chosen to be high dimensional, so that parameter values which could be critical to system recovery but might otherwise have been overlooked could be identified.

2. The methods determine the recovery boundary to arbitrary precision, and are not conservative or inaccurate.

3. Convergence is typically obtained in a small number of iterations, so that application of the algorithms during real-time operation may be feasible.

Furthermore, these techniques address many of the limitations of the classical algorithms:

1. They do not require the existence of suitable energy functions.

2. They compute the actual recovery boundary instead of conservative estimates of it.

3. They work for a class of hybrid systems exhibiting restricted switching and limit behavior.

4. They can consider parameter recovery boundaries for parameters which influence post-disturbance dynamics, thereby obtaining more useful measures of system stability than critical disturbance times.

5. They can still work even if trajectories corresponding to critical parameter values converge to invariant sets which are not equilibrium points.

Additionally, the second class of algorithms, related to minimizing the infimum over time of the inverse trajectory sensitivities, has the following additional advantages:

6. It does not require prior knowledge of the controlling invariant set.

7. For recovery values near the recovery boundary, it approaches a dot product, giving it a linear structure.

In sum, the algorithms address most of the shortcomings of current industry practice and of historical work related to energy function methods. For implementation, the algorithms both require computation of trajectory sensitivities. A modular scheme is introduced for trajectory sensitivity integration which exploits sparse network structure to improve computational efficiency and reduce storage requirements. This enables efficient application of the algorithms developed above.

The algorithms are applied to assess fault vulnerability in two power system test cases: the IEEE 37-bus feeder modified to include induction motor loads, and the IEEE 39-bus system, which includes controller limits. In the former, the algorithms determine the onset of induction motor stalling, representing an important step towards characterizing behavior of Fault Induced Delayed Voltage Recovery (FIDVR). In the latter, the algorithms analyze system stability in the presence of background load with high levels of uncertainty and variable voltage dynamics.

The thesis proposal is organized as follows. Chapter II presents the theoretical development, which includes the counterexample to the historical lemma in Section 2.1. Also included this chapter are the two parts of the proof of RoA continuity for the class of vector fields discussed in Sections 2.2-2.3, along with their extensions to a class of hybrid systems with restricted switching and limit behavior in Section 2.4. Theoretical justification of the maximizing time in the ball around the controlling invariant set algorithm is provided in Section 2.5 and theoretical justification of the minimizing infimum over time of the inverse trajectory sensitivities algorithm in Section 2.6. This will conclude Chapter II Chapter III then presents the numerical algorithms, which includes efficient methods for integration of trajectory sensitivities in Section 3.1. It then presents the class of algorithms corresponding to maximizing time in the ball around the controlling invariant set in Section 3.2, and the class of algorithms corresponding to maximizing the supremum over time of the trajectory sensitivities in Section 3.3. Chapter IV presents the results of application of the above algorithms to assess fault vulnerability in power systems for a modified IEEE 37-bus feeder in Section 4.1 and the IEEE 39-bus system in Section 4.2. Chapter V offers some concluding remarks and future directions.

CHAPTER II

Theory

This chapter is devoted to the theoretical motivation of the algorithms described in Chapter III for numerically computing the recovery boundary, and for finding the closest point on the recovery boundary to some initial set of recovery values. The first step towards providing justification for the algorithms is to show that the recovery boundary consists entirely of critical parameter values and that the closest critical parameter value to any recovery value lies on the recovery boundary. This provides the essential link between the recovery boundary in parameter space and the RoA boundary in state space. To ensure this (assuming the IC is locally continuous) it suffices to show that the RoA varies continuously with small changes in parameter values.

Before turning to the main proofs regarding RoA boundary continuity, Section 2.1 provides a counterexample to a historical lemma which was instrumental in the proof that the RoA boundary is equal to the union of the stable manifolds of the critical elements it contains [CHW88]. To avoid the use of this flawed lemma, and to enable extensions to parameter variation, the assumptions of [CHW88] are strengthened slightly to consider C^1 vector fields which are Morse-Smale along the RoA boundary (with some additional assumptions for the noncompact case).

It is shown that, for this class of vector fields, under small variations in parameter values the RoA boundary varies continuously. The notion of Ω-stable means informally that the structure of the nonwandering set remains the same under perturbations. The continuity proof proceeds in two steps. First, in Section 2.2 the main result is proved under the additional assumption of Ω-stability within the RoA boundary. Then, in Section 2.3 it is shown that Ω-stability within the RoA boundary follows from the original assumptions (plus some additional assumptions in the noncompact case). These RoA boundary continuity results are extended in Section 2.4 to a class of hybrid systems with restricted switching and limit behavior.

In Sections 2.5-2.6, the RoA continuity results are first used to show that the recovery boundary consists entirely of critical parameter values and that the closest critical parameter value to any recovery value lies on the recovery boundary. Then, Section 2.5 develops theoretical motivation for a first class of algorithms by showing that, under suitable assumptions, the time the system trajectory spends in a neighborhood around the controlling invariant set is continuous in parameter and diverges to infinity as any critical parameter value is approached from within the set of recovery values.

Finally, Section 2.6 develops theoretical motivation for a second class of algorithms by showing that, under reasonable assumptions, the supremum over time of the trajectory sensitivity with respect to parameters is continuous over the set of recovery values and diverges to infinity as any critical parameter value is approached from within the set of recovery values.

2.1 Counterexample to Historical Work

2.1.1 Introduction

In the proofs of their superb and groundbreaking theorems, the authors of [CHW88] rely on a lemma which states that if the stable manifold of a first hyperbolic closed orbit intersects transversely the unstable manifold of a second (possibly the same) hyperbolic closed orbit, then the dimension of the unstable manifold of the first is strictly less than the dimension of the unstable manifold of the second. However, we exhibit an example meeting the conditions of the lemma where the dimensions of the unstable manifolds are equal, thereby disproving the lemma. In particular, we present a hyperbolic closed orbit of a smooth (C^∞) vector field over $\mathbb{R}^3$ whose stable and unstable manifolds have nonempty, transversal intersection.

2.1.2 Background

Consider the system:

$$\dot{x} = V(x) \tag{2.1}$$

of a C^1 vector field (V) over Euclidean space ($\mathbb{R}^n$) for some $n > 0$. In [CHW88, Theorem 4.2], the authors claim that if a stable equilibrium point of Eq. 2.1 satisfies their assumptions, then the boundary of its region of attraction (RoA) is equal to the union of the stable manifolds of the equilibrium points and periodic orbits it contains. The main technical result behind their proof of this impressive theorem is [CHW88, Theorem 3-8]. In turn, [CHW88, Lemma 3-5] is crucial in their proof of [CHW88, Theorem 3-8]. This note is devoted to the construction of a counterexample to [CHW88, Lemma 3-5].

The authors of [CHW88] attribute their Lemma 3-5 to a survey paper by Smale [Sma67] and do not provide a proof of the lemma. However, we have not found the lemma in [Sma67]. Furthermore, in another paper by Smale [Sma60] he considers a set of assumptions in which his conditions 1-4 include that all equilibrium points and periodic orbits are hyperbolic, and their stable and unstable manifolds have transversal intersection. Yet Smale also includes condition 5, which states that for any closed orbit γ there does not exist $x \notin \gamma$ such that x converges to γ in both forwards and backwards time. In other words, condition 5 states for every closed orbit γ, its stable and unstable manifolds have empty intersection outside of γ. In addition, Smale writes: "It is true that conditions (1)-(5) are independent." If [CHW88, Lemma 3-5] was correct, then conditions 1-4 would imply condition 5 (because as a consequence of [CHW88, Lemma 3-5] hyperbolic closed orbits could not have nonempty, transversal intersections between their stable and unstable manifolds), so Smale's statement that conditions (1)-(5) are independent would necessarily be incorrect. It is therefore strange that [CHW88, Lemma 3-5] is attributed to Smale, since it would invalidate his own claims in [Sma60]. Recently, [CHW88, Lemma 3-5] was reprinted by one of the original authors [CA15, Lemma 4-4] without modification and without proof, and no reference was provided in this reprinting.

It should be noted that the example presented here does not contradict [CHW88, Theorem 4.2], and that of course the theorem may be correct as stated. However, we have not been able to supply a complete proof. We have recently shown that the result of Theorem 4.2 can be proved analogously to the original proof under a slightly stronger assumption than the original theorem: that the nonwandering set along the RoA boundary consists entirely of critical elements.

Furthermore, it is possible that under the additional assumptions of Theorem 4.2, which were not assumed in the statement of Lemma 3-5, the conclusion of Lemma 3-5 may still be true. However, we have not been able to prove this either and wonder whether, for example, it might be possible for the suspended horseshoe in the transversal intersection of the stable and unstable manifolds of a hyperbolic closed orbit to intersect the RoA boundary only in orbits which converge to critical elements (since a suspended horseshoe has many orbits which could not be present in the RoA boundary under the assumptions of Theorem 4.2).

If such an example exists, then even under the additional assumptions of Theorem 4.2, the conclusion of Lemma 3-5 would be incorrect.

Before presenting [CHW88, Lemma 3-5] in detail, we provide some preliminary definitions. Let a critical element refer to either an equilibrium point or a closed orbit of Eq. 2.1. An equilibrium point (x_e) of Eq. 2.1 is hyperbolic if dV_{x_e} has no imaginary eigenvalues. A periodic orbit γ of Eq. 2.1 is hyperbolic if there exists $x \in \gamma$, a smooth cross section S centered at x, and a C^1 first return map $\hat{\tau} : S \to S$ such that $d\hat{\tau}_x$ has no eigenvalues of norm one. Let γ be a hyperbolic critical element and define the stable manifold of γ ($W^s(\gamma)$) to be the set of all initial conditions which converge to γ in forwards time. Similarly, define the unstable manifold of γ ($W^u(\gamma)$) to be the set of all initial conditions which converge to γ in backwards time. Note that both $W^s(\gamma)$ and $W^u(\gamma)$ are invariant under the flow of V. Consider a differentiable submanifold ($\hat{A}$) of M, let $x \in \hat{A}$, and define the tangent space of $\hat{A}$ at x ($T_x(\hat{A})$). Consider another differentiable submanifold ($\hat{B}$) of M. Then $\hat{A}$ and $\hat{B}$ satisfy the transversality condition, or have transversal intersection, if either they are disjoint or for every $x \in \hat{A} \bigcap \hat{B}$, $T_x\hat{A} + T_x\hat{B} = T_xM$.

The exact statement of [CHW88, Lemma 3-5] is as follows:

Lemma 1. Let x_i and x_j be hyperbolic critical elements of Eq. 2.1. Suppose that the intersection of stable and unstable manifolds of x_i, x_j satisfy the transversality condition and $\{W^u(x_i) - x_i\} \bigcap \{W^s(x_j) - x_j\} \neq \emptyset$. Then dim $W^u(x_i) \geq$ dim $W^u(x_j)$, where the equality sign is true only when x_i is an equilibrium point and x_j is a closed orbit.

2.1.3 Example

To disprove Lemma 1, we will provide an example in which dim $W^u(x_i) =$ dim $W^u(x_j)$ but both x_i and x_j are closed (periodic) orbits. In particular, we will give an example where $x_i = x_j$ is a hyperbolic periodic orbit whose stable and unstable manifolds are transverse and $\{W^u(x_i) - x_i\} \bigcap \{W^s(x_j) - x_j\} \neq \emptyset$. Then dim $W^u(x_i) =$ dim $W^u(x_j)$ because $x_i = x_j$.

So, consider the classic example of the Duffing equation with negative linear stiffness, weak damping, and weak periodic forcing, which is given by the following C^∞ nonautonomous vector field on $\mathbb{R}^2$ [GH83, p. 191]:

$$\begin{aligned}\dot{u} &= v \\ \dot{v} &= u - u^3 + \epsilon(\gamma \cos t - \delta v)\end{aligned}$$

where $\epsilon, \delta, \gamma \geq 0$ are parameters and $u, v \in \mathbb{R}$. At $\epsilon = 0$ this system possesses a hyperbolic

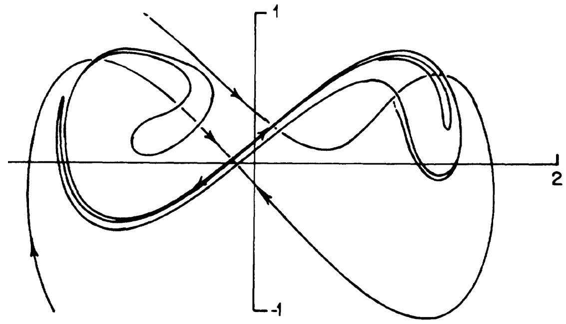

Figure 2.1: This figure originally appeared in [GH83, p. 208] and was computed numerically by Y. Ueda. It shows the stable and unstable manifolds of the first return map of a cross section of the Duffing equation for parameter values $\epsilon\delta = 0.25$ and $\epsilon\gamma = 0.30$. Note that the stable and unstable manifolds have nonempty, transversal intersection.

saddle equilibrium point at $(0, 0)$ whose stable manifold and unstable manifold are equal (they consist of homoclinic orbits). We can rewrite this system as an autonomous vector field on $\mathbb{R}^2 \times \mathbb{S}^1$ by introducing a time coordinate $\mathcal{T}$:

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= u - u^3 + \epsilon(\gamma \cos \mathcal{T} - \delta v) \\ \dot{\mathcal{T}} &= 1. \end{aligned}$$

Letting $w = [u\ v\ \mathcal{T}]^T$, we can write the above vector field as:

$$\dot{w} = V(w). \tag{2.2}$$

Note that V is a C^∞ vector field. For $\epsilon = 0$, this system has a hyperbolic periodic orbit given by $\Gamma_0 = \{(0, 0, \mathcal{T}) : \mathcal{T} \in \mathbb{S}^1\}$ whose stable and unstable manifolds are each two-dimensional. For $\epsilon > 0$ sufficiently small, there exists a unique hyperbolic periodic orbit Γ_ϵ which is C^1-close to Γ_0. Furthermore, there exists a closed curve $\Lambda \subset \mathbb{R}^2$ such that for any $\epsilon \geq 0$ small, the vector field is inward pointing on Λ for all times $\mathcal{T} \in \mathbb{S}^1$ [GH83, Exercise 1.5.5] and $\pi_{\mathbb{R}^2}(\Gamma_\epsilon)$ is contained in the interior of the region bounded by Λ, where $\pi_{\mathbb{R}^2}$ is the projection map onto $\mathbb{R}^2$. This implies that $\pi_{\mathbb{R}^2}(W^u(\Gamma_\epsilon))$ is contained in the interior of the region bounded by Λ.

For any $\mathcal{T} \in \mathbb{S}^1$, let $S_{\mathcal{T}} = \mathbb{R}^2 \times \{\mathcal{T}\}$ be a smooth cross section. Define the Poincaré first return map $(\hat{\tau})$, where $\hat{\tau} : S_{\mathcal{T}} \to S_{\mathcal{T}}$, and note that it is well-defined and C^∞. Let $p_\epsilon^{\mathcal{T}} = \Gamma_\epsilon \bigcap S_{\mathcal{T}}$, which is a single point. Then it is straightforward to see that $p_\epsilon^{\mathcal{T}}$ is a hyperbolic fixed point of V. Let $W^s(p_\epsilon^{\mathcal{T}})$ and $W^u(p_\epsilon^{\mathcal{T}})$ denote its stable and unstable manifolds, respectively, in $S_{\mathcal{T}}$. Using Melnikov's method [GH83, Theorem 4.5.3], it can be shown [GH83, p. 193] that for any $\mathcal{T} \in \mathbb{S}^1$, $\epsilon > 0$ sufficiently small, and $\frac{\gamma}{\delta}$ sufficiently large, $W^s(p_\epsilon^{\mathcal{T}})$ and $W^u(p_\epsilon^{\mathcal{T}})$ are transverse and $\{W^s(p_\epsilon^{\mathcal{T}}) - p_\epsilon^{\mathcal{T}}\} \bigcap \{W^u(p_\epsilon^{\mathcal{T}}) - p_\epsilon^{\mathcal{T}}\} \neq \emptyset$. Fig. 2.1 (originally appearing in [GH83, p. 208], which was computed numerically by Y. Ueda) shows $W^s(p_\epsilon^{\mathcal{T}})$ and $W^u(p_\epsilon^{\mathcal{T}})$ for particular choices of $\mathcal{T}$, $\epsilon > 0$ sufficiently small, and $\frac{\gamma}{\delta}$ sufficiently large such that $\{W^s(p_\epsilon^{\mathcal{T}}) - p_\epsilon^{\mathcal{T}}\}$ and $\{W^u(p_\epsilon^{\mathcal{T}}) - p_\epsilon^{\mathcal{T}}\}$ have nonempty, transversal intersection. As $\mathcal{T} \in \mathbb{S}^1$ was arbitrary, it follows that $W^s(\Gamma_\epsilon)$ and $W^u(\Gamma_\epsilon)$ are transverse and $\{W^s(\Gamma_\epsilon) - \Gamma_\epsilon\} \bigcap \{W^u(\Gamma_\epsilon) - \Gamma_\epsilon\} \neq \emptyset$. Hence, we may fix $\epsilon, \gamma, \delta > 0$ such that $W^s(\Gamma_\epsilon)$ and $W^u(\Gamma_\epsilon)$ are transverse and $\{W^s(\Gamma_\epsilon) - \Gamma_\epsilon\} \bigcap \{W^u(\Gamma_\epsilon) - \Gamma_\epsilon\} \neq \emptyset$.

However, the vector field above is over $\mathbb{R}^2 \times \mathbb{S}^1$, whereas Lemma 1 is stated for a vector field over Euclidean space. So, we will modify the example to obtain a similar vector field over Euclidean space. As $W^s(\Gamma_\epsilon)$ and $W^u(\Gamma_\epsilon)$ are transverse, $W^s_{\text{loc}}(\Gamma_\epsilon)$ and $W^u(\Gamma_\epsilon)$ are transverse, where $W^s_{\text{loc}}(\Gamma_\epsilon)$ is the local stable manifold of Γ_ϵ. As $\{W^s(\Gamma_\epsilon) - \Gamma_\epsilon\} \bigcap \{W^u(\Gamma_\epsilon) - \Gamma_\epsilon\} \neq \emptyset$, by invariance $\{W^s_{\text{loc}}(\Gamma_\epsilon) - \Gamma_\epsilon\} \bigcap \{W^u(\Gamma_\epsilon) - \Gamma_\epsilon\} \neq \emptyset$. For any $s > 0$ let $B(s)$ denote the open ball of radius s centered at the origin in $\mathbb{R}^2$. As Λ is a closed curve in $\mathbb{R}^2$, choose $\mu_1 > \mu_2 > 0$ such that $\Lambda \subset B(\mu_1 - \mu_2)$. As $\pi_{\mathbb{R}^2}(W^s_{\text{loc}}(\Gamma_\epsilon))$ and $\pi_{\mathbb{R}^2}(W^u(\Gamma_\epsilon))$ are contained in the interior of the region bounded by Λ, they are contained in $B(\mu_1 - \mu_2)$ as well. For any $b > a > 0$, let $\mathcal{S}_{(a,b)} : [0, \infty) \to [0, 1]$ denote a C^∞ bump function with $\mathcal{S}^{-1}_{(a,b)}(1) = [0, a]$ and $\mathcal{S}^{-1}_{(a,b)}(0) = [b, \infty)$. Define the following vector field for $w \in \mathbb{R}^2 \times \mathbb{S}^1$:

$$\dot{w} = \mathcal{S}_{(\mu_1 - \mu_2, \mu_1)}\left(\sqrt{u^2 + v^2}\right) V(w). \tag{2.3}$$

As Eq. 2.3 and Eq. 2.2 agree for all points (u, v, t) such that $\pi_{\mathbb{R}^2}(u, v, t) \in B(\mu_1 - \mu_2)$, and since $\pi_{\mathbb{R}^2}(W^s_{\text{loc}}(\Gamma_\epsilon))$ and $\pi_{\mathbb{R}^2}(W^u(\Gamma_\epsilon))$ corresponding to the vector field of Eq. 2.2 are contained in $B(\mu_1 - \mu_2)$, $W^s_{\text{loc}}(\Gamma_\epsilon)$ and $W^u(\Gamma_\epsilon)$ corresponding to the vector field of Eq. 2.3 are equal to those of Eq. 2.2. In particular, this implies that $W^s_{\text{loc}}(\Gamma_\epsilon)$ and $W^u(\Gamma_\epsilon)$ of Eq. 2.3 are transverse, and also that they satisfy $\{W^s_{\text{loc}}(\Gamma_\epsilon) - \Gamma_\epsilon\} \bigcap \{W^u(\Gamma_\epsilon) - \Gamma_\epsilon\} \neq \emptyset$. Hence, for Eq. 2.3, $W^s(\Gamma_\epsilon)$ and $W^u(\Gamma_\epsilon)$ are transverse and $\{W^s(\Gamma_\epsilon) - \Gamma_\epsilon\} \bigcap \{W^u(\Gamma_\epsilon) - \Gamma_\epsilon\} \neq \emptyset$.

Now we construct the desired vector field over $\mathbb{R}^3$. The key idea is to use Eq. 2.3 to construct a vector field over an embedding of $B(\mu_1) \times \mathbb{S}^1$ into $\mathbb{R}^3$, and then to extend it to a C^∞ vector field over $\mathbb{R}^3$ by defining it to be zero on the complement of the embedding of $B(\mu_1) \times \mathbb{S}^1$. This vector field will then be the desired counterexample to Lemma 1. We will use standard cylindrical coordinates $x := (r, \theta, z)$ to construct this vector field. Let

$$r = u + \mu_1 \tag{2.4}$$
$$\theta = \mathcal{T} \tag{2.5}$$
$$z = v. \tag{2.6}$$

Then we define

$$\begin{aligned}\dot{x} &= \mathcal{S}_{(\mu_1 - \mu_2, \mu_1)}\left(\sqrt{u^2 + v^2}\right) V(w) \\ &= \mathcal{S}_{(\mu_1 - \mu_2, \mu_1)}\left(\sqrt{(r - \mu_1)^2 + z^2}\right) V(x)\end{aligned} \tag{2.7}$$

where

$$V(x) = \begin{bmatrix}\dot{r} \\ \dot{\theta} \\ \dot{z}\end{bmatrix} = \begin{bmatrix} z \\ 1 \\ (r - \mu_1) - (r - \mu_1)^3 + \epsilon(\gamma \cos\theta - \delta z)\end{bmatrix}$$

where $V(x)$ was obtained from $V(w)$ using Eqs. 2.4-2.6. Note that Eq. 2.7 defines a C^∞ vector field on $\mathbb{R}^3$. As the supports of Eq. 2.7 and Eq. 2.3 are diffeomorphic, and the two vector fields agree on their supports up to this diffeomorphism, Eq. 2.7 possesses a hyperbolic periodic orbit Γ such that $W^s(\Gamma)$ and $W^u(\Gamma)$ are transverse and $\{W^s(\Gamma) - \Gamma\} \bigcap \{W^u(\Gamma) - \Gamma\} \neq \emptyset$. Therefore, Eq. 2.7 is a counterexample to [CHW88, Lemma 3-5].

2.1.4 Conclusion

This brief section was devoted to the construction of a counterexample to [CHW88, Lemma 3-5], which was necessary in the proof of [CHW88, Theorem 4.2] regarding a decomposition of the region of attraction boundary into a union of stable manifolds. As one goal of the following sections will be to show that this decomposition persists under parameter perturbations, it will first be necessary to complete the original proof of [CHW88, Theorem 4.2]. To do so, the assumptions of [CHW88] will be slightly modified; this will ensure both that the original result of Theorem 4.2 holds, and that it persists under suitable parameter perturbation.

2.2 Continuity of Region of Attraction Boundary Under Perturbation

2.2.1 Introduction

The main technical challenge behind the theoretical justification of the desired algorithms is to show that the RoA boundary varies continuously in an appropriate sense under small changes in parameter values (see Corollaries 2.2.24-2.2.26). For, Example 2.2.1 shows that when the boundary of the RoA is discontinuous at a particular parameter value, then it is possible for the ICs to "jump" over the RoA boundary. In this case, the recovery boundary may not consist of critical parameter values and, in fact, its possible that no critical parameter values even exist. The former implies that computation of critical parameter values may not provide an accurate estimate of the recovery boundary, and hence of the set of recovery values, and the latter implies that any attempt to compute critical parameters must fail since they don't exist; both are problematic for the algorithms which will be developed below. Furthermore, discontinuity of the RoA boundary implies that there may not exist a controlling invariant set in the RoA boundary with the property that as parameter values approach critical values, the time the trajectory spends in a ball around the controlling invariant set diverges to infinity. Similarly, discontinuity of the RoA boundary may imply that as parameter values approach critical values, the supremum over time of the trajectory sensitivites may not diverge to infinity. Hence, if the recovery boundary does not consist of critical parameter values, the central strategy of the algorithms developed below may be unable to compute them. Therefore, establishing continuity of the RoA boundary for a large class of parameter dependent vector fields is crucial for motivating those algorithms.

This section is devoted to part one of the proof of continuity of RoA boundary: assuming the RoA boundary is Ω-stable (and the vector field is Morse-Smale along it), we will show in this section that it varies continuously with respect to small changes in parameter values. Section 2.3 shows that if the vector field is Morse-Smale along the RoA boundary then the RoA boundary is Ω-stable, thereby completing the proof. The stable manifold of a critical element is the set of initial conditions in state space which converge to that critical element in forwards time. The approach that is used to establish continuity of the RoA boundary is to show that at a fixed parameter value the RoA boundary is equal to the union of the stable manifolds of the critical elements it contains, and that this decomposition persists for small changes in parameter values, for a large class of parameter dependent vector fields (see Theorem 2.2.23). Earlier work [CHW88] reported this decomposition result for a large class of fixed parameter C^1 vector fields on Euclidean space. However, their proof relied on a Lemma [CHW88, Lemma 3-5] which has been disproven (see Section 2.1). Therefore, we begin by providing a complete proof for a fixed parameter RoA boundary decomposition result, and then focus on our main goal of extending this work to a continuous (in an appropriate sense - see below) parameterized family of C^1 vector fields on either Euclidean space or a compact Riemannian manifold.

The section is organized as follows. Section 2.2.2 presents relevant background and notation conventions. Section 2.2.3 provides a motivating example for discontinuity of the RoA boundary and the negative implications this can have. Section 2.2.4 gives the main results, focusing on parameter dependent vector fields, although results for parameter independent vector fields are also included. A simple example is provided to illustrate the main theorems. Section 2.2.5 proves the boundary decomposition results for the case where the vector field is parameter indepedent. Section 2.2.6 builds on the foundation of Section 2.2.5 to prove persistence of the boundary decomposition and continuity of the RoA boundaries for a large class of parameter dependent vector fields. Finally, Section 2.2.7 offers some concluding thoughts and future directions.

2.2.2 Notation and Definitions

Let E be a subset of a topological space. Then we define the topological closure of E ($\overline{E}$), the topological boundary of E (∂E), and the topological interior of E ($\text{int } E$). If $\tilde{f} : A \to B$ is any function and $E \subset A$ is any subset of A, define the restriction of the function $\tilde{f}$ to E ($\tilde{f}|_E$) to be $\tilde{f}' : E \to B$ where $\tilde{f}'(e) = \tilde{f}(e)$ for all $e \in E$. For a set D contained in a metric space K, define the ϵ-neighborhood of D (D_ϵ) to be the set of $x \in K$ such that for each x there exists $y \in E$ with $d(x, y) < \epsilon$. Let $\{x_n\}_{n=1}^\infty$ be a sequence. Let $\{n_m\}_{m=1}^\infty$ be any collection of positive integers where we require that $m' > m$ implies that $n_{m'} > n_m$ to ensure the ordering is preserved. Then any subsequence of $\{x_n\}_{n=1}^\infty$ can be written as $\{x_{n_m}\}_{m=1}^\infty$ for some choice of $\{n_m\}_{m=1}^\infty$. Let K be a nonempty, compact metric space. Note that compact Riemmanian manifolds are compact metric spaces, so the discussion below will apply to them. Define the set of nonempty closed subsets of K ($\mathcal{C}(K)$). Let $K_1, K_2 \in \mathcal{C}(K)$. We define the Hausdorff distance (d_h) by

$$d_h(K_1, K_2) = \inf\{r \geq 0 : K_1 \subset (K_2)_r, K_2 \subset (K_1)_r\}.$$

Then d_h is a well-defined metric on $\mathcal{C}(K)$ [Hau57, Section 28] and we say a sequence of sets $A_n \in \mathcal{C}(K)$ converges to $A \in \mathcal{C}(K)$, denoted $A_n \to A$, if $\lim_{n\to\infty} d_h(A_n, A) = 0$. For A, B subsets of a metric space with metric d, define a set distance (d_S) by $d_S(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$. Then if A is compact, B is closed, and $d(A,B) = 0$, A and B must have nonempty intersection. Note that any Riemannian manifold is also a metric space, so this set distance is well-defined on Riemannian manifolds.

Let J be a topological space. For $p \in J$, we say that p has a countable neighborhood basis if there exists a countable collection $\{U_n\}_{n=1}^{\infty}$ of open sets in J that contain p such that for any open set $\hat{U}$ which contains p, there exists n such that $U_n \subset \hat{U}$. Then we say that J is first countable if every point $p \in J$ possesses a countable neighborhood basis. If J is first countable, E is any topological space, and $\tilde{f} : J \to E$ is a function, then $\tilde{f}$ is continuous if and only if for every convergent sequence $\{p_n\}_{n=1}^{\infty}$ in J, say $p_n \to p$, $\tilde{f}(p_n) \to \tilde{f}(p)$. Let J be a first countable topological space and let $\tilde{F} : J \to \mathcal{C}(K)$. We say that the family $\{A_p\}_{p\in J}$ is a Hausdorff continuous family of subsets of K if there exists $\tilde{F} : J \to \mathcal{C}(K)$ such that $\tilde{F}(p) = A_p$ for $p \in J$ and $\tilde{F}$ is continuous. Since J is first countable, $\tilde{F}$ is continuous if and only if for every $p \in J$ and every sequence $p_n \in J$ with $p_n \to p$, $\tilde{F}(p_n) \to \tilde{F}(p)$.

We consider another notion of convergence on $\mathcal{C}(K)$. Let $A_n \in \mathcal{C}(K)$ be a sequence of sets. Define $\liminf_{n\to\infty} A_n$ to be the set of points $x \in K$ such that there exists a sequence $\{a_n\}$, with $a_n \in A_n$ for all n, such that $a_n \to x$. Define $\limsup_{n\to\infty} A_n$ to be the set of points $x \in K$ such that there exist $\{a_{n_m}\}$ with $a_{n_m} \in A_{n_m}$ a subsequence of $\{A_n\}$, such that $a_{n_m} \to x$. Both $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$ are closed [Hau57, Section 28] and $\limsup_{n\to\infty} A_n$ is nonempty since K is sequentially compact, so if $\liminf_{n\to\infty} A_n$ is nonempty then both are elements of $\mathcal{C}(K)$. By definition, $\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n$. If $\limsup_{n\to\infty} A_n \subset \liminf_{n\to\infty} A_n$ then we say the limit exists and $\lim_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$. By statement V of [Hau57, Section 28], since K is compact, if $\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = \lim_{n\to\infty} A_n =: A$ then $d_h(A_n, A) \to 0$. Thus, if there exists $\tilde{F} : J \to \mathcal{C}(K)$ such that for every $p \in J$ and every $p_n \to p$, $\limsup_{n\to\infty} \tilde{F}(p_n) = \liminf_{n\to\infty} \tilde{F}(p_n) = \tilde{F}(p)$, then $d_h(\tilde{F}(p_n), \tilde{F}(p)) \to 0$ so $\{\tilde{F}(p)\}_{p\in J}$ is a Hausdorff continuous family of subsets of K.

Let $M = \mathbb{R}^n$, and let $\mathcal{C}(M)$ be the closed, nonempty subsets of M. The standard Hausdorff distance is not well-defined for unbounded sets, so instead consider the one-point compactification of M, $\mathbb{R}^n \cup \infty \cong \mathbb{S}^n$, where $\mathbb{S}^n$ is the n-sphere. Equip $\mathbb{S}^n$ with the induced Riemannian metric from its inclusion into $\mathbb{R}^{n+1}$, and let its associated distance function be the desired metric on $M \cup \infty$. Then, since $\mathbb{S}^n$ is a compact, nonempty metric space, the Hausdorff distance is well-defined for all closed, nonempty subsets of $\mathbb{S}^n$. Let $\overline{\mathcal{C}}(M) = \{A \cup \{\infty\} : A \in \mathcal{C}(M)\}$. Then all sets in $\overline{\mathcal{C}}(M)$ are closed and nonempty, so the Hausdorff distance is well-defined on $\overline{\mathcal{C}}(M)$, and the metric topology it induces on $\overline{\mathcal{C}}(M)$ is called the Chabauty topology. From the discussion above regarding Hausdorff continuity, it follows that if there exists $\tilde{F} : J \to \overline{\mathcal{C}}(M)$ such that for every $p \in J$ and every $p_n \to p$, $\limsup_{n\to\infty} \tilde{F}(p_n) = \liminf_{n\to\infty} \tilde{F}(p_n) = \tilde{F}(p)$, then $d_h(\tilde{F}(p_n), \tilde{F}(p)) \to 0$ so $\{\tilde{F}(p)\}_{p\in J}$ is a Chabauty continuous family of subsets of M.

Let V be a C^1 vector field on a Riemannian manifold M. An integral curve γ of V is a map from an open subset $U \subset \mathbb{R}$ to M such that for every $t \in U$, $\frac{d}{dt}\gamma(t) = V_{\gamma(t)}$. A flow ($\phi$) is a map $\phi : U \times M \to M$, where $U \subset \mathbb{R}$ is open, such that for any $x \in M$, $\phi(\cdot, x)$ is an integral curve of V. For any C^1 vector field V on a Riemannian manifold M, there exists a C^1 flow ϕ [Lee13, Theorem 9.12]. We say that V is complete if possesses a flow ϕ defined on $\mathbb{R} \times M$. For $t \in \mathbb{R}$, we define the time-t flow (ϕ_t), where $\phi_t : M \to M$, by $\phi_t(x) = \phi(t, x)$. Then acphit is a C^1 diffeomorphism of M for any t since ϕ is C^1 and $\phi_t^{-1} = \phi_{-t}$.

Consider a smooth manifold ($\tilde{M}$). If M and $\tilde{M}$ are Riemannian manifolds, let $C^1(M, \tilde{M})$ denote the set of C^1 maps from M to $\tilde{M}$. There are two common topologies that $C^1(M, \tilde{M})$ can be equipped with: the strong and weak C^1 topologies. Full definitions of these are available in [Hir76, Chapter 2], but the properties of these topologies which are most important for this work are summarized below. The purpose of introducing these topologies is to provide a framework for careful consideration of perturbations to vector fields, and to be able to define a continuous family of vector fields in a suitable way. We will typically equip $C^1(M, \tilde{M})$ with the weak topology, denoted $C^1_W(M, \tilde{M})$. A major benefit of the weak topology is that it has a complete metric, the C^1 distance (d_{C^1}). If V is a C^1 vector field on M then $V \in C^1(M, TM)$. A pair of C^1 vector fields $V, \tilde{V}$ on M are ϵ C^1-close if $d_{C^1}(V, \tilde{V}) < \epsilon$, where d_{C^1} is the C^1 distance on $C^1(M, TM)$. A (weak) C^1 perturbation to the vector field V is a vector field $\tilde{V}$ such that $V, \tilde{V}$ are ϵ C^1-close for sufficiently small $\epsilon > 0$. A parameterized family of C^1 vector fields $\{V_p\}_{p\in J}$ on M is (weakly) C^1 continuous if the induced map $J \to C^1_W(M, TM)$ that sends p to the vector field corresponding to parameter value p (V_p) is continuous. Note that this implies that $V : M \times J \to TM \times TJ$ defined by $V(x, p) = (V_p(x), 0)$ is a C^1 vector field on $M \times J$. In the case of M compact, the strong and weak topologies on $C^1(M, \tilde{M})$ coincide. For M noncompact, a strong C^1 perturbation to a vector field only involves changes to that vector field on a compact set, whereas a weak C^1 perturbation to that vector field could have changes that are unbounded. Hence, weak continuity of vector fields is a weaker assumption than strong continuity.

Let $\hat{A}$ be a Riemannian manifold. Then for each $x \in \hat{A}$, let $T_x\hat{A}$ denote the tangent space to $\hat{A}$ at x. Let $T\hat{A} = \sqcup_{x\in\hat{A}} T_x\hat{A}$ denote the tangent bundle, and note that it is naturally a manifold with dimension twice that of $\hat{A}$. Let the zero section be the subspace of $T\hat{A}$ consisting of the zero vector from each tangent space $T_x\hat{A}$ over $x \in \hat{A}$ (note that it is naturally diffeomorphic to $\hat{A}$ itself). Let $\hat{B} \subset \hat{A}$. For any function $\tilde{f} : \hat{B} \to \hat{A}$ and $x \in \hat{B}$, define the differential of $\tilde{f}$ at x ($d\tilde{f}_x$). Let $i : \hat{B} \to \hat{A}$ by $i(x) = x$ be the inclusion map. We say that i is a C^1 immersion if it is C^1 and for every $y \in \hat{B}$, di_y is injective. Note that a function $\tilde{f} : \hat{A} \to \hat{B}$, where $\hat{A}$ and $\hat{B}$ are C^1 manifolds, is a submersion if $d\tilde{f}_y$ is surjective for every $y \in M$. Then we say $\hat{B}$ is an immersed submanifold if its inclusion map $i : \hat{B} \to \hat{A}$ is a C^1 immersion. Let $T_{\hat{B}}M = \sqcup_{x\in\hat{B}} T_xM$ denote the tangent bundle of M over $\hat{B}$. Consider a pair of C^1 immersed submanifolds $\hat{A}$ and $\hat{B}$. We say that $\hat{A}$ and $\hat{B}$ are transverse at a point $x \in \hat{A} \cap \hat{B}$ if $T_x\hat{A} \oplus T_x\hat{B}$ spans T_xM. Then we say that $\hat{A}$ and $\hat{B}$ are transverse if for every $x \in \hat{A} \cap \hat{B}$, $\hat{A}$ and $\hat{B}$ are transverse at x. Note that if $\hat{A}$ and $\hat{B}$ are disjoint, they are vacuously transverse. A C^1 disk is the image of $i : B \to M$ where $B \subset \mathbb{R}^m$ is a closed ball around the origin in some Euclidean space $\mathbb{R}^m$, and i is a C^1 immersion. A continuous family of C^1 disks is a parameterized family $\{D_x\}_{x\in E}$ where E is a topological space, D_x is a C^1 disk for each x, and $\{D_x\}_{x\in S}$ is a Hausdorff continuous family. Suppose that $\hat{A}$ is a C^1 immersed submanifold of $\hat{B}$, which is a C^1 immersed submanifold of M. By the tubular neighborhood theorem [Lee13, Theorem 6.24], there exists a C^1 continuous family of pairwise disjoint disks $\{D(x)\}_{x\in\hat{B}}$ in M centered along $\hat{B}$ and transverse to it such that their union is an open neighborhood of $\hat{B}$ in M. Taking the restriction $\{D(x)\}_{x\in\hat{A}}$ gives a C^1 continuous family of pairwise disjoint disks in M centered along $\hat{A}$ and transverse to $\hat{B}$. If $\tilde{F} : \hat{A} \to \hat{B}$ is a continuous and injective map between manifolds $\hat{A}$ and $\hat{B}$ of the same dimension, then by invariance of domain [Hat01, Theorem 2B.3], $\tilde{F}$ is an open map, which means the image under $\tilde{F}$ of every open set is open.

There is a notion of a generic C^1 vector field, which is meant to represent typical behavior, similar to the idea of probability one in a probability space. If a property holds for a generic class of C^1 vector fields, it is therefore considered to be typical or usual behavior. As there exist many pathological C^1 vector fields, it is often advantageous to restrict attention to certain classes of generic C^1 vector fields when possible, and to prove results for generic vector fields that often would not hold for arbitrary vector fields. We follow this approach here. In a topological space E, a Baire set is a countable intersection of open, dense subsets of E. A topological space E is metrizable if there exists a metric on E whose metric topology corresponds with the original topology on E. It is completely metrizable if the resultant metric space is complete. The Baire category theorem states that if the topology on E is completely metrizable, then every Baire set in E is dense. By the discussion above, $C^1(M, \tilde{M})$ is a complete metric space for M and $\tilde{M}$ under consideration here, so every Baire set will be dense. Suppose $\mathcal{P}$ is a property that may be possessed by elements of a topological space E. Then $\mathcal{P}$ is called a generic property if the set of elements in M which possess the property $\mathcal{P}$ contains a Baire set in M. So, a property of vector fields is generic (with respect to the weak topology) if the subset of vector fields in $C^1_W(M, TM)$ that possess this property contains a Baire set.

An equilibrium point $x_e \in M$ is a singularity of the vector field, i.e. $V(x_e) = 0$. A periodic orbit $\gamma \subset M$ is an integral curve of V such that there exists $T > 0$ such that each point of γ is a fixed point of ϕ_T. For each point $x \in \gamma$, there exists a codimension-one embedded submanifold transverse to the flow called a Poincaré cross section (S) and a neighborhood $\hat{U}$ of x in S such that the Poincaré first return map $\hat{\tau} : \hat{U} \to S$ is well-defined and C^1 [HS74, Page 281]. Define a critical element (X) in M to be either an equilibrium point or a periodic orbit of V.

A set $E \subset M$ is forward invariant if $\phi_t(E) \subset E$ for all $t > 0$. It is backward invariant if $\phi_t(E) \subset E$ for all $t < 0$, and invariant if it is both forward and backward invariant. Note that critical elements are invariant.

Let x_e be an equilibrium point. Then x_e is hyperbolic if $d(\phi_1)_{x_e}$ is a hyperbolic linear map, i.e. if it has no eigenvalues of modulus one. It is stable if every eigenvalue of $d(\phi_1)_{x_e}$ has modulus less than one. If γ is a periodic orbit then let $x \in \gamma$, S a cross section centered at x, $\hat{U}$ a neighborhood of x in S, and $\hat{\tau} : \hat{U} \to S$ the C^1 first return map. Then γ is hyperbolic if $d\hat{\tau}_x$ is a hyperbolic linear map.

If $X \subset M$ is a hyperbolic critical element then there exist a local stable manifold of X ($W^s_{\text{loc}}(X)$) and a local unstable manifold of X ($W^u_{\text{loc}}(X)$) [KH99, Chapter 6] such that $\phi_t(W^s_{\text{loc}}(X)) \subset W^s_{\text{loc}}(X)$ and $\phi_{-t}(W^u_{\text{loc}}(X)) \subset W^u_{\text{loc}}(X)$ for all $t > 0$. Furthermore, the local stable and unstable manifolds are chosen to be compact. The global stable manifold of X ($W^s(X)$) and unstable manifold of X ($W^u(X)$) are then defined as $W^s(X) = \bigcup_{t \leq 0} \phi_t(W^s_{\text{loc}}(X))$ and $W^u(X) = \bigcup_{t \geq 0} \phi_t(W^u_{\text{loc}}(X))$, respectively. By [KH99, Chapter 6], $W^s(X)$ consists of the set of $x \in M$ such that the forward time orbit of x converges to X, and $W^u(X)$ consists of the set of $x \in M$ such that the backward time orbit of x converges to X. Note that they are invariant under the flow. If X is a hyperbolic periodic orbit and S is a cross section of X with C^1 first return map $\hat{\tau}$, it is often convenient to consider $W^s_{\text{loc}}(X) \cap S$ and $W^u_{\text{loc}}(X) \cap S$. Then $\hat{\tau}(W^s_{\text{loc}}(X) \cap S) \subset W^s_{\text{loc}}(X) \cap S$ and $\hat{\tau}^{-1}(W^u_{\text{loc}}(X) \cap S) \subset W^u_{\text{loc}}(X) \cap S$.

Hence, we abuse notation and let $W^s_{\text{loc}}(X)$ refer either to $W^s_{\text{loc}}(X)$ as defined above or to $W^s_{\text{loc}}(X) \cap S$ for some cross section S of X. The distinction should be clear from context. Define the notation $W^s_{\text{loc}}(X)$ analogously.

Let X be a hyperbolic critical element with $(W^s(X) - X) \cap (W^u(X) - X) \neq \emptyset$. Then the orbit of each $x \in (W^s(X) - X) \cap (W^u(X) - X)$ is called a homoclinic orbit. If, in addition, $W^s(X)$ and $W^u(X)$ have nonempty, transversal intersection, then the orbit of each $x \in (W^s(X) - X) \cap (W^u(X) - X)$ is called a transverse homoclinic orbit. Let X, Y be hyperbolic critical elements with $(W^s(X) - X) \cap (W^u(Y) - Y) \neq \emptyset$. Then the orbit of each $x \in (W^s(X) - X) \cap (W^u(Y) - Y)$ is called a heteroclinic orbit, and it is called a transverse heteroclinic orbit if the intersection is transverse. Let $X^1, ..., X^n$ be a finite set of hyperbolic critical elements with $X^n = X^1$. If $(W^s(X^i) - X^i) \cap (W^u(X^{i+1}) - X^{i+1})$ is nonempty and transverse for each $i \in \{1, ..., n-1\}$, then we call $\{X^i\}_{i=1}^n$ a heteroclinic cycle. If $X^1, X^2, ...$ is a sequence of hyperbolic critical elements with $(W^s(X^i) - X^i) \cap (W^u(X^{i+1}) - X^{i+1})$ nonempty and transverse for all i, then we call $\{X^i\}_{i=1}^\infty$ a heteroclinic sequence.

Let X be a hyperbolic critical element. If X is an equilibrium point, let $B = W^u_{\text{loc}}(X)$, let D be a C^1 disk in M such that D has nonempty, transversal intersection with $W^s(X)$, and let $\hat{f} = \phi_1$ be the time-one flow. If X is a periodic orbit, let $B = W^u_{\text{loc}}(X) \cap S$, where S is a cross section of X, let D be a C^1 disk in S such that D has nonempty, transversal intersection in S with $W^s(X) \cap S$, and let $\hat{f}$ be the C^1 first return map defined on an open subset of S. Suppose dim $D \geq$ dim B. Let $q \in D \cap W^s(X)$. Then the Inclination Lemma, otherwise known as the Lambda Lemma, states [Pal69] that for every $\epsilon > 0$ there exists $n_0 > 0$ such that $n \geq n_0$ implies a submanifold of $\hat{f}^n(D)$ containing $\hat{f}^n(q)$ is ϵ C^1-close to B. For convenience, we often omit the submanifold qualifier and implicitly redefine (shrink) D so that $\hat{f}^n(D)$ itself is ϵ C^1-close to B.

Let V be a C^1 vector field on a Riemannian manifold M with corresponding flow ϕ. A point $x \in M$ is nonwandering for V if for every open neighborhood U of x and every $T > 0$, there exists $t > T$ such that $\phi_t(U) \cap U \neq \emptyset$. Let the nonwandering set under V ($\Omega(V)$) be the set of nonwandering points for V in M where, for brevity, we often omit V and write Ω if the vector field is understood from context. If $y \in M$ and V is a vector field on M, define the ω-limit set of y under V ($\omega(y, V)$) to be the set of points $x \in M$ such that there exists a sequence $t_i \to \infty$ with $\phi_{t_i}(y) \to x$. Let $\omega(V) = \bigcup_{y \in M} \omega(y, V)$. For convenience, we often suppress the dependence on the vector field V when it is clear from context, and write $\omega(x)$ instead. If $\gamma \subset M$ is an orbit, define its ω-limit set to be the ω-limit set of any $y \in \gamma$, and note that this is well-defined because all points on an orbit share the same ω-limit set.

Define the α-limit set of an orbit analogously, for $t_i \to -\infty$. Write $\omega(\gamma)$ and $\alpha(\gamma)$ for the ω-limit set and α-limit set, respectively, of the orbit γ. V is a Morse-Smale vector field if it satisfies:

1. $\Omega(V)$ is a finite union of critical elements.

2. Every critical element is hyperbolic.

3. The stable and unstable manifolds of each individual and all pairs of critical elements have transversal intersection.

By [Sma62, Kup63], Assumptions 2 and 3 are generic, whereas Assumption 1 is not. Note that Morse-Smale vector fields were defined for compact Riemannian manifolds [Sma60]. We will see below that an additional assumption is necessary for Euclidean space.

Define a connected smooth manifold which represents parameter values (J) and fix $p_0 \in J$. Let $J_r = \{p \in J : |p - p_0| < r\}$ and $J_{\bar{r}} = \{p \in J : |p - p_0| \leq r\}$. For $Q \subset J$, let $M_Q = M \times Q$ and let $M_p = M_{\{p\}}$. Let $\{V_p\}_{p \in J}$ be a C^1 continuous family of vector fields on M. Suppose X_{p_0} is a hyperbolic critical element of V_{p_0} for some $p_0 \in J$. Then for J sufficiently small, $p \in J$ implies that there exists a unique hyperbolic critical element corresponding to parameter value p (X_p) of V_p which is C^1-close to X_{p_0}. This defines the family $\{X_p\}_{p \in J}$ of a critical element of the vector fields $\{V_p\}_{p \in J}$. To avoid ambiguity, we reserve the phrase "family of a critical element" to refer to the family obtained from a single critical element as the parameter value p varies over J. In particular, this implies that for each fixed parameter value p, the family of a critical element will possess exactly one critical element of V_p. Throughout the paper, for a fixed parameter value $p \in J$, it will sometimes be convenient to think of a critical element X_p as being a subset of M, and sometimes as a subset of $M \times J$. Therefore, we abuse notation and let X_p denote a critical element of V_p, where sometimes we consider $X_p \subset M$ and sometimes we consider $X_p \subset M \times \{p\} \subset M \times J$. The distinction should be clear from context. For $Q \subset J$, we write $\bigcup_{p \in Q} X_p$ (X_Q) $\subset M \times J$. We write $\sqcup_{p \in Q} W^s(X_p)$ ($W^s(X_Q)$) $\subset M \times J$, and $\sqcup_{p \in Q} W^u(X_p)$ ($W^u(X_Q)$) $\subset M \times J$.

2.2.3 Motivating Example

Example 2.2.1 (Lack of Hausdorff Continuity of Boundaries)**.** We show that every smooth manifold M possesses a family of smooth vector fields which is continuous with respect to the strong C^∞ topology and such that the vector fields have a family of stable equilibria whose boundaries of their regions of attraction are not Hausdorff continuous. As the strong

C^∞ topology is the most restrictive of the standard C^r topologies, this implies that even such a high degree of regularity is not sufficient to prevent a lack of Hausdorff continuity of the boundaries. We define the family of vector fields such that they are supported within a single chart, and then extend them trivially to the entire manifold M by declaring them to be zero outside this chart. So, it suffices to consider $M = \mathbb{R}^n$. Let $\mathcal{S}_{(a,b)} : [0, \infty) \to [0, 1]$ be a smooth bump function with $\mathcal{S}^{-1}_{(a,b)}(1) = [0, a]$ and $\mathcal{S}^{-1}_{(a,b)}(0) = [b, \infty)$. Let $p \in \mathbb{R}$, e_1 denote the first standard basis vector, and define, for $x \in \mathbb{R}^n$,

$$V_p(x) = -x\mathcal{S}_{(0,1.5)}(|x|) + p\mathcal{S}_{(2,3)}(|x|)e_1.$$

Then $\{V_p\}_{p\in(-0.2,0.2)}$ is continuous with respect to the weak C^1 topology. In fact, $\{V_p\}_{p\in(-0.2,0.2)}$ is also continuous with respect to the strong C^∞ topology, the most restrictive of the standard smooth vector field topologies, so $\{V_p\}_{p\in(-0.2,0.2)}$ varies as smoothly as might be desired. Furthermore, for $p \in (-0.2, 0.2)$, the vector field V_p is unchanged outside a fixed compact set. Nevertheless, despite the smoothness of $\{V_p\}_{p\in(-0.2,0.2)}$ and the fact that variations are restricted to a fixed compact set, this family of vector fields exhibits discontinuity in the boundaries of the regions of attraction of a family of stable equilibria.

For each p, V_p has a stable equilibrium point near the origin, call them $\{X^s(p)\}_{p\in(-0.2,0.2)}$. The case of $n = 1$ is illustrated in Fig. 2.2, which shows the vector field V_p for a few values of p. For $p = 0.1$, the vector field is positive for $x \in (-3, 0)$, driving initial conditions in this range towards $X^s_{0.1}$, and is negative for x greater than $X^s_{0.1}$ but less than about 1.1, driving these initial conditions towards $X^s_{0.1}$ as well. So, $W^s(X^s_{0.1}) \approx (-3, 1.1)$ consists of a line segment and $\partial W^s(X^s_{0.1}) \approx \{-3, 1.1\}$ consists of the two points on the boundary of the line segment. In fact, for any $p \in (0, 0.2)$, $W^s(X^s(p))$ will be a line segment that includes $(-3, 0)$, and $\partial W^s(X^s(p))$ will consist of the two points on its boundary, one of which is $\{-3\}$. For $p = -0.1$, the vector field is negative for $x \in (0, 3)$, driving initial conditions in this range towards $X^s_{-0.1}$, and is positive for x less than $X^s_{-0.1}$ but greater than about -1.1, driving these initial conditions towards $X^s_{-0.1}$ as well. So, $W^s(X^s_{-0.1}) \approx (-1.1, 3)$ consists of a line segment and $\partial W^s(X^s_{-0.1}) \approx \{-1.1, 3\}$ consists of the two points on the boundary of the line segment. In fact, for any $p \in (-0.2, 0)$, $W^s(X^s(p))$ will be a line segment that includes $(0, 3)$, and $\partial W^s(X^s(p))$ will consist of the two points on its boundary, one of which is $\{3\}$. Now consider the case where $p = 0$. By analogous reasoning to the above, based on the sign of the vector field, $W^s(X^s_0) = (-1.5, 1.5)$. So, $\partial W^s(X^s_0) = \{-1.5, 1.5\}$. But, we saw that as p approaches zero from above, $\partial W^s(X^s(p))$ contains the point $\{-3\}$, and as p approaches zero from below, $\partial W^s(X^s(p))$ contains the point $\{3\}$, neither of which are contained in $\partial W^s(X^s_0) = \{-1.5, 1.5\}$. Hence, the family $\{\partial W^s(X^s(p))\}_{p\in(-0.2,0.2)}$ is Hausdorff discontinuous at $p = 0$ from both above and below.

Fig. 2.3 illustrates this more clearly, by showing $W^s(X^s(p))$ and $\partial W^s(X^s(p))$ for $p \in (-0.1, 0.1)$, as well as $\partial W^s(X^s_{(-0.1,0.1)})$. Let $M = \mathbb{R}$ and let $J = (-0.1, 0.1)$. Then for $p \in J$ with $p \neq 0$, Fig. 2.3 plots $W^s(X^s(p)) \subset M \times \{p\} \subset M \times J$ in blue. And at $p = 0$, $W^s(X^s_0) \subset M \times \{0\}$ is shown in red. For $p > 0$, $W^s(X^s(p))$ includes $(-3, 0)$, so $\partial W^s(X^s(p))$ contains $\{-3\}$. For $p < 0$, $W^s(X^s(p))$ includes $(0, 3)$, so $\partial W^s(X^s(p))$ contains $\{3\}$. However, for $p = 0$, $W^s(X^s_0) = (-1.5, 1.5)$, so $\partial W^s(X^s_0) = \{-1.5, 1.5\}$, which does not contain $\{3\}$ or $\{-3\}$. Thus, as discussed above, $\{\partial W^s(X^s(p))\}_{p\in J}$ is Hausdorff discontinuous at $p = 0$ from both above and below. Now consider $\partial W^s(X^s_J)$. First note that $\partial W^s(X^s_J)$ contains $\sqcup_{p\in J}\partial W^s(X^s(p))$, so for each $p \in J$ it contains the two points of $\partial W^s(X^s(p))$. However, as $\partial W^s(X^s_J)$ is obtained by taking the topological boundary of $W^s(X^s_J)$ in $M\times J$, it also contains the cyan line segments shown at $p = 0$, which are $[-3, -1.5]$ and $[1.5, 3]$. Hence, $\partial W^s(X^s_J) \bigcap (M \times \{0\})$ contains the line segments $[-3, -1.5]$ and $[1.5, 3]$, whereas $\partial W^s(X^s_0) = \{-1.5, 1.5\}$ consists only of two points. In particular, $\partial W^s(X^s_J)$ is strictly larger than $\sqcup_{p\in J}\partial W^s(X^s(p))$. For a large class of families of C^1 vector fields, Theorem 2.2.23 shows that $\partial W^s(X^s_J) = \sqcup_{p\in J}\partial W^s(X^s(p))$, and Corollaries 2.2.24-2.2.25 show that $\{\partial W^s(X^s(p))\}_{p\in J}$ varies Chabauty or Hausdorff continuously, respectively.

From a practical perspective, we consider an initial condition y_p which is a C^1 function of parameter p and represents the system state after a finite time, parameter-dependent disturbance. In order to provide theoretical motivation for the algorithms which find the recovery boundary by computing critical parameter values, it is essential that there exists a critical parameter value p^* such that $y_{p^*} \in \partial W^s(X^s_{p^*})$. Suppose for some values of p that $y_p \in W^s(X^s_J)$, so the system recovers from the disturbance, and for other values of p that $y_p \notin W^s(X^s_J)$, so the system does not recover from the disturbance. Then since y_p is continuous in p, y_J is connected, so there must exist at least one p^* such that $y_{p^*} \in \partial W^s(X^s_J)$. However, as this example shows, $y_{p^*} \in \partial W^s(X^s_J)$ does not necessarily imply that $y_{p^*} \in \partial W^s(X^s_{p^*})$ as is required for the proof of Theorem 2.5.9. In particular, Fig. 2.3 shows two families of initial conditions y_J: a yellow family of initial conditions which does pass through $\partial W^s(X^s_{p^*})$ for some parameter value p^*, and a green family of initial conditions which does not pass through $\partial W^s(X^s(p))$ for any $p \in J$ but passes through $\partial W^s(X^s_J)$ through one of the cyan line segments. Hence, this example shows that the conclusions of Theorem 2.2.23 are not always satisfied and, if not, then the conclusions of Theorem 2.5.9 may not hold either.

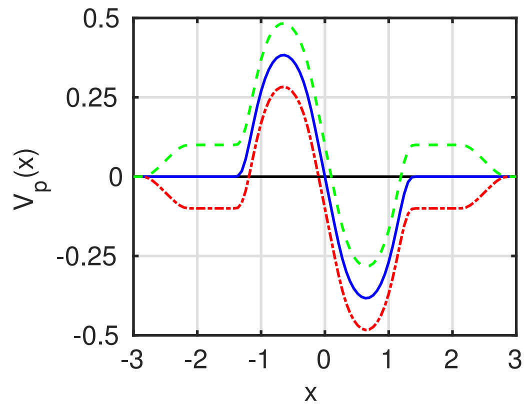

Figure 2.2: The graph of V_p for $p = 0.1$ (green dashed), $p = 0$ (blue solid), and $p = -0.1$ (red dot-dashed). This figure originally appeared in [FH17].

The discussion above generalizes to arbitrary dimension n. Example 2.3.3 of Section 2.3 shows that the assumptions of this paper are tight in a reasonable sense by exhibiting a strong C^1 continuous family of Morse-Smale vector fields on $\mathbb{R}^2$ that has a new equilibrium point entering the boundary of the RoA for p arbitarily close to p_0, and the boundary is Chabauty discontinuous at p_0.

2.2.4 Main Results

2.2.4.1 Vector Field is Parameter Independent

The primary motivation for presenting the results of this Section for parameter independent vector fields is to provide a foundation for, and to improve the clarity of presentation of, the results for parameter dependent vector fields in Section 2.2.4.2. However, the main result here (see Theorem 2.2.12) may also be of some independent interest as it provides a complete proof for parameter independent vector fields of a result for which earlier proofs [CHW88] are incomplete.

Let V be a complete C^1 vector field on M, where M is either a compact Riemannian manifold or $\mathbb{R}^n$. Let X^s be a stable equilibrium point of V. We make the following assumptions.

Assumption 2.2.2. *There exists a neighborhood N of $\partial W^s(X^s)$ such that $\Omega(V) \cap N$ consists of a finite union of critical elements; call them $\{X^i\}_{i \in I}$ where $I = \{1, ..., k\}$.*

Assumption 2.2.3. *For every $x \in \partial W^s(X^s)$, the forward orbit of x under V is bounded.*

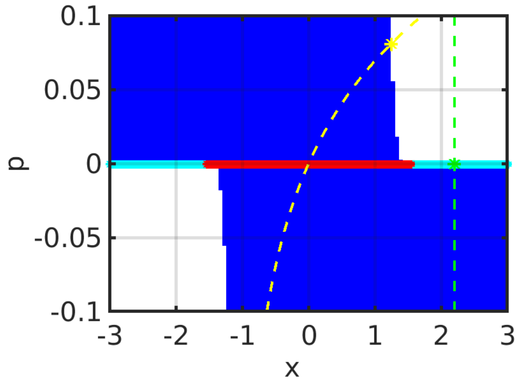

Figure 2.3: The disjoint union of the regions of attraction of the family of stable equilibria of the vector fields $\{V_p\}$ over $p \in [-0.1, 0.1]$ (blue). The region of attraction of the stable equilibrium point of V_0 is shown in red. Then $\partial W^s(X_0^s)$ consists of the two points on the boundary of this red line segment, while $\partial W^s(X_{\bar{J}}^s) \bigcap (\mathbb{R} \times \{0\})$ is equal to the union of the cyan line segments. One family of initial conditions (yellow) begins inside the regions of attraction and passes through one of their boundaries at p is increased. Another family of initial conditions (green) begins inside the regions of attraction and passes outside without passing through one of their boundaries. This occurs because the boundaries of the regions of attraction fail to be Hausdorff continuous at $p = 0$. This figure originally appeared in [FH17].

Assumption 2.2.4. *Every critical element in $\partial W^s(X^s)$ is hyperbolic.*

Assumption 2.2.5. *For each pair of critical elements in $\partial W^s(X^s)$, say X^i and X^j, $W^s(X^i)$ and $W^u(X^j)$ are transversal.*

Remark 2.2.6. Assumptions 2.2.2,2.2.4, and 2.2.5 ensure that V is Morse-Smale along $\partial W^s(X^s)$.

Remark 2.2.7. Assumption 2.2.3 is necessary in the case $M = \mathbb{R}^n$ since Morse-Smale vector fields were defined on compact manifolds [Sma60], whereas for $M = \mathbb{R}^n$ it is necessary to prohibit orbits in $\partial W^s(X^s)$ from diverging to infinity in forwards time.

Remark 2.2.8. By the Kupka-Smale Theorem for M compact [Sma60, Kup63], and its generalization for M σ-compact [KH99, Page 294], Assumptions 2.2.4 and 2.2.5 are generic with respect to the weak C^1 topology.

Remark 2.2.9. By Remark 2.3.7 and Lemma 2.3.8 from Section 2.3, Assumption 2.2.2 can be relaxed to the assumption that there exists a neighborhood of $\partial W^s(X^s)$ in which the number of equilibrium points and periodic orbits is finite, together with an additional assumption that is generic with respect to the strong C^1 topology.

Remark 2.2.10. By Assumption 2.2.4, hyperbolicity of the critical elements implies that their stable and unstable manifolds exist.

Remark 2.2.11. Assumptions 2.2.2-2.2.3 together imply that for any orbit $\gamma \subset \partial W^s(X^s)$, $\omega(\gamma) = X^i$ for some $i \in I = \{1, ..., k\}$.

Theorem 2.2.12 gives a decomposition of the boundary of the region of attraction for a parameter independent vector field as a union of the stable manifolds of the critical elements it contains.

Theorem 2.2.12. *Let M be either a compact Riemannian manifold or Euclidean space, and suppose V is a C^1 vector field on M satisfying Assumptions 2.2.2-2.2.5 Let $\{X^i\}_{i \in I}$ be the critical elements contained in $\partial W^s(X^s)$. Then $\partial W^s(X^s) = \bigcup_{i \in I} W^s(X^i)$.*

Remark 2.2.13. Theorem 2.2.12 was originally reported in [CHW88, Theorem 4-2] under slightly more general assumptions. Namely, our Assumption 2.2.2 was replaced by the assumption that for every $x \in \partial W^s(X^s)$, the trajectory of x converges to a critical element in forwards time. Hence, the number of critical elements in $\partial W^s(X^s)$ was not assumed to be finite, and the set of ω limit points in $\partial W^s(X^s)$, rather than the nonwandering set on a neighborhood of $\partial W^s(X^s)$, was assumed to consist solely of critical elements (in general the nonwandering set may be larger than the closure of the set of ω limit points). The main purpose for presenting Theorem 2.2.12 under these weaker assumptions is that its treatment more closely parallels the results and proofs of Theorem 2.2.23 for the case of parameter dependent vector fields below. For example, a finite number of critical elements is necessary to ensure that all critical elements persist under small perturbations to the vector field. However, the proof of [CHW88, Theorem 4-2] relies on [CHW88, Lemma 3-5], which has been disproven (see Section 2.1).

Therefore, the proof of [CHW88, Theorem 4-2] is incomplete, so the proof of Theorem 2.2.12 presented here represents the first complete proof of this result, and therefore may be of independent interest.

2.2.4.2 Vector Field is Parameter Dependent

Next we generalize the above results to the case where the vector field is parameter dependent. Let J be a connected smooth manifold representing a family of parameters, and let $\{V_p\}_{p \in J}$ be a weak C^1 continuous family of complete C^1 vector fields on M. Let V be the complete C^1 vector field on $M \times J$ defined by $V(x, p) = (V_p(x), 0) \subset T_x M \times T_p J$. Let ϕ

be the C^1 flow of V, ϕ^p be the flow of V_p, and note that $\phi(t,x,p) = (\phi^p(t,x),p)$ for $x \in M$ and $t \in \mathbb{R}$. For fixed t, we often write $\phi_t : M \times J \to M \times J$ by $\phi_t(x,p) = \phi(t,x,p)$ and note that ϕ_t is a C^1 diffeomorphism for each t.

Let $\{X^s(p)\}_{p\in J}$ be a C^1 continuous family of stable equilibria of the vector fields $\{V_p\}_{p\in J}$. Let $X^s_J = \bigcup_{p\in J} X^s(p)$ and let $W^s(X^s_J) = \sqcup_{p\in J} W^s(X^s(p))$. In this setting, there are potentially two different boundaries of regions of attraction to consider. First, for a fixed parameter value $p_0 \in J$ we have $\partial W^s(X^s_{p_0})$, where the topological boundary operation is taken in M. Second, we have $\partial W^s(X^s_J)$, where the topological boundary operation is taken in $M \times J$. In general, $\sqcup_{p\in J} \partial W^s(X^s_{p_0}) \subset \partial W^s(X^s_J)$. Therefore, we make assumptions regarding the behavior of V along $\partial W^s(X^s_J)$ rather than along $\sqcup_{p\in J} \partial W^s(X^s_{p_0})$. For some fixed $p_0 \in J$ we make the following assumptions.

Assumption 2.2.14. *There exists a neighborhood N of $\partial W^s(X^s_J) \cap M_{p_0}$ in M_{p_0} such that $\Omega(V) \bigcap N$ consists of a finite union of critical elements of V_{p_0}; call them $\{X^i_{p_0}\}_{i\in I}$ where $I = \{1,...,k\}$ and $k \geq 1$.*

Assumption 2.2.15. *Every critical element in $\partial W^s(X^s_J) \cap M_{p_0}$ is hyperbolic in M with respect to V_{p_0}.*

Remark 2.2.16. By Assumption 2.2.15, the critical elements $\{X^i_{p_0}\}_{i\in I}$ in $\pi_M\left(\partial W^s(X^s_J) \cap M_{p_0}\right)$ are hyperbolic so, since I is finite, they and their stable and unstable manifolds persist for J sufficiently small. For $p \in J$, define the perturbation of $X^s(p_0)$ corresponding to parameter value p ($X^s(p)$). For $p \in J$ and $i \in I$, define the perturbation of $X^i(p_0)$ corresponding to parameter value p ($X^i(p)$). In M, define the stable manifold of $X^s(p)$ ($W^s(X^s(p))$), and for $i \in I$ define the stable manifold of $X^i(p)$ ($W^s(X^i(p))$) and the unstable manifold of $X^i(p)$ ($W^u(X^i(p))$). In $M \times J$, define $\sqcup_{p\in J} X^s(p)$ (X^s_J), $\sqcup_{p\in J} W^s(X^s(p))$ ($W^s(X^s_J)$), and for $i \in I$ define $\sqcup_{p\in J} X^i(p)$ (X^i_J), $\sqcup_{p\in J} W^s(X^i(p))$ ($W^s(X^i_J)$), and $\sqcup_{p\in J} W^u(X^i(p))$ ($W^u(X^i_J)$)

Assumption 2.2.17. *For each $p \in J$, $\Omega(V_p) \bigcap (\partial W^s(X^s_J) \cap M_p) = \bigcup_{i\in I} X^i(p)$ and for every $x \in \partial W^s(X^s_J) \cap M_p$ its forward orbit under V_p is bounded.*

Assumption 2.2.18. *For each pair of critical elements that are contained in $\partial W^s(X^s_J) \cap M_{p_0}$, say $X^i_{p_0}$ and $X^j_{p_0}$, $W^s(X^i_{p_0})$ and $W^u(X^j_{p_0})$ are transversal in M.*

Remark 2.2.19. Assumptions 2.2.14,2.2.15, and 2.2.18 are straightforward generalizations of Assumptions 2.2.2,2.2.4, and 2.2.5. They ensure that V_{p_0} is Morse-Smale along $\pi_M\left(\partial W^s(X^s_J) \cap M_{p_0}\right)$.

Remark 2.2.20. Assumption 2.2.17 generalizes Assumption 2.2.3 by ensuring that, for every $p \in J$, every orbit in $\partial W^s(X_J^s) \cap M_p$ converges to $X^i(p)$ for some $i \in I$. So, no new critical elements can enter $\partial W^s(X_J^s) \cap M_p$ for $p \in J$, $p \neq p_0$.

Remark 2.2.21. Using Remark 2.3.7 and Lemma 2.3.8 from Section 2.3, Assumption 2.2.14 can be partially relaxed as in Remark 2.2.9.

Remark 2.2.22. In the case of M a compact Riemannian manifold, by Theorem 2.3.9 in Section 2.3, Assumption 2.2.17 is not necessary. In the case of M Euclidean, by Theorem 2.3.20 in Section 2.3, Assumption 2.2.17 can be relaxed, for $\{V_p\}_{p\in J}$ a strong C^1 continuous family of vector fields, to the assumptions that for every $x \in \partial W^s(X_{p_0}^s)$ the forward orbit of x is bounded, and there exists a neighborhood N of infinity such that $\Omega(V_{p_0}) \bigcap N = \emptyset$ and no orbit under V_{p_0} entirely contained in N has empty ω and α limit sets under V_{p_0}.

Theorem 2.2.23 gives a decomposition of $\partial W^s(X_J^s)$ as a disjoint union over parameter values in J of a union of the stable manifolds of its critical elements. Furthermore, it shows that the topological boundary in $M \times J$, $\partial W^s(X_J^s)$, is equal to the disjoint union over $p \in J$ of the topological boundaries in M of the stable manifolds of the stable equilibria. Using Theorem 2.2.23, it is straightforward to then show that $\{\partial W^s(X^s(p))\}_{p\in J}$ is a continuous family of subsets of M (Corollary 2.2.24). Hence, if M is a compact Riemannian manifold, this implies that $\{\partial W^s(X^s(p))\}_{p\in J}$ is a Hausdorff continuous family of subsets of M (Corollary 2.2.25). Finally, if V_{p_0} is Morse-Smale on M a compact Riemannian manifold, using persistence of the so-called phase diagram of Morse-Smale vector fields under perturbation [Pal69], one can show that for any C^1 continuous family of vector fields containing V_{p_0}, and for J sufficiently small, $\{\partial W^s(X^s(p))\}_{p\in J}$ is a Hausdorff continuous family of subsets of M (Corollary 2.2.26).

Theorem 2.2.23. *Let M be either a compact Riemannian manifold or Euclidean space, and let $\{V_p\}_{p\in J}$ be a family of vector fields on M continuous with respect to the weak C^1 topology and satisfying Assumptions 2.2.14-2.2.18. Let $\{X_{p_0}^i\}_{i\in I}$ denote the critical elements of V_{p_0} in $\partial W^s(X_J^s) \cap M_{p_0}$. Then in $M \times J$ for sufficiently small J, $\partial W^s(X_J^s) = \sqcup_{p\in J} \partial W^s(X^s(p)) = \bigcup_{i\in I} W^s(X_J^i)$.*

Corollary 2.2.24. *Let $M = \mathbb{R}^n$ and let $\{V_p\}_{p\in J}$ be a weak C^1 continuous family of vector fields on M satisfying Assumptions 2.2.14-2.2.18. Then $\{\partial W^s(X^s(p))\}_{p\in J}$ is a Chabauty continuous family of subsets of M.*

Corollary 2.2.25. *Let M be a compact Riemannian manifold and let $\{V_p\}_{p\in J}$ be a C^1 continuous family of vector fields on M satisfying Assumptions 2.2.14-2.2.18. Then $\{\partial W^s(X^s(p))\}_{p\in J}$ is a Hausdorff continuous family of subsets of M.*

Corollary 2.2.26. *Let M be a compact Riemannian manifold and let V_{p_0} be a Morse-Smale vector field on M. Then for any C^1 continuous family of vector fields $\{V_p\}_{p\in J}$ on M with $p_0 \in J$, for sufficiently small J, $\{\partial W^s(X^s(p))\}_{p\in J}$ is a Hausdorff continuous family of subsets of M.*

2.2.4.3 Illustrative Example

Example 2.2.27 (Illustration of Main Theorems)**.** To illustrate the results of Theorems 2.2.12, 2.2.23, 2.5.9 and Corollary 2.2.24 we consider the simple example of a damped, driven nonlinear pendulum with constant driving force. The dynamics are given by

$$\dot{x}_1 = x_2 \tag{2.8}$$

$$\dot{x}_2 = -c_1 \sin(x_1) - c_2 x_2 + c_3 \tag{2.9}$$

where $c_1, c_2, c_3 > 0$ are real parameters and $x = (x_1, x_2) \subset \mathbb{R}^2$. Physically, x_1 represents the angle of the pendulum, x_2 its angular velocity, c_1 the natural frequency of the pendulum, c_2 a damping coefficient due to air drag, and c_3 the constant driving torque. Eqs. 2.8-2.9 can also be interpreted as an electrical generator with (x_1, x_2) the angle and angular velocity of the turbine, c_1 a constant determining the electrical torque supplied by the generator, c_2 a damping coefficient due to friction, and c_3 the constant driving mechanical torque. For the demonstration below, we set $c = (c_1, c_2, c_3) = (2, 0.5, 1.5)$ and we restrict x_1 to a single interval of length 2π since x_1 is defined modulo 2π. Although c_3 is initially given the fixed value of 1.5, we let $p = c_3$ and will subsequently treat it as a free parameter, setting $p_0 = 1.5$. At p_0, this system possesses one stable equilibrium point, $X^s_{p_0}$, at $(0.848, 0)$, one unstable equilibrium point, $X^1_{p_0}$, at $(2.294, 0)$, and no other nonwandering elements. Variation of the value of p over a range J that contains p_0 then generates a C^1 continuous family of vector fields, as well as families of equilibria $\{X^s(p)\}_{p\in J}$ and $\{X^1_p\}_{p\in J}$.

We define an initial condition to Eqs. 2.8-2.9 as the output of the following related system.

$$\dot{z}_1 = z_2 \tag{2.10}$$

$$\dot{z}_2 = -c_2 z_2 + c_3 \tag{2.11}$$

Let ϕ_d denote the flow of Eqs. 2.10-2.11 and let $J = (1.3, 2)$. Then the initial condition of Eqs. 2.8-2.9 is given by $y : J \to \mathbb{R}^2$ by $y_p = \phi_d(c_4, X^s(p), p)$, where we set $c_4 = 0.8$ to be the length of time the disturbance is active. If Eqs. 2.8-2.9 are interpreted as an electrical generator, then Eqs. 2.10-2.11 represent a short circuit across the terminals of the generator so it no longer supplies any electrical torque. This is modeled by setting c_1 to zero in Eqs. 2.8-2.9, which then gives Eqs. 2.10-2.11.

Fig. 2.4 shows $\partial W^s(X^s_{p_0})$. Note that the intersection of $\partial W^s(X^s_{p_0})$ with the nonwandering set is X^1, every orbit $\gamma \subset \partial W^s(X^s_{p_0})$ has $\omega(\gamma) = X^1_{p_0}$, $X^1_{p_0}$ is hyperbolic, and the transversality assumption is vacuously true since $X^1_{p_0}$ is the only critical element in $\partial W^s(X^s_{p_0})$. Therefore, the system satisfies Assumptions 2.2.2-2.2.5, so by Theorem 2.2.12 we must have $\partial W^s(X^s_{p_0}) = W^s(X^1_{p_0})$, as can be seen in Fig. 2.4.

Fig. 2.5 shows the boundaries of the regions of attraction of the family of vector fields for several values of the parameter $p = c_3$. For $p > 2$ the stable and unstable equilibria $X^s(p)$ and X^1_p collide in a saddle-node bifurcation and annihilate each other, so we must restrict to sufficiently small $J = (1.3, 2)$, as in Theorem 2.2.23. Fix $p_0 = 1.5$ as above. Then the intersection of the nonwandering set with $\partial W^s(X^s_J) \cap M_{p_0}$ is $X^1_{p_0}$, for every orbit $\gamma \subset \partial W^s(X^s_J)$ we have $\omega(\gamma) \subset X^1_J$, $X^1_{p_0}$ is hyperbolic, and the transversality condition for $\partial W^s(X^s_J) \cap M_{p_0}$ is vacuously satisfied since the only critical element in $\partial W^s(X^s_J) \cap M_{p_0}$ is $X^1_{p_0}$. Therefore, the system satisfies Assumptions 2.2.14-2.2.18, so by Theorem 2.2.23 we must have $\partial W^s(X^s_J) = \sqcup_{p \in J} \partial W^s(X^s(p)) = W^s(X^1_J)$, and by Corollary 2.2.24, $\{\partial W^s(X^s(p))\}_{p \in J}$ is a Chabauty continuous family of subsets of M.

2.2.5 Proof of Theorem 2.2.12

This Section is devoted to the proof of Theorem 2.2.12. Many of the results and proofs of this Section will be recycled for additional use for the parameter dependent vector field results and proofs in Section 2.2.6. Most of the lemmas presented in this Section are similar to results given elsewhere, especially for diffeomorphisms of compact Riemannian manifolds, but the presentation and proofs here are our own unless otherwise specified.

We let M be either a compact Riemannian manifold or Euclidean space unless stated

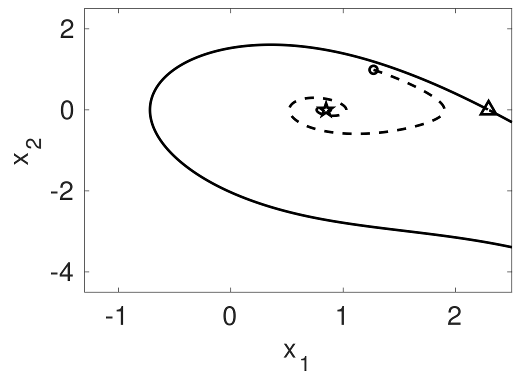

Figure 2.4: The region of attraction boundary $\partial W^s(X^s_{p0})$ (black line) of the stable equilibrium point X^s_{p0} (black star) of Eqs. 2.8-2.9 is shown. It is equal to $W^s(X^1_{p0})$ where X^1_{p0} (black triangle) is the unstable equilibrium point. The orbit (dashed black line) from the initial condition y_{p0} (black circle) is shown.

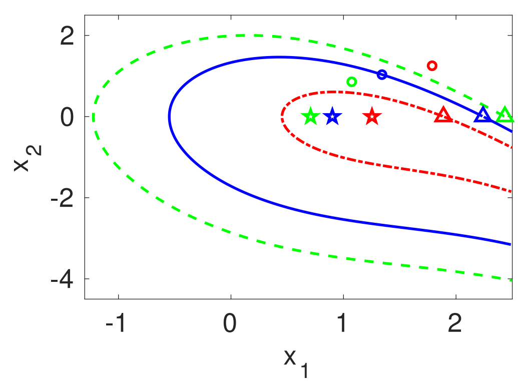

Figure 2.5: The region of attraction boundaries $\partial W^s(X^s(p))$ of the stable equilibrium points $X^s(p)$ (stars) for parameter values $p = 1.3$ (green dashed), $p = 1.568$ (solid blue), and $p = 1.9$ (red dot dashed) are shown. Each boundary is equal to $W^s(X^1_p)$ where X^1_p (triangle) is the unstable equilibrium point corresponding to parameter value p. The initial conditions (circles) are shown.

otherwise.

Since $W^s(X^s)$ is invariant, its topological closure $\overline{W^s}(X^s)$ is invariant. For, if $x \in \overline{W^s}(X^s)$ and $t \in \mathbb{R}$ then there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset W^s(X^s)$ such that $x_n \to x$. By invariance of $W^s(X^s)$, $\phi_t(x_n) \in W^s(X^s)$ for all n. By continuity of ϕ_t, $\phi_t(x_n) \to \phi_t(x)$, so $\phi_t(x) \in \overline{W^s}(X^s)$. Hence, $\overline{W^s}(X^s)$ and $W^s(X^s)$ are invariant, so $\partial W^s(X^s) = \overline{W^s}(X^s) - W^s(X^s)$ is invariant.

Let $\{X^i\}_{i \in I}$ denote the critical elements in $\partial W^s(X^s)$. Then $W^u_{\text{loc}}(X^i)$ and $W^s_{\text{loc}}(X^i)$ are well-defined local unstable and stable manifolds for X^i for all $i \in I$. Lemma 2.2.28 provides a technical construction, for any critical element, of a compact set contained in its unstable manifold such that for any sufficiently small neighborhood N of this compact set in M, the following holds. The union over time of the time-t flow ϕ_t of N over all negative times t, together with the stable manifold of the critical element, contains an open neighborhood of the critical element in M. This result will be instrumental in making the claim below that if a critical element is contained in $\partial W^s(X^s)$ then its unstable manifold intersects $\overline{W^s}(X^s)$. Lemma 2.2.28 is analogous to [Pal69, Corollary 1.2], which states the corresponding result for diffeomorphisms without proof, whereas here the result is shown for vector fields. Fig. 2.6 illustrates the content of Lemma 2.2.28. Recall that if D is a subset of a metric space and $\epsilon > 0$, the notation D_ϵ refers to the subset of the metric space such that for each $x \in D_\epsilon$ there exists $y \in D$ with $d(x, y) < \epsilon$.

Lemma 2.2.28. *For any $i \in I$ and any $\epsilon > 0$ there exists a compact set $D \subset W^u_{loc}(X^i) - X^i$ and an open neighborhood N of D in M disjoint from X^i such that $N \subset D_\epsilon$ and $\bigcup_{t \leq 0} \phi_t(N) \bigcup W^s(X^i)$ contains an open neighborhood of X^i in M.*

Proof of Lemma 2.2.28. If X^i is an equilibrium point let $\hat{f} = \phi_1$ be the time-one flow. If X^i is a periodic orbit, let $\hat{f} = \hat{\tau}$ be the first return map of a cross section S of X^i. Let $D' = W^u_{\text{loc}}(X^i)$ and let D be the topological closure of $D' - (\hat{f})^{-1}(D')$ in M. In order to show the existence of the desired open neighborhood of X^i, the first step will be constructing a C^1 continuous disk family centered along D and contained in an open neighborhood $N \subset D_\epsilon$. Then, this C^1 disk family is extended to a C^1 disk family centered along $W^u_{\text{loc}}(X^i)$ by backward iteration and the inclusion of the disk $W^s_{\text{loc}}(X^i)$. It is shown that this family is in fact C^1 continuous using the Inclination Lemma. Finally, once the C^1 continuous and backward invariant disk family has been constructed, invariance of domain [Hat01, Theorem 2B.3] is applied to conclude that the disk family contains an open neighborhood of X^i. By construction, this implies that $\bigcup_{t \leq 0} \phi_t(N) \bigcup W^s(X^i)$ contains an open neighborhood of X^i. The full proof is provided in Appendix A. □

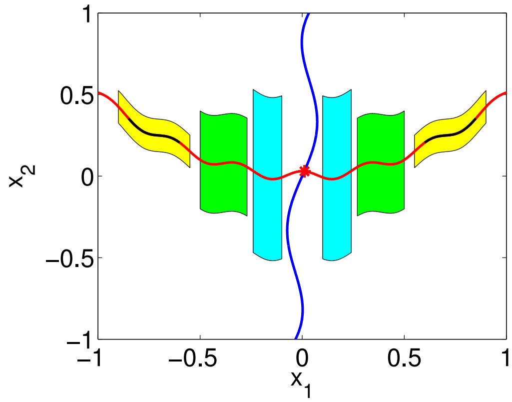

Figure 2.6: The compact set D (black line segments) and the neighborhood N (yellow shapes) mentioned in Lemma 2.2.28 for an equilibrium point (red star). The set D is contained in the unstable manifold (red line). As the neighborhood N is propagated backwards in time (first to the green shapes then to the cyan), it approaches the stable manifold (blue line) of the equilibrium point. From the figure, it appears that the union of the backward flows of N over all negative times, together with the stable manifold, will contain a neighborhood of the equilibrium point, which is the content of Lemma 2.2.28. This figure originally appeared in [FH17].

We will use the technical result of Lemma 2.2.28 to show that the unstable manifold of a critical element in the boundary of the region of attraction must have nonempty intersection with $\overline{W^s}(X^s)$. The following lemma is analogous to the combination of [CHW88, Theorem 3-3] (for equilibrium points in $\partial W^s(X^s)$) and [CHW88, Corollary 3-4] (for periodic orbits in $\partial W^s(X^s)$), although [CHW88, Corollary 3-4] was unproven. Our proof is similar to the proof of [CHW88, Theorem 3-3], although we have explicitly proved Lemma 2.2.28 whereas [CHW88] states a similar technical result without proof, and we also explicity prove the contents of [CHW88, Corollary 3-4].

Lemma 2.2.29. *If $X^i \subset \partial W^s(X^s)$ then $\{W^u(X^i) - X^i\} \bigcap \overline{W^s}(X^s) \neq \emptyset$.*

Proof of Lemma 2.2.29. Using Lemma 2.2.28 we will produce a neighborhood of X^i from its stable and unstable manifolds. Since X^i is in the topological boundary, this neighborhood must intersect $W^s(X^s)$. Then since stable manifolds cannot intersect, by invariance, and by sending ϵ in the statement of Lemma 2.2.28 to zero we will obtain the result.

Let $x \in X^i$ and let $\epsilon > 0$. By Lemma 2.2.28, there exists a compact set $D \subset W^u_{\text{loc}}(X^i) - X^i$ and an open neighborhood N of D in M disjoint from X^i such that $N \subset D_\epsilon$ and $\bigcup_{t \leq 0} \phi_t(N) \bigcup W^s(X^i)$ contains a neighborhood of X^i in M, call it U_ϵ. Then U_ϵ is a neighborhood of $x \in \partial W^s(X^s)$, so $U_\epsilon \cap W^s(X^s) \neq \emptyset$. Since $W^s(X^i) \cap W^s(X^s) = \emptyset$, there must exist some $T \leq 0$ such that $\phi_T(N) \cap W^s(X^s) \neq \emptyset$. Since $W^s(X^s)$ is invariant, this implies that $N \cap W^s(X^s) \neq \emptyset$. Since $N \subset D_\epsilon$, letting d_S be the set distance on the Riemannian manifold M, we have

$$\epsilon \geq d_S(D, W^s(X^s)) = d_S(D, \overline{W^s}(X^s))$$

holds for all $\epsilon > 0$, so $d_S(D, \overline{W^s}(X^s)) = 0$. Since D is compact and $\overline{W^s}(X^s)$ is closed, this implies that $D \bigcap \overline{W^s}(X^s) \neq \emptyset$. Hence, since $D \subset W^u_{\text{loc}}(X^i) - X^i$, $\{W^u(X^i) - X^i\} \bigcap \overline{W^s}(X^s) \neq \emptyset$. □

For X a critical element, let $n^t(X) = 0$ if X is an equilibrium point and let $n^t(X) = 1$ if X is a periodic orbit. Let $n^u(X) = \dim\, W^u(X) - n^t(X)$ and let $n^s(X) = \dim\, W^s(X) - n^t(X)$. Lemma 2.2.30 was proven in [Sma60, Lemma 3.1]. It is reproduced here for clarity of presentation. A slightly different result, that was reported in [CHW88, Lemma 3-5] and was fundamental in the proof of [CHW88, Theorem 4-2], has been disproven (see Section 2.1).

Lemma 2.2.30. *If $W^s(X^i) \bigcap W^u(X^j) \neq \emptyset$ then $n^u(X^i) \leq n^u(X^j) + n^t(X^j) - 1$ and $dim\ W^u(X^i) \leq dim\ W^u(X^j) + n^t(X^i) - 1$.*

Proof of Lemma 2.2.30. Since $W^s(X^i)$ and $W^u(X^j)$ have a point of transversal intersection and are invariant under the flow, they have an orbit δ of transversal intersection. Then $\dot{\delta}(0) \in T_{\delta(0)}W^s(X^i) \cap T_{\delta(0)}W^u(X^j)$. By transversality, this implies that $(\dim\, W^s(X^i) - 1) + (\dim\, W^u(X^j) - 1) \geq n - 1$. Hence, $(n^s(X^i) + n^t(X^i) - 1) + (n^u(X^j) + n^t(X^j) - 1) \geq n - 1$. Since $n^s(X^i) + n^t(X^i) + n^u(X^i) = n$, this implies that $(n - n^u(X^i) - 1) + (n^u(X^j) + n^t(X^j) - 1) \geq n - 1$ so $n^u(X^i) \leq n^u(X^j) + n^t(X^j) - 1$. Hence, $\dim\, W^u(X^i) = n^u(X^i) + n^t(X^i) \leq \dim\, W^u(X^j) + n^t(X^i) - 1$. □

As defined in Section 2.2.2, a heteroclinic sequence is a sequence of hyperbolic critical elements such that the stable manifold of each critical element intersects the unstable manifold of the next element of the sequence. A heteroclinic cycle is a finite heteroclinic sequence where the first and last critical elements are the same. Lemmas 2.2.31-2.2.33 show that Assumptions 2.2.2,2.2.4,and 2.2.5 imply that there are no heteroclinic cycles and, therefore, that all heteroclinic sequences are finite. These are analogous to several Lemmas in [Pal69] for diffeomorphisms, but are proved here for vector fields. Lemma 2.2.31 shows that the intersection of stable and unstable manifolds of critical elements satisfies the transitive property. It was shown in [Pal69, Corollary 1.3] for diffeomorphisms, and is proven here for vector fields.

Lemma 2.2.31. *If* $(W^s(X^i) - X^i) \bigcap (W^u(X^j) - X^j) \neq \emptyset$ *and* $(W^s(X^j) - X^j) \bigcap (W^u(X^k) - X^k) \neq \emptyset$ *then* $(W^s(X^i) - X^i) \bigcap (W^u(X^k) - X^k) \neq \emptyset$.

Proof of Lemma 2.2.31. The proof revolves around the openness of transversal intersection of compact submanifolds which are C^1 close, and the use of the Inclination Lemma to guarantee that the submanifolds are C^1 close.

If X^j is an equilibrium point, let $B = W^u_{\text{loc}}(X^j)$. If X^j is a periodic orbit, let $B = W^u_{\text{loc}}(X^j) \cap S$, where S is any cross section of X^j. By invariance of $W^s(X^i)$ and the assumptions of the Lemma, we have that $W^s(X^i) \bigcap B \neq \emptyset$. We claim that B is transverse to $W^s(X^i)$. This follows trivially if X^j is an equilibrium point, so suppose it is a periodic orbit. For any $x \in W^s(X^i) \bigcap W^u(X^j)$, $T_x W^s(X^i)$ and $T_x W^u(X^j)$ together span $T_x M$ since the intersection is transverse. Then B is obtained by intersecting $W^u_{\text{loc}}(X^j)$ with S, so $T_x W^u(X^j)$ is equal to the span of $T_x B$ and the flow direction $V(x)$. However, as $W^s(X^j)$ is invariant under V, $V(x) \in T_x W^s(X^j)$. Therefore, $T_x W^s(X^i)$ and $T_x B$ together have the same span as $T_x W^s(X^i)$ and $T_x W^u(X^j)$, which implies that $W^s(X^i)$ and B are transverse at x. As x was arbitrary, the claim follows.

By the definition of $W^s(X^i)$, there exists $T < 0$ such that $\phi_T W^s_{\text{loc}}(X^i) \bigcap W^u_{\text{loc}}(X^j) \neq \emptyset$. Note that B is a compact embedded submanifold, and that it is transverse to $\phi_T W^s_{\text{loc}}(X^i)$ since it is transverse to $W^s(X^i)$. Since $\phi_T W^s_{\text{loc}}(X^i)$ and B are compact submanifolds with transversal intersection, by [KH99, Corollary A.3.18] there exists $\epsilon > 0$ such that if D is a compact submanifold which is ϵ C^1-close to B then it has nonempty, transversal intersection with $\phi_T W^s_{\text{loc}}(X^i)$, and hence with $W^s(X^i)$.

Let $y \in (W^s(X^j) - X^j) \bigcap (W^u(X^k) - X^k)$. Since the intersection is transverse, if X^j is an equilibrium point there exists a compact submanifold $D \subset W^u(X^k)$, which we choose to be a C^1 disk centered at y for the purpose of applying the Inclination Lemma, such that D is transverse to $W^s(X^j)$. Similarly, if X^j is a periodic orbit, there exists a C^1 disk $D \subset W^u(X^k) \cap S$ centered at y such that D is transverse to $W^s(X^j) \cap S$ in S. By Lemma 2.2.30, $\dim\, W^u(X^k) \geq \dim\, W^u(X^j)$, so we may choose D such that $\dim\, D = \dim\, B$. Let $\hat{f} = \phi_1$ if X^j is an equilibrium point, and let $\hat{f}$ be a C^1 first return map on S if X^j is a periodic orbit. Then, by the Inclination Lemma for equilibria or periodic orbits, there exists $n_0 > 0$ such that $n \geq n_0$ implies $\hat{f}^n(D)$ is ϵ C^1-close to B. By the choice of ϵ, $\hat{f}^n(D) \cap W^s(X^i) \neq \emptyset$. Since $D \subset W^u(X^k)$ invariant, this implies that $W^s(X^i) \bigcap W^u(X^k) \neq \emptyset$. □

Lemma 2.2.32 shows that there are no homoclinic orbits in $\partial W^s(X^s)$. A similar claim was shown for diffeomorphisms in [Pal69, Corollary 1.4], but the result here is proven for vector fields.

Lemma 2.2.32. *For any X^i, $W^s(X^i) \bigcap W^u(X^i) = X^i$.*

Proof of Lemma 2.2.32. Using transversality and the Inclination Lemma we show that $W^s(X^i) \cap W^u(X^i)$ is nonwandering. By Assumption 2.2.2, this will imply that $W^s(X^i) \cap W^u(X^i) = X^i$.

Clearly $X^i \subset W^s(X^i) \bigcap W^u(X^i)$. Assume towards a contraction that $(W^s(X^i) - X^i) \bigcap (W^u(X^i) - X^i) \neq \emptyset$. If X^i is an equilibrium point, then by Lemma 2.2.30 this implies that $\dim\, W^u(X^i) \leq \dim\, W^u(X^i) - 1 < \dim\, W^u(X^i)$, which is a contradiction. So, suppose X^i is a periodic orbit, let S be a C^1 cross section of X^i, and let $B = W^u_{\text{loc}}(X^i) \bigcap S$. By an analogous argument as in the proof of Lemma 2.2.31, B is transverse to $W^s(X^i)$. By invariance of $W^s(X^i)$, there exists a point $q \in (W^s(X^i) - X^i) \bigcap (B - X^i)$. We claim that q is nonwandering. For, let U be any neighborhood of q in M. Let $\epsilon > 0$ such that the ball of radius ϵ centered at q is contained in U. As B is transverse to $W^s(X^i)$, let $D \subset B$ be a C^1 disk centered at q of the same dimension as B such that $D \subset U$ and D is transverse to $W^s(X^i)$. Note that D is transverse to $W^s(X^i) \cap S$ in S as well. Let $\hat{\tau}$ be a C^1 first return map on S. Then, by the Inclination Lemma there exists $n_0 > 0$ such that $n \geq n_0$ implies that $\tau^n(D)$ is ϵ C^1-close to B. As $q \in B$ and U contains the ball of radius ϵ centered at q, this implies that $\tau^n(D) \bigcap U \neq \emptyset$. Hence, as $D \subset U$, we have that $\tau^n(U) \bigcap U \neq \emptyset$ for $n \geq n_0$, so q is nonwandering.

By Assumption 2.2.2, there exists a neighborhood N of $\partial W^s(X^s)$ such that $\Omega(V) \cap N = \bigcup_{j \in I} X^j$. As $\omega(q) = \alpha(q) = X^i$ and $q \notin X^i$, $q \neq X^j$ for any $j \in I$. But, since $X^i \subset N$ open and $\omega(q) = X^i$, there exists $T > 0$ such that $\phi_T(q) \in N$. As the nonwandering set is invariant, $\phi_t(q)$ is nonwandering in N. As X^j is invariant for each $j \in I$, $\phi_t(q) \notin \bigcup_{j \in I} X^j$, which is a contradiction to the choice of N. □

Lemma 2.2.33 now shows that every heteroclinic sequence has finite length.

Lemma 2.2.33. *There do not exist any heteroclinic cycles. Hence, every heteroclinic sequence has finite length.*

Proof of Lemma 2.2.33. Assume towards a contradiction that $\{X^j\}_{j=1}^m$ is a heteroclinic cycle. By transitivity (Lemma 2.2.31), since $X^n = X^1$ this implies that $(W^s(X^1) - X^1) \cap (W^u(X^1) - X^1) \neq \emptyset$. This contradicts Lemma 2.2.32.

Since $\Omega(V) \cap \partial W^s(X^s)$ consists of a finite number of critical elements, and since there are no heteroclinic cycles, every heteroclinic sequence must be finite. □

Lemma 2.2.34 will be used to complete the proof of Lemma 2.2.35. It is analogous to [Sma67, Lemma 7.1.b.], but for vector fields instead of diffeomorphisms.

Lemma 2.2.34. *Suppose that* $W^u(X^i)\bigcap W^s(X^s)\neq\emptyset$ *and* $W^s(X^i)\bigcap W^u(X^j)\neq\emptyset$*. Then* $W^u(X^j)\bigcap W^s(X^s)\neq\emptyset$*.*

Proof of Lemma 2.2.34. The proof uses the fact that $W^s(X^s)$ is open and, by invariance, intersects $W^u_{\mathrm{loc}}(X^i)$, so any submanifold K which is C^1 close to $W^u_{\mathrm{loc}}(X^i)$ also intersects $W^s(X^s)$. The Inclination Lemma then guarantees that a disk in $W^u(X^j)$ is C^1 close to $W^u_{\mathrm{loc}}(X^i)$.

Since $W^s(X^s)$ is invariant and intersects $W^u(X^i)$, $W^s(X^s)\bigcap W^u_{\mathrm{loc}}(X^i)\neq\emptyset$. So, let $q\in W^s(X^s)\bigcap W^u_{\mathrm{loc}}(X^i)$. By the definition of $W^s(X^s)$, there exists $T<0$ such that $q\in W^u_{\mathrm{loc}}(X^i)\bigcap\phi_T W^s_{\mathrm{loc}}(X^s)$. If X^i is an equilibrium point, let $B=W^u_{\mathrm{loc}}(X^i)$. If X^i is a periodic orbit, let S be a cross section containing q and let $B=W^u_{\mathrm{loc}}(X^i)\bigcap S$. Then it can be shown that B is transverse to $\phi_T W^s_{\mathrm{loc}}(X^s)$ (in S if X^i is a periodic orbit) by an analogous argument as in the proof of Lemma 2.2.31. Since $\phi_T W^s_{\mathrm{loc}}(X^s)$ and B are compact submanifolds with transversal intersection, by [KH99, Proposition A.3.16,Corollary A.3.18] there exists $\epsilon>0$ such that if D' is a compact submanifold which is ϵ C^1-close to B then it has a point of transversal intersection with $\phi_T W^s_{\mathrm{loc}}(X^s)$, hence with $W^s(X^s)$.

Let $x\in W^s(X^i)\bigcap W^u(X^j)$. Since the intersection is transversal by Assumption 2.2.5, if X^i is an equilibrium point there exists a C^1 disk $D\subset W^u(X^j)$ centered at x with D transverse to $W^s(X^i)$. Similarly, if X^i is a periodic orbit there exists a disk $D\subset W^u(X^j)\bigcap S$ centered at x with D transverse to $W^s(X^i)\bigcap S$ in S. By Lemma 2.2.30, $\dim W^u(X^j)\geq\dim W^u(X^i)$, so we may choose D such that $\dim D=\dim B$. If X^i is an equilibrium point let $\hat f=\phi_1$, and if X^i is a periodic orbit let $\hat f$ be a C^1 first return map for S. Then, by the Inclination Lemma for equilibria or periodic orbits, there exists $n_0>0$ such that $n\geq n_0$ implies $\hat f^n(D)$ is ϵ C^1-close to B. By the choice of ϵ, $\hat f^n(D)\cap W^s(X^s)\neq\emptyset$. Since $D\subset W^u(X^j)$ invariant, this implies that $W^u(X^j)\bigcap W^s(X^s)\neq\emptyset$. ☐

Lemma 2.2.35 was reported as [CHW88, Theorem 3-8], where our Assumption 2.2.2 was replaced by the weaker assumption that for every $x\in\partial W^s(X^s)$, the trajectory of x converges to a critical element in forwards time. However, the proof of [CHW88, Theorem 3-8] relies crucially on [CHW88, Lemma 3-5], which has been disproven (see Section 2.1) to show that a particular heteroclinic sequence has finite length. In contrast, the proof of Lemma 2.2.35 shows that an analogous heteroclinic sequence has finite length using Lemma 2.2.33, which relies on Assumption 2.2.2.

Lemma 2.2.35. *If $X^i \subset \partial W^s(X^s)$ then $W^u(X^i) \bigcap W^s(X^s) \neq \emptyset$.*

Proof of Lemma 2.2.35. We first construct a heteroclinic sequence of critical elements, which must be finite by Lemma 2.2.33. Then we show that the unstable manifold of the final critical element in the sequence intersects $W^s(X^s)$. Working backwards, we argue that the unstable manifold of every critical element in the sequence intersects $W^s(X^s)$ using Lemma 2.2.34, which implies the result.

The first step is the construction of the heteroclinic sequence $\{X^j\}_{j\in\tilde{I}}$ where $\tilde{I}$ is an ordered and countable indexing set. As $X^j \subset \partial W^s(X^s)$, by Lemma 2.2.29 there exists $x_j \in (W^u(X^j) - X^j) \bigcap \overline{W^s}(X^s)$. If $x_j \in W^s(X^s)$ then we have finished constructing the heteroclinic sequence, so suppose $x_j \in \partial W^s(X^s)$. Then by Assumptions 2.2.2-2.2.3, $x_j \in W^s(X^{j+1})$ for some critical element $X^{j+1} \subset \partial W^s(X^s)$. Iterating this procedure yields a heteroclinic sequence $\{X^j\}_{j\in\tilde{I}}$. By Lemma 2.2.33 it has finite length. The final element of the sequence, call it X^m, must satisfy $W^u(X^m) \bigcap W^s(X^s) \neq \emptyset$, since otherwise there would be another element X^{m+1} that would be added to the heteroclinic sequence by the procedure above.

We conclude by showing that the unstable manifold of each critical element in the heteroclinic sequence must intersect $W^s(X^s)$, which implies the result. For any $j \in \tilde{I}$, suppose $W^u(X^j) \bigcap W^s(X^s) \neq \emptyset$. By recursion, it suffices to show that this implies $W^u(X^{j-1}) \bigcap W^s(X^s) \neq \emptyset$. However, by the construction of the sequence we have that $W^u(X^{j-1}) \bigcap W^s(X^j) \neq \emptyset$ is a transversal intersection. Hence, the result follows from Lemma 2.2.34. □

Proof of Theorem 2.2.12. Fix $i \in I$. By Lemma 2.2.35, $W^u(X^i) \bigcap W^s(X^s) \neq \emptyset$. To show that $W^s(X^i) \subset \partial W^s(X^s)$, it suffices to show that $W^s_{\text{loc}}(X^i) \subset \partial W^s(X^s)$ since $\partial W^s(X^s)$ is invariant and by the definition of $W^s(X^i)$. Let $x \in W^s_{\text{loc}}(X^i)$. By the proof of Lemma 2.2.34, there exists a disk D centered at x, contained in the ϵ-neighborhood of x in M, and transverse to $W^s_{\text{loc}}(X^i)$, such that $\phi_t(D) \cap W^s(X^s) \neq \emptyset$ for some $t > 0$. By invariance, $D \cap W^s(X^s) \neq \emptyset$. Since D is contained in the ϵ-neighborhood of x in M, $d_S(x, \overline{W^s}(X^s)) = d_S(x, W^s(X^s)) \leq \epsilon$. As this holds for all $\epsilon > 0$, $d_S(x, \overline{W^s}(X^s)) = 0$. Since $\{x\}$ is compact and $\overline{W^s}(X^s)$ is closed, this implies that $x \in \overline{W^s}(X^s)$. However, $x \in W^s_{\text{loc}}(X^i)$ implies that $x \in \partial W^s(X^s)$. Thus, $W^s_{\text{loc}}(X^i) \subset \partial W^s(X^s)$, so $W^s(X^i) \subset \partial W^s(X^s)$. Hence $\bigcup_{i\in I} W^s(X^i) \subset \partial W^s(X^s)$.

By Assumption 2.2.3, if $\gamma \subset \partial W^s(X^s)$ is an orbit then $\omega(\gamma) = X^j$ for some $j \in I$, which implies that $\gamma \subset W^s(X^j)$. Thus, $\partial W^s(X^s) \subset \bigcup_{i\in I} W^s(X^i)$. □

2.2.6 Proof of Theorem 2.2.23 and Corollaries

The proof of Theorem 2.2.23 and its corollaries proceeds by paralleling the treatment of the fixed parameter case in Section 2.2.5. The recurring strategy of the proofs of this section will be to reduce to the fixed parameter case as much as possible, and then to rely on the results and proofs of Section 2.2.5 to complete the arguments.

We begin by defining functions whose images for fixed $p \in J$ are the critical elements and their local stable and unstable manifolds for the vector field V_p. As there are finitely many hyperbolic critical elements
$\{X^i_{p_0}\}_{i\in I}$, we may assume J sufficiently small such that they and their local stable and unstable manifolds are well-defined and vary C^1 continuously with parameter over J. For brevity, define the notation: $W^s_{\text{loc}}(X^s(p_0))$ (S^s), and for $i \in I$ define $X^i(p_0)$ (S^i), $W^s_{\text{loc}}(X^i(p_0))$ (S^i_s), and $W^u_{\text{loc}}(X^i(p_0))$ (S^i_u). As the critical elements and their local stable and unstable manifolds vary C^1 continuously with parameter, for $i \in I$ there exist the following C^1 maps. One maps p onto $X^i(p)$ (F^i), where $F^i : S^i \times J \to M$, as the image of $F^i(\cdot, p)$. Another maps p onto $W^s_{\text{loc}}(X^i(p))$ (F^i_s), where $F^i_s : S^i_s \times J \to M$, as the image of $F^i_s(\cdot, p)$. The next maps p onto $W^u_{\text{loc}}(X^i(p))$ (F^i_u), $F^i_u : S^i_u \times J \to M$, as the image of $F^i_u(\cdot, p)$. The last one maps p onto $W^s_{\text{loc}}(X^s(p))$ (F^s), where $F^s : S^s \times J \to M$, as the image of $F^s(\cdot, p)$. Furthermore, for any $p \in J$, $F^i(\cdot, p)$, $F^i_s(\cdot, p)$, $F^i_u(\cdot, p)$, and $F^s(\cdot, p)$ are C^1 diffeomorphisms onto their images. In other words, F^i, F^i_s, F^i_u, and F^s describe quantitatively how the critical elements and their local stable and unstable manifolds vary C^1 with parameter. Let π_J be the projection onto parameter space: $\pi_J(x, p) = p$. The function above have codomain M, but it will sometimes be convenient to have the codomain be $M \times J$. To this end, let $G^i = (F^i, \pi_J)$, $G^i_s = (F^i_s, \pi_J)$, $G^i_u = (F^i_u, \pi_J)$, and $G^s = (F^s, \pi_J)$, and note that these functions are C^1 injections because for fixed $p \in J$ the functions above are C^1 diffeomorphisms onto their images.

Lemma 2.2.36 shows that $W^s(X^s_J)$ is open and invariant in $M \times J$.

Lemma 2.2.36. *$W^s(X^s_J)$ is open and invariant in $M \times J$.*

Proof of Lemma 2.2.36. Since S^s is C^1 diffeomorphic to $W^s_{\text{loc}}(X^s_{p_0})$, a codimension-zero embedded submanifold with boundary in M, $G^s|_{\text{int } S^s \times J}$ is a continuous injection between manifolds of the same dimension so, by invariance of domain [Hat01, Theorem 2B.3], an open map. Thus, $G^s(\text{int } S^s \times J)$ is an open set in $M \times J$. Hence, by definition of the local stable manifold, $W^s(X^s_J) = \bigcup_{t\le 0} \phi_t(G^s(\text{int } S^s \times J))$ is a union of open sets since ϕ_t is a C^1 diffeomorphism for each t, hence open. Since $W^s(X^s_J) = \bigcup_{p\in J} W^s(X^s(p))$ is a union of invariant sets, it is invariant. □

Let $p_0 \in J$ be a fixed parameter value such that Assumptions 2.2.14-2.2.18 hold. Recall from Section 2.2.2 that the family of a critical element refers here to the family obtained from a single critical element as the parameter value is varied over $p \in J$. Similar to Lemma 2.2.28, Lemma 2.2.37 provides a technical construction, for any critical element contained in $\partial W^s(X_J^s)$, of a compact set contained in its family of unstable manifolds such that for any sufficiently small neighborhood N of this compact set in $M \times J$, the following holds. The union over time of the time-t flow ϕ_t of N over all negative times t, together with the family of stable manifolds of the critical element, contains an open neighborhood of the critical element in $M \times J$. The key difference from the fixed parameter case Lemma 2.2.28 is that here the open neighborhood that is contained in the union is open in $M \times J$, whereas for Lemma 2.2.28 it was open in M alone. This is important because for a critical element contained in $\partial W^s(X_J^s)$, an open neighborhood in $M \times J$ of that critical element is required to guarantee it intersects $W^s(X_J^s)$. This result will be fundamental in proving the claim below that if a critical element in M_{p_0} is contained in $\partial W^s(X_J^s)$ then its unstable manifold intersects $\overline{W^s}(X_J^s)$ in M_{p_0}. Recall that if D is a subset of a metric space and $\epsilon > 0$, the notation D_ϵ refers to the subset of the metric space such that for each $x \in D_\epsilon$ there exists $y \in D$ with $d(x, y) < \epsilon$.

Lemma 2.2.37. *For any $i \in I$ and any $\epsilon > 0$ sufficiently small, there exists a compact set $D \subset W^u_{loc}(X_J^i) - X_J^i$ and an open neighborhood N of D in $M \times J$ such that $N \subset D_\epsilon$, $D_\epsilon \bigcap X_J^i = \emptyset$, and $\bigcup_{t \le 0} \phi_t(N) \bigcup W^s(X_J^i)$ contains an open neighborhood of $X_{p_0}^i$ in $M \times J$.*

Proof of Lemma 2.2.37. Let $\epsilon > 0$. If $X_{p_0}^i$ is an equilibrium point, let $\hat{f} = \phi_1$ be the time-one flow. If $X_{p_0}^i$ is a periodic orbit, let $\hat{f} = \hat{\tau}$ be the first return map of a Poincaré cross section S (note that this map is well-defined and C^1 with respect to parameter value $p \in J$). Let $D'_p = G_u^i(S_u^i \times \{p\})$ for any $p \in J$. Let D_p be the topological closure of $D'_p - (\hat{f})^{-1}(D'_p)$ in M. We will prove the following claim: there exists an open neighborhood N' of D_{p_0} in M and an open neighborhood $\hat{U}$ of $X_{p_0}^i$ in M such that for J sufficiently small, $p \in J$ implies that $D_p \subset N' \subset (D_p)_\epsilon$, $X^i(p) \subset \hat{U}$, and the forward orbit of any point $x \in \hat{U} - W^s(X^i(p))$ under V_p enters N' in finite time. For fixed $p \in J$, this claim is illustrated by Fig. 2.6. From this claim, the main result is straightforward as follows. Choose a subset $J' \subset J$ compact and connected with $p_0 \in \text{int } J'$. Let $D' = G_u^i(S_u^i \times J')$ the continuous image of a compact set, hence compact in $M \times J$. Let D be the topological closure of $D' - (\hat{f})^{-1}(D')$ in $M \times J$. Since D' is contained in the local unstable manifold, $(\hat{f})^{-1}|_{D'}$ is contracting. Hence, $(\hat{f})^{-1}(D') \subset D'$, so $D \subset D'$. As D is closed in D' compact, D is compact. By the claim above, for J sufficiently small,

$D \subset N' \times J \subset D_\epsilon$, D is disjoint from X^i_J, and $\bigcup_{t\leq 0} \phi_t(N' \times J) \bigcup W^s(X^i_J)$ contains $\hat{U} \times J'$ where $\hat{U}$ is an open neighborhood in M containing $X^i_{p_0}$. Letting $N = N' \times J$ completes the proof.

So, it suffices to prove the claim above. We begin with the construction of the C^1 disk family for $\hat{f}_{p_0}$ exactly as in the proof of Lemma 2.2.28. Then it is shown using the Inclination Lemma that for a C^1 perturbation of the diffeomorphism $\hat{f}_{p_0}$, constructing the C^1 disk family for the perturbed diffeomorphism gives a C^1 continuous disk family that is uniformly C^1- close to the original C^1 continuous disk family. Consequently, it is possible to choose $\hat{U}$ an open neighborhood of $W^u_{\text{loc}}(X^i_{p_0})$ sufficiently small such that it is contained in the perturbed disk family and, therefore, the forwards orbit of each point in $\hat{U}$ under the perturbed diffeomorphism either converges to the perturbation of $X^i_{p_0}$ or enters N' in finite time. The full proof is provided in Appendix B. □

The technical construction of Lemma 2.2.37 is used to show that the unstable manifold of any critical element in $\partial W^s(X^s_J)$ must have nonempty intersection with $\overline{W^s}(X^s_J) \bigcap M_{p_0}$. By requiring that the intersection occurs in M_{p_0}, we will be able to reduce to the fixed parameter case of Lemma 2.2.35, which will ensure that the unstable manifold actually intersects $W^s(X^s_J) \bigcap M_{p_0}$ (see Lemma 2.2.39). Although Lemma 2.2.29 and Lemma 2.2.38 both show the intersection of the unstable manifold with the closure of a stable manifold, there is a crucial difference: for Lemma 2.2.29 this closure is taken in M for a fixed parameter, whereas for Lemma 2.2.38 the closure is taken in $M \times J$. As Example 2.2.1 showed, taking the closure in $M \times J$, namely $\overline{W^s}(X^s_J)$, will in general give a larger set than taking the closure in M, namely $\sqcup_{p\in J} \overline{W^s}(X^s(p))$. This motivates the need for Lemma 2.2.37 and Lemma 2.2.38 to explicitly treat the more difficult case where the closure is taken in $M \times J$.

Lemma 2.2.38. *For any $i \in I$, $\{W^u(X^i_{p_0}) - X^i_{p_0}\} \bigcap \overline{W^s}(X^s_J) \neq \emptyset$.*

Proof of Lemma 2.2.38. The proof is similar to that of Lemma 2.2.29, which relied on the technical result of Lemma 2.2.28 to show that the distance between an annulus in $W^u_{\text{loc}}(X^i)$ (denoted by D in that proof) and $W^s(X^s)$ was less than ϵ for any $\epsilon > 0$, and then sent $\epsilon \to 0$ to establish the desired intersection. Here, the goal is to use Lemma 2.2.37 in a similar fashion. The key difference is that, since the critical element $X^i_{p_0}$ lies in $\partial W^s(X^s_J)$, and not necessarily in $\partial W^s(X^s_{p_0})$, it is necessary to consider distances in parameter space J as well. In particular, Lemma 2.2.37 is used to establish that there exists a point $(x_r, p_r) \in W^s(X^s_J)$ such that the distance from an annulus in $W^u_{\text{loc}}(X^i_J)$ (denoted by D in this proof) to (x_r, p_r) is less then ϵ and the distance from p_0 to p_r is less than r for any $\epsilon > 0$ and $r > 0$. Then, sending $r \to 0$ reduces to the fixed parameter argument from the proof of Lemma 2.2.29 (which only requires sending $\epsilon \to 0$). Let $B(p_0, r)$ be the ball of radius r centered at p_0 in J.

Let $\epsilon > 0$. Shrinking ϵ if necessary, by Lemma 2.2.37 there exists a compact set $D \subset W^u(X^i_J) - X^i_J$ and an open neighborhood N of D in $M \times J$ such that $N \subset D_\epsilon$, $D_\epsilon \bigcap X^i_J = \emptyset$, and $\bigcup_{t\leq 0} \phi_t(N) \bigcup W^s(X^i_J)$ contains an open neighborhood of $X^i_{p_0}$ in $M \times J$ - call this open neighborhood N'. Let $N_r = N \bigcap (M \times B(p_0, r))$ and $N'_r = N' \bigcap (M \times B(p_0, r))$ be the intersections of the above neighborhoods with $M \times B(p_0, r)$. Since N'_r is an open neighborhood of $X^i_{p_0} \subset \partial W^s(X^s_J)$, $N'_r \cap W^s(X^s_J) \neq \emptyset$. Since $\bigcup_{t\leq 0} \phi_t(N_r) \bigcup W^s(X^i_J)$ contains N'_r, $N'_r \cap W^s(X^s_J) \neq \emptyset$, and $W^s(X^i_J) \bigcap W^s(X^s_J) = \emptyset$, there exists $T > 0$ such that $\phi_{-T}(N_r) \bigcap W^s(X^s_J) \neq \emptyset$. By invariance of $W^s(X^s_J)$, this implies that $N_r \bigcap W^s(X^s_J) \neq \emptyset$. So, let $(x_r, p_r) \in N_r \bigcap W^s(X^s_J)$ and send r to zero. As $N_r \subset D_\epsilon$ and $W^s(X^s_J) \subset \overline{W^s}(X^s_J)$, $N_r \bigcap W^s(X^s_J) \subset D_\epsilon \bigcap \overline{W^s}(X^s_J)$ compact. Hence, passing to a subsequence if necessary we have that $(x_r, p_r) \to (x, p) \in D_\epsilon \bigcap \overline{W^s}(X^s_J)$. By definition of N_r, since $r \to 0$ we must have $p = p_0$. In the proof of Lemma 2.2.37, D is constructed as the disjoint union of a C^1 continuous family of embedded submanifolds in M. Therefore, $(x, p_0) \in D_\epsilon$ implies that $d_S((x, p_0), D \cap M_{p_0}) \leq \epsilon$. Thus, $d_S(\overline{W^s}(X^s_J), D \cap M_{p_0}) \leq \epsilon$ for all $\epsilon > 0$. Now we have reduced to the fixed parameter case, and the rest of the proof proceeds as in the proof of Lemma 2.2.29 by noting that the above implies (sending $\epsilon \to 0$) that $d_S(\overline{W^s}(X^s_J), D \cap M_{p_0}) = 0$, so that $\overline{W^s}(X^s_J) \bigcap (D \cap M_{p_0}) \neq \emptyset$, which implies the result. □

Thanks to the work of Lemma 2.2.37 and Lemma 2.2.38, the varying parameter case treated in this section was essentially reduced to the fixed parameter case of Section 2.2.5. Hence, Lemma 2.2.39 is exactly analogous to its fixed parameter counterpart Lemma 2.2.35 in both its statement and its proof.

Lemma 2.2.39. *For any $i \in I$, $W^u(X^i_{p_0}) \bigcap W^s(X^s_{p_0}) \neq \emptyset$.*

Proof of Lemma 2.2.39. The proof is identical to the proof of Lemma 2.2.35, substituting Lemma 2.2.38 for Lemma 2.2.29. □

Proof of Theorem 2.2.23. For any $p \in J$ and any $x \in \partial W^s(X^s(p))$, there exist $x_n \in W^s(X^s(p))$ with $x_n \to x$. Hence, $(x_n, p) \in W^s(X^s_J)$ with $(x_n, p) \to (x, p)$, so $(x, p) \in \overline{W^s}(X^s_J)$ closed. As $x \notin W^s(X^s(p))$, this implies that $(x, p) \in \partial W^s(X^s_J)$. Hence,

$$\sqcup_{p\in J} \partial W^s(X^s(p)) \subset \partial W^s(X^s_J). \tag{2.12}$$

We claim that for J sufficiently small, for any $p \in J$ and $i \in I$, $W^u(X^i(p)) \bigcap W^s(X^s(p)) \neq \emptyset$. For, let $i \in I$. Then by Lemma 2.2.39, we have that $W^u(X^i_{p_0}) \bigcap W^s(X^s_{p_0}) \neq \emptyset$. This implies that there exists $T > 0$ such that $\phi_T(W^u_{\text{loc}}(X^i_{p_0})) \bigcap W^s_{\text{loc}}(X^s_{p_0}) \neq \emptyset$. This intersection is trivially transverse since $W^s(X^s_{p_0})$ is an open set in M_{p_0}. Since $\{\phi_T(W^u_{\text{loc}}(X^i(p)))\}_{p\in J}$ and $\{W^s_{\text{loc}}(X^s(p))\}_{p\in J}$ are two C^1 continuous families over J of compact embedded submanifolds with boundary, and since they have a point of transversal intersection at $p = p_0$, for J sufficiently small $p \in J$ implies [KH99, Proposition A.3.16,Corollary A.3.18] that $\phi_T(W^u_{\text{loc}}(X^i(p))) \bigcap W^s_{\text{loc}}(X^s(p)) \neq \emptyset$. Hence, for sufficiently small J, since I is finite, $W^u(X^i(p)) \bigcap W^s(X^s(p)) \neq \emptyset$ for all $i \in I$ and $p \in J$, so the claim above follows. Let $(x, p) \in \partial W^s(X^s_J)$. By Assumption 2.2.17, $x \in W^s(X^j_p)$ for some $j \in I$. By the claim above applied to the particular j here, $W^u(X^j_p) \bigcap W^s(X^s(p)) \neq \emptyset$. Now the argument reduces to the fixed parameter case from earlier, and we can use the proof of Theorem 2.2.12 to show that $W^s(X^j_p) \subset \partial W^s(X^s(p))$. As $(x, p) \in \partial W^s(X^s_J)$ was arbitary, we have

$$\partial W^s(X^s_J) \subset \bigcup_{j\in I} W^s(X^i_J) \subset \sqcup_{p\in J} \partial W^s(X^s(p)). \tag{2.13}$$

Then Eqs. 2.12-2.13 imply the result. □

Proof of Corollary 2.2.24. Recall the definitions of S^i_s and F^i_s from the beginning of Section 2.2.6. Let $p_n \in J$ with $p_n \to p'$ for some $p' \in J$. First, let $x \in \partial W^s(X^s_{p'})$. Then $x \in W^s(X^i_{p'})$ for some $i \in I$. So, there exists $T > 0$ such that $\phi_T(x, p') \in W^s_{\text{loc}}(X^i_{p'})$ and $y \in S^i_s$ such that $F^i_s(y, p') = \phi_T(x, p')$. Let $x_n = \phi_{-T}(F^i_s(y, p_n), p_n) \in W^s(X^i_{p_n})$ by invariance of $W^s(X^i_{p_n})$. Thus, $x_n \in W^s(X^i_{p_n}) \subset \partial W^s(X^s_{p_n})$ by Theorem 2.2.23. Furthermore, $x_n \to x$ since ϕ_{-T} and F^i_s are C^1. Hence, $x \in \liminf_{n\to\infty} \partial W^s(X^s_{p_n})$, so $\partial W^s(X^s_{p'}) \subset \liminf_{n\to\infty} \partial W^s(X^s_{p_n})$.

Next, let $x \in \limsup_{n\to\infty} \partial W^s(X^s_{p_n})$. Then there exist a subsequence $\{p_{n_m}\}_{m=1}^\infty$ of $\{p_n\}_{n=1}^\infty$ and a sequence $\{x_m\}_{m=1}^\infty$ such that $x_m \in \partial W^s(X^s_{p_{n_m}})$ for all m and $x_m \to x$. By Theorem 2.2.23, $\partial W^s(X^s_{p_{n_m}}) \subset \partial W^s(X^s_J)$, so that $(x_m, p_{n_m}) \in \partial W^s(X^s_J)$. As $\partial W^s(X^s_J)$ is closed, $\lim_{m\to\infty}(x_m, p_{n_m}) = (x, p') \in \partial W^s(X^s_J)$. By Theorem 2.2.23, $\partial W^s(X^s_J) = \sqcup_{p\in J} \partial W^s(X^s(p))$, so intersecting both sides with $M_{p'}$ implies that $\partial W^s(X^s_J) \bigcap M_{p'}$ $= \partial W^s(X^s_{p'}) \times \{p'\}$. Hence, $(x, p') \in \partial W^s(X^s_J)$ implies that $x \in \partial W^s(X^s_{p'})$. Thus, $\limsup_{n\to\infty} \partial W^s(X^s_{p_n}) \subset \partial W^s(X^s_{p'})$. Together, these imply that $\lim_{n\to\infty} \partial W^s(X^s_{p_n}) = \partial W^s(X^s_{p'})$. As $p_n \to p'$ were arbitrary, this implies that $\{\partial W^s(X^s(p))\}_{p\in J}$ is a Chabauty continuous family of subsets of M. □

Proof of Corollary 2.2.25. By Corollary 2.2.24, we have that $\{\partial W^s(X^s(p))\}_{p\in J}$ is a Chabauty continuous family of subsets of M. Since M is compact, Hausdorff continuity is equivalent to Chabauty continuity. Hence, $\{\partial W^s(X^s(p))\}_{p\in J}$ is a Hausdorff continuous family of subsets of M. □

Proof of Corollary 2.2.26. Let $\Omega(V_{p_0}) = \{X^i_{p_0}\}_{i=1}^n$ a finite union of hyperbolic critical elements since V_{p_0} is Morse-Smale. Palis showed [Pal69, Theorem 3.5] that for any sufficiently small C^1 perturbation to V_{p_0}, so for J sufficiently small, $p \in J$ implies that V_p is still Morse-Smale with $\Omega(V_p) = \{X^i(p)\}_{i=1}^n$. Let $\{X^i_{p_0}\}_{i=1}^k = \Omega(V_{p_0}) \bigcap (\partial W^s(X^s_J) \cap M_{p_0})$ which is a finite union of critical elements of V_{p_0} since $\Omega(V_{p_0})$ is finite, and $k < n$. Note that $\{V_p\}_{p\in J}$ satisfies Assumption 2.2.15 and Assumption 2.2.18 for J sufficiently small since V_{p_0} is Morse-Smale. Note that both $\bigcup_{i>k} X^i_{p_0}$ and $\partial W^s(X^s_J) \cap M_{p_0}$ are compact, so since M is a normal space there exists an open set N such that $\partial W^s(X^s_J) \cap M_{p_0} \subset N$ and $N \bigcap \bigcup_{i>k} X^i_{p_0} = \emptyset$. As $\Omega(V_{p_0}) = \bigcup_{i=1}^n X^i_{p_0}$, this implies that $\Omega(V_{p_0}) \bigcap N = \bigcup_{i=1}^k X^i_{p_0} = \Omega(V_{p_0}) \bigcap (\partial W^s(X^s_J) \cap M_{p_0})$. Hence, Assumption 2.2.14 is satisfied. So, it suffices to show that $\{V_p\}_{p\in J}$ satisfies Assumption 2.2.17 as well. By Lemma 2.2.39, for any $i \in \{1, .., n\}$ and any $p \in J$, $X^i(p) \subset \partial W^s(X^s_J)$ implies that $W^u(X^i_{p_0}) \cap W^s(X^s_{p_0}) \neq \emptyset$. So, let $x \in W^u(X^i_{p_0}) \cap W^s(X^s_{p_0})$. As $x \in \overline{W^s}(X^s_{p_0})$ closed and invariant, the closure of the orbit of x is contained in $\overline{W^s}(X^s_{p_0})$. Since $\alpha(x) = X^i_{p_0}$ is contained in the closure of the orbit of x, $X^i_{p_0} \subset \overline{W^s}(X^s_{p_0})$. Since $X^i_{p_0}$ does not intersect $W^s(X^s_{p_0})$, $X^i_{p_0} \subset \partial W^s(X^s_{p_0})$. By Theorem 2.2.23, $\partial W^s(X^s_J) = \sqcup_{p\in J} \partial W^s(X^s(p))$, so $X^i_{p_0} \subset \partial W^s(X^s_{p_0}) \subset \partial W^s(X^s(p)) \cap M_{p_0}$. Therefore, by definition of k above, we must have $i \in \{1, ..., k\}$. Hence, for any $i \in \{1, ..., n\}$ and $p \in J$, $X^i(p) \subset \partial W^s(X^s_J)$ implies that $i \in \{1, ..., k\}$, so that $X^i_{p_0} \subset \partial W^s(X^s_J)$. As $\Omega(V_p) = \{X^i(p)\}_{i\in\{1,...,n\}}$, this implies that no new nonwandering elements can enter $\partial W^s(X^s_J)$ for $p \neq p_0$, so Assumption 2.2.17 is satisfied. Thus, $\{V_p\}_{p\in J}$ satisfy Assumptions 2.2.14-2.2.18. Therefore, by Corollary 2.2.25, $\{\partial W^s(X^s(p))\}_{p\in J}$ is a Hausdorff continuous family of subsets of M. □

2.2.7 Conclusion

This work provides theoretical motivation for the development of algorithms which estimate the set of recovery values by estimating its boundary via the computation of critical parameter values. The main technical result required for the algorithms' motivation was that, for the family of vector fields discussed above, which are in essense Morse-Smale along the RoA boundary, the RoA boundary is Hausdorff (for a compact Riemannian manifold) or Chabauty (for Euclidean space) continuous with respect to parameter. This result was established together with a decomposition of the RoA boundary into the union of the stable manifolds of the critical elements it contains, and it was shown that this decomposition persists under small variations in parameter values. Section 2.3 shows that Assumption 2.2.17, which represents Ω-stability of the RoA boundary, is a consequence of the other assumptions. Hence, this will complete the proof of RoA boundary continuity under small parameter variation for vector fields which are in essence Morse-Smale along the RoA boundary.

2.3 Ω-Stability of Region of Attraction Boundary

2.3.1 Introduction

The purpose of this section is to show the second part of the proof of continuity of the RoA boundary under small parameter variation. The goal is to show that if the vector field is Morse-Smale along the RoA boundary, then the RoA boundary is Ω-stable, so the additional assumptions required for Corollaries 2.2.24-2.2.25 beyond the Morse-Smale assumptions at a fixed parameter value are not required. Unfortunately, Example 2.3.3 illustrates a smooth vector field in Euclidean space which is Morse-Smale, but whose RoA boundary varies discontinuously under arbitrarily small changes in parameter value due to the entrance of an additional equilibrium point into the RoA boundary. This is a counterexample to Theorem 5.4 of [CC95], which claimed that under weaker assumptions than the above, for strong C^1 perturbations to the vector field no new nonwandering points could enter the RoA boundary. Hence, while the RoA boundary continuity result holds when the vector field is Morse-Smale along the RoA boundary for compact Riemannian manifolds, as shown in Theorem 2.3.9, the corresponding result for Euclidean space requires additional generic assumptions as well as an assumption regarding behavior near infinity, as given by Theorem 2.3.20.

The section is organized as follows. Section 2.3.2 provides relevant background and notation. Section 2.3.3 introduces several examples in Euclidean space. Section 2.3.4 describes the main results and Section 2.3.5 provides their proofs for each of compact Riemannian manifolds and Euclidean space. Finally, Section 2.3.6 offers some conclusions.

2.3.2 Background

This section collects definitions and notation that will be used throughout the paper. Let M be either a compact Riemannian manifold or Euclidean space. Let V_0 be a C^1 vector field on M. We let ϕ denote its flow, and write $\phi_t(x, V_0)$ for the flow under V_0 at time t starting from initial condition x. For a set $E \subset M$, let $\phi_t(E, V_0) = \bigcup_{x \in E} \phi_t(x, V_0)$. We say that a set

$E \subset M$ is forward (backward) invariant under V_0 if $\phi_t(E, V_0) \subset E$ for all $t > 0$ ($t < 0$). E is invariant if it is both forward and backward invariant. The nonwandering set of V_0, denoted $\Omega(V_0)$, is the set of points $x \in M$ such that for any open neighborhood U of x in M and any $T > 0$, there exists $t \in \mathbb{R}$ with $|t| > T$ such that $\phi_t(U, V_0) \bigcap U \neq \emptyset$. $\Omega(V_0)$ is closed in M and invariant under V_0. A point $x \in M$ is wandering under V_0 if $x \notin \Omega(V_0)$. For any $x \in M$, its ω limit set under V_0, denoted $\omega(x, V_0)$, is the set of points $y \in M$ such that there exists a sequence of times $t_n \to \infty$ such that $\phi_{t_n}(x, V_0) \to y$. For brevity, and when the vector field is clear from context, we omit it and write simply $\omega(x)$. Similarly, for any $x \in M$, its α limit set under V_0, denoted $\alpha(x, V_0)$, is the set of points $y \in M$ such that there exists a sequence of times $t_n \to -\infty$ such that $\phi_{t_n}(x, V_0) \to y$. Note that for any $x \in M$, $\omega(x)$ and $\alpha(x)$ are closed in M, invariant under V_0, and contained in $\Omega(V_0)$. Furthermore, if $x, y \in M$ are part of the same orbit under V_0, then their ω and α limit sets under V_0 are equal.

An equilibrium point x_e of V_0 is hyperbolic if for every $t \in \mathbb{R}$, the differential $d(\phi_t)_{x_e}$ has no eigenvalues with absolute value equal to one. A periodic orbit γ of V_0 is hyperbolic if there exists $x \in \gamma$, a smooth cross section S of γ centered at x, and a C^1 first return map $\hat{\tau} : S \to S$ such that $d\hat{\tau}_x$ has no eigenvalues with absolute value equal to one. Let critical elements refer to the union of equilibrium points and periodic orbits of V_0. A sink denotes a stable hyperbolic equilibrium point, and a source denotes an unstable hyperbolic equilibrium point. By [HPS77], a hyperbolic critical element X of V_0 has local C^1 stable and unstable manifolds, denoted $W^s_{\text{loc}}(X)$ and $W^s_{\text{loc}}(X)$, respectively, and for every $t > 0$ $\phi_t(W^s_{\text{loc}}(X), V_0) \subset W^s_{\text{loc}}(X)$ and $\phi_{-t}(W^u_{\text{loc}}(X), V_0) \subset W^u_{\text{loc}}(X)$. In other words, the local stable and unstable manifolds are contracting under the time-t flow ϕ_t in forwards and backwards time, respectively, and therefore are forward and backward invariant, respectively. The global stable manifold $W^s(X)$ and unstable manifold $W^u(X)$ are obtained as $W^s(X) = \bigcup_{t<0} \phi_t(W^s_{\text{loc}}(X), V_0)$ and $W^u(X) = \bigcup_{t>0} \phi_t(W^u_{\text{loc}}(X), V_0)$, respectively. For $\hat{A} \subset M$ a C^1 submanifold, let $T_x\hat{A}$ denote the tangent space to $\hat{A}$ at x. A pair $\hat{A}$ and $\hat{B}$ of C^1 submanifolds of M are transverse if for every $x \in \hat{A} \bigcap \hat{B}$, $T_x\hat{A} \oplus T_x\hat{B} = T_xM$. A vector field V_0 is Morse-Smale [Sma60] if $\Omega(V_0)$ consists of a finite union of critical elements, each critical element is hyperbolic, and all stable and unstable manifolds are transverse.

If M is Euclidean, there are two standard topologies for the set of C^1 vector fields on M: the strong and weak C^1 topologies [Hir76, p. 34-35]. For M a compact Riemannian manifold, the strong and weak C^1 topologies are the same, and we can unambiguously refer to merely the C^1 topology in this case. If a sequence $\{V_n\}_{n=1}^{\infty}$ of C^1 vector fields on M converges in the strong C^1 topology to a C^1 vector field V_0, then there exists a compact set

$K \subset M$ such that for n sufficiently large, V_n and V_0 agree outside of K. Both the strong and weak C^1 topologies are Baire spaces, meaning that countable intersections of open, dense sets are themselves dense. A residual set in a topological space is a set containing a countable intersection of open, dense sets. A property is called generic in either the strong or weak C^1 topology if the set of vector fields satisfying this property contains a residual set. By the above, this implies that generic properties are satisfied by a dense set in either the strong or weak C^1 topologies. Hyperbolicity of all critical elements and transversality of their stable and unstable manifolds are generic in these topologies [KH99, Ch. 7], although the assumption that $\Omega(V_0)$ consists of a finite union of critical elements is not. A topological space E is first countable if for every $x \in E$ there exists a countable collection $\{N_i^x\}_{i=1}^{\infty}$ of open neighborhoods of x such that for every open neighborhood N of x, there exists i such that $N_i^x \subset N$. Note that every manifold is first countable. If E is first countable, $x \in E$, and there does not exist an open set N containing E such that every point $y \in N$ possesses some property $\mathcal{P}$, then there must exist a sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \to x$ such that x_n does not possess property $\mathcal{P}$ for all n. This fact will be used repeatedly to prove, by means of contradiction, the existence of an open set in parameter space (which will be a manifold) whose corresponding vector fields possess some desired property.

Morse-Smale vector fields on compact Riemannian manifolds were shown to be open in the C^1 topology [Pal69]. Crucial tools in these proofs were fundamental domains and neighborhoods. Let $X = x$ be a hyperbolic equilibrium point with $\hat{f} = \phi_1(\cdot, V_0) : M \to M$ or let X be a hyperbolic periodic orbit with smooth cross section S centered at x, $\hat{\tau} : S \to S$ a C^1 first return map, and let $\hat{f} = \hat{\tau}$. A fundamental domain D of the stable manifold of $\hat{f}$ is given by taking the closure of $W^s_{\text{loc}}(x) - \hat{f}(W^s_{\text{loc}}(x))$. It is straightforward to see that $\bigcup_{n\in\mathbb{Z}} \hat{f}^n(D) = W^s(x)$. Let N be an open neighborhood of D which is disjoint from $W^u(x)$. Then N is called a fundamental neighborhood of the stable manifold of $\hat{f}$. It is shown in Corollary 2 [JPdM82, p. 86] that $\bigcup_{n\geq 0} \hat{f}^n(N) \bigcup W^u_{\text{loc}}(x)$ contains an open neighborhood of x.

Let M be a compact Riemannian manifold and let $\mathcal{M}$ be the set of closed subsets of M. For $A, B \in \mathcal{M}$ nonempty, let $r(A, B)$ be the infimum over $\epsilon \geq 0$ such that the ϵ neighborhood of A, the set of points within a distance ϵ from A, contains B. For $A, B \in \mathcal{M}$ nonempty, define the Hausdorff distance $d_H(A, B)$ to be the maximum of $r(A, B)$ and $r(B, A)$. Then d_h is a metric [Hau57, Section 28] and its corresponding metric topology is called the Hausdorff topology. Furthermore, letting $\mathcal{M}$ denote the set of closed subsets of M equipped with the Hausdorff metric, $\mathcal{M}$ is compact [Hau57, Section 28]. If M is Euclidean, then the Chabauty topology is obtained by first considering the compact Riemannian manifold $M \bigcup \{\infty\}$. For every $A \in \mathcal{M}$, let $\overline{A} = A \bigcup \{\infty\}$. Then we can define a metric d_C on $\mathcal{M}$ such that $d_C(A, B) = d_H(\overline{A}, \overline{B})$ where the Hausdorff distance is taken in the compact manifold $M \bigcup \{\infty\}$. Then the metric topology induced by d_C is called the Chabauty topology on $\mathcal{M}$.

If $\{E_n\}_{n=1}^{\infty}$ is a sequence of subsets of M, define $\liminf_{n\to\infty} E_n$ to be the set of points $x \in M$ such that there exists a sequence $x_n \to x$ with $x_n \in E_n$ for all n. Similarly, define $\limsup_{n\to\infty} E_n$ to be the set of points $x \in M$ such that there exists a subsequence $x_{n_m} \to x$ with $n_m \to \infty$ and $x_{n_m} \in E_{n_m}$ for all m. Then $E_n \to E_0$ in the Hausdorff topology (Chabauty topology if M is Euclidean) if and only if $\liminf_{n\to\infty} E_n = \limsup_{n\to\infty} E_n = E_0$. A parameterized family of subsets of M, $\{E_p\}_{p\in J}$, is Hausdorff (Chabauty) continuous if for every $p_0 \in J$, $p_n \to p_0$ implies that $E_{p_n} \to E_{p_0}$. For M a compact Riemannian manifold, a function $\tilde{f} : \tilde{E} \to \mathcal{M}$, where $\tilde{E}$ is a topological space, is lower semi-continuous if for every convergent sequence $E_n \to E_0$ in $\tilde{E}$, $\liminf_{n\to\infty} \tilde{f}(E_n) \supset \tilde{f}(E_0)$. The set of points of continuity of such a lower semi-continuous function are residual in $\tilde{E}$ by the Remark following Corollary 1 of [Kur68]. This property was used to prove that for generic V_0 in the strong C^1 topology, the closure of the union of equilibrium points and periodic orbits of V_0 contains all points $x \in \Omega(V_0)$ such that $\omega(x, V_0) \bigcup \alpha(x, V_0) \neq \emptyset$ [PR83]. The key technical element of this proof is the C^1 closing lemma for noncompact manifolds, which states that for any $x \in \Omega(V_0)$ such that $\omega(x, V_0) \bigcup \alpha(x, V_0) \neq \emptyset$, there exists V arbitrarily close to V_0 in the strong C^1 topology and y arbitrarily close to x in M such that y is periodic under V [PR83].

For V_0 a vector field on M, write $\{X^i(V_0)\}_{i\in I}$ for a collection of critical elements of V_0. For V near V_0, let $\{X^i(V)\}_{i\in I}$ denote the perturbations of those critical elements. If $\{V_p\}_{p\in J}$ is a strong C^1 continuous family of vector fields on M, let $X^i_J = \sqcup_{p\in J} X^i(V_p) \subset M \times J$. Similarly, let $W^s(X^i_J) = \sqcup_{p\in J} W^s(X^i(V_p)) \subset M \times J$ and let $W^u(X^i_J) = \sqcup_{p\in J} W^u(X^i(V_p)) \subset M \times J$. Let $\partial W^s(X^i)$ denote the boundary of $W^s(X^i)$ in M and let $\overline{W}^s(X^i)$ denote its closure in M. More generally, for a set E let ∂E be its boundary and $\overline{E}$ its closure.

2.3.3 Examples

All of the examples in this section are of C^1 vector fields over $\mathbb{R}^2$. For purposes of illustration, we identify its one point compactification, $\mathbb{R}^2 \bigcup \{\infty\}$, with the sphere $\mathbb{S}^2$ via stereographic projection, where the point infinity is identified with the north pole of the sphere. Each example is shown as a view of the northern hemisphere of the sphere from above. The vector field then extends naturally to the full sphere, and the lower hemisphere of the sphere does not contain any nonwandering points in any of the examples below.

Example 2.3.1 (Nonwandering Points Accumulate on Boundary)**.** One consequence of Theorem 2.3.9 is that for a compact Riemannian manifold possessing a stable hyperbolic equilibrium point $X^s(V_0)$, and such that V_0 is Morse-Smale along $\partial W^s(X^s(V_0))$, there exists an open neighborhood N of $\overline{W}^s(X^s(V_0))$ such that $\Omega(V_0) \bigcap N = \Omega(V_0) \bigcap \overline{W}^s(X^s(V_0))$. In other words, there are no nonwandering points outside of $\overline{W}^s(X^s(V_0))$ that accumulate on $\partial W^s(X^s(V_0))$. The existence of this neighborhood is crucial to the proof of Theorem 2.3.9, which argues that no new nonwandering points can enter $\partial W^s(X^s(V))$ for V close to V_0, and its Corollary 2.3.10, which says that this boundary varies Hausdorff continuously for a C^1 continuous family of vector field. However, this example shows that in the case of Euclidean space, if V_0 is Morse-Smale along $\partial W^s(X^s(V_0)$, there may not exist an open neighborhood N of $\overline{W}^s(X^s(V_0))$ such that $\Omega(V_0) \bigcap N = \Omega(V_0) \bigcap \overline{W}^s(X^s(V_0))$. In particular, Figure 2.7 shows an example in which V_0 is Morse-Smale along $\partial W^s(X^s(V_0))$ but there exists a sequence of periodic orbits outside of $\overline{W}^s(X^s(V_0))$ which accumulate along $\partial W^s(X^s(V_0))$, so that no such neighborhood N exists. Hence, the methods used to prove Theorem 2.3.9 and Corollary 2.3.10 for compact Riemannian manifolds do not apply for Euclidean space.

Example 2.3.2 (Nonwandering Points Enter Boundary Near Infinity)**.** This example shows that even if V_0 is Morse-Smale along $\partial W^s(X^s(V_0))$ and there exists an open neighborhood N of $\overline{W}^s(X^s(V_0))$ in $\mathbb{R}^2$ such that $\Omega(V_0) \bigcap N = \Omega(V_0) \bigcap \overline{W}^s(X^s(V_0))$, there exists V arbitrarily close to V_0 in the strong C^1 topology such that $\partial W^s(X^s(V))$ contains nonwandering points in addition to the perturbations of the nonwandering points in $\partial W^s(X^s(V_0))$. In other words, new nonwandering points enter the boundary under strong C^1 perturbations. By Theorem 2.3.9, this is not possible for compact Riemannian manifolds with V_0 Morse-Smale along $\partial W^s(X^s(V_0))$. Figure 2.8 shows an example satisfying the assumptions above, and such that under C^1 perturbations to V_0 in a compact neighborhood of a hyperbolic saddle equilibrium point, a continuum of nonhyperbolic equilibrium points enter $\partial W^s(X^s(V))$.

Example 2.3.3 (Explosion of Closure of Region of Attraction)**.** This example show that even if V_0 is Morse-Smale along $\partial W^s(X^s(V_0))$, there exists an open neighborhood N' of $\overline{W}^s(X^s(V_0))$ in $\mathbb{R}^2$ such that $\Omega(V_0) \bigcap N' = \Omega(V_0) \bigcap \overline{W}^s(X^s(V_0))$, and there exists an open neighborhood N of ∞ in M such that $\Omega(V_0) \bigcap N = \emptyset$ and every $x \in N$ satisfies $\omega(x) \bigcup \alpha(x) \neq \emptyset$, there still exists V arbitrarily close to V_0 in the strong C^1 topology such that such that $\partial W^s(X^s(V))$ contains nonwandering points in addition to the perturbations of the nonwandering points in $\partial W^s(X^s(V_0))$. Furthermore, $\overline{W}^s(X^s(V))$ explodes in the sense that it grows

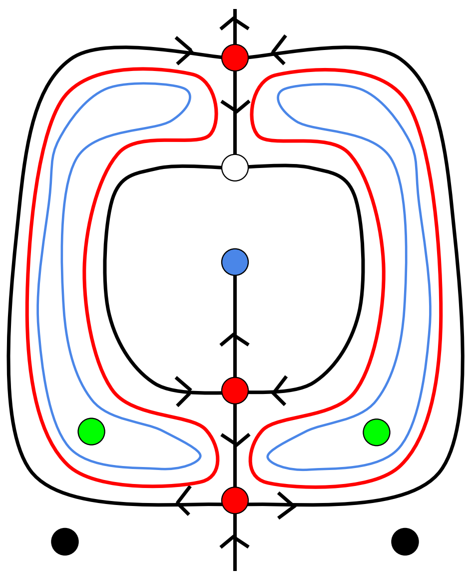

Figure 2.7: Example 2.3.1. A stable hyperbolic equilibrium point (blue dot), saddle hyperbolic equilibrium points (red dots), sinks (black dots), nonhyperbolic equilibrium points (green dots), and a point at infinity (white dot) are shown. A sequence of periodic orbits (red for unstable and blue for stable) accumulates along the boundary of the region of attraction of the stable hyperbolic equilibrium point (blue dot). Black lines indicate stable and unstable manifolds of the saddle equilibrium points, with the arrows indicating direction of flow.

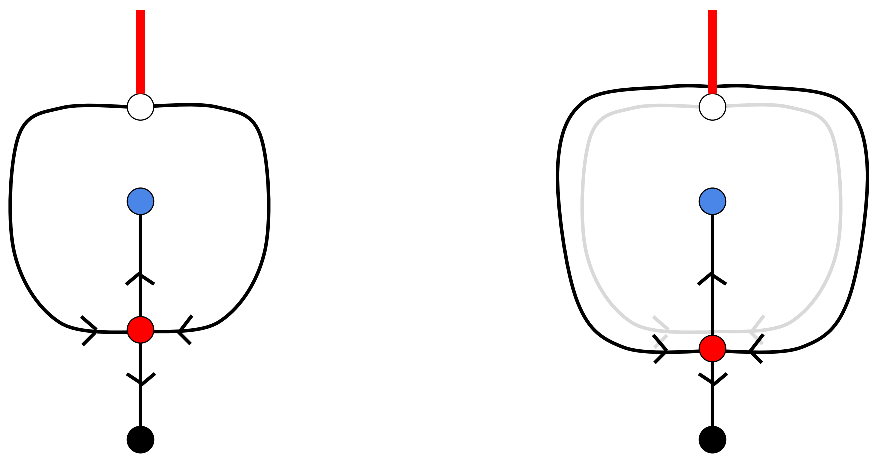

Figure 2.8: Example 2.3.2. A stable hyperbolic equilibrium point (blue dot), saddle hyperbolic equilibrium point (red dot), continuum of nonhyperbolic equilibrium points which are unstable in the transverse direction (red line), a stable sink (black dot), and a point at infinity (white dot) are shown. Black lines indicate stable and unstable manifolds of the saddle equilibrium point, with the arrows indicating direction of flow. The left figure shows the vector field before perturbation and the right figure shows the vector field after a perturbation in a compact neighborhood of its saddle equilibrium point (red dot). A segment of the continuum of nonhyperbolic equilibrium points (red line) enters the boundary of the region of attraction of the stable equilibrium point (blue dot).

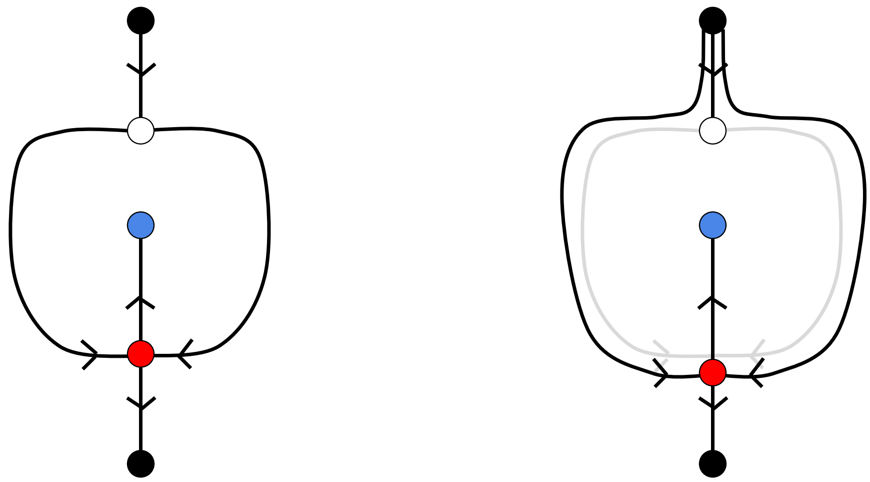

Figure 2.9: Example 2.3.3. A stable hyperbolic equilibrium point (blue dot), saddle hyperbolic equilibrium point (red dot), source and sink (black dots), and a point at infinity (white dot) are shown. Black lines indicate stable and unstable manifolds of the saddle equilibrium point, with the arrows indicating direction of flow. The left figure shows the vector field before perturbation and the right figure shows the vector field after a perturbation in a compact neighborhood of its saddle equilibrium point (red dot). A source (black dot) enters the boundary of the stable equilibrium point (blue dot) arbitrarily far from the boundary before perturbation and from infinity.

discontinuously under arbitarily small perturbations from V_0. Hence, not only are new nonwandering points entering the boundary under perturbation, which is not possible for the compact Riemannian manifold case by Theorem 2.3.9, but also a C^1 continuous family of vector fields $\{V_p\}_{p\in J}$ containing V_0 would have $\partial W^s(X^s(V_p))$ discontinuous in p, which is also not possible for the compact Riemannian manifold case by Corollary 2.3.10. The purpose of Theorem 2.3.20 and Corollary 2.3.21 is to show that this behavior is not possible under the assumptions above for vector fields which are generic in a certain sense. Figure 2.9 shows an example which satisfies the assumptions above, is globally a Morse-Smale vector field on $\mathbb{R}^2$, and such that, for a C^1 perturbation to V_0 in a compact neighborhood of a hyperbolic saddle equilibrium point, a new hyperbolic equilibrium point enters $\partial W^s(X^s(V))$ arbitrarily far away from $\partial W^s(X^s(V_0))$

Theorem 5-4 of [CC95] claims (among other things) that if V_0 is a C^1 vector field on Euclidean space such that $\Omega(V_0)\bigcap\partial W^s(X^s(V_0))$ is a finite union of hyperbolic equilibrium points $\{X^i(V_0)\}_{i\in I}$, $\partial W^s(X^s(V_0)) = \bigcup_{i\in I} W^s(X^i(V_0))$, and if the intersections of their stable and unstable manifolds are transverse, then V sufficiently close to V_0 in the weak C^1 topology implies that $\partial W^s(X^s(V_0)) = \bigcup_{i\in I} W^s(X^i(V))$. Recently, the authors appear to have recognized that their theorem was incorrect [CA15] via a counterexample that uses a weak C^1 perturbation, and have attempted to informally fix their theorem. They claim in this more recent setting that if $\{V_p\}_{p\in(a,b)}$ is a weakly C^1 continuous family of vector fields such that for every $p \in (a,b)$, $\Omega(V_p) \bigcap \partial W^s(X^s(V_p))$ consists of a finite union of hyperbolic equilibrium points, every orbit in $\partial W^s(X^s(V_p))$ converges in forwards time to one of these equilibrium points, the stable and unstable manifolds of these equilibrium points are transverse, and if we call $\{X^i(V_{p_0})\}_{i\in I}$ the equilibrium points in $\partial W^s(X^s(V_{p_0}))$ for some fixed $p_0 \in (a,b)$, then shrinking (a,b) if necessary implies that $\partial W^s(X^s(V_p)) \subset \bigcup_{i\in I} W^s(X^i(V_p))$ unless there is an appearance of a new hyperbolic equilibrium point "at infinity." They go on to say that under these assumptions "and some control at infinity," the boundary "does not suffer from drastic changes with small parameter variation." If we let V_{p_0} be the unperturbed vector field of Figure 2.9, and parametrize its perturbation in that figure in a natural way, then we obtain a strong C^1 continuous family of vector fields $\{V_p\}_{p\in(a,b)}$ satisfying the assumptions above (and more, since they are globally Morse-Smale), and such that there is an open neighborhood of infinity which contains no equilibrium points or nonwandering points for all $p \in (a,b)$ and on which V_p agrees with V_{p_0} for all $p \in (a,b)$. Yet, for $p \in (a,b)$ arbitrarily close to p_0, a new equilibrium point enters the boundary, and the boundary undergoes a drastic expansion. Thus, this example contradicts both the original Theorem 5-4 [CC95] and the attempted revision by the original lead author [CA15]. Furthermore, it illustrates that the conventional paradigm for studying the impact of parameter variations on the boundary by considering only loss of hyperbolicity or transversality overlooks crucial global bifurcations.

2.3.4 Main Results

2.3.4.1 Compact Riemannian Manifolds

For this section, let M be a compact Riemannian manifold and let V_0 be a C^1 vector field on M possessing a stable hyperbolic equilibrium point $X^s(V_0)$. We make the following assumptions.

Assumption 2.3.4. $\Omega(V_0) \bigcap \partial W^s(X^s(V_0))$ *consists of an arbitrary union of critical elements; call them* $\bigcup_{i\in I} X^i(V_0)$ *where* I *is some indexing set.*

Assumption 2.3.5. *For* $i \in I$, $X^i(V_0)$ *is hyperbolic.*

Assumption 2.3.6. *For every $i, j \in I$, $W^s(X^i(V_0))$ and $W^u(X^j(V_0))$ are transverse.*

Remark 2.3.7. Assumptions 2.3.4-2.3.6 ensure that the vector field V_0 is Morse-Smale along $\partial W^s(X^s(V_0))$ (in fact, it is not assumed initially that the index set I is finite, but this will follow as a consequence of the other assumptions by Lemma 2.3.22). Assumptions 2.3.5-2.3.6 are generic in the C^1 topology. By [Pug67], $\Omega(V_0)$ is generically (in the C^1 topology) equal to the closure of the union of equilibrium points and periodic orbits of V_0. Hence, Assumption 2.3.4 could be relaxed to this generic assumption along with the assumption that there exists an open neighborhood of $\partial W^s(X^s(V_0))$ in which the number of equilibrium points and periodic orbits is finite.

Lemma 2.3.8 is a slight modification of the results in [CHW88] to compact Riemannian manifolds under the assumptions above. It shows that $\overline{W}^s(X^s(V_0))$ admits a decomposition into a union of the stable manifolds of the critical elements it contains, that it is contained in the union of their unstable manifolds, and that these critical elements do not possess any homoclinic orbits or heteroclinic cycles.

Lemma 2.3.8. *Let V_0 be a C^1 vector field on a compact Riemannian manifold M which possesses a stable hyperbolic equilibrium point $X^s(V_0)$ and satisfies Assumptions 2.3.4-2.3.6. Then $\overline{W}^s(X^s(V_0)) = \bigcup_{i \in I} W^s(X^i(V_0)) \bigcup W^s(X^s(V_0)) \subset \bigcup_{i \in I} W^u(X^i(V_0)) \bigcup X^s(V_0)$, I and $\partial W^s(X^s(V_0))$ are nonempty, $W^u(X^i(V_0)) \bigcap W^s(X^s(V_0)) \neq \emptyset$ for all $i \in I$, and there are no homoclinic orbits or heteroclinic cycles among the $\{X^i(V_0)\}_{i \in I}$.*

Theorem 2.3.9 is our main technical result for compact Riemannian manifolds. It states that, under the assumptions above, for V a vector field sufficiently C^1 close to V_0, $\overline{W}^s(X^s(V))$ is contained in a fixed neighborhood of $\overline{W}^s(X^s(V_0))$, the only nonwandering points under V in this neighborhood are the perturbations of the critical elements in $\overline{W}^s(X^s(V_0))$, $\overline{W}^s(X^s(V))$ permits a decomposition into the union of stable manifolds of the perturbations of those critical elements, and $\overline{W}^s(X^s(V))$ is contained in the union of unstable manifolds of the perturbations of those critical elements. In particular, under small C^1 perturbations, $\overline{W}^s(X^s(V))$ cannot explode, no nonwandering points appear in $\overline{W}^s(X^s(V))$ other than the perturbations of those critical elements, and the decomposition into stable manifolds persists under perturbation.

Theorem 2.3.9. *Let V_0 be a C^1 vector field on a compact Riemannian manifold M which possesses a stable hyperbolic equilibrium point $X^s(V_0)$ and satisfies Assumptions 2.3.4-2.3.6. Then there exists an open neighborhood W of V_0 in the C^1 topology and an open neighborhood*

U of $\overline{W}^s(X^s(V_0))$ in M such that $V \in W$ implies that
$\overline{W}^s(X^s(V)) = \bigcup_{i \in I} W^s(X^i(V)) \bigcup W^s(X^s(V)) \subset U \subset \bigcup_{i \in I} W^u(X^i(V)) \bigcup X^s(V)$
and $\Omega(V) \bigcap U = \bigcup_{i \in I} X^i(V) \bigcup X^s(V)$.

Using Theorem 2.3.9 and Corollary 2.2.25, it can then be shown that given a C^1 continuous family of vector fields on M which satisfy the assumptions above, the boundaries of their regions of attraction vary Hausdorff continuously with parameter. This is encapsulated in Corollary 2.3.10.

Corollary 2.3.10. *Let J be a smooth manifold and let $\{V_p\}_{p \in J}$ be a C^1 continuous family of vector fields on M possessing a family of stable hyperbolic equilibria $\{X^s(p)\}_{p \in J}$. Suppose there exists $p_0 \in J$ such that V_{p_0} satisfies Assumptions 2.3.4-2.3.6. Then for J sufficiently small such that $p_0 \in J$, $\partial W^s(X^s(V_J)) = \sqcup_{p \in J} \partial W^s(X^s(V_p))$, and $\{\partial W^s(X^s(V_p))\}_{p \in J}$ is Hausdorff continuous.*

2.3.4.2 Euclidean Space

For this section, let M be any Euclidean space and let V_0 be a complete C^1 vector field on M possessing a stable hyperbolic equilibrium point $X^s(V_0)$. We augment the assumptions above for compact Riemannian manifolds with the following.

Assumption 2.3.11. *$\partial W^s(X^s(V_0)) \neq \emptyset$. In particular, I is nonempty.*

Assumption 2.3.12. *For every $x \in \partial W^s(X^s(V_0))$, its forward orbit is bounded.*

Assumption 2.3.13. *There exists a neighborhood N of infinity such that $\Omega(V_0) \bigcap N = \emptyset$ and every orbit under V_0 contained in N has at least one of its ω and α limit sets under V_0 nonempty.*

Remark 2.3.14. It is possible that $W^s(X^s(V_0)) = M$, such as if V_0 is the vector field corresponding to a stable hyperbolic linear system. Assumption 2.3.11 rules out this possibility.

Remark 2.3.15. If any forward orbit in $\partial W^s(X^s(V_0))$ is unbounded then it cannot be contained in $W^s(X^i(V_0))$ for any $i \in I$. Thus, $\partial W^s(X^s(V_0))$ would not admit the desired decomposition into a union of stable manifolds of critical elements. Assumption 2.3.12 rules out this possibility.

Remark 2.3.16. The first part of Assumption 2.3.13, regarding a neighborhood of infinity with no nonwandering points, is required so that Examples 2.3.1-2.3.2 cannot occur, in which there is no neighborhood of $\overline{W}^s(X^s(V_0))$ in $M \bigcup \{\infty\}$ which does not contain additional nonwandering points other than the $X^i(V_0)$ for $i \in I$. The second part of this assumption, that there is a neighborhood of infinity in which every orbit it contains has at least one of its ω and α limit sets nonempty, will be required to show that for V a vector field sufficiently close to V_0, every point in $\overline{W}^s(X^s(V))$ has a nonempty ω limit set under V.

Remark 2.3.17. Assumptions 2.3.5-2.3.6 are still generic in the strong C^1 topology. Assumption 2.3.4 follows from the assumptions that there exists an open neighborhood of $\partial W^s(X^s(V_0))$ which contains only finitely many equilibrium points and periodic orbits, and the strong C^1 generic assumption that, under Assumption 2.3.13, $\Omega(V_0)$ is contained in the closure of the union of the equilibrium points and periodic orbits of V_0.

Lemma 2.3.18 shows that under the assumptions above, $\partial W^s(X^s(V_0))$ admits a decomposition into the union of stable manifolds of the critical elements it contains, those critical elements do not possess any homoclinic orbits or heteroclinic cycles, and that there exists an open neighborhood of $\overline{W}^s(X^s(V_0))$ containing no additional nonwandering points other than the critical elements in $\overline{W}^s(X^s(V_0))$ and such that the backward orbit of every point in this neighborhood converges to either one of those critical elements or to infinity.

Lemma 2.3.18. *Let V_0 be a complete C^1 vector field on some Euclidean space M which possesses a stable hyperbolic equilibrium point $X^s(V_0)$ and satisfies Assumptions 2.3.4-2.3.12. Then $\partial W^s(X^s(V_0)) = \bigcup_{i \in I} W^s(X^i(V_0))$ and there are no homoclinic orbits or heteroclinic cycles among the $\{X^i(V_0)\}_{i\in I}$. If in addition $\overline{W}^s(X^s(V_0))$ is bounded or V_0 satisfies Assumption 2.3.13, then there exists an open neighborhood U of $\overline{W}^s(X^s(V_0))$ in $M \bigcup \{\infty\}$ such that $\Omega(V_0) \bigcap U = \bigcup_{i\in I} X^i(V_0) \bigcup X^s(V_0)$ and for every $x \in U$ its orbit in backwards time either enters $W^u_{loc}(X^j(V_0))$ for some j or the neighborhood of infinity of Assumption 2.3.13.*

In case $\overline{W}^s(X^s(V_0))$ is bounded, by an analogous proof to the case of compact Riemmanian manifolds in Theorem 2.3.9, the results of that Theorem will hold for Euclidean space as well. In particular, Theorem 2.3.19 shows that, under the assumptions above - excluding Assumption 2.3.13 - for V a vector field sufficiently close to V_0, $\overline{W}^s(X^s(V))$ is contained in a fixed neighborhood of $\overline{W}^s(X^s(V_0))$, the only nonwandering points under V in this neighborhood are the perturbations of the critical elements in $\overline{W}^s(X^s(V_0))$, and $\overline{W}^s(X^s(V))$ permits a decomposition into the union of stable manifolds of the perturbations of those critical elements. So, under small C^1 perturbations, $\overline{W}^s(X^s(V))$ cannot explode in size, no nonwandering points appear in $\overline{W}^s(X^s(V))$ other than the perturbations of those critical elements, and the decomposition into stable manifolds persists under perturbation.

Theorem 2.3.19. *Let V_0 be a C^1 vector field on some Euclidean space M which possesses a stable hyperbolic equilibrium point $X^s(V_0)$, satisfies Assumptions 2.3.4-2.3.12, and such that $\overline{W}^s(X^s(V_0))$ is bounded. Then there exists an open neighborhood W of V_0 in the strong C^1 topology and an open neighborhood U of $\overline{W}^s(X^s(V_0))$ in M such that U is bounded and $V \in W$ implies that $\overline{W}^s(X^s(V)) = \bigcup_{i\in I} W^s(X^i(V)) \bigcup W^s(X^s(V)) \subset U$ and $\Omega(V) \bigcap U = \bigcup_{i\in I} X^i(V) \bigcup X^s(V)$.*

The most interesting cases for Euclidean space occur when $\overline{W}^s(X^s(V_0))$ is unbounded. Example 2.3.3 showed that for M Euclidean and V_0 satisfying all of the assumptions above, there may exist V arbitrarily C^1 close to V_0 such that $\overline{W}^s(X^s(V))$ contains nonwandering point in addition to the perturbations of the critical elements in $\overline{W}^s(X^s(V_0))$ and that $\overline{W}^s(X^s(V))$ may escape any open neighborhood of $\overline{W}^s(X^s(V_0))$. In other words, under strong C^1 perturbations new nonwandering points can enter the closure of the region of attraction, and the closure of the region of attraction can grow discontinuously. Theorem 2.3.20 shows that for a strong C^1 continuous family $\{V_p\}_{p\in J}$ of vector fields on M, if V_{p_0} satisfies the assumptions above as well as some additional strong C^1 generic assumptions, then J sufficiently small implies that for $p \in J$, $\overline{W}^s(X^s(V_p))$ is equal to the union of the stable manifolds of the perturbations of the critical elements in $\overline{W}^s(X^s(V_{p_0}))$ and no additional nonwandering points are in $\overline{W}^s(X^s(V_p))$ other than the perturbations of those critical elements.

Theorem 2.3.20. *Let $\hat{V}$ be a C^1 vector field on some Euclidean space M which possesses a stable hyperbolic equilibrium point $X^s(\hat{V})$. Let W' be the open neighborhood of $\hat{V}$ in the strong C^1 topology such that $V \in W'$ implies that $X^s(V)$ exists and is a stable hyperbolic equilibrium point. Then for $V_{p_0} \in W'$ complete satisfying Assumptions 2.3.4-2.3.13, and satisfying some additional generic assumptions, J a sufficiently small open neighborhood of p_0 implies that for every $p \in J$, $\overline{W}^s(X^s(V_p)) = \bigcup_{i\in I} W^s(X^i(V_p)) \bigcup W^s(X^s(V_p))$ and $\Omega(V_p) \bigcap \overline{W}^s(X^s(V_p)) = \bigcup_{i\in I} X^i(V_p) \bigcup X^s(V_p)$.*

Similarly to the case for compact Riemannian manifolds above, using Theorem 2.3.20 and Corollary 2.2.24, it can then be shown that given a strong C^1 continuous family of vector fields on M which satisfy the assumptions above, the boundaries of their regions of attraction vary Chabauty continuously with parameter. This is the content of Corollary 2.3.21.

Corollary 2.3.21. *Let J be a smooth manifold and let $\{V_p\}_{p\in J}$ be a strong C^1 continuous family of vector fields on M possessing a family of stable hyperbolic equilibria $\{X^s(p)\}_{p\in J}$. Suppose there exists $p_0 \in J$ such that V_{p_0} is complete, $\partial W^s(X^s(V_{p_0})) \neq \emptyset$, V_{p_0} satisfies*

Assumptions 2.3.4-2.3.12, and V_{p_0} satisfies some additional generic assumptions. Then for J sufficiently small such that $p_0 \in J$, $\partial W^s(X^s(V_J)) = \sqcup_{p \in J} \partial W^s(X^s(V_p))$, and $\{\partial W^s(X^s(V_p))\}_{p \in J}$ is Chabauty continuous.

2.3.5 Proofs

Let $I^+ := I \bigcup \{s\}$ so that $\bigcup_{i \in I^+} X^i(V_0) = \bigcup_{i \in I} X^i(V_0) \bigcup X^s(V_0)$.

2.3.5.1 Compact Riemannian Manifolds

The purpose of this section is to prove Theorem 2.3.9 and Corollary 2.3.10. Several technical lemmas will be required.

Lemma 2.3.22 shows that Assumptions 2.3.4-2.3.6 imply that the intersection of the nonwandering set with the RoA boundary consists of a finite union of critical elements rather than a possibly infinite union. This is important because any finite collection of hyperbolic critical elements will persist under sufficiently small perturbations to the vector field, whereas for an infinite number of hyperbolic critical elements there may be no perturbation sufficiently small such that all of the critical elements persist.

Lemma 2.3.22. *Under the conditions of Lemma 2.3.8 the number of critical elements of V_0 contained in $\partial W^s(X^s(V_0))$ is finite.*

Proof of Lemma 2.3.22. Let $K = \partial W^s(X^s(V_0))$ for notational convenience. As K is closed in M compact, K is compact. By Assumption 2.3.5, all equilibrium points and periodic orbits of V_0 in K are hyperbolic. Since K is compact, this implies that there are at most countably many equilibrium points and periodic orbits of V_0 in K. So, assume towards a contradiction that there are infinitely many equilibrium points and periodic orbits $\{X^m\}_{m=1}^{\infty}$ in K. The proof proceeds by showing that $\{X^m\}_{m=1}^{\infty}$ converges along a subsequence to a limit X. It then shows that X is nonwandering and invariant under the flow, which implies it is equal to a union of critical elements since it is in K, and that X is compact, connected, and Hausdorff, so it is a continuum. A theorem of Sierpinski then gives that X must be equal to exactly one critical element, which then contradicts that $X^m \to X$ because hyperbolic critical elements are isolated.

As K is compact, $\mathcal{K}$ is compact. Since each X^m is closed and contained in K, $X^m \in \mathcal{K}$ for all m. As $\mathcal{K}$ is compact, passing to a subsequence if necessary implies that $X^m \to X \in \mathcal{K}$. So, $X \subset K$ is closed and nonempty. As K is compact, X is compact. As M is Hausdorff and X is a subspace of M, X is Hausdorff. Let $x \in X$. We claim that x is nonwandering under V_0. For, let U be any open neighborhood of x and let $T > 0$. As $X^m \to X$ and $x \in X$, there exist $x_m \in X^m$ such that $x_m \to x$. As $x_m \to x \in U$ and U is open, choosing M sufficiently large implies that $x_M \in U$. If x_M is an equilibrium point, then fix $t > T$. If x_M is contained in a periodic orbit then let t be a multiple of its period such that $t > T$. Then $\phi_t(U) \bigcap U \supset \{\phi_t(x_M)\} \bigcap \{x_M\} = \{x_M\} \neq \emptyset$. As x was arbitrary, this implies that X is nonwandering under V_0. Since X is contained in K, by Assumption 2.3.4 this implies that $X \subset \bigcup_{m=1}^{\infty} X^m$. Let $x \in X$ and $t \in \mathbb{R}$. Then since $X^m \to X$ and $x \in X$, there exist $x_m \in X^m$ such that $x_m \to x$. As X^m is invariant under V_0 and $x_m \in X^m$ for each m, $\phi_t(x_m) \in X^m$ for each m. By continuity of the flow, $x_m \to x$ implies that $\phi_t(x_m) \to \phi_t(x)$. Then $\phi_t(x_m) \in X^m$ for all m, $X^m \to X$, and $\phi_t(x_m) \to \phi_t(x)$ implies that $\phi_t(x) \in X$. As x and t were arbitrary, this implies that X is invariant under V_0. So, there must exist $E \subset \{1, 2, 3, ...\}$ such that $X = \bigcup_{m \in E} X^m$. We claim that X is connected. For, suppose not. Then there exist A and B nonempty, disjoint subsets of X which are open in X and such that $X = A \cup B$. Hence, A and B are both closed in X. As X is compact, A and B are both compact. As A and B are compact and disjoint in M Hausdorff, there exist U_A and U_B disjoint, open sets in M such that $A \subset U_A$, and $B \subset U_B$. Note that U_A and U_B disjoint and open implies that $\overline{U}_A$ and U_B are disjoint. Hence, ∂U_A is disjoint from U_A and U_B. In particular, this implies that ∂U_A is disjoint from A and B, so ∂U_A is disjoint from $X = A \bigcup B$. Let $y \in A$ and $z \in B$. As $y, z \in X$ and $X^m \to X$, there must exist $y_m, z_m \in X^m$ such that $y_m \to y$ and $z_m \to z$. Hence, since $y_m \to y \in U_A$ open and $z_m \to z \in U_B$ open, for m sufficiently large $y_m \in U_A$ and $z_m \in U_B \subset M - U_A$. As y_m and z_m lie in X^m connected, there must exist $x_m \in X^m$ such that $x_m \in \partial U_A$. As ∂U_A is compact, passing to a subsequence if necessary implies that $x_m \to x \in \partial U_A$. As $X^m \to X$ and $x_m \to x$, we also have that $x \in X$. So, $x \in \partial U_A \bigcap X$. But, this contradicts that ∂U_A is disjoint from X. So, X is connected. As X is connected, compact, and Hausdorff, it is a continuum. Since X is a continuum and $X = \bigcup_{m \in E} X^m$ is a countable union where $\{X^m\}_{m \in E}$ are pairwise disjoint and closed in X, by a theorem of Sierpinski [Eng89, Theorem 6.1.28] E contains at most one element. As X is nonempty, E contains exactly one element. So, there exists m^* such that $X = X^{m*}$. Since X is a hyperbolic equilibrium point or periodic orbit, by the proofs of [JPdM82, Lemmas 2.1-2.2] there exists an open neighborhood $\hat{N}$ of X such that $\hat{N}$ does not entirely contain any equilibrium points or periodic orbits of V_0 other than X. As $X^m \to X$, m sufficiently large implies that $X^m \subset \hat{N}$. As X^m is either an equilibrium point or period orbit for all m, this contradicts the choice of $\hat{N}$. □

By Lemma 2.3.22, the number of critical elements in the RoA boundary is finite. Hence, the set indexing these critical elements, I, is also finite. Therefore, in the remainder of this section we will use that I is finite without explicitly referencing Lemma 2.3.22.

The following lemma is a technical result that states that the presence of a transverse heteroclinic cycle of hyperbolic critical elements implies that the interesections of their stable and unstable manifolds are nonwandering. This lemma appears in [Sma67, Lemma 7.2] for periodic points of diffeomorphisms. The adoption of the proof to critical elements of vector fields is straightforward and is presented below.

Lemma 2.3.23. *Let V_0 be a C^1 vector field on a compact Riemannian manifold M. Let $(X^1, ..., X^m)$ be a transverse heteroclinic cycle of hyperbolic critical elements of V_0 and let $(x_1, ..., x_m)$ be such that $x_m = x_1$ and $x_i \in \{W^u(X^i) - X^i\} \bigcap \{W^s(X^{i+1}) - X^{i+1}\}$. for $i \in \{1, ..., m-1\}$. Then each x_i is nonwandering.*

Proof of Lemma 2.3.23. The idea is to fix one x_k and argue directly that x_k is nonwandering by taking any open neighborhood U of x_k, any $T > 0$, and showing that some forward iterate of $\phi_T(U)$ must intersect U. To do so, first it is shown that $\bigcup_{t \geq T} \phi_t(U) \bigcap W^s(x_k) \neq \emptyset$ by repeated application of Lemma 2.2.34 using the fact that $(X^1, ..., X^m)$ is a transverse heteroclinic cycle. Then the Inclination Lemma [Pal69] is used to show that this implies $\bigcup_{t \geq T} \phi_t(U) \bigcap U \neq \emptyset$.

First we show that $\bigcup_{t \geq T} \phi_t(U) \bigcap W^s(x_k) \neq \emptyset$. Let $x_k \in (x_1, ..., x_m)$, let U be an open neighborhood of x_k, and let $T > 0$. As $x_k \in \{W^u(X^k) - X^k\} \bigcap \{W^s(X^{k+1}) - X^{k+1}\}$, by invariance of the stable and unstable manifolds, $x_k' = \phi_T(x_k) \in \phi_T(U) \bigcap \{W^u(X^k) - X^k\} \bigcap \{W^s(X^{k+1}) - X^{k+1}\}$. Note that $\bigcup_{t \geq T} \phi_t(U)$ is open and forwards invariant.
Lemma 2.2.34 states that, for X^i and X^j hyperbolic critical elements and X^s a hyperbolic stable equilibrium point, if $W^u(X^i) \bigcap W^s(X^s) \neq \emptyset$ and $W^s(X^i) \bigcap W^u(X^j) \neq \emptyset$ then $W^u(X^j) \bigcap W^s(X^s) \neq \emptyset$. In the proof of Lemma 2.2.34 the only properties of $W^s(X^s)$ that were used were that it was open and backwards invariant. Reversing time in that proof and replacing $W^s(X^s)$ with $\bigcup_{t \geq T} \phi_t(U)$ shows that if $W^s(X^i) \bigcap \bigcup_{t \geq T} \phi_t(U) \neq \emptyset$ and $W^u(X^i) \bigcap W^s(X^j) \neq \emptyset$ then $W^s(X^j) \bigcap \bigcup_{t \geq T} \phi_t(U) \neq \emptyset$. Letting $X^j = X^{i+1}$ and proceeding inductively implies that $W^s(X^k) \bigcap \bigcup_{t \geq T} \phi_t(U) \neq \emptyset$.

Next we show that $\bigcup_{t \geq T} \phi_t(U) \bigcap U \neq \emptyset$. If X^k is an equilibrium point let $B = W^u_{\text{loc}}(X^k)$ and if X^k is a periodic orbit let S be a cross section for a C^1 first return map and let $B = S \bigcap W^u_{\text{loc}}(X^k)$. Since $x_k \in W^u(X^k)$, there exist $y \in W^u_{\text{loc}}(X^k)$ and $T' > 0$ such that $x_k = \phi_{T'}(y)$. By continuity of $\phi_{T'}$ and since U is open, there exists $\epsilon > 0$ such that $d(y, z) <$

ϵ implies $\phi_{T'}(z) \in U$. By the Inclination Lemma, since $W^s(X^k) \bigcap \bigcup_{t \geq T} \phi_t(U) \neq \emptyset$ and $\bigcup_{t \geq T} \phi_t(U)$ is open and forward invariant, there exists a C^1 disk $D \subset \bigcup_{t \geq T} \phi_t(U)$ such that D is ϵ C^1-close to B. Then $\phi_{T'}(D) \bigcap U \neq \emptyset$. By forward invariance, this implies that $\bigcup_{t \geq T} \phi_t(U) \bigcap U \neq \emptyset$ so x_k is nonwandering. □

Lemma 2.3.23 is used in the proof of Lemma 2.3.8 to show that the presence of transverse homoclinic orbits or heteroclinic cycles for the critical elements in $\partial W^s(X^s(V_0))$ would generate additional nonwandering points in $\partial W^s(X^s(V_0))$ and, therefore, would contradict Assumption 2.3.4. Hence, no homoclinic orbits or heteroclinic cycles will exist in $\partial W^s(X^s(V_0))$. This will be required in the proof of Lemma 2.3.25.

Proof of Lemma 2.3.8. That $\partial W^s(X^s(V_0))$ and I are nonempty is a straightforward consequence of α limit sets nonempty since M is compact. Similarly, $\overline{W}^s(X^s(V_0))$ contained in both $\bigcup_{i \in I^+} W^u(X^i(V_0))$ and $\bigcup_{i \in I^+} W^s(X^i(V_0))$ follows from similar reasoning. To show that $\bigcup_{i \in I^+} W^s(X^i(V_0)) \subset \overline{W}^s(X^s(V_0))$, a transverse heteroclinic sequence is constructed inductively using $\bigcup_{i \in I} X^i(V_0)$. Using Lemma 2.3.23 it is shown that this sequence cannot contain any cycles or homoclinic orbits so, as a consequence of I finite, it must terminite with a finite number of elements. By the construction of the sequence, and using Lemma 2.2.34, this will imply that the unstable manifold of each element of the sequence intersects $W^s(X^s(V_0))$. By the proof of Theorem 2.2.12, this is used to conclude that $W^s(X^i(V_0)) \subset \partial W^s(X^s(V_0))$ for all $i \in I$.

First we show that $\partial W^s(X^s(V_0))$ and I are nonempty since M compact implies that α limit sets are nonempty. As ϕ_t restricted to $W^s_{\text{loc}}(V_0)$ is a contraction for any $t > 0$, $\Omega(V_0) \bigcap W^s_{\text{loc}}(X^s(V_0)) = X^s(V_0)$. By invariance of the nonwandering set, and since $W^s(X^s(V_0)) = \bigcup_{t \leq 0} \phi_t(W^s_{\text{loc}}(X^s(V_0)))$, $\Omega(V_0) \bigcap W^s(X^s(V_0)) = X^s(V_0)$. Hence, $\Omega(V_0) \bigcap \overline{W}^s(X^s(V_0)) = \bigcup_{i \in I^+} X^i(V_0)$. There exists $x \in W^s_{\text{loc}}(X^s(V_0))$ with $x \neq X^s(V_0)$. As $\alpha(x)$ is nonempty, nonwandering, and contained in $\overline{W}^s(X^s(V_0))$, and since $\alpha(x) \neq X^s(V_0)$ because ϕ_t restricted to $W^s_{\text{loc}}(V_0)$ is a contraction for any $t > 0$, $\alpha(x) \subset \partial W^s(X^s(V_0)) \bigcap \Omega(V_0) = \bigcup_{i \in I} X^i(V_0)$. This implies that both $\partial W^s(X^s(V_0))$ and I are nonempty.

Next we show that $\overline{W}^s(X^s(V_0))$ is contained in both $\bigcup_{i \in I^+} W^u(X^i(V_0))$ and $\bigcup_{i \in I^+} W^s(X^i(V_0))$, again using the fact that the α and ω limit sets are nonempty. Let $x \in \overline{W}^s(X^s(V_0))$. Then $\omega(x)$ and $\alpha(x)$ are closed, nonempty, and nonwandering. Hence, since $\overline{W}^s(X^s(V_0))$ is closed and invariant, $\omega(x), \alpha(x) \subset \Omega(V_0) \bigcap \overline{W}^s(X^s(V_0)) = \bigcup_{i \in I^+} X^i(V_0)$. As $\bigcup_{i \in I^+} X^i(V_0)$ is a finite union of hyperbolic critical elements, they are separable by open sets. Since $\omega(x), \alpha(x)$ are compact they are connected so, since they are also nonempty,

$\omega(x) = X^j(V_0)$ and $\alpha(x) = X^k(V_0)$ for some $j, k \in I^+$. Furthermore, since $\omega(x) = X^j(V_0)$ and $\alpha(x) = X^k(V_0)$ are compact, the forward orbit of x must converge to $X^j(V_0)$ and the backward orbit of x must converge to $X^k(V_0)$. For, if V is any precompact open neighborhood of $X^j(V_0)$ and the forward orbit of x does not converge to $X^j(V_0)$ then it must intersect ∂V at arbitrarily large times, say $\{t_n\}$ with $t_n \to \infty$. As $\bigcup_n \phi_{t_n}(x) \subset \partial V$ compact, passing to a subsequence if necessary we have that $y = \lim_{n\to\infty} \phi_{t_n}(x) \in \omega(x) \bigcap \partial V$, which contradicts that $\omega(x) = X^j(V_0)$. The backwards orbit case is proved analogously. Hence, $x \in W^s(X^j(V_0)) \bigcap W^u(X^k(V_0))$ for some $j, k \in I^+$. Thus, $\overline{W}^s(X^s(V_0)) \subset \bigcup_{i\in I^+} W^u(X^i(V_0))$ and $\overline{W}^s(X^s(V_0)) \subset \bigcup_{i\in I^+} W^s(X^i(V_0))$.

Now we show that $\bigcup_{i\in I^+} W^s(X^i(V_0)) \subset \overline{W}^s(X^s(V_0))$. In particular, it suffices to show that $W^s(X^i(V_0)) \subset \partial W^s(X^s(V_0))$ for every $i \in I$. Let $i_0 \in I$. First we construct a sequence inductively. Given a sequence $(i_0, i_1, ..., i_k)$, since $X^{i_k}(V_0) \subset \partial W^s(X^s(V_0))$ is a hyperbolic critical element, by Lemma 2.2.29, either $W^u(X^{i_k}(V_0)) \bigcap W^s(X^s(V_0)) \neq \emptyset$, in which case we end the sequence, or $\{W^u(X^{i_k}(V_0)) - X^{i_k}(V_0)\} \bigcap \partial W^s(X^s(V_0)) \neq \emptyset$. In the latter case, since $\partial W^s(X^s(V_0)) \subset \bigcup_{i\in I} W^s(X^i(V_0))$, there exists i_{k+1} such that $(W^u(X^{i_k}(V_0)) - X^{i_k}(V_0)) \bigcap (W^s(X^{i_{k+1}}(V_0)) - X^{i_{k+1}}(V_0)) \neq \emptyset$, thus yielding the new sequence $(i_0, i_1, ..., i_k, i_{k+1})$. We iterate this process to construct the sequence. Suppose that the sequence $(i_0, i_1, ...)$ never terminates. Since I is finite, this implies that there must exist a cycle $(i_l, i_{l+1}, ..., i_{l+m}) \subset (i_0, i_1, ...)$ with $i_{l+m} = i_l$. By the construction of the sequence, this implies there there exist $(x_l, ..., x_{l+m})$ with $x_l = x_{l+m}$ such that $x_{l+j} \in (W^u(X^{i_{l+j}}(V_0)) - X^{i_{l+j}}(V_0)) \bigcap (W^s(X^{i_{l+i+j}}(V_0)) - X^{i_{l+i+j}}(V_0)) \bigcap \partial W^s(X^s(V_0))$ for $j \in \{0, 1, ..., m-1\}$. Thus, by Lemma 2.3.23 each x_{l+j} for $j \in \{0, 1, ..., m-1\}$ is nonwandering. By construction, each such x_{l+j} is contained in $\partial W^s(X^s(V_0))$ and not contained in $\bigcup_{i\in I} X^i(V_0)$. However, this contradicts that $\Omega(V_0) \bigcap \partial W^s(X^s(V_0)) = \bigcup_{i\in I} X^i(V_0)$. So, the sequence $(i_0, i_1, ...)$ must terminate in a finite number of steps. Hence, suppose the sequence $(i_0, i_1, ...)$ terminates in a finite number of steps, say k. Then $W^u(X^{i_k}(V_0)) \bigcap W^s(X^s(V_0)) \neq \emptyset$. By repeated application of Lemma 2.2.34, this implies that $W^u(X^{i_l}(V_0)) \bigcap W^s(X^s(V_0)) \neq \emptyset$ for each $i_l \in (i_0, ..., i_k)$. In particular, $W^u(X^{i_0}(V_0)) \bigcap W^s(X^s(V_0)) \neq \emptyset$. By the proof of Theorem 2.2.12, this implies that $W^s(X^{i_0}(V_0)) \subset \partial W^s(X^s(V_0))$. Thus, $\bigcup_{i\in I} W^s(X^i(V_0)) \subset \partial W^s(X^s(V_0))$ so, combining with the above we have that $\overline{W}^s(X^s(V_0)) = \bigcup_{i\in I^+} W^s(X^i(V_0)) \subset \bigcup_{i\in I^+} W^u(X^i(V_0))$. As $\Omega(\overline{W}^s(X^s(V_0))) = \bigcup_{i\in I^+} X^i(V_0)$, there cannot be any homoclinic orbits or heteroclinic cycles among the $\{X_i(V_0)\}_{i\in I^+}$ since transverse homoclinic points are nonwandering (see for example the proof of Lemma 2.2.32) and, by the argument above regarding heteroclinic cycles, both would introduce additional nonwandering points into $\overline{W}^s(X^s(V_0))$. □

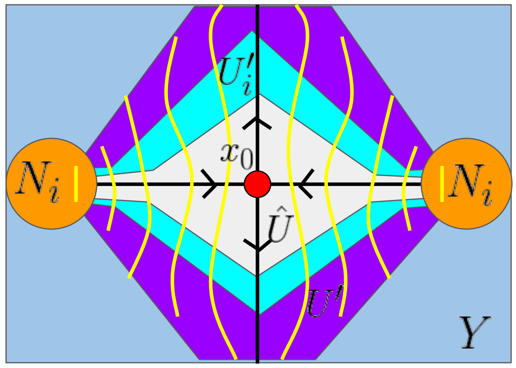

Figure 2.10: Illustration of the key ideas behind the proof of Lemma 2.3.24. The light blue rectangle Y represents the choice of local coordinates, the orange circles compose the neighborhood N_i, the yellow curves represent the disk family centered along $W^s_{\text{loc}}(x_0)$ that is constructed in the proof, the purple neighborhood U' is contained in this disk family, the turquoise neighborhood $U_i' \subset U'$ is contained in the disk family corresponding to perturbations to the vector field, and the gray neighborhood $\hat{U} \subset U_i'$ is such that under perturbations to the vector field every backwards orbit originating in $\hat{U}$ remains in U_i' until it passes through N_i.

Lemma 2.3.24 is a technical construction that is required in the proof of Lemma 2.3.25, which itself is the main result used to prove Theorem 2.3.9. The Lemma states that for any critical element with stable manifold having dimension at least one, for any fundamental neighborhood of that stable manifold there exists an open neighborhood of that critical element such that for any sufficiently small C^1 perturbation of the vector field, the backward orbit of every point in that open neighborhood either converges to the critical element or enters the fundamental neighborhood in finite negative time. A similar result was used in the proof of [Pal69, Theorem 1.9], but a complete proof was not provided.

Lemma 2.3.24. *Let $i \in I$ such that the dimension of $W^s(X^i(V_0))$ is at least one. If $X^i(V_0)$ is an equilibrium point let $D_i(V)$ be the closure of $\{W^s_{loc}(X^i(V)) - \phi_1(W^s_{loc}(X^i(V)))\}$ in M and let N_i be an open neighborhood of $D_i(V_0)$ in M. If $X^i(V_0)$ is a periodic orbit let S be a smooth cross section with $\hat{\tau} : S \to S$ the C^1 first return map, let $D_i(V)$ be the closure of $\{(W^s_{loc}(X^i(V)) \bigcap S) - \hat{\tau}(W^s_{loc}(X^i(V)) \bigcap S)\}$ in S, and let N_i be an open neighborhood of $D_i(V_0)$ in M. Then there exist an open neighborhood U_i of $W^s_{loc}(X^i(V_0))$ in M and an open*

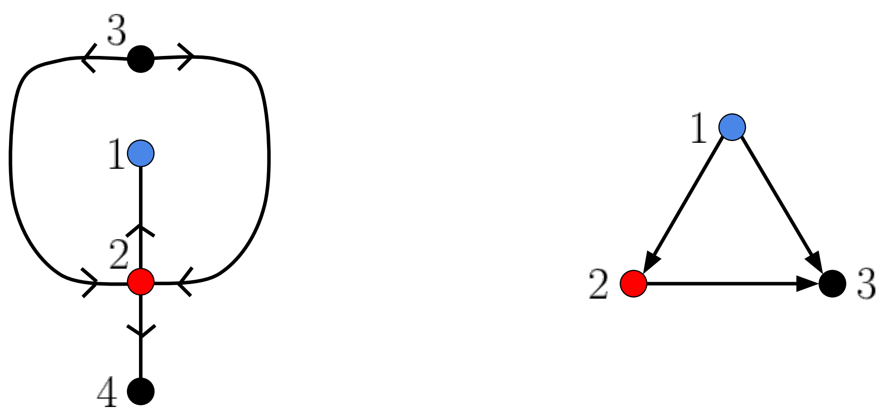

Figure 2.11: Example of the construction of the directed graphs in the proof of Lemma 2.3.25. A vector field on the sphere $\mathbb{S}^2$ (left) and the corresponding directed graph (right) for the closure of the region of attraction of its stable hyperbolic equilibrium point (blue dot) are shown. A saddle hyperbolic equilibrium point (red dot), along with a sink and a source (black dots) are also shown. Note that equilibrium point four does not appear in the directed graph since it is not contained in the closure of the region of attraction.

neighborhood W_i of V_0 in the C^1 topology such that $V \in W_i$ implies that $W^s_{loc}(X^i(V)) \subset U_i$, $D_i(V) \subset N_i$, and for every $x \in U_i - W^u(X^i(V))$, the backwards orbit of x under V intersects N_i in finite negative time. Furthermore, there exists $\hat{U} \subset U_i$ open such that $V \in W$ implies $X^i(V) \subset \hat{U}$ and backwards orbit of any $x \in \hat{U}$ under V is contained in U_i at least until it enters N_i.

Proof of Lemma 2.3.24. The result nearly follows from Lemma 2.2.37, but a slight extension is required. The details are presented in Appendix C. □

Lemma 2.3.25 is the main technical result behind Theorem 2.3.9. It states that there exists an open neighborhood such that for any vector field sufficiently C^1 close to V_0, this open neighborhood contains the union of the stable manifolds of the perturbations of all the critical elements in I^+, and this open neighborhood is contained in the union of their unstable manifolds.

Lemma 2.3.25. *Under the conditions of Lemma 2.3.8 there exists an open neighborhood U of $\overline{W}^s(X^s(V_0))$ in M and an open neighborhood W of V_0 in the C^1 topology such that $V \in W$ implies that $\bigcup_{i \in I^+} W^s(X^i(V)) \subset U \subset \bigcup_{i \in I^+} W^u(X^i(V))$.*

Proof of Lemma 2.3.25. The first step in the proof is the construction of a graph whose vertices are the critical elements in I^+ and whose directed edges indicate the intersection of the stable and unstable manifolds of the vertices they connect. Figure 2.11 provides an example directed graph for a vector field on the sphere $\mathbb{S}^2$. The goal is to construct an open neighborhood of the critical elements such that under the flow of any vector field sufficiently C^1 close to V_0, the backwards orbits of every point in the open neighborhood eventually enter the local unstable manifold of some critical element. This is achieved by constructing the open neighborhood inductively by traversing the directed graph starting from the critical elements which have zero dimensional stable manifolds: call them sources. When each critical element is reached in the induction, an open neighborhood U_i is constructed containing that critical element and an open fundamental neighborhood of its stable manifold is constructed such that under the flow of any vector field V C^1 close to V_0, the backward orbit of any point in U_i will enter either $W^u_{\text{loc}}(X^i(V))$ or the fundamental neighborhood, and the backward orbit of any point in the fundamental neighborhood will enter U_j for some element j which is strictly closer to the sources in the directed graph than i is. Lemma 2.3.24 is used to construct the desired U_i. This will then imply that the union of these neighborhoods is contained in the union of the unstable manifolds of the critical elements. To construct a neighborhood which contains their stable manifolds as well, it is necessary to construct $U_i' \subset U_i$ and $N_i' \subset N_i$ contained in the fundamental neighborhood such that the backward orbit of any point in U_i' does not escape U_i at least until it enters N_i' and the backward orbit of any point in N_i' does not escape N_i at least until it enters U_j for some $j \neq i$. As the local stable manifolds are contained in the union of the U_i by construction, the above construction will imply that the backwards orbits of any point in the fundamental domain of the stable manifold of $X^i(V)$ does not escape the union of the neighborhoods above. Consequently, the entire stable manifold is contained in this neighborhood.

First the directed graph of the critical elements is constructed. By Lemma 2.3.8, $\overline{W}^s(X^s(V_0)) = \bigcup_{i \in I^+} W^s(X^i(V_0)) \subset \bigcup_{i \in I^+} W^u(X^i(V_0))$. Hence, we can draw a directed graph with the elements of I^+ as vertices such that there is an edge from i to j if and only if $W^s(X^i(V_0)) \bigcap W^u(X^j(V_0)) \neq \emptyset$ for any $i, j \in I^+$. By Lemma 2.3.8, I and $\partial W^s(X^s(V_0))$ are nonempty, so this graph contains at least two vertices and at least one edge. By Lemma 2.3.8 there are no homoclinic orbits, so no self edges in the graph, and no heteroclinic cycles, so no cycles in the graph. Thus, since I^+ is finite, every directed path, hereafter referred to simply as paths, in the graph must be finite. We define a forward maximal path to be a path whose final vertex has no outgoing edges. Similarly, a backward maximal path is defined to be a path whose first vertex has no incoming edges. A maximal path is a path which is both forward and backward maximal. Let $E_0 \subset I^+$ be the collection of vertices such that for $i \in E_0$, the dimension of $W^u(X^i(V_0))$ is equal to the dimension of M. We claim that every backward maximal path starts with s and every forward maximal path ends with some element of E_0. For, suppose $(i_1, ..., i_k)$ is backward maximal and assume towards a contradiction that $i_1 \neq s$. Then, by Lemma 2.3.8, $W^s(X^s(V_0)) \bigcap W^u(X^{i_1}(V_0)) \neq \emptyset$. Hence, $(s, i_1, ..., i_k)$ is a path, which contradicts that $(i_1, ..., i_k)$ was backward maximal. Now suppose $(i_1, ..., i_k)$ is forwards maximal and assume towards a contradiction that $i_k \notin E_0$. This implies that the dimension of $W^u(X^{i_k}(V_0))$ is less than the dimension of M. Since $X^{i_k}(V_0)$ is hyperbolic, $W^s(X^{i_k}(V_0))$ must have dimension at least one, so there exists $x \in \{W^s(X^{i_k}(V_0)) - X^{i_k}(V_0)\}$. Then $x \subset \partial W^s(X^s(V_0)) \subset \bigcup_{i \in I} W^u(X^i(V_0))$ so $x \in W^u(X^j(V_0))$ for some $j \in I$. Hence, $(i_1, ..., i_k, j)$ is a path, which contradicts that $(i_1, ..., i_k)$ is forwards maximal.

Next some notation is introduced which will be useful below. Let $b(i, j)$ be the behavior of i to j [Pal69], which is defined to be the path of maximum length from i to j, where we define $b(i, j) = 0$ if there is no path from i to j. For example, in the directed graph of Figure 2.11, $b(1, 2) = 1$, $b(2, 3) = 1$, and $b(1, 3) = 2$. For any $E \subset I^+$, define $b(i, E)$ to be the maximum of $b(i, j)$ over $j \in E$. Let $s_k = \{i \in I^+ : b(i, E_0) = k\}$ and let $E_k = \bigcup_{l=0}^{k} s_l$. For each $i \in I^+ - E_0$, if $X^i(V_0)$ is an equilibrium point let $D_i(V_0)$ be the closure of $\{W^s_{\text{loc}}(X^i(V_0)) - \phi_1(W^s_{\text{loc}}(X^i(V_0))\}$, and if $X^i(V_0)$ is a periodic orbit let S be a smooth cross section of $X^i(V_0)$ with C^1 first return map $\hat{\tau}$ and let $D_i(V_0)$ be the closure of $\{W^s_{\text{loc}}(X^i(V_0)) \bigcap S - \hat{\tau}(W^s_{\text{loc}}(X^i(V_0)) \bigcap S)\}$. Then $D_i(V_0)$ is called a fundamental domain of $X^i(V_0)$ [Pal69] and satisfies $\bigcup_{t \in \mathbb{R}} \phi_t(D_i(V_0)) = W^s(X^i(V_0))$. Note that $D_i(V_0)$ is compact since it is closed in M compact. For each $i \in E_0$, let $U_i' \subset U_i$ be open neighborhoods of $X^i(V_0)$ in M and W_i an open neighborhood of V_0 in the C^1 topology such that $V \in W_i$ implies that $U_i \subset W^u_{\text{loc}}(X^i(V))$ and the backwards orbit of U_i' does not escape U_i under V. This is possible since $U_i' \subset U_i \subset W^s_{\text{loc}}(X^i(V))$ and the time-t flow ϕ_t of V restricted to $W^s_{\text{loc}}(X^i(V))$ is a contraction for any $t > 0$.

The main core of the argument involves proving the claim below, which constructs the pieces needed to give the neighborhood U in the statement of the Lemma with the desired properties. We claim that for every $i \in I^+ - E_0$, say $b(i, E_0) = l > 0$, there exist open neighborhoods $N_i' \subset N_i$ of $D_i(V_0)$ in M, open neighborhoods $U_i' \subset U_i$ of $W^s_{\text{loc}}(X^i(V_0))$ in M, $T_i > 0$, and an open neighborhood W_i of V_0 in the C^1 topology such that $V \in W_i$ implies that $W^s_{\text{loc}}(X^i(V)) \subset U_i'$, $D_i(V) \subset N_i'$, for every $x \in N_i$ there exists $t \in [0, T_i]$ such that $\phi_{-t}(x) \in \bigcup_{j \in E_{l-1}} U_j$, for every $x \in N_i'$ there exists $t \in [0, T_i]$ such that $\phi_{[-t,0]}(x, V) \subset N_i$ and

$\phi_{-t}(x) \in \bigcup_{j \in E_{l-1}} U_j'$, every backward orbit of points in U_i is either in $W^u(X^i(V))$ or enters N_i in finite negative time, and every backward orbit of points in U_i' is contained in U_i at least until it enters N_i.

We proceed by induction in k on the sets E_k of the directed graph for V_0. Suppose that for every $i \in E_k$, say $b(i, E_0) = l \leq k$, there exist open neighborhoods $N_i' \subset N_i$ of $D_i(V_0)$, open neighborhoods $U_i' \subset U_i$ of $W^s_{\text{loc}}(X^i(V_0))$, $T_i > 0$, and an open neighborhood W_i of V_0 such that $V \in W_i$ implies that $W^s_{\text{loc}}(X^i(V)) \subset U_i'$, $D_i(V) \subset N_i'$, for every $x \in N_i$ there exists $t \in [0, T_i]$ such that $\phi_{-t}(x) \in \bigcup_{j \in E_{l-1}} U_j$, for every $x \in N_i'$ there exists $t \in [0, T_i]$ such that $\phi_{[-t,0]}(x, V) \subset N_i$ and $\phi_{-t}(x) \in \bigcup_{j \in E_{l-1}} U_j$, every backward orbit of points in U_i is either in $W^u(X^i(V))$ or enters N_i in finite negative time, and every backward orbit of points in U_i' is contained in U_i at least until it enters N_i. Let $i \in s_{k+1}$. Note that this implies that every path starting from i consists only of vertices in E_k (other than i itself) since otherwise $b(i, E_0)$ would be greater than $k+1$. Hence, $W^s(X^i(V_0)) \bigcap W^u(X^j(V_0)) \neq \emptyset$ is possible only if $j \in E_k$. Consequently, $D_i(V_0) = D_i(V_0) \bigcap \bigcup_{j \in E_k} W^u(X^j(V_0))$ since $D_i(V_0) \subset W^s(X^i(V_0)) \subset \partial W^s(X^s(V_0)) \subset \bigcup_{j \in I^+} W^u(X^j(V_0))$ and by the above remark. Let $x \in D_i(V_0) \bigcap W^u(X^j(V_0))$ for some $j \in E_k$. As U_j' is an open neighborhood of $X^j(V_0)$, there exists $T_x > 0$ such that $\phi_{-T_x}(x) \in U_j'$. By continuity of the flow, there exists an open neighborhood U_x of x such that $\phi_{-T_x}(U_x) \subset U_j'$. Repeating this for each $j \in E_k$, we have that the collection $\{U_x\}_{x \in D_i(V_0)}$ is an open cover of $D_i(V_0)$, so there exists a finite subcover, say $\{U_{x_n}\}$. Choose U_{x_n}' and U_{x_n}'' open such that $\overline{U}_{x_n}'' \subset U_{x_n}' \subset \overline{U}_{x_n}' \subset U_{x_n}$ and $D_i(V_0) \subset \bigcup_n U_{x_n}''$. Let $N_i' = \bigcup_n U_{x_n}''$ and let $T_i = \max_n T_{x_n}$. Then N_i' is an open neighborhood of $D_i(V_0)$ such that for each $x \in N_i'$ there exists $t \in [0, T_i]$ such that $\phi_{-t}(x) \in \bigcup_{j \in E_k} U_j'$. Since N_i' is an open neighborhood of $D_i(V_0)$, by Lemma 2.3.24 there exist an open neighborhood U_i of $W^s_{\text{loc}}(X^i(V_0))$ in M, an open neighborhood W_i of V_0 in the C^1 topology, and an open neighborhood U_i' of $X^i(V_0)$ in M such that $V \in W_i$ implies that $W^s_{\text{loc}}(X^i(V)) \subset U_i$, $X^i(V) \in U_i'$, $D_i(V) \subset N_i'$, for every $x \in U_i - W^u(X^i(V))$, the backwards orbit of x under V intersects N_i' in finite negative time, and for every $x \in U_i'$ its backwards orbit is contained in U_i at least until it enters N_i'. Shrink W_i if necessary so that $V \in W_i$ implies that $\phi_{-T_{x_n}}(U_{x_n}, V) \subset \bigcup_{j \in E_k} U_j'$ for all n, which is possible by continuity of the flow and since n is finite. Let $\epsilon_i > 0$ such that the Hausdorff distance from $\partial \phi_{[-T_{x_n},0]}(U_{x_n}', V_0)$ to $\partial \phi_{[-T_{x_n},0]}(U_{x_n}, V_0)$ is greater than ϵ_i, which is possible since the closure of the former is contained in the latter. Similarly, shrink ϵ_i if necessary so that the Hausdorff distance from $\partial \phi_{[-T_{x_n},0]}(U_{x_n}', V_0)$ to $\partial \phi_{[-T_{x_n},0]}(U_{x_n}'', V_0)$ is greater than ϵ_i. Shrink W_i if necessary so that $V \in W_i$ implies that $\phi_{[-T_{x_n},0]}(U_{x_n}, V)$ is ϵ_i C^1-close to $\phi_{[-T_{x_n},0]}(U_{x_n}, V_0)$ and $\phi_{[-T_{x_n},0]}(U_{x_n}'', V)$ is ϵ_i C^1-close to $\phi_{[-T_{x_n},0]}(U_{x_n}'', V_0)$. Let

$N_n := \phi_{[-T_{x_n},0]}(U'_{x_n}, V_0)$. The above implies that for any $V \in W_i$, $\phi_{[-T_{x_n},0]}(U''_{x_n}, V) \subset N_n \subset \phi_{[-T_{x_n},0]}(U_{x_n}, V)$. Let $N_i = \bigcup_n N_n$. Then N_i is open and for every $V \in W_i$ and $x \in N_i$ there exists $t \in [0, T_i]$ such that $\phi_{-t}(x, V) \in \bigcup_{j \in E_k} U'_j$. Furthermore, for every $V \in W_i$ and $x \in D_i(V)$ there exists $t \in [0, T_i]$ such that $\phi_{-t}(x, V) \in \bigcup_{j \in E_k} U'_j$ and $\phi_{[-t,0]}(x, V) \subset N_i$. This satisfies the induction step for i. Repeating for all $i \in s_{k+1}$ completes the induction.

Now we prove the claim. Let $W = \bigcap_{i \in I^+} W_i$. Let $U = \bigcup_{i \in I^+} U_i \bigcup_{i \in I^+ - E_0} N_i$. First we show that U is contained in the union of unstable manifolds of the critical elements. Now let $V \in W$ and let $x \in U$. By construction, the backwards orbit of x passes from U_i to either $W^u(X^i(V))$ or to N'_i and then to U'_j with $b(j, E_0) < b(i, E_0)$ and repeats this process. Note that for any $x \in N_i$, its backwards orbit also passes to U_j with $b(j, E_0) < b(i, E_0)$. Since $b(i, E_0)$ is finite, repeating this process the orbit must ultimately enter $W^u(X^j(V))$ for some $j \in I^+ - E_0$ or must enter $U_k \subset W^u(X^k(V))$ for some $k \in E_0$. Hence, $U \subset \bigcup_{i \in I^+} W^u(X^i(V))$. Next we show that U contains the union of stable manifolds of the critical elements. This is trivially true for sources since for any $V \in W$ and $i \in E_0$, $W^s(X^i(V)) = X^i(V) \subset U_i \subset U$. So, let $i \in I^+ - E_0$. Then $W^s_{\text{loc}}(X^i(V)) \subset U_i \subset U$. Therefore, it suffices to show that for any $x \in D_i(V)$ the backwards orbit of x under V is contained in U. So, let $x \in D_i(V)$. Then $x \in N'_i$ so there exists $t \in [0, T_i]$ such that $\phi_{[-t,0]}(x, V) \subset N_i$, $\phi_{-t}(x, V) \in U'_j$ with $b(j, E_0) < b(i, E_0)$, and either $\phi_{-t}(x, V) \in W^u_{\text{loc}}(X^j(V))$ or the backwards orbit of $\phi_{-t}(x, V)$ remains in U_j until it enters N'_j, say at y. Then there exists $t \in [0, T_j]$ such that $\phi_{[-t,0]}(y, V) \subset N_j$ and $\phi_{-t}(y, V) \in U'_k$ with $b(k, E_0) < b(j, E_0)$. As $b(i, E_0)$ is finite, this process must terminate in a finite number of steps with the backwards orbit of x entering $W^u_{\text{loc}}(X^j(V)) \cap U'_j$ for some $j \in I^+$, say at z. By construction, the backwards orbit of z under V does not escape U_j. As $W^s_{\text{loc}}(X^i(V)) \bigcup_{t<0} \phi_t(D_i(V)) = W^s(X^i(V))$, this implies that $W^s(X^i(V)) \subset U$. Hence, $U \supset \bigcup_{i \in I^+} W^s(X^i(V))$. Thus, by Lemma 2.3.8, $U \supset \bigcup_{i \in I^+} W^s(X^i(V_0)) = \overline{W}^s(X^s(V_0))$. $\square$

Lemma 2.3.26 shows that under the conditions of Lemma 2.3.25, for any vector field sufficiently C^1 close to V_0, the only nonwandering points in U for this vector field are the perturbations of the critical elements of V_0 in U.

Lemma 2.3.26. *Under the conditions of Lemma 2.3.25, for $V \in W$,*
$\Omega(V) \bigcap U = \bigcup_{i \in I^+} X^i(V)$.

Proof of Lemma 2.3.26. Assuming the existence of a nonwandering point other than the perturbations of the critical elements in I^+, the C^1 Closing Lemma implies that there is a nearby vector field and a nearby point in U which is periodic. However, by Lemma 2.3.25 this contradicts that U is contained in the union of unstable manifolds of the perturbed critical elements. Let $V \in \mathcal{W}$ and assume towards a contradiction that $x \in \Omega(V) \bigcap U$ such that $x \notin \bigcup_{i \in I^+} X^i(V)$. As $\bigcup_{i \in I^+} X^i(V)$ is a finite union of hyperbolic critical elements, they are separable by open sets, say $\bigcup_{i \in I^+} U_i'$ with $X^i(V) \subset U_i'$ for each $i \in I^+$ and the U_i' pairwise disjoint. Let U_0' be an open neighborhood of x which, by shrinking the set of U_i', can be made disjoint from $\bigcup_{i \in I^+} U_i'$. For each $i \in I^+$ let $U_i = U_i' \cap U$ and let $U_0 = U_0' \cap U$. Let $\mathcal{W}' \subset \mathcal{W}$ such that $V \in \mathcal{W}'$ and $V' \in \mathcal{W}'$ implies $X^i(V') \subset U_i$ for all $i \in I^+$. By the C^1 Closing Lemma [Pug67] there exists $V' \in \mathcal{W}'$ and $y \in U_0$ such that y is contained in a periodic orbit. By construction, $y \notin X^i(V')$ for any $i \in I^+$. As $V' \in \mathcal{W}' \subset \mathcal{W}$, $y \in U_0 \subset U \subset \bigcup_{i \in I^+} W^u(X^i(V'))$, which contradicts that y is periodic. □

Lemma 2.3.27 shows that the closure of the stable manifold of the perturbed stable equilibrium point of any vector field sufficiently close to V_0 is equal to the union of the stable manifolds of the perturbed critical elements in I^+.

Lemma 2.3.27. *Under the conditions of Lemma 2.3.25, shrinking $\mathcal{W}$ if necessary, for $V \in \mathcal{W}$ $\overline{W}^s(X^s(V)) = \bigcup_{i \in I^+} W^s(X^i(V))$.*

Proof of Lemma 2.3.27. It suffices to show that $\partial W^s(X^s(V)) = \bigcup_{i \in I} W^s(X^i(V))$. That $\bigcup_{i \in I} W^s(X^i(V)) \subset \partial W^s(X^s(V))$ will follow from the proof of Theorem 2.2.23. For the other direction, choosing $U' \subset \overline{U}' \subset U$ from Lemma 2.3.25, we show that $\overline{W}^s(X^s(V)) \subset U$ which implies, by Lemma 2.3.26, that its nonwandering set is equal to $\bigcup_{i \in I^+} X^i(V)$. The result then follows.

First we show that $\bigcup_{i \in I} W^s(X^i(V)) \subset \partial W^s(X^s(V))$. By the argument in the proof of Theorem 2.2.23, shrinking $\mathcal{W}$ if necessary implies that for $V \in \mathcal{W}$ and $i \in I$, $W^u(X^i(V)) \bigcap W^s(X^s(V)) \neq \emptyset$. By the proof of Theorem 2.2.12, this implies that $W^s(X^i(V)) \subset \partial W^s(X^s(V))$ for every $i \in I$. Hence, $\bigcup_{i \in I} W^s(X^i(V)) \subset \partial W^s(X^s(V))$.

Next we show that $\partial W^s(X^s(V)) \subset \bigcup_{i \in I} W^s(X^i(V))$. Observing the proof of Lemma 2.3.25, we may choose $U' \subset U$ open in M and $\mathcal{W}' \subset \mathcal{W}$ open in the C^1 topology such that $\overline{U}' \subset U$ and the conclusions of Lemma 2.3.25 hold for U' and $\mathcal{W}'$. Hence, $V \in \mathcal{W}'$ implies that $W^s(X^s(V)) \subset U'$, which implies that $\overline{W}^s(X^s(V)) \subset \overline{U}' \subset U$. So, $\partial W^s(X^s(V)) \subset U$. By Lemma 2.3.26, $\Omega(V) \bigcap U = \bigcup_{i \in I^+} X^i(V)$, so $\Omega(V) \bigcap \partial W^s(X^s(V)) = \bigcup_{i \in I^+} X^i(V)$. For $x \in \partial W^s(X^s(V))$, $\omega(x)$ is closed, invariant, and nonwandering, so $\omega(x) \subset \bigcup_{i \in I} X^i(V)$. As in the proof of Lemma 2.3.8, this implies that $\omega(x) = X^j(V)$ for some $j \in I$ and that x converges to $X^j(V)$ in forwards time. Hence, $x \in W^s(X^j(V))$. Thus, for $V \in \mathcal{W}'$, $\partial W^s(X^s(V)) \subset \bigcup_{i \in I} W^s(X^i(V))$. □

Proof of Theorem 2.3.9. Follows immediately from Lemmas 2.3.25-2.3.27. □

Lemma 2.3.28 states that the nonwandering points in the boundary of the union of stable manifolds of the stable equilibrium point, where the boundary is taken in $M \times J'$, is equal to the union of the perturbations of the critical elements in I. In addition, this boundary in $M \times J'$ does not contain any additional nonwandering points beyond those contained in the union of boundaries of the stable manifolds taken in M.

Lemma 2.3.28. *Under the conditions of Corollary 2.3.10, for J sufficiently small with $p_0 \in J$,*

$$\begin{aligned} &\Omega(V_J) \bigcap \partial W^s(X^s(V_J)) \\ &= \sqcup_{p\in J} \Omega(V_p) \bigcap \partial W^s(X^s(V_p)) \\ &= \bigcup_{i \in I} X^i_J. \end{aligned}$$

Lemma 2.3.28. The proof uses Lemma 2.3.25 to get neighborhoods $U' \subset \overline{U}' \subset U$ such that $W^s(X^s(V_p)) \subset U'$ for all $p \in J$, so $W^s(X^s(V_J)) \subset U' \times J$. Then $\overline{W}^s(X^s(V_J)) \subset \overline{U}' \times J \subset U \times J$ and the result follows from Lemma 2.3.26.

By Lemma 2.3.25 and Lemma 2.3.27, there exist U an open neighborhood of $W^s(X^s(p_0))$ and J sufficiently small such that $p_0 \in J$ and for any $p \in J$, $\overline{W}^s(X^s(V_p)) = \bigcup_{i\in I^+} W^s(X^i(V_p)) \subset U \subset \bigcup_{i \in I^+} W^u(X^i(V_p))$. Observing the proof of Lemma 2.3.25, we may choose $U' \subset U$ open in M and shrink J further such that $\overline{U}' \subset U$ and the conclusions of Lemma 2.3.25 hold for U' and $p \in J$. In particular, for any $p \in J$, $W^s(X^s(V_p)) \subset U'$. Hence, $W^s(X^s(V_J)) \subset U' \times J$, so $\overline{W}^s(X^s(V_J)) \subset \overline{U}' \times J$ where the closure is taken in $M \times J$. As $\overline{U}' \subset U$, this implies that $\overline{W}^s(X^s(V_J)) \subset U \times J$. By Lemma 2.3.26, for each $p \in J$ $\Omega(V_p) \bigcap U = \bigcup_{i \in I^+} X^i(V_p)$. Thus, $\Omega(V_J) \bigcap U \times J = \bigcup_{i\in I^+} X^i(V_J)$. So, $\overline{W}^s(X^s(V_J)) \subset U \times J$ implies that $\Omega(V_J) \bigcap \overline{W}^s(X^s(V_J)) \subset \bigcup_{i \in I^+} X^i(V_J)$. By Lemma 2.3.27, we have $\bigcup_{i\in I^+} X^i(V_J) \subset \overline{W}^s(X^s(V_J))$, so $\Omega(V_J) \bigcap \overline{W}^s(X^s(V_J)) = \bigcup_{i \in I^+} X^i(V_J)$. Since $\bigcup_{i\in I} X^i(V_J) \subset \partial W^s(X^s(V_J))$, this implies that $\Omega(V_J) \bigcap \partial W^s(X^s(V_J)) = \bigcup_{i \in I} X^i(V_J)$. By Lemma 2.3.26,
$\sqcup_{p \in J} \Omega(V_p) \bigcap \partial W^s(X^s(V_p)) = \sqcup_{p\in J} \bigcup_{i\in I} X^i(V_p) = \bigcup_{i \in I} X^i_J$. □

Proof of Corollary 2.3.10. The conclusions of the corollary will follow immediately from Theorem 2.2.23 and Corollary 2.2.25 once the assumptions behind them are verified. In particular, Assumptions 2.2.14-2.2.15 and Assumptions 2.2.17-2.2.18 must be verified. Let U be the neighborhood of Lemma 2.3.26, so that $\Omega(V_J) \bigcap (U \times \{p_0\}) = \Omega(V_p) \bigcap U = \bigcup_{i \in I} X^i(p_0)$. By the proof of Lemma 2.3.28, $\partial W^s(X^s(V_J)) \bigcap (M \times \{p_0\}) \subset U \times \{p_0\}$. In particular, U is an open neighborhood of $\partial W^s(X^s(V_J)) \bigcap (M \times \{p_0\})$ in $(M \times \{p_0\})$ such that

$\Omega(V_J)\bigcap(U \times \{p_0\}) = \bigcup_{i\in I} X^i(p_0)$ is a finite union of hyperbolic critical elements in M, thus satisfying Assumptions 2.2.14-2.2.15. The above implies that the critical elements contained in $\partial W^s(X^s(V_J))\bigcap(M \times \{p_0\})$ are at most $\{X^i_{p_0}\}_{i\in I}$. By Assumption 2.3.6, the stable and unstable manifolds of each pair of critical elements in $\{X^i_{p_0}\}_{i\in I}$ are transverse, thus verifying Assumption 2.2.18. Let $p \in J$. For each orbit γ contained in $\partial W^s(X^s(V_J))\cap(M\times\{p\})$, since M is compact, $\omega(\gamma)$ is nonempty, nonwandering, compact, and connected. By Lemma 2.3.28, $\Omega(V_J)\bigcap(\partial W^s(X^s(V_J)) \cap (M \times \{p\})) = \bigcup_{i\in I} X^i(p)$. Hence, $\omega(\gamma) \subset \bigcup_{i\in I} X^i(p)$ as it is nonwandering. As $\{X^i(p)\}_{i\in I}$ are a finite collection of hyperbolic critical elements, they are separable by open sets. As $\omega(\gamma)$ is connected, this implies that $\omega(\gamma) = X^i(p)$ for some $i \in I$. The above verifies Assumption 2.2.17. □

2.3.5.2 Euclidean Space

Similar to Lemma 2.3.22, Lemma 2.3.29 shows that under Assumptions 2.3.4-2.3.13, the intersection of the nonwandering set with the RoA boundary consists of a finite union of critical elements, as opposed to an arbitrary union. This is important because a finite union of hyperbolic critical elements persists under perturbations to the vector field, whereas an arbitrary union may not.

Lemma 2.3.29. *Under the conditions of Lemma 2.3.18 the number of critical elements of V_0 contained in $\partial W^s(X^s(V_0))$ is finite.*

Proof of Lemma 2.3.29. By the assumptions of Lemma 2.3.18, either $\partial W^s(X^s(V_0))$ is bounded or by Assumption 2.3.13 there exists a neighborhood N of ∞ such that $\Omega(V_0)\bigcap N = \emptyset$. In either case, there exists a compact set $K \subset \partial W^s(X^s(V_0))$ such that $\Omega(V_0)\bigcap\partial W^s(X^s(V_0)) = \Omega(V_0)\bigcap K$. The rest of the proof proceeds exactly as in the proof of Lemma 2.3.22. □

Because of Lemma 2.3.29, henceforth we assume that I, the set indexing the critical elements in the RoA boundary as in Assumption 2.3.4, is finite.

We first prove the main result in the case that the RoA is bounded. This reduces to the same proof as for compact Riemannian manifolds above.

Proof of Theorem 2.3.19. As $\overline{W}^s(X^s(V_0))$ is bounded, it is compact. Then the proofs of Lemma 2.3.8 and Lemmas 2.3.23-2.3.27 follow exactly as for compact Riemannian manifolds. The theorem then follows from Lemmas 2.3.25-2.3.27. □

Proof of Lemma 2.3.18. That I is nonempty will follow from Assumptions 2.3.11-2.3.12. That $\overline{W}^s(X^s(V_0)) = \bigcup_{i\in I^+} W^s(X^i(V_0))$ follows from the same argument as in the proof of Lemma 2.3.8. If $\overline{W}^s(X^s(V_0))$ is bounded, then the proof of Lemma 2.3.25 can be used to find the desired neighborhood U. Otherwise, if Assumption 2.3.13 is satisfied, it can be shown there is an open, invariant, wandering set U_∞ containing ∞, and the proof of Lemma 2.3.25 can be modified by including ∞ in the directed graph of the critical elements in the natural way.

First we show that I is nonempty. By Assumption 2.3.11, $\partial W^s(X^s(V_0)) \neq \emptyset$, so there exists $x \in \partial W^s(X^s(V_0))$. By Assumption 2.3.12, the closure of the forward orbit of x is compact, so $\omega(x)$ is nonempty. As $\omega(x)$ is nonwandering, by Assumption 2.3.4 this implies that I must be nonempty.

Next we show that $\overline{W}^s(X^s(V_0)) = \bigcup_{i\in I^+} W^s(X^i(V_0))$. By Assumption 2.3.12, for every $x \in \partial W^s(X^s(V_0))$, $\omega(x)$ is nonempty and compact. Hence, the same argument as in the proof of Lemma 2.3.8 shows that $\overline{W}^s(X^s(V_0)) \subset \bigcup_{i\in I^+} W^s(X^i(V_0))$. Similarly, the same argument as in the proof of Lemma 2.3.8 shows that $\bigcup_{i\in I^+} W^s(X^i(V_0)) \subset \overline{W}^s(X^s(V_0))$ and that there are no homoclinic orbits or heteroclinic cycles among the critical elements in I^+ for V_0.

If $\overline{W}^s(X^s(V_0))$ is bounded, then the construction in Lemma 2.3.25 can be applied to show, as in the proof of Lemma 2.3.26, that there exists an open neighborhood U of $\overline{W}^s(X^s(V_0))$ in M such that $\Omega(V_0) \bigcap U = \bigcup_{i\in I^+} X^i(V_0)$.

If Assumption 2.3.13 holds then let N be an open neighborhood of $\{\infty\}$ in $M \times \{\infty\}$ such that $\Omega(V_0) \bigcap N = \emptyset$. Define, for the sake of convenience, $W^u(\infty) := \{x \in M : \alpha(x) = \emptyset\}$. As in the proof of Lemma 2.3.25, define a directed graph of $I^+ \bigcup\{\infty\}$. As, by the above, there are no homoclinic orbits or cycles, and $I \bigcup \{\infty\}$ is finite, every maximal forward path must end in either some $i \in E_0$ or in ∞. Let $U_\infty = \bigcup_{t\in\mathbb{R}} \phi_t(N, V)$. Then U_∞ is an open invariant neighborhood of ∞ which consists of wandering points since N does and the wandering set is invariant. Therefore, repeating the construction of U in Lemma 2.3.25 for the case of a fixed parameter vector field only yields a neighborhood U of $\overline{W}^s(X^s(V_0))$ such that $\Omega(V_0) \bigcap U = \bigcup_{i\in I^+} X^i(V_0)$. Furthermore, for every $x \in U$ its orbit in backwards time enters either $W^u_{\text{loc}}(X^j(V_0))$ for some $j \in I^+$ or the neighborhood N. □

Remark 2.3.30. Despite the similarities between the proof of Lemma 2.3.25 and the proof of Lemma 2.3.18, even if there exists a neighborhood of ∞ containing no nonwandering points, the result of Lemma 2.3.25 does not hold for M Euclidean. The reason is that U_∞ in the argument above can have nonwandering points in its closure, as well as nonwandering points which enter it under perturbation away from ∞, so that perturbed orbits in $\partial W^s(X^s(V))$

can approach these nonwandering points through U_∞. Alternatively, setting $U_\infty = N'$ for N' a sufficiently small neighborhood of ∞ containing no nonwandering points implies that N' will not have nonwandering points in its closure or under perturbation, but N' is not backwards (or forwards) invariant, so the perturbed orbits in $\partial W^s(X^s(V))$ can exit N' and approach other nonwandering points.

Let $\mathcal{M}$ be the set of closed subsets of M with the Chabauty metric. Let $\{K_n\}_{n=1}^{\infty}$ be an exhaustion of M by compact sets and let $\mathcal{K}_n$ be the set of closed subsets of K_n with the Hausdorff metric. Define a family of functions $G_n^1 : W' \to \mathcal{K}_n$ by sending V to the closure of $\overline{W}^s(X^s(V)) \bigcap \text{int } K_n$. Define another family of functions $G_n^2 : W' \to \mathcal{K}_n$ by sending V to the closure of the intersection of int K_n with the closure of the union of the equilibrium points and periodic orbits of V. Define the family of functions $G_n^3 : W' \to \mathcal{K}_n$ by sending V to the closure of the intersection of int K_n with the closure of the union of points in M whose ω limit sets under V are empty.

Lemma 2.3.31 shows that each of the functions defined above is lower semi-continuous over W'. As $\mathcal{K}_n$ is a compact metric space for each n, and the set of continuity points of semi-continuous functions with codomain a compact metric space is residual by the Remark following Corollary 1 of [Kur68], these functions are continuous for generic vector fields in W'. This fact will be the basis of the proof of Theorem 2.3.20.

Lemma 2.3.31. *For each n, the functions G_n^1, G_n^2, and G_n^3 are lower semi-continuous.*

Proof of Lemma 2.3.31. Lower semi-continuity of G_n^2 will follow from the proof of [PR83, Theorem 11.3]. Lower semi-continuity of G_n^1 is shown by noting that int $W^s_{\text{loc}}(X^s(V))$ is C^1 continuous and open, and for any point in $W^s(X^s(V_0))$ there exists some time at which it enters int $W^s_{\text{loc}}(X^s(V))$. Lower semi-continuity of G_n^3 is proven by recognizing that convergence in the strong C^1 topology implies that vector fields agree outside some compact set, and that if a point has an empty ω limit set then its forward orbit must exit and remain outside this compact set for all future time.

Fix $n > 0$, let $V_0 \in W'$, and let $V_m \in W'$ with $V_m \to V_0$. First we show that G_n^1 is lower semi-continuous. It suffices to show that $\liminf_{m\to\infty} G_n^1(V_m) \supset G_n^1(V_0)$. First suppose there exists $x \in G_n^1(V_0) \bigcap W^s(X^s(V_0)) \bigcap \text{int } K_n$. Then there exists $T > 0$ such that $\phi_T(x, V_0) \in \text{int } W^s_{\text{loc}}(X^s(V_0))$. As $V_m \to V_0$, int $W^s_{\text{loc}}(X^s(V))$ is C^1 continuous with respect to V, int $W^s_{\text{loc}}(X^s(V_0))$ is open, and the flow is C^1 continuous, there exists Z such that $m \geq Z$ implies that $\phi_T(x, V_m) \in \text{int } W^s_{\text{loc}}(X^s(V_0)) \subset W^s_{2\epsilon}(X^s(V_m))$. Thus, $x \in \text{int } K_n \bigcap W^s(X^s(V_m)) \subset G_n^1(V_m)$ for $m \geq M$, so $x \in \liminf_{m\to\infty} G_n^1(V_m)$. Next suppose there exists $x \in G_n^1(V_0) \bigcap \partial W^s(X^s(V_0)) \bigcap \text{int } K_n$. Since $x \in \partial W^s(X^s(V_0))$ and int K_n is open, there exist $\{x_k\}_{k=1}^{\infty} \subset \text{int } K_n \bigcap W^s(X^s(V_0))$ such that $x_k \to x$. By the above, $\{x_k\}_{k=1}^{\infty} \subset \liminf_{m\to\infty} G_n^1(V_m)$. As $\liminf_{m\to\infty} G_n^1(V_m)$ is closed [Hau57, Section 28], this implies that $x \in \liminf_{m\to\infty} G_n^1(V_m)$. Now suppose there exists $x \in G_n^1(V_0) \bigcap \partial K_n$. Then by definition of G_n^1, there exist $\{x_k\}_{k=1}^{\infty} \subset \text{int } K_n \bigcap \overline{W}^s(X^s(V_0))$ such that $x_k \to x$. By the above, $\{x_k\}_{k=1}^{\infty} \subset \liminf_{m\to\infty} G_n^1(V_m)$. As $\liminf_{m\to\infty} G_n^1(V_m)$ is closed, this implies that $x \in \liminf_{m\to\infty} G_n^1(V_m)$. Hence, $\liminf_{m\to\infty} G_n^1(V_m) \supset G_n^1(V_0)$.

That G_n^2 is lower semi-continuous is shown in the proof of [PR83, Theorem 11.3] in which $\overline{\Gamma}_i$ plays the role of G_n^2.

Next we show that G_n^3 is lower semi-continuous. Since $V_m \to V_0$ in the strong C^1 topology there exists a compact set K such that for m sufficently large V_m and V_0 are equal outside of K. Suppose $x \in G_n^3(V_0) \bigcap \text{int } K_n$. Since $\omega(x) = \emptyset$ and K is compact, there must exist $T > 0$ such that $\phi_t(x, V_0) \notin K$ for $t \geq T$. Let $y = \phi_T(x, V_0)$ and for each m let $x_m = \phi_{-T}(y, V_m)$. Since V_m and V_0 agree outside of K, $\phi_t(x_m, V_m) \notin K$ for $t \geq T$ and the foward orbits of y under V_0 and $\phi_t(x_m, V_m)$ under V_m agree. Hence, their ω limit sets must be equal, so $\omega(x_m) = \omega(y) = \emptyset$. Then for m sufficiently large, $x_m \in \text{int } K_n$ since int K_n is open. Clearly $x_m \to \phi_{-T}(y, V_0) = x$. Thus, $x \in \liminf_{m\to\infty} G_n^3(V_m)$. Now suppose there exists $x \in G_n^3(V_0) \bigcap \partial K_n$. Then by definition of G_n^3, there exist $\{x_k\}_{k=1}^{\infty} \subset \text{int } K_n$ such that $x_k \to x$ and $\omega(x_k) = \emptyset$ for each k. By the above, $\{x_k\}_{k=1}^{\infty} \subset \liminf_{m\to\infty} G_n^3(V_m)$. As $\liminf_{m\to\infty} G_n^3(V_m)$ is closed, this implies that $x \in \liminf_{m\to\infty} G_n^3(V_m)$. Hence, $\liminf_{m\to\infty} G_n^3(V_m) \supset G_n^3(V_0)$. □

Lemma 2.3.32 shows that the assumption that a vector field in W' is a point of continuity of G_n^1, G_n^2, and G_n^3 for all n is generic.

Lemma 2.3.32. *Let W be an open set in the strong C^1 topology. Then for generic $V_0 \in W$, V_0 is a point of continuity of G_n^1, G_n^2, and G_n^3 for all n.*

Proof of Lemma 2.3.32. By the Remark following Corollary 1 of [Kur68], for fixed n, since K_n is compact and therefore $\mathcal{K}_n$ is a compact metric space, the set of continuity points C_n of G_n^1, G_n^2, and G_n^3 is residual in W'. Let $\tilde{C} = \bigcap_{n=1}^{\infty} C_n$. Then $\tilde{C}$ is residual in W' and, since the space of C^1 vector fields with the strong C^1 topology has the Baire property, $\tilde{C}$ is dense in W'. □

Lemma 2.3.33 shows that for p sufficiently close to p_0, every point x in $\overline{W}^s(X^s(V_p))$ has $\omega(x, V_p)$ nonempty. This ensures that under small perturbation, no orbits are in $\overline{W}^s(X^s(V_p))$ which flow to infinity in forward time. This is one of the key lemmas behind the proof of

Theorem 2.3.20 because it will help ensure that every orbit in $\overline{W}^s(X^s(V_p))$ converges to a critical element.

Lemma 2.3.33. *If $V_{p_0} \in W'$ complete, satisfies Assumptions 2.3.4-2.3.13, and V_{p_0} is a point of continuity of G_n^1, G_n^2, and G_n^3 for all n, then for J a sufficiently small open neighborhood of p_0, $p \in J$ implies that for all $x \in \overline{W}^s(X^s(V_p))$, $\omega(x, V_p) \neq \emptyset$.*

Proof of Lemma 2.3.33. We claim that there exists an open neighborhood of p_0 in J such that for every p in this neighborhood and every $x \in \overline{W}^s(X^s(V_p))$, $\omega(x, V_p)$ is nonempty. Assume towards a contradiction than no such open neighborhood exists. As J is a manifold it is first countable, so the above implies that there exists a sequence $\{p_m\}_{m=1}^{\infty}$ such that $p_m \to p_0$ and there exist $x_m \in \overline{W}^s(X^s(V_{p_m}))$ such that $\omega(x_m, V_{p_m}) = \emptyset$. As $x_m \in M \bigcup\{\infty\}$ compact, passing to a subsequence if necessary we have that $x_m \to x \in M \bigcup\{\infty\}$. We will see that if $x \in M$ we arrive at a contradiction via continuity of G_n^1 and G_n^3 at V_{p_0}. If $x \in \infty$, then either we are able to reduce to the case of $x \in M$ by following each x_n along its orbit or, as V_{p_0} agrees with V_{p_n} outside a compact set since $V_{p_n} \to V_{p_0}$ in the strong C^1 topology, we obtain a contradiction to Assumption 2.3.13.

First assume that $x \in M$. Then there exists n such that $x \in \operatorname{int} K_n$. By continuity of G_n^1 at V_{p_0}, $x \in \overline{W}^s(X^s(V_{p_0}))$. By continuity of G_n^3 at V_{p_0}, $\omega(x, V_{p_0}) = \emptyset$. By Lemma 2.3.18, $\overline{W}^s(X^s(V_{p_0})) = \bigcup_{i \in I^+} W^s(X^i(V_0))$ so that $x \in W^s(X^i(V_{p_0}))$ for some $i \in I^+$. Thus, $\omega(x, V_{p_0}) = X^i(V_{p_0})$ which contradicts that $\omega(x, V_{p_0}) = \emptyset$. Next assume that $x = \infty$. Let O_m denote the orbit in both forwards and backwards time of x_m under V_{p_m}. Assume towards a contradiction that there exists a subsequence $\{O_{m_k}\}_{k=1}^{\infty}$ and $K' \subset M$ compact such that $O_{m_k} \bigcap K' \neq \emptyset$ for all k. Letting $y_{m_k} \in O_{m_k} \bigcap K'$ for all k implies that, since K' is compact, passing to a subsequence if necessary $y_{m_k} \to y \in K' \subset M$. Hence, as the ω limit set is invariant along an orbit, this reduces to the case of $x \in M$ above, which leads to a contradiction. So, there must exist a subsequence $\{O_{m_k}\}_{k=1}^{\infty}$ such that $O_{m_k} \bigcap K_{m_k} = \emptyset$ for all k. Let N be the neighborhood of Assumption 2.3.13 so that $\Omega(V_{p_0}) \bigcap N = \emptyset$ and no orbit under V_{p_0} contained in N has both its ω and α limit sets under V_{p_0} empty. As $V_{p_{m_k}} \to V_{p_0}$, there exists $K \subset M$ compact such that $V_{p_{m_k}}$ and V_{p_0} agree outside K. Then for k sufficiently large, the closure of O_{m_k} is contained in $(M - K) \bigcap N$. As V_{p_0} and $V_{p_{m_k}}$ agree on $M - K$, this implies that O_{m_k} is also the orbit of x_{m_k} under V_{p_0}, so that $\omega(O_{m_k}, V_{p_0}) = \omega(O_{m_k}, V_{p_{m_k}}) = \emptyset$. As $O_{m_k} \subset N$, this implies that $\alpha(O_{m_k}, V_{p_0}) \neq \emptyset$. As the closure of O_{m_k} is contained in N, $\alpha(O_{m_k}, V_{p_0}) \subset N$. However, since $\alpha(O_{m_k}, V_{p_0})$ is nonempty and nonwandering under V_{p_0}, this contradicts that $\Omega(V_{p_0}) \bigcap N = \emptyset$. Hence, $x_m \to x = \infty$ results in a contradiction. As $x \notin M$ and $x \neq \infty$, we obtain a contradiction to the main assertion above. ☐

Lemma 2.3.34 is the main technical result behind Theorem 2.3.20. It states that under the conditions of the theorem, for p sufficiently close to p_0, the only nonwandering points that are present in $\overline{W}^s(X^s(V_p))$ are the perturbations of the original critical elements contained in $\overline{W}^s(X^s(V_{p_0}))$.

Lemma 2.3.34. *If $V_{p_0} \in W'$ complete, satisfies Assumptions 2.3.4-2.3.13, and V_{p_0} is a point of continuity of G_n^1, G_n^2, and G_n^3 for all n, then for J a sufficiently small open neighborhood of p_0, $p \in J$ implies that $\Omega(V_p) \bigcap \overline{W}^s(X^s(V_p)) = \bigcup_{i \in I^+} X^i(V_p)$.*

Proof of Lemma 2.3.34. In the case where $\overline{W}^s(X^s(V_{p_0}))$ is bounded, the result follows by Theorem 2.3.19 under weaker assumptions than stated in this lemma. So, we may assume that $\overline{W}^s(X^s(V_{p_0}))$ is unbounded. We claim that there exists an open neighborhood of p_0 in J such that for every p in this neighborhood, $\Omega(V_p) \bigcap \overline{W}^s(X^s(V_p)) = \bigcup_{i \in I^+} X^i(V_p)$. Assume towards a contradiction that no such open neighborhood exists. As J is a manifold it is first countable, so the above implies that there exists a sequence $\{p_m\}_{m=1}^{\infty}$ such that $p_m \to p_0$ and there exist $x_m \in \overline{W}^s(X^s(V_{p_m}))$ with $x_m \in \Omega(V_{p_m})$ such that $x_m \notin \bigcup_{i \in I^+} X^i(V_{p_m})$. By Lemma 2.3.33 we may shrink J if necessary so that $p \in J$ and $x \in \overline{W}^s(X^s(V_p))$ implies that $\omega(x, V_p) \neq \emptyset$. As $p_m \to p_0$, we may assume $p_m \in J$ for all m. In particular, $\omega(x_m, V_{p_m}) \neq \emptyset$ for all m. As $\{V_p\}_{p \in J}$ is strong C^1 continuous and $p_m \to p_0$, $V_{p_m} \to V_{p_0}$ in the strong C^1 toplogy. This implies that there exists K compact such that for m sufficiently large, V_{p_m} and V_{p_0} agree on $M - K$. By Assumption 2.3.13, there exists a neighborhood N of ∞ such that $\Omega(V_{p_0}) \bigcap N = \emptyset$.

The proof will proceed as follows. First we show that there exists a point y_m in the orbit of x_m such that $\omega(y_m, V_{p_m}) \bigcap (K \cup (M - N)) \neq \emptyset$ for all m and $\{y_m\}_{n=1}^{\infty}$ are contained in a compact set. Passing to a subsequence if necessary implies that $y_m \to y \in M$, and continuity of G_n^1 at V_{p_0} implies that $y \in \overline{W}^s(X^s(V_{p_0}))$. Then we will apply the C^1 Closing Lemma to obtain a sequence of vector fields $\{V_m'\}_{m=1}^{\infty}$ with $V_m' \to V_{p_0}$ in the strong C^1 topology such that V_m' and V_{p_0} agree outside the compact set $K \bigcup (M - N)$ for all m, and there exist points z_m near y_m with $z_m \to y$ and z_m periodic under V_m'. Continuity of G_n^2 at V_{p_0} then implies that y is periodic. Hence, $y \in X^j(V_{p_0})$ for some j. It is then shown that the orbits of z_m under V_m' converge with respect to the Hausdorff metric to $X^j(V_{p_0})$. As $X^j(V_m') \to X^j(V_{p_0})$ with respect to the Hausdorff metric as well, and $X^j(V_m')$ is disjoint from the orbit of z_m under V_m', by hyperbolicity of $X^j(V_{p_0})$ we will obtain a contradiction.

First, assume towards a contradiction that there exists m such that $\omega(x_m, V_{p_m}) \subset (M - K) \bigcap N$. Since $(M - K) \bigcap N$ is open, this implies that, after some time $T_m > 0$, the forward orbit of x_m under V_{p_m} remains in $(M-K)\bigcap N$ for all future time. Let $y'_m = \phi_{T_m}(x_m, V_{p_m})$. Then the above implies that the forward orbit of y'_m under V_{p_m} agrees with the forward orbit of y'_m under V_{p_0}. So, $\omega(y'_m, V_{p_0}) = \omega(y'_m, V_{p_m}) = \omega(x_m, V_{p_m}) \subset (M-K)\bigcap N$ is nonempty since $\omega(x_m, V_{p_m})$ is. So, $\omega(y'_m, V_{p_0})$ is nonempty, nonwandering, and contained in N. This contradicts that $\Omega(V_{p_0})\bigcap N = \emptyset$. Therefore, we must have $\omega(x_m, V_{p_m})\bigcap(K\bigcup(M-N)) \neq \emptyset$ for all m. As $K\bigcup(M-N)$ is compact, there exists n such that $K\bigcup(M-N) \subset \text{int } K_{n-1}$. As $\omega(x_m, V_{p_m})\bigcap \text{int } K_{n-1} \neq \emptyset$, int K_{n-1} is open, and $\omega(x_m, V_{p_m})$ is in the closure of the forward orbit of x_m under V_{p_m}, there exists y_m a point in the forward orbit of x_m under V_{p_m} such that $y_m \in \text{int } K_{n-1}$ for each m. Note that $y_m \in \overline{W}^s(X^s(V_{p_m}))$ and is nonwandering since x_m is and both $\overline{W}^s(X^s(V_{p_m}))$ and $\Omega(V_{p_m})$ are invariant. Furthermore, $y_m \notin \bigcup_{i\in I^+} X^i(V_{p_m})$ as x_m is not and these sets are also invariant. As K_{n-1} is compact and $y_m \in K_{n-1}$ for all m, passing to a subsequence if necessary we have that $y_m \to y \in K_{n-1} \subset \text{int } K_n$. Note that $y_m \in G^1_n(V_{p_m})\bigcap \text{int } K_n$ for each m since $y_m \in \overline{W}^s(X^s(V_{p_m}))\bigcap \text{int } K_n$ for each m. As G^1_n is continuous at V_{p_0}, $V_{p_m} \to V_{p_0}$, $y_m \to y$, and $y_m \in G^1_n(V_{p_m})$ for all m, $y \in \overline{W}^s(X^s(V_{p_0}))$. Furthermore, $\omega(y_m, V_{p_m}) = \omega(x_m, V_{p_m})$, so, by the above, $\omega(y_m, V_{p_m})\bigcap(K\bigcup(M-N)) \neq \emptyset$ for all m. Then by the C^1 Closing Lemma for noncompact manifolds [PR83, p. 311] there exists a sequence $\{V'_m\}_{m=1}^{\infty}$ of vector fields on M such that V'_m and V_{p_m} agree on $M-(K\bigcup(M-N))$ and are $(1/m)$ C^1-close on $(K\bigcup(M-N))$, and such that there exists a point z_m with distance less than $(1/m)$ from y_m such that z_m is periodic under V'_m. As $y_m \notin \bigcup_{i\in I^+} X^i(V_{p_m})$, each $X^i(V_{p_m})$ is compact for $i \in I^+$, and I^+ is finite, y_m and $\bigcup_{i\in I^+} X^i(V_{p_m})$ are separable by open sets. As z_m can be chosen arbitrarily close to y_m and each $X^i(V_p)$ varies C^1 continuously with respect to p, for m sufficiently large we may choose z_m such that $z_m \notin \bigcup_{i\in I^+} X^i(V'_m)$. As V'_m and V_{p_m} agree outside the compact set $K\bigcup(M-N)$ for all m, and as their C^1 distance on $K\bigcup(M-N)$ tends to zero, $V'_m \to V_{p_0}$ in the strong C^1 topology. Furthermore, $z_m \to y$, and z_m periodic with distance less than $(1/m)$ from y_m implies that for m sufficiently large, $z_m \in \text{int } K_n$ and $z_m \in G^2_n(V'_m)$. Since G^2_n is continuous at V_{p_0}, $z_m \in G^2_n(V'_m)$ for m sufficiently large, $V'_m \to V_{p_0}$, and $z_m \to y$, $y \in G^2_n(V_{p_0})$. As $y \in \text{int } K_n$ as well, this implies that y is contained in the union of equilibria and periodic orbits of V_{p_0}. As $y \in \overline{W}^s(X^s(V_{p_0}))$ by the argument above, and by Lemma 2.3.18 $\Omega(V_{p_0})\bigcap\overline{W}^s(X^s(V_{p_0})) = \bigcup_{i\in I^+} X^i(V_{p_0})$, this implies that $y \in X^j(V_{p_0})$ for some $j \in I^+$.

Let O_m denote the orbit of z_m under V'_m for each m. Let $\mathcal{M}^\infty$ denote the metric space of closed subsets of $M\bigcup\{\infty\}$ equipped with the Hausdorff metric. As $M\bigcup\{\infty\}$ is compact, $\mathcal{M}^\infty$ is compact Then $\{O_m\}_{m=1}^{\infty}$ is a sequence in $\mathcal{M}^\infty$ compact, so passing to a subsequence if necessary implies that $O_m \to O \subset \mathcal{M}^\infty$. Note that since $y \in X^j(V_{p_0})$ periodic, the orbit of y under V_{p_0} is $X^j(V_{p_0})$. Let $\phi_t(y, V_{p_0}) \in X^j(V_{p_0})$ for any $t \in \mathbb{R}$. Then, since $z_m \to y$ and $V'_m \to V_{p_0}$, $\phi_t(z_m, V'_m) \to \phi_t(y, V_{p_0})$ by continuity of the flow. As $\phi_t(z_m, V'_m) \in O_m$ and $O_m \to O$, this implies that $\phi_t(y, V_{p_0}) \in O$ for all $t \in \mathbb{R}$, so $X^j(V_{p_0}) \subset O$. Assume towards a contradiction that there exists $x \in O - X^j(V_{p_0})$. As $\{x\}$ and $X^j(V_{p_0})$ are compact and disjoint, there exists an open neighborhood U' containing $X^j(V_{p_0})$ such that $\overline{U}'$ is disjoint from x. By Lemma 2.3.18, there exists an open neighborhood U of $\overline{W}^s(X^s(V_{p_0})) \supset X^j(V_{p_0})$ such that $\Omega(V_{p_0}) \bigcap U = \bigcup_{i \in I^+} X^i(V_{p_0})$. Shrinking U' if necessary, we may have that $\overline{U}' \subset U$ and, since I^+ is finite and each $X^i(V_{p_0})$ is compact, $\overline{U}'$ is disjoint from $X^i(V_{p_0})$ for each $i \neq j$. The above implies that $\partial U' \bigcap \Omega(V_{p_0}) = \emptyset$. As $x \in O$ there exists a sequence $x'_m \in O_m$ such that $x'_m \to x$. So, m sufficiently large implies that $x'_m \in O_m \bigcap (M - \overline{U}')$ and $z_m \in O_m \bigcap U'$ since $z_m \to y \in X^j(V_{p_0}) \subset U'$. Then O_m periodic implies it is connected, so there exists $y'_m \in O_m \bigcap \partial U'$. As $\partial U'$ is compact, passing to a subsequence if necessary we have that $y'_m \to y' \in \partial U' \bigcap O$. Choose n sufficiently large such that $y' \in \text{int } K_n$. Then m sufficiently large implies that $y'_m \in \text{int } K_n$ periodic, so $y'_m \in G_n^2(V'_m)$ for m sufficiently large. By continuity of G_n^2 at V_{p_0}, since $y'_m \to y'$, $V'_m \to V_{p_0}$, and $y'_m \in G_n^2(V'_m)$ for m sufficiently large, $y' \in G_n^2(V'_m) \bigcap \text{int } K_n$. So, y' is periodic under V_{p_0}, which implies that $y' \in \Omega(V_{p_0}) \bigcap \partial U'$. But $\Omega(V_{p_0}) \bigcap \partial U' = \emptyset$, which is a contradiction. So, $O = X^j(V_{p_0})$. By the proofs of [JPdM82, Lemmas 2.1-2.2] there exists an open neighborhood N_j of $X^j(V_{p_0})$ and an open neighborhood $\hat{W}$ of V_{p_0} in the strong C^1 topology such that $N_j \subset U'$ and for every $V \in \hat{W}$, N_j does not entirely contain any equilibrium points or periodic orbits of V other than $X^j(V)$. As $V'_m \to V_0$ in the strong C^1 topology, m sufficiently large implies that $V'_m \in \hat{W}$. As $O_m \to O = X^j(V_{p_0}) \subset N_j$, m sufficiently large implies that $O_m \subset N_j$. As O_m is periodic and disjoint from $\bigcup_{i \in I^+} X^i(V'_m)$, this contradicts the choice of $\hat{W}$ and N_j. ☐

Lemma 2.3.35 shows that there exists an open neighborhood U of $\overline{W}^s(X^s(V_{p_0}))$ in $M \bigcup \{\infty\}$ such that for p sufficiently close to p_0, $\overline{W}^s(X^s(V_p))$ is contained in U. This will be used in the proof of Corollary 2.3.21.

Lemma 2.3.35. *If $V_{p_0} \in W'$ complete, satisfies Assumptions 2.3.4-2.3.13, and V_{p_0} is a point of continuity of G_n^1, G_n^2, and G_n^3 for all n, then for J a sufficiently small open neighborhood of p_0, $p \in J$ implies that $\overline{W}^s(X^s(V_p)) \subset U$, where U is the open neighborhood of $\overline{W}^s(X^s(V_{p_0}))$ constructed in Lemma 2.3.18.*

Proof of Lemma 2.3.35. In the case where $\overline{W}^s(X^s(V_{p_0}))$ is bounded, the result follows by Theorem 2.3.19 under weaker assumptions than stated in this lemma. So, we may assume that $\overline{W}^s(X^s(V_{p_0}))$ is unbounded. Let U be the neighborhood of Lemma 2.3.18. Then U

is an open neighborhood of $\overline{W}^s(X^s(V_{p_0}))$ in $M \bigcup \{\infty\}$. We claim that there exists an open neighborhood of p_0 in J such that for every p in this neighborhood, $\overline{W}^s(X^s(V_p)) \subset U$. Assume towards a contradiction that no such open neighborhood exists. As J is a manifold it is first countable, so the above implies that there exists a sequence $\{p_m\}_{m=1}^{\infty}$ such that $p_m \to p_0$ and there exist $x_m \in \overline{W}^s(X^s(V_{p_m}))$ with $x_m \notin U$. As $M \bigcup \{\infty\}$ is compact, passing to a subsequence if necessary we have that $x_m \to x \in M \bigcup \{\infty\}$. First suppose $x = \infty$. As $\overline{W}^s(X^s(V_{p_0}))$ is unbounded and U is an open neighborhood of $\overline{W}^s(X^s(V_{p_0}))$ in $M \bigcup \{\infty\}$, $\infty \in U$. As U is open and $x_m \to \infty \in U$, m sufficiently large implies that $x_m \in U$, which is a contradiction. So, we must have $x \in M$. Then there exists n such that $x \in \text{int } K_n$. As $x_m \to x$, m sufficiently large implies that $x_m \in \overline{W}^s(X^s(V_{p_m})) \bigcap \text{int } K_n \subset G_n^1(V_{p_m})$. By continuity of G_n^1 at V_{p_0}, since $x_m \to x$, $V_{p_m} \to V_{p_0}$, and $x_m \in G_n^1(V_{p_m})$ for m sufficiently large, $x \in G_n^1(V_{p_0}) \bigcap \text{int } K_n$. So, $x \in \overline{W}^s(X^s(V_{p_0})) \subset U$. As U is open and $x_m \to x \in U$, m sufficiently large implies that $x_m \in U$, which is a contradiction. □

Proof of Theorem 2.3.20. First it is shown using Lemma 2.3.33 and Lemma 2.3.34 that for J sufficiently small, $p \in J$ implies that $\overline{W}^s(X^s(V_p)) \subset \bigcup_{i \in I^+} W^s(X^i(V_p))$. Then by Lemma 2.3.18, $W^u(X^i(V_{p_0})) \bigcap W^s(X^s(V_{p_0})) \neq \emptyset$ is a transversal intersection, which persists by openness of transversality for J sufficiently small, and this implies by the proof of Theorem 2.2.12 that for $p \in J$, $\bigcup_{i \in I^+} W^s(X^i(V_p)) \subset \overline{W}^s(X^s(V_p))$.

By Lemma 2.3.32 the additional assumptions present in the statement of Lemma 2.3.34 that do not appear in the statement of the theorem hold for generic $V_{p_0} \in W'$. By Lemma 2.3.34, shrinking J if necessary implies it is an open neighborhood of p_0 such that $p \in J$ implies that $\Omega(V_p) \bigcap \overline{W}^s(X^s(V_p)) = \bigcup_{i \in I^+} X^i(V_p)$. By Lemma 2.3.33, shrinking J further if necessary implies it is an open neighborhood of p_0 such that $p \in J$ and $x \in \overline{W}^s(X^s(V_p))$ implies that $\omega(x, V_p) \neq \emptyset$. So, let $p \in J$ and $x \in \overline{W}^s(X^s(V_p))$. As $\overline{W}^s(X^s(V_p))$ is closed and invariant under V_p, and since $\omega(x, V_p)$ is contained in the closure of the orbit of x under V_p, $\omega(x, V_p) \subset \overline{W}^s(X^s(V_p))$. Then $\omega(x, V_p)$ is nonempty, nonwandering, closed, and contained in $\overline{W}^s(X^s(V_p))$, so by the above this implies that $\omega(x, V_p) \subset \bigcup_{i \in I^+} X^i(V_p)$. As $\bigcup_{i \in I^+} X^i(V_p)$ is compact and $\omega(x, V_p)$ is closed, $\omega(x, V_p)$ is compact. Since $\omega(x, V_p)$ is compact, it is connected. Hence, $\omega(x, V_p) = X^j(V_p)$ for some $j \in I^+$. As $\omega(x, V_p)$ is compact, the forward orbit of x must converge to $X^j(V_p) = \omega(x, V_p)$. Thus, $x \in W^s(X^j(V_p))$. Therefore, $\overline{W}^s(X^s(V_p)) \subset \bigcup_{i \in I^+} W^s(X^i(V_p))$. By the proof of Lemma 2.3.18, $W^u(X^i(V_{p_0})) \bigcap W^s(X^s(V_{p_0})) \neq \emptyset$ for all $i \in I$. So, for each i there exists $T_i > 0$ such that $\phi_{T_i}(W^u_{\text{loc}}(X^i(V_{p_0}))) \bigcap W^s_{\text{loc}}(X^s(V_{p_0})) \neq \emptyset$. This intersection is trivially transverse since $W^s_{\text{loc}}(X^s(V_{p_0}))$ is codimension zero in M. Thus, $\{\phi_{T_i}(W^u_{\text{loc}}(X^i(V_p)))\}_{p \in J}$

and $\{W^s_{\mathrm{loc}}(X^s(V_p))\}_{p\in J}$ are C^1 continuous families over J of compact embedded submanifolds with a point of transversal intersection, so shrinking J if necessary implies that $\phi_{T_i}(W^u_{\mathrm{loc}}(X^i(V_p))) \bigcap W^s_{\mathrm{loc}}(X^s(V_p)) \neq \emptyset$ for all $p \in J$ [KH99, Proposition A.3.16, Corollary A.3.18]. Hence, as I is finite, shrinking J if necessary gives that for all $i \in I$ and $p \in J$, $W^u(X^i(V_p)) \bigcap W^s(X^s(V_p)) \neq \emptyset$. By the proof of Theorem 2.2.12, this implies that for all $p \in J$, $\bigcup_{i\in I^+} W^s(X^i(V_p)) \subset \overline{W}^s(X^s(V_p))$. □

Theorem 2.3.20 showed results for the family $\{\overline{W}^s(X^s(V_p))\}_{p\in J}$ in M. In contrast, Corollary 2.3.21 and the lemmas below will derive results for $\overline{W}^s(X^s_J)$ in $M \times J$, which a priori is potentially larger than $\sqcup_{p\in J}\overline{W}^s(X^s(V_p))$. It is $\overline{W}^s(X^s_J)$ that turns out to be relevant in the applications of interest, although the Corollary will show that under the assumptions of Theorem 2.3.20 they are equal. Lemma 2.3.36 shows that if $\overline{W}^s(X^s(V_{p_0}))$ is bounded, then for J sufficiently small, $p \in J$ and $x \in \overline{W}^s(X^s_J)$ implies that $\omega(x, V_p)$ is nonempty.

Lemma 2.3.36. *Let J be a smooth manifold and let $\{V_p\}_{p\in J}$ be a strong C^1 continuous family of vector fields on M possessing a family of stable hyperbolic equilibria $\{X^s(p)\}_{p\in J}$. Suppose there exists $p_0 \in J$ such that V_{p_0} is complete, V_{p_0} satisfies Assumptions 2.3.4-2.3.12, and $\overline{W}^s(X^s(V_{p_0}))$ is bounded. Then for J open and sufficiently small, and such that $p_0 \in J$, for every $(x, p) \in \overline{W}^s(X^s_J)$, $\omega(x, V_p) \neq \emptyset$.*

Proof of Lemma 2.3.36. By Theorem 2.3.19, for every $p \in J$ and every $x \in \overline{W}^s(X^s(V_p))$, $\omega(x, V_p) \neq \emptyset$. Assume towards a contradiction that there exists $(x, p) \in \overline{W}^s(X^s_J)$ with $\omega(x, V_p) = \emptyset$. By Theorem 2.3.19, there exists an open neighborhood U of $\overline{W}^s(X^s(V_{p_0}))$ in M such that U is bounded and, for J sufficiently small, $q \in J$ implies that $\overline{W}^s(X^s(V_q)) \subset U$. Hence, there exists $K \subset M$ compact such that $U \subset K$. As $\omega(x, V_p) = \emptyset$, there exists y in the forward orbit of x under V_p such that $y \in M - K$. As $\overline{W}^s(X^s(V_p))$ is V_p-invariant, $y \in \overline{W}^s(X^s(V_p))$. However, this contradicts that $\overline{W}^s(X^s(V_p)) \subset U \subset K$. □

Lemma 2.3.37 extends the results of Lemma 2.3.33 to show that for J sufficiently small, $p \in J$ and $(x, p) \in \overline{W}^s(X^s_J)$ implies that $\omega(x, V_p)$ is nonempty.

Lemma 2.3.37. *Let J be a smooth manifold and let $\{V_p\}_{p\in J}$ be a strong C^1 continuous family of vector fields on M possessing a family of stable hyperbolic equilibria $\{X^s(p)\}_{p\in J}$. Suppose there exists $p_0 \in J$ such that V_{p_0} is complete, V_{p_0} satisfies Assumptions 2.3.4-2.3.13, and V_{p_0} is a point of continuity of G^1_n, G^2_n, and G^3_n for all n. Then for J open and sufficiently small, and such that $p_0 \in J$, for every $(x, p) \in \overline{W}^s(X^s_J)$, $\omega(x, V_p) \neq \emptyset$.*

Proof of Lemma 2.3.37. First it is shown that for every $x \in \overline{W}^s(X_J^s)\bigcap(M\times\{p_0\})$, $\omega(x,V_{p_0}) \neq \emptyset$. This is proven by contradiction, which first assumes the existence of $(x,p_0) \in \overline{W}^s(X_J^s)$ such that $\omega(x,V_{p_0})$ is empty. Then continuity of G_n^1 at V_{p_0} will imply that $x \in \overline{W}^s(X^s(V_{p_0}))$, but this contradicts Lemma 2.3.33. Next, it is shown that for J sufficiently small, for every $(x,p) \in \overline{W}^s(X_J^s)$, $\omega(x,V_p) \neq \emptyset$. This is again proven by obtaining a contradiction using the first claim above and Assumption 2.3.13, using an argument similar to the proof of Lemma 2.3.33.

First we claim that for every $x \in \overline{W}^s(X_J^s)\bigcap(M \times \{p_0\})$, $\omega(x,V_{p_0}) \neq \emptyset$. For, assume towards a contradiction that there exists $x \in \overline{W}^s(X_J^s)\bigcap(M \times \{p_0\})$ with $\omega(x,V_{p_0}) = \emptyset$. Then there exists n such that $x \in \text{int } K_n$. Furthermore, by Lemma 2.3.33, $x \notin \overline{W}^s(X^s(V_{p_0}))$ since for every $y \in \overline{W}^s(X^s(V_{p_0}))$, $\omega(y,V_{p_0}) \neq \emptyset$. As $(x,p_0) \in \overline{W}^s(X_J^s)$, there exist $(x_m,p_m) \in W^s(X_J^s)$ with $(x_m,p_m) \to (x,p_0)$. As $\text{int } K_n$ is open and $x_m \to x$, m sufficiently large implies that $x_m \in \text{int } K_n$. By continuity of G_n^1 at V_{p_0}, $x_m \in W^s(X^s(V_{p_m}))\bigcap \text{int } K_n \subset G_n^1(V_{p_m})$ implies that $x \in G_n^1(V_{p_0})$. As $x \in \text{int } K_n$, this implies that $x \in \overline{W}^s(X^s(V_{p_0}))$, which is a contradiction. So, we must have $\omega(x,V_{p_0}) \neq \emptyset$.

Next we claim that for J sufficiently small, for every $(x,p) \in \overline{W}^s(X_J^s)$, $\omega(x,V_p) \neq \emptyset$. Assume towards a contradiction that no such open J containing p_0 exists. Then there must exist sequences $p_m \to p_0$ and $(x_m,p_m) \in \overline{W}^s(X_J^s)$ such that $\omega(x_m,V_{p_m}) = \emptyset$ for all m. As $M\bigcup\{\infty\}$ is compact, passing to a subsequence if necessary we have that $x_m \to x \in M\bigcup\{\infty\}$. First suppose $x \in M$. As $(x_m,p_m) \in \overline{W}^s(X_J^s)$ closed and $(x_m,p_m) \to (x,p_0)$, $(x,p_0) \in \overline{W}^s(X_J^s)$. Choose n such that $x \in \text{int } K_n$ and note that for m sufficiently large, $x_m \in \text{int } K_n$. By continuity of G_n^3 at V_{p_0}, $\omega(x_m,V_{p_m}) = \emptyset$ implies that $\omega(x,V_{p_0}) = \emptyset$. But, this contradicts the claim proved above. So, suppose $x = \infty$. Let O_m denote the orbit in both forwards and backwards time of x_m under V_{p_m}. Assume towards a contradiction that there exists a subsequence $\{O_{m_k}\}_{k=1}^\infty$ and $K' \subset M$ compact such that $O_{m_k}\bigcap K' \neq \emptyset$ for all k. Letting $y_{m_k} \in O_{m_k}\bigcap K'$ for all k implies that, since K' is compact, passing to a subsequence if necessary $y_{m_k} \to y \in K' \subset M$. Hence, as the ω limit set is invariant along an orbit, this reduces to the case of $x \in M$ above, which leads to a contradiction. So, there must exist a subsequence $\{O_{m_k}\}_{k=1}^\infty$ such that $O_{m_k}\bigcap K_{m_k} = \emptyset$ for all k. Let N be the neighborhood of Assumption 2.3.13 so that $\Omega(V_0)\bigcap N = \emptyset$ and no orbit under V_0 contained in N has both its ω and α limit sets under V_0 empty. As $V_{m_k} \to V_0$ in the strong C^1 topology, there exists $K \subset M$ compact such that V_{m_k} and V_0 agree outside K. Then for k sufficiently large, the closure of O_{m_k} is contained in $(M-K)\bigcap N$. As V_0 and V_{m_k} agree on $M-K$, this implies that O_{m_k} is also the orbit of x_{m_k} under V_0, so that $\omega(O_{m_k},V_0) = \omega(O_{m_k},V_{m_k}) = \emptyset$. As $O_{m_k} \subset N$, this implies that $\alpha(O_{m_k}, V_0) \neq \emptyset$. As the closure of O_{m_k} is contained in N, $\alpha(O_{m_k}, V_0) \subset N$. However, since $\alpha(O_{m_k}, V_0)$ is nonempty and nonwandering under V_0, this contradicts that $\Omega(V_0) \bigcap N = \emptyset$. Hence, $x_m \to x = \infty$ results in a contradiction. As $x \notin M$ and $x \neq \infty$, we obtain a contradiction to the main assertion above. □

Lemma 2.3.38 extends the results of Lemma 2.3.34 to show that for J sufficiently small, $p \in J$ implies that the nonwandering points under V_p in $M \times \{p\}$ that are contained in $\overline{W}^s(X^s_J)$ are precisely $\bigcup_{i \in I^+} X^i(V_p)$.

Lemma 2.3.38. *Let J be a smooth manifold and let $\{V_p\}_{p \in J}$ be a strong C^1 continuous family of vector fields on M possessing a family of stable hyperbolic equilibria $\{X^s(p)\}_{p \in J}$. Suppose there exists $p_0 \in J$ such that V_{p_0} is complete, V_{p_0} satisfies Assumptions 2.3.4-2.3.13, and V_{p_0} is a point of continuity of G^1_n, G^2_n, and G^3_n for all n. Then for J open and sufficiently small, and such that $p_0 \in J$, for every $p \in J$, $\Omega(V_p) \bigcap (\overline{W}^s(X^s_J) \cap (M \times \{p\})) = \bigcup_{i \in I^+} X^i(V_p)$.*

Proof of Lemma 2.3.38. First it is shown that $\Omega(V_{p_0}) \bigcap (\overline{W}^s(X^s_J) \cap (M \times \{p_0\})) = \bigcup_{i \in I^+} X^i(V_{p_0})$. This is proven by contradiction, which first assumes the existence of $(x, p_0) \in \overline{W}^s(X^s_J)$ such that x is nonwandering under V_{p_0} and x is not in $\bigcup_{i \in I^+} X^i(V_{p_0})$. Then continuity of G^1_n at V_{p_0} will imply that $x \in \overline{W}^s(X^s(V_{p_0}))$, but this contradicts Lemma 2.3.34. Next, it is shown that for J sufficiently small, $p \in J$ implies that $\Omega(V_p) \bigcap (\overline{W}^s(X^s_J) \cap (M \times \{p\})) = \bigcup_{i \in I^+} X^i(V_p)$. This is again proven by obtaining a contradiction using the first claim above, Lemma 2.3.37, and Assumption 2.3.13, using an argument similar to the proof of Lemma 2.3.34.

First we claim that $\Omega(V_{p_0}) \bigcap (\overline{W}^s(X^s_J) \cap (M \times \{p_0\})) = \bigcup_{i \in I^+} X^i(V_{p_0})$. For, assume towards a contradiction that there exists $x \in \Omega(V_{p_0})$ with $x \notin \bigcup_{i \in I^+} X^i(V_{p_0})$ and such that $(x, p_0) \in \overline{W}^s(X^s_J)$. Then there exists n such that $x \in \text{int } K_n$. Furthermore, by Lemma 2.3.34, $x \notin \overline{W}^s(X^s(V_{p_0}))$ since $\Omega(V_{p_0}) \bigcap \overline{W}^s(X^s(V_{p_0})) = \bigcup_{i \in I^+} X^i(V_{p_0})$. As $(x, p_0) \in \overline{W}^s(X^s_J)$, there exist $(x_m, p_m) \in W^s(X^s_J)$ with $(x_m, p_m) \to (x, p_0)$. Then m sufficiently large implies $x_m \in \text{int } K_n$. Hence, $x_m \in G^1_n(V_{p_m})$ for m sufficiently large. Since G^1_n is continuous at V_{p_0}, $V_{p_m} \to V_{p_0}$, $x_m \to x$, and $x_m \in G^1_n(V_{p_m})$ for m sufficiently large, $x \in G^1_n(V_{p_0}) \bigcap \text{int } K_n$. Hence, $x \in \overline{W}^s(X^s(V_{p_0}))$, which is a contradiction. So, we must have $\Omega(V_{p_0}) \bigcap (\overline{W}^s(X^s_J) \cap (M \times \{p_0\})) = \bigcup_{i \in I^+} X^i(V_{p_0})$.

Next we claim that for J sufficiently small, for every $p \in J$, $\Omega(V_p) \bigcap (\overline{W}^s(X^s_J) \cap (M \times \{p\})) = \bigcup_{i \in I^+} X^i(V_p)$. Assume towards a contradiction that no such open J containing p_0 exists. Then there must exist sequences $p_m \to p_0$ and $(x_m, p_m) \in \overline{W}^s(X^s_J)$ such that $x_m \in \Omega(V_{p_m})$ but $x_m \notin \bigcup_{i \in I^+} X^i(V_{p_m})$. By Lemma 2.3.37, for every $(x, p) \in \overline{W}^s(X^s_J)$, $\omega(x, V_p) \neq \emptyset$. In particular, $\omega(x_m, V_{p_m}) \neq \emptyset$ for all m. As $V_{p_m} \to V_{p_0}$ in the strong C^1

topology, there exists K compact such that for m sufficiently large, V_{p_m} and V_{p_0} agree on $M-K$. By Assumption 2.3.13, there exists a neighborhood N of ∞ such that $\Omega(V_{p_0})\bigcap N=\emptyset$. Then, by the same argument as in the proof of Lemma 2.3.34, $\omega(x_m,V_{p_m})\bigcap(K\bigcup(M-N))\neq\emptyset$ for all m. As $K\bigcup(M-N)$ is compact, there exists n such that $K\bigcup(M-N)\subset \text{int } K_{n-1}$. As $\omega(x_m,V_{p_m})\bigcap \text{int } K_{n-1}\neq\emptyset$, int K_{n-1} is open, and $\omega(x_m,V_{p_m})$ is in the closure of the forward orbit of x_m under V_{p_m}, there exists y_m a point in the forward orbit of x_m under V_{p_m} such that $y_m\in \text{int } K_{n-1}$ for each m. Note that $(y_m,p_m)\in\overline{W}^s(X^s_J)$ and is nonwandering under V_{p_m} since x_m is and both $\overline{W}^s(X^s_J)$ and $\Omega(V_{p_m})$ are invariant. Furthermore, $y_m\notin\bigcup_{i\in I^+}X^i(V_{p_m})$ as x_m is not and these sets are also invariant. As K_{n-1} is compact and $y_m\in K_{n-1}$ for all m, passing to a subsequence if necessary we have that $y_m\to y\in K_{n-1}\subset \text{int } K_n$. As $(y_m,p_m)\in\overline{W}^s(X^s_J)$ closed and $(y_m,p_m)\to(y,p_0)$, $(y,p_0)\in\overline{W}^s(X^s_J)$. Note that $V'_m\to V_{p_0}$ with respect to the strong C^1 topology. Then, by the same argument as in the proof of Lemma 2.3.34, there exists a sequence $\{V'_m\}_{m=1}^{\infty}$ of vector fields on M such that V_{p_m} and V'_m agree on $M-(K\bigcup(M-N))$, are $(1/m)$ C^1-close on $(K\bigcup(M-N))$, and such that there exist points z_m with distance less than $(1/m)$ from y_m such that z_m is periodic under V'_m and $z_m\notin\bigcup_{i\in I^+}X^i(V'_m)$. Then, by the same arguments as in the proof of Lemma 2.3.34, we have that y is periodic under V_{p_0}. As $(y,p_0)\in\overline{W}^s(X^s_J)$ and since it was shown above that $\Omega(V_{p_0})\bigcap(\overline{W}^s(X^s_J)\cap(M\times\{p_0\}))=\bigcup_{i\in I^+}X^i(V_{p_0})$, we must have $y\in X^j(V_{p_0})$ for some $j\in I$. Let O_m denote the orbit of z_m under V'_m. Then, by the same argument as in the proof of Lemma 2.3.34, passing to a subsequence if necessary we have that $O_m\to X^j(V_{p_0})$ with respect to the Hausdorff metric. Finally, by the same argument as in the proof of Lemma 2.3.34, there exists an open neighborhood N_j of $X^j(V_{p_0})$ and an open neighborhood $\hat{W}$ of V_{p_0} in the strong C^1 topology such that for every $V\in\hat{W}$, N_j does not entirely contain any equilibrium points or periodic orbits of V. As $V_{p_0}\in\hat{W}$ and $V'_m\to V_{p_0}$ with respect to the strong C^1 topology, m sufficiently large implies that $V'_m\in\hat{W}$. As $O_m\to X^j(V_0)\subset N_j$, m sufficiently large implies that $O_m\subset N_j$. As O_m is periodic and disjoint from $\bigcup_{i\in I^+}X^i(V'_m)$, this contradicts the choice of $\hat{W}$ and N_j. □

Proof of Corollary 2.3.21. The conclusions of the corollary will follow immediately from Theorem 2.2.23 and Corollary 2.2.24 once the assumptions behind them are verified. In particular, Assumptions 2.2.14-2.2.15 and Assumptions 2.2.17-2.2.18 must be verified.

Let U be the neighborhood of Lemma 2.3.18. Then $\Omega(V_{p_0})\bigcap U=\bigcup_{i\in I^+}X^i(V_{p_0})$. Let U' be an open neighborhood of $\overline{W}^s(X^s(V_{p_0}))$ in $M\bigcup\{\infty\}$ such that $\overline{U}'\subset U$. Then by Lemma 2.3.35, for $p\in J$, $\overline{W}^s(X^s(V_p))\subset U'$. Thus, $W^s(X^s_J)\subset U'\times J$, so $\overline{W}^s(X^s_J)\subset\overline{U}'\times J\subset U\times J$. In particular, U is an open neighborhood of $\partial W^s(X^s_J)\bigcap(M\times\{p_0\})$ in $(M\times\{p_0\})$.

As $\Omega(V_{p_0})\bigcap U$ consists of a finite union of hyperbolic critical elements, Assumptions 2.2.14-2.2.15 are satisfied. By Assumption 2.3.6, the stable and unstable manifolds of each pair of critical elements in $\partial W^s(X_J^s)\bigcap(M\times\{p_0\})$ are transverse, satisfying Assumption 2.2.18. By Lemma 2.3.38, for every $p\in J$, $\Omega(V_p)\bigcap(\overline{W}^s(X_J^s)\cap(M\times\{p\}))=\bigcup_{i\in I^+}X^i(V_p)$. By Lemma 2.3.37, for every $(x,p)\in\overline{W}^s(X_J^s)$, $\omega(x,V_p)\neq\emptyset$. As the orbit of x under V_p is contained in $\overline{W}^s(X_J^s)$ closed and invariant, $\omega(x,V_p)\subset\overline{W}^s(X_J^s)$. But, $\omega(x,V_p)$ nonempty and nonwandering implies by the above that $\omega(x,V_p)\subset\bigcup_{i\in I^+}X^i(V_p)$. As $\bigcup_{i\in I^+}X^i(V_p)$ is compact and $\omega(x,V_p)$ is closed, $\omega(x,V_p)$ is compact, so it is also connected. Thus, $\omega(x,V_p)=X^j(V_p)$ for some $j\in I^+$. The above verifies Assumption 2.2.17. □

2.3.6 Conclusion

This section completed the final step to the proof of RoA boundary continuity under parameter perturbation for vector fields which are in essence Morse-Smale along the RoA boundary. This step was to show that these assumptions imply that the RoA boundary is Ω-stable. In particular, for compact Riemannian manifolds, it was shown that if the vector field is Morse-Smale along the RoA boundary then under sufficiently small strong C^1 perturbations the RoA boundary is Ω-stable. Initially, Example 2.3.3 showed that in Euclidean space the RoA boundary can vary discontinuously even for a globally Morse-Smale vector field under a strong C^∞ perturbation. This was a counterexample to [CC95, Theorem 5.4], which claimed that under weaker assumptions the nonwandering set in the RoA boundary could not grow under strong C^1 perturbations. Hence, for Euclidean space it is necessary to make additional generic assumptions, and assumptions regarding the behavior near infinity, along with that the vector field is Morse-Smale along the RoA boundary, to show that the RoA boundary is Ω-stable. These results, together with the results of Section 2.2, imply that under these assumptions, the RoA boundary varies continuously under sufficiently small changes in parameter values.

2.4 Extensions to Vector Fields with Event-Selected Discontinuities

2.4.1 Introduction

Practical models in many application areas, including power systems, involve switching events, which lead to vector fields that are not C^1. In power systems, these may include clipping limits on control devices and switching due to tap changing transformers. Furthermore, it has been observed that such switching is fundamentally coupled with loss of system recoverability in practice [His04, VJSZ94]. The purpose of this section is to extend the RoA boundary continuity results of Section 2.2 to the setting of a large class of nonsmooth vector fields, closely related to what have been called vector fields with C^1 event-selected discontinuities [BSKR15], which can exhibit various forms of switching. In turn, this will make it possible to provide theoretical justification for the extension of numerical algorithms for computing critical parameter values to systems exhibiting switching. In particular, Sections 2.5-2.6 provide justification for algorithms for vector fields with C^1 event-selected discontinuities, and the algorithms developed in Chapter III can be applied to these nonsmooth systems as well. Vector fields with event-selected discontinuities have received attention recently due to their applicability to many physical and engineering system models which possess a finite number of triggering hypersurfaces where the vector field is discontinuous.

A sketch of a proof was made in [VJSZ94] for a decomposition of the boundary of the RoA for systems with clipping limits and fixed parameter values. In that work, the vector field was locally Lipschitz. Generalizations to systems exhibiting switching, or any discontinuous vector fields, have not been made even in the case of constant parameters. Therefore, the classification of the boundary of the RoA for vector fields with C^1 event-selected discontinuities, presented here, may be of interest for the fixed parameter case, as well as for its behavior under small variations in parameter. The primary application, though, is the theoretical justification of algorithms for determining the recovery boundary.

The section is organized as follows. Section 2.4.2 gives a motivating example. Section 2.4.3 provides some dynamical systems background. Then Section 2.4.4 discusses the main results. Section 2.4.5 provides a sketch of the key proofs, although some are nearly identical to the corresponding results in Section 2.2 and, therefore, omitted for brevity. Finally, Section 2.4.6 offers some concluding remarks.

2.4.2 Example

Example 2.4.1. The example below serves to illustrate a mechanism whereby the RoA boundary can fail to vary continuously for a parameterized vector field, and how this can lead to situations where no critical parameter value exists. In this case, it will not be possible to determine the recovery boundary by computation of critical parameter values, so the algorithms described in Chapter III would not be successful. The theory developed in subsequent sections will prove that this example is not typical; in particular, a large class of hybrid dynamical systems show continuity of their RoA boundaries under small changes in parameter values, thereby ensuring that the recovery boundary consists of critical parameter values (see Section 2.5).

Let $J = (-1, 1)$ and for $p \in J$ define V_p as a vector field on $\mathbb{R}^2$ piecewise as follows. Let $r := r(x, y) = \sqrt{x^2 + y^2}$. Let $\mathcal{S}_{(a,b)} : \mathbb{R} \to \mathbb{R}$ be a smooth bump function with $\mathcal{S}^{-1}(1) = [0, a]$ and $\mathcal{S}^{-1}(0) = [b, \infty)$ (see [Lee13, Lemma 2.21] for a specific example). For $|r| \leq 1$, let $V_p(x, y) = (-x, -y) - p\frac{\mathcal{S}_{(2.5,3)}(r)}{r}(x, y)$, and for $|r| > 1$, let $V_p(x, y) = \frac{\mathcal{S}_{(1,2)}(r)}{2r}(-x - y, x - y) - p\frac{\mathcal{S}_{(2.5,3)}(r)}{r}(x, y)$. This family of vector fields are each piecewise C^1 with a switching surface on the circle $|r| = 1$. For $p \in J$, let $s_p(x, y) = (r - 1)^2$. Then $s_p^{-1}(0) = \{(x, y) : |r(x, y)| = 1\}$. Thus, for each $p \in J$, $\{V_p, s_p\}$ defines a vector field with event-selected C^1 discontinuities, as will be defined below. Furthermore, $\{V_p, s_p\}_{p\in J}$ is a strong C^1 continuous family of event-selected C^1 vector fields, also defined below.

Figure 2.12 shows the vector field V_p for $p = 0.3$. It has a stable equilibrium point at the origin whose RoA is the open ball of radius $r = 3$ with boundary the circle of radius $r = 3$. This qualitative picture, in particular the RoA and its boundary, remain the same for any $p > 0$. The red and green curves in the figure show the images of two sets of parameter dependent initial conditions (ICs). The ICs corresponding to $p = 0.3$ are shown as red and green circles, and the ICs move along the red and green lines as p decreases. Note that both the red and green ICs lie inside the RoA for $p = 0.3$ (and any $p > 0$).

Figure 2.13 shows the vector field V_p for $p = 0$. Note that the RoA of its stable equilibrium point is the open ball of radius $r = 2$, with boundary the circle of radius $r = 2$. Here the red circle, denoting the red IC, lies inside the RoA whereas the green circle, denoting the green IC, lies outside the RoA.

Figure 2.14 shows the vector field V_p for $p = -0.3$. Note that the RoA of its stable equilibrium point is the open ball of radius $r = 2$, with boundary the circle (now a periodic orbit) of radius $r = 2$. Here the red circle, denoting the red IC, lies on the boundary of the RoA, and hence $p = -0.3$ is a critical parameter value for the red ICs, whereas the green circle, denoting the green IC, lies outside the RoA.

In summary, the red ICs intersect the RoA boundary at $p = -0.3$, so that $p = -0.3$ is a critical parameter value for them. However, the green ICs pass from inside the RoA (for $p > 0$) to outside the RoA (for $p \leq 0$) without ever lying on the RoA boundary. So, there is no critical parameter value for the green ICs. This is possible because the RoA boundary varies discontinuously at $p = 0$. The theory developed in this paper will prove that discontinuous variation of the RoA boundary under small parameter variation is not possible for a large

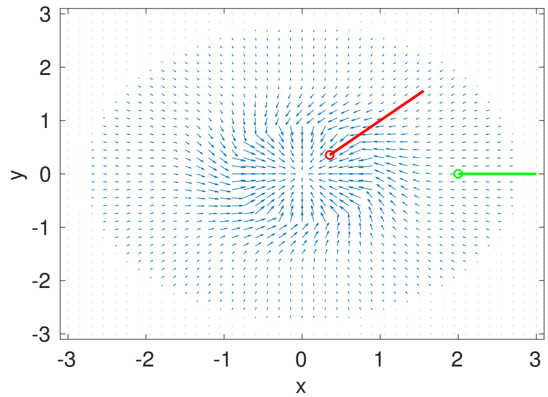

Figure 2.12: The vector field V_p for $p = 0.3$. The red and green lines show two sets of parameter dependent initial conditions, with the initial conditions for $p = 0.3$ shown as red and green circles.

class of practical hybrid dynamical systems, that these systems do possess critical parameter values for a particular disturbance, and that a previously designed algorithm can be used to numerically compute those critical values.

Example 2.4.2. The example shown here motivates the need to employ a slightly different definition of vector fields with C^1 event-selected discontinuities than in [BSKR15]. In particular, the definition in [BSKR15] allows switching surfaces to have nontranversal intersection. Under strong C^1 perturbations, as considered here, it is possible for switching surfaces to pull apart, and this could result in a vector field which is no longer event-selected discontinuous. This is problematic for developing the RoA boundary continuity results that are desired, because arbitrarily small variations in parameter values could lead to loss of existence and/or uniqueness of the flow, which complicates the possible behaviors of the RoA boundary. Therefore, to avoid this possibility, we will make the generic assumption below that the switching surfaces intersect transversely.

Figure 2.15 shows an example of a vector field with C^∞ event-selected discontinuities that has a pair of nontransversal switching surfaces. The example is defined by the following

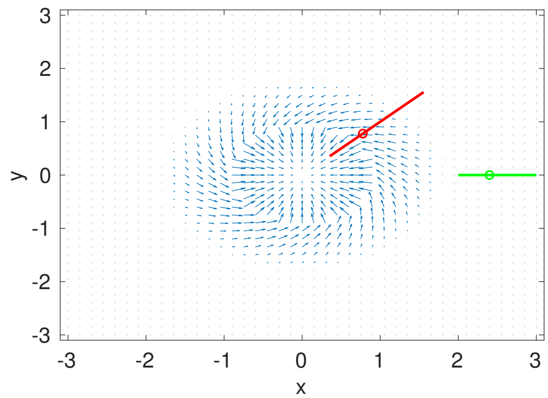

Figure 2.13: The vector field V_p for $p = 0$. The red and green lines show two sets of parameter dependent initial conditions, with the initial conditions for $p = 0$ shown as red and green circles.

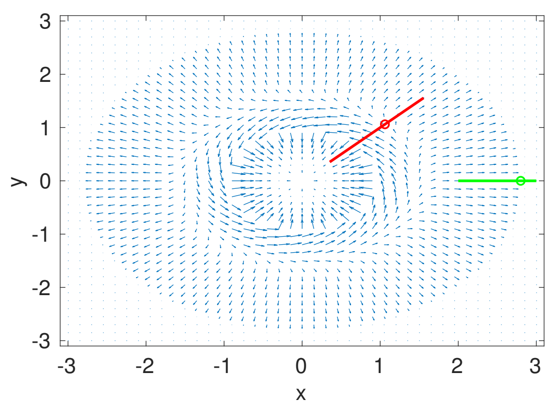

Figure 2.14: The vector field V_p for $p = -0.3$. The red and green lines show two sets of parameter dependent initial conditions, with the initial conditions for $p = -0.3$ shown as red and green circles.

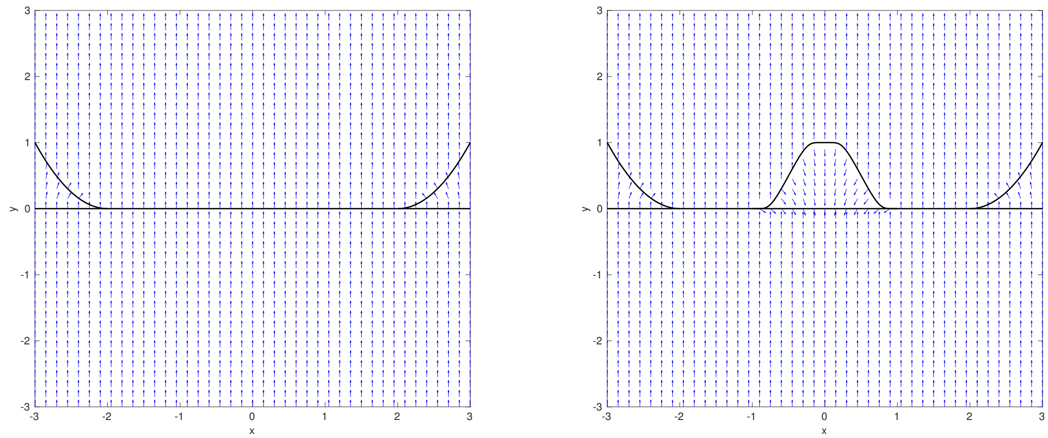

Figure 2.15: The vector field V of Example 2.4.2 is shown. It shows how a vector field with C^∞ event-selected discontinuities can become no longer event-selected discontinuous under strong C^∞ perturbations due to the presence of two nontransversal switching surfaces (black curves).

two smooth vector fields:

$$\omega = 2\pi/6 \tag{2.14}$$

$$V_1(x, y) = (0, 1) \tag{2.15}$$

$$V_2(x, y) = (\sin(\omega(x + 3)), \cos(\omega(x + 3))). \tag{2.16}$$

$$\tag{2.17}$$

There are two switching surfaces, shown as black lines, in the example. Define the rough vector field V as follows. Above the top switching surface and below the bottom switching surface, let $V = V_1$. In between the two switching surfaces, let $V = V_2$. Note that V is a vector field with C^∞ event-selected discontinuities and, therefore, posssess a global flow which exists, is unique, and is piecewise C^∞. However, under arbitrarily small C^∞ perturbations to the top switching surface, the two switching surfaces separate along a portion of their tangency, the resulting vector field is no longer event-selected discontinuous, and it exhibits loss of existence and uniqeness of local flows. This motivates the need to assume below that the switching surfaces are transverse.

2.4.3 Background

We will assume that all vector fields, C^1 or not, are defined on $M := \mathbb{R}^n$ for some $n > 0$. First we review some terminology of nonlinear systems with a C^1 vector field. Let $J \subset \mathbb{R}$ be an open interval containing p_0 and let $\{V_p\}_{p \in J}$ be a parameterized family of vector fields on M. Define V a vector field on $M \times J$ by $V(x,p) = (V_p(x), 0)$. Let ϕ denote the flow of V such that $\phi(t,x,p)$ is the flow of the system from initial condition x under the vector field V_p for a time t. Let $\phi_{(t,p)} : M \to M$ by $\phi_{(t,p)}(x) = \phi(t,x,p)$. If $X(p) \subset M$ is an equilibrium point or periodic orbit of V_p for some $p \in J$, we call $X(p)$ a critical element of V_p.

An equilibrium point $X(p)$ of V_p is hyperbolic if $d(\phi_{(1,p)})_{X(p)}$ has no imaginary eigenvalues. If $X(p)$ is a periodic orbit and $x \in X(p)$ then there exists a hypersurface $S \supset \{x\}$ and a C^1 map $\hat{\tau} : S \to S$ such that $\hat{\tau}$ is the Poincaré first return map. The periodic orbit $X(p)$ is hyperbolic if $d\hat{\tau}_x$ has no imaginary eigenvalues. For a hyperbolic critical element, let $n^u(X(p_0))$ denote the unstable dimension of $T_{X(p_0)}M$, and let $n^s(X(p_0))$ denote the stable dimension of $T_{X(p_0)}M$. A hyperbolic critical element $X(p)$ possesses local stable and unstable manifolds, denoted $W^s_{\text{loc}}(X(p))$ and $W^u_{\text{loc}}(X(p))$, respectively, such that the flow restricted to $W^s_{\text{loc}}(X(p))$ is a contraction in forwards time, and the flow restricted to $W^u_{\text{loc}}(X(p))$ is a contraction in backwards time. The stable and unstable manifolds, $W^s(X(p))$ and $W^u(X(p))$, respectively, are then constructed by flowing $W^s_{\text{loc}}(X(p))$ and $W^u_{\text{loc}}(X(p))$ backwards and forwards in time, respectively. The RoA of a stable hyperbolic equilibrium point is its stable manifold, and the RoA boundary is the boundary of its stable manifold.

A point $x \in M$ is wandering under V_p if there exists an open neighborhood U of x in M and some $T > 0$ such that $|t| > T$ implies that $\phi(t,U,p) \bigcap U = \emptyset$. A point $x \in M$ is nonwandering if it is not wandering. Let $\Omega(V_p)$ denote the set of all nonwandering points of V_p. These include all critical elements of V_p. A pair of C^1 immersed submanifolds X and Y are transverse if for every $q \in X \cap Y$, $T_qX + T_qY = T_qM \cong M$. A Morse-Smale vector field $\tilde{V}$ is a C^1 vector field such that $\Omega(\tilde{V})$ is equal to a finite union of hyperbolic critical elements whose stable and unstable manifolds intersect transversely [Sma60].

Let $C^1(M, \tilde{M})$ denote C^1 maps from M to $\tilde{M}$ C^1 manifolds. The strong C^1 topology on $C^1(M, \tilde{M})$ is defined in [Hir76, Chapter 2]. A parameterized family of vector fields $\{V_p\}_{p \in J}$ is strong C^1 continuous if the map $p \to V_p$ is continuous as a map from J to $C^1(M, TM)$ equipped with the strong C^1 topology. In particular, if $V_p \to \tilde{V}$ as $p \to \hat{p}$ in the strong C^1 topology then there exists $K \subset M$ compact such that for any $\epsilon > 0$, p sufficiently close to $\hat{p}$ implies that V_p and $\tilde{V}$ agree outside of K and that V_p and $\tilde{V}$ along with their first derivatives are ϵ-close on K. A property is generic for vector fields on M if the set of vector fields possessing this property contains a countable intersection of open, dense sets in $C^1(M, TM)$. A family of critical elements $\{X(p)\}_{p\in J}$ of $\{V_p\}_{p\in J}$ is C^1 continuous if there exists a C^1 function $\tilde{F} : X(p_0) \times J \to M$ such that $\tilde{F}|_{X(p_0)\times\{p\}}$ is injective onto its image $X(p)$ for all $p \in J$. We say $\{E_p\}_{p\in J}$ is a Chabauty continuous family of subsets of $\mathbb{R}^n$ if for every $p_0 \in J$ and every $\{p_n\}_{n=1}^{\infty} \subset J$ such that $p_n \to p_0$, for every $x \in E_{p_0}$ there exists a sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in E_{p_n}$ and $x_n \to x$, and every sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in E_{p_n}$ has all of its limit points contained in E_{p_0}.

Next we introduce some additional notation. For $Q \subset J$, write $X(Q) = \bigcup_{p\in Q} X(p) \subset M$ and $X_Q = \bigcup_{p\in Q} X_p \subset M \times J$. For any set E, let $\sqcup_{p\in J} E := E \times J$. If $D \subset M$, let $D_\epsilon := \{x \in M : d(x, D) < \epsilon\}$ where d is the Euclidean distance.

In order to incorporate the effects of clipping limits and switching, a large class of hybrid dynamical systems, which we will call vector fields with event-selected C^1 discontinuities although their definition in [BSKR15] is slightly more general, is employed. This class of vector fields, which is defined below, has several desirable properties including the existence of a global flow which is piecewise C^1.

A rough vector field is a vector field that is not assumed to be C^1 or even C^0; this includes vector fields exhibiting limits or switching behavior, but also includes C^1 vector fields. The following definition of event functions, whose zero level sets will be the points of discontinuity of the vector field and represent switching surfaces, ensures that the flow of the vector field will cross switching surfaces transversely. This definition rules out grazing of the switching surfaces and Zeno type behavior.

Definition 2.4.3. Let V be a rough vector field on M, $s \in C^1(M, \mathbb{R}^m)$, and U an open set in M. Then define it to be an event or switching function (s) for V on U if for each component s_j of s, either $s_j(U)$ is disjoint from zero or, if not, then there exists $c > 0$ such that $d(s_j)_x(V(x)) \geq c$ for all $x \in U$. Such a U is referred to as an event neighborhood of V.

Note that the latter part of the definition does not restrict the sign of the event functions in the sense that if an event function s^j satisfied $d(s^j)_y(V(y)) \leq -c^j < 0$ for all $y \in U^j$, then defining the new event function $-s^j$ would satisfy the definition given above.

Let $B_m = \{-1, +1\}^m$ be m copies of $\{-1, +1\}$, meant to denote whether each event function is positive or negative to indicate the status of each switching event. For each $j \in \{1, ..., m\}$, define the switching surface corresponding to s^j by letting $Z_j = (s^j)^{-1}(0)$. Let $Z = \bigcup_{j=1}^{m} Z_j$ be the union of the switching surfaces. Let $\sigma_j : M \to \{-1, +1\}$ that sends x to the sign of $s^j(x)$, or to 1 if $s^j(x) = 0$. Define a function that gives the switching state status (σ), where $\sigma : M \to \{-1, +1\}^m$, by the product of σ_j for $j \in \{1, ..., m\}$. Then $\sigma(x)$ denotes the switching states of the system at x.

The following definition of event-selected C^1 ensures both that the flow encounters switching surfaces transversely and that the flow through points of intersection of multiple switching surfaces is well-defined and piecewise C^1.

Definition 2.4.4. Let V be a rough vector field on M and $s \in C^1(M, \mathbb{R}^m)$. Then V and s determine an event-selected C^1 vector field if there exists a set of C^1 vector fields $\{V^b\}_{b \in B_m}$ on M, called selection functions, and the following conditions are satisfied:

1. (event functions) for every $x \in M$, there exists an open neighborhood $U_x \subset M$ containing x such that s is an event function for V on U_x.

2. (C^1 extension) for every $x \in M$, there exists an open neighborhood $U_x \subset M$ containing x such that for every $b \in B_m = \{-1, +1\}^m$, with

$$D^b = \{y \in U_x : \sigma(y) = b\},$$

$V|_{\text{int } D^b} = V^b|_{\text{int } D^b}$.

3. (transversality) for every $j, j' \in \{1, ..., m\}$, Z_j and $Z_{j'}$ are transverse.

The definition of vector fields with C^1 event-selected discontinuities ensures that orbits of the vector field encounter switching surfaces transversely, and that the vector fields on each side of a switching surface are consistent in such a way that local flows exist and are unique. However, it is important to note that Definition 2.4.4 differs from the original definition in [BSKR15]. Nevertheless, the event functions and C^1 extension parts of Definition 2.4.4 imply that any vector field satisfying Definition 2.4.4 must also satisfy the definition of event-selected discontinuous in [BSKR15]. In particular, the event functions and C^1 extension parts of Definition 2.4.4 require that the vector field is event-selected discontinuous at $x \in M$, and as this holds for all $x \in M$, the definition in [BSKR15] is satisfied. Hence, Definition 2.4.4 is a special case of the definition in [BSKR15], so all the results developed for the class of vector fields considered in [BSKR15] apply to vector fields satisfying Definition 2.4.4.

In addition to the definition in [BSKR15], Definition 2.4.4 includes an additional transversality condition which is required to ensure that the vector field remains event-selected C^1 under perturbations to the vector field. Example 2.4.2 showed that, without assuming transversality of the switching surfaces, arbitrarily small C^∞ perturbations could cause the vector field to no longer be event-selected C^1, and to no longer have existence of a local flow. The problem is that, for some $b \in B_m = \{-1, +1\}^m$, it is possible that int D^b is empty, but then, under perturbation, int D^b may suddenly become nonempty, and the selection function V^b may be inconsistent with the selection functions of the neighboring regions. The transversality condition ensures that, under perturbations to the vector field, no region int D^b goes from being empty to being nonempty. It turns out this is the key property required to ensure persistence of C^1 event-selected vector fields under perturbation.

An example of an event-selected C^1 vector field was shown in Section 2.3.3 with event function $s_1(x, y) = (\sqrt{(x^2 + y^2)} - 1)^2$. By [BSKR15, Corollary 1], if V and s determine an event-selected C^1 vector field over M then there exists a piecewise C^1 global flow.

Definition 2.4.5. A piecewise C^1 immersed submanifold is a topological manifold $T \subset M$ together with a C^1 immersed submanifold, denoted $\tilde{T}$, in $M \backslash Z$ which is an open and dense topological submanifold of T.

Definition 2.4.6. Two piecewise C^1 immersed submanifolds T and T' are transverse if $\tilde{T}$ and $\tilde{T}'$ are transverse in $M \backslash Z$.

If $\{s_p\}_{p \in J}$ is a family of C^1 functions from M to $\mathbb{R}^m$, for each $p \in J$ we define $\sigma_p : M \to \{0, 1\}^m$ such that $(\sigma_p)_j$ sends x to the sign of $(s_p(x))_j$, or to 1 if $(s_p(x))_j = 0$, for all $j \in \{1, ..., m\}$.

Definition 2.4.7. Let $\{V_p\}_{p \in J}$ be a family of rough vector fields on M and let $\{s_p\}_{p \in J}$ be a family of C^1 functions from M to $\mathbb{R}^m$. We say that $\{V_p, s_p\}_{p \in J}$ is a strong C^1 continuous family of event-selected C^1 vector fields on M if there exists a collection $\{V_p^b\}_{b \in B^m, p \in J}$ such that V_p^b is a C^1 vector field on M for all $b \in B^m$ and $p \in J$, and the following conditions are satisfied:

1. (event-selected at one parameter value) there exists $p_0 \in J$ such that V_{p_0} and s_{p_0} determine an event-selected discontinuous C^1 vector field on M with the selection functions $\{V_{p_0}^b\}_{b \in B^m}$.

2. (continuity of event functions) $\{s_p\}_{p \in J}$ is a strong C^1 continuous family of functions in $C^1(M, \mathbb{R}^m)$.

3. (continuity of selection functions) for each $b \in B^m$, $\{V_p^b\}_{p \in J}$ is a strong C^1 continuous family of C^1 vector fields on M.

4. (consistency of vector fields) for all $p \in J$ and $b \in B_m$, with

$$D_p^b = \{y \in M : \sigma_p(y) = b\},$$

$V|_{\text{int } D_p^b} = V_p^b|_{\text{int } D_p^b}$.

Motivating the terminology of Definition 2.4.7, we will see in Lemma 2.4.15 that if $\{V_p, s_p\}_{p\in J}$ is a strong C^1 continuous family of event-selected C^1 vector fields on M then for J sufficiently small, (V_p, s_p) determines an event-selected C^1 vector field for each $p \in J$. Define V a rough vector field on $M \times J$ by $V(x,p) = (V_p(x), 0)$. Then Lemma 2.4.15 will show that for J sufficiently small, V has a piecewise C^1 flow.

For $j \in \{1, ..., m\}$ and $p \in J$, define the event or switching function j corresponding to parameter value p (s_p^j) and the switching surface j corresponding to parameter value p (Z_j^p), where $Z_j^p = (s_p^j)^{-1}(0)$. For $p \in J$ define $\bigcup_{j=1}^m Z_j^p$ (Z^p). For $j \in \{1, ..., m\}$ define $\sqcup_{p\in J} Z_j^p$ (Z_j). Finally, define $\sqcup_{p\in J} Z^p$ (Z^J) and $(M \times J)\backslash Z^J$ (C^J).

2.4.4 Results

Let $J \subset \mathbb{R}$ be an open interval and let $p_0 \in J$. Let $\{V_p, s_p\}_{p\in J}$ be a strong C^1 continuous family of event-selected C^1 vector fields on M. Let $X^s(p_0)$ be a stable hyperbolic equilibrium point of V_{p_0} which lies in an open neighborhood on which V_{p_0} is C^1. For J sufficiently small we make the following assumptions:

Assumption 2.4.8. *Every equilibrium point of V_{p_0} is disjoint from Z^{p_0}.*

Assumption 2.4.9. *There exists a neighborhood N of $\pi_M\left(\partial W^s(X_J^s) \bigcap M_{p_0}\right)$ such that $\Omega(V) \bigcap N$ consists of a finite union of critical elements of V_{p_0}; call them $\{X^i(p_0)\}_{i=1}^k$. Shrink V if necessary so that $X_{p_0}^i \subset \partial W^s(X_J^s)$ for every $i \in I := \{1, ..., k\}$.*

Assumption 2.4.10. *Let $\gamma \subset \partial W^s(X_J^s)$ be an orbit. Then γ converges to X^i for some $i \in I$.*

Assumption 2.4.11. *With respect to V_{p_0}, every equilibrium point $X^i(p_0)$ is hyperbolic in the sense that it is hyperbolic in its neighborhood on which V_{p_0} is C^1. For any $X^i(p_0)$ a periodic orbit, we will see below that it possesses a point $x \in M\backslash Z^{p_0}$ and a C^1 cross section S with a C^1 first return map with x as a fixed point. Then for any periodic orbit $X^i(p_0)$, assume that x is a hyperbolic fixed point of its first return map.*

Assumption 2.4.12. *We will see below that every $X^i(p_0)$ possesses stable and unstable manifolds, call them $W^s(X^i(p_0))$ and $W^u(X^i(p_0))$, respectively, which are piecewise C^1 immersed submanifolds. For every $i, j \in I$, $W^u(X^i(p_0))$ and $W^s(X^j(p_0))$ are transversal in M.*

Assumptions 2.4.9, 2.4.11, and 2.4.12 ensure that V_{p_0} is Morse-Smale along $\partial W^s(X^s_J) \bigcap M \times \{p_0\}$. Assumpton 2.4.10 ensures that no trajectories in $\partial W^s(X^s_J)$ escape to infinity, which is required since M is not compact, and that no new nonwandering elements enter $\partial W^s(X^s_J)$ for $p \in J$, although we will see that the perturbations of the critical elements $\{X^i(p_0)\}_{i\in I}$ will be contained in the boundary. Assumptions 2.4.8, 2.4.11, and 2.4.12 are generic. For the components on which V_{p_0} is C^1, it is generically true that $\Omega(V_{p_0})$ is equal to the closure of the union of critical elements of V_{p_0}. However, it is not generic that there exists a neighborhood V containing a finite number of critical elements, so Assumption 2.4.9 is not generic.

Theorem 1 shows that the boundary of the RoA is equal to the union over $p \in J$ of the family of boundaries of the RoAs, and that the decomposition of the boundary of the RoA into a union of stable manifolds persists for $p \in J$. Corollary 1 then states that the family of boundaries varies continuously with $p \in J$.

Theorem 2.4.13. *For J sufficiently small, $\partial W^s(X^s_J) = \sqcup_{p\in J} \partial W^s(X^s(p)) = \bigcup_{i\in I} W^s(X^i_J)$.*

Corollary 2.4.14. *$\{\partial W^s(X^s(p))\}_{p\in J}$ is a Chabauty continuous family of subsets of M.*

The proofs of these results rely on a large number of technical lemmas. These are provided in the next section, with many proofs omitted because they are nearly identical to prior proofs in Section 2.2.

2.4.5 Proofs

Lemma 2.4.15. *If $\{V_p, s_p\}_{p\in J}$ is a strong C^1 continuous family of event-selected vector fields on M, then for J sufficiently small, $p \in J$ implies that V_p and s_p determine an event-selected C^1 vector field over M.*

Lemma 2.4.15. We begin by establishing that it suffices to verify the definition of event-selected C^1 on a compact subset of M. For, as $\{V_p, s_p\}_{p\in J}$ is a strong C^1 continuous family of event-selected vector fields on M, by definition for each $j \in I_m := \{1, ..., m\}$ and each $b \in B_m$, $\{s^j_p\}_{p\in J}$ and $\{V^b_p\}_{p\in J}$ are strong C^1 continuous. As I_m and B_m are finite, this implies that there exists a compact set $K \subset M$ such that shrinking J if necessary implies that $(s_p^j)|_{M\backslash K} = (s_{p_0}^j)|_{M\backslash K}$ and $(V_p^b)|_{M\backslash K} = (V_{p_0}^b)|_{M\backslash K}$ for all $p \in J$, $j \in I_m$, and $b \in B_m$. By the consistency of vector fields in Definition 2.4.7, this implies that $(V_p)|_{M\backslash K} = (V_{p_0})|_{M\backslash K}$ for all $p \in J$. As $M\backslash K$ is open and V_{p_0} and s_{p_0} are event-selected C^1, they satisfy the event function and C^1 extension conditions of Definition 2.4.4 at every $x \in M\backslash K$. This implies that, for every $p \in J$, V_p and s_p satisfy the event function and C^1 extension conditions at every $x \in M\backslash K$ as well. Furthermore, as $M\backslash K$ is open and $Z_{p_0}^j$, $Z_{p_0}^{j'}$ are transverse on $M\backslash K$ for all $j, j' \in I_m$, the above implies that Z_p^j, $Z_p^{j'}$ are transverse on $M\backslash K$ for all $j, j' \in I_m$ and $j \in J$. Let K' be compact such that $K \subset \text{int } K'$. Hence, by the above, to verify the claim it suffices to show that for J sufficiently small, $p \in J$ implies that V_p and s_p satisfy the event function and C^1 extension properties at every $x \in K'$, and that for any $p \in J$ and any $j, j' \in I_m$, Z_p^j and $Z_p^{j'}$ are transverse over K'.

First we verify the event function condition. As V_{p_0} and s_{p_0} are event-selected C^1, for each $j \in I_m$ and each $\rho \in Z_{p_0}^j$, $s_{p_0}^j$ is an event function for V_{p_0} over an open neighborhood of ρ. This implies that $d(s_{p_0}^j)_\rho$ is nonzero for each $j \in I_m$. As this holds for all $\rho \in Z_{p_0}^j$, zero is a regular value of $s_{p_0}^j$, so $Z_{p_0}^j$ is a properly embedded C^1 submanifold with codimension one for each $j \in I_m$. Fix $\rho \in K'$. For each $j \in I_m$ such that $s_{p_0}^j(\rho) \neq 0$, by continuity of $s_{p_0}^j$ there exists an open neighborhood U_j of ρ in M such that $s_{p_0}^j(U_j)$ has constant sign and is disjoint from zero. For each such $j \in I_m$, as $\{s_p^j\}_{p \in J}$ is strong C^1 continuous, shrinking J if necessary implies that $s_p^j(U_j)$ has constant sign and is disjoint from zero for all $p \in J$. Let $I' \subset I_m$ consist of all j such that $s_{p_0}^j(\rho) = 0$. Let m' be the size of I and relabel I to be $\{1, ..., m'\}$. Let B_m' be the subset of B_m such that for each $b \in B_m'$ and each $j' \in I_m$ such that $s_{p_0}^{j'}(\rho) \neq 0$, $b_{j'} = \sigma_{p_0}^{j'}(\rho)$. Fix $j \in I'$. As V_{p_0} and s_{p_0} are event-selected C^1, by the event function property of Definition 2.4.4 there exists an open neighborhood U_j of ρ such that $s_{p_0}^j$ is an event function for V_{p_0} over U_j and that $V_{p_0}|_{\text{int } D_{p_0}^b} = V_{p_0}^b|_{\text{int } D_{p_0}^b}$ for each $b \in B_m'$. Hence, there exists a constant $c > 0$ such that $d(s_{p_0}^j)_x(V_{p_0}(x)) \geq c$ for all $x \in U_j$. As ρ is a point of transversal intersection of C^1 embedded submanifolds $\{Z_{p_0}^j\}_{j \in I'}$, this intersection can be locally brought into a normal form using C^1 adapted coordinates [KH99, Lemma A.3.17] where each embedded submanifold is locally diffeomorphic to a linear subspace of Euclidean space, and all of these linear subspaces are transverse. This implies that for each $b \in B_m'$ and $p \in J$, int $D_{p_0}^b$ is nonempty and that there exists a sequence $x_k^b \in \text{int } D_{p_0}^b$ with $x_k^b \to \rho$ as $k \to \infty$ for each $b \in B_m'$. As mentioned in the text immediately following Definition 2.4.4, this condition is what is required to ensure persistence of C^1 event-selected vector fields under perturbation. Fix $b \in B_m'$. As V_{p_0} and $V_{p_0}^b$ agree on int $D_{p_0}^b$, this implies that $d(s_{p_0}^j)_x(V_{p_0}^b(x)) \geq c$ for all $x \in \text{int } D_{p_0}^b$. As $V_{p_0}^b$ and $d(s_{p_0}^j)$ are continuous, $x_k^b \to \rho$, and $d(s^j_{p_0})_{x^b_k}(V^b_{p_0}(x^b_k)) \geq c$ for all k, $d(s^j_{p_0})_\rho(V^b_{p_0}(\rho)) \geq c$. As $\{V_p, s_p\}_{p\in J}$ is a strong C^1 continuous family of event-selected vector fields on M, by definition for each $j \in I_m$ and each $b \in B_m$, $\{s^j_p\}_{p\in J}$ and $\{V^b_p\}_{p\in J}$ are strong C^1 continuous. Fix $c' \in (0, c)$. Then this implies that there exists an open neighborhood $U_{(j,b)}$ of ρ in M such that for J sufficiently small, $d(s^j_p)_x(V^b_p(x)) \geq c'$ for all $x \in U_{(j,b)}$ and for all $p \in J$. Shrink U_j from above if necessary so that it is contained in $\bigcap_{b\in B'_m} U_{(j,b)}$. Note that we may still take U_j to be an open neighborhood of ρ because B'_m is finite. Finally, let $U = \bigcap_{j\in I_m} U_j$ and note that it is an open neighborhood of ρ. By construction of U we must have $V_p(x) \in \{V^b_p(x)\}_{b\in B'_m}$ for all $x \in U$ and all $p \in J$ because for each $j \notin I'$, which corresponds to each j such that $s^j_{p_0}(\rho) \neq 0$, s^j_p is constant in sign and disjoint from zero over U for all $p \in J$. Then, by construction of U, for each $j \in I_m$ and each $p \in J$, either $s^j_p(U)$ is constant in sign and disjoint from zero (true if $j \notin I'$) or $d(s^j_p)_x(V^b_p(x)) \geq c' > 0$ for all $x \in U$ and each $b \in B'_m$ (true if $j \in I'$). The latter case implies, since $V_p(x) \in \{V^b_p(x)\}_{b\in B'_m}$ for all $x \in U$ and all $p \in J$, that $d(s^j_p)_x(V_p(x)) \geq c' > 0$ for all $x \in U$, $p \in J$, and $j \in I'$. Hence, for each $j \in I_m$ and each $p \in J$, s_p is an event function for V_p over U. As such a U exists for every $\rho \in K'$, let $\{U_\rho\}_{\rho\in K'}$ be their collection, and let J_ρ denote the size of J (possibly shrunk) required for each $\rho \in K'$ to satisfy the event function part of the definition. As K' is compact, there exists a finite subcover of $\{U_\rho\}_{\rho\in K'}$, say $\{U_{\rho_k}\}^{k'}_{k=1}$ with k' finite. Let $J = \bigcap^{k'}_{k=1} J_{\rho_k}$ and note that it is open and contains p_0. Then the event function part of Definition 2.4.4 is satisfied by V_p and s_p at every $\rho \in K'$ and for every $p \in J$.

Next, we show that V_p and s_p satisfy the C^1 extension part of Definition 2.4.4 for every $p \in J$. Fix $\rho \in K'$. As $\{V_p, s_p\}_{p\in J}$ is a strong C^1 continuous family of event-selected vector fields on M, consistency of vector fields implies that for each $b \in B'_m$, $V_p|_{\text{int } D^b_p} = V^b_p|_{\text{int } D^b_p}$. In particular, restricting to U from the proof of satisfying the event function condition above implies that V_p admits the C^1 extension from int $D^b_p \bigcap U$ to U given by $(V^b_p)|_U$ for each $b \in B'_m$. As $\rho \in K'$ was arbitary, this implies that the C^1 extension part of Definition 2.4.4 is satisfied by V_p and s_p for every $p \in J$.

For each $p \in J$, to show that V_p and s_p determine an event-selected C^1 vector field it suffices to show that Z^j_p and $Z^{j'}_p$ are transverse over K' for all $j, j' \in I_m$. By the arguments above for the event function condition, for each $j \in I_m$, $p \in J$, and each $\rho \in Z^j_p$, $d(s^j_p)_\rho$ is nonzero. As this holds for all $\rho \in Z^j_p$, zero is a regular value of s^j_p, so $Z^j_p = (s^j_p)^{-1}(0)$ is a properly embedded C^1 submanifold with codimension one for each $j \in I_m$ and $p \in J$. Furthermore, this implies that $T_\rho Z^j_p = \ker d(s^j_p)_\rho$ for each $j \in I_m$, $p \in J$, and $\rho \in Z^j_p$. Additionally, it follows that the span of $d(s^j_p)_\rho$ is the unique one-dimensional linear subspace orthogonal to $\ker d(s_p^j)_\rho$ for each $j \in I_m$, $p \in J$, and $\rho \in Z_p^j$. Hence, for any $j, j' \in I_m$, $p \in J$, and $\rho \in Z_p^j \bigcap Z_p^{j'}$, Z_p^j and $Z_p^{j'}$ are transverse at ρ in M if and only if $T_\rho Z_p^j \oplus T_\rho Z_p^{j'} = T_\rho M$ which holds if and only if $\ker d(s_p^j)_\rho \oplus \ker d(s_p^{j'})_\rho = T_\rho M$ which holds (by orthogonality) if and only if $d(s_p^j)_\rho$ and $d(s_p^{j'})_\rho$ are linearly independent. Thus, for any $j, j' \in I_m$, $p \in J$, and $\rho \in Z_p^j \bigcap Z_p^{j'}$, Z_p^j and $Z_p^{j'}$ are transverse at ρ if and only if $d(s_p^j)_\rho$ and $d(s_p^{j'})_\rho$ are linearly independent. Fix $\rho \in K'$. Construct the open neighborhood U of ρ as in the argument above verifying the event function condition. Let $I' \subset I_m$ be the set of indeces j such that $s_{p_0}^j(\rho) = 0$. As $\{V_p, s_p\}_{p \in J}$ is a strong C^1 continuous family of event-selected vector fields on M, the event surfaces which pass through ρ are pairwise transverse at ρ. So, by the above, this implies that for every $j, j' \in I'$, $d(s_{p_0}^j)_\rho$ and $d(s_{p_0}^{j'})_\rho$ are linearly independent. As linear indepedence is an open condition and $s_{p_0}^j$ is C^1 for all $j \in I'$, there exists an open neighborhood $U_\rho \subset U$ such that $x \in U_\rho$ implies that $d(s_{p_0}^j)_x$ and $d(s_{p_0}^{j'})_x$ are linearly independent for all $j, j' \in I'$. As linear indepedence is an open condition and $\{s_p^j\}_{p \in J}$ is strong C^1 continuous for any $j \in I'$, shrinking J if necessary implies that $d(s_p^j)_x$ and $d(s_p^{j'})_x$ are linearly independent for all $x \in U_\rho$, $p \in J$, and $j, j' \in I'$. By the discussion above, this implies that Z_p^j and $Z_p^{j'}$ are transverse over U_ρ for all $p \in J$ and all $j, j' \in I'$. As $U_\rho \subset U$, by the construction of U we have that Z_p^j is disjoint from U_ρ for each $j \notin I'$, so these are vacuously transverse to $Z_p^{j'}$ over U_ρ for all $j' \in I_m$. Thus, Z_p^j and $Z_p^{j'}$ are transverse over U_ρ for all $p \in J$ and all $j, j' \in I_n$. Let $\{U_\rho\}_{\rho \in K'}$ be the collection of such neighborhoods, and let J_ρ denote the size of J (possibly shrunk) required for each $\rho \in K'$ to satisfy the transversality condition above for all $p \in J$ and all $x \in U_\rho$. As K' is compact, there exists a finite subcover of $\{U_\rho\}_{\rho \in K'}$, say $\{U_{\rho_k}\}_{k=1}^{k'}$ with k' finite. Let $J = \bigcap_{k=1}^{k'} J_{\rho_k}$ and note that it is open and contains p_0. Then, for any $p \in J$ and any $j, j' \in I_m$, Z_p^j and $Z_p^{j'}$ are transverse over K'. Hence, the transversality part of Definition 2.4.4 is satisfied by V_p and s_p at every $\rho \in K'$ and for every $p \in J$. Thus, V_p and s_p are event-selected C^1 for every $p \in J$. □

Lemma 2.4.16. *For J sufficiently small, V possesses a piecewise C^1 flow $\phi : M \times \mathbb{R} \times J \to M$.*

Lemma 2.4.16. We construct a flow analogously to the construction in [BSKR15, Section 3]. The only novel contributions here are the inclusion of the explicit dependence on parameter $p \in J$, and the use of Lemma 2.4.15 which required the additional assumption that switching surfaces are transverse. The idea is to construct a local flow, and then to piece the local flows together in the natural way, as done in [BSKR15, Corollary 1] in an analogous manner to the smooth case [Lee13, Theorem 9.12]. Let $(x, p) \in M \times \mathbb{R} \times J$. If $x \notin Z^p$ then there exists a neighborhood U^x which is not contained in Z^p (see the proof of Lemma 2.4.15). As $\{V_p^{\sigma(x)}\}_{p\in J}$ is a strong C^1 continuous family of vector fields, shrinking U^x if necessary, the local flow $\phi^{\sigma(x)}$ of $V^{\sigma(x)}$ is C^1 continuous over $U^x \times T^x \times J$ where T^x is an open interval containing zero. Then the local flow of V is given by $(x, t, p) \to (\phi^{\sigma(x)}(x, t, p), p)$, which is C^1 over its domain.

So, suppose $x \in Z^p$. First we construct a local flow candidate, and then show that it is in fact a flow of the vector field V. Let $K' \subset K$ such that $x \in Z_j^p$ if and only if $j \in K'$. Let B^x consist of all $b \in B$ such that for $j \in K - K'$, $b_j = \sigma(s_p^j(x))$. Then B^x consists of all possible combinations of switching states for the switching surfaces that intersect U^x.

By Lemma 2.4.15, V_p is event-selected discontinuous for any $p \in J$. This implies that, if a piecewise C^1 local flow ϕ exists (which we will soon establish), it must satisfy $\frac{d}{dt}s_p^j \circ \phi(t, y, p) = d(s_p^j)_y(V_p^k(y)) \geq c_j > 0$ almost everywhere. In particular, $s_p^j \circ \phi$ is monotonically increasing along the flow in forwards time, and monotonically decreasing along the flow in backwards time, for all $j \in K'$. This implies that, in forwards time, the flow can only cross Z_j^p by passing from $s_p^j \circ \phi$ negative to $s_p^j \circ \phi$ positive. In other words, along an orbit, σ_j can only pass from -1 to $+1$ for each $j \in K'$, not the other way around. Hence, the only order in which the switching surfaces Z_j^p can be encountered, if they are encountered at all, is in lexicographic order in forwards time, and reverse lexicographic order in backwards time. This observation (due to [BSKR15]) will be crucial in the construction of the local flow below. For each $k \in B^x$, the functions $(x, t, p) \to s_p^j \circ \phi^k(x, t, p)$ are C^1 for all $j \in K$ and $k \in B$ since $\{s_p^j\}_{p\in J}$ is a strong C^1 continuous family and ϕ^k is C^1. Furthermore, for each $j \in K'$ and $b \in B^x$, $\frac{\partial}{\partial t}|_{t=0} s_p^j \circ \phi^k(x, t, p) = d(s_p^j)_x(V_p^k(x)) \geq c_j > 0$ by Lemma 2.4.15. Hence, as $s_p^j \circ \phi^k(x, 0, p) = 0$, since $x \in Z_j^p$, by the implicit function theorem, shrinking U^x and J if necessary, there exist C^1 functions $\tau^{(j,k)} : U^x \times J \to \mathbb{R}$ such that for $(x, p) \in U^x \times J$, $s_p^j \circ \phi^k(x, \tau^{(j,k)}(x, p), p) = 0$. In other words, $\phi^k(x, \tau^{(j,k)}(x, p), p) \in Z_j^p$. Then $\tau^{(j,k)}(x, p)$ represents the time (positive or negative) for (x, p) to flow under ϕ^k until it hits Z_j^p. For $k \in B^x$, define the budgeted time-to-boundaries by

$$\tau_k^+(t, x, p) = \max\{0, \min(\{t\} \cup \{\tau^{(j,k)}(x, p) : j \in K', k_j < 0\})\},$$
$$\tau_k^-(t, x, p) = \min\{0, \max(\{t\} \cup \{\tau^{(j,k)}(x, p) : j \in K', k_j > 0\})\}.$$

Then for $t > 0$, $\tau_k^+(t, x, p)$ is the minimum forward time for ϕ_p^k to flow x to Z^p, or t, whichever is smaller. Similarly, for $t < 0$, $\tau_k^-(t, x, p)$ is the minimum backward time for ϕ_p^k to flow x to Z^p, or t, whichever is smaller in magnitude. Note that τ_k^+ and τ_k^- are piecewise C^1 because they consist of a finite number of maxima and minima over C^1 functions. Next, for $k \in B^x$, define candidate parts of the local flow by

$$\phi_k^+(t,x,p) = (\phi^k(x, \tau_k^+(t,x,p), p), t - \tau_k^+(t,x,p), p),$$
$$\phi_k^-(t,x,p) = (\phi^k(x, \tau_k^-(t,x,p), p), t - \tau_k^-(t,x,p), p).$$

Then ϕ_k^+ flows (t,x,p) forwards in time under ϕ^k until its first intersection with Z^p at time $\tau_k^+(t,x,p)$, and deducts the time $\tau_k^+(t,x,p)$ it takes to reach the intersection from t. Similarly, ϕ_k^- flows (t,x,p) backwards in time under ϕ^k until its first intersection with Z^p at time $\tau_k^-(t,x,p)$, and deducts the time $\tau_k^-(t,x,p)$ it takes to reach the intersection from t. Next, define the local flow candidate to be

$$\phi = \pi_1 \circ \prod_{k-} \phi_k^+ \circ \prod_{k+} \phi_k^- \tag{2.18}$$

$$\tag{2.19}$$

where π_1 is the projection onto the first coordinate M, $\prod_{k-} \phi_k^+$ refers to the composition of the functions ϕ_k^+ for $k \in B^x$ in lexicographic order, and $\prod_{k+} \phi_k^-$ refers to the composition of the functions ϕ_k^- for $k \in B^x$ in reverse lexicographic order. Note that ϕ is a finite composition of piecewise C^1 functions, hence piecewise C^1. Then for $t > 0$, $\phi(t,x,p)$ first flows x forwards in time along $\phi^{\sigma(x)}$ until it hits Z^p. Then, for each event surface encountered in the intersection, the corresponding entry in $\sigma(x)$ is flipped from -1 to $+1$, yielding a new k, and then the state flows forward under ϕ^k until the next intersection with Z^p, etc, until the time runs to zero. As discussed above, this process gives the local flow precisely because V_p is event-selected discontinuous for $p \in J$; otherwise, the lexicographic and reverse lexicographic orders would not be guaranteed to capture all possible orbits under V. Furthermore, note that by construction ϕ is C^1 away from switching surfaces. For each $p \in J$, V_p is event-selected discontinuous and so, by [BSKR15, Corollary 1] it possesses a global flow ϕ_p which satisfies $\frac{\partial}{\partial t}\phi_p(x,t) = V_p(x,t) = V(x,t,p)$ almost everywhere in $M \times \mathbb{R}$ [BSKR15, Lemma 1] (everywhere other than switching surfaces). For fixed $p \in J$, the formula for $\phi(\cdot,\cdot,p)$ in Equation 2.19 agrees with the formula for ϕ_p [BSKR15, Equation 12]. In particular, this implies that $\phi(\cdot,\cdot,p)$ is differentiable with respect to time almost everywhere and $\frac{\partial}{\partial t}\phi(t,x,p) = V_p(x,t) = V(x,t,p)$ almost everywhere (everywhere other than switching surfaces) in an open neighborhood of x and t in $M \times \mathbb{R}$. Hence, $\frac{\partial}{\partial t}\phi(t,x,p) = V(x,t,p)$ on

C^J and, thus, is differentiable almost everywhere (everywhere other than switching surfaces) in an open neighborhood of (x, t, p) in $M \times \mathbb{R} \times J$. The verification that ϕ is in fact a local flow of V follows analogously to the proof of [BSKR15, Theorem 1] using the above time derivative $\frac{\partial}{\partial t}\phi$. Finally, the local flows are pieced together to form a global flow in an analogous manner to the fixed parameter case [BSKR15, Corollary 1], which is itself analogous to the construction of global flows for smooth vector fields as in [Lee13, Theorem 9.12]. □

Lemma 2.4.17. *Z^J is closed and C^J is open in $M \times J$.*

Proof of Lemma 2.4.17. For each $j \in K$, since $\{s_p^j\}_{p \in J}$ is strong C^1 continuous, there exists $H : M \times J \to \mathbb{R}$ such that $H(x, p) = s_p^j(x)$ and H is C^1. Since H is continuous, $Z_j^J = H^{-1}(0)$ is closed in $M \times J$. As K is finite, $Z^J = \bigcup_{j \in K} Z_J^j$ is closed in $M \times J$. Hence, $C^J = (M \times J) - Z^J$ is open in $M \times J$. □

Lemma 2.4.18. *For any $(x, t, p) \in M \times \mathbb{R} \times J$ such that $x, \phi(t, x, p) \in C^J$, there exist open neighborhoods U^x, T^x, and J^x of x, t, and p, respectively, such that $\phi|_{U^x \times T^x \times J^x}$ is C^1.*

Proof of Lemma 2.4.18. By Lemma 2.4.17, C^J is open in $M \times J$, so there exist $U' \subset M$ and $J' \subset J$ open neighborhoods of $\phi(t, x, p)$ and p, respectively, such that $U' \times J' \subset C^J$. As $\phi(\cdot, x, p)$ is continuous, $T^x = \phi(\cdot, x, p)^{-1}(U' \times J')$ is an open neighborhood of t in $\mathbb{R}$. As $\phi(t, \cdot, \cdot)$ is continuous, $\hat{U} = (\bigcup_{t \in T^x} \phi(t, \cdot, \cdot)^{-1}(U' \times J')) \bigcap C^J$ is an open neighborhood of (x, p) in $M \times J$. Let U^x and J^x be open neighborhoods of x and p, respectively, such that $U^x \times J^x \subset \hat{U}$. Then $\phi|_{U^x \times T^x \times J^x}$ maps an open subset of C^J into an open subset of C^J. By the proof of Lemma 2.4.16, the derivative of ϕ exists and is continuous on C^J; discontinuities are confined to Z^J. Thus, $\phi|_{U^x \times T^x \times J^x}$ is C^1. □

Lemma 2.4.19. *For any $p \in J$, $x \in M$, and $T > 0$ finite, $\phi([0, T], x, p)$ intersects Z^p in only finitely many isolated points.*

Proof of Lemma 2.4.19. For any $(x, p) \in M \times J$ and any $T > 0$ finite, let $\gamma = \phi(\cdot, x, p)^{-1}(\phi([0, T], x, p) \bigcap Z^p)$ denote the intersection times of the flow with the switching surfaces Z^p. First note that γ is closed since $\phi([0, T], x, p)$ is compact, Z^p is closed, and $\phi(\cdot, x, p)$ is continuous. As γ is closed in $[0, T]$ compact, γ itself is compact hence sequentially compact. Assume towards a contradiction that γ has infinite cardinality. Then there exists a monotonically increasing sequence of distinct points $(t_n)_{n=1}^{\infty}$ in γ. Passing to a subsequence if necessary, we may assume $t_n \to \hat{t} \in \gamma$ since γ is sequentially compact. Let $z := \phi(\hat{t}, x, p)$ and let $K' \subset K$ such that $j \in K'$ if and only if $z \in Z_j^p$. As in the proof of Lemma 2.4.15, there exists an open neighborhood U^z of z in M such that U^z is an event neighborhood for each $j \in K'$ and U^z does not intersect $Z^p_{j'}$ for any $j' \in K - K'$. Hence, for $j \in K'$, $\frac{\partial}{\partial t} s^j_p \circ \phi(0, y, p) = d(s^j_p)_z(V_p(z)) \geq c_j > 0$ almost everywhere. Let $T^z = \phi(\cdot, x, p)^{-1}(U^z)$, which is open since U^z is open and $\phi(\cdot, x, p)$ is continuous. Then for each $j \in K'$ and every $t \in T^z - \{\hat{t}\}$, $\frac{\partial}{\partial t} s^j_p \circ \phi(t, x, p) \geq c_j > 0$. This implies that the restriction of $s^j_p \circ \phi(\cdot, x, p)$ to T^z equals zero if and only if $t = \hat{t}$ for each $j \in K'$. Hence, for $t \in T^z$, along the flow of $\phi(\cdot, x, p)$, Z^p_j for $j \in K'$ is only encountered at $t = \hat{t}$, and Z^p_j for $j \in K - K'$ is not encountered. As T^z is an open neighborhood of $\hat{t}$ such that $\phi(\cdot, x, p)^{-1}(\phi(T^z, x, p) \bigcap Z^p) = \{\hat{t}\}$, this contradicts that $t_n \to \hat{t} \in \gamma$. ☐

Lemma 2.4.20. *Let $X^i(p_0)$ be a periodic orbit for V_{p_0}. Then there exists $x \in X^i(p_0) \bigcap C^J$ and a C^1 cross section S containing x such that the first return map is well-defined and C^1 on S. Furthermore, for J and S sufficiently small, $p \in J$ implies that the first return map is well-defined and C^1 on S, that it varies C^1 with p, and that V_p is transverse to S.*

Proof of Lemma 2.4.20. By Lemma 2.4.19, since $X^i(p_0)$ has finite period T, its intersection with Z^{p_0} is finite. So, there exists (in particular) at least one $x \in X^i(p_0) - Z^{p_0}$. By Lemma 2.4.18, there exist open neighborhoods U^x, T^x, and J^x of x, t, and p, respectively, such that $\phi|_{U^x \times T^x \times J^x}$ is C^1. Let S be a smooth cross section contained in $Z^{p_0}_i \bigcap C^J \bigcap U^x$, transverse to V_{p_0}, and with $x \in S$. Shrink J^x if necessary so that V_p is transverse to S for all $p \in J$. Since S is a smooth embedded submanifold, shrinking S if necessary there exists a smooth submersion $s : M \to \mathbb{R}$ such that $S = s^{-1}(0)$. Hence, $(s \circ \phi)|_{U^x \times T^x \times J^x}$ is C^1 and its derivative with respect to time is full rank since V_{p_0} is transverse to $S = s^{-1}(0)$. Thus, by the implicit function theorem there exist open neighborhoods $U' \subset U^x$, $J' \subset J^x$, and a C^1 function $\tau : U' \times J' \to T^x$ such that $s \circ \phi(tau(z, p), z, p) = 0$ for all $(z, p) \in U' \times J'$, so that $\hat{\tau}$ represents the first intersection time with S. For $(z, p) \in S \times J^x$, let $y(z, p) = \phi(\tau(z, p), z, p)$. Then y is C^1 as it is a composition of C^1 maps, and sends $(z, p) \in S \times J^x$ to its first intersection with S. ☐

Corollary 2.4.21. *For J sufficiently small and for any $i \in I$ and $p \in J$, there exists a unique critical element $X^i(p)$ C^0-close to $X^i(p_0)$ such that $X^i(p)$ is hyperbolic in the sense described by Assumption 2.4.11.*

Proof of Corollary 2.4.21. As I is finite, it suffices to prove the claim for fixed $i \in I$. If $X^i(p_0)$ is an equilibrium point then it lies in C^J, which is open by Lemma 2.4.17. As $X^i(p_0)$ is hyperbolic and $V|_{C^J}$ is C^1, for J sufficiently small and $p \in J$, there exists a hyperbolic equilibrium point $X^i(p) \in C^J$ which is C^1 close to $X^i(p_0)$. If $X^i(p_0)$ is a periodic orbit then by Lemma 2.4.20, for J sufficiently small, there exist $x \in X^i(p_0) \bigcap C^J$ and a smooth cross section S with C^1 first return map $y : S \times J \to S$. Then x is a hyperbolic fixed point of $y|_{S\times\{p_0\}}$ by Assumption 2.4.11. As y is C^1, for J sufficiently small and $p \in J$, there exists a unique (hyperbolic) fixed point $x_p \in S$ of $y_{S\times\{p\}}$ which is C^1 close to x. Under the flow of V_p this gives rise to a unique periodic orbit $X^i(p)$ which is C^0 close to $X^i(p_0)$. □

Lemma 2.4.22. *For every $i \in I$ and $p \in J$, $W^s(X^i(p))$ and $W^u(X^i(p))$ are V_p-invariant, piecewise C^1 immersed submanifolds with $\tilde{W}^s(X^i(p)) = W^s(X^i(p)) \bigcap C^J$ and $\tilde{W}^u(X^i(p)) = W^u(X^i(p)) \bigcap C^J$.*

Proof of Lemma 2.4.22. Fix $i \in I$ and $p \in J$. By Lemma 2.4.15, V_p is event-selected discontinuous. Without loss of generality, consider $W^s(X^i(p))$; the proof for $W^u(X^i(p))$ is analogous. If $X^i(p)$ is an equilibrium point (periodic orbit) then $X^i(p)$ (then there exists a point $x \in X^i(p)$ that) lies in C^J. In the latter case, choose x to be the point that possesses a cross section S with a C^1 first return map by Lemma 2.4.20. By hyperbolicity, if $X^i(p)$ is an equilibrium point (periodic orbit) there exists a local stable manifold $W^s_{\text{loc}}(X^i(p))$ ($W^s_{\text{loc}}(X^i(p)) \bigcap S$) which is contained in C^J. If $X^i(p)$ is an equilibrium point, define the notation: $W^s_{\text{loc}}(X^i(p))$ ($S^i_s(p)$). If $X^i(p)$ is a periodic orbit, since V_p is transverse to S let $S^i_s(p)$ denote a flowout of $W^s_{\text{loc}}(X^i(p)) \bigcap S$ such that $S^i_s(p) \subset C^J$. Then $W^s(X^i(p)) = \bigcup_{t\le 0} \phi_{(t,p)}(S^i_s(p))$, so since $S^i_s(p)$ is forward V_p-invariant, $W^s(X^i(p))$ is V_p-invariant. Furthermore, for any point $y \in W^s(X^i(p))$, there exists $T > 0$ such that $\phi(T, y, p) \in \text{int } S^i_s(p)$. Let $U^y = \phi(T, \cdot, p)^{-1}(\text{int } S^i_s(p))$. Then $\phi(T, \cdot, p)|_{U^y} : U^y \to \text{int } S^i_s(p)$ is a C^0 homeomorphism between a neighborhood of U^y in $W^s(X^i(p))$ and the C^1 embedded submanifold int $S^i_s(p)$. As y was arbitrary, $W^s(X^i(p))$ is a topological manifold. Let $\tilde{W}^s(X^i(p)) = W^s(X^i(p)) \bigcap C^J$ and let $y \in \tilde{W}^s(X^i(p))$. There exists $T > 0$ such that $\phi(T, y, p) \in \text{int } S^i_s(p) \subset C^J$. By Lemma 2.4.18, $\phi(T, \cdot, p)$ is a C^1 diffeomorphism from an open neighborhood of y in C^J onto an open neighborhood of $\phi(T, y, p)$ in C^J. As int $S^i_s(p)$ is a C^1 embedded submanifold, $\phi(T, \cdot, p)$ restricts to a C^1 diffeomorphism from a neighborhood of y in $W^s(X^i(p)) \bigcap C^J$ onto a neighborhood of $\phi(T, y, p)$ in $S^i_s(p)$. As y was arbitrary, $\tilde{W}^s(X^i(p))$ is a C^1 immersed submanifold. Since $\tilde{W}^s(X^i(p))$ and $W^s(X^i(p))$ have the same dimension, $\tilde{W}^s(X^i(p))$ is open in $W^s(X^i(p))$. Let $y \in W^s(X^i(p)) - \tilde{W}^s(X^i(p))$. Then $y \in Z^p$. By Lemma 2.4.19, intersections with Z^p are isolated in time, so for some open interval T^y containing zero, $\phi(\cdot, y, p)^{-1}(\phi(T^y, y, p) \bigcap Z^p) = \{0\}$. Then the sequence $y_n = \phi(\frac{1}{n}, y, p) \in \tilde{W}^s(X^i(p))$ for n sufficiently large and $y_n \to y$ by continuity of ϕ. Hence, $\tilde{W}^s(X^i(p))$ is dense in $W^s(X^i(p))$. □

Lemma 2.4.23. *$W^s(X^s_J)$ is open and V-invariant in $M \times J$.*

Proof of Lemma 2.4.23. By Lemma 2.4.22, $W^s(X^s(p))$ is V_p-invariant for each $p \in J$, which implies that $W^s(X^s_J)$ is V-invariant. Note that $W^s_{\text{loc}}(X^s_J)$ is open in $M \times J$ by Lemma 2.2.36. As $W^s(X^s_J) = \bigcup_{t \le 0} \phi_t(W^s_{\text{loc}}(X^s_J))$ is a union of open sets in $M \times J$ - since $\phi(t, \cdot, \cdot)$ is a C^0 homeomorphism hence an open map - $W^s(X^s_J)$ is itself open in $M \times J$. $\square$

Lemma 2.4.24. *For any $i \in I$ such that $X^i(p_0)$ is an equilibrium point, (periodic orbit with $x \in X^i(p_0)$ possessing a cross section S with a C^1 first return map by Lemma 2.4.20) and for $\epsilon > 0$ sufficiently small, there exists a compact set $D \subset W^u_{loc}(X^i_{J'}) - X^i_{J'}$ ($D \subset W^u_{loc}(X^i_{J'}) \bigcap S - X^i_{J'}$) and an open neighborhood N of D in $M \times J$ such that $N \subset D_\epsilon$, $D_\epsilon \bigcap X^i_J = \emptyset$, and $\bigcup_{t \le 0} \phi_t(N) \bigcup W^s(X^i_J)$ contains an open neighborhood of $X^i_{p_0}$ (x) in $M \times J$.*

Proof of Lemma 2.4.24. As C^J is open by Lemma 2.4.17, and $X^i(p_0)$ (x) is contained in C^J, choose J sufficiently small such that $p \in J$ implies that $X^i(p)$ (x) is contained in C^J. Then since Lemma 2.2.37 is a local result, and $V|_{C^J}$ is C^1, the claim follows. $\square$

Lemma 2.4.25. *For any $i \in I$, $(W^u(X^i_{p_0}) - X^i_{p_0}) \bigcap \overline{W}^s(X^s_J) \neq \emptyset$.*

Proof of Lemma 2.4.25. As C^J is open by Lemma 2.4.17, and $X^i(p_0)$ (x) is contained in C^J, choose J sufficiently small such that $p \in J$ implies that $X^i(p)$ (x) is contained in C^J. Then, since the proof of this result in Lemma 2.2.38 for a weakly C^1 continuous family of vector fields followed from Lemma 2.2.37, the same proof carries over without modification to prove the claim here. $\square$

Lemma 2.4.26. *If $W^s(X^i(p_0)) \bigcap W^u(X^j(p_0)) \neq \emptyset$ then $n^u(X^j(p_0)) \ge n^u(X^i(p_0))$ for any $i, j \in I$. If $X^j(p_0)$ is an equilibrium point, the inequality is strict.*

Proof of Lemma 2.4.26. Since $W^s(X^i(p_0)) \bigcap W^u(X^j(p_0)) \neq \emptyset$ and they are V_{p_0}-invariant, there exists $q \in S^i_s(p_0) \bigcap W^u(X^j(p_0))$, where $S^i_s(p)$ is either $W^s_{\text{loc}}(X^i(p)$ if $X^i(p_0)$ is an equilibrium point or a flowout of $W^s_{\text{loc}}(X^i(p)) \bigcap S$ contained in C^J if $X^i(p_0)$ is a periodic orbit, where in the latter case S is a smooth cross section of $X^i(p_0)$ contained in C^J. As $S^i_s(p_0) \subset C^J$, $q \in C^J \bigcap W^s(X^i(p_0)) \bigcap W^u(X^j(p_0))$. As $W^s(X^i(p_0))$ and $W^u(X^j(p_0))$ are V_{p_0}-invariant, and C^J is open by Lemma 2.4.17, there exists an open integral curve γ of V_{p_0} containing q such that $\gamma \subset C^J \bigcap W^s(X^i(p_0)) \bigcap W^u(X^j(p_0))$. By Lemma 2.4.22, $\tilde{W}^s(X^i(p_0)) = W^s(X^i(p_0)) \bigcap C^J$ and $\tilde{W}^u(X^j(p_0)) = W^u(X^j(p_0)) \bigcap C^J$, so $\gamma \subset C^J \bigcap \tilde{W}^s(X^i(p_0)) \bigcap \tilde{W}^u(X^j(p_0))$. As $\tilde{W}^s(X^i(p_0))$ and $\tilde{W}^u(X^j(p_0))$ are transverse C^1 immersed submanifolds, the claim follows from the transversality condition together with the flow dimension, exactly as in the proof of Lemma 2.2.30. $\square$

Lemma 2.4.27. *If* $(W^s(X^i(p_0)) - X^i(p_0)) \bigcap (W^u(X^j(p_0)) - X^j(p_0)) \neq \emptyset$ *and* $(W^s(X^j(p_0)) - X^j(p_0)) \bigcap (W^u(X^k(p_0)) - X^k(p_0)) \neq \emptyset$ *then* $(\tilde{W}^s(X^i(p_0)) - X^i(p_0)) \bigcap (\tilde{W}^u(X^k(p_0)) - X^k(p_0)) \neq \emptyset$.

Proof of Lemma 2.4.27. If $X^j(p_0)$ is an equilibrium point, define the notation: $W^u_{\text{loc}}(X^j(p))$ $(S^j_u(p))$. If $X^j(p_0)$ is a periodic orbit, since V_{p_0} is transverse to S let $S^j_u(p_0)$ denote a flowout of $W^u_{\text{loc}}(X^j(p_0)) \bigcap S$ such that $S^j_u(p_0) \subset C^J$, where S is a smooth cross section of $X^j(p_0)$ as in Lemma 2.4.20. Define $S^j_s(p_0)$ analogously for the stable manifold of $X^j(p_0)$. As $(W^s(X^i(p_0)) - X^i(p_0)) \bigcap (W^u(X^j(p_0)) - X^j(p_0)) \neq \emptyset$ and $(W^s(X^j(p_0)) - X^j(p_0)) \bigcap (W^u(X^k(p_0)) - X^k(p_0)) \neq \emptyset$, and by V_{p_0}-invariance of the stable and unstable manifolds, there exist $y \in S^j_u(p_0) \bigcap (W^s(X^i(p_0)) - X^i(p_0))$ and $z \in S^j_s(p_0) \bigcap (W^u(X^k(p_0)) - X^k(p_0))$. As $S^j_s(p_0), S^j_u(p_0) \subset C^J$, by Lemma 2.4.22 $y \in S^j_u(p_0) \bigcap (\tilde{W}^s(X^i(p_0)) - X^i(p_0))$ and $z \in S^j_s(p_0) \bigcap (\tilde{W}^u(X^k(p_0)) - X^k(p_0))$. Let $D \subset (\tilde{W}^s(X^i(p_0)) - X^i(p_0))$ be a closed C^1 disk containing y that is transversal to $S^j_u(p_0)$. As D and $S^j_u(p_0)$ are compact C^1 embedded submanifolds with transversal intersection, by [KH99, Corollary A.3.18], there exists $\epsilon > 0$ such that if B' is a compact C^1 embedded submanifold which is ϵ C^1-close to $S^j_u(p_0)$ then it has nonempty, transversal intersection with D. Let $B \subset (\tilde{W}^u(X^k(p_0)) - X^k(p_0))$ be a closed C^1 disk containing z that is transversal to $S^j_s(p_0)$. By Lemma 2.4.26, $\dim \tilde{W}^u(X^k(p_0)) \geq \dim S^j_u(p_0)$, so we may choose B such that $\dim B = \dim S^j_u(p_0)$. As C^J is open, $V|_{C^J}$ is C^1, and the Inclination Lemma [Pal69, Lemma 1.1] is a local result for C^1 vector fields, shrinking δ if necessary we may apply the Inclination Lemma to conclude that there exists $T' > 0$ such that $t \geq T'$ implies $\phi_{(t,p_0)}(B)$ is ϵ C^1-close to $S^j_u(p_0)$. Hence, by the choice of ϵ, $\phi_{(t,p_0)}(B) \bigcap D \neq \emptyset$. So, let $q \in \phi_{(t,p_0)}(B) \bigcap D$. As $D \subset (\tilde{W}^s(X^i(p_0)) - X^i(p_0))$, $q \in (\tilde{W}^s(X^i(p_0)) - X^i(p_0)) \bigcap C^J$. As $B \subset (W^u(X^k(p_0)) - X^k(p_0))$, and $W^u(X^k(p_0))$ is V_{p_0}-invariant, $q \in (\tilde{W}^u(X^k(p_0)) - X^k(p_0))$. □

Corollary 2.4.28. *If* $W^u(X^i(p_0)) \bigcap W^s(X^s(p_0)) \neq \emptyset$ *and* $W^s(X^i(p_0)) \bigcap W^u(X^j(p_0)) \neq \emptyset$ *then* $\tilde{W}^u(X^j(p_0)) \bigcap \tilde{W}^s(X^s(p_0)) \neq \emptyset$.

Proof of Corollary 2.4.28. Follows from Lemma 2.4.27 with $X^i(p_0) = X^s(p_0)$, $X^j(p_0) = X^i(p_0)$, and $X^k(p_0) = X^j(p_0)$. □

Lemma 2.4.29. *For any* $i \in I$, $W^s(X^i(p_0)) \bigcap W^u(X^i(p_0)) = X^i(p_0)$.

Proof of Lemma 2.4.29. The proof is identical to the proof of Lemma 2.2.32. □

Lemma 2.4.30. *There do not exist any transverse heteroclinic cycles of critical elements contained in $\partial W^s(X^s(p_0))$. Hence, every heteroclinic cycle of critical elements contained in $\partial W^s(X^s(p_0))$ has finite length.*

Proof of Lemma 2.4.30. Assume towards a contradiction that $\{X^j(p_0)\}_{j=1}^m$ is a heteroclinic cycle contained in $\partial W^s(X^s(p_0))$. By transitivity (Lemma 2.4.27), since $X^n(p_0) = X^1(p_0)$, $(W^s(X^1(p_0)) - X^1(p_0)) \bigcap (W^u(X^1(p_0)) - X^1(p_0)) \neq \emptyset$. This contradicts Lemma 2.4.29. Since, by Assumption 2.4.9, the intersection of the nonwandering set of V_{p_0} with $\partial W^s(X^s(p_0))$ consists of a finite number of critical elements, and since there are no heteroclinic cycles, every heteroclinic sequence must be finite. □

Lemma 2.4.31. *For any $i \in I$, $W^u(X^i(p_0)) \bigcap W^s(X^s(p_0)) \neq \emptyset$.*

Proof of Lemma 2.4.31. The proof is identical to the proof of Lemma 2.2.35 using Lemmas 2.4.25-2.4.30 here. □

Lemma 2.4.32. *If $W^u(X^i(p)) \bigcap W^s(X^s(p)) \neq \emptyset$ for any $p \in J$ then $W^s(X^i(p)) \subset \partial W^s(X^s(p))$.*

Proof of Lemma 2.4.32. If $X^i(p)$ is an equilibrium point, let $G^s = W^s_{\text{loc}}(X^i(p))$ and let $G^u = W^u_{\text{loc}}(X^i(p))$. If $X^i(p)$ is a periodic orbit, let S be a smooth cross section with C^1 first return map as in Lemma 2.4.20, let $G^s = W^s_{\text{loc}}(X^i(p)) \bigcap S$, and let $G^u = W^u_{\text{loc}}(X^i(p)) \bigcap S$. As $W^s(X^i(p)) = \bigcup_{t \leq 0} \phi_{(t,p)}(G^s)$ and $\partial W^s(X^s(p))$ is V_p-invariant, it suffices to show that $G^s \subset \partial W^s(X^s(p))$. So, let $q \in G^s$. As $W^u(X^i(p)) \bigcap W^s(X^s(p)) \neq \emptyset$ and $W^s(X^s(p))$ is V_p-invariant, $G^u \bigcap W^s(X^s(p)) \neq \emptyset$. Hence, there exists $\gamma > 0$ such that if D is γ C^1-close to G^u then $D \bigcap W^s(X^s(p)) \neq \emptyset$. Let $\epsilon > 0$ and let D_ϵ be a closed C^1 disk centered at q, contained in the open ball of radius ϵ centered at q, and transverse to $\tilde{W}^s(X^i(p))$ such that $\dim D_\epsilon = \dim G^u$. As C^J is open, $V|_{C^J}$ is C^1, and the Inclination Lemma [Pal69, Lemma 1.1] is a local result for C^1 vector fields, shrinking δ if necessary we may apply the Inclination Lemma to conclude that there exists $T' > 0$ such that $t \geq T'$ implies $\phi_{(t,p)}(D_\epsilon)$ is γ C^1-close to G^u. Hence, by the choice of γ, $\phi_{(t,p)}(D_\epsilon) \bigcap W^s(X^s(p)) \neq \emptyset$. By V_p-invariance of $W^s(X^s(p))$, $D_\epsilon \bigcap W^s(X^s(p)) \neq \emptyset$. Hence, $d(q, \overline{W}^s(X^s(p))) \leq \epsilon$. As this holds for all $\epsilon > 0$, $d(D, \overline{W}^s(X^s(p))) = 0$. Since q is compact and $\overline{W}^s(X^s(p))$ is closed, $q \in \overline{W}^s(X^s(p))$. But $q \in G^s \subset W^s(X^i(p))$ implies that $q \in \partial W^s(X^s(p))$. □

Proof of Theorem 2.4.13. Let $x \in \partial W^s(X^s(p))$. Then there exists a sequence $x_n \in W^s(X^s(p))$ with $x_n \to x$. Hence, there exists a sequence $(x_n, p) \in W^s(X^s_J)$ with $(x_n, p) \to (x, p)$. Thus, $(x, p) \in \partial W^s(X^s_J)$. So, $\partial W^s(X^s_J) \supset \sqcup_{p \in J} \partial W^s(X^s(p))$.

By Assumption 2.4.10, $\partial W^s(X_J^s) \subset \bigcup_{i\in I} W^s(X_J^i)$. Fix $i \in I$. If $X^i(p_0)$ is an equilibrium point, let $S_u^i(p_0) = W_{\text{loc}}^u(X^i(p_0))$. If $X^i(p_0)$ is a periodic orbit, let S be a smooth cross section with C^1 first return map as in Lemma 2.4.20, and let $S_u^i(p_0) = W_{\text{loc}}^u(X^i(p_0)) \bigcap S$. By Lemma 2.4.31, $W^u(X^i(p_0)) \bigcap W^s(X^s(p_0)) \neq \emptyset$. As $W^s(X^s(p_0))$ is V_{p_0}-invariant, $S_u^i(p_0) \bigcap W^s(X^s(p_0)) \neq \emptyset$. As $\{S_u^i(p)\}_{p\in J}$ is C^1 continuous and $W^s(X_J^s)$ is open by Lemma 2.4.23, J sufficiently small implies that for $p \in J$, $S_u^i(p) \bigcap W^s(X^s(p)) \neq \emptyset$. By Lemma 2.4.32, $W^s(X^i(p)) \subset \partial W^s(X^s(p))$. Hence, as I is finite, for J sufficiently small $\bigcup_{i\in I} W^s(X_J^i) \subset \sqcup_{p\in J} \partial W^s(X^s(p))$. The claim follows by combining with the inclusions above. □

Proof of Corollary 2.4.14. The proof is identical to the proof of Corollary 2.2.24 except for $X^i(p_0)$ a periodic orbit, $W_{\text{loc}}^s(X^i(p_0)) \bigcap S$ is substituted for $W_{\text{loc}}^s(X^i(p_0))$ and the first return map is used in place of the flow. □

2.4.6 Conclusion

Theory presented in this section extends the RoA boundary continuity results of Section 2.2 to rough vector fields which exhibit clipping limits and switching, such as power systems. In particular, for vector fields with C^1 event-selected discontinuities which are in essence Morse-Smale along the RoA boundary, it is shown that, for a sufficiently small change in parameter values, the RoA boundary varies Chabauty continuously and it is equal to the union of the stable manifolds of the critical elements it contains. This result will be used to show that the recovery boundary consists of critical parameter values, and that the closest critical parameter value to any recover parameter value lies on the recovery boundary. In turn, this will be crucial in providing theoretical motivation for the algorithms introduced in Chapter III.

2.5 Theoretical Motivation for Maximizing Time in Ball Algorithm

2.5.1 Introduction

The purpose of this section is to provide theoretical motivation for the class of algorithms described in Section 3.2 for computing points on the recovery boundary. The first step is to establish, using the RoA boundary continuity results of Sections 2.2-2.4, that the recovery boundary consists of critical parameter values, and that starting from any recovery value the closest critical parameter value to it is also the closest point to it on the recovery boundary. This provides justification for the computation of the (closest point on the) recovery boundary via the computation of (the closest) critical parameter values.

The remainder of this section provides theoretical motivation for the algorithms of Section 3.2. In particular, it is shown that for each point in the recovery boundary there exists a controlling invariant set possessing the property that the time the trajectory spends in a neighborhood of the controlling invariant set is continuous with respect to parameter values and diverges to infinity as the parameter values approach that point, thereby providing justification for the algorithms.

2.5.2 Results

For each $p \in J$, define initial condition corresponding to parameter value p ($\hat{y}_p$), where $\hat{y} : J \to M$ sends p to the initial condition of V_p, and we write $\hat{y}_p = \hat{y}(p)$. Assume that $\hat{y}$ is C^1 over J and, as with critical elements above, sometimes consider $\hat{y}_p \in M$ and sometimes $\hat{y}_p \in M \times J$; the distinction should be clear from context. Then a parameter value $p^* \in J$ is a critical parameter value if and only if $\hat{y}_{p^*} \in \partial W^s(X^s_{p^*})$. Define $R = \{p \in J : \hat{y}_p \in W^s(X^s(p))\}$ to be the set of recovery parameter values (R), define the recovery parameter boundary (∂R), and let $C = \{p \in J : \hat{y}_p \in \partial W^s(X^s(p))\}$ be the set of critical parameter values (C). Theorem 2.5.1 shows, under the assumptions of Section 2.4.4 (for hybrid systems) or of Section 2.3.4 (for C^1 vector fields), that $\partial R \subset C$ and that the closest critical parameter value to any recovery value is also the closest point in ∂R to that recovery value.

Theorem 2.5.1. *Let M be either a compact Riemannian manifold or Euclidean space, and let $\{V_p\}_{p \in J}$ be a family of vector fields on M satisfying the conclusions of Theorem 2.4.13 (for vector fields with C^1 event-selected discontinuities) or Theorem 2.2.23 (for C^1 vector fields). Let $\hat{y} : J \to M$ be C^1. Then $\partial R \subset C$. Assume there exist $p_1, p_2 \in J$ such that $\hat{y}_{p_1} \in W^s(X^s_{p_1})$ and $\hat{y}_{p_2} \notin W^s(X^s_{p_2})$. Fix any $p_0 \in J$ and let $J_0 = \{p^* \in C : d(p_0, p^*) = d_S(p_0, C)\}$. Then J_0 is nonempty, $J_0 \subset \partial R$, and $J_0 = \{p^* \in \partial R : d(p_0, p^*) = d_S(p_0, \partial R)\}$.*

Theorem 2.5.1 justifies the method of determining or finding the closest point on the recovery boundary by computing the closest critical parameter values. Then Corollary 2.5.2 shows that for each point p^* in the recovery boundary, there exists a critical element, called the controlling critical element, such that $\hat{y}_{p^*}$ is in its stable manifold.

Corollary 2.5.2. *Assume the conditions of Theorem 2.5.1. Fix any $p^* \in \partial R$. Then there exists a unique critical element $X_J^* \subset \partial W^s(X_J^s)$, called the controlling critical element corresponding to p^*, such that $\hat{y}_{p^*} \in W^s(X_{p^*}^*)$.*

Remark 2.5.3. It is straightforward to generalize the notion of controlling critical element to controlling invariant set, where the controlling invariant set would be the ω limit set of $\hat{y}_{p^*}$ for $p^* \in \partial R$ (assuming it is nonempty). This approach is not taken here in order to preserve consistency with the settings of Sections 2.2-2.4, in which the intersection of the nonwandering set with the RoA boundary consists entirely of critical elements, but we will outline the trivial extension to the more general case of controlling invariant sets below.

We make the following assumption:

Assumption 2.5.4. *Let γ be a C^1 path in J such that $\gamma([0,1)) \subset R$ and $\gamma(1) \in \partial R$. Let X_J^* be the unique controlling critical element corresponding to $\gamma(1)$, as in Corollary 2.5.2. There exists a compact codimension-zero smooth embedded submanifold with boundary N in M such that for $p \in \gamma([0,1])$, X_p^* is contained in the interior of N, $X^s(p)$ and $\hat{y}_p$ are disjoint from N, and the orbit of $\hat{y}_p$ under V_p has nonempty, transversal intersection with ∂N.*

Remark 2.5.5. We modify Assumption 2.5.4 slightly in the case where the vector fields are C^1 event-selected discontinuous. If the controlling critical element is a periodic orbit which has nonempty intersection with at least one switching surface, then let $x \in X^*(p_0) \bigcap C^J$ as in Lemma 2.4.20, and assume that x is in the interior of N and that N intersects X_p^* for $p \in J$ (instead of assuming that N contains X_p^* for $p \in J$). More generally, assume further that $N \times J \subset C^J$ lies in the same connected component of C^J as X_J^* (in the case where $X_J^* \subset C^J$) or x (in the case where $X_J^* \bigcap Z^J \neq \emptyset$).

Remark 2.5.6. If it is desirable to consider a more general controlling invariant set, the vector fields have C^1 event-selected discontinuities, and the controlling invariant set intersects at least one switching surface, choose a point x in the controlling invariant set such that $x \in C^J$, and make the assumptions of Remark 2.5.5.

Remark 2.5.7. Unlike Assumption 2.2.18, the transversality condition of Assumption 2.5.4 can be easily checked directly by numerical simulation, and the neighborhood N adjusted accordingly if necessary. In applications, N is typically taken to be a closed ball and its radius is adjusted to ensure the transversality condition of Assumption 2.5.4 holds.

Remark 2.5.8. Assumption 2.5.4 also ensures that the initial conditions and the stable equilibria do not intersect the neighborhood N, and that the controlling critical element X_J^j is contained in N.

Let γ and N be as in Assumption 2.5.4. Let $t_{ball} : \gamma([0,1]) \to [0,\infty]$ by $t_{ball}(p) = \int_0^\infty 1_N(\phi(t,\hat{y}_p,p))dt$ where 1_N is the indicator function of N. So, $1_N(x) = 1$ if $x \in N$ and $1_N(x) = 0$ if $x \notin N$. Therefore, $t_{ball}(p)$ measures the length of time the orbit of V_p with initial condition $\hat{y}_p$ spends in N. Theorem 2.5.9 shows that t_{ball} is well-defined and continuous over $\gamma([0,1])$. Since $\hat{y}_{\gamma(1)}$ converges to $X^*_{\gamma(1)} \subset N$, $t_{ball}(\gamma(1)) = \infty$, so continuity implies that $t_{ball}(p)$ diverges to infinity as p approaches $\gamma(1)$ along the path γ.

Theorem 2.5.9. *Assume the conditions of Theorem 2.5.1. Fix any $p_0 \in J$ and any $p^* \in \partial R$. By Corollary 2.5.2, there exists a unique controlling critical element $X^*_J \subset \partial W^s(X^s_J)$ such that $\hat{y}_{p^*} \in W^s(X^*_{p^*})$. Let $\gamma : [0,1] \to J$ be a C^1 path satisfying Assumption 2.5.4 and such that $\gamma(0) = p_0$ and $\gamma(1) = p^*$. Then $t_{ball} : \gamma([0,1]) \to [0,\infty]$ is well-defined and continuous. In particular, $\lim_{p\in\gamma([0,1]), p\to p^*} t_{ball}(p) = t_{ball}(p^*) = \infty$.*

Remark 2.5.10. It is straightforward to generalize Theorem 2.5.9 to the case where the orbit of $\hat{y}_{p^*}$ converges to a more general controlling invariant set, rather than a controlling critical element. The proof is analogous because t_{ball} depends only on the choice of neighborhood N, as given in Assumption 2.5.4, and not on the nature of the controlling invariant set.

Remark 2.5.11. It is tempting here to attempt to derive the form of a particular function of $t_{ball}(p)$ with a certain (ex. linear) structure, as in Theorem 2.6.13. In particular, computation of the hitting times in Lemma 2.6.18 indicates that a good choice for this function may be $e^{-\mathrm{Re}(\lambda_1(p))t}$ where $\lambda_1(p)$ is the unstable eigenvalue with largest real part of the linearization of the vector field at $X^*(p)$. Unfortunately, the technique used in the proof of Theorem 2.6.13 to conjugate the nonlinear system with the linear system is merely a diffeomorphism, not an isometry, so it is not time preserving. Consequently, it is not clear how to translate these hitting time results back to the nonlinear system using that method. Therefore, this remains an open problem.

2.5.3 Example

We continue Example 2.2.27 of Section 2.2.4.3. Choose two values of p, call them p_1 and p_2, such that $\hat{y}_{p_1} \in W^s(X^s_J)$ but $\hat{y}_{p_2} \notin W^s(X^s_J)$. In particular, we may choose $p_1 = p_0 = 1.5$ and $p_2 = 2$. Then $\hat{y}_{p_1} = (1.27, 0.99) \in W^s(X^s_J)$ and $\hat{y}_{p_2} = (2.13, 1.32) \notin W^s(X^s_J)$, as could be verified, for example, by numerical integration. Furthermore, $\hat{y}$ is C^1 since ϕ_d is.

Hence, by Theorem 2.5.2 there must exist a point p^* in the recovery boundary such that $\hat{y}_{p^*} \in \partial W^s(X^s_{p^*})$ and $d(p_0, p^*) = d_S(p_0, \partial R)$. We will see that $p^* = 1.568$ is the desired closest point on the recovery boundary. Since $\partial W^s(X^s_{p^*}) = W^s(X^1_{p^*})$, this implies that

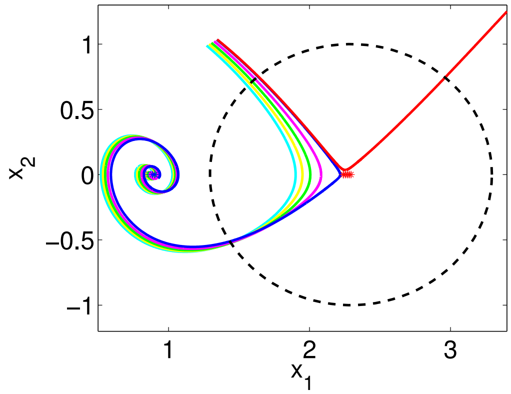

Figure 2.16: The transversal intersection of several orbits with the ball N containing the unstable equilibria (red stars). Orbits are shown for parameter values (driving torques) of 1.5 (cyan), 1.516 (yellow), 1.532 (green), 1.55 (magenta), 1.568 (blue), and 1.57 (red), where only the initial condition corresponding to the final parameter value lies outside the regions of attraction of the stable equilibria.

$\hat{y}_{p^*} \in W^s(X^1_{p^*})$, so $X^*_J = X^1_J$. Let $\gamma : [0,1] \to J$ by $\gamma(s) = (1-s)p_0 + sp^*$. Then γ is C^1, $\gamma(0) = p_0$, $\gamma(1) = p^*$, and $\gamma(s) \notin \partial R$ for $s \in (0,1)$ because γ is a minimal geodesic and $d(p_0, p^*) = d_S(p_0, C) = d_S(p_0, \partial R)$ by Theorem 2.5.1. As $\gamma([0,1))$ is connected, it does not intersect ∂R, and $\gamma(0) \in R$, we must have $\gamma([0,1)) \subset R$.

Let N be the closed ball centered at $X^*_{p_1} = X^1_{p_1}$ of radius $r = 1$ in $\mathbb{R}^2$. Fig. 2.16 shows X^*_p and the orbit of Eqs. 2.8-2.9 for a range of initial conditions $\hat{y}_p$ for $p \in [p_1, p_2]$. In particular, one can infer that the orbit has nonempty, transversal intersection with ∂N for $p \in [p_1, p_2]$. Furthermore, $X^*_J \subset N$ and $\hat{y}_{[p_1,p_2]}$, X^s_J are disjoint from N. Therefore, the path γ defined above satisfies Assumption 2.5.4 so by Theorem 2.5.9 we must have that the time the orbit spends in the neighborhood N, t_{ball}, is well-defined and continuous over $\gamma([0,1]) = [p_0, p^*]$. Fig. 2.17 illustrates the dependence of t_{ball} on $p \in \gamma([0,1]) = [p_0, p^*]$. One observes that t_{ball} is continuous and that t_{ball} diverges to infinity as p converges to a fixed value p^*. For $p = p^*$ Fig. 2.5 shows (solid blue) that $\hat{y}_{p^*} \in \partial W^s(X^s_J)$. Furthermore, $p \in \gamma([0,1)) = [p_0, p^*)$ implies that $\hat{y}_p \in W^s(X^s_J)$. Although t_{ball} is monotonic in this example, this need not be true in general.

2.5.4 Proofs

Proof of Theorem 2.5.1. Note that since the vector fields satisfy either Theorem 2.4.13 (for vector fields with C^1 event-selected discontinuities) or Theorem 2.2.23 (for C^1 vector fields),

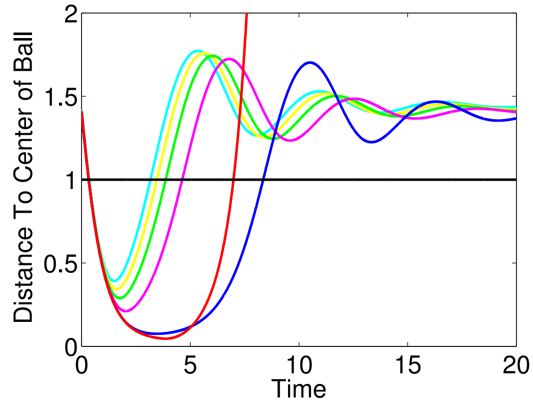

Figure 2.17: Distance from the center of the ball N as a function of time for several orbits. The line $r = 1$ marks the boundary of the ball ∂N, so the time in the ball equals the difference in time between the intersections of the orbit with this line. The orbits shown correspond to those in Fig. 2.16. As the parameter value approaches its critical value from below, the time in the neighborhood N increases. The final parameter value, which is greater than the critical parameter value, has an orbit (red) which spends less time in N than that corresponding to the critical parameter value (blue).

we have that $\partial W^s(X^s_J) = \sqcup_{p\in J}\partial W^s(X^s(p))$, which we will call equality of the boundaries. First we show that R, C, and ∂R are nonempty by connectedness of any path from p_1 to p_2. Then, we prove that every parameter value p^* in ∂R is a critical parameter value since we will see that $\hat{y}_{p^*} \in \partial W^s(X^s_J)$ which will imply, using equality of the boundaries, that $p^* \in C$. Next it is shown that J_0 is nonempty by considering a compact subset of C and arguing that the minimum distance from C to J_0 is achieved by a point in this compact subset. Finally, we argue that $J_0 = \{p \in \partial R : d(p_0, p) = d_S(p_0, \partial R)\}$ by choosing a minimal geodesic from p_0 to any fixed $p^* \in J_0$, and arguing by connectedness that all points of the geodesic other than p^* must lie in R.

First we show that R, C, and ∂R are nonempty. Since $\hat{y}_{p_1} \in W^s(X^s_{p_1})$, $p_1 \in R$ so R is nonempty. Let $\delta : [0,1] \to J$ be any continuous path in J from p_1 to p_2, so $\delta(0) = p_1$ and $\delta(1) = p_2$. Such a path exists because J is a connected manifold, hence pathwise connected. As $\hat{y}$ and δ are continuous and $[0,1]$ is connected, $\hat{y}_{\delta([0,1])}$ is connected. Since $\hat{y}_{\delta([0,1])}$ is connected and intersects both $W^s(X^s_J)$ (at $\hat{y}_{p_1}$) and $M \times J - W^s(X^s_J)$ (at $\hat{y}_{p_2}$), it must intersect $\partial W^s(X^s_J)$. Hence, there must exist $p^* \in \delta([0,1]) \subset J$ such that $\hat{y}_{p^*} \in \partial W^s(X^s_J)$. By equality of the boundaries, $\partial W^s(X^s_J) = \sqcup_{p\in J}\partial W^s(X^s(p))$. Hence, $\hat{y}_{p^*} \in \partial W^s(X^s_{p^*})$, so $p^* \in C$. Thus, C is nonempty. As $C \bigcap R = \emptyset$ and R is nonempty, this implies that ∂R is nonempty.

Next we show that $\partial R \subset C$. So, let $p^* \in \partial R$. Then there exists a sequence $p_n \in R$ with $p_n \to p^*$. Hence, by definition of R, $(\hat{y}_{p_n}, p_n) \in W^s(X^s_{p_n})$ for all n with $(\hat{y}_{p_n}, p_n) \to (\hat{y}_{p^*}, p^*)$ since $\hat{y}$ is C^1 and $p_n \to p^*$. In particular, $(\hat{y}_{p_n}, p_n) \in W^s(X^s_{p_n}) \subset \overline{W^s}(X^s_J)$ for all n. As $\overline{W^s}(X^s_J)$ is closed and $(\hat{y}_{p_n}, p_n) \to (\hat{y}_{p^*}, p^*)$, this implies that $(\hat{y}_{p^*}, p^*) \in \overline{W^s}(X^s_J)$. By equality of the boundaries, $\overline{W^s}(X^s_J) = \sqcup_{p \in J} \overline{W^s}(X^s(p))$. Hence, $\hat{y}_{p^*} \in \overline{W^s}(X^s_{p^*})$. First assume towards a contradiction that $\hat{y}_{p^*} \in W^s(X^s_{p^*})$. Let U be an open neighborhood of $X^s_{p^*}$ such that $\overline{U} \subset \text{int } W^s_{\text{loc}}(X^s_{p^*})$. Then there exists $T > 0$ such that $\phi_T(\hat{y}_{p^*}) \in U$. As int $W^s_{\text{loc}}(X^s(p))$ varies C^1 with parameter p, there exists an open neighborhood J' of p^* in J such that $p \in J'$ implies that $U \subset \text{int } W^s_{\text{loc}}(X^s(p))$. As U is open in M and both ϕ_T and $\hat{y}$ are C^1, shrinking J' if necessary implies that for $p \in J'$, $\phi_T(\hat{y}_p) \in U \subset \text{int } W^s_{\text{loc}}(X^s(p))$. Hence, J' is an open neighborhood of p^* in J such that $J' \subset R$. But, this contradicts that $p^* \in \partial R$. So, since $p^* \in \overline{W^s}(X^s_{p^*})$ but $p^* \notin W^s(X^s_{p^*})$, we must have $p^* \in \partial W^s(X^s_{p^*})$. Hence, $p^* \in C$.

We proceed by showing that J_0 is nonempty. So, fix $p_0 \in J$ and let J_0 be the set of critical parameter values $p^* \in C$ such that $d(p_0, p^*) = d_S(p_0, C)$, so $J_0 = \{p^* \in C : d(p_0, p^*) = d_S(p_0, C)\}$. First we will show that J_0 is nonempty. By the above, C is nonempty, so let $\hat{p} \in C$. Then $d_S(p_0, C) \leq d(p_0, \hat{p}) < \infty$. Shrink J slightly if necessary so that $\overline{J}$ is contained in what was originally J, and still $p_0, \hat{p} \in J$. Let B be the closed ball centered at p_0 of radius $d(p_0, \hat{p})$ in $\overline{J}$, and note that B is compact. As $\hat{y}$ is continuous, $\hat{y}_B$ is compact, so $\hat{y}_B$ is closed in $M \times J$. As $\partial W^s(X^s_J)$ is closed in $M \times J$, $\partial W^s(X^s_J) \bigcap \hat{y}_B$ is closed in $M \times J$. Let $C' = \hat{y}^{-1}(\partial W^s(X^s_J) \bigcap \hat{y}_B)$. Since $\hat{y}$ is continuous, C' is closed in J. As $\hat{p} \in C'$, $d_S(p_0, C') \leq d(p_0, \hat{p}) < \infty$. So, since C' is closed and p_0 is compact, there exists $p^* \in C'$ such that $d(p_0, p^*) = d_S(p_0, C')$. However, we claim that $C' = \{p' \in C : d(p_0, p') \leq d(p_0, \hat{p})\}$. For, $p' \in C'$ implies that $\hat{y}_{p'} \in \partial W^s(X^s_J)$. But, by equality of the boundaries, $\partial W^s(X^s_J) = \sqcup_{p \in J} \partial W^s(X^s(p))$. So, $\hat{y}_{p'} \in \partial W^s(X^s_{p'})$ and, hence, $p' \in C$. Furthermore, $p' \in C'$ implies that $p' \in B$ and therefore, by definition of B, that $d(p_0, p') \leq d(p_0, \hat{p})$. Similarly, if $p' \in C$ with $d(p_0, p') \leq d(p_0, \hat{p})$, then $p' \in B$ by definition of B and $\hat{y}_{p'} \in \partial W^s(X^s_{p'}) \subset \partial W^s(X^s_J)$. Hence, $\hat{y}_{p'} \in \partial W^s(X^s_J) \bigcap \hat{y}_B$ so $p' \in C'$. Since $C' = \{p' \in C : d(p_0, p') \leq d(p_0, \hat{p})\}$, this implies that $d_S(p_0, C) = d_S(p_0, C') = d(p_0, p^*)$ and that $p^* \in C' \subset C$. Hence, $p^* \in J_0$ so J_0 is nonempty.

Finally, we show that $J_0 = \{p \in \partial R : d(p_0, p) = d_S(p_0, \partial R)\}$. Let $p^* \in J_0$. Let $\gamma : [0, 1] \to J$ be a minimal geodesic from p_0 to p^*, so $\gamma(0) = p_0$, $\gamma(1) = p^*$, and the length of γ is equal to $d(p_0, p^*)$. For example, if J was a convex subset of Euclidean space then the image of γ would be the line segment between p_0 and p^*. For every $x \in [0, 1)$, by definition of a minimal geodesic, $d(p_0, \gamma(x)) < d(p_0, \gamma(1)) = d(p_0, p^*) = d_S(p_0, C)$, where the last equality follows since $p^* \in J_0$. This implies that for every $x \in [0, 1)$, $\gamma(x) \notin C$, since otherwise we would have that $d_S(p_0, C) \leq d(p_0, \gamma(x)) < d(p_0, p^*)$, which would contradict that $p^* \in J_0$ (so $d_S(p_0, C) = d(p_0, p^*)$). Hence, $\gamma([0,1)) \bigcap C = \emptyset$. Furthermore, since $[0,1)$ is connected and both γ and $\hat{y}$ are continuous, $\hat{y}_{\gamma([0,1))}$ is connected. Assume towards a contradiction that there exists $x \in [0,1)$ such that $\hat{y}_{\gamma(x)} \notin W^s(X^s_{\gamma(x)})$. As $\hat{y}_{p_1} \in W^s(X^s_J)$, $\hat{y}_{\gamma(x)} \notin W^s(X^s_J)$, and $\hat{y}_{p_1}, \hat{y}_{\gamma(x)} \in \hat{y}_{\gamma([0,1))}$ connected, we must have $\hat{y}_{\gamma([0,1))} \bigcap \partial W^s(X^s_J) \neq \emptyset$. So, there exists $x' \in [0,1)$ such that $\hat{y}_{\gamma(x')} \in \partial W^s(X^s_J)$. By equality of the boundaries, $\partial W^s(X^s_J) = \sqcup_{p \in J} \partial W^s(X^s(p))$. In particular, $\hat{y}_{\gamma(x')} \in \partial W^s(X^s_{\gamma(x')})$. This implies that $\gamma(x') \in C$. But, this contradicts that $\gamma([0,1)) \bigcap C = \emptyset$. So, we must have $\hat{y}_{\gamma(x)} \in W^s(X^s_{\gamma(x)})$ for all $x \in [0,1)$. Hence, $\gamma([0,1)) \subset R$. Let $p_n = \gamma\left(1 - \frac{1}{n}\right)$. Then $p_n \in R$ with $p_n \to p^*$, so $p^* \in \partial R$. As $\partial R \subset C$ by the above, $d_S(p_0, \partial R) \geq d_S(p_0, C) = d(p_0, p^*)$. As $p^* \in \partial R$, $d_S(p_0, \partial R) \leq d(p_0, p^*)$. Hence, combining these inequalities we have that $d_S(p_0, \partial R) = d(p_0, p^*)$. As $p^* \in J_0$ was arbitrary, this implies that $J_0 = \{p^* \in \partial R : d(p_0, p^*) = d_S(p_0, \partial R)\}$. □

Proof of Corollary 2.5.2. Fix $p^* \in \partial R$. By Theorem 2.5.1, $\partial R \subset C$ so $p^* \in C$. Hence, by definition of C, $\hat{y}_{p^*} \in \partial W^s(X^s_{p^*})$. By Theorem 2.2.23 (for C^1 vector fields) or Theorem 2.4.13 (for vector fields with C^1 event-selected discontinuities), $\partial W^s(X^s_{p^*}) = \bigcup_{i \in I} W^s(X^i_{p^*})$. Thus, $\hat{y}_{p^*} \in \partial W^s(X^s_{p^*})$ implies there exists a unique $j \in I$ such that $\hat{y}_{p^*} \in W^s(X^j_{p^*})$. Let $X^*_J = X^j_J$ be the controlling critical element. Then $\hat{y}_{p^*} \in W^s(X^*_{p^*})$. □

Lemma 2.5.12. *Let $\gamma : [0,1] \to J$ be a path that satisfies Assumption 2.5.4 with embedded submanifold N. For $p \in \gamma([0,1])$ let $T_p \subset [0,\infty)$ denote the set of times $\{t \in [0,\infty) : \phi_t(\hat{y}_p) \in N\}$. Then for $p \in \gamma([0,1))$, T_p consists of a finite union of closed intervals, so $t_{ball}(p)$ is well-defined and finite. For $p = \gamma(1)$, T_p consists of a finite union of closed intervals together with an interval of the form $[t', \infty)$ for some $t' > 0$, so $t_{ball}(p) = \infty$ is well-defined. For $p = \gamma(1)$, if the vector fields are C^1 event-selected discontinuous and the controlling critical element intersects at least one switching surface, then T_p consists of a countably infinite union of closed intervals, and $t_{ball}(p) = \infty$ is well-defined.*

Proof of Lemma 2.5.12. We begin by proving the results in the case that the vector fields are C^1, and subsequently illustrate the changes necessary in the case where the vector fields are C^1 event-selected discontinuous. First we show that the forward orbit of $\hat{y}_p$ under V_p is a one-dimensional C^1 embedded submanifold. This will imply, since this orbit is transverse to ∂N and N, that its intersection with N is a one-dimensional C^1 embedded submanifold with boundary equal to its intersection with ∂N, which is a zero-dimensional C^1 embedded submanifold. By compactness, this manifold boundary consists of a finite number of points.

Then the formulas for T_p are obtained by considering the connected components of a one-dimensional manifold in $[0,\infty)$.

First we show that for any $p \in \gamma([0,1])$, $\phi(\cdot,\hat{y}_p,p)^{-1}\left(\phi_{(0,T)}(\hat{y}_p)\bigcap N\right)$ is a one-dimensional embedded submanifold with boundary equal to $\phi(\cdot,\hat{y}_p,p)^{-1}\left(\phi_{(0,T)}(\hat{y}_p)\bigcap \partial N\right)$, which consists of a finite union of points. So, let $p \in \gamma([0,1])$. If $p = \gamma(1)$ then let $X_J = X_J^*$ where X_J^* is the controlling critical element corresponding to $\gamma(1)$ as in Corollary 2.5.2, and choose $W^s_{\text{loc}}(X^*_{\gamma(1)})$ sufficiently small so that it is contained in N. Otherwise, let $X_J = X_J^s$ and choose $W^s_{\text{loc}}(X^s_J)$ sufficiently small so that it is disjoint from N. Then the orbit of $\hat{y}_p$ under V_p converges to X_p, so there exists $T > 0$ such that $\phi_T(\hat{y}_p) \in \text{int } W^s_{\text{loc}}(X_p)$. Hence, by definition of the local stable manifold, $t \geq T$ implies that $\phi_t(\hat{y}_p) \in \text{int } W^s_{\text{loc}}(X_p)$. As $\hat{y}_p \in W^s(X_p)$ but $\hat{y}_p \notin X_p$, the forward orbit of $\hat{y}_p$ under V_p does not contain any critical elements, so $\phi(\cdot,\hat{y}_p,p)$ is an injective C^1 immersion from $[0,\infty)$ into M. As $\phi(\cdot,\hat{y}_p,p)$ is a continuous bijection onto its image, $[0,T]$ is compact, and M is Hausdorff, $\phi(\cdot,\hat{y}_p,p)$ is a homeomorphism from $[0,T]$ onto $\phi_{[0,T]}(\hat{y}_p)$. Hence, $\phi(\cdot,\hat{y}_p,p)$ is a C^1 embedding from $[0,T]$ onto its image, so $\phi_{(0,T)}(\hat{y}_p)$ is a C^1 embedded submanifold in M and $\phi(\cdot,\hat{y}_p,p)$ is a C^1 diffeomorphism from $(0,T)$ onto $\phi_{(0,T)}(\hat{y}_p)$. By Assumption 2.5.4, $\phi_{(0,T)}(\hat{y}_p)$ is transverse to ∂N, and it is trivially transverse to the interior of N since the dimension of N is equal to the dimension of M. Therefore, $\phi_{(0,T)}(\hat{y}_p)\bigcap N$ is a one dimensional C^1 embedded submanifold with boundary equal to $\phi_{(0,T)}(\hat{y}_p)\bigcap \partial N$ a zero dimensional C^1 embedded submanifold. Since $\hat{y}_p, \phi_T(\hat{y}_p) \notin \partial N$, $\phi_{(0,T)}(\hat{y}_p)\bigcap \partial N = \phi_{[0,T]}(\hat{y}_p)\bigcap N$. Hence, N compact and $\phi_{[0,T]}(\hat{y}_p)$ compact implies that their intersection $\phi_{[0,T]}(\hat{y}_p)\bigcap N$ is compact. So, the above implies that $\phi_{(0,T)}(\hat{y}_p)\bigcap \partial N$ is a compact zero dimensional embedded submanifold. As zero dimensional manifolds are discrete, this implies that $\phi_{(0,T)}(\hat{y}_p)\bigcap \partial N$ consists of a finite union of points. As $\phi(\cdot,\hat{y}_p,p)$ is a C^1 diffeomorphism from $(0,T)$ onto $\phi_{(0,T)}(\hat{y}_p)$, $\phi(\cdot,\hat{y}_p,p)^{-1}\left(\phi_{(0,T)}(\hat{y}_p)\bigcap N\right)$ is a one-dimensional embedded submanifold with boundary equal to $\phi(\cdot,\hat{y}_p,p)^{-1}\left(\phi_{(0,T)}(\hat{y}_p)\bigcap \partial N\right)$, which consists of a finite union of points.

Next, we show that t_{ball} is well-defined and finite for $p \in \gamma([0,1))$. So, suppose $p \in \gamma([0,1))$. Then $\hat{y}_p, \phi_T(\hat{y}_p) \notin N$, so $\phi_{(0,T)}(\hat{y}_p)\bigcap N = \phi_{[0,T]}(\hat{y}_p)\bigcap N$ is compact as $\phi_{[0,T]}(\hat{y}_p)$ and N are compact. Thus, as $\phi(\cdot,\hat{y}_p,p)$ is a C^1 diffeomorphism from $(0,T)$ onto $\phi_{(0,T)}(\hat{y}_p)$, this implies that $\phi(\cdot,\hat{y}_p,p)^{-1}\left(\phi_{(0,T)}(\hat{y}_p)\bigcap N\right)$ is a compact one-dimensional embedded submanifold in $(0,T)$ with boundary consisting of a finite number of points. Since it is a compact one-dimensional manifold, it has finitely many connected components and each contains its manifold boundary. Hence, $\phi(\cdot,\hat{y}_p,p)^{-1}\left(\phi_{(0,T)}(\hat{y}_p)\bigcap N\right)$ consists of a finite union of closed intervals. As $\phi_t(\hat{y}_p) \notin N$ for all $t \geq T$, this implies that $T_p = \{t \in [0,\infty) : \phi_t(\hat{y}_p) \in N\}$

consists of this finite union of closed intervals. So, $t_{ball}(p) = \lambda(T_p)$, where λ is the Lebesgue measure, is well-defined and is equal to the sum of the lengths of each such interval, which is finite since the intervals are contained in $[0, T]$ which has finite length T.

Finally, we show that $t_{ball}(\gamma(1)) = \infty$. So, let $p = \gamma(1)$. For $t \geq T$, $\phi_T(\hat{y}_p) \in \text{int } W^s_{\text{loc}}(X^*_{\gamma(1)}) \subset \text{int } N$, so $\phi_t(\hat{y}_p) \in \text{int } N$ for all $t \geq T$. In particular, $\phi_t(\hat{y}_p) \notin \partial N$ for all $t \geq T$. As $\phi(\cdot, \hat{y}_p, p)^{-1} \left(\phi_{(0,T)}(\hat{y}_p) \bigcap \partial N\right)$ consists of a finite number of points, let t' be the largest value in this set. Then t' represents the final intersection of the forward orbit of $\hat{y}_p$ under V_p with ∂N since, by the above, no further interesections occur for $t \geq T$. We claim that for all $t \in [t', T]$, $\phi_t(\hat{y}_p) \in N$. For, assume towards a contradiction that the claim is false. Then there exists $\hat{t} \in (t', T)$ with $\phi_{\hat{t}}(\hat{y}_p) \notin N$. As $\phi_{[\hat{t},T]}(\hat{y}_p)$ is connected with $\phi_{\hat{t}}(\hat{y}_p) \notin N$, $\phi_T(\hat{y}_p) \in N$, and N connected, there must exist $t'' \in [\hat{t}, T)$ such that $\phi_{t''}(\hat{y}_p) \in \partial N$. But, $t'' > t'$ with $\phi_{t''}(\hat{y}_p) \in \partial N$, so this contradicts that t' was the final intersection of the forward orbit of $\hat{y}_p$ under V_p with ∂N. Hence, $[t', \infty) \subset T_p$. As t' is a manifold boundary point for $\phi(\cdot, \hat{y}_p, p)^{-1} \left(\phi_{(0,T)}(\hat{y}_p) \bigcap N\right)$, there exists $\epsilon > 0$ such that $[t' - \epsilon, t') \bigcap T_p = \emptyset$. Hence, $\phi_{(0,t')}(\hat{y}_p) \bigcap N = \phi_{[0,t'-\epsilon]}(\hat{y}_p) \bigcap N$ is an intersection of two compact sets, hence compact. Thus, $\phi(\cdot, \hat{y}_p, p)^{-1} \left(\phi_{(0,t')}(\hat{y}_p) \bigcap N\right)$ is a compact one-dimensional embedded submanifold in $(0, t')$ with boundary consisting of a finite number of points. Hence, $\phi(\cdot, \hat{y}_p, p)^{-1} \left(\phi_{(0,t')}(\hat{y}_p) \bigcap N\right)$ is a finite union of closed intervals. Therefore, T_p is the union of $[t', \infty)$ with a finite union of closed intervals. So, $t_{ball}(p)$ is well-defined with $t_{ball}(p) = \infty$.

Consider now the case where the vector fields are event-selected discontinuous. First suppose that the controlling critical element does not intersect any switching surfaces. Then, by Remark 2.5.5, $N \times J \subset C^J$ is contained in the same connected component of C^J as X^*_J. By Lemma 2.4.16, the flow restricted to this component is C^1. Thus, by intersecting the orbit with this component of C^J, the proof reduces to the corresponding proof for C^1 vector fields shown above.

Now consider the case where the controlling critical element is a periodic orbit which has nonempty intersection with at least one switching surface. For $p \in \gamma([0, 1))$, the proof proceeds exactly as for the case where $X^*_J \subset C^J$ discussed above. For $p = \gamma(1)$, the argument above shows that for any $T > 0$, $T_p \bigcap [0, T]$ consists of a finite union of closed intervals. As the orbit corresponding to $\gamma(1)$ converges to $X^*_{\gamma(1)}$, and since $X^*_{\gamma(1)}$ is not contained in N, for any $T > 0$ there exists $t > T$ such that $\phi_t(\hat{y}_p) \in \partial N$. Together, the above imply that $T_{\gamma(1)}$ is a countably infinite union of closed intervals. Hence, $t_{ball}(\gamma(1))$ is well-defined and equal to the countable sum of the lengths of these intervals. Let $\hat{t}$ be the length of time the orbit $X^*_{\gamma(1)}$ spends in N during one period and let $\epsilon > 0$ small. For $p = \gamma(1)$, as the orbit of

$\phi_t(\hat{y}_p)$ converges to X_p^*, for $T > 0$ sufficiently large, the infimum of the lengths of the closed intervals in $T_p \bigcap [T, \infty)$ is at least $\hat{t} - \epsilon$. Since there are infinitely many such intervals in $[T, \infty)$, $t_{ball}(\gamma(1)) \geq \int_T^\infty 1_N(\phi(t, \hat{y}_p, p))dt \geq (\hat{t} - \epsilon)n$ for all $n > 0$. Thus, $t_{ball}(\gamma(1)) = \infty$. □

Lemma 2.5.13. *Let $\gamma : [0, 1] \to J$ be a path that satisfies Assumption 2.5.4 with embedded submanifold N. Then $\lim_{p \to \gamma(1), p \in \gamma([0,1])} t_{ball}(p) = \infty$.*

Proof of Lemma 2.5.13. By Lemma 2.5.12, $t_{ball}(\gamma(1)) = \infty$. Let $K > 0$. First we prove the result here for the case of C^1 vector fields or vector fields with C^1 event-selected discontinuities when the controlling critical element does not intersect any switching surfaces. We then prove the case where the controlling critical element does intersect at least one switching surface. So, assume that $X_J^* \subset C^J$. By the proof of Lemma 2.5.12, there exists a final time $t' \in [0, \infty)$ such that $\phi_{t'}(\hat{y}_{\gamma(1)}) \in \partial N$, and $t > t'$ implies that $\phi_t(\hat{y}_{\gamma(1)}) \in \text{int } N$. Choose any $\epsilon > 0$. Then $\phi_{[t'+\epsilon, t'+\epsilon+K]}(\hat{y}_{\gamma(1)}) \subset \text{int } N$. As ∂N and $\phi_{[t'+\epsilon, t'+\epsilon+K]}(\hat{y}_{\gamma(1)})$ are compact and disjoint in M a normal space, there exist disjoint open neighborhoods U and $\hat{U}$ in M such that $\partial N \subset U$ and $\phi_{[t'+\epsilon, t'+\epsilon+K]}(\hat{y}_{\gamma(1)}) \subset \hat{U}$. Shrink $\hat{U}$ if necessary so that $\hat{U} \subset \text{int } N$. As $\hat{U}$ is open in M and $\phi([t' + \epsilon, t' + \epsilon + K], \hat{y}_p, p)$ is compact and C^1 continuous with respect to p, then for $\delta > 0$ sufficiently small, $p \in \gamma((1 - \delta, 1])$ implies that $\phi_{[t'+\epsilon, t'+\epsilon+K]}(\hat{y}_p) \subset \hat{U} \subset \text{int } N$. So, $[t' + \epsilon, t' + \epsilon + K] \subset T_p$. By the proof of Lemma 2.5.12, T_p consists of a finite union of closed intervals, and $t_{ball}(p)$ is equal to the sum of the lengths of these intervals. Hence, $t_{ball}(p)$ is at least as large as the length of the closed interval that contains $[t' + \epsilon, t' + \epsilon + K]$, which is at least length K. As $t_{ball}(p) \geq K$ for all $p \in \gamma((1 - \delta), 1]$ and $K > 0$ was arbitrary, $\lim_{p \in \gamma([0,1]), p \to \gamma(1)} t_{ball}(p) = \infty$.

Suppose now that the controlling critical element intersects at least one switching surface. By Lemma 2.5.12 $T_{\gamma(1)}$ consists of a countably infinite union of closed intervals whose total length is infinite. Choose any $\epsilon > 0$ and let $T > 0$ such that $\lambda(T_{\gamma(1)} \bigcap [0, T]) \geq K + \epsilon$, where λ is the Lebesgue measure. By an analogous argument to the proof of Lemma 2.5.14, using Lemma 2.4.18 to ensure the intersection times of the flow with ∂N depend C^1 on parameter, it can be shown that there exists $\delta > 0$ sufficiently small such that the following holds. For $p \in \gamma((1-\delta, 1])$, the intersection times of $\phi_t(\hat{y}_p)$ with ∂N are the same number and sufficiently close to the intersection times of $\phi_t(\hat{y}_{\gamma(1)})$ with ∂N that $\lambda(T_p \bigcap [0, T]) \geq K$. Hence, $t_{ball}(p) \geq \lambda(T_p \bigcap [0, T]) \geq K$ for all $p \in \gamma((1 - \delta, 1))$. As $t_{ball}(p) \geq K$ for all $p \in \gamma((1 - \delta), 1]$ and $K > 0$ was arbitrary, $\lim_{p \in \gamma([0,1]), p \to \gamma(1)} t_{ball}(p) = \infty$. □

Lemma 2.5.14. *Let $\gamma : [0, 1] \to J$ be a path that satisfies Assumption 2.5.4 with embedded submanifold N. Then t_{ball} is continuous over $\gamma([0, 1))$*

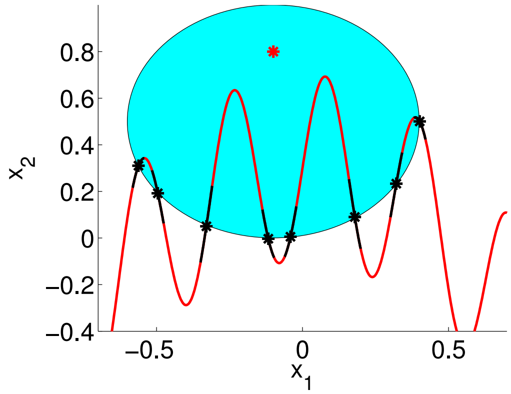

Figure 2.18: For a fixed parameter value $p \in \gamma([0,1))$, the intersection of the orbit of $\hat{y}_p$ (red and black line segments) with the embedded submanifold with boundary N (cyan ellipse) containing X_p^* (red star), an equilibrium point in this example, is shown. There are a finite number of intersections of the orbit of $\hat{y}_p$ with ∂N (black stars). The orbit is a union of line segments of the form $\phi(T_i, \hat{y}_p, p)$ (black), which contain the intersection points, and line segments of the form $\phi(T_i', \hat{y}_p, p)$ (red), which are compact, contain no intersection points, and intersect the black line segments on each end (although this intersection is not visible in the figure). This figure originally appeared in [FH17].

Proof of Lemma 2.5.14. We first prove the result in the case where the vector fields are C^1, and then extend it to vector fields with C^1 event-selected discontinuities below. So, assume the vector fields are C^1. Fix $s \in [0,1)$. To show continuity of t_{ball} over $\gamma([0,1))$ it suffices to show that it is continuous over a neighborhood of $\gamma(s)$ in $\gamma([0,1))$. Let $\epsilon > 0$. The proof procedes by first showing that there exists $T > 0$ such that $T_{\gamma(s')} \subset [0,T]$ for s' close to s since $\phi_T(\hat{y}_{\gamma(s')}) \in W^s_{\text{loc}}(X^s_{\gamma(s')})$. Then, by stability of transversal intersections and the implicit function theorem, it is shown that for every intersection point of the orbit of $\hat{y}_{\gamma(s)}$ under $V_{\gamma(s)}$ with ∂N, s' close to s implies that there exists a unique intersection point of the orbit of $\hat{y}_{\gamma(s')}$ under $V_{\gamma(s')}$ with ∂N near the original intersection point. It then argues that for s' close to s, no new intersection points appear beyond the perturbations of the original intersection points. As T_p is equal to a finite union of closed intervals whose boundaries are equal to these intersection points by Lemma 2.5.12, it will be shown that there is a one-to-one correspondence between the closed intervals in $T_{\gamma(s)}$ and the closed intervals in $T_{\gamma(s')}$. As $t_{ball}(\gamma(s'))$ is equal to the sum of the lengths of these closed intervals, there are finitely many of them, and their lengths vary continuously with parameter value since their endpoints (the intersection times) vary continuously with parameter value, it will follow that $|t_{ball}(\gamma(s')) - t_{ball}(\gamma(s))| < \epsilon$ for s' close to s. This proof is illustrated with the aid of Fig. 2.18.

First we show that $T_p \subset [0, T)$ for some $T > 0$ and p close to $\gamma(s)$. As $\gamma(s) \in R$, the forward orbit of $\hat{y}_{\gamma(s)}$ under $V_{\gamma(s)}$ converges to $X^s_{\gamma(s)}$. So, there exists $T > 0$ such that $\phi_T(\hat{y}_{\gamma(s)}) \in \text{int } W^s_{\text{loc}}(X^s_{\gamma(s)})$. As $W^s_{\text{loc}}(X^s(p))$ is open and varies C^1 with $p \in J$, and as $\phi_T(\hat{y}_p)$ varies C^1 with $p \in J$, there exists $\delta > 0$ such that $p \in \gamma((s-\delta, s+\delta))$ implies that $\phi_T(\hat{y}_p) \in \text{int } W^s_{\text{loc}}(X^s(p))$. Similarly, choosing $W^s_{\text{loc}}(X^s_{\gamma(s)})$ sufficiently small implies that for $p \in \gamma((s-\delta, s+\delta))$, $W^s_{\text{loc}}(X^s(p))$ is disjoint from N. Hence, $p \in \gamma((s-\delta, s+\delta))$ implies that for any $t \geq T$, $\phi_t(\hat{y}_p) \in \text{int } W^s_{\text{loc}}(X^s(p))$ which is disjoint from N. Therefore, $T_p \subset [0, T)$.

Next, persistence of the original intersection points is shown under small changes in parameter values. By Lemma 2.5.12, there are a finite number of intersections of $\phi_{[0,T]}(\hat{y}_{\gamma(s)})$ with ∂N. Let t_i denote the ith intersection time of the orbit of $\hat{y}_{\gamma(s)}$ under $V_{\gamma(s)}$ with ∂N. Note that $\{\phi_{[0,T]}(\hat{y}_p)\}_{p \in \gamma([0,1])}$ is a C^1 continuous family of compact embedded submanifolds with boundary in M with $\phi_{[0,T]}(\hat{y}_{\gamma(s)})$ transverse to ∂N. From the proof that points of transversal intersection between compact embedded submanifolds with boundary persist under C^1 perturbations [KH99, Proposition A.3.16], it follows by the implicit function theorem that for $\delta > 0$ sufficiently small, there exist open neighborhoods $T_i \subset [0, T]$ and C^1 functions $\tau_i : \gamma((s-\delta, s+\delta)) \to T_i$ such that the following holds. For $p \in \gamma((s-\delta, s+\delta))$, $\phi(\cdot, \hat{y}_p, p)^{-1}\left(\phi_{T_i}(\hat{y}_p) \bigcap \partial N\right) = \{\tau_i(p)\}$ and $\tau_i(\gamma(s)) = t_i$ for each i. In other words, for each $p \in \gamma((s-\delta, s+\delta))$ and for each i, there exists a unique intersection of $\phi_{[0,T]}(\hat{y}_p)$ with ∂N that occurs in the time interval T_i.

It is shown next that for δ sufficiently small, and for $p \in \gamma((s-\delta, s+\delta))$, the number of intersection times of $\phi_{[0,T]}(\hat{y}_p)$ with ∂N is constant and each intersection time varies continuously with the parameter values. Let m be the number of intersections of $\phi_{[0,T]}(\hat{y}_{\gamma(s)})$ with ∂N. Let $T'_i \subset [0, T]$ be a connected closed interval for each i such that $T'_i \bigcap T_{i-1} \neq \emptyset$, $T'_i \bigcap T_i \neq \emptyset$, and T'_i does not contain any times of intersection of $\phi_{[0,T]}(\hat{y}_{\gamma(s)})$ with ∂N, where we let $T_0 = \{0\}$ and $T_{m+1} = \{T\}$. For each i, as $\phi_{T'_i}(\hat{y}_{\gamma(s)})$ and ∂N are compact and disjoint, and since $\phi(T'_i, \hat{y}_p, p)$ varies C^1 with respect to p, shrinking δ further if necessary implies that for $p \in \gamma((s-\delta, s+\delta))$, $\phi_{T'_i}(\hat{y}_p)$ is disjoint from ∂N for all i. More specifically, $\phi_{T'_i}(\hat{y}_p) \subset N$ if and only if $\phi_{T'_i}(\hat{y}_{\gamma(s)}) \subset N$. We can write $[0, T] = \bigcup_{i=1}^m T_i \bigcup_{i=1}^{m+1} T'_i$, as shown in Fig. 2.18. Hence, for $p \in \gamma((s-\delta, s+\delta))$, the only intersection times of $\phi_{[0,T]}(\hat{y}_p)$ with ∂N occur in $\bigcup_{i=1}^m T_i$. But, by the choice of the T_i above, for $p \in \gamma((s-\delta, s+\delta))$ this implies that the only intersection times of $\phi_{[0,T]}(\hat{y}_p)$ with ∂N are $\bigcup_{i=1}^m \tau_i(p)$. By the choice of T above, this implies that for $p \in \gamma((s-\delta, s+\delta))$, the only intersection times of the orbit of $\hat{y}_p$ under V_p with ∂N are $\bigcup_{i=1}^m \tau_i(p)$. Hence, for $p \in \gamma((s-\delta, s+\delta))$, the number of intersections of the orbit of $\hat{y}_p$ under V_p with ∂N is constant.

Finally, we show that for $p \in \gamma((s-\delta, s+\delta))$, there is a one-to-one correspondence between the closed intervals in T_p and the closed intervals in $T_{\gamma(s)}$, where the interval lengths can be brought arbitrarily close to each other for sufficiently small δ, so we will conclude that $|t_{ball}(p) - t_{ball}(\gamma(s))| < \epsilon$. By Lemma 2.5.12, as $(s-\delta, s+\delta) \subset [0,1)$, for any p in $\gamma((s-\delta, s+\delta))$, T_p consists of a finite union of closed intervals whose boundary points are the intersection times. Then for each i, $[\tau_{i-1}(p), \tau_i(p)] \subset T_p$ if and only if $\phi_{T_i'}(\hat{y}_p) \subset N$ since $T_i' \subset [\tau_{i-1}(p), \tau_i(p)]$, $\phi_{[\tau_{i-1}(p), \tau_i(p)]}(\hat{y}_p)$ is connected, and there are no intersections of the orbit of $\hat{y}_p$ under V_p with ∂N in the time interval $(\tau_{i-1}(p), \tau_i(p))$. Hence, for each i, $[\tau_{i-1}(p), \tau_i(p)] \subset T_p$ if and only if $\phi_{T_i'}(\hat{y}_p) \subset N$ if and only if $\phi_{T_i'}(\hat{y}_{\gamma(s)}) \subset N$ if and only if $[\tau_{i-1}(\gamma(s)), \tau_i(\gamma(s))] \subset T_{\gamma(s)}$. Therefore, since the number of intersection times is constant over $p \in \gamma((s-\delta, s+\delta))$, for each $i \in \{1, ..., m\}$, $[\tau_{i-1}(\gamma(s)), \tau_i(\gamma(s))] \subset T_{\gamma(s)}$ if and only if $[\tau_{i-1}(p), \tau_i(p)] \subset T_p$. Therefore, T_p and $T_{\gamma(s)}$ consist of the same finite number of corresponding closed intervals which differ only slightly in their endpoints, which are the intersection times $\{\tau_i(p)\}_{i=1}^m$. Shrink δ such that for $p \in \gamma((s-\delta, s+\delta))$ and each i, $|\tau_i(p) - t_i| < \frac{\epsilon}{2m}$. Since for $p \in \gamma((s-\delta, s+\delta))$ the ith boundary point $\tau_i(p)$ is within $\frac{\epsilon}{2m}$ of $t_i = \tau_i(\gamma(s))$ for each i, and since T_p consists of the same number of corresponding closed intervals as $T_{\gamma(s)}$, each closed interval in T_p has length within $\frac{\epsilon}{m}$ of the length of the corresponding interval in $T_{\gamma(s)}$. For $p \in \gamma((s-\delta, s+\delta))$, as $t_{ball}(p)$ is equal to the sum of the lengths of the closed intervals in T_p, there are m such closed intervals in T_p, and the length of each closed interval in T_p is within $\frac{\epsilon}{m}$ of the corresponding interval in $T_{\gamma(s)}$, $|t_{ball}(p) - t_{ball}(\gamma(s))| < \epsilon$. Hence, t_{ball} is continuous at $\gamma(s)$.

Next, suppose the vector fields are C^1 event-selected discontinuous. As $N \times J \subset C^J$ is contained in a single connected component of C^J, the flow restricted to C^J is C^1 by Lemma 2.4.16, and by Lemma 2.4.18 the behavior of the flow after a switching surface depends C^1 on the behavior before the switching surface, the implicit function theorem can be applied analogously to the above to conclude that the intersection times $\{\tau_i(p)\}_{i=1}^m$ are C^1 with respect to parameter. The rest of the proof proceeds analogously. □

Proof of Theorem 2.5.9. Fix $p_0 \in J$, $p^* \in \partial R$, and let $\gamma : [0,1] \to J$ be a C^1 path satisfying Assumption 2.5.4 and such that $\gamma(0) = p_0$ and $\gamma(1) = p^*$. By Lemma 2.5.12, $t_{ball} : \gamma([0,1]) \to [0, \infty]$ is well-defined and $t_{ball}(\gamma(1)) = \infty$. By Lemma 2.5.13, $\lim_{p \in \gamma([0,1]), p \to p^*} t_{ball}(p) = \infty = t_{ball}(\gamma(1))$, so t_{ball} is continuous at $p^* = \gamma(1)$. By Lemma 2.5.14, t_{ball} is continuous over $\gamma([0,1))$. Hence, t_{ball} is continuous over $\gamma([0,1])$. □

2.5.5 Conclusion

This section provided motivation for the algorithms described in Section 3.2 both for the case of C^1 vector fields and for vector fields with C^1 event-selected discontinuities. It was first shown that the recovery boundary consists entirely of critical parameter values, and that the closest critical parameter value to any recovery value lies on the recovery boundary. This serves to justify the approach of computing (often the closest) critical parameter values in order to determine the recovery boundary. Furthermore, it was shown that, under the assumption of transversal intersection, for any point in the recovery boundary, the time the orbit spends in a neighborhood of (at least one point in) the controlling invariant set is continuous over the set of recovery values, and diverges to infinity as that point is approached from within the set of recovery values. This provides motivation for algorithms which compute critical parameter values by varying parameter values so as to maximize the time the orbit spends in a neighborhood of the controlling invariant set.

2.6 Theoretical Motivation for Maximizing Trajectory Sensitivities Algorithm

2.6.1 Introduction

The purpose of this section is to provide theoretical motivation for the algorithms described in Section 3.3. The initial results of Section 2.5 motivate the approach of computing critical parameter values in order to determine the recovery boundary. The class of algorithms mentioned above compute critical parameter values by varying parameter values so as to maximize the sensitivity of the system trajectory with respect to parameters. For numerical purposes, it is advantageous to accomplish this by varying parameter values so as to minimize the inverse trajectory sensitivities. To motivate these algorithms, it is shown here that the infimum over time of the inverse trajectory sensitivities is continuous and strictly positive over the set of recovery values, is identically zero on the recovery boundary, and converges to zero as it approaches the recovery boundary from within the set of recovery values. Furthermore, as the recovery boundary is approached from within the set of recovery values, it is shown that the infimum of the inverse trajectory sensitivities approaches a dot product, motivating algorithms which exploit this linear structure.

2.6.2 Results

Consider the same setting as in Section 2.5, as described in detail in Section 2.5.2. Then Theorem 2.5.1 and Corollary 2.5.2 apply here as well.

Assumption 2.6.1. *By Theorem 2.4.13 (for vector fields with C^1 event-selected discontinuities) or Theorem 2.2.23 (for C^1 vector fields), $\partial W^s(X^s_J) = \bigcup_{i\in I} W^s(X^i_J)$. Assume that $\hat{y}_J$ is transverse to $W^s(X^i_J)$ for all $i \in I$.*

Remark 2.6.2. Assumption 2.6.1 is generic because C^1 submanifolds are generically transverse [KH99, Theorem A.3.20].

Remark 2.6.3. If it is desirable to consider a more general controlling invariant set, as opposed to the hyperbolic controlling critical elements considered above, we assume that this controlling invariant set is hyperbolic and that $\hat{y}_J$ is transverse to its stable manifold.

By Lemma 2.4.16, V possesses a piecewise C^1 flow ϕ. By [Sch12, Chapter 3], this implies that the flow possesses a first-order approximation called the *Bouligand derivative* which is piecewise linear and continuous, rather than being purely linear like the derivative, at every point in $M \times J$. We write $d\phi$ to represent the Bouligand derivative. This does not conflict with earlier uses of this notaton, because the Bouligand derivative agrees with the (standard) derivative over C^J.

Recall that R refers to the set of recovery values, and ∂R to the recovery boundary. Note that $\frac{d\phi_t(\hat{y}(p))}{dp}$ is the derivative of the flow respect to parameter, and represents the sensitivity of the trajectory with respect to parameter. Section 3.1 discusses trajectory sensitivities in more detail. Define a function $H : [0,\infty) \times \overline{R} \to [0,\infty]$ by $H(t,p) = ||\frac{d\phi_t(\hat{y}(p))}{dp}||^{-1}$ where $||\cdot||$ is any norm on M. Then $H(t,p)$ represents the inverse of the norm of the trajectory sensitivity at time t and for parameter value p. Define the function $G : \overline{R} \to [0,\infty)$ by $G(p) = \inf_{t\in[0,\infty)} H(t,p)$. Then G gives the infimum over time of the inverse of the norm of the trajectory sensitivity.

Theorem 2.6.4. *Assume the conditions of Theorem 2.5.1 and Assumption 2.6.1. Then G is well-defined and continuous over $\overline{R}$, strictly positive on R, and identically zero on ∂R. In particular, for any $p^* \in \partial R$, $\lim_{p\in R, p\to p^*} G(p) = 0$.*

Remark 2.6.5. This theorem holds as well for a more general hyperbolic controlling invariant set. The proof for this setting is sketched immediately following the main proof of the theorem.

Under additional assumptions it will be possible to show that as $p \to p^*$, $G(p)$ approaches a dot product (in particular, a linear function). To achieve this we will require the following additional assumptions.

Assumption 2.6.6. *The vector field V_{p^*} is C^∞, $\{V_p\}_{p \in J}$ is strong C^2 continuous, and $\hat{y}$ is C^2.*

Assumption 2.6.7. *The matrix $\frac{\partial V_{p^*}(X^*(p^*))}{\partial x}$ has distinct eigenvalues and satisfies a non-resonance condition - known as the strong Sternberg condition for Q with Q sufficiently large - which will be discussed in more detail in Section 2.6.3.*

Assumption 2.6.8. *The unstable manifold $W^u(X^*(p^*))$ is one dimensional.*

Assumption 2.6.9. *By the proof of Theorem 2.2.23, $W^u(X^*(p^*)) \bigcap W^s(X^s(p^*)) \neq \emptyset$. So, let $\hat{x} \in W^u_{loc}(X^*(p^*)) \bigcap W^s(X^s(p^*))$. By the proof of Theorem 2.6.4, the function $t \to \left|\left|d(\phi_{(t,p^*)})_{\hat{x}} V_{p^*}(\hat{x})\right|\right|_1$ achieves a maximum on $\{t : t \geq 0\}$. Assume that this function achieves a unique maximum on $\{t : t \geq 0\}$.*

Remark 2.6.10. The C^∞ smoothness of Assumption 2.6.6 is more than is needed - in particular, C^k for k sufficiently large would suffice - but is adapted for simplicity of presentation.

Remark 2.6.11. The strong Sternberg condition for Q of Assumption 2.6.7 will be discussed in detail in Section 2.6.3. It is generically true that the eigenvalues of a matrix are distinct.

Remark 2.6.12. By the generic Assumption 2.6.1, $W^s(X^*(p^*))$ has codimension one. If $X^*(p^*)$ is an equilibrium point, this then implies Assumption 2.6.8. Also note that under Assumption 2.6.8, $X^*(p^*)$ must be an equilibrium point.

Theorem 2.6.13. *Assume the conditions of Theorem 2.5.1 and that Assumptions 2.6.6-2.6.9 are satisfied. Then for any $p^* \in \partial R$, there exists a constant nonzero vector c such that for $p \in R$, as $p \to p^*$, $G(p) \to |c^T(p - p^*)|$.*

Remark 2.6.14. Theorem 2.6.13 can also be extended to vector fields with C^∞ event-selected discontinuities. A brief description of this extension is included in a remark following the main proof.

2.6.3 Proofs

Lemma 2.6.15. *G is well-defined, strictly positive, and continuous on R.*

Proof of Lemma 2.6.15. Define the functions $\mathcal{F} : [0,\infty)\times\overline{R} \to [0,\infty)$ by $\mathcal{F}(t,p) = ||\frac{d\phi_t(\hat{y}(p))}{dp}||$ and $\hat{F} : \overline{R} \to [0,\infty]$ by $\hat{F}(p) = \sup_{t\in[0,\infty)} \mathcal{F}(t,p)$. Then $G = \frac{1}{\hat{F}}$, so to prove the claim it suffices to show that $\hat{F}$ is well-defined, finite, and continuous over R. Note that, by the chain rule, $\frac{d\phi_t(\hat{y}(p))}{dp} = d(\phi_t)_{\hat{y}(p)}(\hat{y}'(p))$, where $\hat{y}'(p) = \frac{d\hat{y}(p)}{dp}$.

Fix $\hat{p} \in R$. Since $\hat{y}(\hat{p}) \in W^s(X^s(\hat{p}))$, there exists $T > 0$ such that $\phi_T(\hat{y}_{\hat{p}}) \in \text{int } W^s_{\text{loc}}(X^s_J)$ (note that $W^s_{\text{loc}}(X^s_J) \subset C^J$). Since $W^s_{\text{loc}}(X^s_J)$ is open, there exists $\delta > 0$ such that $|p - \hat{p}| < \delta$ implies $\phi_T(\hat{y}(p)) \in \text{int } W^s_{\text{loc}}(X^s_J)$. As $W^s_{\text{loc}}(X^s_J)$ is forward invariant, $\phi_{t+T}(\hat{y}(p)) \in W^s_{\text{loc}}(X^s_J)$ for all $t \geq 0$. This implies that for any $t > 0$,

$$\begin{aligned} ||d(\phi_{t+T})_{\hat{y}(p)}(\hat{y}'(p))|| &= ||d(\phi_t)_{\phi_T(\hat{y}(p))} \circ d(\phi_T)_{\hat{y}(p)}(\hat{y}'(p))|| \\ &\leq ||d(\phi_t)_{\phi_T(\hat{y}(p))}||\ ||d(\phi_T)_{\hat{y}(p)}(\hat{y}'(p))|| < ||d(\phi_T)_{\hat{y}(p)}(\hat{y}'(p))|| \end{aligned}$$

since $\phi_t|_{W^s_{\text{loc}}(X^s(p))}$ is a contraction so $||d(\phi_t)_{\phi_T(\hat{y}(p))}|| < 1$. Thus, for $p \in B_\delta(\hat{p})$, $\hat{F}(p) = \sup_{t\in[0,\infty)} ||d(\phi_t)_{\hat{y}(p)}(\hat{y}'(p))|| = \sup_{t\in[0,T]} \mathcal{F}(t,p)$. Since $\hat{y}'$ is continuous and the Bouligand derivative is piecewise continuous, $\mathcal{F}(t,p)$ is piecewise continuous over $[0,T]\times\overline{R}$. As $[0,T]$ is compact, this implies that, for $p \in B_\delta(\hat{p})$, $\hat{F}(p) = \sup_{t\in[0,T]} \mathcal{F}(t,p) = \max_{t\in[0,T]} \mathcal{F}(t,p) < \infty$. Hence, as $\hat{p} \in R$ was arbitrary, $\hat{F}$ is well-defined and finite over R.

It suffices to show that $\hat{F}$ is continuous at $\hat{p}$. As $\mathcal{F}$ is piecewise continuous over $[0,T] \times B_\delta(\hat{p})$, there exists a finite set of continuous selection functions $H^j : [0,T] \times B_\delta(\hat{p}) \to [0,\infty)$ for $j \in I_m$ with m finite such that $\mathcal{F}(t,p) \in \{H^j(t,p)\}_{j\in I_m}$ for all $(t,p) \in [0,T] \times B_\delta(\hat{p})$. By Lemma 2.4.19, since $[0,T]$ is compact, $\phi(\hat{y}(p),[0,T],p)$ intersects Z^p in only finitely many isolated points. In particular, there are only finitely many intersection times at which discontinuities of $\mathcal{F}(\cdot,p)$ can occur. Using the implicit function theorem as in the proof of Lemma 2.5.14, shrinking δ if necessary implies that for $p \in B_\delta(\hat{p})$, the number of intersection times of the orbit of $\hat{y}(p)$ with Z^p is constant, and that the intersection times vary C^1 with parameter over $B_\delta(\hat{p})$. This implies that for any $p \in B_\delta(\hat{p})$, we can write $[0,T]$ as a finite union of closed intervals on which $\mathcal{F}(\cdot,p)$ is continuous, that the number of such intervals is constant over $B_\delta(\hat{p})$, and that the endpoints of the intervals vary continuously over $B_\delta(\hat{p})$. For $p \in B_\delta(\hat{p})$, as $\mathcal{F}(\cdot,p)$ is continuous over each such closed interval and the intervals are compact, it achieves a maximum on each closed interval. Moreover, since the endpoints of each interval vary continuously over $B_\delta(\hat{p})$, and $\mathcal{F}$ is equal to a continuous selection function on each interval, the maximum of $\mathcal{F}$ over each interval varies continuously over $B_\delta(\hat{p})$. Note that, for $p \in B_\delta(\hat{p})$, $\hat{F}(p)$ is given by the maximum over each interval of the maximum of $\mathcal{F}(\cdot,p)$ on that interval. Hence, over $B_\delta(\hat{p})$, $\hat{F}$ is the maximum of a finite number of continuous functions, so $\hat{F}$ is continuous. □

Lemma 2.6.16. *G is identically zero on ∂R and for any $p^* \in \partial R$, $\lim_{p\to p^*, p\in R} G(p) = 0$.*

Proof of Lemma 2.6.16. Define the functions $\mathcal{F} : [0,\infty)\times\overline{R} \to [0,\infty)$ by $\mathcal{F}(t,p) = ||\frac{d\phi_t(\hat{y}(p))}{dp}||$ and $\hat{F} : \overline{R} \to [0,\infty]$ by $\hat{F}(p) = \sup_{t\in[0,\infty)} \mathcal{F}(t,p)$. Then $G = \frac{1}{\hat{F}}$, so to prove the claim it suffices to show that $\hat{F}$ is identically infinite over ∂R, and that for any $p^* \in \partial R$, $\lim_{p\to p^*, p\in R} \hat{F}(p) = \infty$. Note that, by the chain rule, $\frac{d\phi_t(\hat{y}(p))}{dp} = d(\phi_t)_{\hat{y}(p)}(\hat{y}'(p))$, where $\hat{y}'(p) = \frac{d\hat{y}(p)}{dp}$.

Let $p^* \in \partial R$. By Theorem 2.5.1, p^* is a critical parameter value, and so by Corollary 2.5.2 there exists a unique controlling critical element X_J^* corresponding to p^*. First we show that $\hat{F}(p^*) = \infty$. Let $K > 0$. It suffices to show that there exists $T > 0$ such that $\mathcal{F}(T,p^*) > K$. If $X^*(p^*)$ is an equilibrium point (periodic orbit) let $x = X^*(p^*)$ ($x \in X^*(p^*)$ such that $x \notin Z^{p^*}$), let S be a neighborhood of x (S be a C^1 cross section containing x) such that $S \cap Z^{p^*} = \emptyset$, and let $g = \phi(\cdot, 1, p^*)|_S$ be the C^1 time-one map ($g : S \to S$ be a C^1 first return map as in Lemma 2.4.20). Then x is a hyperbolic fixed point of g C^1 so it possesses a local stable manifold $W^s_{\text{loc}}(x)$ contained in M (S). As $\hat{y}(p^*) \in W^s(X^*(p^*))$, there exists t' such that $x_0 := \phi(\hat{y}(p^*), t', p^*) \in W^s_{\text{loc}}(x)$. By Lemma 2.4.18, the flow is C^1 on a neighborhood of $\hat{y}(p^*)$ which implies, since by Assumption 2.6.1 $\hat{y}'(p^*)$ is transversal to $W^s(X^*(p^*))$ at $\hat{y}(p^*)$, and $W^s(X^*(p^*))$ is invariant under the flow, that $v_0 := d(\phi(\cdot, t', p^*))_{\hat{y}(p^*)}(\hat{y}'(p^*))$ is transversal to $W^s_{\text{loc}}(x)$ at $\phi(\hat{y}(p^*), t', p^*)$ in M (S). If $X^*(p^*)$ is a periodic orbit, let $\hat{\tau} : S \to [0,\infty)$ send x to the time it takes for the flow from initial condition x to intersect S again. Let $v_n = dg_{g^n(x_0)} \circ dg_{g^{n-1}(x_0)} \circ ... \circ dg_{x_0}(v_0)$. If $X^*(p^*)$ is an equilibrium point (periodic orbit) then $||v_n|| = \mathcal{F}(t'+n, p^*)$ ($||v_n|| = \mathcal{F}(t' + \sum_{i=0}^{n-1} \hat{\tau}(g^i(x_0)), p^*)$). So it suffices to show - using $T = t' + n$ ($T = t' + \sum_{i=0}^{n-1} \hat{\tau}(g^i(x_0))$) - that there exists n such that $||v_n|| > K$. By the Inclination Lemma [Pal69, Lemma 1.1], $\frac{||v_{n+1}||}{||v_n||} \to b' > 1$, where b' is a constant, and $||v_n|| \neq 0$ for all n. Hence, for N sufficiently large, $n \geq N$ implies that $\frac{||v_{n+1}||}{||v_n||} \geq b > 1$. Thus,

$$||v_n|| = \frac{||v_n||}{||v_{n-1}||}\frac{||v_{n-1}||}{||v_{n-2}||}...\frac{||v_{N+1}||}{||v_N||}||v_N|| \geq b^{n-N}||v_N||.$$

So, taking n sufficiently large implies that $||v_n|| > K$.

Next we show that $\lim_{p\to p^*, p\in R} \hat{F}(p) = \infty$. Let $K > 0$. It suffices to show that there exists $\delta > 0$ such that $p \in B_\delta(\hat{p}) \bigcap R$ implies that $\hat{F}(p) > K$. By the argument above, there exists $T > 0$ such that $\mathcal{F}(T,p^*) > K$ and $\phi(\hat{y}(p^*), T, p^*) \notin Z^{p^*}$. As $\phi(\hat{y}(p^*), T, p^*) \in C^J$, by Lemma 2.4.17 C^J is open, and $\hat{y}$ is C^1, there exists $\delta > 0$ such that $p \in B_\delta(p^*)$ implies that $\phi(\hat{y}(p), T, p) \in C^J$. In particular, this implies that $\mathcal{F}(T, \cdot)$ is continuous over $B_\delta(p^*)$ since the flow is C^1 over C^J by Lemma 2.4.16. As $\mathcal{F}(T, p^*) > K$, shrinking δ if necessary implies, by continuity of $\mathcal{F}(T, \cdot)$ over $B_\delta(p^*)$, that for $p \in B_\delta(p^*)$, $\mathcal{F}(T, p) > K$. Hence, for $p \in B_\delta(p^*)$, $\hat{F}(p) \geq \mathcal{F}(T, p) > K$. □

Proof of Theorem 2.6.4. Follows immediately from the conclusions of Lemma 2.6.15 and Lemma 2.6.16. □

Remark 2.6.17. We sketch a brief summary of the proof of Theorem 2.6.4 in the case of a more general controlling invariant set, under the assumption that this controlling invariant set is hyperbolic and that $\hat{y}_J$ is transverse to its stable manifold (see Remark 2.6.5). First, we note that the proof of Lemma 2.6.15 proceeds identically to the earlier case. So, it suffices to prove Lemma 2.6.16. Since the controlling invariant set is hyperbolic, there exists a continuous splitting of the ambient manifold's tangent bundle into stable and unstable bundles such that the bundles are invariant under $d\phi_1$ and the norm of $d\phi_1$ on the unstable bundle is bounded below by some constant $b > 1$. As the controlling invariant set is closed, these bundles extend to stable and unstable bundles over an open neighborhood N of the controlling invariant set such that, possibly shrinking b slightly, the norm of $d\phi_1$ on the unstable bundle is bounded below by $b > 1$. As $\hat{y}(p^*)$ converges to the controlling invariant set, after some finite time t' it enters the local stable manifold of the controlling invariant set and remains there for all times $t \geq t'$. In particular, we may assume that $\phi(\hat{y}(p^*), t, p^*) \in N$ for $t \geq t'$. Let $g = \phi_1$. Construct x_0 and v_0 as in the proof of Lemma 2.6.16, and note that v_0 is transversal to the stable bundle, so it must have a nonzero component in the unstable direction. Define v_n analogously to the proof of Lemma 2.6.16 as well. Then $\frac{||v_{n+1}||}{||v_n||} \geq b$ for sufficiently large n (since the component in the stable direction will converge to zero), so by an analogous argument to that in the proof of Lemma 2.6.16, $\hat{F}(p^*) = \infty$. That $\lim_{p \to p^*, p \in R} \hat{F}(p) = \infty$ follows by an analogous argument to that in the proof of Lemma 2.6.16.

Let A be a square matrix, and let $\Sigma(A) = \{\lambda_1, ..., \lambda_n\}$ be its eigenvalues. Let m be any vector whose entries are nonnegative integers. Define

$$\gamma(\lambda, m) = \lambda - \sum_{i=1}^{n} m_i \lambda_i.$$

Let $|m| = \sum_{i=1}^n m_i$. Then A satisfies the Sternberg condition of order N if $\gamma(\lambda, m) \neq 0$ for all $\lambda \in \Sigma(A)$ and all m such that $2 \leq |m| \leq N$. Furthermore, A satisfies the strong Sternberg condition of order N if A satisfies the Sternberg condition of order N and Re $\gamma(\lambda, m) \neq 0$

for all $\lambda \in \Sigma(A)$ and all m such that $|m| = N$. Note that if A satisfies the strong Sternberg condition then A is hyperbolic.

If A is hyperbolic, let $\Sigma^+(A)$ denote its eigenvalues with positive real part, and let $\Sigma^-(A)$ denote its eigenvalues with negative real part. For $i \in \{+, -\}$, let $\rho^i = \frac{\max\{|\text{Re } \lambda| : \lambda \in \Sigma^i(A)\}}{\min\{|\text{Re } \lambda| : \lambda \in \Sigma^i(A)\}}$. Let Q be a positive integer. Then the Q-smoothness of A is the largest integer $K \geq 0$ such that there exist positive integers Q^+ and Q^- with $Q = Q^+ + Q^-$, $Q^+ - K\rho^+ \geq 0$, and $Q^- - K\rho^- \geq 0$. In particular, the Q-smoothness of A approaches infinity as $Q \to \infty$, so for Q sufficiently large we may assume it is at least two. Assumption 2.6.7 can be stated as assuming that the matrix $\frac{\partial V_{p^*}(X^*(p^*))}{\partial x}$ satisfies the strong Sternberg condition for Q with Q sufficiently large such that the Q-smoothness of $\frac{\partial V_{p^*}(X^*(p^*))}{\partial x}$ is at least two.

Lemma 2.6.18. *Consider the vector field $\dot{x} = A(p)x$ for $p \in J$ where $A(p)$ varies C^1 with respect to p and is linear for each p. Assume that $A(0)$ is hyperbolic, has distinct eigenvalues, and has exactly one unstable eigenvalue. Let $\hat{y}(p)$ be a C^1 initial condition for this linear vector field, assume that $\hat{y}(0)$ is in the stable eigenspace of $A(0)$, and that $\hat{y}(J)$ is transverse to the disjoint union of the stable eigenspaces of $A(p)$ for $p \in J$. Let $u > 0$ small and let $t_{hit} = t_{hit}(p)$ be the time at which the projection of the forward orbit of $\hat{y}(p)$ onto the unstable eigenspace has magnitude u (note this is defined only for a connected component $J^T \subset J$ where $\hat{y}(J^T)$ is disjoint from the stable manifolds of $A(p)$). Let $\hat{G}(p) = \left(\sum_{j=1}^{n} \left|\left|\frac{\partial x(t_{hit}(p))}{\partial p_j}\right|\right|_1\right)^{-1}$. Then there exists a constant nonzero vector c such that for $p \in J^T$, as $p \to p^*$, $\hat{G}(p) \to |c^T p|$.*

Proof of Lemma 2.6.18. The proof proceeds primarily by computation. The main intuition is that the trajectory sensitivities in the direction of the (one dimensional) unstable eigenspace dominate the other sensitivities in the directions of the stable eigenspace. As $p \to p^*$, this will imply that the structure of $\hat{G}(p)$ is dominated by this one dimensional behavior, which in turn is governed by a dot product. The dot product structure will follow from taking the explicit solution of the linear system, computing $t_{\text{hit}}(p)$ directly, computing the trajectory sensitivities explicitly, and then deriving a formula for $\hat{G}(p)$.

Let $x \in \mathbb{R}^n$ and $p \in J$. Consider the vector field $\dot{x} = A(p)x$ where $A(p)$ is linear for each $p \in J$. We have that for all $p \in J$, $A(p)$ is hyperbolic, all its eigenvalues are distinct, and its unstable eigenspace is one dimensional. Let $\{\lambda_1(p), ..., \lambda_n(p)\}$ be the eigenvalues of $A(p)$, and order them by their real component in descending order so that λ_1 is the unstable eigenvalue. As the eigenvalues of A are distinct, A is diagonalizable with a basis of eigenvectors; denote them by $\{v_1, ..., v_n\}$ and assume they are normalized so that $\sum_j |v_i|_j = 1$ for all i. Let $\Lambda(p)$

be the diagonal matrix of the eigenvalues of $A(p)$ and let $V(p)$ be the matrix whose columns are the eigenvectors of $A(p)$. Note that $A(p) = V(p)\Lambda(p)V(p)^{-1}$. Let $z(t) = V(p)^{-1}x(t)$ so that

$$\dot{z} = \Lambda(p)z. \tag{2.20}$$

Then the solution to Eq. 2.20 is given by

$$z(t) = e^{\Lambda(p)t}z(0)$$
$$z_i(t) = e^{\lambda_i(p)t}z_i(0).$$

We have that the initial condition and eigenvalues are at least C^1 so that we may write $z(0) = Bp + b + o(||p||_1)$, which implies that $z_i(0) = B_{in}p_n + b_i + o(||p||_1)_i$, and $\lambda_i(p) = \lambda_i^0 + c_{in}p_n + o(||p||_1)_i$ for all i. We have that $z_i(0)$ at $p = 0$ is in the stable eigenspace of the origin, which implies that $b_1 = 0$ since the stable eigenspace is given by $\{z : z_1 = 0\}$. Furthermore, we have that $z_0(J)$ is transverse to the stable eigenspace at $p = 0$. The solution to Eq. 2.20 can be written as

$$z_i(t) = e^{(\lambda_i^0 + c_{in}p_n + o(||p||_1)_i)t}(B_{in}p_n + b_i + o(||p||_1)_i).$$

Let $u > 0$ small. We wish to compute the time t_{hit} at which the first coordinate (the unstable coordinate) reaches u, This is merely the hitting time of the hypersurface $\{z : \ z_1 = u\}$. So, we wish to solve $|z_1(t_{\text{hit}}(p))| = u$. This gives

$$t_{\text{hit}}(p) = \frac{1}{\text{Re } \lambda_1(p)} \log \frac{u}{|z_1(0)|} = \frac{1}{\text{Re } (\lambda_i^0) + \text{Re } (c_{in})p_n + o(||p||_1)_i} \log \frac{u}{|B_{1n}p_n + o(||p||_1)_1|}$$

which is well-defined for all $p \in J$ such that $z(0)$ is not in the stable eigenspace of the origin since the stable eigenspace is given by $\{z : z_1 = 0\}$. Note that the set on which t_{hit} is well-defined is nonempty because $z_0(J)$ is transverse to the stable eigenspace at $p = 0$. We compute

$$\left|\frac{\partial z_i(t)}{\partial p_j}\right| = e^{\text{Re}(\lambda_i(p))t} \left|(c_{ij} + o(1)_{ij})t(B_{in}p_n + b_i + o(||p||_1)_i) + (B_{ij} + o(1)_{ij})\right|$$

since $\frac{\partial o(||p||_1)}{\partial p_j} = o(1)$. Evaluating at $t = t_{hit}(p)$ gives

$$\left|\frac{\partial z_i(t_{\text{hit}}(p))}{\partial p_j}\right| = \left(\frac{u}{|B_{1n}p_n + o(||p||_1)_1|}\right)^{\frac{\text{Re}(\lambda_i(p))}{\text{Re}(\lambda_1(p))}}$$
$$* \left|(c_{ij} + o(1)_{ij})\frac{B_{in}p_n + b_i + o(||p||_1)_i}{\text{Re}(\lambda_1(p))} \log \frac{u}{|B_{1n}p_n + o(||p||_1)_1|} + B_{ij} + o(1)_{ij}\right|.$$

Note that

$$\lim_{||p||_1 \to 0} (B_{1n}p_n + o(||p||_1)_1) \log \frac{u}{|B_{1n}p_n + o(||p||_1)_1|} = 0$$

as $\lim_{x\to 0} x \log \frac{u}{x} = 0$. So,

$$(B_{1n}p_n + o(||p||_1)_1) \log \frac{u}{|B_{1n}p_n + o(||p||_1)_1|} = o(1). \tag{2.21}$$

In particular, this implies that at $i = 1$, and since $b_1 = 0$, we have

$$\left|\frac{\partial z_1(t_{\text{hit}}(p))}{\partial p_j}\right| = \frac{u}{|B_{1n}p_n + o(||p||_1)_1|}$$
$$* \left|\left(\sum_j c_{1j} + o(1)_{1j}\right) \frac{B_{1n}p_n + o(||p||_1)_1}{\text{Re}(\lambda_1(p))} \log \frac{u}{|B_{1n}p_n + o(||p||_1)_1|} + \left(\sum_j B_{1j} + o(1)_{1j}\right)\right|$$
$$= \frac{u}{|B_{1n}p_n + o(||p||_1)_1|} \left|\left(\sum_j c_{1j} + o(1)_{1j}\right) \frac{o(1)}{\text{Re}(\lambda_1(p))} + \left(\sum_j B_{1j} + o(1)_{1j}\right)\right|$$
$$= \frac{u}{|B_{1n}p_n + o(||p||_1)_1|} \left|\sum_j B_{1j} + o(1)\right|$$

where we use that $\lambda_1^0 \neq 0$. As $\lim_{||p||_1 \to 0} \frac{|B_{1n}p_n + o(||p||_1)_1|}{u} = 0$ and $\frac{\text{Re}\lambda_i(p)}{\text{Re}\lambda_1(p)} < 0$ for all $i \geq 2$ and $p \in J$,

$$\lim_{||p||_1 \to 0} \left(\frac{u}{|B_{1n}p_n + o(||p||_1)_1|}\right)^{\frac{\text{Re}\lambda_i(p)}{\text{Re}\lambda_1(p)}} = 0$$

so $\left(\frac{u}{|B_{1n}p_n + o(||p||_1)_1|}\right)^{\frac{\text{Re}\lambda_i(p)}{\text{Re}\lambda_1(p)}} = o(1)_i$. Hence, for $i \geq 2$ we have

$$\left|\frac{\partial z_i(t_{\text{hit}}(p))}{\partial p_j}\right| = o(1)_i \left|(c_{ij} + o(1)_{ij})\frac{B_{in}p_n + b_i + o(||p||_1)_i}{\text{Re}(\lambda_1(p))} \log \frac{u}{|B_{1n}p_n + o(||p||_1)_1|} + B_{ij} + o(1)_{ij}\right|.$$

Define

$$\hat{G}^z(p) = \sum_i \left|\left|\frac{\partial z_i(t_{\text{hit}}(p))}{\partial p}\right|\right|_1 = \sum_{i,j} \left|\frac{\partial z_i(t_{\text{hit}}(p))}{\partial p_j}\right|.$$

We compute

$$\begin{aligned}
\hat{G}^z(p) &= \frac{u}{|B_{1n}p_n + o(||p||_1)_1|}\left|\sum_j B_{1j} + o(1)\right| \\
&+ \sum_{i=2}^{n} o(1)_i \left|\left(\sum_j c_{ij} + o(1)_{ij}\right)\frac{B_{in}p_n + b_i + o(||p||_1)_i}{\text{Re}(\lambda_1(p))}\log\frac{u}{|B_{1n}p_n + o(||p||_1)_1|} + \left(\sum_j B_{ij} + o(1)_{ij}\right)\right| \\
&= \frac{u}{|B_{1n}p_n + o(||p||_1)_1|}\left|\sum_j B_{1j} + o(1)\right| \\
&+ \frac{u}{|B_{1n}p_n + o(||p||_1)_1|}\sum_{i=2}^{n} o(1)_i \\
&* \left|\left(\sum_j c_{ij} + o(1)_{ij}\right)\frac{B_{in}p_n + b_i + o(||p||_1)_i}{\text{Re}(\lambda_1(p))}\frac{|B_{1n}p_n + o(||p||_1)_1|}{u}\log\frac{u}{|B_{1n}p_n + o(||p||_1)_1|}\right. \\
&\left. + \frac{|B_{1n}p_n + o(||p||_1)_1|}{u}\left(\sum_j B_{ij} + o(1)_{ij}\right)\right| \\
&= \frac{u}{|B_{1n}p_n + o(||p||_1)_1|}\left(\left|\sum_j B_{1j} + o(1)\right| + o(1)\right)
\end{aligned}$$

by Eq. 2.21 and standard rules for arithmetic with $o(1)$ functions.

We have that $x(t) = V(p)z(t)$, so that $x_i(t) = V(p)_{ik}z_k(t)$. This implies that

$$\begin{aligned}
\frac{\partial x_i(t)}{\partial p_j} &= \frac{\partial V(p)_{ik}}{\partial p_j}z_k(t) + V(p)_{ik}\frac{\partial z_k(t)}{\partial p_j} \\
\frac{\partial x_i(t_{\text{hit}}(p))}{\partial p_j} &= \frac{\partial V(p)_{ik}}{\partial p_j}z_k(t_{\text{hit}}(p)) + V(p)_{ik}\frac{\partial z_k(t_{\text{hit}}(p))}{\partial p_j}
\end{aligned}$$

Let $V(p)_{ik} = d_{ik} + D_{ikn}p_n + o(||p||_1)_{ik}$. Then $\frac{\partial V(p)_{ik}}{\partial p_j} = D_{ikj} + o(1)_{ikj}$ is bounded over $p \in J$. Furthermore, as $z_k(t_{\text{hit}}(p))$ is contained in a small neighborhood of the origin for all $p \in J$

along the target surface $\{z : z_1 = u\}$, $z_k(t_{\text{hit}}(p))$ is uniformly bounded over $p \in J$. Hence, $\frac{\partial V(p)_{ik}}{\partial p_j} z_k(t_{\text{hit}}(p))$ is uniformly bounded over $p \in J$. By an analogous computation to the derivation of $\hat{G}^z(p)$ above, we can show that for any fixed i such that $|V(p)_{i1}| \neq 0$,

$$\begin{aligned}\sum_k \left| V(p)_{ik} \frac{\partial z_k(t_{\text{hit}}(p))}{\partial p_j} \right| &= \sum_k |V(p)_{ik}| \left| \frac{\partial z_k(t_{\text{hit}}(p))}{\partial p_j} \right| \\ &= |V(p)_{i1}| \frac{u}{|B_{1n}p_n + o(||p||_1)_1|} \left(\left| \sum_j B_{1j} + o(1) \right| + o(1) \right).\end{aligned}$$

As $\frac{\partial V(p)_{ik}}{\partial p_j} z_k(t_{\text{hit}}(p))$ is uniformly bounded over $p \in J$, we have

$$\begin{aligned}\frac{\partial V(p)_{ik}}{\partial p_j} z_k(t_{\text{hit}}(p)) &= \frac{u}{|B_{1n}p_n + o(||p||_1)_1|} \left(\frac{\partial V(p)_{ik}}{\partial p_j} z_k(t_{\text{hit}}(p)) \frac{|B_{1n}p_n + o(||p||_1)_1|}{u} \right) \\ &= \frac{u}{|B_{1n}p_n + o(||p||_1)_1|} \left(o(1) \right). \end{aligned} \tag{2.22}$$

Define

$$\begin{aligned}\hat{G}(p) &= \sum_i \left\| \frac{\partial x_i(t_{\text{hit}(p)})}{\partial p} \right\|_1 \\ &= \sum_{i,j} \left| \frac{\partial x_i(t_{\text{hit}}(p))}{\partial p_j} \right|.\end{aligned}$$

Then

$$\begin{aligned}\hat{G}(p) &= \sum_i \sum_j \left| \frac{\partial x_i(t_{\text{hit}}(p))}{\partial p_j} \right| \\ &= \left(\sum_i |V(p)_{i1}| \right) \frac{u}{|B_{1n}p_n + o(||p||_1)_1|} \left(\left| \sum_j B_{1j} + o(1) \right| + o(1) \right) \\ &= \frac{u}{|B_{1n}p_n + o(||p||_1)_1|} \left(\left| \sum_j B_{1j} + o(1) \right| + o(1) \right)\end{aligned}$$

since by Eq. 2.22 the terms $\frac{\partial V(p)_{ik}}{\partial p_j} z_k(t_{\text{hit}}(p))$ can be incorporated into the additive $o(1)$ term in the parentheses, as can the contributions from the eigenvectors for $i \geq 2$ (as in the derivation of $\hat{G}^z(p)$ above), and since $\sum_i |V(p)_{i1}| = 1$ by the normalization above. Thus, we have

$$\frac{1}{\hat{G}(p)} = \frac{|B_{1n}p_n + o(||p||_1)_1|}{u\left(\left|\sum_j B_{1j} + o(1)\right| + o(1)\right)}$$

which approaches a dot product - an affine function - as $||p||_1 \to 0$. □

Remark 2.6.19. It will be useful for Chapter III to note the relationship between $t_{\text{hit}}(p)$ and $\hat{G}(p)$ in the proof of Lemma 2.6.18. We note that as $p \to p^* = 0$, $t_{\text{hit}}(p) \to c_1 \log \frac{1}{c_2^T p}$ while $\hat{G}(p) \to \frac{1}{c_3^T p}$ for nonzero constant vectors c_2 and c_3, and for real nonzero constant c_1. In particular, this implies that $\hat{G}(p)$ grows exponentially faster than $t_{\text{hit}}(p)$ as $p \to p^* = 0$.

Proof of Theorem 2.6.13. For the intuition behind this proof, consider partitioning each trajectory of the system into four parts: the initial segment from $\hat{y}(p)$ into a neighborhood of $X^*(p)$, the behavior in this neighborhood of $X^*(p)$, a segment from the neighborhood of $X^*(p)$ to $W^s_{\text{loc}}(X^s(p))$, and then the forward orbit inside $W^s_{\text{loc}}(X^s(p))$. As the flow restricted to $W^s_{\text{loc}}(X^s(p))$ is a contraction, trajectory sensitivities must achieve their maximum before entering $W^s_{\text{loc}}(X^s(p))$, so we can restrict our attention to the first three parts mentioned above. For the initial segment from $\hat{y}(p)$ into a neighborhood of $X^*(p)$, we may neglect this segment and instead absorb this behavior into new parameter dependent initial conditions in the neighborhood of $X^*(p)$. Taking the neighborhood of $X^*(p)$ to be sufficiently small, we may apply a Theorem of George Sell which is similar to Sternberg's Theorem to obtain a smooth (for our purposes C^2 will suffice) conjugacy between the nonlinear vector field V_p and its linearization at $X^*(p)$ for all $p \in J$. This will allow us to translate the results from the linear dynamics of Lemma 2.6.18 into the nonlinear setting here. Finally, we show that G achieves its minimum in the segment from the neighborhood of $X^*(p)$ to $W^s_{\text{loc}}(X^s(p))$ and that the time at which this minimum is achieved approaches a constant as $p \to p^*$ for $p \in R$. From this, it is shown that the map from the trajectory sensitivities as the trajectory is leaving the neighborhood of $X^*(p)$ to the trajectory sensitivities when G achieves its minimum is nearly constant. Consequently, the dominant effect in the structure of G under the nonlinear vector field comes from the linear dynamics near $X^*(p)$, and this dot product structure then follows from Lemma 2.6.18.

Fix $p^* \in \partial R$. By Theorem 2.5.1, $p^* \in C$. So, there exists a critical element X^* such that $\hat{y}(p^*) \in W^s(X^*(p^*))$. We have that $X^*(p^*)$ is an equilibrium point and $W^u(X^*(p^*))$ is one dimensional. Furthermore, we assume that $\frac{\partial \hat{y}}{\partial p_j}$ is transverse to $W^s(X^*(p^*))$ at $\hat{y}(p^*)$ for all j.

For $p \in J$ let $A(p) = \frac{\partial V_p(X^*(p))}{\partial x}$. Assume that $A(p^*)$ satisfies the strong Sternberg condition for Q sufficiently large such that the Q-smoothness of $A(p^*)$ is at least two. Then by [Sel85, Theorem 7], for J sufficiently small there exists a neighborhood U of $X^*(p^*)$ in M and a conjugacy Θ between $\dot{x} = V_p(x)$ on U and $\dot{x} = A(p)x$ on a neighborhood of the origin in $\mathbb{R}^n$ such that the conjugacy is at least C^2 in both p and x. Let $T > 0$ such that $\phi(\hat{y}(p^*), T, p^*) \in U$. Shrink J if necessary so that $\phi(\hat{y}(p), T, p) \in U$ for $p \in J$. As in the proof of Lemma 2.6.18, let $z(t) = V(p)^{-1}x(t)$, and let $z(0) = V(p)^{-1}\Theta(\phi(\hat{y}(p), T, p), p)$. By the proof of Theorem 2.2.23, $W^u(X^*(p^*)) \bigcap W^s(X^s(p^*)) \neq \emptyset$. As $W^u(X^*(p^*))$ is one dimensional, this implies that one connected component of $W^u(X^*(p^*))$ lies inside $W^s(X^s(p^*))$. Let $u > 0$ small, let s be either $+1$ or -1, and let $\hat{x} = \Theta^{-1}(V(p) * (su, 0), p^*)$. Note that $\hat{x} \in W^u(X^*(p^*))$ and, by the proper choice of s, we may have that $\hat{x} \in W^s(X^s(p^*))$.

Hence, there exists $T' > 0$ such that $\phi(\hat{x}, p^*, T') \in W^s_{\text{loc}}(X^s(p^*))$. Let N be a neighborhood of $\hat{x}$ in M and shrink J if necessary such that $p \in J$ and $x \in N$ implies that $\phi(x, p, T') \in W^s_{\text{loc}}(X^s(p))$. Shrink J further if necessary so that $p \in J$ implies that $\Theta^{-1}(x(t_{\text{hit}}(p)), p) \in N$.

For each j, define

$$G^j(t,p) = \left|\left| d(\phi_t)_{\Theta^{-1}(x(t_{\text{hit}}(p)),p)} d\Theta^{-1}_{x(t_{\text{hit}}(p)),p} \frac{\partial x(t_{\text{hit}}(p))}{\partial p_j} \right|\right|_1 .$$

As the flow restricted to $W^s_{\text{loc}}(X^s(p^*))$ is a contraction, for any $p \in J$, $G^j(\cdot, p)$ must achieve its maximum on $[0, t_{\text{hit}}(p) + T']$. As $G^j(\cdot, p)$ achieves its maximum over $[0, t_{\text{hit}}(p)]$ at $t = t_{\text{hit}}(p)$, $G^j(\cdot, p)$ must achieve its maximum on $[t_{\text{hit}}(p), t_{\text{hit}}(p) + T']$. Define

$$G^{\text{lim}}(t) = ||d(\phi_t)_{\hat{x}} V_{p^*}(\hat{x})||_1 .$$

As the forward orbit of $\hat{x}$ under V_{p^*} eventually enters $W^s_{\text{loc}}(X^s(p^*))$ on which the flow of V_{p^*} is a contraction, $G^{\text{lim}}(t)$ must attain a maximum on $[0, T']$. Assume that $G^{\text{lim}}(t)$ attains a unique global maximum for $t \geq 0$. Let $\delta > 0$. Note that $G^j(t + t_{\text{hit}}(p), p) \to c^j(p)G^{\text{lim}}(t)$ as $p \to p^*$ since $d(\phi_t)_{\Theta^{-1}(x(t_{\text{hit}}(p)),p)} \to d(\phi_t)_{\hat{x}}$, $d\Theta^{-1}_{x(t_{\text{hit}}(p)),p} \to d\Theta^{-1}_{(u,0),p^*}$, and $d\Theta^{-1}_{(u,0),p^*} \frac{\partial x(t_{\text{hit}}(p))}{\partial p_j} \to c^j(p)V_{p^*}(\hat{x})$ by the Inclination Lemma. As $[0, T']$ is compact, $G^j(t + t_{\text{hit}}(p), p)$ can be brought arbitrarily close to $c^j(p)G^{\text{lim}}(t)$ on this time interval. In particular, let $\epsilon > 0$ sufficiently small such that if d is the largest local maximum of $G^{\text{lim}}(t)$ other than $G^{\text{lim}}(t_{\max})$ (if such d exists) then $G^{\text{lim}}(t_{\max}) - \epsilon > d$. Then there exists $h(t)$ such that $G^j(t + t_{\text{hit}}(p), p) - c^j(p)G^{\text{lim}}(t) = c^j(p)h(t)$ and $|h(t)| < \epsilon$ for all $t \in [0, T']$. We wish to determine $t^j_{\max}(p) = \text{argmax}_t G^j(t + t_{\text{hit}}(p), p)$. Note that $t_{\max}(p) \in [0, T']$ by the argument above. As $c^j(p)$ is constant in time, we have

$$\begin{aligned} t_{\max}^{j}(p) &= \operatorname{argmax}_{t\in[0,T']} G^{j}(t+t_{\text{hit}}(p),p) \\ &= \operatorname{argmax}_{t\in[0,T']} \frac{1}{c^{j}(p)} G^{j}(t+t_{\text{hit}}(p),p) \\ &= \operatorname{argmax}_{t\in[0,T']} G^{\lim}(t)+h(t). \end{aligned}$$

By construction, shrinking ϵ further if desired implies that $t_{\max}^{j}(p)$ can be brought arbitrarily close to $t_{\max}$ for p sufficiently close to p^*. In particular, we have shown that $\lim_{p\to p^*} t_{\max}^{j}(p) = t_{\max}$ for all j.

Define

$$\hat{G}(t,p) = \sum_j G^{j}(t,p)$$

and let $t_{\max}(p) = \operatorname{argmax}_{t\in[0,T']} \hat{G}(t+t_{\text{hit}}(p),p)$. Then $\lim_{p\to p^*} t_{\max}^{j}(p) = t_{\max}$ for all j implies that $\lim_{p\to p^*} t_{\max}(p) = t_{\max}$.

As $\hat{G}(t,p)$ converges to $\sum_j c^{j}(p)G^{\lim}(t)$ uniformly on $[0,T']$ as $p\to p^*$, and as $\lim_{p\to p^*} t_{\max}(p) = t_{\max}$, we have

$$\begin{aligned} \lim_{p\to p^*} \hat{G}(t_{\max}(p),p) &= \lim_{p\to p^*} \hat{G}(t_{\max},p) \\ &= \sum_j c^{j}(p)G^{\lim}(t_{\max}). \end{aligned}$$

Hence,

$$\hat{G}(t_{\max}(p),p) = \left(\sum_j c^{j}(p)\right) G^{\lim}(t_{\max}) + o(1).$$

In the notation of the proof of Lemma 2.6.18, $\sum_j c^{j}(p) = \hat{G}(p)$, so substituting the expression for $\hat{G}(p)$ in the proof of Lemma 2.6.18 for $\sum_j c^{j}(p)$ here gives

$$\hat{G}(t_{\max}(p),p) = \frac{G^{\lim}(t_{\max})}{|V_{p^*}(\hat{x})|} \frac{u}{|B_{1n}(p_n-p_n^*)+o(||p-p^*||_1)_1|} \left(\left|\sum_j B_{1j}+o(1)\right|+o(1)\right)+o(1).$$

Thus,

$$\begin{aligned}
G(p) &= \frac{1}{\hat{G}(t_{\max}(p), p)} \\
&= \frac{|V_{p^*}(\hat{x})|\ |B_{1n}(p_n - p_n^*) + o(||p-p^*||_1)_1|}{uG^{\lim}(t_{\max})\left(\left|\sum_j B_{1j} + o(1)\right| + o(1)\right) + |V_{p^*}(\hat{x})|\ |B_{1n}(p_n - p_n^*) + o(||p-p^*||_1)_1| o(1)} \\
&= \frac{|V_{p^*}(\hat{x})|\ |B_{1n}(p_n - p_n^*) + o(||p-p^*||_1)_1|}{uG^{\lim}(t_{\max})\left(\left|\sum_j B_{1j} + o(1)\right| + o(1)\right) + o(||p-p^*||_1)}
\end{aligned}$$

which approaches a dot product - an affine function - as $||p - p^*||_1 \to 0$, which corresponds to $p \to p^*$. □

Remark 2.6.20. The proof of Theorem 2.6.13 reveals that the result can be extended to vector fields with C^∞ event-selected discontinuities (or C^k with k sufficiently large). In particular, close examination of the above proof indicates that while smoothness was required on an arbitrarily small neighborhood of $X^*(p^*)$ in $M \times J$, a piecewise C^k would be sufficient to complete the remainder of the proof. As equilibria are disjoint from switching surfaces for vector fields with event-selected discontinuities, and as the flow is piecewise C^k (k equal to the smoothness of the vector field selection functions) for vector fields with event-selected discontinuities, the proof of Theorem 2.6.13 for vector fields with C^∞ event-selected discontinuities proceeds analogously to the above

2.6.4 Conclusions

This section developed theoretical motivation for the algorithms of Section 3.3 for vector fields with C^1 event-selected discontinuities. It was shown that the infimum over time of the inverse trajectory sensitivity is continuous and strictly positive over the set of recovery values, and converges to zero as it approaches the recovery boundary. This provides justification for algorithms which determine the recovery boundary by varying parameter values so as to minimize the infimum over time of the inverse trajectory sensitivities. Furthermore, it was shown that as parameter values approach the recovery boundary, the infimum over time of the inverse trajectory sensitivity approaches a dot product - a linear structure - motivating algorithms in Section 3.3.7.

CHAPTER III

Algorithms

This chapter develops the algorithms which were theoretically motivated by Chapter II. The numerical backbones of these algorithms are trajectory sensitivities, which are first and higher order derivatives of the system trajectory with respect to parameters. Therefore, to apply the algorithms it is important that these be computed efficiently. Section 3.1 introduces modular techniques for numerical integration of first and higher order trajectory sensitivities for hybrid dynamical systems, which include switching and limit behavior, that exploit sparse network structure to reduce computation time and storage requirements. In particular, the necessary equations for numerical integration of first, second, and third order trajectory sensitivities, including derivations of initial conditions and behavior at discrete switching events, are presented. Although first and second order trajectory sensitivity integration for hybrid systems has been developed previously [HP00, GH19], the use of modularity to improve computational efficiency is novel, as are the formulation introduced here for computation of sensitivities at discrete events and the extension of the above to third order sensitivities.

Two classes of algorithms are designed for numerical computation of critical parameter values based on the theory of Chapter II. Section 3.2 presents algorithms which vary parameter values so as to maximize the time the system trajectory spends inside a ball around the controlling invariant set (typically an equilibrium point), thereby driving them to their critical values. Section 3.3 presents algorithms which vary parameter values so as to maximize the supremum over time of the norm of the trajectory sensitivities with respect to parameters, thereby driving the parameter values to their critical values. In the latter case, the maximization is performed by minimizing inverse trajectory sensitivities for numerical reasons. Within each class of algorithms, several algorithm types are developed. In the case of one-dimensional parameter space, the unique closest critical parameter value is numerically computed. For two-dimensional parameter space, the recovery boundary is traced numerically via continuation methods. And for arbitrary dimensional parameter space, algorithms are introduced to find the nearest point on the recovery boundary to some initial set of recovery values.

3.1 Efficient Computation of Nonsmooth, Higher Order Trajectory Sensitivities

3.1.1 Introduction

The algorithms described in Sections 3.2-3.3 require trajectory sensitivities for implementation. Trajectory sensitivities are partial derivatives of the flow with respect to components of the initial condition or parameters. This section develops methods for efficient numerical integration of trajectory sensitivities that exploit sparse network structure to reduce computational cost and storage requirements. In particular, efficient integration schemes for computation of first, second, and third order trajectory sensitivities are presented for systems described by differential algebraic equations (DAEs) that can also exhibit discrete impulse and switching behavior. First [HP00] and second [GH19] order trajectory sensitivity integration schemes have been previously developed for this setting. The novel contributions here are the exploitation of network structure to improve efficiency and reduce memory usage, the integration of third order trajectory sensitivities, and a novel formulation for sensitivity computation at discrete events.

This section is organized as follows. Section 3.1.2 provides background and notation for the model of interest and trajectory sensitivities. Section 3.1.3 develops the time derivatives of the trajectory sensitivities away from discrete events. Section 3.1.4 provides the equations for numerical integration of the trajectory sensivities away from discrete events. Section 3.1.5 gives the initial conditions for trajectory sensitivities with respect to parameters. Section 3.1.6 develops the equations for trajectory sensitivities at discrete impulse and switching events. Section 3.1.7 introduces techniques to exploit sparse network topology to reduce computational cost and storage requirements. Finally, Section 3.1.8 offers some concluding remarks.

3.1.2 Background

The model of interest away from discrete events is a coupled set of C^1 differential and algebraic equations (DAEs). Let $x \in \mathbb{R}^n$ be dynamic states, $y \in \mathbb{R}^m$ be algebraic states, and write

$$\begin{aligned} \dot{x} &= f(x, y) \\ 0 &= g(x, y) \end{aligned} \tag{3.1}$$

where $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ and $g : \mathbb{R}^{n+m} \to \mathbb{R}^m$ are both C^1, f represents the differential equations, and g represents the algebraic equations. In general, we let dots denote derivatives (typically of states) with respect to time. Under reasonable assumptions [BBCK08] this system of DAEs can be treated as an equivalent system of ordinary differential equations (ODEs), so that the theory developed in Chapter II may be applied. Let ν denote the current time step of numerical integration, and let Δ denote the step size of each integration time step. Let the superscript of ν on a term denote the corresponding value of that term evaluated at $(x(\nu), y(\nu))$, and similarly for superscripts of $\nu+1$. Away from switching events, Eq. 3.1 is numerically integrated by the following trapezoidal integration scheme:

$$\begin{aligned} x^{\nu+1} &= x^{\nu} + \frac{\Delta}{2}\left(f^{\nu+1} + f^{\nu}\right) \\ 0 &= g(x^{\nu+1}, y^{\nu+1}). \end{aligned} \tag{3.2}$$

This system of nonlinear equations is typically solved numerically for the unknown variables $(x^{\nu+1}, y^{\nu+1})$ via Newton-Raphson.

Triggering of each discrete event is defined by an associated switching function $s : \mathbb{R}^{n+m} \to \mathbb{R}$ such that a switch occurs precisely when $s(x, y) = 0$. Let (x^-, y^-) denote the states at the time instant just before the switch, and let (x^+, y^+) denote the states at the time instant just after the switch. Let τ denote the intersection time of the orbit with the hypersurface $s(x, y) = 0$, which is also the time of the event, and is an implicit function of initial conditions. The dynamics of the switch are given by

$$\begin{aligned} x^{-}(x(0), y(0)) &= \phi(x(0), y(0), \tau(x(0), y(0))) \\ 0 &= s(x^{-}, y^{-}) \\ 0 &= g^{-}(x^{-}, y^{-}) \\ x^{+} &= h(x^{-}, y^{-}) \\ 0 &= g^{+}(x^{+}, y^{+}) \end{aligned} \tag{3.3}$$

where $h : \mathbb{R}^{n+m} \to \mathbb{R}^{n}$ is a C^1 impulse function, $g^{-} : \mathbb{R}^{n+m} \to \mathbb{R}^{n}$ are the set of algebraic equations at the time instant before the switch, and $g^{+} : \mathbb{R}^{n+m} \to \mathbb{R}^{n}$ are the set of algebraic equations at the time instant just after the switch. Hence, in this model formulation, switching of the vector field occurs indirectly through the switching of the algebraic equations from g^{-} to g^{+}. Let $\phi(x, y, t)$ denote the flow of Eqs. 3.1 3.3 at time t from initial condition (x, y). Let $\hat{\phi}(x, y, t)$ denote the dynamic states of $\phi(x, y, t)$, and let $\tilde{\phi}(x, y, t)$ denote the algebraic states of $\phi(x, y, t)$.

Trajectory sensitivities are partial derivatives of the flow $\phi(x, t)$ with respect to components of the initial condition (x, y). Trajectory sensitivities are therefore functions of both time and initial condition. The order of a trajectory sensitivity is the number of partial derivatives of the flow that it involves. To consider trajectory sensitivities with respect to parameters, we introduce each parameter p as a dynamic state with trivial dynamics given by $\dot{p} = 0$. Then the trajectory sensitivity with respect to parameter p is simply the trajectory sensitivity with respect to the initial condition of the dynamic state p.

We introduce the following notation for first-order trajectory sensitivities, where $i, j \in \{1, ..., n\}$ and $\alpha, \beta \in \{1, ..., m\}$.

$$\Phi_{ij}(t) := \frac{\partial \hat{\phi}_i(x, y, t)}{\partial x_j}, \qquad \Phi_{i\alpha}(t) := \frac{\partial \hat{\phi}_i(x, y, t)}{\partial y_\alpha},$$

$$\Psi_{\alpha j}(t) := \frac{\partial \tilde{\phi}_\alpha(x, y, t)}{\partial x_j}, \qquad \Psi_{\alpha\beta}(t) := \frac{\partial \tilde{\phi}_\alpha(x, y, t)}{\partial y_\beta}.$$

We often suppress the dependence on time for notational convenience, and write Φ_{ij}, $\Phi_{i\alpha}$, $\Psi_{\alpha j}$, and $\Psi_{\alpha\beta}$ to refer implicitly to the trajectory sensitivities as functions of time. More generally, let $i, j, k, l, m, n, o \in \{1, ..., n\}$ and let $\alpha, \beta, \gamma, \delta \in \{1, ..., m\}$. For higher order trajectory sensitivities of ϕ, given a sequence of subscripts we let the first subscript denote the component of ϕ, the remaining English letter subscripts denote partial derivatives with respect to components of the dynamic states x, and the remaining Greek letter subscripts denote partial derivatives with respect to components of the algebraic states y. For example,

$$\Phi_{i\alpha j\beta} := \frac{\partial^3 \hat{\phi}_i(x, y, t)}{\partial y_\alpha \partial x_j \partial y_\beta}.$$

Higher order trajectory sensitivities of $\tilde{\phi}$ are defined analogously, for example:

$$\Psi_{\alpha ij\beta} := \frac{\partial^3 \tilde{\phi}_\alpha(x, y, t)}{\partial x_i \partial x_j \partial y_\beta}.$$

Finally, the partial derivatives of the functions f, g, and h are defined analogously, with the first subscript denoting the component of the function, and the remaining subscripts denoting partial derivatives with respect to dynamic variables (English letters) and algebraic variables (Greek letters).

For clarity and brevity in presenting the equations of the following sections, we use Einstein summation notation, which holds that for each subscript index that appears on the right hand side of an equation but not on the left, that index should be summed over on the right hand side. For example, the equation $a_i = b_{ij} + c_{ijk}$ in Einstein summation notation means $a_i = \sum_j b_{ij} + \sum_{j,k} c_{ijk}$. Many of the subsequent equations will involve tensor contractions, which are a generalization of matrix multiplication. For example, the following is a tensor contraction: $a_{ijk} = \sum_l b_{ijl} c_{lk}$, which we write using the above notation as $a_{ijk} = b_{ijl} c_{lk}$. While matrix multiplication is linear, tensor contractions are multilinear (linear in each dimension, known as modes). This property is used to justify the following procedure for performing tensor contractions in MATLAB: reshaping tensors into matrices, performing matrix multiplication along the desired mode of contraction, and then reshaping the result back into a tensor. Similarly, when solving a system of multilinear equations for a tensor, multilinearity implies that the tensor can be reshaped into a matrix, the ensuing linear system solved in the standard way by MATLAB for a matrix solution, and then the result reshaped back into a tensor.

3.1.3 Time Derivatives

Trajectory sensitivities are computed in practice through numerical integration, as shown in Section 3.1.4. To perform the integration, the variational equations describing their evolution away from discrete events are required. This section is devoted to their presentation.

The first- and second-order sensitivity variational equations have been found previously [HP00, GH19], while the third-order sensitivity variational equations are novel. They are obtained by repeated differentiation of Eq. 3.1 with respect to components of the initial condition of the dynamic states, and application of the chain rule.

The first-order variational equations are:

$$\begin{aligned} \dot{\Phi}_{ij} &= f_{im}\Phi_{mj} + f_{i\alpha}\Psi_{\alpha j} \\ 0 &= g_{\alpha m}\Phi_{mj} + g_{\alpha\beta}\Psi_{\beta j}. \end{aligned} \tag{3.4}$$

The second-order variational equations are given by:

$$\begin{aligned} \dot{\Phi}_{ijk} &= f_{imn}\Phi_{nk}\Phi_{mj} + f_{im\alpha}\Psi_{\alpha k}\Phi_{mj} + f_{im}\Phi_{mjk} + f_{i\alpha m}\Phi_{mk}\Psi_{\alpha j} + f_{i\alpha\beta}\Psi_{\beta k}\Psi_{\alpha j} + f_{i\alpha}\Psi_{\alpha jk} \\ 0 &= g_{\alpha mn}\Phi_{nk}\Phi_{mj} + g_{\alpha m\beta}\Psi_{\beta k}\Phi_{mj} + g_{\alpha m}\Phi_{mjk} + g_{\alpha\beta m}\Phi_{mk}\Psi_{\beta j} + g_{\alpha\beta\gamma}\Psi_{\gamma k}\Psi_{\beta j} + g_{\alpha\beta}\Psi_{\beta jk}. \end{aligned} \tag{3.5}$$

The third-order variational equations are:

$$\begin{aligned} \dot{\Phi}_{ijkl} &= f_{imno}\Phi_{ol}\Phi_{nk}\Phi_{mj} + f_{imn\alpha}\Psi_{\alpha l}\Phi_{nk}\Phi_{mj} + f_{imn}\Phi_{nkl}\Phi_{mj} + f_{imn}\Phi_{nk}\Phi_{mjl} + f_{im\alpha n}\Phi_{nl}\Psi_{\alpha k}\Phi_{mj} \\ &+ f_{im\alpha\beta}\Psi_{\beta l}\Psi_{\alpha k}\Phi_{mj} + f_{im\alpha}\Psi_{\alpha kl}\Phi_{mj} + f_{im\alpha}\Psi_{\alpha k}\Phi_{mjl} + f_{imn}\Phi_{nl}\Phi_{mjk} + f_{im\alpha}\Psi_{\alpha l}\Phi_{mjk} \\ &+ f_{im}\Phi_{mjkl} + f_{i\alpha mn}\Phi_{nl}\Phi_{mk}\Psi_{\alpha j} + f_{i\alpha m\beta}\Psi_{\beta l}\Phi_{mk}\Psi_{\alpha j} + f_{i\alpha m}\Phi_{mkl}\Psi_{\alpha j} + f_{i\alpha m}\Phi_{mk}\Psi_{\alpha jl} \\ &+ f_{i\alpha\beta m}\Phi_{ml}\Psi_{\beta k}\Psi_{\alpha j} + f_{i\alpha\beta\gamma}\Psi_{\gamma l}\Psi_{\beta k}\Psi_{\alpha j} + f_{i\alpha\beta}\Psi_{\beta kl}\Psi_{\alpha j} + f_{i\alpha\beta}\Psi_{\beta k}\Psi_{\alpha jl} + f_{i\alpha m}\Phi_{ml}\Psi_{\alpha jk} \\ &+ f_{i\alpha\beta}\Psi_{\beta l}\Psi_{\alpha jk} + f_{i\alpha}\Psi_{\alpha jkl} \\ 0 &= g_{\alpha mno}\Phi_{ol}\Phi_{nk}\Phi_{mj} + g_{\alpha mn\beta}\Psi_{\beta l}\Phi_{nk}\Phi_{mj} + g_{\alpha mn}\Phi_{nkl}\Phi_{mj} + g_{\alpha mn}\Phi_{nk}\Phi_{mjl} + g_{\alpha m\beta n}\Phi_{nl}\Psi_{\beta k}\Phi_{mj} \\ &+ g_{\alpha m\beta\gamma}\Psi_{\gamma l}\Psi_{\beta k}\Phi_{mj} + g_{\alpha m\beta}\Psi_{\beta kl}\Phi_{mj} + g_{\alpha m\beta}\Psi_{\beta k}\Phi_{mjl} + g_{\alpha mn}\Phi_{nl}\Phi_{mjk} + g_{\alpha m\beta}\Psi_{\beta l}\Phi_{mjk} \\ &+ g_{\alpha m}\Phi_{mjkl} + g_{\alpha\beta mn}\Phi_{nl}\Phi_{mk}\Psi_{\beta j} + g_{\alpha\beta m\gamma}\Psi_{\gamma l}\Phi_{mk}\Psi_{\beta j} + g_{\alpha\beta m}\Phi_{mkl}\Psi_{\beta j} + g_{\alpha\beta m}\Phi_{mk}\Psi_{\beta jl} \\ &+ g_{\alpha\beta\gamma m}\Phi_{ml}\Psi_{\gamma k}\Psi_{\beta j} + g_{\alpha\beta\gamma\delta}\Psi_{\delta l}\Psi_{\gamma k}\Psi_{\beta j} + g_{\alpha\beta\gamma}\Psi_{\gamma kl}\Psi_{\beta j} + g_{\alpha\beta\gamma}\Psi_{\gamma k}\Psi_{\beta jl} + g_{\alpha\beta m}\Phi_{ml}\Psi_{\beta jk} \\ &+ g_{\alpha\beta\gamma}\Psi_{\gamma l}\Psi_{\beta jk} + g_{\alpha\beta}\Psi_{\beta jkl}. \end{aligned} \tag{3.6}$$

An additional set of time derivatives that will be useful for the algorithms described in Section 3.3 are given here. They are obtained by differentiating Eq. 3.1 with respect to time, and then with respect to components of the initial condition of the dynamic states.

$$\begin{aligned}
g_{\alpha\beta}\dot{y}_\beta &= -g_{\alpha m}f_m \\
g_{\alpha\beta}\dot{\Psi}_{\beta j} &= -\left(g_{\alpha mn}\Phi_{nj}f_m + g_{\alpha m\beta}\Psi_{\beta j}f_m + g_{\alpha m}f_{mn}\Phi_{nj} + g_{\alpha m}f_{m\beta}\Psi_{\beta j} + g_{\alpha\beta m}\Phi_{mj}\dot{y}_\beta + g_{\alpha\beta\gamma}\Psi_{\gamma j}\dot{y}_\beta\right) \\
\ddot{\Phi}_{ij} &= f_{imn}f_n\Phi_{mj} + f_{im\beta}\dot{y}_\beta\Phi_{mj} + f_{im}\dot{\Phi}_{mj} + f_{i\beta m}f_m\Psi_{\beta j} + f_{i\beta\gamma}\dot{y}_\gamma\Psi_{\beta j} + f_{i\beta}\dot{y}_{\beta j}, \qquad (3.7)
\end{aligned}$$

where

$$\dot{y}_\beta = \frac{\partial\tilde{\phi}_\beta(x,y,t)}{\partial t},$$

dots denote first derivatives with respect to time, and double dots denote second derivatives with respect to time. Note that under reasonable assumptions [BBCK08], $\dot{y}_\beta$ and $\dot{\Psi}_{\beta j}$ exist away from discrete events and are given by the equations above.

3.1.4 Numerical Integration

Earlier work has shown that, using a trapezoidal integration scheme to numerically integrate the underlying dynamics of Eq. 3.1, the first [HP00] and second [GH19] order trajectory sensitivities can be efficiently computed as a byproduct of this underlying integration. Here, this efficient computation is extended to third order trajectory sensitivities. Let ν denote the current time step of numerical integration, and let Δ denote the step size of each integration time step. Given the trajectory sensitivities at ν, we will compute their values at $\nu+1$. For each term in the equations, let the superscript of ν denote the corresponding value of that term evaluated at $(x(\nu), y(\nu), t(\nu))$, and similarly for superscripts of $\nu+1$. The trajectory sensitivity variational equations of Eqs. 3.4-(3.6) can be numerically integrated using trapezoidal integration as in Eq. 3.2. We formulate this below for first, second, and third order trajectory sensitivities. For the first order trajectory sensitivities, this gives:

$$\begin{bmatrix} \frac{\Delta}{2}f_{im}^{\nu+1} - I_{im} & \frac{\Delta}{2}f_{i\beta}^{\nu+1} \\ g_{\alpha m}^{\nu+1} & g_{\alpha\beta}^{\nu+1} \end{bmatrix} \begin{bmatrix} \Phi_{mj}^{\nu+1} \\ \Psi_{\beta j}^{\nu+1} \end{bmatrix} = -\begin{bmatrix} \Phi_{ij}^{\nu} \\ 0 \end{bmatrix} - \frac{\Delta}{2}\begin{bmatrix} f_{im}^{\nu}\Phi_{mj}^{\nu} + f_{i\alpha}^{\nu}\Psi_{\alpha j}^{\nu} \\ 0 \end{bmatrix}. \qquad (3.8)$$

Trapezoidal integration of the second order sensitivities is given by:

$$
\begin{aligned}
&\begin{bmatrix} \frac{\Delta}{2} f_{im}^{\nu+1} - I_{im} & \frac{\Delta}{2} f_{i\beta}^{\nu+1} \\ g_{\alpha m}^{\nu+1} & g_{\alpha\beta}^{\nu+1} \end{bmatrix} \begin{bmatrix} \Phi_{mjk}^{\nu+1} \\ \Psi_{\beta jk}^{\nu+1} \end{bmatrix} = - \begin{bmatrix} \Phi_{ijk}^{\nu} \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{\Delta}{2} \left(f_{imn}^{\nu} \Phi_{nk}^{\nu} \Phi_{mj}^{\nu} + f_{im\alpha}^{\nu} \Psi_{\alpha k}^{\nu} \Phi_{mj}^{\nu} \right) \\ g_{\alpha mn}^{\nu+1} \Phi_{nk}^{\nu+1} \Phi_{mj}^{\nu+1} + g_{\alpha m\beta}^{\nu+1} \Psi_{\beta k}^{\nu+1} \Phi_{mj}^{\nu+1} \end{bmatrix} \\
&\quad - \begin{bmatrix} \frac{\Delta}{2} \left(f_{im}^{\nu} \Phi_{mjk}^{\nu} + f_{i\alpha m}^{\nu} \Phi_{mk}^{\nu} \Psi_{\alpha j}^{\nu} + f_{i\alpha\beta}^{\nu} \Psi_{\beta k}^{\nu} \Psi_{\alpha j}^{\nu} + f_{i\alpha}^{\nu} \Psi_{\alpha jk}^{\nu} + f_{imn}^{\nu+1} \Phi_{nk}^{\nu+1} \Phi_{mj}^{\nu+1} \right) \\ g_{\alpha\beta m}^{\nu+1} \Phi_{mk}^{\nu+1} \Psi_{\beta j}^{\nu+1} + g_{\alpha\beta\gamma}^{\nu+1} \Psi_{\gamma k}^{\nu+1} \Psi_{\beta j}^{\nu+1} \end{bmatrix} \\
&\quad - \begin{bmatrix} \frac{\Delta}{2} \left(f_{im\alpha}^{\nu+1} \Psi_{\alpha k}^{\nu+1} \Phi_{mj}^{\nu+1} + f_{i\alpha m}^{\nu+1} \Phi_{mk}^{\nu+1} \Psi_{\alpha j}^{\nu+1} + f_{i\alpha\beta}^{\nu+1} \Psi_{\beta k}^{\nu+1} \Psi_{\alpha j}^{\nu+1} \right) \\ 0 \end{bmatrix}
\end{aligned} \tag{3.9}
$$

The third order sensitivity integration is given by:

$$
\begin{aligned}
&\begin{bmatrix} \frac{\Delta}{2} f_{im}^{\nu+1} - I_{im} & \frac{\Delta}{2} f_{i\beta}^{\nu+1} \\ g_{\alpha m}^{\nu+1} & g_{\alpha\beta}^{\nu+1} \end{bmatrix} \begin{bmatrix} \Phi_{mjkl}^{\nu+1} \\ \Psi_{\beta jkl}^{\nu+1} \end{bmatrix} = - \begin{bmatrix} \Phi_{ijkl}^{\nu} \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{\Delta}{2} \left(f_{imno}^{\nu} \Phi_{ol}^{\nu} \Phi_{nk}^{\nu} \Phi_{mj}^{\nu} + f_{imn\alpha}^{\nu} \Psi_{\alpha l}^{\nu} \Phi_{nk}^{\nu} \Phi_{mj}^{\nu} \right) \\ g_{\alpha mno}^{\nu+1} \Phi_{ol}^{\nu+1} \Phi_{nk}^{\nu+1} \Phi_{mj}^{\nu+1} + g_{\alpha mn\beta}^{\nu+1} \Psi_{\beta l}^{\nu+1} \Phi_{nk}^{\nu+1} \Phi_{mj}^{\nu+1} \end{bmatrix} \\
&- \begin{bmatrix} \frac{\Delta}{2} \left(f_{imn}^{\nu} \Phi_{nkl}^{\nu} \Phi_{mj}^{\nu} + f_{imn}^{\nu} \Phi_{nk}^{\nu} \Phi_{mjl}^{\nu} + f_{im\alpha n}^{\nu} \Phi_{nl}^{\nu} \Psi_{\alpha k}^{\nu} \Phi_{mj}^{\nu} + f_{im\alpha\beta}^{\nu} \Psi_{\beta l}^{\nu} \Psi_{\alpha k}^{\nu} \Phi_{mj}^{\nu} + f_{im\alpha}^{\nu} \Psi_{\alpha kl}^{\nu} \Phi_{mj}^{\nu} \right) \\ g_{\alpha mn}^{\nu+1} \Phi_{nkl}^{\nu+1} \Phi_{mj}^{\nu+1} + g_{\alpha mn}^{\nu+1} \Phi_{nk}^{\nu+1} \Phi_{mjl}^{\nu+1} + g_{\alpha m\beta n}^{\nu+1} \Phi_{nl}^{\nu+1} \Psi_{\beta k}^{\nu+1} \Phi_{mj}^{\nu+1} + g_{\alpha m\beta\gamma}^{\nu+1} \Psi_{\gamma l}^{\nu+1} \Psi_{\beta k}^{\nu+1} \Phi_{mj}^{\nu+1} \end{bmatrix} \\
&- \begin{bmatrix} \frac{\Delta}{2} \left(f_{im\alpha}^{\nu} \Psi_{\alpha k}^{\nu} \Phi_{mjl}^{\nu} + f_{imn}^{\nu} \Phi_{nl}^{\nu} \Phi_{mjk}^{\nu} + f_{im\alpha}^{\nu} \Psi_{\alpha l}^{\nu} \Phi_{mjk}^{\nu} + f_{im}^{\nu} \Phi_{mjkl}^{\nu} + f_{i\alpha mn}^{\nu} \Phi_{nl}^{\nu} \Phi_{mk}^{\nu} \Psi_{\alpha j}^{\nu} \right) \\ g_{\alpha m\beta}^{\nu+1} \Psi_{\beta kl}^{\nu+1} \Phi_{mj}^{\nu+1} + g_{\alpha m\beta}^{\nu+1} \Psi_{\beta k}^{\nu+1} \Phi_{mjl}^{\nu+1} + g_{\alpha mn}^{\nu+1} \Phi_{nl}^{\nu+1} \Phi_{mjk}^{\nu+1} + g_{\alpha m\beta}^{\nu+1} \Psi_{\beta l}^{\nu+1} \Phi_{mjk}^{\nu+1} \end{bmatrix} \\
&- \begin{bmatrix} \frac{\Delta}{2} \left(f_{i\alpha m\beta}^{\nu} \Psi_{\beta l}^{\nu} \Phi_{mk}^{\nu} \Psi_{\alpha j}^{\nu} + f_{i\alpha m}^{\nu} \Phi_{mk}^{\nu} \Psi_{\alpha jl}^{\nu} + f_{i\alpha\beta m}^{\nu} \Phi_{ml}^{\nu} \Psi_{\beta k}^{\nu} \Psi_{\alpha j}^{\nu} + f_{i\alpha\beta\gamma}^{\nu} \Psi_{\gamma l}^{\nu} \Psi_{\beta k}^{\nu} \Psi_{\alpha j}^{\nu} \right) \\ + g_{\alpha\beta mn}^{\nu+1} \Phi_{nl}^{\nu+1} \Phi_{mk}^{\nu+1} \Psi_{\beta j}^{\nu+1} + g_{\alpha\beta m\gamma}^{\nu+1} \Psi_{\gamma l}^{\nu+1} \Phi_{mk}^{\nu+1} \Psi_{\beta j}^{\nu+1} + g_{\alpha\beta m}^{\nu+1} \Phi_{mkl}^{\nu+1} \Psi_{\beta j}^{\nu+1} + g_{\alpha\beta m}^{\nu+1} \Phi_{mk}^{\nu+1} \Psi_{\beta jl}^{\nu+1} \end{bmatrix} \\
&- \begin{bmatrix} \frac{\Delta}{2} \left(f_{i\alpha m}^{\nu} \Phi_{mkl}^{\nu} \Psi_{\alpha j}^{\nu} + f_{i\alpha\beta}^{\nu} \Psi_{\beta kl}^{\nu} \Psi_{\alpha j}^{\nu} + f_{i\alpha\beta}^{\nu} \Psi_{\beta k}^{\nu} \Psi_{\alpha jl}^{\nu} + f_{i\alpha m}^{\nu} \Phi_{ml}^{\nu} \Psi_{\alpha jk}^{\nu} + f_{i\alpha\beta}^{\nu} \Psi_{\beta l}^{\nu} \Psi_{\alpha jk}^{\nu} + f_{i\alpha}^{\nu} \Psi_{\alpha jkl}^{\nu} \right) \\ g_{\alpha\beta\gamma m}^{\nu+1} \Phi_{ml}^{\nu+1} \Psi_{\gamma k}^{\nu+1} \Psi_{\beta j}^{\nu+1} + g_{\alpha\beta\gamma\delta}^{\nu+1} \Psi_{\delta l}^{\nu+1} \Psi_{\gamma k}^{\nu+1} \Psi_{\beta j}^{\nu+1} + g_{\alpha\beta\gamma}^{\nu+1} \Psi_{\gamma kl}^{\nu+1} \Psi_{\beta j}^{\nu+1} + g_{\alpha\beta\gamma}^{\nu+1} \Psi_{\gamma k}^{\nu+1} \Psi_{\beta jl}^{\nu+1} \end{bmatrix} \\
&- \begin{bmatrix} \frac{\Delta}{2} \left(f_{imno}^{\nu+1} \Phi_{ol}^{\nu+1} \Phi_{nk}^{\nu+1} \Phi_{mj}^{\nu+1} + f_{imn\alpha}^{\nu+1} \Psi_{\alpha l}^{\nu+1} \Phi_{nk}^{\nu+1} \Phi_{mj}^{\nu+1} + f_{imn}^{\nu+1} \Phi_{nkl}^{\nu+1} \Phi_{mj}^{\nu+1} + f_{imn}^{\nu+1} \Phi_{nk}^{\nu+1} \Phi_{mjl}^{\nu+1} \right) \\ g_{\alpha\beta m}^{\nu+1} \Phi_{ml}^{\nu+1} \Psi_{\beta jk}^{\nu+1} + g_{\alpha\beta\gamma}^{\nu+1} \Psi_{\gamma l}^{\nu+1} \Psi_{\beta jk}^{\nu+1} \end{bmatrix} \\
&- \begin{bmatrix} \frac{\Delta}{2} \left(f_{im\alpha n}^{\nu+1} \Phi_{nl}^{\nu+1} \Psi_{\alpha k}^{\nu+1} \Phi_{mj}^{\nu+1} + f_{im\alpha\beta}^{\nu+1} \Psi_{\beta l}^{\nu+1} \Psi_{\alpha k}^{\nu+1} \Phi_{mj}^{\nu+1} + f_{im\alpha}^{\nu+1} \Psi_{\alpha kl}^{\nu+1} \Phi_{mj}^{\nu+1} + f_{im\alpha}^{\nu+1} \Psi_{\alpha k}^{\nu+1} \Phi_{mjl}^{\nu+1} \right) \\ 0 \end{bmatrix} \\
&- \begin{bmatrix} \frac{\Delta}{2} \left(f_{imn}^{\nu+1} \Phi_{nl}^{\nu+1} \Phi_{mjk}^{\nu+1} + f_{im\alpha}^{\nu+1} \Psi_{\alpha l}^{\nu+1} \Phi_{mjk}^{\nu+1} + f_{i\alpha mn}^{\nu+1} \Phi_{nl}^{\nu+1} \Phi_{mk}^{\nu+1} \Psi_{\alpha j}^{\nu+1} + f_{i\alpha m\beta}^{\nu+1} \Psi_{\beta l}^{\nu+1} \Phi_{mk}^{\nu+1} \Psi_{\alpha j}^{\nu+1} \right) \\ 0 \end{bmatrix} \\
&- \begin{bmatrix} \frac{\Delta}{2} \left(f_{i\alpha m}^{\nu+1} \Phi_{mkl}^{\nu+1} \Psi_{\alpha j}^{\nu+1} + f_{i\alpha m}^{\nu+1} \Phi_{mk}^{\nu+1} \Psi_{\alpha jl}^{\nu+1} + f_{i\alpha\beta m}^{\nu+1} \Phi_{ml}^{\nu+1} \Psi_{\beta k}^{\nu+1} \Psi_{\alpha j}^{\nu+1} + f_{i\alpha\beta\gamma}^{\nu+1} \Psi_{\gamma l}^{\nu+1} \Psi_{\beta k}^{\nu+1} \Psi_{\alpha j}^{\nu+1} \right) \\ 0 \end{bmatrix} \\
&\quad - \begin{bmatrix} \frac{\Delta}{2} \left(f_{i\alpha\beta}^{\nu+1} \Psi_{\beta kl}^{\nu+1} \Psi_{\alpha j}^{\nu+1} + f_{i\alpha\beta}^{\nu+1} \Psi_{\beta k}^{\nu+1} \Psi_{\alpha jl}^{\nu+1} + f_{i\alpha m}^{\nu+1} \Phi_{ml}^{\nu+1} \Psi_{\alpha jk}^{\nu+1} + f_{i\alpha\beta}^{\nu+1} \Psi_{\beta l}^{\nu+1} \Psi_{\alpha jk}^{\nu+1} \right) \\ 0 \end{bmatrix} .
\end{aligned} \tag{3.10}
$$

Efficient methods to perform the integration will be discussed in Section 3.1.7.

3.1.5 Initial Conditions

To perform the trajectory sensitivity integration, it is necessary to find the proper initial conditions. As the application of trajectory sensitivities here is to study the response of a system to large disturbances, it is natural to consider systems which are initially at a stable equilibrium point (before being subjected to the disturbance). Typically such equilibrium points are functions of parameter. Therefore, we assume that $(x, y) = (x_e(p), y_e(p))$ is an equilibrium point and a function of parameter. Hence,

$$\begin{aligned} f(x_e(p), y_e(p)) &= 0 \\ g(x_e(p), y_e(p)) &= 0. \end{aligned} \tag{3.11}$$

In the equations below, all trajectory sensitivities shown are assumed to be the values at time $t = 0$. The initial conditions for the trajectory sensitivities are obtained by repeated differentiation of Eq. 3.11 with respect to parameters. For first order trajectory sensitivity initial conditions this yields:

$$\begin{bmatrix} f_{im} & f_{i\beta} \\ g_{\alpha m} & g_{\alpha\beta} \end{bmatrix} \begin{bmatrix} \Phi_{mj} \\ \Psi_{\beta j} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.12}$$

which can be rewritten as:

$$\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} \begin{bmatrix} \tilde{\Phi}_{mj} \\ \Psi_{\beta j} \end{bmatrix} = - \begin{bmatrix} f_{param} \\ g_{param} \end{bmatrix} \tag{3.13}$$

where f_x and g_x are derivatives of f and g with respect to the dynamic states with nontrivial dynamics (the dynamic states which are not parameters), f_{param} and g_{param} are derivatives of f and g with respect to the parameters p, and $\tilde{\Phi}_{mj}$ consists only of the trajectory sensitivities of the dynamic states which are not parameters.

The second order trajectory sensitivity initial conditions are given by:

$$\begin{bmatrix} f_{im} & f_{i\beta} \\ g_{\alpha m} & g_{\alpha\beta} \end{bmatrix} \begin{bmatrix} \Phi_{mjk} \\ \Psi_{\beta jk} \end{bmatrix} = - \begin{bmatrix} f_{imn}\Phi_{nk}\Phi_{mj} + f_{im\alpha}\Psi_{\alpha k}\Phi_{mj} + f_{i\alpha m}\Phi_{mk}\Psi_{\alpha j} + f_{i\alpha\beta}\Psi_{\beta k}\Psi_{\alpha j} \\ g_{\alpha mn}\Phi_{nk}\Phi_{mj} + g_{\alpha m\beta}\Psi_{\beta k}\Phi_{mj} + g_{\alpha\beta m}\Phi_{mk}\Psi_{\beta j} + g_{\alpha\beta\gamma}\Psi_{\gamma k}\Psi_{\beta j} \end{bmatrix} \tag{3.14}$$

The third order trajectory sensitivity initial conditions are given by:

$$\begin{bmatrix} f_{im} & f_{i\beta} \\ g_{\alpha m} & g_{\alpha\beta} \end{bmatrix} \begin{bmatrix} \Phi_{mjkl} \\ \Psi_{\beta jkl} \end{bmatrix} = - \begin{bmatrix} f_{imno}\Phi_{ol}\Phi_{nk}\Phi_{mj} + f_{imn\alpha}\Psi_{\alpha l}\Phi_{nk}\Phi_{mj} + f_{imn}\Phi_{nkl}\Phi_{mj} \\ g_{\alpha mno}\Phi_{ol}\Phi_{nk}\Phi_{mj} + g_{\alpha mn\beta}\Psi_{\beta l}\Phi_{nk}\Phi_{mj} + g_{\alpha mn}\Phi_{nkl}\Phi_{mj} \end{bmatrix}$$
$$- \begin{bmatrix} f_{imn}\Phi_{nk}\Phi_{mjl} + f_{im\alpha n}\Phi_{nl}\Psi_{\alpha k}\Phi_{mj} + f_{im\alpha\beta}\Psi_{\beta l}\Psi_{\alpha k}\Phi_{mj} + f_{im\alpha}\Psi_{\alpha kl}\Phi_{mj} + f_{im\alpha}\Psi_{\alpha k}\Phi_{mjl} \\ g_{\alpha mn}\Phi_{nk}\Phi_{mjl} + g_{\alpha m\beta n}\Phi_{nl}\Psi_{\beta k}\Phi_{mj} + g_{\alpha m\beta\gamma}\Psi_{\gamma l}\Psi_{\beta k}\Phi_{mj} + g_{\alpha m\beta}\Psi_{\beta kl}\Phi_{mj} + g_{\alpha m\beta}\Psi_{\beta k}\Phi_{mjl} \end{bmatrix}$$
$$- \begin{bmatrix} f_{imn}\Phi_{nl}\Phi_{mjk} + f_{im\alpha}\Psi_{\alpha l}\Phi_{mjk} + f_{i\alpha mn}\Phi_{nl}\Phi_{mk}\Psi_{\alpha j} + f_{i\alpha m\beta}\Psi_{\beta l}\Phi_{mk}\Psi_{\alpha j} + f_{i\alpha m}\Phi_{mkl}\Psi_{\alpha j} \\ g_{\alpha mn}\Phi_{nl}\Phi_{mjk} + g_{\alpha m\beta}\Psi_{\beta l}\Phi_{mjk} + g_{\alpha\beta mn}\Phi_{nl}\Phi_{mk}\Psi_{\beta j} + g_{\alpha\beta m\gamma}\Psi_{\gamma l}\Phi_{mk}\Psi_{\beta j} + g_{\alpha\beta m}\Phi_{mkl}\Psi_{\beta j} \end{bmatrix}$$
$$- \begin{bmatrix} f_{i\alpha m}\Phi_{mk}\Psi_{\alpha jl} + f_{i\alpha\beta m}\Phi_{ml}\Psi_{\beta k}\Psi_{\alpha j} + f_{i\alpha\beta\gamma}\Psi_{\gamma l}\Psi_{\beta k}\Psi_{\alpha j} + f_{i\alpha\beta}\Psi_{\beta kl}\Psi_{\alpha j} \\ g_{\alpha\beta m}\Phi_{mk}\Psi_{\beta jl} + g_{\alpha\beta\gamma m}\Phi_{ml}\Psi_{\gamma k}\Psi_{\beta j} + g_{\alpha\beta\gamma\delta}\Psi_{\delta l}\Psi_{\gamma k}\Psi_{\beta j} + g_{\alpha\beta\gamma}\Psi_{\gamma kl}\Psi_{\beta j} \end{bmatrix}$$
$$- \begin{bmatrix} f_{i\alpha\beta}\Psi_{\beta k}\Psi_{\alpha jl} + f_{i\alpha m}\Phi_{ml}\Psi_{\alpha jk} + f_{i\alpha\beta}\Psi_{\beta l}\Psi_{\alpha jk} \\ g_{\alpha\beta\gamma}\Psi_{\gamma k}\Psi_{\beta jl} + g_{\alpha\beta m}\Phi_{ml}\Psi_{\beta jk} + g_{\alpha\beta\gamma}\Psi_{\gamma l}\Psi_{\beta jk} \end{bmatrix} \quad (3.15)$$

3.1.6 Discrete Events

This section presents a novel formulation of trajectory sensitivity computations at discrete impulse and switching events. The dynamics at the discrete event are given by Eq. 3.3. Appendix D shows the derivation of the computation of trajectory sensitivities at the discrete event. The final results are summarized below. Let τ_j, τ_{jk}, and τ_{jkl} denote derivatives of the intersection time τ with respect to components of the initial condition of the dynamic states. Let terms with minus superscripts denote the value of that term just before the discrete event, and let terms with plus superscripts denote the value of that term just after the discrete event.

The equations for computation of first order trajectory sensitivities after a discrete event are:

$$\begin{bmatrix} s_{im}f_m^- & s_{i\beta} \\ g_{\alpha m}^- f_m^- & g_{\alpha\beta}^- \end{bmatrix} \begin{bmatrix} \tau_j \\ \Psi_{\beta j}^- \end{bmatrix} = - \begin{bmatrix} s_{im}\Phi_{mj} \\ g_{\alpha m}^-\Phi_{mj} \end{bmatrix} \quad (3.16)$$

$$\Phi_{ij}^- = \Phi_{ij} + f_i^-\tau_j \quad (3.17)$$

$$\begin{bmatrix} -I_{im} & 0 \\ g_{\alpha m}^+ & g_{\alpha\beta}^+ \end{bmatrix} \begin{bmatrix} \Phi_{mj}^+ \\ \Psi_{\beta j}^+ \end{bmatrix} = - \begin{bmatrix} h_{im}\Phi_{mj}^- + h_{i\alpha}\Psi_{\alpha j}^- \\ 0 \end{bmatrix}. \quad (3.18)$$

The second order trajectory sensitivity equations at a discrete event are given by:

$$\begin{bmatrix} s_{im}f_m^- & s_{i\beta} \\ g_{\alpha m}^- f_m^- & g_{\alpha\beta}^- \end{bmatrix} \begin{bmatrix} \tau_{jk} \\ \Psi_{\beta jk}^- \end{bmatrix} = -\begin{bmatrix} s_{imn}\Phi_{nk}^-\Phi_{mj}^- + s_{im\alpha}\Psi_{\alpha k}^-\Phi_{mj}^- + s_{i\alpha m}\Phi_{mk}^-\Psi_{\alpha j}^- + s_{i\alpha\beta}\Psi_{\beta k}^-\Psi_{\alpha j}^- \\ g_{\alpha mn}^-\Phi_{nk}^-\Phi_{mj}^- + g_{\alpha m\beta}^-\Psi_{\beta k}^-\Phi_{mj}^- + g_{\alpha\beta m}^-\Phi_{mk}^-\Psi_{\beta j}^- + g_{\alpha\beta\gamma}^-\Psi_{\gamma k}^-\Psi_{\beta j}^- \end{bmatrix}$$
$$- \begin{bmatrix} s_{im}\left(\Phi_{mjk} + f_{mn}^-\Phi_{nk}\tau_j + f_{m\alpha}^-\Psi_{\alpha k}\tau_j\right) \\ g_{\alpha m}^-\left(\Phi_{mjk} + f_{mn}^-\Phi_{nk}\tau_j + f_{m\beta}^-\Psi_{\beta k}\tau_j\right) \end{bmatrix} \quad (3.19)$$

$$\Phi_{ijk}^- = \Phi_{ijk} + f_{im}^-\Phi_{mk}\tau_j + f_{i\alpha}^-\Psi_{\alpha k}\tau_j + f_i^-\tau_{jk} \quad (3.20)$$

$$\begin{bmatrix} -I_{im} & 0 \\ g_{\alpha m}^+ & g_{\alpha\beta}^+ \end{bmatrix} \begin{bmatrix} \Phi_{mjk}^+ \\ \Psi_{\beta jk}^+ \end{bmatrix} = -\begin{bmatrix} h_{imn}\Phi_{nk}^-\Phi_{mj}^- + h_{im\alpha}\Psi_{\alpha k}^-\Phi_{mj}^- + h_{im}\Phi_{mjk}^- + h_{i\alpha m}\Phi_{mk}^-\Psi_{\alpha j}^- \\ g_{\alpha mn}^+\Phi_{nk}^+\Phi_{mj}^+ + g_{\alpha m\beta}^+\Psi_{\beta k}^+\Phi_{mj}^+ + g_{\alpha\beta m}^+\Phi_{mk}^+\Psi_{\beta j}^+ + g_{\alpha\beta\gamma}^+\Psi_{\gamma k}^+\Psi_{\beta j}^+ \end{bmatrix}$$
$$- \begin{bmatrix} h_{i\alpha\beta}\Psi_{\beta k}^-\Psi_{\alpha j}^- + h_{i\alpha}\Psi_{\alpha jk}^- \\ 0 \end{bmatrix}. \quad (3.21)$$

The third order trajectory sensitivity equations at a discrete event are:

$$\begin{bmatrix} s_{im}f_m^- & s_{i\beta} \\ g_{\alpha m}^- f_m^- & g_{\alpha\beta}^- \end{bmatrix} \begin{bmatrix} \tau_{jkl} \\ \Psi_{\beta jkl}^- \end{bmatrix} = -\begin{bmatrix} s_{imno}\Phi_{ol}^-\Phi_{nk}^-\Phi_{mj}^- + s_{imn\alpha}\Psi_{\alpha l}^-\Phi_{nk}^-\Phi_{mj}^- + s_{imn}\Phi_{nkl}^-\Phi_{mj}^- \\ g_{\alpha mno}^-\Phi_{ol}^-\Phi_{nk}^-\Phi_{mj}^- + g_{\alpha mn\beta}^-\Psi_{\beta l}^-\Phi_{nk}^-\Phi_{mj}^- + g_{\alpha mn}^-\Phi_{nkl}^-\Phi_{mj}^- \end{bmatrix}$$
$$- \begin{bmatrix} s_{imn}\Phi_{nk}^-\Phi_{mjl}^- + s_{im\alpha n}\Phi_{nl}^-\Psi_{\alpha k}^-\Phi_{mj}^- + s_{im\alpha\beta}\Psi_{\beta l}^-\Psi_{\alpha k}^-\Phi_{mj}^- + s_{im\alpha}\Psi_{\alpha kl}^-\Phi_{mj}^- + s_{im\alpha}\Psi_{\alpha k}^-\Phi_{mjl}^- \\ g_{\alpha mn}^-\Phi_{nk}^-\Phi_{mjl}^- + g_{\alpha m\beta n}^-\Phi_{nl}^-\Psi_{\beta k}^-\Phi_{mj}^- + g_{\alpha m\beta\gamma}^-\Psi_{\gamma l}^-\Psi_{\beta k}^-\Phi_{mj}^- + g_{\alpha m\beta}^-\Psi_{\beta kl}^-\Phi_{mj}^- + g_{\alpha m\beta}^-\Psi_{\beta k}^-\Phi_{mjl}^- \end{bmatrix}$$
$$- \begin{bmatrix} s_{imn}\Phi_{nl}^-\Phi_{mjk}^- + s_{im\alpha}\Psi_{\alpha l}^-\Phi_{mjk}^- + s_{i\alpha mn}\Phi_{nl}^-\Phi_{mk}^-\Psi_{\alpha j}^- + s_{i\alpha m\beta}\Psi_{\beta l}^-\Phi_{mk}^-\Psi_{\alpha j}^- + s_{i\alpha m}\Phi_{mkl}^-\Psi_{\alpha j}^- \\ g_{\alpha mn}^-\Phi_{nl}^-\Phi_{mjk}^- + g_{\alpha m\beta}^-\Psi_{\beta l}^-\Phi_{mjk}^- + g_{\alpha\beta mn}^-\Phi_{nl}^-\Phi_{mk}^-\Psi_{\beta j}^- + g_{\alpha\beta m\gamma}^-\Psi_{\gamma l}^-\Phi_{mk}^-\Psi_{\beta j}^- + g_{\alpha\beta m}^-\Phi_{mkl}^-\Psi_{\beta j}^- \end{bmatrix}$$
$$- \begin{bmatrix} s_{i\alpha m}\Phi_{mk}^-\Psi_{\alpha jl}^- + s_{i\alpha\beta m}\Phi_{ml}^-\Psi_{\beta k}^-\Psi_{\alpha j}^- + s_{i\alpha\beta\gamma}\Psi_{\gamma l}^-\Psi_{\beta k}^-\Psi_{\alpha j}^- + s_{i\alpha\beta}\Psi_{\beta kl}^-\Psi_{\alpha j}^- + s_{i\alpha\beta}\Psi_{\beta k}^-\Psi_{\alpha jl}^- \\ g_{\alpha\beta m}^-\Phi_{mk}^-\Psi_{\beta jl}^- + g_{\alpha\beta\gamma m}^-\Phi_{ml}^-\Psi_{\gamma k}^-\Psi_{\beta j}^- + g_{\alpha\beta\gamma\delta}^-\Psi_{\delta l}^-\Psi_{\gamma k}^-\Psi_{\beta j}^- + g_{\alpha\beta\gamma}^-\Psi_{\gamma kl}^-\Psi_{\beta j}^- + g_{\alpha\beta\gamma}^-\Psi_{\gamma k}^-\Psi_{\beta jl}^- \end{bmatrix}$$
$$- \begin{bmatrix} s_{i\alpha m}\Phi_{ml}^-\Psi_{\alpha jk}^- + s_{i\alpha\beta}\Psi_{\beta l}^-\Psi_{\alpha jk}^- + s_{im}\left(\Phi_{mjkl} + f_{mno}^-\Phi_{ol}\Phi_{nk}\tau_j + f_{mn\alpha}^-\Psi_{\alpha l}\Phi_{nk}\tau_j\right) \\ g_{\alpha\beta m}^-\Phi_{ml}^-\Psi_{\beta jk}^- + g_{\alpha\beta\gamma}^-\Psi_{\gamma l}^-\Psi_{\beta jk}^- + g_{\alpha m}^-\left(\Phi_{mjkl} + f_{mno}^-\Phi_{ol}\Phi_{nk}\tau_j + f_{mn\beta}^-\Psi_{\beta l}\Phi_{nk}\tau_j\right) \end{bmatrix}$$
$$- \begin{bmatrix} s_{im}\left(f_{mn}^-\Phi_{nkl}\tau_j + f_{mn}^-\Phi_{nk}\tau_{jl} + f_{m\alpha n}^-\Phi_{nl}\Psi_{\alpha k}\tau_j + f_{m\alpha\beta}^-\Psi_{\beta l}\Psi_{\alpha k}\tau_j + f_{m\alpha}^-\Psi_{\alpha kl}\tau_j\right) \\ g_{\alpha m}^-\left(f_{mn}^-\Phi_{nkl}\tau_j + f_{mn}^-\Phi_{nk}\tau_{jl} + f_{m\beta n}^-\Phi_{nl}\Psi_{\beta k}\tau_j + f_{m\beta\gamma}^-\Psi_{\gamma l}\Psi_{\beta k}\tau_j + f_{m\beta}^-\Psi_{\beta kl}\tau_j\right) \end{bmatrix}$$
$$- \begin{bmatrix} s_{im}\left(f_{m\alpha}^-\Psi_{\alpha k}\tau_{jl} + f_{mn}^-\Phi_{nl}\tau_{jk} + f_{m\alpha}^-\Psi_{\alpha l}\tau_{jk}\right) \\ g_{\alpha m}^-\left(f_{m\beta}^-\Psi_{\beta k}\tau_{jl} + f_{mn}^-\Phi_{nl}\tau_{jk} + f_{m\beta}^-\Psi_{\beta l}\tau_{jk}\right) \end{bmatrix} \quad (3.22)$$

$$\begin{aligned} \Phi_{ijkl}^- = \Phi_{ijkl} &+ f_{imn}^-\Phi_{nl}\Phi_{mk}\tau_j + f_{im\alpha}^-\Psi_{\alpha l}\Phi_{mk}\tau_j + f_{im}^-\Phi_{mkl}\tau_j + f_{im}^-\Phi_{mk}\tau_{jl} \\ &+ f_{i\alpha m}^-\Phi_{ml}\Psi_{\alpha k}\tau_j + f_{i\alpha\beta}^-\Psi_{\beta l}\Psi_{\alpha k}\tau_j + f_{i\alpha}^-\Psi_{\alpha kl}\tau_j + f_{i\alpha}^-\Psi_{\alpha k}\tau_{jl} \\ &+ f_{im}^-\Phi_{ml}\tau_{jk} + f_{i\alpha}^-\Psi_{\alpha l}\tau_{jk} + f_i^-\tau_{jkl} \end{aligned} \quad (3.23)$$

$$
\begin{aligned}
\begin{bmatrix} -I_{im} & 0 \\ g^{+}_{\alpha m} & g^{+}_{\alpha\beta} \end{bmatrix} \begin{bmatrix} \Phi^{+}_{mjkl} \\ \Psi^{+}_{\beta jkl} \end{bmatrix} = &- \begin{bmatrix} h_{imno}\Phi^{-}_{ol}\Phi^{-}_{nk}\Phi^{-}_{mj} + h_{imn\alpha}\Psi^{-}_{\alpha l}\Phi^{-}_{nk}\Phi^{-}_{mj} + h_{imn}\Phi^{-}_{nkl}\Phi^{-}_{mj} \\ g^{+}_{\alpha mno}\Phi^{+}_{ol}\Phi^{+}_{nk}\Phi^{+}_{mj} + g^{+}_{\alpha mn\beta}\Psi^{+}_{\beta l}\Phi^{+}_{nk}\Phi^{+}_{mj} + g^{+}_{\alpha mn}\Phi^{+}_{nkl}\Phi^{+}_{mj} \end{bmatrix} \\
&- \begin{bmatrix} h_{imn}\Phi^{-}_{nk}\Phi^{-}_{mjl} + h_{im\alpha n}\Phi^{-}_{nl}\Psi^{-}_{\alpha k}\Phi^{-}_{mj} + h_{im\alpha\beta}\Psi^{-}_{\beta l}\Psi^{-}_{\alpha k}\Phi^{-}_{mj} + h_{im\alpha}\Psi^{-}_{\alpha kl}\Phi^{-}_{mj} + h_{im\alpha}\Psi^{-}_{\alpha k}\Phi^{-}_{mjl} \\ g^{+}_{\alpha mn}\Phi^{+}_{nk}\Phi^{+}_{mjl} + g^{+}_{\alpha m\beta n}\Phi^{+}_{nl}\Psi^{+}_{\beta k}\Phi^{+}_{mj} + g^{+}_{\alpha m\beta\gamma}\Psi^{+}_{\gamma l}\Psi^{+}_{\beta k}\Phi^{+}_{mj} + g^{+}_{\alpha m\beta}\Psi^{+}_{\beta kl}\Phi^{+}_{mj} + g^{+}_{\alpha m\beta}\Psi^{+}_{\beta k}\Phi^{+}_{mjl} \end{bmatrix} \\
&- \begin{bmatrix} h_{imn}\Phi^{-}_{nl}\Phi^{-}_{mjk} + h_{im\alpha}\Psi^{-}_{\alpha l}\Phi^{-}_{mjk} + h_{im}\Phi^{-}_{mjkl} + h_{i\alpha mn}\Phi^{-}_{nl}\Phi^{-}_{mk}\Psi^{-}_{\alpha j} + h_{i\alpha m\beta}\Psi^{-}_{\beta l}\Phi^{-}_{mk}\Psi^{-}_{\alpha j} \\ g^{+}_{\alpha mn}\Phi^{+}_{nl}\Phi^{+}_{mjk} + g^{+}_{\alpha m\beta}\Psi^{+}_{\beta l}\Phi^{+}_{mjk} + g^{+}_{\alpha\beta mn}\Phi^{+}_{nl}\Phi^{+}_{mk}\Psi^{+}_{\beta j} + g^{+}_{\alpha\beta m\gamma}\Psi^{+}_{\gamma l}\Phi^{+}_{mk}\Psi^{+}_{\beta j} + g^{+}_{\alpha\beta m}\Phi^{+}_{mkl}\Psi^{+}_{\beta j} \end{bmatrix} \\
&- \begin{bmatrix} h_{i\alpha m}\Phi^{-}_{mkl}\Psi^{-}_{\alpha j} + h_{i\alpha m}\Phi^{-}_{mk}\Psi^{-}_{\alpha jl} + h_{i\alpha\beta m}\Phi^{-}_{ml}\Psi^{-}_{\beta k}\Psi^{-}_{\alpha j} + h_{i\alpha\beta\gamma}\Psi^{-}_{\gamma l}\Psi^{-}_{\beta k}\Psi^{-}_{\alpha j} + h_{i\alpha\beta}\Psi^{-}_{\beta kl}\Psi^{-}_{\alpha j} \\ g^{+}_{\alpha\beta m}\Phi^{+}_{mk}\Psi^{+}_{\beta jl} + g^{+}_{\alpha\beta\gamma m}\Phi^{+}_{ml}\Psi^{+}_{\gamma k}\Psi^{+}_{\beta j} + g^{+}_{\alpha\beta\gamma\delta}\Psi^{+}_{\delta l}\Psi^{+}_{\gamma k}\Psi^{+}_{\beta j} + g^{+}_{\alpha\beta\gamma}\Psi^{+}_{\gamma kl}\Psi^{+}_{\beta j} + g^{+}_{\alpha\beta\gamma}\Psi^{+}_{\gamma k}\Psi^{+}_{\beta jl} \end{bmatrix} \\
&- \begin{bmatrix} h_{i\alpha\beta}\Psi^{-}_{\beta k}\Psi^{-}_{\alpha jl} + h_{i\alpha m}\Phi^{-}_{ml}\Psi^{-}_{\alpha jk} + h_{i\alpha\beta}\Psi^{-}_{\beta l}\Psi^{-}_{\alpha jk} + h_{i\alpha}\Psi^{-}_{\alpha jkl} \\ g^{+}_{\alpha\beta m}\Phi^{+}_{ml}\Psi^{+}_{\beta jk} + g^{+}_{\alpha\beta\gamma}\Psi^{+}_{\gamma l}\Psi^{+}_{\beta jk} \end{bmatrix}.
\end{aligned}
\tag{3.24}
$$

3.1.7 Efficient Computation and Storage

Improving computational efficiency and reducing memory requirements are important for facilitating trajectory sensitivity integration in practice. The key property we wish to exploit is that for many practical applications that consider dynamics over a sparse network, including power systems, the model given by Eq. 3.1 admits a decomposition as follows into a modular structure. Suppose there are $\mathbb{N} > 0$ components that are connected through the network. For each $i \in \mathbb{N}$, let x_i and y_i refer to the dynamic states and algebraic states associated with that component, respectively. Then assume that Eq. 3.1 can be written in the form:

$$
\begin{aligned}
f(x,y) &= \begin{bmatrix} f_1(x_1, y_1) \\ f_2(x_2, y_2) \\ \dots \\ f_{\mathbb{N}}(x_{\mathbb{N}}, y_{\mathbb{N}}) \end{bmatrix} \\
g(x,y) &= \begin{bmatrix} g_1(x_1, y_1) \\ g_2(x_2, y_2) \\ \dots \\ g_{\mathbb{N}}(x_{\mathbb{N}}, y_{\mathbb{N}}) \\ Ly \end{bmatrix}
\end{aligned}
\tag{3.25}
$$

where L is a constant matrix. This structure implies that while the individual components may have complicated, nonlinear and nonsmooth dynamics, their coupling through the network (via the algebraic variables) is linear. For example, this is true in electrical networks such as power systems in which the network coupling equations are given by Kirchoff's Laws, which are linear in currents and voltages.

Assuming the structure of Eq. 3.25, consider first the integration of trajectory sensitivities away from discrete events. Inspection of the right hand sides of Eqs. 3.8-3.10, used for numerical integration of the sensitivities, reveals that every term either consists of the contraction of derivatives of f against sensitivities, or of the contraction of at least second order derivatives of g against sensitivities. However, taking derivatives of f preserves its modular structure, while taking at least second (partial) derivatives of g both preserves the modular part of its structure and cancels out the network part since the second derivative of a linear function is zero. In particular, this implies that the computations required to construct the right hand sides of Eqs. 3.8-3.10 can be carried out in a modular fashion. Hence, both runtimes and storage requirements for these computations grow linearly with the number of network components $\mathbb{N}$ (assuming a bound on the dimension of the network components, which is typical in practice). Furthermore, as a result of this modularity, these computations are easily parallelizable, and could take advantage of parallel computing resources.

In fact, the discussion above also applies to the computation of trajectory sensitivities at discrete events. Each term in the right hand side of Eqs. 3.16-3.24 consists either of the contraction of f or its derivatives against sensitivities, the tensor product of f or its derivatives with derivatives of the intersection time, or of the contraction of at least second order derivatives of g against sensitivities. In all of these cases, the computation can be carried out in a modular fashion. As in the case of integration away from discrete events, this significantly decreases computation time but, perhaps more importantly, it dramatically reduces the memory usage required for the computations. Similarly, these computations could take advantage of parallel computing as well.

Once the computations of the right hand side of Eqs. 3.8-3.10 are complete, it is necessary in each case to solve a system of linear equations. Note that all three linear systems have the same matrix on the left hand side. It was observed [HP00] that this matrix is merely the Jacobian of the system, and is obtained as a byproduct of the underlying trapezoidal integration. In particular, an LU decomposition of this matrix is performed during the underlying integration of the system dynamics so that, as pointed out by [HP00], it is only necessary to solve triangular linear systems for the trajectory sensitivity integration steps, which can be solved very efficiently. Furthermore, at discrete events, the two pairs of linear systems for each order of trajectory sensitivity in Eqs. 3.16-3.24 have the same matrices on the left hand side. As was observed by [GH19], this implies that once the LU decomposition is performed for the first order sensitivity computation, this can be used to solve triangular linear systems for the second (and, we add, third) order sensitivity integration as well.

3.1.8 Conclusion

This section developed trapezoidal integration schemes for first, second, and third order trajectory sensitivities for differential algebraic equations with discrete impulse and switching events. These trajectory sensitivities will be crucial in implementing the algorithms described in Sections 3.2-3.3. The novel contributions presented here are the exploitation of sparse network structure to develop modular and parallel algorithms to drastically reduce computational cost and storage requirements, an integration scheme for third order trajectory sensitivities, and a novel formulation of trajectory sensitivity computation at discrete events.

3.2 Maximizing Time in Ball Around Controlling Equilibrium Point Algorithms

3.2.1 Introduction

Algorithms are developed based on the theoretical motivation of Section 2.5 for determining the recovery boundary via computation of critical parameter values. The key idea is to vary parameter values so as to maximize the time the system trajectory spending inside a ball around the controlling UEP, thereby driving parameter values to their critical values. For the purposes of this discussion, it is assumed that the controlling UEP has already been found by some other method, such as those of [TVK96, Chi11]. Algorithms are developed to find the unique closest critical parameter value in one dimensional parameter space, trace the recovery boundary in two dimensional parameter space, and find the nearest point on the recovery boundary to some initial set of recovery values in higher dimensional parameter space. The algorithms are illustrated on a test case of a network of three synchronous machines.

3.2.2 One Dimensional Parameter Space

The first step of the algorithm is to find the parameter value corresponding to the system spending a fixed target time inside the ball. Let $\hat{x}$ denote the controlling UEP, r the radius of the ball around it, t_{ball} the target time inside the ball, and $\phi(t,p)$ the system state at time t for parameter value p. Furthermore, let t_1, t_2 denote the initial and final crossings of the boundary of the ball by the system trajectory, and let x_1, x_2 denote the system dynamic states at times t_1, t_2, respectively. Similarly, let y_1, y_2 denote the system algebraic states at times t_1, t_2, respectively. Letting $z = [x_1\ y_1\ t_1\ x_2\ y_2\ t_2\ p]^\intercal$, these conditions can be described algebraically by,

$$F(z) = \begin{bmatrix} \phi(t_1,p) - x_1 \\ g(x_1,y_1) \\ ||x_1 - \hat{x}||_2^2 - r^2 \\ \phi(t_2,p) - x_2 \\ g(x_2,y_2) \\ ||x_2 - \hat{x}||_2^2 - r^2 \\ t_2 - t_1 - t_{ball} \end{bmatrix}. \tag{3.26}$$

The desired conditions are obtained by solving,

$$F(z) = 0. \tag{3.27}$$

A solution implies that $||x_1 - \hat{x}||_2 = ||x_2 - \hat{x}||_2 = r$, so x_1, x_2 are on the boundary of the ball around the controlling UEP. Also, $\phi(t_1,p) = x_1$ and likewise for t_2 ensures that t_1, t_2 are the initial and final times at which the trajectory intersects the boundary of the ball. Furthermore, $t_{ball} = t_2 - t_1$ enforces the time constraint between the first and final intersections. Finally, $g(x_1,y_1) = g(x_2,y_2) = 0$ ensures that y_1, y_2 are the algebraic states corresponding to the dynamic states x_1, x_2.

This system of equations can be solved using Newton-Raphson, which requires the corresponding Jacobian matrix,

$$J(z) = \frac{\partial F(z)}{\partial z} =$$

$$\begin{bmatrix} -I & 0 & f|_1 & 0 & 0 & 0 & \frac{\partial \phi(t_1,p)}{\partial p} \\ \frac{\partial g}{\partial x}|_1 & \frac{\partial g}{\partial y}|_1 & 0 & 0 & 0 & 0 & 0 \\ (x_1 - \hat{x})^\intercal & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & f|_2 & \frac{\partial \phi(t_2,p)}{\partial p} \\ 0 & 0 & 0 & \frac{\partial g}{\partial x}|_2 & \frac{\partial g}{\partial y}|_2 & 0 & 0 \\ 0 & 0 & 0 & (x_2 - \hat{x})^\intercal & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{bmatrix} \tag{3.28}$$

where $f|_1 := f(x_1, y_1)$ and $f|_2 := f(x_2, y_2)$. To form this Jacobian matrix, it is necessary to compute the partial derivatives $\frac{\partial \phi(t_1,p)}{\partial p}$ and $\frac{\partial \phi(t_2,p)}{\partial p}$. They can be obtained from the numerical integration of the trajectory sensitivities, as described in Section 3.1.

The Newton-Raphson update has the usual form,

$$z^{\nu+1} = z^{\nu} - J(z^{\nu})^{-1} F(z^{\nu}) \tag{3.29}$$

where ν is the iteration number. For purposes of computational efficiency, this system is usually solved using either LU or QR decomposition of $J(z^{\nu})$. The result of this Newton-Raphson algorithm is a parameter value p which drives the system to spend a target time t_{ball} inside the ball around the controlling UEP.

The next step is to vary a parameter so as to maximize the time spent by the trajectory in the ball around the controlling UEP. This is accomplished by now treating t_{ball} as a variable in (3.26), giving $z_{cont} = [x_1\ y_1\ t_1\ x_2\ y_2\ t_2\ p\ t_{ball}]^\intercal$. Then z_{cont}, together with the constraints $F(z_{cont}) = 0$, locally defines a curve since there is one more variable in z_{cont} than the number of equations in $F(z_{cont}) = 0$. A continuation method, similar in concept to those in [DH06], can be employed to track the solution as t_{ball} is increased. In general, $F(z_{cont}) = 0$ may not be a smooth curve, but empirical observations indicate that it typically is smooth in practice. Further study is required to determine the reasons for this smoothness.

The key idea of this continuation method is to iteratively trace a sequence of points along a curve defined by a set of algebraic equations, as illustrated in Figure 3.1. At each step a prediction is made by moving along the tangent to the curve. Then the predicted point is corrected by projecting back onto the curve. This process repeats until the curve is complete. The unit tangent vector η to the curve can be found by letting

$$J_{cont}(z_{cont}) = \begin{bmatrix} J(z_{cont}) & \frac{\partial F}{\partial t_{ball}}(z_{cont}) \end{bmatrix}. \tag{3.30}$$

Then η satisfies

$$J_{cont}(z_{cont})\eta = 0 \tag{3.31}$$

$$||\eta||_2^2 = 1. \tag{3.32}$$

The solution is given by taking a QR decomposition of $J_{cont}(z_{cont})^{\intercal}$, and setting η equal to the final column of the orthogonal matrix in the decomposition. Starting from the ν-th point on the curve, the next predicted point is,

$$z_p = z_{cont}^{\nu} + \kappa\eta \tag{3.33}$$

where κ is the step size. For the correction step, which involves projecting back onto the curve, we require that $z_{cont}^{\nu+1} - z_p$ be orthogonal to η, so

$$\eta^{\intercal}(z_{cont}^{\nu+1} - z_p) = 0. \tag{3.34}$$

From (3.33), this implies that,

$$\eta^{\intercal}(z_{cont}^{\nu+1} - z_{cont}^{\nu}) - \kappa = 0. \tag{3.35}$$

This corrector step is accomplished by setting,

$$F_{proj}(z_{cont}^{\nu+1}) = \begin{bmatrix} F(z_{cont}^{\nu+1}) \\ \eta^{\intercal}(z_{cont}^{\nu+1} - z_{cont}^{\nu}) - \kappa \end{bmatrix} \tag{3.36}$$

with the corresponding Jacobian,

$$J_{proj}(z_{cont}^{\nu+1}) = \begin{bmatrix} J_{cont}(z_{cont}^{\nu+1}) \\ \eta^{\intercal}. \end{bmatrix} \tag{3.37}$$

Then we can obtain a solution to $F_{proj}(z_{cont}^{\nu+1}) = 0$ using the standard Newton-Raphson update. This method varies the parameter value p so as to move in the direction of maximizing

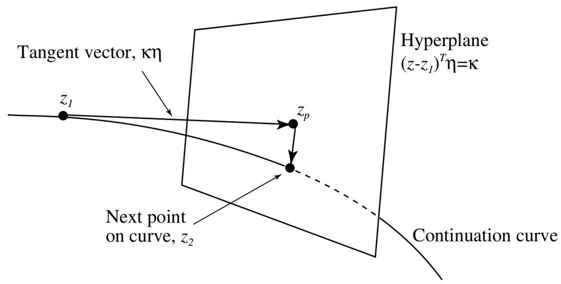

Figure 3.1: A continuation method for iteratively tracing a sequence of points along a curve defined by a set of algebraic equations. At each step a prediction is made by moving along the tangent to the curve. Then the predicted point is projected back onto the curve. This process repeats until the curve is traced.

t_{ball}. As a consequence, p is driven to its critical value.

3.2.3 Two Dimensional Parameter Space

Once a critical parameter value has been found, a pair of parameters can be chosen, and their recovery boundary computed. Fix t_{ball} to its value from the end of the continuation process described above. Let $z_{bound} = [z^\intercal \; p_1 \; p_2]^\intercal$ for a pair of parameters p_1, p_2. Then $F(z_{bound}) = 0$ locally describes a curve in parameter space which lies arbitrarily close to the desired recovery boundary. Using a continuation method like the one described in the previous step of Section 3.2.2, this curve can be traced progressively to obtain the desired recovery boundary. Note that the recovery boundary may not be a smooth curve in general, but numerical experiments have suggested that it is smooth in applications of intersest. The reasons for this remain unknown.

3.2.4 Higher Dimensional Parameter Space

Given a set of $P > 0$ parameters, let $z = [x_1^\intercal \; y_1^\intercal \; t_1 \; x_2^\intercal \; y_2^\intercal \; t_2]^\intercal$ and let $p \in \mathbb{R}^P$ be parameter values. Then $F(z, p) = 0$, where F is given by Eq. 3.26, locally defines a $(P-1)$-manifold. Fix any recover parameter value $p_0 \in \mathbb{R}^P$. The goal is to find the closest point p^* (it may not be unique) on the recovery boundary to p_0. This can be expressed by the following optimization problem:

$$\begin{aligned} &\min_{z,p} \frac{1}{2}||p - p_0||_2^2 \\ &\text{s.t.} \;\; F(z,p) = 0. \end{aligned} \tag{3.38}$$

The Lagrangian for this problem is:

$$L(z, p, \lambda) = ||p - p_0||_2^2 + \lambda^\intercal F(z, p)$$

where λ is a vector of Lagrange multipliers. Any stationary point, such as a local minimum, must satisfy:

$$\nabla L(z, p, \lambda) = 0 = \begin{bmatrix} \left(\frac{\partial F}{\partial z}\right)^\intercal \lambda \\ (p - p_0) + \left(\frac{\partial F}{\partial p}\right)^\intercal \lambda \\ F(z, p) \end{bmatrix}. \tag{3.39}$$

The matrices $\frac{\partial F}{\partial z}$ and $\frac{\partial F}{\partial p}$ can be computed analogously to the computation of J in Eq. 3.28, except that $\frac{\partial F}{\partial p}$ has P columns corresponding to the P dimensions of parameter space instead of just the one column devoted to the one dimensional parameter space in Eq. 3.28.

To solve Eq. 3.39, we use an optimization algorithm similar to [DH06] which was adapted from [Dob03]. This algorithm typically requires local convexity of the recovery boundary in order to converge, which does not hold on general. On the Modified IEEE 37-bus feeder of Section 4.1 the method converges rapidly, but this may not be true for other test cases, especially in the absence of local convexity. We begin each iteration ν with a unit vector ρ^ν in parameter space $\mathbb{R}^P$. Let $p = p_0 + \gamma\rho^\nu$, where γ is a scalar and is initialized to zero. Then a continuation process is used to vary γ from zero until the recovery boundary is encountered at $p^\nu = p_0 + \gamma^\nu\rho^\nu$. Let z^ν be the values of z obtained at p^ν. As p^ν lies on the recovery boundary, and hence is a critical parameter value, this implies that $F(z^\nu, p^\nu) = 0$. The Lagrange multipliers λ^ν at each iteration are chosen to satisfy the first line of Eq. 3.39, so that $\left(\frac{\partial F(z^\nu,p^\nu)}{\partial z}\right)^\intercal \lambda^\nu = 0$. Hence, λ^ν can be computed by performing a QR decomposition of $\frac{\partial F(z^\nu,p^\nu)}{\partial z}$ and setting λ^ν equal to the last column of the orthogonal matrix in the decomposition. Then let $\rho^{\nu+1} = \left(\frac{\partial F(z^\nu,p^\nu)}{\partial p}\right)^\intercal \lambda^\nu$ and normalize.

The iterations described above are repeated until ρ^ν converges, say to ρ^*. This implies that γ^ν, $p^\nu = p_0 + \gamma^\nu\rho^\nu$, z^ν, and λ^ν must also converge, say to γ^*, p^*, z^*, and λ^*, respectively. As p^* is a critical parameter value, $F(z^*, p^*) = 0$, so the last line of Eq. 3.39 is satisfied. Note that the method above for choosing λ^ν immediately implies that $\left(\frac{\partial F(z^*,p^*)}{\partial z}\right)^\intercal \lambda^* = 0$, so the first line of Eq. 3.39 is satisfied. Furthermore, multiplying λ by any constant implies this equation is still satisfied. The method of updating ρ implies that $\left(\frac{\partial F(z^*,p^*)}{\partial p}\right)^\intercal \lambda^* = c\rho^*$ where c is a nonzero constant. Choose $\hat{\lambda}^* = -\frac{\gamma^*}{c}\lambda^*$ and note that the first and third lines of

Eq. 3.39 are still satisfied by $\hat{\lambda}^*$. However, the second line of Eq. 3.39 now reads:

$$\begin{aligned}(p^* - p_0) + \Big(\frac{\partial F(z^*, p^*)}{\partial p}\Big)^\intercal \hat{\lambda}^* &= (p^* - p_0) - \frac{\gamma^*}{c}\Big(\frac{\partial F(z^*, p^*)}{\partial p}\Big)^\intercal \lambda^* \\ &= (p^* - p_0) - \frac{\gamma^*}{c}(c\rho^*) = (p^* - p_0) - \gamma^*\rho^* = 0\end{aligned}$$

since $p^* = p_0 + \gamma^*\rho^*$. Thus, convergence of the above method implies that Eq. 3.39 is satisfied, and at least a local solution of Eq. 3.38 has been obtained.

3.2.5 Choice of Norms for Measuring Distance

The minimization of Section 3.2.4 was performed using the norm $||p - p_0||_2^2$ in parameter space. However, more generally the optimization could be performed by minimizing the objective function $(p - p_0)^T A(p - p_0)$ where A is a positive semidefinite matrix. This would introduce only minor changes to the disucssion and derivation of Section 3.2.4, but could have a large impact on the obtained solutions. A natural choice for the weighting matrix A would be so as to reflect the uncertainty in measurements or estimation of system parameter values. In particular, using a Gaussian model for parameter uncertainty, the covariance matrix for the uncertainty would be a reasonable choice for the weighting matrix. Future work will consider this choice of weighting matrix.

3.2.6 Example: Three Synchronous Machines

The algorithms presented above were applied to a test case of three synchronous machines arranged in a loop. The machines are described by two-axis fourth-order synchronous machine models, with complete details given by equations (6.110)-(6.113) in [SP97]. All three machines have constant field voltage, constant mechanical torque, and mechanical damping $D\omega$ where D is a parameter implicitly representing the damping due to the governor. The parameters of the generators are provided in Table 3.1. The moments of inertia for the three generators are 0.077 pu, 0.105 pu, and 0.084 pu, respectively. The constant field voltages were chosen to ensure that the terminal bus voltages are all at 1.05 pu in steady state. The generator active powers were initialized with $P_1 = 0.4$ pu, $P_2 = 0.5$ pu, and $P_3 = -0.9$ pu. The line resistances are assumed to be zero, and the reactances are $X_{13} = 1.0000$ pu, $X_{12} = 1.2500$ pu and $X_{23} = 0.8333$ pu.

The disturbance considered is a fault applied at the terminals of generator one. The fault is modelled as a switched, constant shunt reactance with $X_{fault} = 0.001$ pu. The

Table 3.1: Synchronous machine parameters.

r_d	0	x_d	0.5880	x'_d	0.0913	T'_{do}	6.59
x_q	0.5880	x'_q	0.1	T'_{qo}	1.0	D	0.2653

main difference between the three generators in this example is their moments of inertia. Accordingly, the fault was located at generator one because it has the lowest inertia, and hence is most vulnerable to the fault. A fault duration time of 1.0 sec has been used throughout the example. Whilst this is an unrealistically long fault clearing time, it is a useful choice for illustrating algorithm characteristics.

The generator moments of inertia are parameters of interest in this system as they play an important role in recovery from the fault. We introduce a scaling factor which multiplies the moments of inertia of all three synchronous machines simultaneously. Lower values of the scaling factor lead to less inertia in the system, which reduces its ability to recover from faults.

The controlling UEP for this test case was identified through a two step process. First, a continuation method was used to identify four power flow solutions. Then, for each of those solutions, all possible choices of machine states were determined. Once those equilibria were known, it was straightforward to determine the controlling UEP by observing the trajectory as the system was driven to instability.

The first step is to drive the trajectory to spend a fixed length of time t_{ball} inside the ball around the controlling UEP. This is illustrated in Figure 3.2, which depicts the distance from a point on the trajectory to the controlling UEP as a function of time. Given a ball of radius $r = 10.5$, the moment of inertia scaling factor was determined using Eq. 3.27 such that the trajectory spent the target time $t_{ball} = 6.5$ sec inside the ball.

Next, the time t_{ball} in the ball was increased to drive the moment of inertia scaling factor to its critical value. This effectively moved the post-fault initial conditions onto the post-fault RoA boundary. This was accomplished via the predictor-corrector continuation process presented in Section 3.2.2. Figure 3.3 shows the moment of inertia scaling factor as the time in the ball is increased. Note that the scaling factor converges to its critical value quite quickly as the time inside the ball increases, as suggested by the theory. This process yields a critical moment of inertia scaling factor whose associated post-fault initial conditions lie on the RoA boundary.

Now that a critical value of the parameter has been found, a pair of parameters can be treated as free variables, and their recovery boundary computed. For this test case, the mo-

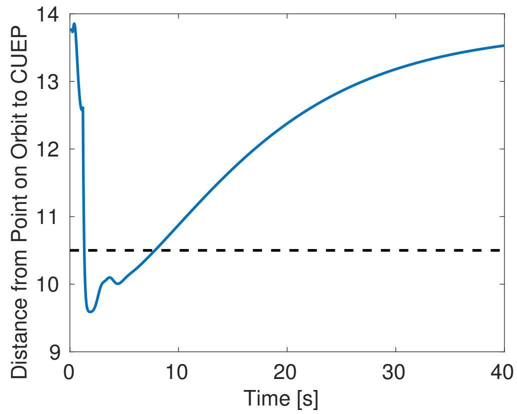

Figure 3.2: The distance of the trajectory from the controlling unstable equilibrium point as a function of time. The radius of the ball around it is $r = 10.5$. The moment of inertia scaling factor was varied to drive the trajectory to spend a fixed time $t_{ball} = 6.5$ sec inside the ball.

ments of inertia of generators one and two were selected as the two free parameters. Using an analogous continuation method, as described in Section 3.2.3, the recovery boundary is numerically computed. Figure 3.4 depicts this boundary. For any parameter values above the boundary, the system will recover from the fault, returning to the original SEP. Conversely, for any parameter values below the boundary, the system will not recover from the specified disturbance. Although the recovery boundary turns out to be almost linear for this example, the continuation method is capable of tracing curves that have high curvature [FH16]. Further investigations are required to determine the reason for the apparent linearity in this example.

3.2.7 Conclusion

Algorithms are developed to find the unique closest point on the recovery boundary for one dimensional parameter space, trace the recovery boundary for two dimensional parameter space, and find the closest point on the recovery boundary to any initial recovery value in arbitrary dimensional parameter space. The main idea behind the approaches is to vary parameter values so as to maximize the time the system trajectory spends in a ball around the controlling UEP, thereby driving parameter values to the recovery boundary. Continuation methods are used to perform the maximizations and to trace the recovery boundary. The algorithms are illustrated on a network of three synchronous machines, and are applied to a large power system in Section 4.1.

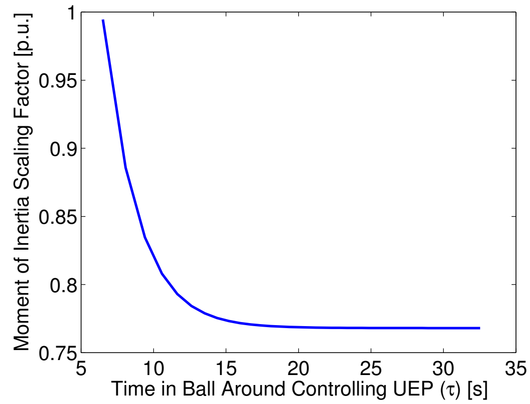

Figure 3.3: The moment of inertia scaling factor is varied to maximize the time spent by the trajectory inside the ball around the controlling unstable equilibrium point. As this time increases, the moment of inertia scaling factor converges to its critical value.

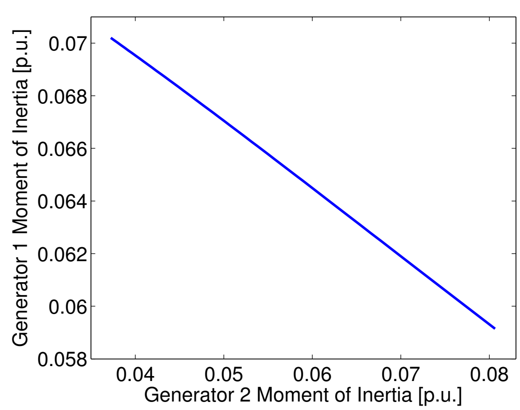

Figure 3.4: The recovery boundary for the moments of inertia of generators one and two. All parameter values above the boundary lead to fault recovery, whereas all parameter values below the boundary lead to instability.

3.3 Maximizing Trajectory Sensitivities Algorithms

3.3.1 Introduction

The purpose of this section is to develop methods for determining the recovery boundary based on the theoretical motivation of Section 2.6. Unlike the techniques presented in Section 3.2, the algorithms here do not require prior knowledge of the controlling UEP, the determination of which is computationally intractable for realistic system models. Furthermore, unlike those in Section 3.2, the algorithms here do not require the assumption of transversal intersection of the system trajectory with a ball in state space, which was often violated in practice, and do not have to worry about solution degeneracy resulting from many intersections of the system trajectory with that ball, both of which led to challenges in practical implementations of those algorithms. The approach behind the algorithms here is that parameter values are varied so as to minimize the infimum over time of the inverse of the norm of the trajectory sensitivities, thereby driving parameter values onto the recovery value. Algorithms are presented for numerically computing the recovery boundary in two parameter dimensions, and for finding the nearest point on the recovery boundary from any initial set of recovery values in higher parameter dimensions. The algorithms are illustrated on the simple test case of two classical machines. The algorithms are then modified so as to exploit the linear structure discussed in Section 2.6.

3.3.2 One Dimensional Parameter Space

Let p be a one-dimensional real-valued parameter, let j be the index of the dynamic state corresponding to p, and let i be an index that runs through the set of dynamic states. We use the notation of Section 3.1 for trajectory sensitivities and related computations. Initially, it was observed through numerical experiments that minimizing inverse trajectory sensitivities instead of directly maximizing them led to faster convergence rates, and that root finding methods such as Newton-Raphson converged rapidly for these inverse sensitivities. More recently, it was shown (see Theorem 2.6.13) that the inverse trajectory sensitivities approach a dot product as parameter values approach the recovery boundary. This indicates that inverse sensitivities possess an approximately linear structure, which implies they can be rapidly solved by root finder methods such as Newton-Raphson. See Section 3.3.7 for further discussion.

Define

$$H(t,p) = \frac{1}{\Phi_{ij}(t,p)\Phi_{ij}(t,p)}$$
$$G(p) = \inf_{t\geq 0} H(t,p).$$

Then by Theorem 2.6.4, p^* in the recovery boundary implies that it satisfies:

$$G(p^*) = 0. \tag{3.40}$$

To find the critical parameter p^*, Eq. 3.40 will be solved using Newton-Raphson, which will require the derivative $\frac{dG(p)}{dp}$. To compute this, first note that

$$\frac{\partial H(t,p)}{\partial p} = -2\frac{\Phi_{ijj}\Phi_{ij}}{(\Phi_{ij}\Phi_{ij})^2} \tag{3.41}$$

where the dependence of the trajectory sensitivities on (t,p) is understood, but has been supressed in the notation for clarity of presentation. Let $\hat{t}(p) = \text{argmin}_{t\geq 0} H(t,p)$. Then, by the proof of Lemma 2.6.15, if p is a recovery value then $\hat{t}(p)$ is finite so $G(p) = H(\hat{t}(p),p)$. Differentiating with respect to p yields

$$\begin{aligned} DG(p) = \frac{dG(p)}{dp} &= \frac{\partial}{\partial p} H(\hat{t}(p),p) \\ &= \frac{\partial H}{\partial t}(\hat{t}(p),p)\frac{d\hat{t}(p)}{dp}(p) + \frac{\partial H}{\partial p}(\hat{t}(p),p) \\ &= \frac{\partial H}{\partial p}(\hat{t}(p),p) \end{aligned} \tag{3.42}$$

where the final equality follows since $\frac{\partial H}{\partial t}(\hat{t}(p),p) = 0$ because $\hat{t}(p)$ is the time when $H(t,p)$ achieves a maximum in time, and $\frac{\partial H}{\partial t} = 0$ at an extremum point. Therefore, Eq. 3.40 is solved iteratively by the following standard Newton-Raphson update, where p^ν denotes the parameter value of the current iteration, and $p^{\nu+1}$ the value of the next iteration:

$$p^{\nu+1} = p^\nu - DG(p^\nu)^{-1}G(p^\nu). \tag{3.43}$$

As $DG(p) = \frac{\partial H}{\partial p}(\hat{t}(p),p)$ and $G(p) = H(\hat{t}(p),p)$, by the formulas for H and $\frac{\partial H}{\partial p}$, computation of G and DG requires only knowledge of trajectory sensitivities, which can be computed efficiently by the methods of Section 3.1. Then $\hat{t}(p)$ is observed from the output of the integration, and the values of $G(p)$ and $DG(p)$ can be computed. As the algorithm proceeds, $G(p^\nu)$ approaches zero, causing p^ν to approach the recovery boundary.

3.3.3 Two Dimensional Parameter Space

Let p be a parameter that takes values in $\mathbb{R}^2$, let j, k run through the two indeces of the two dynamic states corresponding to the parameter p, and let i be an index that runs through all the dynamic states. Define

$$H(t,p) = \frac{1}{\Phi_{ij}\Phi_{ij}} \tag{3.44}$$

$$G(p) = \inf_{t\geq 0} H(t,p). \tag{3.45}$$

By Theorem 2.6.4, the recovery boundary for p is given by $\{p : G(p) = 0\}$, which typically describes a curve in practice. However, it may be nonsmooth in general, and determining why it is typically smooth in practice is a subject of future work. Let $\epsilon > 0$ small. To establish consistent precision, we choose to approximate the recovery boundary with the curve $\{p : G(p) = \epsilon\}$. The goal is to use a continuation method to iteratively construct a sequence of points that traces this manifold. At each step a prediction is made by moving along the tangent to the curve. Then the predicted point is corrected by projecting back onto the curve. This process repeats until the curve is complete. Let p^ν denote the current point on the curve. The unit tangent vector η to the curve at p^ν is given by $\eta = DG(p^\nu)^\intercal/||DG(p^\nu)||_2$ where, by an analogous argument to the derivation of Eq. 3.42, we have

$$DG(p) = \frac{\partial H}{\partial p}(\hat{t}(p), p) \tag{3.46}$$

$$\frac{\partial H(t,p)}{\partial p_j} = -2\frac{\Phi_{ijk}\Phi_{ik}}{(\Phi_{ik}\Phi_{ik})^2}. \tag{3.47}$$

In the predictor step, for some choice of $\kappa > 0$ we let

$$p^{\nu+1}_{pred} = p^\nu + \kappa\eta.$$

In the corrector step we solve

$$0 = (p^{\nu+1}_{pred} - p^\nu)^\intercal (p^{\nu+1} - p^{\nu+1}_{pred})$$

which implies, after substituting for $p_{pred}^{\nu+1}$ above,

$$0 = \eta^T(p^{\nu+1} - p^{\nu}) - \kappa. \tag{3.48}$$

Hence, we obtain

$$F_{cont}(p^{\nu+1}) = \begin{bmatrix} G(p^{\nu+1}) - \epsilon \\ \eta^T(p^{\nu+1} - p^{\nu}) - \kappa. \end{bmatrix}$$

The corrector step must solve:

$$F_{cont}(p^{\nu+1}) = 0. \tag{3.49}$$

To do so, we require the derivative DF which is given by

$$DF_{cont}(p^{\nu+1}) = \begin{bmatrix} DG(p^{\nu+1}) \\ \eta^T \end{bmatrix}. \tag{3.50}$$

We then solve Eq. 3.49 iteratively using Newton-Raphson, analogously to Eq. 3.43. This process of prediction and correction is then applied iteratively to generate a sequence of points along $\{p : G(p) = \epsilon\}$, thereby computing the recovery boundary to arbitrarily close precision.

3.3.4 Higher Dimensional Parameter Space: Newton's Method

Let p be a parameter that takes values in $\mathbb{R}^P$ for $P \geq 1$. Let j, k, l run through the indeces of the P dynamic states corresponding to the parameter p, and let i be an index that runs through all the dynamic states. Define

$$H(t,p) = \frac{1}{\Phi_{il}\Phi_{il}}$$
$$G(p) = \inf_{t\geq 0} H(t,p) = H(\hat{t}(p), p)$$

where $\hat{t}(p) = \text{argmin}_{t\geq 0} H(t,p)$. Fix any $p_0 \in \mathbb{R}^P$ a recover parameter value. As the dimension P is arbitrary, there may be many points in the recovery boundary that could be found by varying p. Therefore, we seek the point on the recovery boundary that is closest to p_0, which represents the smallest change in parameter space that could lead to failure to recover from the disturbance. So, we wish to solve

$$\min_{p \in \mathbb{R}^P} \frac{1}{2}(p - p_0)^\intercal A(p - p_0) \tag{3.51}$$

$$\text{s.t. } G(p) - \epsilon = 0 \tag{3.52}$$

where A is a positive semidefinite weighting matrix (often set to the identity matrix) and $\epsilon > 0$ is small and will be required to ensure feasibility. The weighting matrix could be chosen as in Section 3.2.5. In general, the solution to this problem does not necessarily depend smoothly (or even continuously) with respect to p_0. Nevetheless, when the technique for solving this problem, as described below, was applied to the simple test case of Section 3.3.6 it converged rapidly to the desired minimum. However, in general this problem is very difficult and the method offers no guarantees of convergence or that convergence will be to the global (or even a local) minimum. Further experimentation is required to assess its potential.

To solve Eqs. 3.51-3.52, we form the Lagrangian $\mathcal{L}(p, \lambda) = \frac{1}{2}(p-p_0)^\intercal A(p-p_0)+\lambda(G(p)-\epsilon)$ where λ is a Lagrange multiplier. Let $F(p, \lambda) := D\mathcal{L}(p, \lambda)$. Any stationary point of $\mathcal{L}(p, \lambda)$, such as a local minimum, must satisfy:

$$0 = D\mathcal{L}(p, \lambda) = \begin{bmatrix} A(p - p_0) + \lambda DG(p)^\intercal \\ G(p) - \epsilon \end{bmatrix} =: F(p, \lambda). \tag{3.53}$$

Eq. 3.53 is solved iteratively by the following standard Newton-Raphson update, where (p^ν, λ^ν) denote the corresponding values at the current iteration, and $(p^{\nu+1}, \lambda^{\nu+1})$ the values at the next iteration:

$$\begin{bmatrix} p^{\nu+1} \\ \lambda^{\nu+1} \end{bmatrix} = \begin{bmatrix} p^\nu \\ \lambda^\nu \end{bmatrix} - DF(p^\nu, \lambda^\nu)^{-1} F(p^\nu, \lambda^\nu) \tag{3.54}$$

where $F(p^\nu, \lambda^\nu) = D\mathcal{L}(p, \lambda)$ is given in Eq. 3.53 and DF is given by:

$$DF(p, \lambda) = \begin{bmatrix} A + \lambda D(DG(p)^\intercal) & DG(p)^\intercal \\ DG(p) & 0 \end{bmatrix} \tag{3.55}$$

Computation of $DG(p)$ and $D(DG(p)^\intercal)$ first requires several additional derivatives, which are obtained via repeated differentiation of $H(t, p)$ with respect to parameter components and time:

$$\frac{\partial H}{\partial p_j}(t,p) = -2\frac{\Phi_{ijl}\Phi_{il}}{(\Phi_{il}\Phi_{il})^2} \tag{3.56}$$

$$\frac{\partial^2 H}{\partial p_j \partial p_k}(t,p) = -2H^2\Big(\Phi_{ijkl}\Phi_{il} + \Phi_{ijl}\Phi_{ikl}\Big) + 8H^3\Big(\Phi_{ikl}\Phi_{il}\Big)\Big(\Phi_{ijl}\Phi_{il}\Big) \tag{3.57}$$

$$\frac{\partial^2 H}{\partial p_j \partial t}(t,p) = -2H^2\Big(\dot{\Phi}_{ijl}\Phi_{il} + \Phi_{ijl}\dot{\Phi}_{il}\Big) + 8H^3\Big(\dot{\Phi}_{il}\Phi_{il}\Big)\Big(\Phi_{ijl}\Phi_{il}\Big) \tag{3.58}$$

$$\frac{\partial^2 H}{\partial t^2}(t,p) = 8H^3\Big(\dot{\Phi}_{il}\Phi_{il}\Big)^2 - 2H^2\Big(\Phi_{il}\ddot{\Phi}_{il} + \dot{\Phi}_{il}\dot{\Phi}_{il}\Big) \tag{3.59}$$

where $H = H(t,p)$ and $\dot{\Phi}_{il}$, $\dot{\Phi}_{ijl}$ are given by Eqs. 3.4-3.5 in Section 3.1.3, and $\ddot{\Phi}_{il}$ is given by Eq. 3.7 of that section. Next we compute the derivatives $DG(p)$ and $D(DG(p)^\intercal)$. First, note that $(DG(p))_j = \frac{\partial G(p)}{\partial p_j}$. By an analogous argument as in the derivation of Eq. 3.42 for the one dimensional parameter case in Section 3.3.2, $(DG(p))_j = \frac{\partial H}{\partial p_j}(\hat{t}(p),p)$. Next, differentiating this equation with respect to p_k gives

$$\begin{aligned}(D(DG(p)^\intercal))_{jk} &= \frac{\partial^2 G(p)}{\partial p_k \partial p_j} \\ &= \frac{\partial}{\partial p_k}(DG(p)^\intercal)_j = \frac{\partial}{\partial p_k}\frac{\partial H}{\partial p_j}(\hat{t}(p),p) \\ &= \frac{\partial^2 H}{\partial p_j \partial t}(\hat{t}(p),p)\frac{\partial \hat{t}(p)}{\partial p_k} + \frac{\partial^2 H}{\partial p_k \partial p_j}(\hat{t}(p),p).\end{aligned}$$

Note that computation of $D(DG(p)^\intercal)$ requires $\frac{\partial \hat{t}(p)}{\partial p_k}$. We derive this as follows. Since $\hat{p} = \text{argmin}_{t\geq 0} H(t,p)$, $\frac{\partial}{\partial t}H(\hat{t}(p),p) = 0$. Hence, as this function of p is identically zero, differentiating with respect to p_k gives:

$$\begin{aligned}0 =& \frac{\partial}{\partial p_k}\frac{\partial}{\partial t}H(\hat{t}(p), p) = \frac{\partial}{\partial t}\frac{\partial}{\partial p_k}H(\hat{t}(p), p)\\ =& \frac{\partial}{\partial t}\left(\frac{\partial H}{\partial t}(\hat{t}(p), p)\frac{\partial \hat{t}(p)}{\partial p_k} + \frac{\partial H}{\partial p_k}(\hat{t}(p), p)\right)\\ =& \frac{\partial^2 H}{\partial t^2}(\hat{t}(p), p)\frac{\partial \hat{t}(p)}{\partial p_k} + \frac{\partial^2 H}{\partial p_k \partial t}(\hat{t}(p), p).\end{aligned}$$

Hence, solving for $\frac{\partial \hat{t}(p)}{\partial p_k}$ yields

$$\frac{\partial \hat{t}(p)}{\partial p_k} = -\frac{\partial^2 H}{\partial p_k \partial t}(\hat{t}(p), p)\left(\frac{\partial^2 H}{\partial t^2}(\hat{t}(p), p)\right)^{-1}.$$

Substituting this back into the expression for $D(DG(p)^\intercal)_{jk}$, we obtain

$$\begin{aligned}(DG(p))_j &= \frac{\partial H}{\partial p_j}(\hat{t}(p), p)\\ (D(DG(p)^\intercal))_{jk} &= \frac{\partial^2 H}{\partial p_k \partial p_j}(\hat{t}(p), p)\\ &\quad - \frac{\partial^2 H}{\partial p_j \partial t}(\hat{t}(p), p)\frac{\partial^2 H}{\partial p_k \partial t}(\hat{t}(p), p)\left(\frac{\partial^2 H}{\partial t^2}(\hat{t}(p), p)\right)^{-1}\end{aligned}$$

which can be computed from Eqs. 3.56-3.59. In turn, the expressions for $DG(p)$ and $D(DG(p)^\intercal)$ given here can be used to compute F and DF in Eq. 3.55 and Eq. 3.53, respectively. Finally, F and DF are used to perform the Newton-Raphson updates of Eq. 3.54, which drive p^ν towards one of the (possibly many locally) closest points on the recovery boundary to p_0.

3.3.5 Higher Dimensional Parameter Space: Alternate Method

Consider the setting of Section 3.3.4. The goal is to solve Eq. 3.53 to obtain the (possibly one of many) closest point on the recovery boundary to some initial recovery value p_0. We present an alternative solution strategy to that of Section 3.3.4 which does not require third order trajectory sensitivities and is similar to that of Section 3.2.4. As remarked in Section 3.2.4, this method requires local convexity of the recovery boundary to guarantee convergence, but the recovery boundary is not in general convex. So, further testing is required to determine the practicality of this technique.

P_2	P_3	V_2	V_3	D_2	D_3	H_2	H_3	X_{12}	X_{23}
1	0.5	1	0.9	0.3	0.2	0.3	0.2	0.4	0.5

Table 3.2: Test Case Parameter Values

Define an iterative algorithm as follows. Let ν denote the current iteration. Given a unit vector ρ^ν in $\mathbb{R}^P$, find γ^ν such that $G(p_0 + A^{-1}\gamma^\nu\rho^\nu) = \epsilon$. Then set $p^\nu = p_0 + A^{-1}\gamma^\nu\rho^\nu$. This can be accomplished by using the one dimensional method of Section 3.3.2 to solve for γ^ν using Newton-Raphson and noting that

$$\begin{aligned} \Phi_{i\gamma} &= \Phi_{ij}\rho_j^\nu \\ \Phi_{i\gamma\gamma} &= \Phi_{ijk}\rho_j^\nu\rho_k^\nu \end{aligned} \tag{3.60}$$

where $\Phi_{i\gamma} = \frac{d\phi_i}{d\gamma}$ and $\Phi_{i\gamma\gamma} = \frac{d^2\phi_i}{d\gamma^2}$. Once γ^ν, and hence p^ν, is obtained, set $\rho^{\nu+1} = DG(p^\nu)^T/||DG(p^\nu)||_2$ where $DG = \frac{\partial H}{\partial p}(\hat{t}(p), p)$ and $\frac{\partial H}{\partial p}$ is given by substituting Eq. 3.60 into Eq. 3.41 of Section 3.3.2.

The process described above is then repeated iteratively until ρ^ν converges, say to ρ^*. Then γ^ν and p^ν also converge, say to γ^* and p^*, respectively. Let $\lambda = -\gamma^*/||DG(p^*)||_2$. By the method for computing γ^ν, we must have that $G(p^*) = G(p_0 + A^{-1}\gamma^* p^*) = \epsilon$ and $p^* = p_0 + A^{-1}\gamma^*\rho^*$. Furthermore, by the method for determining $\rho^{\nu+1}$, we have that $\rho^* = DG(p^*)^T/||DG(p^*)||_2$. Hence,

$$\begin{aligned} A(p^* - p_0) + \lambda DG(p^*)^T &= A(p^* - p_0) + (-\gamma^*/||DG(p^*)||_2)(||DG(p^*)||_2\rho^*) \\ &= A(p^* - p_0) - \gamma^*\rho^* \\ &= A(p^* - p_0) - A(p^* - p_0) \\ &= 0. \end{aligned}$$

Hence, convergence of the algorithm implies that Eq. 3.53 is satisfied, so a local minimum of Eqs. 3.51-3.52 has been found.

3.3.6 Example: Two Classical Machines

The test case considered is a simple model of a power system which consists of two synchronous generators connected to two buses arranged radially, with an infinite bus at the head of the feeder, as shown in Fig. 3.5. The infinite bus is held at constant voltage

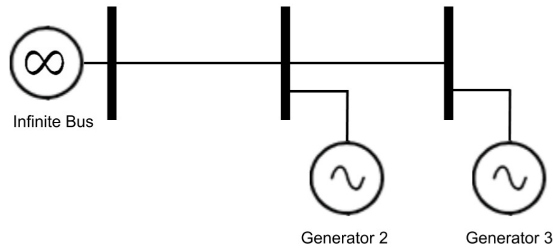

Figure 3.5: Power system model used as the test case in Section 2.3.3. An infinite bus, which is held at constant voltage and can draw arbitrarily high power, is connected to two synchronous generators.

of 1 p.u. (per unit) and can draw arbitrarily high power to match the power generation of the two generators. Let the infinite bus be connected to bus one, generator 2 to bus 2, and generator 3 to bus 3. Let δ_2 and ω_2 be the rotor angle and its angular velocity for generator 2, and define δ_3 and ω_3 analogously for generator 3. Let $y = [\delta_2\ \omega_2\ \delta_3\ \omega_3]^\intercal$ be the vector of state variables. Let P_2, D_2, H_2, and V_2 be the active power, damping coefficient, moment of inertia, and voltage magnitude at generator 2, and define P_3, D_3, H_3, and V_3 analogously for generator 3. Let X_{ij} denote the impedance on the line from bus i to bus j. Parameter values used are given in Table 3.2. Let $p = [P_2\ P_3\ V_2\ V_3\ D_2\ D_3\ H_2\ H_3]^\intercal$ be the vector of chosen parameters, and let $x = [y^\intercal\ p^\intercal]^\intercal$. Then the dynamics of this system are given by:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
H_2\dot{x}_2 &= P_2 - D_2x_2 - \sin(x_1)\frac{V_2}{X_{12}} - (\sin(x_1)\cos(x_3) - \cos(x_1)\sin(x_3))\frac{V_2V_3}{X_{23}} \\
\dot{x}_3 &= x_4 \\
H_3\dot{x}_4 &= P_3 - D_3x_4 - (\sin(x_3)\cos(x_1) - \cos(x_3)\sin(x_1))\frac{V_2V_3}{X_{23}},
\end{aligned}
$$

which can be written more succinctly as

$$\dot{x} = V(x) \tag{3.61}$$

Let p_0 be the parameter values given in Table 3.2. There exists a stable equilibrium point of Eq. 3.61 of $y^0(p_0) = [0.6435\ 0\ 0.9250\ 0]^\intercal$. For p reasonably near p_0 there exists a stable equilibrium point $y^0(p)$ of Eq. 3.61 close to $y^0(p_0)$ which can be found, for example, by solving $V(y^0(p)) = 0$ using Newton-Raphson with initial condition p_0. After the system begins at the stable equilibrium point $x^0(p)$, a fault is applied at the head of the feeder, lasts for a

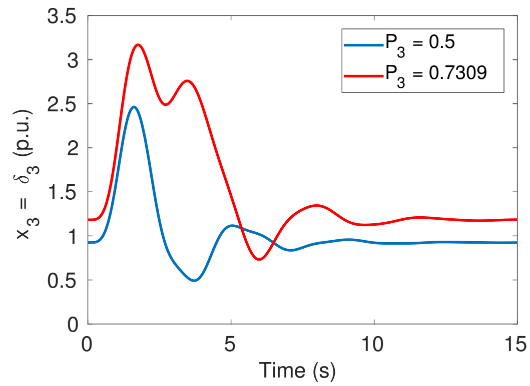

Figure 3.6: For the test case of Section 2.3.3, $x_3 = \delta_3$ is shown as a function of time for (a) the initial value $P_3 = 0.5$ (blue line) and (b) the critical value $P_3 = 0.7309$ (red line).

duration of 0.8 s, and then is cleared. The fault is modeled by setting the term $-\sin(x_1)\frac{V_2}{X_{12}}$ to zero in equation V_2, leaving the other components of V unchanged, and integrating this modified system for 0.8 s. Before and after the fault, the system dynamics are given by Eq. 3.61.

Fig. 3.6 shows $x_3 = \delta_3$ as a function of time for $p = p_0$. Note that the system is initially at a stable equilibrium point. Then, a fault is applied, and the angle δ_3 deviates far from equilibrium. After 0.8 s the fault is cleared, normal system dynamics are restored, and the system evolves until δ_3 returns to its prior equilibrium value. This picture is typical for cases in which the system is able to recover from the fault.

For each $p_i \in p$, the algorithm detailed in Section 3.3.2 was applied to find the nearest point on the recovery boundary of p_i. There were some parameters, namely H_2, H_3, and D_3, where varying just one of them was not sufficient to make the system fail to recover from the fault (for positive values of the parameters, as is required physically). In these cases, the algorithm attempted to send the parameter values towards zero or negative, so it was quickly clear that they were unable to make the system fail to recover.

Among the remaining parameters, $p = P_3$ is representative of the observed behavior of the algorithm. Fig. 3.7 shows the convergence of the one-dimensional parameter space algorithm for $p = P_3$. Although the critical value of P_3 is about 50% larger than its initial value, convergence to its critical value is rapid and monotonic. At the final iterate, $G(P_3) < 10^{-5}$. More generally, for each $p_i \in p$ (other than those mentioned above which could not drive the system to non-recovery), rapid, monotonic convergence to the corresponding critical value was observed.

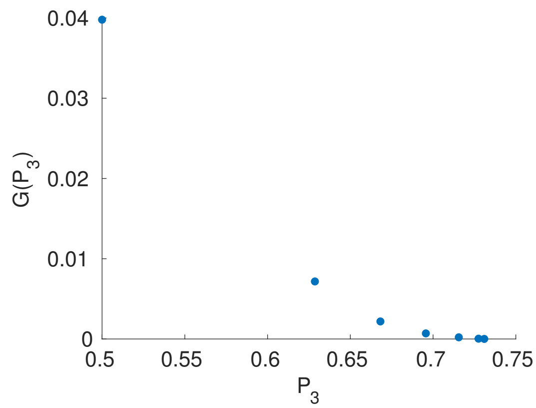

Figure 3.7: Computation of critical parameter value for P_3 using the one-dimensional parameter space algorithm of Section 3.3.2. Iterations begin in the top left and proceed towards the bottom right. Rapid convergence to the critical parameter value is observed.

Several choices of multi-dimensional parameter spaces were tested. The weighting matrix A was set to the identity matrix in all cases. The initial parameter values p_0 were taken from Table 3.2. First, the choice $p = p_i$ for each i was made to confirm that the higher dimensional parameter space algorithm of Section 3.3.4 gave the same critical parameter values as the one-dimensional parameter space algorithm of Section 3.3.2. Next, the following three sets of parameters were considered:

$$\begin{aligned} P_4 &:= \{P_2, P_3, V_2, V_3\} \\ P_6 &:= \{P_2, P_3, V_2, V_3, D_2, D_3\} \\ P_8 &:= \{P_2, P_3, V_2, V_3, D_2, D_3, H_2, H_3\}. \end{aligned}$$

Note that P_4 consists of all the parameter values that power system operators can select during real-time operation, P_6 consists of P_4 together with the parameter values that can be tuned by control engineers (namely, the damping that is a result of controller design), and P_8 consists of P_6 together with the remaining generator parameter values.

We chose $\epsilon = 10^{-5}$ for the constraint of Eq. 3.52. Then, the higher dimensional parameter space algorithm of Section 3.2.4 converged for P_4 in 9 iterations, for P_6 in 9 iterations, and for P_8 in 8-iterations. Let p_4^*, p_6^*, and p_8^* denote the critical parameter values obtained via

Parameter Set	Objective Function Value
P_4	0.0075
P_6	0.0062
P_8	0.0053

Table 3.3: Values of the objective function of (3.51) for the critical parameter values obtained by the higher dimensional optimization algorithm for the parameter sets P_4, P_6, and P_8.

this algorithm for P_4, P_6, and P_8, respectively. Then

$$\begin{aligned} p_4^* &= \begin{bmatrix} 1.0657 & 0.5651 & 0.9198 & 0.8986 \end{bmatrix}^\intercal \\ p_6^*(1:3) &= \begin{bmatrix} 1.0555 & 0.5549 & 0.9352 \end{bmatrix}^\intercal \\ p_6^*(4:6) &= \begin{bmatrix} 0.8991 & 0.2618 & 0.1736 \end{bmatrix}^\intercal \\ p_8^*(1:4) &= \begin{bmatrix} 1.0466 & 0.5462 & 0.9489 & 0.8992 \end{bmatrix}^\intercal \\ p_8^*(5:8) &= \begin{bmatrix} 0.2648 & 0.1751 & 0.2767 & 0.1647 \end{bmatrix}^\intercal \end{aligned}$$

The parameter values satisfy $G(p_4^*) = G(p_6^*) = G(p_8^*) = \epsilon$, so they all lie on their respective recovery recovery boundarys.

The goal was to obtain parameter values on the recovery boundary which minimized the objective function of Eq. 3.51: $\frac{1}{2}(p - p_0)^\intercal A(p - p_0)$. Table 3.3 shows the values of the objective functions at p_4^*, p_6^* and p_8^*. As additional parameters are introduced, more options for values on the recovery boundary become available, some of which may be closer to p_0 than the previous options. Consequently, the objective function value, which measures the distance of the chosen value on the recovery boundary from p_0, decreased from P_4 to P_6, and again from P_6 to P_8. This suggests that the result of the higher dimensional parameter space algorithm for all of the cases above is indeed the value on the recovery boundary at a minimum distance from p_0, as desired. Overall, for all the combinations of parameters chosen above, the higher dimensional parameter space algorithm converged rapidly to a value on the recovery boundary which appears to be the closest such value to p_0 in each parameter space.

3.3.7 Algorithm Adjustments Based on Linearization Theory

Empirical observations [NPH02b] and analysis for parameter-independent linear vector fields [RH12] indicated that, in the case of one dimensional parameter space, the infimum over time of the inverse trajectory sensitivities is approximately linear in parameter value. Section 2.6 has extended the analysis to parameter-dependent nonlinear vector fields with higher dimensional parameter spaces. In particular, Theorem 2.6.13 shows that the infimum of the one norm of inverse trajectory sensitivities approaches a dot product as parameter values approach the recovery boundary. Motivated by this theory, the algorithms of this section, which were developed using the two norm (Euclidean norm) above, are modified for use with the one norm instead. By doing so, the algorithms are then able to exploit the linear structure revealed by the theory.

3.3.7.1 One Dimensional Parameter Space

We preserve the notational conventions used previously in Section 3.3. Let p be a one-dimensional real-valued parameter, let j be the index of the dynamic state corresponding to p, and let i be an index that runs through the set of dynamic states. Motivated by Theorem 2.6.13 of Section 2.6, we use inverse trajectory sensitivities with one norms to obtain a function that is approximately linear. Consequently, we solve for $G(p) = 0$ using Newton-Raphson because of its ability to rapidly solve linear and approximately linear systems. Define

$$H(t,p) = \frac{1}{|\Phi_{ij}(t,p)|}$$
$$G(p) = \inf_{t\geq 0} H(t,p).$$

The goal is to solve Eq. 3.40: $G(p^*) = 0$. This requires the derivative

$$\frac{\partial H(t,p)}{\partial p} = -H^2(t,p)\Phi_{ijj}\text{sign}(\Phi_{ij})$$

where for any vector v, $(\text{sign}(v))_i = 1$ if $v_i \geq 0$ and $(\text{sign}(v))_i = -1$ if $v_i < 0$ for all components i. Then, as in Section 3.3.2, we have

$$DG(p) = \frac{\partial H}{\partial p}(\hat{t}(p), p)$$

and solve Eq. 3.40 iteratively using Newton-Raphson. Newton-Raphson is used because it efficiently solves linear or approximately linear systems, and $G(p)$ approaches a dot product as $p \to p^*$.

3.3.7.2 Two Dimensional Parameter Space

Let p be a parameter that takes values in $\mathbb{R}^2$, let j, k run through the two indeces of the two dynamic states corresponding to the parameter p, and let i be an index that runs through all the dynamic states. Define

$$H(t,p) = \frac{1}{|\Phi_{ij}(t,p)|}$$
$$G(p) = \inf_{t \geq 0} H(t,p)$$

as above (note that now j runs through two values, not just one as in Section 3.3.7.1). We require the derivative

$$\frac{\partial H(t,p)}{\partial p_k} = -H^2(t,p)\Phi_{ijk}\text{sign}(\Phi_{ij}).$$

Then

$$DG(p) = \frac{\partial H}{\partial p}(\hat{p}, p)$$

has two components corresponding to the derivatives with respect to p_1 and p_2. A predictor-corrector continuation method can then be formulated analogously to that in Section 3.3.3.

3.3.7.3 Higher Dimensional Parameter Space

Let p be a parameter that takes values in $\mathbb{R}^P$ for $P \geq 1$. Let j, k run through the indeces of the P dynamic states corresponding to the parameter p, and let i be an index that runs through all the dynamic states. Define

$$H(t,p) = \frac{1}{|\Phi_{ij}(t,p)|}$$
$$G(p) = \inf_{t \geq 0} H(t,p) = H(\hat{t}(p), p)$$

as above (note that now j runs through P values). We will require the derivatives

$$\frac{\partial H(t,p)}{\partial p_k} = -H^2(t,p)\Phi_{ijk}\text{sign}(\Phi_{ij})$$
$$DG(p) = \frac{\partial H}{\partial p}(\hat{p},p).$$

As in Section 3.3.4, for fixed $\epsilon > 0$ small and A positive semidefinite (where the weighting matrix A could be chosen as in Section 3.2.5), we wish to solve

$$\min_{p\in\mathbb{R}^P} \frac{1}{2}(p-p_0)^\intercal A(p-p_0) \tag{3.62}$$
$$\text{s.t. } G(p) - \epsilon = 0. \tag{3.63}$$

Motivated by the linearity theory of Section 2.6 (especially Theorem 2.6.13) we approximate $G(p) - \epsilon$ by its linearization and solve a succession of quadratic programs. In particular, at each iteration ν we replace the nonlinear constraint $G(p^{\nu+1}) - \epsilon = 0$ with the corresponding linear constraint $DG(p^\nu)(p^{\nu+1} - p^\nu) + G(p^\nu) - \epsilon = 0$. This gives the quadratic program

$$\min_{p^{\nu+1}\in\mathbb{R}^P} \frac{1}{2}(p^{\nu+1}-p_0)^\intercal A(p^{\nu+1}-p_0) \tag{3.64}$$
$$\text{s.t. } DG(p^\nu)(p^{\nu+1} - p^\nu) + G(p^\nu) - \epsilon = 0. \tag{3.65}$$

The solution is given by solving the following linear system

$$\begin{bmatrix} A & DG(p^\nu)^T \\ DG(p^\nu) & 0 \end{bmatrix} \begin{bmatrix} p^{\nu+1} \\ \lambda^{\nu+1} \end{bmatrix} = \begin{bmatrix} Ap_0 \\ DG(p^\nu)p^\nu - G(p^\nu) + \epsilon \end{bmatrix}$$

where $\lambda^{\nu+1}$ is a Lagrange multiplier but is not needed by the algorithm. If $p^{\nu+1}$ lies on the other side of the recovery boundary, which is determined as a byproduct of the simulation required to compute $G(p^{\nu+1})$ and $DG(p^{\nu+1})$, then we still move in the direction $p^{\nu+1} - p^\nu$ but only half the distance as before. This distance is repeatedly halved until a value for $p^{\nu+1}$ is found such that $p^{\nu+1}$ is a recovery value. This process is then repeated iteratively until the algorithm converges. As the solution to Eqs. 3.64-3.65 (in which $G(p)$ is replaced by a linear constraint) exists and is unique, and since $G(p)$ approaches a linear function as p approaches the recovery boundary by Theorem 2.6.13, p_0 sufficiently close to the recovery boundary implies the existence of a unique solution to Eqs. 3.62-3.63. So, unlike the methods of

Section 3.2.4, Section 3.3.4, and Section 3.3.5, the techniques of this section have local guarantees (for p_0 sufficiently close to the recovery boundary) of existence and uniqueness of a solution, and of converging to that solution rapidly, due to the linear structure discussed in Section 2.6.

3.3.7.4 Implementation Concern: Divergence of Runtimes

From an implementation perspective, it is important that the runtimes for algorithms which compute critical parameter values do not diverge rapidly to infinity as the critical parameter values are approached. Otherwise, computing these critical parameter values to desired accuracy quickly becomes infeasible as the accuracy is increased due to unreasonably high runtimes. Several results from Chapter II suggest that rapid divergence of runtimes as critical parameter values are approached may be a serious concern. Theorem 2.5.9 shows that as a critical parameter value is approached, the time to escape a ball of fixed radius around the controlling critical element diverges to infinity. By the proof of Lemma 2.6.18, for a parameter-dependent linear system, as a critical parameter value is approached, the time to escape to a fixed distance in the unstable direction is approximately logarithmic in the inverse of the distance to the critical parameter value. In particular, this implies that it diverges to infinity at a logarithmic rate as the parameter approaches its critical value, and one would expect (although it remains to be proven) similar behavior for the nonlinear system as well.

Fortunately, Theorem 2.6.13 shows that as any critical parameter value is approached, the norm of the trajectory sensitivities is approximately linear in the inverse of the distance to the critical parameter value. Thus, as discussed in Remark 2.6.19, as parameters approach a critical value the norm of the trajectory sensitivities is expected to grow exponentially faster than the escape time. Since the algorithms of this section proceed by maximizing the norm of the trajectory sensitivities (or, equivalently, minimizing their inverse), critical parameter values are computed to sufficient accuracy for these algorithms once the trajectory sensitivity norms are sufficiently large. The above then implies that, as parameters approach a critical value, it is expected that the norm of the trajectory sensitivity will converge to a sufficiently large size exponentially faster than the rate at which the escape times diverge towards infinity. From the proof of Theorem 2.6.13 it is clear that as parameters approach a critical value, the runtime is dominated by the time the system spends in a neighborhood of the controlling critical element: the escape time. Consequently, it is expected, and numerical experiments confirm, that even for high levels of accuracy in computing critical parameter values, runtimes do not rapidly diverge towards infinity, and practical implementation of the algorithms remains feasible.

3.3.8 Conclusion

Algorithms are developed to find the unique closest point on the recovery boundary in one parameter dimension, compute the recovery boundary in two parameter dimensions, and find the nearest point on the recovery boundary to any initial set of recover parameter values in arbitrary parameter dimensions. Unlike the algorithms of Section 3.2, the algorithms presented here do not require prior knowledge of the controlling UEP. The algorithms proceed by varying parameter values so as to minimize the infimum over time of the norm of the inverse trajectory sensitivities. They are illustrated on the test case of two classical machines, and are applied to a larger power system in Section 4.2. Recently, the algorithms have been updated to exploit the linear structure discussed in Section 2.6.

CHAPTER IV

Application to Power Systems

The algorithms developed in Chapter III are applied in this chapter to assess fault vulnerability in power systems. There are two main power system test cases of interest. The first, as discussed in Section 4.1, takes the IEEE 37-bus feeder, which is a model power distribution system, and modifies it by replacing 60% of the background load with induction motors. The purpose of this test case is a first step towards the study of the onset of Fault Induced Delayed Voltage Recovery (FIDVR), which is a phenomenon, likely driven by induction motors in residential air conditioners, in which voltages are slow or fail to recover in distribution grids following a fault. Results are presented for analyzing the onset of induction motor stalling. The second test case, as discussed in Section 4.2, is the IEEE 39-bus power system, which is a network of synchronous machines and includes the nonsmooth controller limits on the Active Voltage Regulators (AVRs) and Power System Stabilizers (PSSs). The purpose of this test case is to analyze the stability of a network of synchronous machines with uncertain background load and load dynamics. Results reveal an unexpected negative influence of constant impedance load on system recovery that would not have been identified otherwise.

4.1 Modified IEEE 37-Bus Feeder

4.1.1 Introduction

Distribution networks that serve large numbers of air conditioners are vulnerable to Fault Induced Delayed Voltage Recovery (FIDVR), [NER09]. FIDVR events are typically initiated by a fault near a substation, triggering a temporary drop in voltage that affects the associated distribution grid. When the local voltage drops, the induction motors driving air-conditioner compressors are affected; the electrical torque drops without a corresponding decrease in mechanical torque, causing the induction motor to decelerate. Induction motors draw more current when they reaccelerate, leading to a further reduction in local voltage. As a result, restoration of pre-fault voltages can be significantly delayed. Furthermore, if the fault is not cleared quickly enough, some motors will stall and the system may not be able to recover to pre-fault conditions.

Although the root cause for FIDVR events is well understood in a general sense [NER09], rigorous analysis of the instability mechanisms is still required. Whether induction motors stall or recover depends on system parameters, such as motor moments of inertia and fault clearing times. We consider a fixed disturbance, such as a three-phase fault on a particular line, and a particular set of parameters that affect induction motor stalling. The algorithms of Section 3.2 are applied to determine the set of recovery values for which all induction motors recover to pre-fault conditions. This provides a first step towards understanding the parameter dependence of cascading induction motor stalling during FIDVR events by characterizing the onset of induction motor stalling in a distribution network.

4.1.2 Numerical Experiments

The algorithms presented above were applied to a modified version of the IEEE 37-bus test feeder shown in Figure 4.1. The dimension of the dynamic states for this system is 100. In order to induce stressed conditions, the total load on the system was doubled. Then 60% of the load at each bus was replaced by an induction motor with the equivalent real power, leaving 40% background load. This is consistent with the peak air-conditioning load penetration of Southern California Edison inland load, [NER09]. The moments of inertia of the induction motors were chosen to be proportional to their real power, with a common proportionality constant used for all motors. This proportionality constant will be refer to as the moment of inertia scaling factor, SF, and forms one of the parameters of interest in the following studies, along with the fault duration and the voltage at the infinite bus during the fault.

We consider a nominal steady state in which no motors are stalled, apply a fault on the infinite bus at the head of the feeder, and then clear the fault after a specified duration. Stressed conditions led to cascaded stalling indicative of FIDVR events. Also note that the test cases below share a common controlling UEP, and that they appear to satisfy all the technical assumptions discussed in Chapter II.

A full 3-phase model could be used for the algorithm, but results shown here accurately

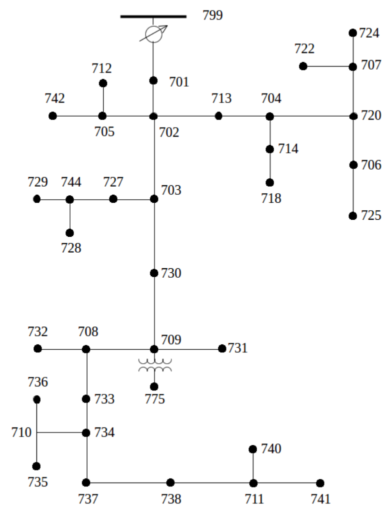

Figure 4.1: IEEE 37-bus test feeder.

capture the qualitative tradeoffs inherent in distribution grids with high penetration of AC load.

4.1.3 Results

Each of the algorithms presented above was applied to the modified 37-bus IEEE test feeder. First, Figure 4.2 depicts the (Euclidean) distance from the trajectory to the controlling UEP over time. The dashed line indicates a fixed distance of $r = 190$ from the UEP. Using the Newton-Raphson algorithm described earlier, the moment of inertia scaling factor SF was varied to ensure that the trajectory spent a fixed amount of time $t_{ball} = 0.2$ sec inside the specified ball. Close inspection of the figure shows that the intersections of the trajectory with the ball occur exactly 0.2 sec apart.

The next step was to run a continuation process, by treating t_{ball} as a second free parameter, in order to reach a parameter value that was arbitrarily close to the recovery boundary. Figure 4.3 illustrates this continuation process. Here, t_{ball} was increased from 0.2 sec inside the ball to 1.5 sec, leading to a decrease in SF of just over 0.02. When the slope of the curve approaches zero it implies that further increasing t_{ball} will have negligible effect on the value of the free parameter SF. Practically, this implies that SF is near its critical value $SF^{\star}$, indicating that the algorithm has succeeded in approaching arbitrarily closely to the recovery boundary.

After finding a point arbitrarily close to the recovery boundary, t_{ball} is fixed and a pair of parameters can be freed, often resulting in a 1-manifold in parameter space. Figure 4.4

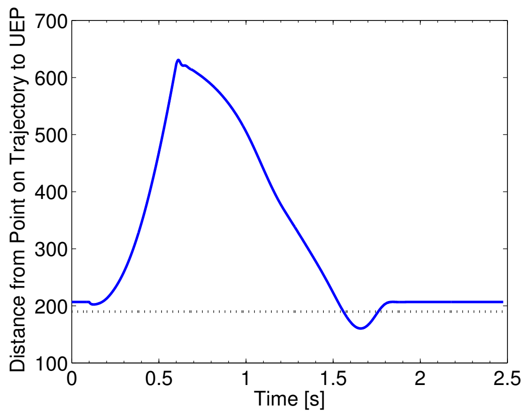

Figure 4.2: Distance from the trajectory to the unstable equilibrium point as a function of time, for a trajectory that has been driven inside a ball of radius 190 for a time $t_{ball} = 0.2$s. The radius 190 is depicted by the dotted line.

depicts the recovery boundary for bus 740 motor moment of inertia and bus 741 motor moment of inertia, the two motors at the end of the feeder. Points above the curve correspond to parameter sets that lead to recovery of all motors after the fault clears while points below the curve are parameter sets that cause one or more motors to stall after the fault.

In contrast, Figure 4.5 shows close to the recovery boundary for bus 740 motor moment of inertia and bus 725 motor moment of inertia. Note that bus 740 is at the end of the feeder whereas bus 725 is on another branch of the feeder with a tolerance of less than $5e-4$ from the recovery boundary in their parameter space.

Comparison of the figures illustrates that, for a fixed bus 740 motor moment of inertia, a much larger increase in motor moment of inertia is required for the distant motor at bus 725 than for the near motor at bus 741 to lead to recover of all motors after the fault clears. So, in the parameter space of the pairs of motor moments of inertia, the near motor has a much stronger influence on motor recovery than the one further away. This is representative of typical behavior observed in the network, where stalling cascades from the ends of the feeder upstream towards the substation.

Finally, we considered an arbitrary number P of free parameters p, specified values p_0, and seek the point $\hat{p}$ on the recovery boundary that is closest to p_0. For this example, the free parameters used were fault duration time and voltage at the infinite bus during the fault. The specified values p_0 are 0.35 sec and 0.55 pu, respectively. Figure 4.6 shows the iterations of the optimization algorithm. Note that convergence was achieved in just three iterations, and that the points generated at each iteration lie on the recovery boundary in

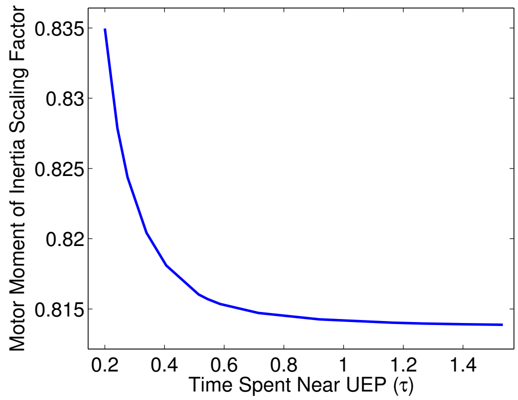

Figure 4.3: Time t_{ball} spent within a fixed radius of the unstable equilibrium point versus moment of inertia scaling factor SF.

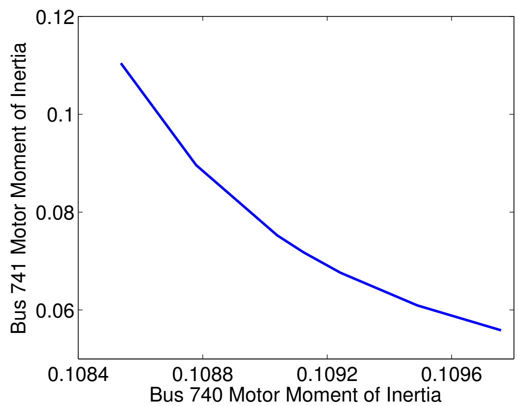

Figure 4.4: Recovery boundary for the moments of inertia at buses 740 and 741.

the directions successively estimated by the algorithm.

We also considered a case where the background active and reactive power load at every bus was freed, a total of 50 free parameters. This gave the smallest change in background load that would lead the system to be arbitrarily close to the recovery boundary. The optimization algorithm in this case converged in two iterations.

Explicit computation of recovery boundarys and knowledge of the point on the border of minimum distance from a given initial parameter set can be used to better understand the dependence of FIDVR events on parameters, and motivate approaches for reducing the likelihood and severity of these events in practice.

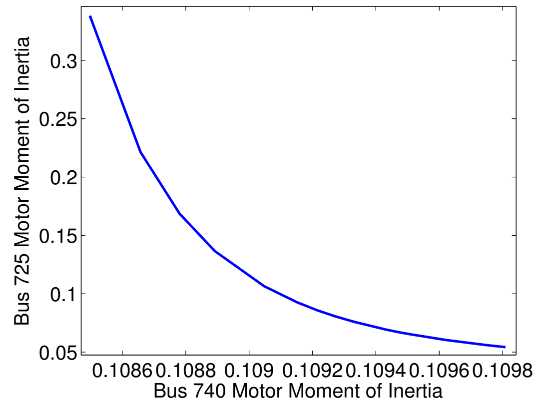

Figure 4.5: Recovery boundary for the moments of inertia at buses 740 and 725.

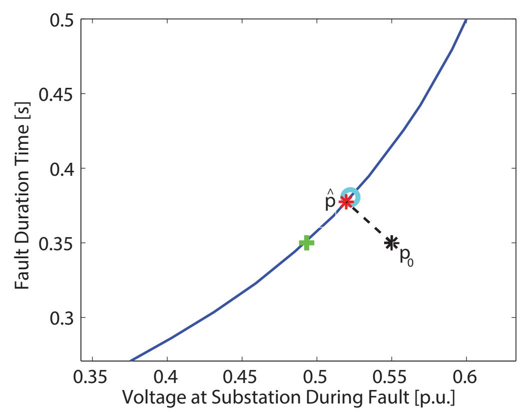

Figure 4.6: Optimization algorithm to find the nearest point on the recovery boundary. The specified value p_0, first iteration, second iteration, and third iteration are given by the black star, green plus, cyan circle, and red star, respectively. The black dashed line connects p_0 with the final solution, and is orthogonal to the recovery boundary.

4.1.4 Conclusion

The algorithms of Section 3.2 are applied to analyze the onset of induction motor stalling in a distribution network. The recovery boundary was computed in term of pairs of moments of inertia of various motors. The optimization algorithm was used to compute the nearest point on the recovery boundary in the parameter space of fault duration time and fault voltage dip. The optimization algorithm was also applied to find the smallest change in background load that resulted in instability.

4.2 IEEE 39-Bus New England Power System

4.2.1 Introduction

The algorithms of Section 3.3 are applied to the IEEE 39-bus New England power system. This is a 39-bus network containing 10 synchronous machines, and subject to a three-phase fault at the terminals of one of the generators. The purpose of this test case is to demonstrates the effectiveness of the algorithms on a more realistic network of synchronous machines with fourth order machine models, and including the nonsmooth PSS and AVR controller limits. Whereas the algorithms of Section 3.2 have already been demonstrated on a system of induction motors in Section 4.2 which exhibit dynamics similar to a gradient system, synchronous machines behave more like Hamiltonian systems. The algorithms of interest work best for systems which exhibit strong damping and minimal oscillations. As induction motors contribute real eigenvalues to their linearization and resemble a gradient system, they represent an easier test case for these algorithms. In contrast, a network of synchronous machines is typically weakly damped and possesses complex eigenvalues, hence oscillations, which represents a challenge to these algorithms. The aim of this work is to illustrate that the algorithms of Section 3.3 can overcome these challenges and, furthermore, can succeed without knowledge of the controlling UEP. Parameters of interest for assessing fault vulnerability are primarily background real and reactive power load, as well as the exponents governing the voltage dependence of these loads, as these are typically uncertain and time-varying in realistic power systems.

4.2.2 Numerical Experiments

The IEEE 39-bus New England power system test case shown in Fig. 4.7 will be used to illustrate the algorithms of Section 3.3. Generators are modeled using a 4th order machine

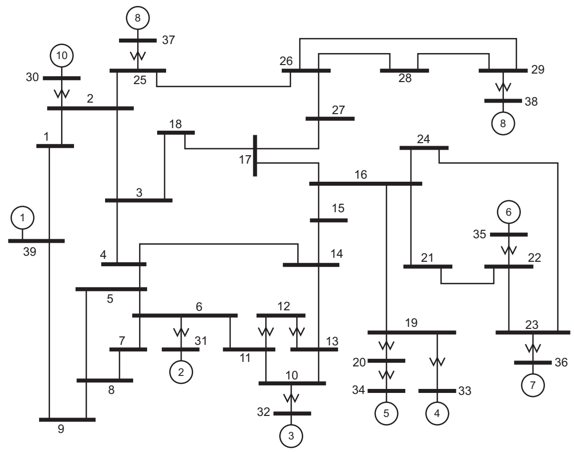

Figure 4.7: IEEE 39-bus New England power system.

model as given in [SP97], AVR and PSS models are based on the IEEE standard models PSS 1A and Exciter ST1 [IEE82], and the full set of dynamic equations and system parameters are given in [fSCTF15]. These models include the nonsmooth limits on AVR and PSS controller states. The dimensional of the dynamic states for this system is 111. A fault occurs at bus 16, and is cleared after $0.2s$. The fault is modeled as a switched, constant shunt reactance with $X_{fault} = 0.001$p.u.

Many model parameters of the system are of interest for recovery considerations. A background load scaling factor (SF) is introduced which multiplies the active and reactive power background loads at every bus in the network. As background load is time-varying and uncertain, it is a natural choice for assessing system recoverability. An AVR gain SF multiplies the AVR gain for every generator, and helps to capture the impact of controller tuning on system stability. The load active and reactive power are represented by the standard exponential voltage load model. The voltage exponents are set equal for all background loads, with one set of exponents for active power and another set for reactive power. As load dynamics are notoriously difficult to model, these parameters serve to quantize the impact of uncertain load behavior on system recoverability.

4.2.3 Results

The algorithms of Section 3.3 were demonstrated on the test case of Section 4.2.2. The dynamic states appearing in the first and second order trajectory sensitivities used in Section 3.3 were restricted to the generator dynamic states to avoid the possibility for the sensitivities of internal controller states to overshadow the sensitivities of the physical gen-

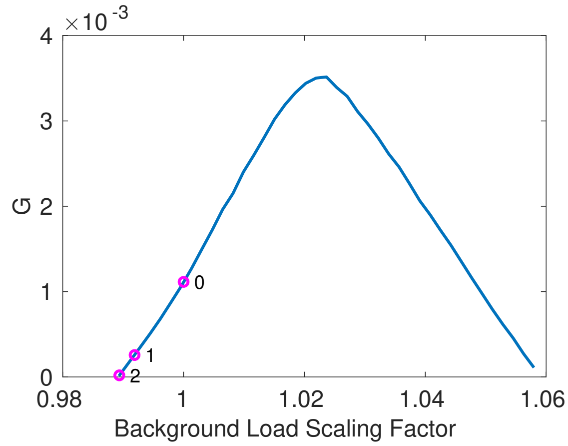

Figure 4.8: G as a function of background load scaling factor (blue line). The iterations of the one dimensional parameter space algorithm are labeled in order (circles).

erator states. The algorithm for one dimensional parameter space, which finds the closest point on the recovery boundary to an initial given parameter value, was applied with the parameter of interest being the background load SF.

Fig. 4.8 illustrates that the algorithm converged to the recovery boundary in just 2 iterations (3 simulations total). Furthermore, it shows that G is approximately linear near the recovery boundary, both when approaching the recovery boundary for high load and for low load, as predicted by the theory in Section 2.6. Hence, this algorithm rapidly and accurately determines the nearest point on the recovery boundary in one dimensional parameter space.

To observe the approach to instability, note that under stressed conditions the first generator to go unstable is generator 5. Fig. 4.9 shows the relative angle of generator 5 as a function of time for all values of the background load SF which were attained during the algorithm of Fig. 4.8. Observe that as the background load SF approaches the recovery boundary, the initial fluctuations in the relative angle of generator 5 grow larger, indicative of proximity to instability.

Once a point on the recovery boundary has been identified, the continuation method described in Section 3.3.3 can be applied to numerically trace the recovery boundary in two dimensional parameter space. This algorithm was applied for the two dimensional parameter space consisting of the AVR gain SF and the reactive load voltage exponent. The tolerance was set to $\epsilon = 10^{-5}$. Fig. 4.10 shows the set of recovery values, which we term the recovery region (RR), in this parameter space. If the AVR gains are reduced about 20% less than nominal (SF=1.0), the system will recover regardless of the reactive load exponent. For

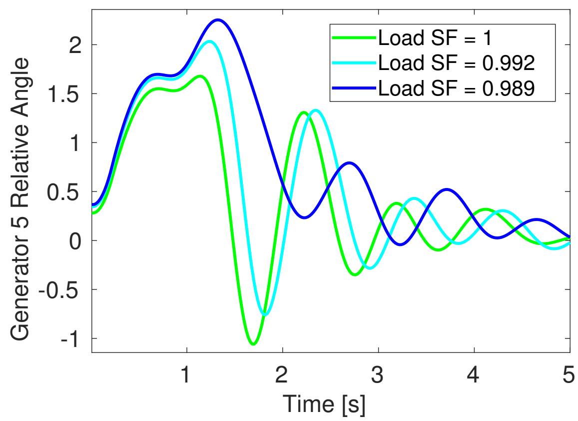

Figure 4.9: The angle of generator 5 relative to the angle of generator 2 as a function of time for the values of the background load scaling factor that correspond to the iterations of the one dimensional algorithm of Fig. 4.8.

slightly higher AVR gain SFs, a nonempty region outside the set of recovery values emerges. In this regime of the AVR gain SFs, there is a wide range of reactive load voltage exponent values which lie outside the set of recovery values. In particular, as the loads approach constant current and constant impedance loads, for nominal and slightly above nominal AVR gains the system will not be able to recover from the fault. In contrast, as the loads approach constant power loads, the system becomes more resilient and is able to recover. These observations run counter to standard intuition that constant power loads have a more detrimental impact on system stability than constant impedance loads. This behavior is counterintuitive, and serves as an example of how these algorithms have the potential to reveal unexpected dynamic behaviors which would not have been observed otherwise.

To observe the influence of variations in reactive load voltage exponents on system recovery, recall that generator 5 is the first generator to go unstable as a result of the fault. Fig. 4.11 shows the relative angle of generator 5 as a function of time for a fixed AVR gain SF and for several reactive load voltage exponent values. For sufficiently high or sufficiently low exponents, the angle shows smaller initial fluctuations and a more rapid return to synchrony. For exponents near the recovery boundary, initial fluctuations are larger and take a longer time to damp before resynchronization. For exponent values between these recovery boundary values, the system goes unstable and is unable to recover from the fault. Hence, for intermediary voltage exponents, angle fluctuations grow and lead to instability, whereas for high or low load voltage exponents the angles are able to recover from the transient disturbance.

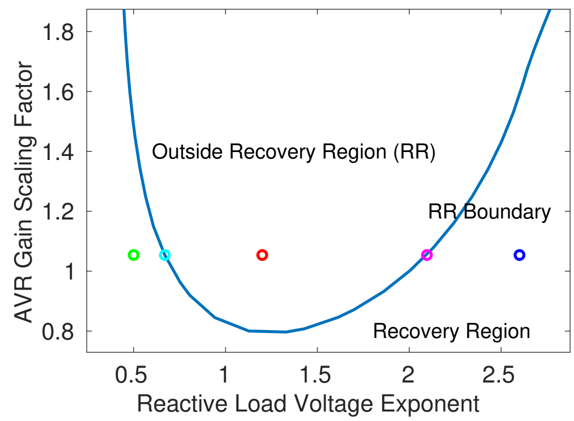

Figure 4.10: The recovery boundary and the set of recovery values (recovery region) in the two dimensional parameter space of AVR gain SF and reactive load voltage exponent. Colored circles indicate the parameter values whose corresponding dynamic behavior is shown in Fig. 4.11.

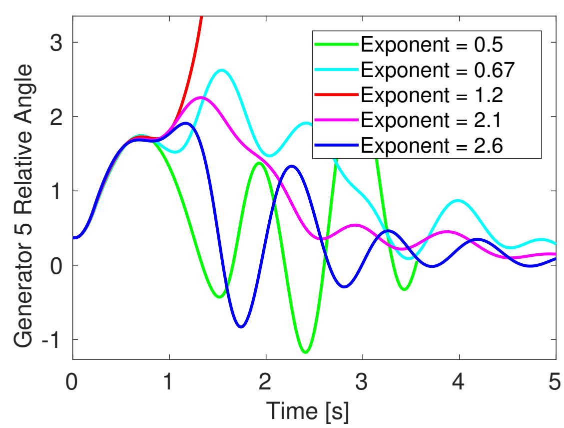

Figure 4.11: The angle of generator five relative to the angle of generator two as a function of time for an AVR gain SF of 1.054 and a range of reactive load voltage dynamics exponents.

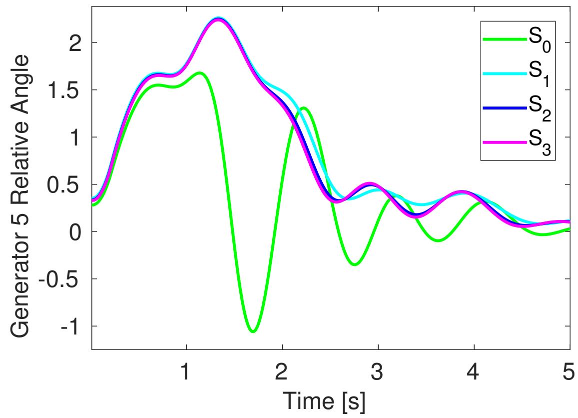

Figure 4.12: The angle of generator five relative to the angle of generator two as a function of time for the solutions of the optimization algorithm applied to the parameter sets S_1, S_2, and S_3, as well as for the nominal parameter values S_0.

The optimization algorithm described in Section 3.3.7.3 was applied to find the smallest changes in background load, load voltage exponents, and AVR gain SFs that would result in inability to recover from the disturbance. In particular, the following sets of parameters were considered. Let S_1 denote the set of real and reactive background loads, representing uncertainty in background load. Let S_2 consist of S_1 together with the set of real and reactive load voltage exponents, which captures the uncertainty not only in the level of background load but also its dynamics. Define S_3 to be the union of S_2 and the set of AVR gain SFs, for the purpose of illustrating the influence of controller setpoints on system recoverability. Note that S_1 consists of 38 parameters, S_2 of 76 parameters, and S_3 of 86 parameters. The optimization method was applied to each of these sets of parameter values, with $\epsilon = 10^{-5}$ chosen for the constraint $G(p) = \epsilon$. The algorithm converged on all of the parameter sets to within a tolerance of 10^{-6}. Fig. 4.12 shows the dynamic response of the relative angle of generator 5 for the trajectories corresponding to these optimal solutions, as well as for the set of nominal parameter values (denoted by S_0). Note that the relative angles for the optimal solutions are in close agreement with each other, and show larger initial fluctuations than for the nominal parameter values, indicative of proximity to instability.

The goal was to obtain parameter values which minimized the (square of the) Euclidean distance from the nominal parameter values p_0, while satisfying the constraint $G(p) = \epsilon$. In other words, these are the closest parameter values to p_0 which lie on the recovery boundary. Table 4.1 shows the value of this objective function - the Euclidean distance from p_0 squared - for the optimal solution of each set of parameter values. As additional parameters are

Parameter Set	Objective Function Value
S_1	0.0107
S_2	0.0079
S_3	0.0071

Table 4.1: Values of the objective function of Eq. 3.64 for the parameter values on the recovery boundary obtained as the optimal solutions to the higher dimensional parameter space optimization algorithm of Section 3.3.7.3 for the parameter sets S_1, S_2, and S_3.

introduced, more options for parameter values on the recovery boundary become available, some of which may be closer to p_0 than the previous options. Consequently, the objective function value, which measures the distance of the chosen optimal parameter value from p_0, decreased from S_1 to S_2, and again from S_2 to S_3. Overall, for all the chosen combinations of parameters, the higher dimensional parameter space algorithm converged to a parameter value on the recovery boundary which appears to be the closest such value to p_0.

4.2.4 Conclusion

The algorithms of Section 3.3 were successfully applied to the IEEE 39-bus New England power system to find the closest value of the background load SF that lies on the recovery boundary, to numerically trace the recovery boundary for reactive load voltage exponent and AVR gain SF, and to find the smallest change in background loads, load voltage exponents, and AVR gain SFs that lies on the recovery boundary. Unlike current industry practice, these algorithms are not approximate or conservative, and are able to efficiently compute the recovery boundary to arbitrary precision, or to find the closest point on it to some nominal set of parameter values. Furthermore, these algorithms do not require prior knowledge of the controlling unstable equilibrium point, unlike those applied in Section 4.1. The algorithms reveal an unexpected negative influence of constant impedance loads, and positive influence of constant power loads, on system recovery that would not have been identified otherwise.

CHAPTER V

Conclusions and Future Work

This work introduces a new method for assessing stability of nonlinear systems in the presence of parameter uncertainty. The approach is to consider a parameter dependent family of deterministic vector fields posssessing parameter dependent ICs. The set of recovery values consists of the parameter values whose corresponding trajectories converge to a particular SEP representing desired operation. The objective is to determine the recovery boundary: the boundary in parameter space of the set of recovery values. For many applications, such as power systems, this provides a quantitative measure of the margins for safe operation.

Critical parameter values, which are the parameter values whose corresponding ICs lie on the boundary of the RoA of the corresponding SEP, provide an explicit link between parameter space and state space behavior. If the recovery boundary consisted of critical parameter values, this could be exploited for algorithm design. Unfortunately, Example 2.2.1 illustrated that, even when the IC is continuous with respect to parameter, it is possible that there are no critical parameter values on the recovery boundary because the RoA boundary could vary discontinuously with parameter. It is shown that, for a C^1 vector field on a compact Riemannian manifold, if the vector field is Morse-Smale along the RoA boundary, then for sufficiently small changes in parameter values, the RoA boundary varies continuously with respect to parameter and permits a decomposition into a union of stable manifolds of the critical elements it contains. Example 2.3.3 gives a globally Morse-Smale vector field on Euclidean space in which the RoA varies discontinuously under arbitrarily small parameter variations. Hence, the above continuity and decomposition result is obtained for Euclidean space under additional generic assumptions and assumptions about the nonwandering set in a neighborhood of infinity. Using these results, it is shown that the closest point on the recovery boundary to an initial set of recovery values is a critical parameter value. The above results are extended to a class of hybrid systems which are vector fields with C^1 event-selected discontinuities that also have transversal intersection of switching surfaces.

Two collections of algorithms are devised to numerically compute critical parameter values, thereby computing points on the recovery boundary. To motivate the first group of algorithms, it is shown that, under transversality assumptions, for any critical parameter value there exists a controlling invariant set in the RoA boundary and a neighborhood of it such that the time the system trajectory spends inside that neighborhood is continuous over the set of recovery values, and diverges to infinity as the critical parameter value is approached from within the set of recovery values. To motivate the second collection of algorithms, it is shown that, assuming the controlling invariant set is hyperbolic, the supremum over time of the norm of the trajectory sensitivities with respect to parameters is continuous over the set of recovery values, and diverges to infinity as any critical parameter value is approached from within the set of recovery values. Hence, the first class of algorithms computes critical parameter values by maximizing the time the system trajectory spends in the neighborhood of the controlling invariant set, and the second class of algorithms compute critical parameter values by maximizing the supremum over time of the norm of the trajectory sensitivities. Thus, the abstract problem of computing critical parameter values is reduced in both cases to a concrete optimization problem which can be solved efficiently. Moreover, in the latter case it is shown that as parameter values approach the recovery boundary, the infimum over time of the inverse trajectory sensitivities approaches a dot product. This linearity is exploited to improve algorithm design and theoretical guarantees.

Both collections of algorithms require trajectory sensitivities for implementation. Therefore, modular methods are introduced to numerically integrate first, second, and third order trajectory sensitivities for a hybrid dynamical system model by exploiting sparse network structure to reduce computational time and storage requirements. These are used to develop algorithms for finding the unique point on the recovery boundary in the case of one parameter dimension, numerically computing the recovery boundary for two-dimensional parameter space, and finding the nearest point on the recovery boundary to some initial set of recover parameter values for arbitrary parameter dimensions. The algorithms represent significant improvements over current industry state-of-the-art by providing computationally efficient means of considering high dimensional parameter spaces for determining fault recoverability, computing points on the recovery boundary to arbitrary precision (without any conservativeness), and offer the potential for real-time fault vulnerability assessment during power system operations. The algorithms address major drawbacks of historical methods by not requiring energy functions, eliminating the conservativeness of earlier estimation methods, working for a collection of vector fields which exhibit restricted switching and limit behavior, incorporating parameters which influence post-disturbance dynamics, handling cases where the controlling invariant set is not an equilibrium point, and, for the second collection of algorithms, not requiring computation of the controlling invariant set.

The algorithms are applied to determine the margins for recovery of power systems from faults. For the test case of an IEEE 37-bus feeder modified to include induction motor load, the algorithm traced the recovery boundary for motor moment of inertia scaling factor and fault clearing time, and found the nearest point on the recovery boundary in the parameter space of background load. For the case of a modified IEEE 39-bus system of synchronous machines, the algorithm traced the recovery boundary for AVR gain scaling factor and reactive load voltage exponents, uncovering the counterintuitive result that constant power loads improve system resiliency while constant impedance loads reduce it for this case. The nearest point on the recovery boundary was found for parameter spaces including background loads, load voltage exponents, and AVR gain scaling factors, indicating the margins for safe operation.

Considerable challenges remain before industrial use of these algorithms is possible. Of these, one is the scaling of the computational difficulties as the size of the network approaches realistic industrial systems. In particular, computing trajectory sensitivities increases both the runtime and the storage requirements of a single time domain simulation compared to merely integrating the system dynamics. The idea is that, while each individual simulation takes longer when computing trajectory sensitivities, the algorithms presented here can converge in a small number of iterations to the desired recovery boundary, which dramatically reduces the total number of simulations required, thereby reducing the total computational cost. While work here has begun to exploit the structure of the differential-algebraic equations to improve efficiency (see Section 3.1.7), there is more work to be done in this direction. Furthermore, future work will involve demonstrating the ability to apply these algorithms to larger, more realistic power systems. In the process, further testing and refinement of the algorithms on a diverse set of cases should be undertaken.

Several additional questions have arisen as a result of this work. A common theme is attempting to provide theoretical understanding or justification for observations from numerical experiments. Future work should investigate the causes behind the empirical result that the recovery boundary has been smooth in all cases explored thus far, when one might expect to observe nonsmoothness. Another question is related to the linearity of the inverse trajectory sensitivities. Although the theory of Section 2.6 shows that they resemble a dot product for parameter values sufficiently close to the recovery boundary, numerical experiments have indicated that this linearity holds for large ranges in parameter values. Further work is required to elucidate the explanation for this behavior. Finally, the empirical effectiveness of the higher dimensional parameter space optimization methods should be investigated in greater detail.

While the focus of this thesis was on assessing vulnerability of the system to one particular disturbance, such as one particular fault, it is a natural next step to consider vulnerability to a collection of potential disturbances. In theory one could, for a fixed parameter space, perform the algorithms described in this thesis for every possible disturbance in the collection, identifying for each disturbance the closest point on the recovery boundary for that disturbance. This would give, for each disturbance, the radius of a ball in parameter space around the initial parameter values that is entirely contained in the set of recovery values for that disturbance. Choosing the smallest such radius (and recall that the norm used to measure parameter space distance can be defined with any positive semidefinite weighting matrix) yields a ball in parameter space in which recovery is guaranteed from all disturbances in the collection. Unfortunately, large collections of disturbances, as expected in practice, could lead to prohibitively large computation times. One possibility for mitigating this computational cost would be to attempt to identify offline a relatively small subset of disturbances to which the system is most vulnerable. Then, once these most dangerous disturbances have been identified, the above technique could be applied to this small subset during operational time scales to obtain more accurate vulnerability assessment. The most dangerous disturbances could be identified by applying the above technique offline and identifying the disturbances with the smallest radii for which recovery was guaranteed. Another potential means of reducing computational effort in the case of multiple disturbances would be to employ approximation techniques, such as approximating the derivative of the function G using low rank updates, a generalization of the secant approximation in one dimension. For the case of one dimensional parameter space, historical applications of this technique have shown strong performance [NPH02b, NPH02a] and extensions to arbitrary dimensional parameter space may be possible.

There are many other avenues for potential future work as well. In terms of dynamics, these techniques can be used to quantify the impact of nonsmooth controller limits and switching behavior on fault recovery, as well as the impact of replacing conventional generation with distributed resources interfaced via power converters. It appears that, as a consequence of convergence, the trajectory sensitivity method should reveal the unstable eigenvalue of the controlling UEP. In turn, this provides information about the primary mode of instability for a particular fault, and could be useful both for vulnerability assessment and control design. Finally, the methods used here could be applied to develop fault recovery-constrained optimal power flow algorithms, and to aid in controller design by varying controller parameters so as to enlarge the region of attraction in the directions of the most dangerous faults.

APPENDICES

APPENDIX A

Proof of Lemma 2.2.28

Figure 2.10 shows a visual depiction of the disk family in the proof of Lemma 2.3.24, which is an extension of Lemma 2.2.28 to incorporate perturbations to the vector field. Although the notation is slightly different, the figure is useful for following the proof of Lemma 2.2.28 here as well.

The claim is simple in the case that $W^s(X^i)$ has dimension zero. If the dimension of $W^s(X^i)$ is zero, let $\hat{U} = W^s_{\text{loc}}(X^i)$. This satisfies the claim, so assume that the dimension of $W^s(X^i)$ is greater than zero.

We begin by constructing a C^1 continuous disk family along D using the vector field V. Below this disk family will be extended to all of $W^u_{\text{loc}}(X^i)$ using the diffeomorphism $\tilde{f}$. Let $A = \partial W^u_{\text{loc}}(X^i)$. Then A is a C^1 immersed submanifold of W^u_{loc} of codimension one, and $\bigcup_{t<0} \phi_t(A) = W^u_{\text{loc}}(X^i)$. As the time-$t$ flow restricted to $W^u_{\text{loc}}(X^i)$ is a contraction for any $t < 0$, for each $y \in D$ there exists a unique $x = x(y) \in A$ and $t = t(y) > 0$ such that $\phi(t(y), y) = x(y) \in A$. By the tubular neighborhood theorem [Lee13, Theorem 6.24], as discussed in Section 2.2.2, there exists a C^1 continuous family of pairwise disjoint disks $\{D(x)\}_{x\in A}$ centered along A and transverse to $W^u(X^i)$. For each $y \in D$, let $D(y) = \phi_{-t(y)}(D(x(y)))$. Since A is a C^1 immersed submanifold of $W^u(X^i)$ of codimension one, there exists a real vector-valued function s defined on a neighborhood of $W^u_{\text{loc}}(X^i)$ in $W^u(X^i)$ such that s is a C^1 submersion, $A = s^{-1}(0)$, and the codomain of s has dimension one less than the dimension of $W^u(X^i)$. Let $y \in D$ and choose t such that $\phi(t, y) \in A$. Then $s \circ \phi(t, y) = 0$. Furthermore, $\frac{\partial}{\partial t} s \circ \phi(t, y) = ds_{\phi(t,y)}(V(\phi(t, y)))$. But, since s is constant on A, and by dimensionality the kernel of $ds_{\phi(t,y)}$ has dimension one, the kernel of $ds_{\phi(t,y)}$ is exactly $T_{\phi(t,y)}A$. However, since the time-t flow restricted to $W^u_{\text{loc}}(X^i)$ is a contraction,

V is transverse to A, so $V_{\phi(t,y)} \notin T_{\phi(t,y)}A$. Thus, $V_{\phi(t,y)}$ is not in the kernel of $ds_{\phi(t,y)}$, so $\frac{\partial}{\partial t} s \circ \phi(t,y) = ds_{\phi(t,y)}(V(\phi(t,y))) \neq 0$. Hence, by the implicit function theorem there exists a neighborhood N' of y in $W^u_{\text{loc}}(X^i)$ and a C^1 function $t : N' \to \mathbb{R}$ such that $y' \in N'$ implies that $s \circ \phi(t(y'), y') = 0$ or, equivalently, $\phi(t(y'), y') \in A$. Thus, the function $t = t(y)$ is C^1, and $x = x(y) = \phi(t(y), y)$ is also C^1 since ϕ and $t(y)$ are. Therefore, by the construction of the disk family above, the disk family $\{D(y)\}_{y \in D}$ is C^1 continuous. As D is compact, we may shrink $W^u_{\text{loc}}(X^i)$, shrink the disk family $\{D(x)\}_{x \in A}$, and choose N an open neighborhood in M such that $D \subset N \subset D_\epsilon$ and for each $y \in D$, $D(y) \subset N$.

Next the C^1 continuous disk family along $W^u_{\text{loc}}(X^i)$ is constructed for $\tilde{f}$ by backward iteration of the disk family above, and it is shown to contain an open neighborhood of X^i. If X^i is an equilibrium point, let $x_0 = X^i$ and let $S = M$. If X^i is a periodic orbit, let $x_0 \in X^i$ such that $\tilde{f}$ is the first return map for a cross section S centered at x_0. Let Y be a neighborhood of x_0 in S such that Y is diffeomorphic to the C^1 local coordinates of [JPdM82, p. 80-81] so that we can write $Y = W^s_{\text{loc}}(x_0) \times W^u_{\text{loc}}(x_0)$. In these coordinates, we can have $Y \subset \mathbb{R}^n = E^s \oplus E^u$, where E^s and E^u are the stable and unstable eigenspaces of $d\tilde{f}_{x_0}$, respectively, $W^s_{\text{loc}}(x_0) \subset E^s$, and $W^u_{\text{loc}}(x_0) \subset E^u$. Next we extend the disk family $\{D(x)\}_{x \in D}$ to all of $W^u_{\text{loc}}(x_0)$. For each $x \in W^u_{\text{loc}}(x_0) - \{x_0\}$, let $n(x)$ be the smallest $n > 0$ such that $\tilde{f}^{n(x)}(x) \in D$. Note that $n(x)$ is finite since such an intersection always exists by the construction of D. Let $x \in W^u_{\text{loc}}(x_0) - \{x_0\}$. Let $D^0(x) = D(\tilde{f}^{n(x)}(x))$ and for each $m \in 1, 2, ..., n(x)$, let $D^m(x)$ be the connected component of $(\tilde{f})^{-1}(D^{m-1}(x)) \bigcap Y$ that contains $\tilde{f}^{n(x)-m}(x)$ where we set $\tilde{f}^0(x) = x$. Then let $D(x) = D^{n(x)}(x)$. Let $D(x_0) = W^s_{\text{loc}}(x_0)$. This gives a family of connected C^1 disks centered along $W^u_{\text{loc}}(x_0)$. Let B_0 and B_I be the outer and inner topological boundaries of D, respectively. Let $y \in W^u_{\text{loc}}(x_0) - \{x_0\}$. Then $\tilde{f}^{n(y)} \in D$. Since the family $\{D(x)\}_{x \in D}$ is C^1 continuous, $(\tilde{f})^{-1}$ maps B_0 onto B_I, and for $x \in B_0$ $(\tilde{f})^{-1}(D(x)) = D((\tilde{f})^{-1}(x))$, the disk family $\{D(x)\}_{x \in W^u_{\text{loc}}(x_0)}$ is C^1 continuous on a neighborhood of $\tilde{f}^{n(y)}$. Hence, applying the diffeomorphism $\tilde{f}^{-n(y)}$, by the construction of the disk family above this implies that $\{D(x)\}_{x \in W^u_{\text{loc}}(x_0)}$ is C^1 continuous on a neighborhood of y. Since the family $\{D(x)\}_{x \in D}$ is C^1 continuous and transverse to $W^u_{\text{loc}}(x_0)$, by the Inclination Lemma [Pal69], the family $\{D(\tilde{f}^n(x))\}_{x \in D}$ converges uniformly in the C^1 topology to $W^s_{\text{loc}}(x_0)$ as $n \to \infty$. Hence, $\{D(x)\}_{x \in W^u_{\text{loc}}(x_0) - \{x_0\}}$ converges uniformly to $W^s_{\text{loc}}(x_0)$ as $x \to x_0$. This shows that the family $\{D(x)\}_{x \in W^u_{\text{loc}}(x_0)}$ defined above is C^1 continuous at x_0. Thus, $\{D(x)\}_{x \in W^u_{\text{loc}}(x_0)}$ is a C^1 continuous family of disks transverse to $W^u_{\text{loc}}(x_0)$ and such that $x \neq y$ with $x, y \in W^u_{\text{loc}}(x_0)$ implies that $D(x) \cap D(y) = \emptyset$. Hence, there exists a C^1 injective function $\tilde{F} : B^s_r(0) \times W^u_{\text{loc}}(x_0) \to M$ such that $\tilde{F}(B^s_r(0) \times \{x\}) = D(x)$ for every $x \in W^u_{\text{loc}}(x_0)$. Thus, since $\tilde{F}$ is a continuous injection between manifolds of the same dimension, by invariance of domain [Hat01, Theorem 2B.3] $\hat{U} = \tilde{F}(\text{int } B^s_r(0) \times \text{int } W^u_{\text{loc}}(x_0))$ is an open neighborhood of int $W^u_{\text{loc}}(x_0)$ in M. By construction, for every $x \in \hat{U} - W^s_{\text{loc}}(x_0)$, its forward orbit intersects $D(y)$ for some $y \in D$ in finite positive time. But by construction, $D(y) \subset N$ for all $y \in D$. Hence, $\bigcup_{t<0} \phi_t(N) \bigcup W^s_{\text{loc}}(x_0)$ contains $\hat{U}$ an open neighborhood of X^i.

APPENDIX B

Proof of Lemma 2.2.37

Figure 2.10 shows a visual depiction of the disk family in the proof of Lemma 2.3.24, which is a slight extension of Lemma 2.2.37. Although the notation is slightly different, the figure is useful for following the proof of Lemma 2.2.37 here as well.

Let $\tilde{f}_p$ denote the time-one flow (if $X^i_{p_0}$ is an equilibrium point) or a C^1 first return map (if $X^i_{p_0}$ is a periodic orbit) corresponding to parameter value p. First note that $W^s_{\text{loc}}(X^i(p))$, $W^u_{\text{loc}}(X^i(p))$, the exponential maps, and the flows are C^1 continuous with respect to parameter value, so that p close to p_0 implies that the disk family $\{D(x)\}_{x\in D_p}$ of the perturbed vector field will be uniformly C^1-close to that of the original. Hence, $(D_p)_\epsilon$ will be C^1-close to $(D_{p_0})_\epsilon$. So, J sufficiently small implies that there exists N' open such that for $p \in J$, $D_p \subset N' \subset (D_p)_\epsilon$. Let $x_0(p)$ be defined in the natural way so that it is C^1 with respect to parameter, $x_0(p) = X^i(p)$ if $X^i(p)$ is an equilibrium point, and if $X^i(p)$ is a periodic orbit then $x_0(p) \in X^i(p)$. Construct the full disk family $\{D(x)\}_{x\in W^u_{\text{loc}}(x_0(p))}$ for the perturbed vector field as in the proof of Lemma 2.2.28. In the proof of the Inclination Lemma, $\hat{g}$ sufficiently C^1-close to $\hat{f}$ implies that the same bounds on $\hat{g}$ and its partial derivatives will hold as for $\hat{f}$. Hence, the uniform bounds on the inclinations (slopes) obtained for the transverse disk family $\{D(x)\}_{x\in D_{p_0}}$ can also be taken to apply to $\{D(x)\}_{x\in D_p}$ for p sufficiently close to p_0. In particular, there exists $Z_N > 0$ such that $n \geq Z_N$ implies that for any $x \in D_{p_0}$, $D(\tilde{f}^{-n}_{p_0}(x))$ is δ C^1-close to $W^s_{\text{loc}}(x_0(p_0))$ and for any $x \in D_p$, $D(\tilde{f}^{-n}_p(x))$ is δ C^1-close to $W^s_{\text{loc}}(x_0(p))$. This implies that for every $x \in D_{p_0}$ and $x' \in D_p$, for every $n \geq Z_N$, $D(\tilde{f}^{-n}_{p_0}(x))$ is 3δ C^1-close to $D(\tilde{f}^{-n}_p(x'))$. As $\{D(x)\}_{x\in D_{p_0}}$ and $\{D(x)\}_{x\in D_p}$ are uniformly C^1-close and Z_N is finite, for p sufficiently close to p_0 we may have that the two disk families $\{D(\tilde{f}^{-n}_{p_0}(x))\}_{x\in D_{p_0}}$ and $\{D(\tilde{f}^{-n}_p(x))\}_{x\in D_p}$ are uniformly 3δ C^1-close for $n \leq Z_N$. Hence, combining the above, we have that the disk families $\{D(x)\}_{x \in W^u_{\text{loc}}(x_0(p_0))}$ and $\{D(x)\}_{x \in W^u_{\text{loc}}(x_0(p))}$ are uniformly 3δ C^1-close. Constructing $\tilde{F}_p$ for $\tilde{f}_p$ analogously to the construction in Lemma 2.2.28 of $\tilde{F}$, this implies that $\tilde{F}_p(\text{int } B^s_r(0) \times \text{int } W^u_{\text{loc}}(x_0(p)))$ has Hausdorff distance no greater than 3δ from $\tilde{F}_{p_0}(\text{int } B^s_r(0) \times \text{int } W^u_{\text{loc}}(x_0(p_0)))$. In particular, this implies that $\delta > 0$ and $\hat{U}$ can be chosen sufficiently small, such that for J sufficiently small, $p \in J$ implies that $\tilde{F}_p(\text{int } B^s_r(0) \times \text{int } W^u_{\text{loc}}(x_0(p))) \supset \hat{U}$. Hence, for every $x \in \hat{U} - W^s_{\text{loc}}(x_0(p))$, the forward orbit of x intersects N' in finite time.

APPENDIX C

Proof of Lemma 2.3.24

Figure 2.10 shows a visual depiction of the disk family in the proof of Lemma 2.3.24 here. The majority of the proof, including the construction of a disk family and its persistence under perturbation, follows from the proofs of Lemma 2.2.28 and Lemma 2.2.37 in Appendices A-B.

We construct $\hat{U} \subset U_i'$ such that the backwards orbit of any point in $\hat{U}$ under $\hat{g}$ sufficiently close to $\hat{f}$ remains in $\hat{U}$ at least until it enters N_i. Choose $r' \in (0, r)$ such that the closure of $\hat{F}(\text{int } B^u_{r'}(0) \times \text{int } W^s_{\text{loc}}(x_0))$ is contained in U_i'. Then there exists $\delta > 0$ such that the distance between their boundaries is greater than δ. The argument of Appendix B shows that for γ sufficiently small, $\hat{F}'(\text{int } B^u_{r'}(0) \times \text{int } W^{s'}_{\text{loc}}(x_0')) \subset U_i'$ since its Hausdorff distance from $\hat{F}(\text{int } B^u_{r'}(0) \times \text{int } W^s_{\text{loc}}(x_0))$ can be made less than δ. Repeating that argument again, one can choose $\hat{U} \subset \hat{F}(\text{int } B^u_{r'}(0) \times \text{int } W^s_{\text{loc}}(x_0))$ open such that $x_0' \in \hat{U}$ for γ sufficiently small. As $\hat{U} \subset \hat{F}'(\text{int } B^u_{r'}(0) \times \text{int } W^{s'}_{\text{loc}}(x_0')) \subset U_i'$, this implies that for every point in $\hat{U}$ its backwards orbit is contained in U_i' at least until it enters N_i.

Finally, we return from diffeomorphisms to vector fields. Let W be an open neighborhood of V_0 such that $V \in W$ implies that g, constructed for V analogously to the construction of $\hat{f}$ for V_0 above, is γ C^1-close to $\hat{f}$. If $X^i(V_0)$ is an equilibrium point, this completes the proof. If $X^i(V_0)$ is a periodic orbit, let $T > 0$ finite be such that for every point $x \in S$, the first return time of x is less than T. Redefine $U_i := \bigcup_{t \in (-T,T)} \phi_t(U_i)$ and $\hat{U} := \bigcup_{t \in (-T,T)} \phi_t(\hat{U})$. Then U_i is open in M and contains $W^s_{\text{loc}}(X^i(V_0))$.

APPENDIX D

Derivation of Trajectory Sensitivities at Discrete Events

Differentiate Eqs. 3.3 with respect to the components of the initial condition of the dynamic states (components of $x(0)$). For first derivatives, differentiate with respect to x_j to obtain

$$\Phi_{ij}^- = \Phi_{ij} + f_i^- \tau_j \tag{D.1}$$

$$0 = s_{im}\Phi_{mj}^- + s_{i\beta}\Psi_{\beta j}^- \tag{D.2}$$

$$0 = g_{\alpha m}^- \Phi_{mj}^- + g_{\alpha\beta}^- \Psi_{\beta j}^- \tag{D.3}$$

$$\Phi_{ij}^+ = h_{im}\Phi_{mj}^- + h_{i\beta}\Psi_{\beta j}^- \tag{D.4}$$

$$0 = g_{\alpha m}^+ \Phi_{mj}^+ + g_{\alpha\beta}^+ \Psi_{\beta j}^+. \tag{D.5}$$

Then Eq. D.1 is equal to Eq. 3.17. Substituting the expression for Φ_{ij}^- from Eq. D.1 into Eqs. D.2-D.3 and rearranging yields Eq. 3.16. Combining Eqs. D.4-D.5 and rearranging yields Eq. 3.18. This gives the first order trajectory sensitivity equations.

For second order, differentiate Eqs. D.1-D.5 with respect to x_k to get

$$\Phi^-_{ijk} = \Phi_{ijk} + f^-_{im}\Phi_{mk}\tau_j + f^-_{i\alpha}\Psi_{\alpha k}\tau_j + f^-_i\tau_{jk} \tag{D.6}$$

$$0 = s_{imn}\Phi^-_{nk}\Phi^-_{mj} + s_{im\alpha}\Psi^-_{\alpha k}\Phi^-_{mj} + s_{im}\Phi^-_{mjk} + s_{i\alpha m}\Phi^-_{mk}\Psi^-_{\alpha j} + s_{i\alpha\beta}\Psi^-_{\beta k}\Psi^-_{\alpha j} + s_{i\alpha}\Psi^-_{\alpha jk} \tag{D.7}$$

$$0 = g^-_{\alpha mn}\Phi^-_{nk}\Phi^-_{mj} + g^-_{\alpha m\beta}\Psi^-_{\beta k}\Phi^-_{mj} + g^-_{\alpha m}\Phi^-_{mjk} + g^-_{\alpha\beta m}\Phi^-_{mk}\Psi^-_{\beta j} + g^-_{\alpha\beta\gamma}\Psi^-_{\gamma k}\Psi^-_{\beta j} + g^-_{\alpha\beta}\Psi^-_{\beta jk} \tag{D.8}$$

$$\Phi^+_{ijk} = h_{imn}\Phi^-_{nk}\Phi^-_{mj} + h_{im\alpha}\Psi^-_{\alpha k}\Phi^-_{mj} + h_{im}\Phi^-_{mjk} + h_{i\alpha m}\Phi^-_{mk}\Psi^-_{\alpha j} + h_{i\alpha\beta}\Psi^-_{\beta k}\Psi^-_{\alpha j} + h_{i\alpha}\Psi^-_{\alpha jk} \tag{D.9}$$

$$0 = g^+_{\alpha mn}\Phi^+_{nk}\Phi^+_{mj} + g^+_{\alpha m\beta}\Psi^+_{\beta k}\Phi^+_{mj} + g^+_{\alpha m}\Phi^+_{mjk} + g^+_{\alpha\beta m}\Phi^+_{mk}\Psi^+_{\beta j} + g^+_{\alpha\beta\gamma}\Psi^+_{\gamma k}\Psi^+_{\beta j} + g^+_{\alpha\beta}\Psi^+_{\beta jk}. \tag{D.10}$$

Then Eq. D.6 is equal to Eq. 3.20. Substituting the expression for Φ^-_{ijk} from Eq. D.6 into Eqs. D.7-D.8 and rearranging yields Eq. 3.19. Combining Eqs. D.9-D.10 and rearranging yields Eq. 3.21. This gives the second order trajectory sensitivity equations.

For third order, differentiate Eqs. D.6-D.10 with respect to x_k to get

$$\begin{aligned}\Phi^-_{ijkl} &= \Phi_{ijkl} + f^-_{imn}\Phi_{nl}\Phi_{mk}\tau_j + f^-_{im\alpha}\Psi_{\alpha l}\Phi_{mk}\tau_j + f^-_{im}\Phi_{mkl}\tau_j + f^-_{im}\Phi_{mk}\tau_{jl} + f^-_{i\alpha m}\Phi_{ml}\Psi_{\alpha k}\tau_j\\ &\quad + f^-_{i\alpha\beta}\Psi_{\beta l}\Psi_{\alpha k}\tau_j + f^-_{i\alpha}\Psi_{\alpha kl}\tau_j + f^-_{i\alpha}\Psi_{\alpha k}\tau_{jl} + f^-_{im}\Phi_{ml}\tau_{jk} + f^-_{i\alpha}\Psi_{\alpha l}\tau_{jk} + f^-_i\tau_{jkl}\end{aligned} \tag{D.11}$$

$$\begin{aligned}0 &= s_{imno}\Phi^-_{ol}\Phi^-_{nk}\Phi^-_{mj} + s_{imn\alpha}\Psi^-_{\alpha l}\Phi^-_{nk}\Phi^-_{mj} + s_{imn}\Phi^-_{nkl}\Phi^-_{mj} + s_{imn}\Phi^-_{nk}\Phi^-_{mjl}\\ &\quad + s_{im\alpha n}\Phi^-_{nl}\Psi^-_{\alpha k}\Phi^-_{mj} + s_{im\alpha\beta}\Psi^-_{\beta l}\Psi^-_{\alpha k}\Phi^-_{mj} + s_{im\alpha}\Psi^-_{\alpha kl}\Phi^-_{mj} + s_{im\alpha}\Psi^-_{\alpha k}\Phi^-_{mjl} + s_{imn}\Phi^-_{nl}\Phi^-_{mjk}\\ &\quad + s_{im\alpha}\Psi^-_{\alpha l}\Phi^-_{mjk} + s_{im}\Phi^-_{mjkl} + s_{i\alpha mn}\Phi^-_{nl}\Phi^-_{mk}\Psi^-_{\alpha j} + s_{i\alpha m\beta}\Psi^-_{\beta l}\Phi^-_{mk}\Psi^-_{\alpha j} + s_{i\alpha m}\Phi^-_{mkl}\Psi^-_{\alpha j}\\ &\quad + s_{i\alpha m}\Phi^-_{mk}\Psi^-_{\alpha jl} + s_{i\alpha\beta m}\Phi^-_{ml}\Psi^-_{\beta k}\Psi^-_{\alpha j} + s_{i\alpha\beta\gamma}\Psi^-_{\gamma l}\Psi^-_{\beta k}\Psi^-_{\alpha j} + s_{i\alpha\beta}\Psi^-_{\beta kl}\Psi^-_{\alpha j}\\ &\quad + s_{i\alpha\beta}\Psi^-_{\beta k}\Psi^-_{\alpha jl} + s_{i\alpha m}\Phi^-_{ml}\Psi^-_{\alpha jk} + s_{i\alpha\beta}\Psi^-_{\beta l}\Psi^-_{\alpha jk} + s_{i\alpha}\Psi^-_{\alpha jkl}\end{aligned} \tag{D.12}$$

$$
\begin{aligned}
0 &= g^-_{\alpha mno}\Phi^-_{ol}\Phi^-_{nk}\Phi^-_{mj} + g^-_{\alpha mn\beta}\Psi^-_{\beta l}\Phi^-_{nk}\Phi^-_{mj} + g^-_{\alpha mn}\Phi^-_{nkl}\Phi^-_{mj} + g^-_{\alpha mn}\Phi^-_{nk}\Phi^-_{mjl} \\
&+ g^-_{\alpha m\beta n}\Phi^-_{nl}\Psi^-_{\beta k}\Phi^-_{mj} + g^-_{\alpha m\beta\gamma}\Psi^-_{\gamma l}\Psi^-_{\beta k}\Phi^-_{mj} + g^-_{\alpha m\beta}\Psi^-_{\beta kl}\Phi^-_{mj} + g^-_{\alpha m\beta}\Psi^-_{\beta k}\Phi^-_{mjl} + g^-_{\alpha mn}\Phi^-_{nl}\Phi^-_{mjk} \\
&+ g^-_{\alpha m\beta}\Psi^-_{\beta l}\Phi^-_{mjk} + g^-_{\alpha m}\Phi^-_{mjkl} + g^-_{\alpha\beta mn}\Phi^-_{nl}\Phi^-_{mk}\Psi^-_{\beta j} + g^-_{\alpha\beta m\gamma}\Psi^-_{\gamma l}\Phi^-_{mk}\Psi^-_{\beta j} + g^-_{\alpha\beta m}\Phi^-_{mkl}\Psi^-_{\beta j} \\
&+ g^-_{\alpha\beta m}\Phi^-_{mk}\Psi^-_{\beta jl} + g^-_{\alpha\beta\gamma m}\Phi^-_{ml}\Psi^-_{\gamma k}\Psi^-_{\beta j} + g^-_{\alpha\beta\gamma\delta}\Psi^-_{\delta l}\Psi^-_{\gamma k}\Psi^-_{\beta j} + g^-_{\alpha\beta\gamma}\Psi^-_{\gamma kl}\Psi^-_{\beta j} \\
&+ g^-_{\alpha\beta\gamma}\Psi^-_{\gamma k}\Psi^-_{\beta jl} + g^-_{\alpha\beta m}\Phi^-_{ml}\Psi^-_{\beta jk} + g^-_{\alpha\beta\gamma}\Psi^-_{\gamma l}\Psi^-_{\beta jk} + g^-_{\alpha\beta}\Psi^-_{\beta jkl} \qquad \text{(D.13)} \\
\Phi^+_{ijkl} &= h_{imno}\Phi^-_{ol}\Phi^-_{nk}\Phi^-_{mj} + h_{imn\alpha}\Psi^-_{\alpha l}\Phi^-_{nk}\Phi^-_{mj} + h_{imn}\Phi^-_{nkl}\Phi^-_{mj} + h_{imn}\Phi^-_{nk}\Phi^-_{mjl} \\
&+ h_{im\alpha n}\Phi^-_{nl}\Psi^-_{\alpha k}\Phi^-_{mj} + h_{im\alpha\beta}\Psi^-_{\beta l}\Psi^-_{\alpha k}\Phi^-_{mj} + h_{im\alpha}\Psi^-_{\alpha kl}\Phi^-_{mj} + h_{im\alpha}\Psi^-_{\alpha k}\Phi^-_{mjl} + h_{imn}\Phi^-_{nl}\Phi^-_{mjk} \\
&+ h_{im\alpha}\Psi^-_{\alpha l}\Phi^-_{mjk} + h_{im}\Phi^-_{mjkl} + h_{i\alpha mn}\Phi^-_{nl}\Phi^-_{mk}\Psi^-_{\alpha j} + h_{i\alpha m\beta}\Psi^-_{\beta l}\Phi^-_{mk}\Psi^-_{\alpha j} + h_{i\alpha m}\Phi^-_{mkl}\Psi^-_{\alpha j} \\
&+ h_{i\alpha m}\Phi^-_{mk}\Psi^-_{\alpha jl} + h_{i\alpha\beta m}\Phi^-_{ml}\Psi^-_{\beta k}\Psi^-_{\alpha j} + h_{i\alpha\beta\gamma}\Psi^-_{\gamma l}\Psi^-_{\beta k}\Psi^-_{\alpha j} + h_{i\alpha\beta}\Psi^-_{\beta kl}\Psi^-_{\alpha j} \\
&+ h_{i\alpha\beta}\Psi^-_{\beta k}\Psi^-_{\alpha jl} + h_{i\alpha m}\Phi^-_{ml}\Psi^-_{\alpha jk} + h_{i\alpha\beta}\Psi^-_{\beta l}\Psi^-_{\alpha jk} + h_{i\alpha}\Psi^-_{\alpha jkl} \qquad \text{(D.14)} \\
0 &= g^+_{\alpha mno}\Phi^+_{ol}\Phi^+_{nk}\Phi^+_{mj} + g^+_{\alpha mn\beta}\Psi^+_{\beta l}\Phi^+_{nk}\Phi^+_{mj} + g^+_{\alpha mn}\Phi^+_{nkl}\Phi^+_{mj} + g^+_{\alpha mn}\Phi^+_{nk}\Phi^+_{mjl} \\
&+ g^+_{\alpha m\beta n}\Phi^+_{nl}\Psi^+_{\beta k}\Phi^+_{mj} + g^+_{\alpha m\beta\gamma}\Psi^+_{\gamma l}\Psi^+_{\beta k}\Phi^+_{mj} + g^+_{\alpha m\beta}\Psi^+_{\beta kl}\Phi^+_{mj} + g^+_{\alpha m\beta}\Psi^+_{\beta k}\Phi^+_{mjl} + g^+_{\alpha mn}\Phi^+_{nl}\Phi^+_{mjk} \\
&+ g^+_{\alpha m\beta}\Psi^+_{\beta l}\Phi^+_{mjk} + g^+_{\alpha m}\Phi^+_{mjkl} + g^+_{\alpha\beta mn}\Phi^+_{nl}\Phi^+_{mk}\Psi^+_{\beta j} + g^+_{\alpha\beta m\gamma}\Psi^+_{\gamma l}\Phi^+_{mk}\Psi^+_{\beta j} + g^+_{\alpha\beta m}\Phi^+_{mkl}\Psi^+_{\beta j} \\
&+ g^+_{\alpha\beta m}\Phi^+_{mk}\Psi^+_{\beta jl} + g^+_{\alpha\beta\gamma m}\Phi^+_{ml}\Psi^+_{\gamma k}\Psi^+_{\beta j} + g^+_{\alpha\beta\gamma\delta}\Psi^+_{\delta l}\Psi^+_{\gamma k}\Psi^+_{\beta j} + g^+_{\alpha\beta\gamma}\Psi^+_{\gamma kl}\Psi^+_{\beta j} \\
&+ g^+_{\alpha\beta\gamma}\Psi^+_{\gamma k}\Psi^+_{\beta jl} + g^+_{\alpha\beta m}\Phi^+_{ml}\Psi^+_{\beta jk} + g^+_{\alpha\beta\gamma}\Psi^+_{\gamma l}\Psi^+_{\beta jk} + g^+_{\alpha\beta}\Psi^+_{\beta jkl}. \qquad \text{(D.15)}
\end{aligned}
$$

Then Eq. D.11 is equal to Eq. 3.23. Substituting the expression for Φ^-_{ijkl} from Eq. D.11 into Eqs. D.12-D.13 and rearranging yields Eq. 3.22. Combining Eqs. D.14-D.15 and rearranging yields Eq. 3.24. This gives the third order trajectory sensitivity equations.

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