Topics in Stochastic Analysis and Control

by

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ABSTRACT

In this dissertation, problems in stochastic analysis and control are investigated, which include mathematical finance, online learning, and mean field game. For mathematical finance, 1) a martingale optimal transport problem with bounded volatility is studied, which allows to calibrate not only current observation (option prices) but also historical data (stock prices); see Chapter II, 2) the embedding problem in multi-dimension is solved via excursion theory in probability; see Chapter III, 3) size of most stable subgraphs of random graphs, k-core, is determined by using branching processes; see Chapter IV. For online learning, 1) an unprecedented solution to the 4-expert problem with finite stopping is provided, via an explicit construction of the solution to a nonlinear partial differential equation; see Chapter V 2) prediction problems with a limited adversary are studied using partial differential equation tools; see Chapter VI and VII. For mean field game, 1) the convergence phenomenon of $N+1$-player Nash equilibrium is studied by the entropy solution to scalar conservative laws; see Chapter VIII, 2) infinite horizon mean field type control and game are solved via McKean-Vlasov forward backward stochastic differential equations; see Chapter IX.

CHAPTER I

Introduction

This thesis is devoted to several problems in stochastic analysis and optimal control.

Chapter II is based on [34]. It focuses on martingale optimal transport problems when the martingales are assumed to have bounded quadratic variation. First, we give a result that characterizes the existence of a probability measure satisfying some convex transport constraints in addition to having given initial and terminal marginals. Several applications are provided: martingale measures with volatility uncertainty, optimal transport with capacity constraints, and Skorokhod embedding with bounded times. Next, we extend this result to multi-marginal constraints. Finally, we consider an optimal transport problem with constraints and obtain its Kantorovich duality. A corollary of this result is a monotonicity principle which gives a geometric way of identifying the optimizer.

Chapter III is based on [32]. It investigates an embedding problem of Walsh Brownian motion. Let (Z, κ) be a Walsh Brownian motion with spinning measure κ. Suppose μ is a probability measure on $\mathbb{R}^n$. We first provide a necessary and sufficient condition for μ to be a stopping distribution of (Z, κ). Then if the stopped process is required to be uniformly integrable, we show that such a stopping time exists if and only if μ is *balanced*. Next, under the assumption of being balanced, we identify the minimal stopping times with those τ such that the stopped process Z^τ is uniformly integrable. Finally, we generalize Vallois' embedding, and prove that it minimizes the expectation $\mathbb{E}[\Psi(L^Z_\tau)]$ among all the admissible solutions τ, where Ψ is a strictly convex function and $(L^Z_t)_{t\geq 0}$ is the local time of the Walsh Brownian motion at the origin.

Chapter IV is based on [23]. We determine the size of k-core in a large class of dense graph sequences. Let G_n be a sequence of undirected, n-vertex graphs with edge weights $\{a^n_{i,j}\}_{i,j\in[n]}$ that converges to a kernel $W : [0,1]^2 \to [0,+\infty)$ in the cut metric. Keeping an edge (i,j) of G_n with probability $\min\{a^n_{i,j}/n, 1\}$ independently, we obtain a sequence of random graphs $G_n(\frac{1}{n})$. Denote by $C_k(G)$ the size of k-core in graph G, by X^W the branching process associated with the kernel W, by $\mathcal{A}$ the property of a branching process that the initial particle has at least k children, each of which has at least $k-1$ children, each of which has at least $k-1$ children, and so on. Using branching process and theory of dense graph limits, under mild assumptions we obtain the size of k-core of random graphs $G_n(\frac{1}{n})$,

$$C_k\left(G_n\left(\frac{1}{n}\right)\right) = n\mathbb{P}_{X^W}\left(\mathcal{A}\right) + o_p(n).$$

Our result can also be used to obtain the threshold of appearance of a k-core of order n. In addition, we obtain a probabilistic result concerning cut-norm and branching process which might be of independent interest.

Chapter V is based on [26]. We explicitly solve the nonlinear PDE that is the continuous limit of dynamic programming equation of the *expert prediction problem* in finite horizon setting with $N = 4$ experts. The *expert prediction problem* is formulated as a zero sum game between a player and an adversary. By showing that the solution is $\mathcal{C}^2$, we are able to show that the comb strategies, as conjectured in [107], form an asymptotic Nash equilibrium. We also prove the "Finite vs Geometric regret" conjecture proposed in [106] for $N = 4$, and show that this conjecture in fact follows from the conjecture that the comb strategies are optimal for all N.

Chapter VI is based on [30]. We consider a prediction problem with two experts and a forecaster. We assume that one of the experts is honest and makes correct prediction with probability μ at each round. The other one is malicious, who knows true outcomes at each round and makes predictions in order to maximize the loss of the forecaster. Assuming the forecaster adopts the classical multiplicative weights algorithm, we find an upper bound (6.5) for the value function of the malicious expert, and also a lower bound (6.19). Our results imply that the multiplicative weights algorithm cannot resist the corruption of malicious experts. We also show that an adaptive multiplicative weights algorithm is asymptotically optimal for the forecaster, and hence more resistant to the corruption of malicious experts.

Chapter VII is based on [27]. We study the problem of prediction with expert advice with adversarial corruption where the adversary can at most corrupt one expert. Using tools from viscosity theory, we characterize the long-time behavior of the value function of the game between the forecaster and the adversary. We provide lower and upper bounds for the growth rate of regret without relying on a comparison result. We show that depending on the description of regret, the limiting behavior of the game can significantly differ.

Chapter VIII is based on [31]. We analyze an $N+1$-player game and the corresponding mean field game with state space $\{0, 1\}$. The transition rate of j-th player is the sum of his control α^j plus a minimum jumping rate η. Instead of working under monotonicity conditions, here we consider an anti-monotone running cost. We show that the mean field game equation may have multiple solutions if $\eta < \frac{1}{2}$. We also prove that that although multiple solutions exist, only the one coming from the entropy solution is charged (when $\eta = 0$), and therefore resolve a conjecture of [109].

Chapter IX is based on [33]. We show existence and uniqueness of solutions of the infinite horizon McKean-Vlasov FBSDEs using two different methods, which lead to two different sets of assumptions. We use these results to solve the infinite horizon mean field type control problems and mean field games.

CHAPTER II

Transport Plans with Domain Constraints

2.1 Introduction

Martingale optimal transport has been an active research area in the past decade due to its applications in robust hedging problems in Mathematical Finance. In this set-up one is only given vanilla option prices at certain maturities, which thanks to a result by [53] corresponds to fixing the marginals of the martingale measures at these maturities, and tries to obtain model independent no-arbitrage price bounds. Mathematically, given two probability measures α, β on $\mathbb{R}^d$ and a cost function c on $\mathbb{R}^d \times \mathbb{R}^d$, one wants to minimize $\mathbb{E}^P[c(X,Y)]$ among all joint distributions P on $\mathbb{R}^d \times \mathbb{R}^d$ such that P has initial marginal α, terminal marginal β and $\mathbb{E}^P[Y|X] = X$. However, it is not clear whether there exists such a P satisfying both the marginal and martingale constraints. This question was answered by Strassen [179]: assume α and β have finite first moments,

$$\begin{aligned} &\exists\, P \text{ s.t. } P \circ X^{-1} = \alpha;\ P \circ Y^{-1} = \beta;\ \mathbb{E}^P[Y|X] = X \\ \iff\ &\alpha(f) \leq \beta(f),\ \forall \text{ convex functions } f. \end{aligned}$$

For martingale optimal transport and its application in Mathematical Finance, we refer readers e.g. to [36],[101],[90],[37],[85], and the references therein.

Another strand of literature considered pricing and hedging problems under volatility uncertainty (volatility is not known but is assumed to belong to a bounded interval): [9, 147, 153, 152]. This is also related to the notion of G-expectations; see e.g. [162, 154, 89]. However, in the volatility uncertainty literature, only the underlying stock price process is assumed to be observable and no liquid option prices are given as in the martingale optimal transport problem described above.

In this chapter our aim is to combine these two different ideas of model uncertainty and analyze the martingale transport problem with bounded volatility. Another motivating factor for us is the fact that without the volatility restriction, the hedging prices obtained from the martingale optimal transport are all the same for large classes of European, American, Asian, Bermudan options with similar forms of payoff functions (as observed in [25] and proved in [113]), which is of course not financially realistic. On the other hand, there are results indicating that once we have the bounded volatility restriction, these prices are generally not equal (see e.g., [5] and [29]), which is more practically viable.

First, we determine when there exists such a martingale measure satisfying the given volatility constraints and the marginals. Using [179, Theorem 7] together with a measurable selection argument, we obtain Proposition 2.2.3. Based on this proposition, we prove a general result Theorem 2.2.2. After giving a financial interpretation of this theorem (see Remark 2.2.5), we provide several examples: 1) martingale measures with volatility uncertainty, see subsection 2.2.1; 2) optimal transport with capacity constraints, see subsection 2.2.2; 3) Skorokhod embedding with bounded times, see subsection 2.2.3.

Subsequently, we extend Theorem 2.2.2 to the case of finitely many marginals using a pasting argument; see Theorem 2.3.3. By taking weak limits, we obtain the corresponding in continuous time when all one-dimensional marginals are given, which characterizes the existence of peacocks under constraints; see Theorem 2.4.1

and Remark 2.4.4. We also provide examples concerning the existence of martingale measures with volatility uncertainty in the case of finitely many marginals and one-dimensional marginals; see Example 2.3.2 and Example 2.4.3.

Finally, we consider the optimization problem (2.15), and obtain a duality result. It is a natural generalization of [105, Theorem 9.5] in our setup. Using the duality result established, we prove a general monotonicity principle which characterizes the geometric structure of the optimizer.

The rest of the chapter is organized as follows. In the next section, we establish the existence result when there are only two marginals given. In Section 3, we obtain the result when there are finitely many marginals given. In Section 4 we have the result with all the 1-D marginals in continuous time. In Section 5, we obtain the Kantorovich duality. Finally, in Section 6, we deduce the monotonicity principle from the duality result.

2.2 Result with two marginals

We will let Ω be one of the following three spaces:

- $\mathbb{X}^{N+1}$, where $\mathbb{X}$ is a polish space, and $N \in \mathbb{N}$;
- $C[0,1]$, the space of continuous functions $f : [0,1] \mapsto \mathbb{X}$, endowed with the uniform distance metric, where $\mathbb{X} \subset \mathbb{R}^d$ is connected and closed;
- $D[0,1]$, the space of RCLL functions $g : [0,1] \mapsto \mathbb{X}$, endowed with the Skorokhod metric, where $\mathbb{X} \subset \mathbb{R}^d$ is connected and closed.

Let $T = N$ if $\Omega = \mathbb{X}^{N+1}$ and $T = 1$ if $\Omega = C[0,1], D[0,1]$. For any probability measure P and random variable Y, $\mathbb{E}^P[Y] := \mathbb{E}^P[Y^+] - \mathbb{E}^P[Y^-]$ with the convention $\infty - \infty = -\infty$.

The spaces of probability measures in this chapter are endowed with the relativized weak* topology (see e.g. [99, Appendix 6], [179, Section 6]) as we describe next.

Let G and H be continuous functions on $\mathbb{X}$ that are positive and bounded away from 0. For $F = G, H$, let

$$\mathfrak{P}_F := \{\mu \in \mathfrak{P}(\mathbb{X}) : \ \mu(F) < \infty\} \text{ (simply } \mathfrak{P} \text{ if } F = 1),$$

and $\mathfrak{C}_F$ (simply $\mathfrak{C}$ if $F = 1$) be the Banach space of continuous functions f on $\mathbb{X}$ such that

$$\sup_{x \in \mathbb{X}} \frac{|f(x)|}{F(x)} < \infty.$$

Define $J := G \oplus H$ the continuous function on $\mathbb{X}^2$,

$$J(x_0, x_1) := G \oplus H(x_0, x_1) := G(x_0) + H(x_1).$$

Let

$$\mathfrak{P}_J := \{\mu \in \mathfrak{P}(\mathbb{X}^2) : \ \mu(J) < \infty\},$$

and $\mathfrak{C}_J$ be set of continuous functions f on $\mathbb{X}^2$ such that

$$\sup_{x \in \mathbb{X}^2} \frac{|f(x)|}{J(x)} < \infty.$$

For $L = G, H, J$, we say a subset of probability measures $\Lambda \subset \mathfrak{P}_L$ is L-closed, if for any $(P_n) \subset \Lambda$ and P with

$$\mathbb{E}^{P_n}[l] \to \mathbb{E}^P[l], \quad \forall l \in \mathfrak{C}_L, \tag{2.1}$$

we have $P \in \Lambda$. That is, we will endow spaces of probability measures with the topology generated by (2.1). When no such L is specified (e.g., we simply say a probability set is closed or weakly closed), then by default we endow the underlying space of probabiliy measures with weak topology, i.e., the topology generated by (2.1) with $\mathfrak{C}_L$ being the set of bounded and continuous functions.

Let X be the canonical process on Ω and $(\mathcal{F}_t)_t$ be the filtration generated by X. Let $\Gamma : \mathbb{X} \mapsto 2^{\mathfrak{P}(\Omega)}$ be such that $\emptyset \neq \Gamma(x) \subset \mathfrak{P}(\Omega^x)$, where $\mathfrak{P}(\Omega)$ is the set of Borel probability measures on Ω, and

$$\Omega^x := \{\omega \in \Omega : \ \omega_0 = x\}.$$

Here $\Gamma(x)$ represents the set of admissible transport plans given $X_0 = x$. We assume that the graph of Γ,

$$\mathrm{Gr}(\Gamma) := \{(x, P') : \ x \in \mathbb{X}, P' \in \Gamma(x)\}$$

is analytic. Denote

$$C := \{P \in \mathfrak{P}(\Omega) : \ P|_{X_0=\omega_0} \in \Gamma(\omega_0), \ P\text{-a.s. } \omega\}.$$

Let $\alpha \in \mathfrak{P}_G$ and $\beta \in \mathfrak{P}_H$ be two probability measures on $\mathbb{X}$. Let

$$A := \{P \in \mathfrak{P}(\Omega) : \ P \circ X_0^{-1} = \alpha\} \quad \text{and} \quad B := \{P \in \mathfrak{P}(\Omega) : \ P \circ X_T^{-1} = \beta\}. \tag{2.2}$$

Remark 2.2.1. Thanks to the analyticity assumption of $\mathrm{Gr}(\Gamma)$, by the Jankov-von Neumann Theorem (see, e.g., [40, Proposition 7.49]), there exists a universally measurable selector $P'(\cdot)$ such that $P'(x) \in \Gamma(x)$ for any $x \in \mathbb{X}$. Then $P_0 \otimes P' \in C$ for any probability measure P_0 on $\mathbb{X}$, where

$$P_0 \otimes P'(I) := \int_I P_0(d\omega_0) P'(\omega_0, d\omega), \quad I \in \mathcal{B}(\Omega).$$

In particular, this implies that $A \cap C \neq \emptyset$.

Our aim is to find a necessary and sufficient condition for $A \cap B \cap C \neq \emptyset$. In particular, Γ here is treated as a transport constraint from time 0 to time T, which is different from the marginal constraints. Below is the main result of this section.

Theorem 2.2.2. *Assume $\alpha \in \mathfrak{P}_G$, $\beta \in \mathfrak{P}_H$, and*

$$(A \cap C)_{0,T} := \{P \circ (X_0, X_T)^{-1} : \ P \in A \cap C\}$$

is convex and J-closed. Then

$$A \cap B \cap C \neq \emptyset \iff \alpha(f^\Gamma) \leq \beta(f), \ \forall f \in \mathfrak{C}_H, \tag{2.3}$$

where $\beta(f) := \int_{\mathbb{X}} f \ \beta(dx)$, and $f^\Gamma(x)$ is defined by

$$f^\Gamma(x) := \inf_{Q \in \Gamma(x)} \mathbb{E}^Q[f(X_T)]. \tag{2.4}$$

We will prove this result at the end of this section. In the case of $\Omega = \mathbb{X}^2$, we have Proposition 2.2.3, which will be useful in proving Theorem 2.2.2. The proof Proposition 2.2.3 essentially follows [179, Theorem 7] together with a measurable selection argument.

Proposition 2.2.3. *Assume $\Omega = \mathbb{X}^2$, $\alpha \in \mathfrak{P}_G$ and $\beta \in \mathfrak{P}_H$. Moreover let $A \cap C$ be convex and J-closed. Then*

$$A \cap B \cap C \neq \emptyset \iff \alpha(f^\Gamma) \leq \beta(f), \ \forall f \in \mathfrak{C}_H.$$

Proof. "$\Longrightarrow$". Take $P \in A \cap B \cap C$. For any $f \in \mathfrak{C}_H$, we have $\mathbb{E}^P|f(X_1)| < \infty$. Hence,

$$\beta(f) = \mathbb{E}^P[f(X_1)] = \mathbb{E}^P[\mathbb{E}^P[f(X_1)|X_0]] \geq \mathbb{E}^P[f^\Gamma(X_0)] = \alpha(f^\Gamma).$$

"$\Longleftarrow$". Using a measurable selection argument, we can show that

$$\alpha(f^\Gamma) = \inf_{P\in A\cap C} \mathbb{E}^P[f], \quad \forall f \in \mathfrak{C}_H. \tag{2.5}$$

Let

$$(A\cap C)\circ X_1^{-1} := \{P\circ X_1^{-1} : \; P\in A\cap C\}.$$

Then β is in the H-closure of $(A\cap C)\circ X_1^{-1}\cap\mathfrak{P}_H$, for otherwise by the Hahn-Banach theorem (see e.g.[127, Corollary 14.4]) there would exist $f\in\mathfrak{C}_H$ such that

$$\beta(f) < \inf_{P\in A\cap C} \mathbb{E}^P[f(X_1)] = \alpha(f^\Gamma),$$

a contradiction.

Let $P_n \in A\cap C$ with $\beta_n := P_n\circ X_1^{-1}$ such that $\beta_n \to \beta$ in the sense of (2.1). It can be shown that the sequence (P_n) is relatively J-compact (see [179]). Then there exists $P_\infty \in \mathfrak{P}_J$ such that up to a subsequence $P_n \to P_\infty$ in the sense of (2.1). As $A\cap C$ is J-closed, $P_\infty \in A\cap C$. Moreover, $P_n\circ X_1^{-1} = \beta^n \to \beta$ implies that $P_\infty\in C$. The conclusion follows. □

Let us discuss our assumptions in the following remarks. In what follows, we will give natural examples where these assumptions are satisfied.

Remark 2.2.4. The closedness of $A\cap C$ cannot imply the closedness of $(A\cap C)_{0,T}$. For instance, let $\Omega=\mathbb{R}^3$, $\alpha=\delta_0$,

$$\Gamma(x) = \{P\in\mathfrak{P}(\Omega^x) : \; P\circ(X_1,X_2)^{-1} = \delta_{(x_1,x_2)}, \; (x_1,x_2)\in S\},$$

where $S=\{(x_1,x_2) : \; x_1>0, x_2>0, x_1x_2\geq 1\}$. Then $A\cap C$ is weakly closed, but $(A\cap C)_{0,2} = \{\delta_0\otimes\delta_x : \; x>0\}$ is not.

Moreover, in the above theorem the assumption $(A\cap C)_{0,T}$ being closed cannot be replaced by $A \cap C$ being closed. Consider again the above with $\beta = \delta_0$. Then obviously $A \cap B \cap C = \emptyset$. However, for any continuous function f,

$$\alpha(f^\Gamma) = f^\Gamma(0) = \inf_{(x_1,x_2)\in S} f(x_2) \leq f(0) = \beta(f).$$

Remark 2.2.5 (**Financial interpretation**). Suppose Γ contains the martingale constraint, i.e.,

$$\Gamma(x) \subset \{Q \in \mathfrak{P}(\Omega^x) : \ Q \text{ martingale measure}\}, \quad x \in \mathbb{X}.$$

Suppose X represents the stock price, and f is the payoff of an option written on X_T. Assume $\alpha = \delta_x$.

Then $f^\Gamma(x)$ represents the sub-hedging price of the option f given the current stock price $X_0 = x$, and $\beta(f)$ is market price of f (which is consistent with the vanilla option prices). Then the right-hand-side of (2.3) means that the sub-hedging price is smaller than the market price. By symmetry, the super-hedging price is larger than the market price. On the other hand, the left-hand-side of (2.3) means there is a measure consistent with the constraints. As a result, both sides of (2.3) represent no arbitrage. For the role martingale optimal transport plays in finance see [36].

The following lemma gives a useful sufficient condition for closedness of $(A\cap C)_{0,T}$.

Lemma 2.2.6. *Let $G = H = 1$, so that the topology generated by* (2.1) *is the weak topology in the usual sense. If $A \cap C$ is weakly compact, then $(A \cap C)_{0,T}$ is weakly closed.*

Proof. Let $Q_n \in (A \cap C)_{0,T}$ such that $Q_n \xrightarrow{w} Q$ for some $Q \in \mathfrak{P}(\mathbb{X}^2)$. Then there exists $P_n \in A \cap C$ such that

$$P_n \circ (X_0, X_T)^{-1} = Q_n.$$

Since $A \cap C$ is weakly compact, there exists some $P \in A \cap C$ such that $P_n \xrightarrow{w} P$. Obviously $P \circ (X_0, X_T)^{-1} = Q$, and thus $Q \in (A \cap C)_{0,T}$. $\square$

2.2.1 Examples of volatility uncertainty

Our starting point is to consider C as the set of martingale measures with volatility uncertainty. With some compact constraints on the volatility, we can show $A \cap C$ is indeed weakly compact and thus weakly closed (Lemma 2.2.6). Here are some examples.

Example 2.2.7 (Volatility uncertainty in one period)**.** Let $\mathbb{X} = \mathbb{R}^d$ and $\Omega = \mathbb{X}^2$. Assume α has a finite first moment (i.e., $\alpha(|x|) < \infty$), and let

$$\Gamma(x) = \left\{ Q \in \mathfrak{P}(\Omega^x) : \ \mathbb{E}^Q[X_T] = x, \ Q\{(x,y) : \ |y - x| \leq a(x)\} = 1 \right\}, \tag{2.6}$$

where $a(\cdot)$ is a nonnegative, bounded and continuous function on $\mathbb{X}$. It can be shown that $\mathrm{Gr}(\Gamma)$ is Borel measurable.

Proposition 2.2.8. *In this example, $A \cap C$ is convex and weakly compact.*

Proof. Convexity is obvious. Now for any $\varepsilon > 0$, there exists a compact set $K \subset \mathbb{X}$ such that $\alpha(K) \geq 1 - \varepsilon$. Then for any $P \in A \cap C$,

$$P(X \in K^\varepsilon) \geq 1 - \varepsilon,$$

where

$$K^\varepsilon := \left\{ (x,y) : \ x \in K, |y - x| \leq \sup_{z \in \mathbb{X}} a(z) \right\}$$

is a compact set in $\mathbb{X}^2$. Therefore, $A \cap C$ is tight and thus relatively compact by Prokhorov's theorem (see e.g. [65, Theorem 3.5.13]).

Assume $P_n \in A \cap C$ such that $P_n \xrightarrow{w} P$. Then by the Portmanteau Theorem (see e.g. [167, Theorem 1.2]),

$$P(\{(x,y):\ |y-x|\le a(x)\})\ge \limsup_{n\to\infty} P_n(\{(x,y):\ |y-x|\le a(x)\})=1.$$

Now, let us show the martingale property under the limiting measure. Let g be any continuous and bounded function on $\mathbb{X}$. Define the compact subset $U^\epsilon := \{(x,y)\in\mathbb{X}^2 : d((x,y),K^\epsilon)\le\epsilon\}$. Let f^ϵ be a continuous function on $\mathbb{X}^2$ such that $0\le f^\epsilon\le 1$, f^ϵ is compactly supported by U^ϵ and $f^\epsilon|_{K^\epsilon}=1$. Since $|X_1-X_0|\le \sup_{z\in\mathbb{X}} a(z)<\infty$ P_n-a.s. and P-a.s., the function $(x,y)\mapsto (y-x)g(x)f^\epsilon(x,y)$ is continuous and bounded for any $\epsilon>0$. According to the definition of weak convergence, we have that

$$\mathbb{E}^P[(X_1-X_0)g(X_0)f^\epsilon(X_0,X_1)]=\lim_{n\to\infty}\mathbb{E}^{P_n}[(X_1-X_0)g(X_0)f^\epsilon(X_0,X_1)]=0.$$

As the random variable $|(X_1-X_0)g(X_0)|$ is bounded P-a.s., we can conclude by the dominated convergence theorem,

$$\mathbb{E}^P[(X_1-X_0)g(X_0)]=\lim_{\epsilon\to 0}\mathbb{E}^P[(X_1-X_0)g(X_0)f^\epsilon(X_0,X_1)]=0.$$

This implies P is a martingale measure. As a result, $P\in A\cap C$, and thus $A\cap C$ is weakly compact. $\square$

With Γ defined in (2.6), it can be shown that for any function $f:\mathbb{X}\mapsto\mathbb{R}$,

$$\begin{aligned} f^\Gamma(x) &= \mathcal{C}(f|_{\bar{O}(x,a(x))})(x) \\ &= \inf\left\{\sum_{i=0}^d \lambda_i f(y_i):\ |y_i-x|\le a(x), \lambda_i\ge 0, i=0,\dots,d, \sum_{i=0}^d\lambda_i=1, \sum_{i=0}^d \lambda_i y_i = x\right\}, \end{aligned}$$

where $\mathcal{C}(f|_{\bar{O}(x,b)})(x)$ is given by the convex envelope of f restricted on $\bar{O}(x,b):=\{y\in\mathbb{X}:\ |y-x|\le b\}$ and then evaluating at x.

Example 2.2.9 (Volatility uncertainty in multiple periods)**.** Let $\mathbb{X} = \mathbb{R}^d$ and $\Omega = \mathbb{X}^{N+1}$, $N \geq 1$. Assume α has a finite first moment, and let

$$\Gamma(x) = \left\{ Q \in \mathfrak{P}(\Omega^x) : \begin{array}{l} Q \text{ martingale measure,} \\ Q\{|X_n - X_{n-1}| \leq a_{n-1}(X_{n-1})\} = 1, \; n = 1, \dots, N \end{array} \right\},$$

where a_{n-1} is a nonnegative, bounded and continuous function on $\mathbb{X}$ for $n = 1, \dots, N$.

Proposition 2.2.10. *In this example $A \cap C$ is convex and weakly compact, and f^Γ can be calculated recursively as follows:*

$$g_N = f, \quad g_{n-1}(x) = \mathcal{C}(g_n|_{\bar{O}(x, a_{n-1}(x))})(x), \; n = 1, \dots, N, \quad f^\Gamma = g_0.$$

Proof. The proof is similar to Proposition 2.2.8. It only remains to show $\mathbb{E}^P[X_n|X_{n-1}] = X_{n-1}$ for $n = 2, \dots, N$. Let us show that $\mathbb{E}^P[X_2|X_1] = X_1$, and the rest can be proved by induction. Denote by α_1 the distribution of X_1 under P. Since $|X_1 - X_0| \leq \max_{z \in \mathbb{X}} a_0(z) < +\infty$, α_1 has finite first moment. Replacing α with α_1 in the proof of Proposition 2.2.8, we directly obtain that $\mathbb{E}^P[(X_2 - X_1)g(X_1)] = 0$ for any bounded continuous function g, which implies that $\mathbb{E}^P[X_2|X_1] = X_1$. $\square$

Example 2.2.11 (Volatility uncertainty in continuous time)**.** Let $\mathbb{X} = \mathbb{R}^d$ and $\Omega = C[0,1]$. Assume α has a finite first moment, and let

$$\Gamma(x) = \left\{ Q \in \mathfrak{P}(\Omega^x) : \; Q \text{ martingale measure, } \frac{d\langle X \rangle_t}{dt} \in \mathbb{D}, \; dt \times Q\text{-a.e.} \right\},$$

where $\mathbb{D} \subset \mathbb{R}^{d \times d}$ is some fixed convex and compact set of matrices. In this case, f^Γ is the G-expectation of f (see [89]).

Proposition 2.2.12. *In this example, $A \cap C$ is convex and weakly compact.*

Proof. First, we show $A \cap C$ is tight. We have

$$\lim_{L\to\infty} \sup_{P\in A\cap C} P(|X_0| > L) = \lim_{L\to\infty} \alpha\left(\{x : |x| > L\}\right) = 0.$$

Moreover, for any $s, t \in [0,1]$, since $\mathbb{D}$ is bounded, by the Burkholder-Davis-Gundy inequality (see e.g. [124, Theorem 3.3.28]) there exists some constant K independent of s and t such that

$$\sup_{P\in A\cap C} \mathbb{E}^P[|X_t - X_s|^4] \leq \sup_{P\in A\cap C} \mathbb{E}^P\left[\mathbb{E}^P\left[\sup_{s\leq r\leq t} |X_r - X_s|^4 \Big| X_s\right]\right] \leq K|t-s|^2. \quad (2.7)$$

By the moment criterion, $A \cap C$ is tight (see e.g. [124, Problem 2.4.11]).

Next we show $A \cap C$ is closed. Let $P^n \in A \cap C$ such that $P^n \xrightarrow{w} P$. Obviously $P \in A$. Then using almost the same argument as in the proof of [155, Lemma 3.2], we can show that $P \in C$. $\square$

2.2.2 Example of capacity constraint

In [134], Korman and McCann studied the optimal transport problem with capacity constraints. Suppose f and g are two probability density functions on $\mathbb{R}^d$, c is a cost function on $\mathbb{R}^d \times \mathbb{R}^d$, and $\bar{h} \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ is a capacity constraint. Define $\Gamma^{\bar{h}}(f,g) := \{h \in L^1(\mathbb{R}^d \times \mathbb{R}^d) :\ h \text{ has } f, g \text{ as its marginals, and } h \leq \bar{h}\}$. Under the assumption $\Gamma^{\bar{h}}(f,g) \neq \emptyset$, Korman and McCann proved that any optimizer h_0 of the problem,

$$\inf_{h\in\Gamma^{\bar{h}}(f,g)} \int c(x,y)h(x,y)dxdy,$$

is geometrically extreme, i.e., $h_0 = \mathbb{1}_W \bar{h}$ for some measurable set $W \subset \mathbb{R}^d \times \mathbb{R}^d$.

In this subsection, we give one more criterion for weak closedness of $(A \cap C)_{0,T}$. In doing so, we can apply Theorem 2.2.2 and describe when this non-emptiness assumption $\Gamma^{\bar{h}}(f,g) \neq \emptyset$ is satisfied. Actually we can deal with more general capacity constraints.

Let $R : \mathbb{X} \mapsto \mathfrak{P}(\Omega)$ be a transition kernel, and

$$\Gamma(x) := \left\{ Q \in \mathfrak{P}(\Omega^x) : \ \frac{Q(dy)}{R(x,dy)} \leq a(x,y) \right\},$$

where $a(\cdot,\cdot) \geq 0$ is a bounded and Borel measurable function. For any Borel measurable set $A \in \mathcal{B}(\Omega)$, according to [43, Lemma 4.6], the function $Q \mapsto \mathbb{E}^Q[1_A]$ is Borel measurable. Since the function $x \mapsto \int_\Omega 1_A a(x,y) R(x,dy)$ is also Borel measurable, so is the set

$$L_A := \left\{ (x,Q) \in \mathbb{X} \times \mathfrak{P}(\Omega) : \ \mathbb{E}^Q[1_A] \leq \int\limits_\Omega 1_A a(x,y) R(x,dy) \right\}.$$

It can be easily checked that $\{(x,Q) : \ \ Q \in \mathfrak{P}(\Omega^x)\}$ is closed, and hence the set

$$\mathcal{L}_A := L_A \cap \{(x,Q) : \ \ Q \in \mathfrak{P}(\Omega^x)\}$$

is Borel measurable. Now let $(A_i)_{i=1}^\infty$ be a countable algebra generating $\mathcal{B}(\Omega)$. Then

$$\mathrm{Gr}(\Gamma) = \cap_{i=1}^\infty \mathcal{L}_{A_i}$$

is Borel measurable, and hence analytic.

Proposition 2.2.13. *In this example, $A \cap C$ is weakly compact, and thus $(A \cap C)_{0,T}$ is weakly closed.*

Proof. By the boundedness of $a(\cdot,\cdot)$, the subset of $\mathfrak{P}(\mathbb{X} \times \Omega)$

$$\Lambda := \{\alpha \times Q : \ Q \text{ is any transition kernel such that } Q(\cdot) \in \Gamma(\cdot) \ \alpha\text{-a.s.}\}$$

is relatively compact. If we can show Λ is weakly compact, then the subset of $\mathfrak{P}(\Omega)$,

$$A \cap C = \{\bar{P} \circ \pi_2^{-1} : \bar{P} \in \Lambda \ \text{ and } \ \pi_2(x,y) := y, \ \forall \ (x,y) \in \mathbb{X} \times \Omega\},$$

is also weakly compact. Take $\alpha \times Q_n \in \Lambda$ such that $\alpha \times Q_n \xrightarrow{w} \bar{P}^*$. By the definition of $\Gamma(x)$, there exist Borel measurable functions b_n with $0 \leq b_n(\cdot,\cdot) \leq a(\cdot,\cdot)$ such that for $(x,y) \in \mathbb{X} \times \Omega$,

$$b_n(x,y)R(x,dy) = Q_n(x,y).$$

Consider $L^2(\mathbb{X} \times \Omega)$ over the probability space $(\mathbb{X} \times \Omega, \alpha \times R)$. Since L^2 is reflexive, the weak* topology and weak topology coincide. Now because b_n is uniformly bounded, by Banach-Alaoglu theorem (see e.g. [127, Theorem17.4]), there exists a Borel measurable function b^* on $\mathbb{X} \times \Omega$ such that $b_n \xrightarrow{w} b^*$, i.e., for any measurable function f on $\mathbb{X} \times \Omega$ with $\mathbb{E}^{\alpha \times R}|f|^2 < \infty$,

$$\mathbb{E}^{\alpha \times R}[f b_n] \to \mathbb{E}^{\alpha \times R}[f b^*]. \tag{2.8}$$

In particular, the above holds for bounded and continuous functions f, which implies that

$$\alpha \times b_n R = \alpha \times Q_n \xrightarrow{w} \alpha \times b^* R.$$

So we conclude $\alpha \times b^* R = \bar{P}^*$.

Note that for any bounded, nonnegative, and measurable function f,

$$\mathbb{E}^{\alpha \times R}[f b_n] \leq \mathbb{E}^{\alpha \times R}[f a].$$

By (2.8),

$$\mathbb{E}^{\alpha \times R}[f b^*] \leq \mathbb{E}^{\alpha \times R}[f a].$$

This implies that $b^* \leq a$, $\alpha \times R$-a.s., and thus $\bar{P}^* = \alpha \times b^* R \in \Lambda$. $\square$

2.2.3 Application to Skorokhod embedding with bounded times

Theorem 2.2.2 and Example 2.2.11 provide a necessary and sufficient condition for the existence of a Skorokhod embedding in bounded time. We will rely on a time change argument to make a connection to Skorohod embedding; see e.g. Hobson [115]. To wit, let $\Omega = C[0,1]$ with $\mathbb{X} = \mathbb{R}$. Let $\alpha, \beta \in \mathfrak{P}(\mathbb{X})$ with finite first moments and $\sigma > 0$ be a constant. For $u, r > 0$, define

$$\mathcal{Q}_{u,r} := \left\{ Q \in \mathfrak{P}(\bar{\Omega}) : \ Q \text{ martingale measure}, \ \frac{d\langle \bar{X} \rangle_t}{dt} \leq u, \ 0 \leq t \leq r, \ dt \times Q\text{-a.e.} \right\},$$

where $\bar{\Omega} := C_0[0, \infty)$ is the set of continuous paths $[0, \infty) \to \mathbb{X}$ starting from position 0, and $\bar{X}$ is the canonical process on $\overline{\Omega}$. For any function $f \in \mathfrak{C}$ and $u, r > 0$, define

$$f^{u,r}(x) := \inf_{Q \in \mathcal{Q}_{u,r}} \mathbb{E}^Q[f(x + \bar{X}_r)].$$

We have the following.

Proposition 2.2.14. *For Brownian motion $\mathfrak{B}$ with initial distribution $\mathfrak{B}_0 \stackrel{d}{=} \alpha$, there exists a stopping time τ such that*

$$\tau \leq \sigma \quad \textit{and} \quad \mathfrak{B}_\tau \stackrel{d}{=} \beta,$$

if and only if for any $f \in \mathfrak{C}$,

$$\alpha(f^{\sigma,1}) \leq \beta(f).$$

Proof. "$\Longrightarrow$". For $f \in \mathfrak{C}$, we have that

$$\beta(f) = \mathbb{E}^W[f(\mathfrak{B}_\tau)] = \mathbb{E}^W[\mathbb{E}^W[f(\mathfrak{B}_\tau)|\mathcal{B}_0]] \geq \alpha(f^{1,\sigma}) = \alpha(f^{\sigma,1}),$$

where W is the probability measure associated with the Brownian motion, and the third (in)equality follows from $\frac{d\langle \bar{X}_{\cdot\wedge\tau}\rangle_t}{dt} = 0$ for $t > \tau$, and the fourth (in)equality follows from a change of the time scale.

"$\Longleftarrow$". Take $d = 1, \mathbb{D} = [0, \sigma]$ in Example 2.2.11, and

$$\Gamma(x) := \left\{ Q \in \mathfrak{P}(\bar{\Omega}^x) : \ Q \text{ martingale measure}, \ \frac{d\langle \bar{X}\rangle_t}{dt} \leq \sigma, \ 0 \leq t \leq 1, \ dt \times Q\text{-a.e.} \right\},$$

where $\bar{\Omega}^x$ is the set of continuous paths starting from position x. Then we have $f^\Gamma(x) = f^{\sigma,1}(x)$. Applying Theorem 2.2.2 and Example 2.2.11, there exists $Q \in \mathcal{Q}_{\sigma,1}$ such that

$$Q \circ X_0^{-1} = \alpha \quad \text{and} \quad Q \circ X_1^{-1} = \beta.$$

By the Dambis-Dubins-Schwarz theorem (see e.g. [124, Theorem 3.4.6], we can extend X and Q to the time interval $[0, \infty)$ so that the condition of the theorem is satisfied), $\mathfrak{B}_s := X_{\mathfrak{T}(s)}$ is a Brownian motion w.r.t. the filtration $\mathcal{G}_s := \overline{\mathcal{F}}_{\mathfrak{T}(s)}$, having the initial distribution $\mathfrak{B}_0 \overset{d}{=} \alpha$, and $X_t = \mathfrak{B}_{\langle X\rangle_t}$, where $\overline{\mathcal{F}}_t$ is given by $\cap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon}$ completed by Q, and

$$\mathfrak{T}_s := \inf\{t \geq 0 : \ \langle X\rangle_t > s\}.$$

In particular,

$$X_1 = \mathfrak{B}_{\langle X\rangle_1} \overset{d}{=} \beta \quad \text{and} \quad \tau := \langle X\rangle_1 \leq \sigma.$$

□

2.2.4 Proof of Theorem 2.2.2

Proof. "$\Longrightarrow$". The argument is similar to the one for Proposition 2.2.3.

"$\Longleftarrow$". Let

$$\Gamma_{0,T}(x) := \{P \circ (X_0, X_T)^{-1} : \ P \in \Gamma(x)\}.$$

Then

$$f^{\Gamma}(x) = \inf_{Q \in \Gamma_{0,T}(x)} \mathbb{E}^{Q}[f(Y_1)],$$

where $Y = (Y_0, Y_1) := (X_0, X_T)$ is the canonical process on $\mathbb{X}^2$ (starting from position x). By Proposition 2.2.3, there exists $P^* \in \mathfrak{P}(\mathbb{X}^2)$ such that

$$P^* \circ Y_0^{-1} = \alpha,\ \ P^* \circ Y_1^{-1} = \beta,\ \ P^*|_{Y_0} \in \Gamma_{0,T}(Y_0),\ \ P^*\text{-a.s..}$$

Let

$$P^* = \alpha \otimes Q^*,$$

be the disintegration of P^*, where $Q^*(\cdot)$ is Borel measurable. By restricting to a Borel set $L \in \sigma(Y_0) = \mathcal{B}(\mathbb{X})$ with $P^* \circ Y_0^{-1}(L) = 1$, we may without loss of generality assume that $Q^*(x) \in \Gamma(x)$ for all $x \in \mathbb{X}$. Then the set

$$I_1 := \{(x, P, Q) :\ x \in \mathbb{X}, P \in \mathfrak{P}(\Omega), Q = Q^*(x)\}$$

is Borel measurable. Moreover, since $\mathrm{Gr}(\Gamma)$ is analytic, the set

$$I_2 := \{(x, P, Q) :\ x \in \mathbb{X}, P \in \Gamma(x), Q = P \circ X_T^{-1}\}$$

is also analytic. Then the set

$$I_1 \cap I_2 = \{(x, P, Q) :\ x \in \mathbb{X}, P \in \Gamma(x),\ P \circ X_T^{-1} = Q = Q^*(x)\}$$

is analytic. By the Jankov-von Neumann Theorem (see e.g., [40, Proposition 7.49]), there exists a univerally measurable selector $(P', Q') :\ \mathbb{X} \mapsto \mathfrak{P}(\Omega) \times \mathfrak{P}(\mathbb{X})$ such that

$$P' \in \Gamma(x),\ \ (P'(x)) \circ X_T^{-1} = Q'(x) = Q^*(x).$$

Define

$$\bar{P} = \alpha \otimes P'.$$

It can be seen that $\bar{P} \in A \cap B \cap C$. $\square$

Remark 2.2.15 (**Extension to moment constraints**). Let $\mathcal{A} \subset \mathfrak{P}_G(\mathbb{X})$ be convex and G-compact, $\mathcal{B} \subset \mathfrak{P}_H(\mathbb{X})$ be convex and H-closed. Define

$$\mathbb{A} := \{P \in \mathfrak{P}(\Omega) : \; P \circ X_0^{-1} \in \mathcal{A}\} \quad \text{and} \quad \mathbb{B} := \{P \in \mathfrak{P}(\Omega) : \; P \circ X_T^{-1} \in \mathcal{B}\}.$$

Using almost the same argument as above, we have the following. Assume $(\mathbb{A} \cap C)_{0,T}$ is convex and J-closed. Then

$$\mathbb{A} \cap \mathbb{B} \cap C \neq \emptyset \iff \inf_{\alpha \in \mathcal{A}} \alpha(f^\Gamma) \leq \sup_{\beta \in \mathcal{B}} \beta(f), \; \forall f \in \mathfrak{C}_H.$$

2.3 Result for multiple marginals

We still use the three cases of Ω from the last section. Assume $0 = t_0 < t_1 < \ldots < t_n = T$ such that for $i = 1, \ldots, n-1$, $t_i \in \{1, \ldots, N-1\}$ if $\Omega = \mathbb{X}^{N+1}$, and $t_i \in [0,1]$ if $\Omega = C[0,1]$ or $D[0,1]$. For $i = 0, \ldots, n-1$, let $\Omega_i = \mathbb{X}^{t_{i+1}-t_i+1}, C[0, t_{i+1} - t_i], D[0, t_{i+1} - t_i]$, and $\overline{\Omega}_i = \mathbb{X}^{N-t_i+1}, C[0, 1-t_i], D[0, 1-t_i]$, if $\Omega = \mathbb{X}^{N+1}, C[0,1], D[0,1]$ respectively. Let $\Omega_i^x \subset \Omega_i(\cdot)$ be the space of the paths starting from $x \in \mathbb{X}$. Denote $X_{[0,t]}$ the path from time 0 to time t.

Let $\Gamma_i : \mathbb{X} \mapsto 2^{\mathfrak{P}(\Omega_i)}$ such that $\emptyset \neq \Gamma_i(x) \subset \mathfrak{P}(\Omega_i^x)$ for any $x \in \mathbb{X}$, and assume $\mathrm{Gr}(\Gamma_i)$ is analytic, $i = 0, \ldots, n-1$. Define $P^{t_i,\omega}$ to be the conditional probability of P given ω up to time t_i, i.e., for any Borel measurable function f on Ω,

$$\mathbb{E}^{P^{t_i,\omega}}[f(\omega \otimes_{t_i} \cdot)] = \mathbb{E}^P[f|\mathcal{F}_t](\omega), \quad P\text{-a.s. } \omega,$$

where for $\omega' \in \overline{\Omega}_i$ such that $\omega'_0 = \omega_{t_i}$,

$$(\omega \otimes_{t_i} \omega')_s = \begin{cases} \omega_s, & s < t_i, \\ \omega'_{s-t_i}, & s \geq t_i. \end{cases}$$

Let

$$C_i := \{P \in \mathfrak{P}(\Omega_i) : \; P|_{X_0=\omega_0} \in \Gamma_i(\omega_0), \; P\text{-a.s. } \omega\}, \tag{2.9}$$

and

$$\overline{C}_i := \left\{P \in \mathfrak{P}(\Omega) : \; P^{t_i,\omega,} \circ X^{-1}_{[0,t_{i+1}-t_i]} \in \Gamma_i(\omega_{t_i}), \; P\text{-a.s. } \omega\right\}, \tag{2.10}$$

where $P^{t_i,\omega,} \circ X^{-1}_{[0,t_{i+1}-t_i]}$ represents the marginal probability distribution of $P^{t_i,\omega}$ from time 0 to time $t_{i+1} - t_i$.

Remark 2.3.1. Here Γ_i represents the restriction of probability measures from time t_i to time t_{i+1}. Note that the restriction only depends on the current location instead of the whole history (i.e., path). This property is critical for the construction of probability measures with multiple marginals later on. Also note that it does not imply the underlying probability measure is Markovian.

Example 2.3.2. Assume $\Omega = C[0,1]$ with $\mathbb{X} = \mathbb{R}^d$. Let $P \in \mathfrak{P}(\Omega)$ be a martingale measure such that

$$\frac{d\langle X \rangle_t}{dt} \in \mathbb{D}, \quad dt \times P\text{-a.e.}, \tag{2.11}$$

where $\mathbb{D} \subset \mathbb{R}^{d \times d}$ is some bounded set of matrices. Then this martingale and volatility uncertainty restriction satisfies the property mentioned above. To be more specific, let

$$\Gamma_i(x) := \left\{Q \in \mathfrak{P}(\Omega_i^x) : \; Q \text{ martingale measure, } \frac{d\langle X \rangle_t}{dt} \in \mathbb{D}, \; dt \times Q\text{-a.e.}\right\}.$$

Then P satisfies (2.11) if and only if $P \in \cap_{i=0}^{n-1} \overline{C}_i$.

Let $\alpha_i \in \mathfrak{P}(\mathbb{X})$, and

$$A_i := \{P \in \mathfrak{P}(\Omega_i) : \ P \circ X_0^{-1} = \alpha_i\} \quad \text{and}$$
$$\overline{A}_i := \{P \in \mathfrak{P}(\Omega) : \ P \circ X_{t_i}^{-1} = \alpha_i\}, \quad i = 0, \dots, n.$$

Recall f^Γ defined in (2.4). The following is the main result of this section.

Theorem 2.3.3. *Let $G = H$. Assume $\alpha_i \in \mathfrak{P}_H$ and $(A_i \cap C_i)_{0, t_{i+1}-t_i}$ is convex and J-closed for $i = 0, \dots, n$. Then*

$$\bigcap_{i=0}^{n} \overline{A}_i \cap \bigcap_{j=0}^{n-1} \overline{C}_j \neq \emptyset \iff \alpha_i(f^{\Gamma_i}) \leq \alpha_{i+1}(f), \ \forall\, f \in \mathfrak{C}_H, \ i = 0, \dots, n-1.$$

Proof. "$\Longrightarrow$". Take $P \in (\bigcap_{i=0}^{n} \overline{A}_i) \cap (\bigcap_{j=0}^{n-1} \overline{C}_j)$. For $i = 0, \dots, n-1$,

$$\alpha_{i+1}(f) = \mathbb{E}^P[f(X_{t_{i+1}})] = \mathbb{E}^P[\mathbb{E}^P[f(X_{t_{i+1}})|\mathcal{F}_{t_i}]] \geq \mathbb{E}^P[f^{\Gamma_i}(X_{t_i})] = \alpha_i(f^{\Gamma_i}),$$

where the inequality follows from the definition in (2.4), and the fact that the conditional probability associated with $\mathbb{E}^P[\cdot|\mathcal{F}_{t_i}](\omega)$ is an element of $\Gamma_i(\omega_{t_i})$ for P-a.s. ω (see (2.9) and (2.10)).

"$\Longleftarrow$". By Theorem 2.2.2 there exists a probability measure $P_i \in A_i \cap B_i \cap C_i$ on Ω_i for $i = 0, \dots, n-1$, where

$$B_i := \{P \in \mathfrak{P}(\Omega_i) : \ P \circ X_{t_{i+1}-t_i}^{-1} = \alpha_{i+1}\}.$$

Let $P := P_0 \otimes \ldots \otimes P_{n-1}$. That is,

$$P(I) := \int_I P_0(d\omega_{[t_0,t_1]}) P_1(\omega_{t_1}, d\omega_{[t_1,t_2]}) \ldots P_{n-1}(\omega_{t_{n-1}}, d\omega_{[t_{n-1},t_n]}), \quad I \in \mathcal{B}(\Omega).$$

where for $x \in \mathbb{X}$,

$$P_i(x, \cdot) := P_i|_{\omega_0 = x} \tag{2.12}$$

is the conditional probability of P_i given $\omega_0 = x$. It can be shown that P indeed is a probability measure on Ω. Moreover, $P^{t_i,\omega} \circ X^{-1}_{[0,t_{i+1}-t_i]} = P_i(\omega_{t_i}, \cdot) \in \Gamma_i(\omega_{t_i})$ for P-a.s. ω, and thus $P \in \overline{C}_i$ for $i = 0, \dots, n-1$.

It remains to show that

$$P \circ X^{-1}_{t_i} = \alpha_i, \quad i = 0, \dots, n. \tag{2.13}$$

We prove the above by induction. Obviously (2.13) holds for $i = 0$. Assume it holds for $i = k$ with $0 \leq k \leq n-1$, and consider the case when $i = k+1$. For any bounded and measurable function f on $\mathbb{X}$, we have that

$$\begin{aligned}
\mathbb{E}^P[f(X_{t_{k+1}})] &= \mathbb{E}^P[\mathbb{E}^P[f(X_{t_{k+1}})|X_{t_k}]] \\
&= \int_{\mathbb{X}} \alpha_k(dx) \int_{\Omega^x_k} f(\omega_{t_{k+1}-t_k}) P_k(x, d\omega) \\
&= \int_{\Omega_k} f(\omega_{t_{k+1}-t_k}) \alpha_k(dx) P_k(x, d\omega) \\
&= \mathbb{E}^{P_k}[f(X_{t_{k+1}-t_k})] \\
&= \alpha_{k+1}(f),
\end{aligned}$$

where the second equality follows from the induction hypothesis $P \circ X^{-1}_{t_k} = \alpha_k$ and (2.12), the fourth equality follows from $P_k \in A_k$, and the fifth from $P_k \in B_k$. □

2.4 Result with all the 1-D marginals in continuous time

In this section, we consider two cases $\Omega = C[0,1]$ or $D[0,1]$. For $t \in [0,1]$, let $\Omega_t = C[0,t], D[0,t]$ when $\Omega = C[0,1], D[0,1]$ respectively, $\Omega^x_t \subset \Omega_t$ be the set of paths starting from position $x \in \mathbb{X}$. We are given a class of maps $\Gamma_{[s,t]} : \mathbb{X} \mapsto 2^{\mathfrak{P}(\Omega_{t-s})}$ for $0 \leq s < t \leq 1$. Each $\Gamma_{[s,t]}$ will represent the restriction of probability measures to the time interval $[s,t]$. In particular, this restriction is Markovian in the sense that

$\Gamma_{[s,t]}(\cdot)$ only depends on the current value $\omega_s \in \mathbb{X}$ instead of the whole history $\omega_{[0,s]}$. Again we assume that for any $0 \le s < t \le 1$, $\emptyset \ne \Gamma_{[s,t]}(x) \subset \mathfrak{P}(\Omega^x_{t-s})$ for $x \in \mathbb{X}$, and $\mathrm{Gr}(\Gamma_{[s,t]})$ is analytic.

For $0 \le s < t \le 1$, let

$$C_{[s,t]} := \{P \in \mathfrak{P}(\Omega): \ P^{s,\omega} \circ X^{-1}_{[s,t]} \in \Gamma_{[s,t]}(\omega_s), \ P\text{-a.s. } \omega\}.$$

We assume $\{\Gamma_{[s,t]}\}_{0 \le s < t \le 1}$ is such that the following consistency property holds:

$$C_{[s,t]} \cap C_{[s',t']} = C_{[s\wedge s', t\vee t']}, \quad \text{if } [s,t] \cap [s',t'] \ne \emptyset. \tag{2.14}$$

Let $(\alpha_t)_{t\in[0,1]} \subset \mathfrak{P}(\mathbb{X})$. We will consider probability measures on Ω with marginals $(\alpha_t)_{t\in[0,1]}$. We assume the map $t \mapsto \alpha_t$ is continuous if $\Omega = C[0,1]$, and is right continuous if $\Omega = D[0,1]$ (otherwise $(\alpha_t)_{t\in[0,1]}$ cannot be the marginals of any $P \in \mathfrak{P}(\Omega)$). Define

$$A_t := \{P \in \mathfrak{P}(\Omega): \ P \circ X_t^{-1} = \alpha_t\}, \quad t \in [0,1].$$

Below is the main result of this section.

Theorem 2.4.1. *Assume $A_s \cap C_{[s,t]}$ is weakly compact for any $0 \le s < t \le T$. Then*

$$\bigcap_{0\le r\le 1} A_r \cap \bigcap_{0\le s<t\le 1} C_{[s,t]} \ne \emptyset \iff \alpha_s(f^{\Gamma_{[s,t]}}) \le \alpha_t(f), \ \forall\, f \in \mathfrak{C}, \ 0 \le s < t \le 1.$$

Proof. "$\Longrightarrow$" follows from the same argument used in the proof of Proposition 2.2.3.

"$\Longleftarrow$" By Theorem 2.3.3, there exists $P^n \in \Lambda^n$, where

$$\Lambda^n := \bigcap_{i=0}^{2^n} A_{i/2^n} \cap \bigcap_{j=0}^{2^n-1} C_{[j/2^n,(j+1)/2^n]}.$$

According to our assumption $A_s \cap C_{[s,t]}$ is weakly compact for any $0 \le s < t \le T$, it is easy to show that Λ^n is weakly compact for any $n \in \mathbb{N}$. By the consistency assumption (2.14), it follows that

$$\bigcap_{j=0}^{2^n-1} C_{[j/2^n,(j+1)/2^n]} = C_{[0,1]},$$

and hence

$$\Lambda^{n+1} = \bigcap_{i=0}^{2^{n+1}} A_{i/2^{n+1}} \cap C_{[0,1]} \subset \bigcap_{i=0}^{2^n} A_{i/2^n} \cap C_{[0,1]} = \Lambda^n.$$

Therefore, $P^m \in \Lambda^n$ for any $m \geq n$. In particular, $P^m \in \Lambda^1$ with Λ^1 weakly compact. Then there exists $P \in \mathfrak{P}(\Omega)$ such that

$$P^m \xrightarrow{w} P.$$

It can be seen that $P \in \Lambda^n$ for any $n \in \mathbb{N}$.

The proof of $P \in \bigcap_{0\leq r\leq 1} A_r \cap \bigcap_{0\leq s<t\leq 1} C_{[s,t]}$ goes as follows. By (2.14), $P \in C_{[0,1]} \subset C_{[s,t]}$ for any $0 \leq s < t \leq 1$. If $t \in \mathcal{T}$, where

$$\mathcal{T} := \{k/2^n : \; k = 0, \ldots, 2^n, \; n \in \mathbb{N}\},$$

then $P \circ X_t^{-1} = \alpha_t$, since $P_n \circ X_t^{-1} = \alpha_t$ for n large enough. In general, for $t \in [0,1]$, let $t^k \in \mathcal{T}$ such that $t^k \searrow t$. Since X is right continuous,

$$\alpha_{t^k} = P \circ X_{t^k}^{-1} \xrightarrow{w} P \circ X_t^{-1}.$$

As $t \mapsto \alpha_t$ is right continuous, we have $P \circ X_t^{-1} = \alpha_t$. □

Remark 2.4.2. The result still holds and the proof still goes through with minor adjustments, if we weaken/replace the assumption by: (1) there exists $\mathfrak{T} \subset [0,1]$ that is dense in $[0,1]$, such that $(A_s \cap C_{[s,t]}) \circ (X_s, X_t)^{-1}$ is convex and closed for any $s,t \in \mathfrak{T}$ with $s < t$; (2) $A_s \cap C_{[s,t]}$ is weakly compact for any $s,t \in \mathfrak{T}$ with $s < t$; (3)

the consistency assumption (2.14).

Example 2.4.3 (Martingale measures with volatility uncertainty)**.** Let $\Omega = C[0,1]$ with $\mathbb{X} = \mathbb{R}^d$. Assume α_t has a finite first moment for any $t \in [0,1]$. Let

$$\Lambda := \left\{ P \in \mathfrak{P}(\Omega) : \begin{array}{l} P \circ X_t^{-1} = \alpha_t,\ t \in [0,1],\ P \text{ martingale measure}, \\ \dfrac{d\langle X\rangle_t}{dt} \in \mathbb{D},\ dt \times P\text{-a.e.} \end{array} \right\},$$

where $\mathbb{D} \subset \mathbb{R}^{d\times d}$ is a convex and compact set of matrices.

Then it can be seen that

$$\Lambda = \bigcap_{0\le r\le 1} A_r \cap \bigcap_{0\le s<t\le 1} C_{[s,t]},$$

with $\Gamma_{[s,t]}$ defined by

$$\begin{aligned} \Gamma_{t-s} &:= \Gamma_{[s,t]}(x) \\ &:= \left\{ Q \in \mathfrak{P}(\Omega^x_{t-s}) :\ Q \text{ martingale measure},\ \frac{d\langle X\rangle_r}{dr} \in \mathbb{D},\ dr \times Q\text{-a.e.} \right\}. \end{aligned}$$

Moreover, with Γ defined above, the consistency condition (2.14) is obviously satisfied, and $A_s \cap C_{[s,t]}$ is weakly compact for any $0 \le s < t \le T$. Therefore, by Theorem 2.4.1,

$$\Lambda \neq \emptyset \iff \alpha_s(f^{\Gamma_{t-s}}) \le \alpha_t(f),\ \forall f \in \mathfrak{C},\ 0 \le s < t \le 1.$$

Again,

$$f^{\Gamma_r}(x) = \inf_{Q\in\Gamma_r(x)} \mathbb{E}^Q[f(X_T)]$$

is the G-expectation of f (see [89]).

Remark 2.4.4. When $\mathbb{X} = \mathbb{R}^d$, the existence of a martingale measure without volatility constraint with given marginals $(\mu_t)_t$ is characterized by Kellerer in [126], Hirsch and Roynette in [114]. For any stochastic process X, denote by $\mathcal{F}^X$ the filtration

$\mathcal{F}^X(t) := \sigma(X_s, s \le t)$. Then

$$\begin{aligned} &\exists \text{ martingale } X \text{ w.r.t. } \mathcal{F}^X \text{ s.t. } X_t \overset{d}{=} \mu_t \\ &\iff t \mapsto \mu_t(f) \text{ is increasing, } \forall \text{ convex functions } f. \end{aligned}$$

In particular for $d = 1$, Kellerer showed that the martingale can be Markov.

2.5 Kantorovich duality

In this section, we will provide the Kantorovich duality with our domain constraint as in section 2. Our proof idea is similar to [105, Theorem 9.5] where it proved an unconstrained result. Here we use the usual weak topology, but the results can be easily generalized to relativized case.

Consider the optimization problem

$$\mathcal{T}_c^\Gamma(\alpha, \beta) = \inf_{\pi \in \Pi_\Gamma(\alpha,\beta)} \int_{\mathbb{X}} c(x, \pi|_{X_0=x})\alpha(dx), \tag{2.15}$$

where

$$\Pi_\Gamma(\alpha, \beta) := \{\pi \in \mathfrak{P}(\Omega) : \ \pi \circ X_0^{-1} = \alpha, \ \pi \circ X_T^{-1} = \beta, \ \pi|_{X_0=x} \in \Gamma(x) \ \pi\text{-a.s.}\}$$

is the set of probability measures with marginals α, β and domain constraint Γ. We make the following assumption.

Assumption 2.5.1. *(i) The cost function $c : \mathbb{X} \times \mathfrak{P}(\Omega) \to [0, \infty]$ is lower-semicontinuous with respect to product topology.*

(ii) The function $Q \mapsto c(x, Q)$ is convex for all $x \in \mathbb{X}$.

Example 2.5.2. If c is given by

$$c(x,Q) = \int\limits_{y\in\mathbb{X}} \mathcal{C}(x,y)\; Q\circ X_T^{-1}(dy), \tag{2.16}$$

where $\mathcal{C} : \mathbb{X}\times\mathbb{X} \to [0,\infty]$ is continuous. Then

$$\mathcal{T}_c^{\Gamma}(\alpha,\beta) = \inf_{\pi\in\Pi_\Gamma(\alpha,\beta)} \int \mathcal{C}(x,y)\;\pi(dx,dy). \tag{2.17}$$

In this case, c is linear with respect to Q and Assumption 2.5.1 is satisfied.

Remark 2.5.3. By a slight modification of [13, Proposition 2.8], it can be seen that the function

$$\pi \mapsto I_c[\pi] := \int c(x, \pi|_{X_0=x})\alpha(dx)$$

is lower-semicontinuous under Assumption 2.5.1.

Remark 2.5.4. Assume $\Omega = \mathbb{X}^2$ and $A\cap C$ is weakly closed. Proposition 2.2.3 provides a necessary and sufficient condition for the non-emptiness of the weakly compact set $A\cap B\cap C$. Then under Assumption 2.5.1, the infimum in (2.15) is attained.

We use Φ (resp. $\Phi_b(\mathbb{X})$) to denote the set of continuous (resp. continuous and bounded from below) functions $\phi : \mathbb{X} \to \mathbb{R}$ satisfying the linear growth condition

$$|\phi(x)| \le a + b\, d(x,x_0), \forall x\in\mathbb{X},$$

for some $a, b \ge 0$ and some (and hence all) $x_0 \in \mathbb{X}$. Below is the Kantorovich duality with the domain constraint.

Theorem 2.5.5. *Assume $\Omega = \mathbb{X}^2$, $A\cap C$ is convex and weakly closed, and let Assumption 2.5.1 hold. Then*

$$\mathcal{T}_c^{\Gamma}(\alpha,\beta) = \sup_{\phi\in\Phi_b(\mathbb{X})} \left\{ \int R_c^{\Gamma}\phi(x)\alpha(dx) - \int \phi(y)\beta(dy) \right\}, \tag{2.18}$$

where

$$R_c^\Gamma \phi(x) := \inf_{Q\in\Gamma(x)} \{\mathbb{E}^Q[\phi(X_T)] + c(x,Q)\}, \quad x\in\mathbb{X}, \phi\in\Phi_b(\mathbb{X}).$$

Proof. We will apply Fenchel-Moreau theorem (see e.g. [52, Theorem 4.2.1]). For the rest of the proof, let $\mathfrak{M}(\mathbb{X})$ be the space of all Borel signed measures with finite first moments. We equip it with weak topology.

Consider $F : \mathfrak{M}(\mathbb{X}) \mapsto [0,\infty]$ defined by

$$F(m) = \mathcal{T}_c^\Gamma(\alpha, m) = \inf_{\pi\in\Pi_\Gamma(\alpha,m)} \int_{\mathbb{X}} c(x, \pi|_{X_0=x})\alpha(dx),$$

with the convention $\inf\emptyset = +\infty$. As $A\cap C$ is convex and weakly closed, first we show that the set

$$\mathfrak{Im} := \{P\circ X_T^{-1} : P\in A\cap C\}$$

is also convex and weakly closed. Take any convergent sequence $\{m_n\}_{n\in\mathbb{N}} \subset \mathfrak{Im}$, with $\{\pi^n\}_{n\in\mathbb{N}} \subset A\cap C$ such that $\pi^n\circ X_T^{-1} = m_n$. For any $\epsilon > 0$, since $\{m_n\}_{n\in\mathbb{N}}$ is relatively compact, we could find a compact set $K_\epsilon \subset \mathbb{X}$ such that $m_n(K_\epsilon) \geq 1-\epsilon$ for each n. Let $L_\epsilon \subset \mathbb{X}$ be a compact set such that $\alpha(L_\epsilon) \geq 1-\epsilon$. We get that $\pi^n(L_\epsilon\times K_\epsilon) \geq 1-2\epsilon$ for each n and therefore conclude $\{\pi^n\}_{n\in\mathbb{N}}$ is relatively compact by Prokhorov's Theorem. By the closedness of $A\cap C$, the limit π of the sequence $\{\pi^n\}_{n\in\mathbb{N}}$ (up to a subsequence) is in $A\cap C$. It is clear that $\{m_n\}_{n\in\mathbb{N}}$ converges to $\pi\circ X_T^{-1}\in\mathfrak{Im}$ and we conclude.

Next, we show that F is convex. Take $m_0, m_1 \in \mathfrak{P}(\mathbb{X})$. If either one of $F(m_0)$ and $F(m_1)$ is positive infinity, then we trivially have

$$F(tm_0 + (1-t)m_1) \leq tF(m_0) + (1-t)F(m_1), \quad \forall\, t\in(0,1).$$

Thus we assume $m_0, m_1 \in \mathfrak{Im}$ without loss of generality. Take $\pi^i \in \Pi_\Gamma(\alpha, m_i), i = 0, 1$.

Since the cost function c is convex in its second argument, it holds that

$$\begin{aligned} F(tm_0 + (1-t)m_1) &\le \int c(x, t\pi^0|_{X_0=x} + (1-t)\pi^1|_{X_0=x})\alpha(dx) \\ &\le t\int c(x, \pi^0|_{X_0=x})\alpha(dx) + (1-t)\int c(x, \pi^1|_{X_0=x})\alpha(dx). \end{aligned}$$

Optimizing over π^0, π^1, we get that

$$F(tm_0 + (1-t)m_1) \le tF(m_0) + (1-t)F(m_1), \quad \forall t \in (0,1),$$

which implies the convexity of F.

Then we prove that F is lower semicontinuous. Let $\{m_n\}_{n\in\mathbb{N}}$ converges to m in the weak topology. If $m \notin \mathfrak{Im}$, then by the closedness of $\mathfrak{Im}$, we have that $m_n \notin \mathfrak{Im}$ for n large enough. This implies that

$$\liminf_{n\to\infty} F(m_n) = +\infty = F(m).$$

Now consider the case $m \in \mathfrak{Im}$. Without loss of generality, we assume the limit $\lim_{n\to\infty} F(m_n)$ exists and is finite. Let $\pi^n \in \Pi_\Gamma(\alpha, m_n) \subset A \cap C$ such that

$$\int c(x, \pi^n|_{X_0=x})\alpha(dx) \le F(m_n) + \frac{1}{n}.$$

By the same argument as in the second paragraph, we know $\{\pi^n\}_{n\in\mathbb{N}}$ is relatively compact. Extracting a subsequence, we can assume $\{\pi^n\}_{n\in\mathbb{N}}$ converges to π without loss of generality. It is easily seen that $\pi \in \Pi_\Gamma(\alpha, m)$. By Assumption 2.5.1,

$$F(m) \le I_c[\pi] \le \liminf_{n\to\infty} I_c[\pi_n] = \lim_{n\to\infty} F(m_n).$$

Notice that $\mathfrak{M}^*(\mathbb{X})$ can be identified with $\Phi(\mathbb{X})$(see e.g. [105, Lemma 9.8]), i.e., for any $l \in \mathfrak{M}^*(\mathbb{X})$, there is one corresponding $\phi \in \Phi(\mathbb{X})$ such that

$$l(m) = \int_{\mathbb{X}} \phi(x) m(dx), \quad \forall\, m \in \mathfrak{M}(\mathbb{X}).$$

Therefore Fenchel-Legendre transform $F^*(l) := \sup_{m \in \mathfrak{M}} \{l(m) - F(m)\}$ is equivalent to $F^*(\phi) = \sup_{m \in \mathfrak{M}} \{\int \phi dm - F(m)\}$. Applying Fenchel-Moreau theorem, we get that

$$F(m) = \sup_{\phi \in \Phi(\mathbb{X})} \left\{ \int \phi \, dm - F^*(\phi) \right\} = \sup_{\phi \in \Phi(\mathbb{X})} \left\{ \int -\phi \, dm - F^*(-\phi) \right\}.$$

Replacing ϕ by $\phi \vee k$ and letting $k \to -\infty$, we can restrict the last supremum to $\Phi_b(\mathbb{X})$.

To conclude the proof, we show that

$$F^*(-\phi) = -\int R_c^\Gamma \phi(x) \alpha(dx), \quad \forall \phi \in \Phi_b(\mathbb{X}).$$

Since F is positive infinity outside $\mathfrak{Im}$, we have that

$$\begin{aligned}
F^*(-\phi) &= \sup_{m \in \mathfrak{Im}} \left\{ \int -\phi \, dm - F(m) \right\} \\
&= \sup_{m \in \mathfrak{Im}} \sup_{\pi \in \Pi_\Gamma(\alpha, m)} \left\{ \int -\phi \, dm - I_c[\pi] \right\} \\
&= - \inf_{\pi \in A \cap C} \left\{ \int [\mathbb{E}^{\pi|_{X_0 = x}}[\phi(X_T)] + c(x, \pi|_{X_0 = x})] \, \alpha(dx) \right\} \\
&\leq \int \left(- \inf_{Q \in \Gamma(x)} \{[\mathbb{E}^Q[\phi(X_T)] + c(x, Q)]\} \right) \alpha(dx) \\
&= - \int R_c^\Gamma \phi(x) \alpha(dx).
\end{aligned}$$

On the other hand, for any $\varepsilon > 0$, by [40, Proposition 7.50] there exists a universally measurable probability kernel $P^\varepsilon : \mathbb{X} \times \mathfrak{P}(\Omega) \to \mathbb{R}$ such that

$$\mathbb{E}^{P^\varepsilon(x,\cdot)}[\phi(X_T)] + c(x, P^\varepsilon(x, \cdot)) \leq R_c^\Gamma \phi(x) + \varepsilon.$$

Therefore,

$$F^*(-\phi) \geq -\int [\mathbb{E}^{P^\varepsilon(x,\cdot)}[\phi(X_T)] + c(x, P^\varepsilon(x,\cdot))]\alpha(dx) \geq -\int R_c^\Gamma \phi(x)\alpha(dx) - \varepsilon.$$

Taking $\varepsilon \to 0$, we conclude the result. $\square$

Remark 2.5.6. For $\Omega \neq \mathbb{X}^2$, weak closedness of $A \cap C$ cannot imply closedness of $\mathfrak{Im}$ (see Remark 2.2.4). But if we assume $A \cap C$ is convex and weakly compact, and let Assumption 2.5.1 hold, we still have (2.18) by using the same argument as above.

Corollary 2.5.7. *Let $\Omega = \mathbb{X}^2$. Assume $A \cap C$ is convex and weakly closed, and c is given by* (2.16)*. Then*

$$\mathcal{T}_c^\Gamma(\alpha, \beta) = \sup_{(f,g)\in\mathcal{F}^\Gamma(\alpha,\beta)} \left\{ \int f(x)\ \alpha(dx) + \int g(y)\ \beta(dy) \right\},$$

where

$$\mathcal{F}^\Gamma(\alpha,\beta) = \left\{ (f,g) : \begin{array}{l} -g \in \Phi_b(\mathbb{X}); \\ f(x) + \int g(y)\ p(dy) \leq \int C(x,y)\ p(dy),\ \forall x \in \mathbb{X}, p \in \Gamma(x) \end{array} \right\}$$

In particular, if we take $\Gamma(x) = \mathfrak{P}(\mathbb{X})$, $\forall x \in \mathbb{X}$, then it is easy to see that $(f,g) \in \mathcal{F}^\Gamma(\alpha,\beta)$ iff $f(x) + g(y) \leq C(x,y)$, $\forall (x,y) \in \mathbb{X} \times \mathbb{X}$. In this case, we recover the classical duality result (see e.g. [174, Theorem 1.42]).

2.6 Monotonicity principle

In this section, we provide a monotonicity principle and an application. We again use the usual weak topology. The monotonicity principle is as follows.

Theorem 2.6.1. *Let Assumption 2.5.1 hold. Assume $A \cap C$ is convex and weakly compact (or convex and weakly closed when $\Omega = \mathbb{X}^2$), and $\mathcal{T}_c^\Gamma(\alpha,\beta)$ defined in* (2.15) *is finite. Let π^* be an optimizer of $\mathcal{T}_c^\Gamma(\alpha,\beta)$. Then there exists a Borel set $\Lambda \subset \mathbb{X}$ with $\alpha(\Lambda)=1$, such that if $x, x' \in \Lambda$, $m_x \in \Gamma(x), m_{x'} \in \Gamma(x')$, and*

$$m_x + m_{x'} = \pi^*|_{X_0=x} + \pi^*|_{X_0=x'}, \tag{2.19}$$

then

$$c(x, \pi^*|_{X_0=x}) + c(x', \pi^*|_{X_0=x'}) \le c(x, m_x) + c(x, m_{x'}). \tag{2.20}$$

Proof. Take an optimizing sequence $\{\phi_n\}_n \in \Phi_b(\mathbb{X})$ for the right-hand-side of (2.18). Note that

$$\int \phi_n(y)\beta(dy) = \int_{x\in\mathbb{X}} \mathbb{E}^{\pi^*|_{X_0=x}}[\phi_n(X_T)]\ \alpha(dx).$$

We define

$$\begin{aligned} f_n(x) &:= R_c^\Gamma \phi_n(x) - \mathbb{E}^{\pi^*|_{X_0=x}}[\phi_n(X_T)] \\ &= \inf_{Q\in\Gamma(x)} \{\mathbb{E}^Q[\phi_n(X_T)] + c(x,Q) - \mathbb{E}^{\pi^*|_{X_0=x}}[\phi_n(X_T)]\}. \end{aligned} \tag{2.21}$$

Then it is clear that

$$\int c(x,\pi_x^*)\alpha(dx) = \lim_{n\to\infty} \int f_n(x)\alpha(dx).$$

Since $\pi^*|_{X_0=x} \in \Gamma(x)$ α- a.e, we have that $f_n(x) \le c(x,\pi^*|_{X_0=x})$ by taking $p = \pi^*|_{X_0=x}$ on the right hand side of equation (2.21). Because

$$\lim_{n\to\infty} \left(\int c(x,\pi^*|_{X_0=x}) - f_n(x) \right) \alpha(dx) = 0$$

and $c(x,\pi^*|_{X_0=x}) - f_n(x) \ge 0$, we can find a Borel set $\Lambda \subset \mathbb{X}$ and a subsequence $f_{n(k)}$ such that $\alpha(\Lambda)=1$ and

$$\lim_{k\to\infty} f_{n_k}(x) = c(x,\pi^*|_{X_0=x}) \quad \text{on } \Lambda.$$

It remains to show that Λ has the monotonicity property. Let $x, x' \in \Lambda$ and $m_x \in \Gamma(x), m_{x'} \in \Gamma(x')$ satisfy (2.19). By (2.21),

$$
\begin{aligned}
f_n(x) + f_n(x') &\leq \mathbb{E}^{m_x}[\phi_n(X_T)] + c(x, m_x) - \mathbb{E}^{\pi^*|X_0=x}[\phi_n(X_T)] \\
&\quad + \mathbb{E}^{m_{x'}}[\phi_n(X_T)] + c(x', m_{x'}) - \mathbb{E}^{\pi^*|X_0=x'}[\phi_n(X_T)] \\
&= c(x, m_x) + c(x', m_{x'}).
\end{aligned}
$$

Then (2.20) follows by sending $n \to \infty$ $\square$

Remark 2.6.2. If we take $\Omega = \mathbb{X}^2$ and $\Gamma(x) = \mathfrak{P}(\Omega^x), \ \forall x \in \mathbb{X}$, then our result recovers [12, Proposition 4.1]. While we use Kantorovich duality in the proof, [12] uses a measurable selection argument.

2.6.1 Left-monotonicity when $\Omega = \mathbb{R}^2$

In this part, we provide an application of Theorem 2.6.1. It can be thought of as an extension of [37, Theorem 6.1].

Let $\Omega = \mathbb{R}^2$. Then $\Omega^x = \{x\} \times \mathbb{R}$ can be identified with $\mathbb{R}$, and $\{P|_{X_0=x}\}_{x\in\mathbb{R}}$ is the disintegration $\{P_x\}_{x\in\mathbb{R}}$. Let

$$
\Gamma(x) = \left\{ Q \in \mathfrak{P}(\mathbb{R}) : \ Q\{y : |y - x| \leq a(x)\} = 1, \ \int y \, Q(dy) = x \right\}, \tag{2.22}
$$

where $a(\cdot)$ is a nonnegative, bounded and continuous function on $\mathbb{R}$.

Definition 2.6.3. A subset $\Delta \subset \mathbb{R}^2$ is called Γ-left monotone, if for every triple $(x, y^-), (x, y^+), (x', y') \in \Delta$ we cannot have the situation

$$
x < x', y^- < y' < y^+, |y' - x| \leq a(x), |y^- - x'| \leq a(x'), |y^+ - x'| \leq a(x'). \tag{2.23}
$$

And a transport plan $\pi \in \mathfrak{P}(\mathbb{R}^2)$ is said to be Γ-left monotone if it concentrates on a Γ-left monotone set.

Proposition 2.6.4. *Assume the cost function c is given by*

$$c(x, Q) = \int_{y\in\mathbb{R}} h(y-x)\, Q(dy),$$

where h is a differentiable function on $\mathbb{R}$ with h' strictly convex. Then any minimizer of the problem (2.15) *is Γ-left monotone.*

Proof. By Proposition 2.2.8, $A \cap C$ is convex and weakly compact. Let π^* be a minimizer of (2.15). Let Λ be given in Theorem 2.6.1, and

$$\Delta = \cup_{x\in\Lambda}\{(x, y) : y \in \operatorname{supp}(\pi^*_x)\}.$$

It is clear that $\pi^*(\Delta) = 1$. Suppose there exists a triple $(x, y^-), (x, y^+), (x', y') \in \Delta$ violates Γ-left monotonicity. We strive for a contradiction.

Because

$$y^- < y' < y^+, \quad \{y^-, y^+\} \subset \operatorname{supp}(\pi^*_x), \quad y' \in \operatorname{supp}(\pi^*_{x'}),$$

we can construct two measures μ, ν together with real numbers l, r satisfying the following property:

$$\{y^-, y^+\} \subset \operatorname{supp}(\mu) \subset \{y : |y - x'| \le a(x')\}, \quad \mu \le \pi^*_x;$$

$$y' \in \operatorname{supp}(\nu) \subset \{y : |y - x| \le a(x)\}, \quad \nu \le \pi^*_{x'};$$

$$\mu \text{ and } \nu \text{ have the same barycenter and the same mass;} \tag{2.24}$$

$$\mu \text{ is concentrated on } \mathbb{R} \setminus (l, r) \text{ while } \nu \text{ is concentrated on } [l, r]. \tag{2.25}$$

Let

$$m_x := \pi^*_x - \mu + \nu \quad \text{and} \quad m_{x'} := \pi^*_{x'} + \mu - \nu.$$

It is clear that $m_x + m_{x'} = \pi^*_x + \pi^*_{x'}$ and $m_x \in \Gamma(x), m_{x'} \in \Gamma_{x'}$. Thanks to (2.24), (2.25) and the strict convexity of h', we can apply [37, Example 2.4] and get that

$$\int h'(y-x)\,\mu(dy) > \int h'(y-x)\,\nu(dy).$$

Now we have

$$\begin{aligned}
\int h(y-x)\,\pi^*_x(dy) + \int h(y-x')\,\pi^*_{x'}(dy) - \int h(y-x)\,m_x(dy) - \int h(y-x')\,m_{x'}(dy) \\
= \int h(y-x)\,(\mu-\nu)(dy) - \int h(y-x')\,(\mu-\nu)(dy) \\
= \int_x^{x'} dz \int_{y\in\mathbb{R}} h'(y-z)(\mu-\nu)(dy) > 0,
\end{aligned}$$

which contradicts (2.20). □

Here is an example such that Γ-left monotone transport plans may not be left monotone.

Example 2.6.5. Take $\alpha = \frac{1}{2}(\delta_0 + \delta_5), \beta = \frac{1}{4}(\delta_{-2} + \delta_0 + \delta_2 + \delta_{10})$, and

$$\Gamma(x) = \left\{ Q \in \mathfrak{P}(\mathbb{R}) : Q\{y : |y-x| \le 6\} = 1, \int y\,Q(dy) = x \right\}.$$

It can be easily checked that $\frac{1}{4}(\delta_{(0,-2)} + \delta_{(0,2)} + \delta_{(5,0)} + \delta_{(5,10)})$ is the unique Γ-left monotone transport plan, while $\frac{1}{8}(2\delta_{(0,0)} + \delta_{(0,-2)} + \delta_{(0,2)} + \delta_{(5,-2)} + \delta_{(5,2)} + 2\delta_{(5,10)})$ is the left-curtain coupling (i.e. the unique left monotone transport plan; see [37]).

Next, we will prove that the minimizer of the problem (2.15) is unique if the initial distribution α concentrates on two points.

Proposition 2.6.6. *Under the assumption of Proposition 2.6.4, if the initial distribution α concentrates on two points, then there exists at most one optimizer of problem* (2.15).

Proof. Without loss of generality, we assume that $\alpha = p\delta_0 + (1-p)\delta_1$, where $p \in (0,1)$. Assuming that there are two optimizers π and $\tilde{\pi}$, we prove the proposition by contradiction. Take

$$A_0 := [-a(0), a(0)], \quad A_1 := [1-a(1), 1+a(1)],$$

where $a(.)$ defines the constraint in (2.22). Define

$$\beta_0 = \beta|_{\text{supp}(\beta)\setminus A_1}, \quad \beta_1 = \beta|_{\text{supp}(\beta)\setminus A_0}, \quad \tilde{\beta} = \beta|_{A_0 \cap A_1} = \beta - \beta_0 - \beta_1.$$

Note that the mass at initial position 0 cannot be transported to $\text{supp}(\beta)\setminus A_0$. Therefore the mass of β_1 must be transported from position 1. Hence we have

$$\pi_1|_{\text{supp}\beta\setminus A_0} = \tilde{\pi}_1|_{\text{supp}\beta\setminus A_0} = \beta_1/(1-p),$$

and similarly,

$$\pi_0|_{\text{supp}\beta\setminus A_1} = \tilde{\pi}_0|_{\text{supp}\beta\setminus A_1} = \beta_0/p. \tag{2.26}$$

Since π and $\tilde{\pi}$ are different, $\pi_0 - \tilde{\pi}_0 = \sigma^+ - \sigma^-$ is a nontrivial signed measure with positive part σ^+ and negative part σ^-. Using (2.26), the martingale condition, and the fact that $\pi_0(\mathbb{R}) = \tilde{\pi}_0(\mathbb{R}) = 1$, we obtain that $\text{supp}(\sigma^+) \cup \text{supp}(\sigma^-) \subset A_0 \cap A_1$, and that

$$\int\limits_{A_0\cap A_1} x \, \sigma^+(dx) = \int\limits_{A_0\cap A_1} x \, \sigma^-(dx), \quad \sigma^+(A_0 \cap A_1) = \sigma^-(A_0 \cap A_1). \tag{2.27}$$

Without loss of generality, assume that $y^+ := \max\{y : y \in \text{supp}(\sigma^+)\} \geq \max\{y : y \in \text{supp}(\sigma^-)\}$, and that $\sigma^-(\{y^+\}) = 0$ if these two maximums are equal. Take $y^- := \min\{y : y \in \text{supp}(\sigma^+)\}$. As a result of (2.27), there exists some $y' \in \text{supp}(\sigma^-)$

such that $y^- < y' < y^+$. Therefore, we can find two positive measures μ, ν together with two real numbers l, r satisfying the following property:

$$\{y^-, y^+\} \subset \mathrm{supp}(\mu) \subset \mathrm{supp}(\sigma^+), \quad \mu \leq \sigma^+;$$

$$y' \in \mathrm{supp}(\nu) \subset \mathrm{supp}(\sigma^-), \quad \nu \leq \sigma^-;$$

μ and ν have the same barycenter and the same mass;

μ is concentrated on $\mathbb{R} \setminus (l, r)$ while ν is concentrated on $[l, r]$.

Since π and $\tilde{\pi}$ have the same terminal distribution, i.e., $p\pi_0 + (1-p)\pi_1 = p\tilde{\pi}_0 + (1-p)\tilde{\pi}_1$, we can deduce that $\pi_1 - \tilde{\pi}_1 = \frac{p}{1-p}(\sigma^- - \sigma^+)$, and hence $\frac{p}{1-p}\nu \leq \pi_1$. Construct a new coupling π^* via $\pi_0^* = \pi_0 - \mu + \nu$ and $\pi_1^* = \pi_1 + \frac{p}{1-p}(\mu - \nu)$. Then by the same argument used in the last part of the proof of Proposition 2.6.4, it can be seen that

$$pc(0, \pi_0^*) + (1-p)c(1, \pi_1^*) < pc(0, \pi_0) + (1-p)c(1, \pi_1),$$

which contradicts our assumption that π is an optimizer. $\square$

CHAPTER III

Embedding of Walsh Brownian Motion

3.1 Introduction

The Skorokhod embedding problem was formulated and solved by Skorokhod in 1961 [175]. For a centered target distribution μ with finite first moment, one looks for a stopping time τ such that $B_\tau \sim \mu$, where $(B_t)_{t\geq 0}$ is a standard Brownian motion. Over fifty years, various solutions have been proposed, and some of them have been shown to have particular optimality properties. There is a large number of literature on this problem, but we will only mention a few of them that are related to our own work. For more detailed information, we refer the reader to [156] for a nice survey, to [35] for the Skorokhod embedding's connection with optimal transport, and to [115] for its application to mathematical finance.

Although the embedding problem for one-dimensional Brownian motion has been well studied, there are not many results in higher dimensions (see e.g. [96], [140]). As stated in [156, Section 3.10], if we consider measures concentrated on the unit circle, only the uniform distribution can be embedded by means of an integrable stopping time. The main challenge is that a multi-dimensional Brownian motion does not visit points anymore. This motivates us to consider the embedding of Walsh Brownian motion, and our result shows that any $\mu \in \mathcal{P}(\mathbb{R}^n)$ can be embedded using this alternative, where $\mathcal{P}(\mathbb{R}^n)$ is the set of Borel probability measures on $\mathbb{R}^n$.

Walsh Brownian motion is a singular diffusion with state space $\mathbb{R}^n$, which behaves like a one-dimensional Brownian motion on each ray away from $\mathbf{0}$. Once it hits the origin, it is kicked away from $\mathbf{0}$ like a reflected Brownian motion, and is assigned a random direction according to some given distribution $\kappa \in \mathcal{P}(\mathcal{S}^{n-1})$ (see e.g. [18], [98], [181]), where $\mathcal{S}^{n-1}$ is the unit sphere of $\mathbb{R}^n$. To be more precise, let us give the following definition.

Definition 3.1.1. Let $(R_t)_{t\geq 0}$ be a reflected Brownian motion and $\tau_0 := \inf\{t \geq 0 : \; R_t = 0\}$. A process $(Z_t)_{t\geq 0}$ is an n-dimensional Walsh Brownian motion with spinning measure $\kappa \in \mathcal{P}(\mathcal{S}^{n-1})$ if

(i) $Z_t = (\Gamma_t, R_t)$ in polar coordinates, where Γ_t is a $\mathcal{S}^{n-1}$-valued process with the convention that $\Gamma_t = (1, 0, \ldots, 0)$ when $R_t = 0$;

(ii) If $Z_0 = \mathbf{0}$, then for each $t > 0$, the random variable Γ_t has distribution κ and is independent of R_t;

(iii) If $Z_0 = (\gamma, r)$ with $r > 0$, then $\Gamma_t = \gamma$ on the set $\{t < \tau_0\}$, and on the set $\{t > \tau_0\}$, Γ_t has distribution κ independent of R_t.

In case $n = 1$, $\mathcal{S}^0 = \{-, +\}$, (Z, κ) becomes a skew Brownian motion. We will discuss this simplest case and then the general case. The discussion of the skew Brownian motion will let us see what to expect. Before that, let us introduce some notation. Consider any $\mu \in \mathcal{P}(\mathbb{R}^n)$ as a measure on the product space $\mathcal{S}^{n-1} \times \mathbb{R}_+$. Denote its marginal on $\mathcal{S}^{n-1}$ by $\tilde{\mu}^\sigma$, and its disintegration with respect to $\tilde{\mu}^\sigma$ by $(\tilde{\mu}_\gamma)_{\gamma \in \mathcal{S}^{n-1}}$. Also define $m^\mu = \int |z| \; \mu(dz), m^\mu_\gamma = \int_{\mathbb{R}_+} r \; \tilde{\mu}_\gamma(dr)$.

Let us now report the three observations for skew Brownian motion. First, for a skew Brownian motion, suppose there exists a stopping time τ such that $Z_\tau \sim \mu$. Then if $\mu \in \mathcal{P}(\mathbb{R})$ charges $(-\infty, 0)$, we must have $\kappa(-) > 0$. Similarly, if μ charges $(0, +\infty)$, then $\kappa(+) > 0$. Hence it is necessary that $\tilde{\mu}^\sigma \ll \kappa$, i.e., $\tilde{\mu}^\sigma$ is absolutely continuous with respect to κ. It can also be shown that $\tilde{\mu}^\sigma \ll \kappa$ is a sufficient condition for the existence of such τ that $Z_\tau \sim \mu$ (see the discussion of Section 3.3).

The second observation is about the minimality of stopping time τ and the uniform integrability of stopped process $(Z_{\tau\wedge t})_{t\geq 0}$. By using scale functions and speed measures, one can construct a one-to-one correspondence between the embedding problem for one-dimensional diffusions and for Brownian motion (see [11]). In the case of skew Brownian motion, the corresponding scale function is

$$s_\kappa(x) = \begin{cases} \kappa(-)x & \text{if } x \geq 0, \\ \kappa(+)x & \text{if } x < 0. \end{cases}$$

Then the scaled process $M_t := s_\kappa(Z_t)$ is a martingale, and there exists some standard Brownian motion $(B_t)_{t\geq 0}$ such that $M_t = B_{\langle M\rangle_t}$ (see e.g. [111], [181]). Therefore, $Z_\tau \sim \mu$ is equivalent to that $B_{\langle M\rangle_\tau} \sim \mu \circ s_\kappa^{-1}$, where $\mu \circ s_\kappa^{-1}$ is the pushforward measure of μ along s_κ. According to [71] and [150], if $\int_{\mathbb{R}} x \; \mu \circ s_\kappa^{-1}(dx) = 0$, then the stopping time $\langle M\rangle_\tau$ is minimal if and only if the stopped process $(B_{\langle M\rangle_\tau \wedge t})_{t\geq 0}$ is uniformly integrable. Note that $\int_{\mathbb{R}} x \; \mu \circ s_\kappa^{-1}(dx) = \kappa(-)\tilde{\mu}^\sigma(+)m_+^\mu - \kappa(+)\tilde{\mu}^\sigma(-)m_-^\mu$. Therefore, $\mu \circ s_\kappa^{-1}$ being centered is equivalent to $\kappa(\pm) = \frac{m_\pm^\mu}{m^\mu}\tilde{\mu}^\sigma(\pm)$. It can also be seen that the minimality of τ and $\langle M\rangle_\tau$ and the uniform integrability of $(Z_{\tau\wedge t})_{t\geq 0}$ and $(B_{\langle M\rangle_\tau \wedge t})_{t\geq 0}$ are equivalent. Putting them all together we see that if $Z_\tau \sim \mu$ and $\kappa(\pm) = \frac{m_\pm^\mu}{m^\mu}\tilde{\mu}^\sigma(\pm)$, then τ is minimal if and only if $(Z_{\tau\wedge t})_{t\geq 0}$ is uniformly integrable.

Third observation is about the optimal embedding problems. In [83], Cox and Hobson reconstructed Vallois' embedding τ^v of Brownian motion. They also reproved that τ^v minimizes $\mathbb{E}[\Psi(L_\tau)]$ among all the admissible stopping times τ, where Ψ is a strictly convex function and L is the local time of Brownian motion. By using the scale function approach mentioned above, the results can be easily generalized to skew Brownian motion.

Now let us report our corresponding results in the case of Walsh Brownian motion, which motivated from the discussion above. Our first result, Theorem 3.3.2, shows that one can find a stopping time τ such that $Z_\tau \sim \mu$ if and only if $\tilde{\mu}^\sigma \ll \kappa$. It is proved by an application of potential theoretic results in [173]. In particular, the proof is done by first characterizing α-excessive functions of Walsh Brownian motion (see Proposition 3.2.8), and verifying assumption (3.7).

In our second result, we show the relationship between the minimality of τ and the uniform integrability of $(Z_{\tau\wedge t})_{t\geq 0}$. In Proposition 3.4.1 we prove that if there exists a stopping time τ such that $Z_\tau \sim \mu$ and the stopped process is uniformly integrable, then we must have the balanced condition, i.e.,

$$\kappa(d\gamma) = \frac{m^\mu_\gamma}{m^\mu}\tilde{\mu}^\sigma(d\gamma). \tag{$*$}$$

In Proposition 3.4.4, we prove that the uniform integrability implies the minimality. To show the converse to this last proposition, we first develop properties of potential functions of Walsh Brownian motion, then characterize the uniform integrability using these potential functions. Finally, we prove the results in Proposition 3.4.14 and Theorem 3.4.15 by adopting the method of [71], which relies on the so-called standard stopping times and identifying their relevant properties.

Third, considering Brownian motion as a Poisson point process on its excursion space, we generalize the construction of [83] to Walsh Brownian motion under the condition $(*)$; see Theorem 3.5.4. As a corollary of this result and Proposition 3.4.1, we obtain that there exists a stopping time τ such that $Z_\tau \sim \mu$ and $(Z_{\tau\wedge t})_{t\geq 0}$ uniformly integrable if and only if μ is balanced. The other corollary of Theorem 3.5.4 is that there exists an integrable stopping time τ such that $Z_\tau \sim \mu$ if and only if (μ, κ) is balanced and the second moment of μ is finite; see Corollary 3.5.6. Also, we show that the Vallois type embedding we constructed here solves the optimization problem

(see Theorem 3.5.7)

$$\inf_{\tau \in \mathcal{T}} \mathbb{E}[\Psi(L_\tau^Z)], \tag{3.1}$$

where Ψ is a strictly convex function, $\mathcal{T}$ the collection of minimal stopping times τ such that $Z_\tau \sim \mu$, and $(L_t^Z)_{t\geq 0}$ is the local time at the origin.

We would like to mention that there are two equivalent viewpoints of our results. The first is, given a Walsh Brownian motion (Z, κ), which μ can be embedded. The other is, given μ, how to choose κ such that μ can be embedded in (Z, κ).

The rest of the chapter is organized as follows. In Section 3.2, we present some auxiliary results about excursion theory and stochastic calculus of Walsh Brownian motion and its α-excessive functions as well as one-dimensional potential theory. In Section 3.3, we prove the existence of almost surely finite solutions. In Section 3.4 , we identify minimal stopping times with those τ such that $(Z_{\tau\wedge t})_{t\geq 0}$ is uniformly integrable. In Section 3.5, we construct the Vallois type embedding and prove that it solves the optimization problem (3.1).

In the rest of this section, we provide frequently used notation.

3.1.1 Notation.

Denote the left-, right-hand derivative by ∂_-, ∂_+, respectively, and denote the origin of $\mathbb{R}^n$ by $\mathbf{0}$. For any $x \in \mathbb{R}$, define $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$. Denote the probability law of Walsh Brownian motion starting from position z by $\mathbb{P}^z$ (simply by $\mathbb{P}$ if $z = \mathbf{0}$), and the expectation with respect to Walsh Brownian motion starting from position z by $\mathbb{E}^z$ (simply by $\mathbb{E}$ if $z = \mathbf{0}$). Denote a Walsh Brownian motion by $(Z_t)_{t\geq 0}$ or $(\Gamma_t, R_t)_{t\geq 0}$ in polar coordinates, and its local time at the origin by $(L_t^Z)_{t\geq 0}$. For any subsets $A, B \subset \mathcal{S}^{n-1}$, we define scalar processes

$$R_t^A := R_t \mathbb{1}_{\{\Gamma_t \in A\}}, \quad h_{A,B}(Z_t) := \kappa(A) R_t^B - \kappa(B) R_t^A. \tag{3.2}$$

Define a map Φ from punctured Euclidean space $\mathbb{R}^n \setminus \{\mathbf{0}\}$ to $\mathcal{S}^{n-1} \times \mathbb{R}_+$ as the following,

$$\Phi : z \mapsto (\gamma, r),$$

where $z = r\gamma$. Denote $\tilde{\mu} = \mu|_{\mathbb{R}^n \setminus \{\mathbf{0}\}} \circ \Phi^{-1}$, the pushforward measure of $\mu|_{\mathbb{R}^n \setminus \{\mathbf{0}\}}$. We extend $\tilde{\mu}$ to a probability measure on $\mathcal{S}^{n-1} \times [0, +\infty)$ by distributing the mass $\mu(\mathbf{0})$ to $\mathcal{S}^{n-1} \times \{0\}$ in proportion to $\gamma \mapsto \tilde{\mu}(\{\gamma\} \times \mathbb{R}_+)$. Take $k = 1 - \tilde{\mu}(\mathcal{S}^{n-1} \times \mathbb{R}_+) = \mu(\{\mathbf{0}\})$ and assign mass $\frac{k}{1-k}\tilde{\mu}(A \times \mathbb{R}_+)$ to $A \times \{0\}$ for any Borel subset $A \subset \mathcal{S}^{n-1}$. Denote the first marginal of $\tilde{\mu}$ by $\tilde{\mu}^\sigma$, the disintegration of $\tilde{\mu}$ with respect to $\tilde{\mu}^\sigma$ by $(\tilde{\mu}_\gamma)_{\gamma \in \mathcal{S}^{n-1}}$.

For any $\mu \in \mathcal{P}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} |z| \ \mu(dz) < +\infty$, define

$$\begin{aligned}
m^\mu &= \int\limits_{\mathbb{R}^n} |z| \ \mu(dz), \\
m^\mu_\gamma &= \int\limits_0^\infty r \ \tilde{\mu}_\gamma(dr), \quad \gamma \in \mathcal{S}^{n-1}, \\
c^\mu_\gamma(x) &= \frac{\int_{\mathbb{R}} \Big| m^\mu y / m^\mu_\gamma - x \Big| \ \tilde{\mu}_\gamma(dy) + x + m^\mu}{2}, \quad x \in \mathbb{R}.
\end{aligned} \tag{3.3}$$

When μ is clear from the context, we may write m for m^μ, m_γ for m^μ_γ. In the special case of $\mu = \delta_{\mathbf{0}}$, we define $c^\mu_\gamma(x) = |x|$, $\tilde{\mu}^\sigma = \kappa$, $\frac{m^\mu}{m^\mu_\gamma} = 1$ for convention. Foy any stopping time τ, denote the distribution of Z_τ by $\mu(\tau)$. For simplicity, we also denote

$$m^\tau = m^{\mu(\tau)}, \quad m^\tau_\gamma = m^{\mu(\tau)}_\gamma, \quad c^\tau_\gamma = c^{\mu(\tau)}_\gamma, \quad \gamma \in \mathcal{S}^{n-1}.$$

3.2 Preliminaries

In this section, we present some auxiliary results.

3.2.1 Excursions of Walsh Brownian motion

The excursion space for the Walsh Brownian motion $(Z_t)_{t\geq 0}$ is given by

$$\mathcal{U}_Z = \mathcal{S}^{n-1} \times \mathcal{U}_R,$$

where $\mathcal{U}_R = \{e \in \mathcal{C}([0,+\infty)) : e^{-1}(0,+\infty) = (0,\xi),\ \text{for some } \xi > 0\}$ is the excursion space of reflected Brownian motion. Here $\xi \in (0,+\infty)$ is the lifetime of excursion e. We can associate $(Z_t)_{t\geq 0}$ with a Poisson point process on $\mathcal{U}_Z$. To see this, let $(L_t^Z)_{t\geq 0}$ denote the local time of $(Z_t)_{t\geq 0}$ at the origin. It characterizes the amount of time spent by $(Z_t)_{t\geq 0}$ at $\mathbf{0}$ and is just the local time of $(R_t)_{t\geq 0}$ at 0. Take $(I_l)_{l\geq 0}$ to be the right continuous inverse of $(L_t^Z)_{t\geq 0}$. We "label" excursions using the local time at $\mathbf{0}$.

Definition 3.2.1. The excursion point process is the process $(e_l)_{l\geq 0}$, defined with values in $\mathcal{U}_Z$ by

(i) if $I_l - I_{l-} > 0$, then e_l is the map

$$t \mapsto \mathbb{1}_{\{t \leq I_l - I_{l-}\}} Z_{I_{l-}+t};$$

(ii) if $I_l - I_{l-} = 0$, then e_l is the identically zero function.

It can be shown that this excursion point process is a Poisson point process, with intensity function given by a unique σ-finite measure η on the excursion space $\mathcal{U}_Z$ (refer to [169, Chapter XII], [171, Chapter VI, Section 8] for details). For any $\mathcal{U} \subset \mathcal{U}_Z$ and $l > 0$, we set $\mathcal{U}_l := (0,l) \times \mathcal{U}$, and

$$N^{\mathcal{U}_l} = \sum_{0<s<l} \mathbb{1}_{\mathcal{U}}(e_s),$$

which is the number of excursions in $\mathcal{U}$ before local time l. It can be shown that $N^{\mathcal{U}_l}$ is a Poisson random variable with parameter $l\eta(\mathcal{U})$. According to our construction, the measure η is the product $\kappa \times n$, where n is the excursion measure for reflected Brownian motion. We recall one important property of the measure n, which is used in Section 3.5 (see e.g. [169, Chapter XII, Exercise 2.10]).

Lemma 3.2.2. *For every $x > 0$, we have*

$$n\left(\left\{e \in \mathcal{U}_R : \sup_{t \geq 0} e(t) \geq x\right\}\right) = \frac{1}{x}.$$

3.2.2 Stochastic calculus

First we state the change of variable formula for Walsh Brownian motion (see [110], [119], [125]), and then prove a simple proposition which is used many times in the chapter.

Definition 3.2.3. Let $\mathfrak{D}$ be the class of Borel measurable functions $g : \mathbb{R}^n \to \mathbb{R}$, such that

(i) For every $\gamma \in \mathcal{S}^{n-1}$, the function $r \mapsto g_\gamma(r)$ is differentiable on $[0, +\infty)$, and the derivative $r \mapsto g'_\gamma(r)$ is absolutely continuous on $[0, +\infty)$;

(ii) The function $\gamma \mapsto g'_\gamma(0)$ is bounded;

(iii) There exist a real number $\xi > 0$ and a Lebesgue-integrable function $\iota : (0, \xi] \to [0, +\infty)$ such that $|g''_\gamma(r)| \leq \iota(r)$ holds for all $\gamma \in \mathcal{S}^{n-1}$ and $r \in (0, \xi]$.

Now define

$$B_t^Z = R_t - R_0 - L_t^Z,$$

which is a Brownian motion according to [18, Lemma 2]. We have the following change of variable formula (see [125, Theorem 2.12]).

Lemma 3.2.4. *Let $(Z_t)_{t \geq 0}$ be a Walsh Brownian motion with spinning measure κ. Then for any $g \in \mathfrak{D}$, the process $g(Z_t)_{t \geq 0}$ is a continuous semimartingale and satisfies* *the identity*

$$g(Z_t) = g(Z_0) + \int_0^t \mathbb{1}_{\{R_s \neq 0\}} g'_{\Gamma_s}(R_s)\ dB_s^Z + \int_0^t \mathbb{1}_{\{R_s \neq 0\}} g''_{\Gamma_s}(R_s)\ ds + V_g^Z(t),$$

where $V_g^Z(t) := \big(\int_{\gamma\in\mathcal{S}^{n-1}} \partial_+ g_\gamma(0)\kappa(d\gamma)\big) L_t^Z$.

Proposition 3.2.5. *Suppose* $\rho : \mathcal{S}^{n-1} \to (0, +\infty)$ *is bounded, and define the hitting time of* ρ,

$$\tau(\rho) = \inf\{t \geq 0 : \rho(\Gamma_t) = R_t\}.$$

Then we have $\tau(\rho) < +\infty$, *and*

$$\mathbb{P}[\Gamma_{\tau(\rho)} \in d\gamma] = \frac{\frac{1}{\rho(\gamma)}}{\int_{\beta\in\mathcal{S}^{n-1}} \frac{1}{\rho(\beta)}\kappa(d\beta)}\kappa(d\gamma), \qquad \mathbb{E}[\tau(\rho)] = \frac{\int_{\gamma\in\mathcal{S}^{n-1}} \rho(\gamma)\kappa(d\gamma)}{\int_{\gamma\in\mathcal{S}^{n-1}} \frac{1}{\rho(\gamma)}\kappa(d\gamma)}.$$

Proof. For any disjoint Borel subsets $A, B \subset \mathcal{S}^{n-1}$, recall the scalar process $h_{A,B}(Z_t)$. Applying Lemma 3.2.4, we see that $(h_{A,B}(Z_t))_{t\geq 0}$ is a martingale. By the optional sampling theorem, we obtain

$$0 = \mathbb{E}[h_{A,B}(Z_{\tau(\rho)})] = \kappa(A) \int_{\gamma\in B} \rho(\gamma)\mathbb{P}[\Gamma_{\tau(\rho)} \in d\gamma] - \kappa(B) \int_{\gamma\in A} \rho(\gamma)\mathbb{P}[\Gamma_{\tau(\rho)} \in d\gamma].$$

Since choices of A and B are arbitrary, it can be see that $\frac{\rho(\gamma)\mathbb{P}[\Gamma_{\tau(\rho)}\in d\gamma]}{\kappa(d\gamma)}$ is a constant, from which we can deduce the first part of the lemma.

Take $g(\gamma, r) = r^2$. Again by Lemma 3.2.4, we know that $(g(Z_t) - t)_{t\geq 0}$ is a martingale. By employing the optional sampling theorem, we conclude

$$\mathbb{E}[\tau(\rho)] = \mathbb{E}[g(Z_{\tau(\rho)})] = \int_{\gamma\in\mathcal{S}^{n-1}} \rho^2(\gamma)\mathbb{P}[\Gamma_{\tau(\rho)} \in d\gamma] = \frac{\int_{\gamma\in\mathcal{S}^{n-1}} \rho(\gamma)\kappa(d\gamma)}{\int_{\gamma\in\mathcal{S}^{n-1}} \frac{1}{\rho(\gamma)}\kappa(d\gamma)}.$$

□

3.2.3 α-excessive functions

Here we characterize bounded α-excessive functions of Walsh Brownian motion.

Definition 3.2.6. Let $\alpha \geq 0$, and $(P_t)_{t\geq 0}$ be the semigroup of Walsh Brownian motion $(Z_t)_{t\geq 0}$ (see [18]). A non-negative universally measurable function g is called α-excessive relative to $(Z_t)_{t\geq 0}$ if

(i) $g \geq e^{-\alpha t} P_t g$ for every $t \geq 0$;

(ii) $e^{-\alpha t} P_t g \to g$ pointwise as $t \to 0$.

The characterization of α-excessive functions for Brownian motion is well-known. The following lemma can be deduced easily from [51, Chapter II], after which we present the result for Walsh Brownian motion.

Lemma 3.2.7. *Let $g : \mathbb{R} \to [0, +\infty)$ be an α-excessive function of Brownian motion. Then there exists a non-negative concave function W such that $g(x) = e^{-\sqrt{2\alpha}x} W(e^{2\sqrt{2\alpha}x})$.*

Proposition 3.2.8. *Suppose g is a bounded α-excessive function of $(Z_t)_{t\geq 0}$, then there exists a family of functions $(W_\gamma)_{\gamma\in\mathcal{S}^{n-1}}$ such that*

(i) The function $W_\gamma : [1, +\infty) \to [0, +\infty)$ is concave and $W_\gamma(1) = g(\mathbf{0})$;

(ii) For $\gamma \in \mathcal{S}^{n-1}$, we have $g_\gamma(r) = e^{-\sqrt{2\alpha}r} W_\gamma(e^{2\sqrt{2\alpha}r})$;

(iii) $\int_{\gamma\in\mathcal{S}^{n-1}} \partial_+ W_\gamma(1)\ \kappa(d\gamma) \leq \frac{g(\mathbf{0})}{2}$.

Proof. By the strong Markov property for Walsh Brownian motion and the definition of α-excessive function, we have, for all $s \leq t$

$$\mathbb{E}^z[e^{-\alpha t} g(Z_t)|\mathcal{F}_s] = e^{-\alpha s}\mathbb{E}^{Z_s}[e^{-\alpha(t-s)} g(Z_{t-s})] \leq e^{-\alpha s} g(Z_s).$$

Therefore, $(e^{-\alpha t} g(Z_t))_{t\geq 0}$ is a supermartingale, and hence for any stopping time τ,

$$g(z) \geq \mathbb{E}^z[e^{-\alpha\tau} g(Z_\tau)].$$

By restricting g to a single ray in E, we obtain that g_γ is α-excessive for Brownian motion for each $\gamma \in \mathcal{S}^{n-1}$. Using Lemma 3.2.7, we can conclude (i)&(ii) of the proposition.

Take $\tau(\epsilon) := \inf\{t \geq 0 : R_t = \epsilon\}$. Employing Proposition 3.2.5, we obtain that $\mathbb{E}[\tau(\epsilon)] = \epsilon^2$ and $\mathbb{P}[\Gamma_{\tau(\epsilon)} \in d\gamma] = \kappa(d\gamma)$. By the optional sampling theorem, we get $g(\mathbf{0}) \geq \mathbb{E}[e^{-\alpha\tau(\epsilon)} g(Z_{\tau(\epsilon)})]$. Subtracting both sides by $g(\mathbf{0})$ and dividing the inequality by ϵ, we obtain

$$\begin{aligned} 0 \geq \mathbb{E}\left[\frac{g(Z_{\tau(\epsilon)}) - g(\mathbf{0})}{\epsilon}\right] - \mathbb{E}\left[\frac{1 - e^{-\alpha\tau(\epsilon)}}{\epsilon} g(Z_{\tau(\epsilon)})\right] \\ = \int\limits_{\gamma \in \mathcal{S}^{n-1}} \frac{g(\gamma, \epsilon) - g(\mathbf{0})}{\epsilon} \kappa(d\gamma) - \mathbb{E}\left[\frac{1 - e^{-\alpha\tau(\epsilon)}}{\epsilon} g(Z_{\tau(\epsilon)})\right]. \end{aligned} \tag{3.4}$$

Since function g is bounded, we can obtain the following inequality,

$$\begin{aligned} \lim_{\epsilon \to 0} \mathbb{E}\left[\frac{1 - e^{-\alpha\tau(\epsilon)}}{\epsilon} g(Z_{\tau(\epsilon)})\right] \leq \|g\|_\infty \lim_{\epsilon \to 0} \mathbb{E}\left[\frac{1 - e^{-\alpha\tau(\epsilon)}}{\epsilon}\right] \\ \leq \|g\|_\infty \lim_{\epsilon \to 0} \mathbb{E}\left[\frac{\alpha\tau(\epsilon)}{\epsilon}\right] = \|g\|_\infty \lim_{\epsilon \to 0} \alpha\epsilon = 0. \end{aligned} \tag{3.5}$$

As to the first term on the right-hand side of (3.4), we rewrite $\frac{g_\gamma(\epsilon) - g(\mathbf{0})}{\epsilon} = \int_0^\epsilon \frac{g'_\gamma(r)}{\epsilon}\, dr$ as a sum of two integrals

$$-\frac{\int_0^\epsilon \sqrt{2\alpha} g_\gamma(r)\, dr}{\epsilon} + \frac{2(e^{\sqrt{2\alpha}\epsilon} - 1)}{\epsilon} \frac{\int_1^{e^{\sqrt{2\alpha}\epsilon}} W'_\gamma(x^2)\, dx}{e^{\sqrt{2\alpha}\epsilon} - 1}.$$

Denote the first term by $-u_\gamma(\epsilon)$, and the second term by $\frac{2(e^{\sqrt{2\alpha}\epsilon}-1)}{\epsilon} v_\gamma(\epsilon)$. In conjunction with (3.5), (3.4) becomes

$$\int\limits_{\gamma \in \mathcal{S}^{n-1}} v_\gamma(\epsilon)\kappa(d\gamma) \leq \frac{\epsilon}{2(e^{\sqrt{2\alpha}\epsilon} - 1)} \left(\int\limits_{\gamma \in \mathcal{S}^{n-1}} u_\gamma(\epsilon)\kappa(d\gamma) + \|g\|_\infty \lim_{\epsilon \to 0} \mathbb{E}\left[\frac{\alpha\tau(\epsilon)}{\epsilon}\right] \right). \tag{3.6}$$

Since g is non-negative, the derivative $W'_\gamma(r)$ is non-negative. In addition, function

$v_\gamma(\epsilon)$ increases as ϵ decreases to 0. Therefore, we can apply monotone convergence theorem to the left-hand side of (3.6). Since the boundedness of g implies the boundedness of $u_\gamma(\epsilon)$, we can apply bounded convergence theorem to the first integral on the right-hand side of (3.6). Letting ϵ decrease to 0 in (3.6), we obtain

$$\int_{\gamma\in\mathcal{S}^{n-1}} \partial_+ W_\gamma(1)\kappa(d\gamma) \leq \frac{1}{2}g(\mathbf{0}).$$

□

3.2.4 One-dimensional potential theory

Here we state [70, Lemmas 2.5, 3.2] and the related notion of being standard for stopping times of martingales, which will be used in Section 3.4.

Lemma 3.2.9. *Suppose X_l is a sequence of random variables such that X_l weakly converges to X and $\lim\limits_{l\to\infty}\mathbb{E}[|x-X_l|]$ exists for one $x\in\mathbb{R}$. If there exists a random variable Y such that $\mathbb{E}[|x-X_l|]\leq\mathbb{E}[|x-Y|]$ for any $x\in\mathbb{R}$, $l\in\mathbb{N}$, then $\lim\limits_{l\to\infty}\mathbb{E}[|x-X_l|]=\mathbb{E}[|x-X|]$ for all $x\in\mathbb{R}$.*

Definition 3.2.10. Let $(Y_t)_{t\geq0}$ be a martingale on a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq0},\mathbb{P})$. A stopping time τ with respect to the filtration $(\mathcal{F}_t)_{t\geq0}$ is said to be standard for $(Y_t)_{t\geq0}$ if $\mathbb{E}[|Y_\tau|]<+\infty$, and there exists a sequence of bounded stopping time $(\tau_l)_{l\geq0}$ such that $\lim\limits_{l\to\infty}\tau_l=\tau$ a.s., and

$$\lim_{l\to\infty}\mathbb{E}[|x-Y_{\tau_l}|]=\mathbb{E}[|x-Y_\tau|],\quad \forall x\in\mathbb{R}.$$

Lemma 3.2.11. *Let $(Y_t)_{t\geq0}$ be a martingale on a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq0},\mathbb{P})$. If τ is a stopping time with respect to the filtration $(\mathcal{F}_t)_{t\geq0}$, then the following conditions are equivalent:*

(i) τ is a standard stopping time;

(ii) $\lim_{l\to\infty} \mathbb{E}[|x - Y_{\tau\wedge l}|] = \mathbb{E}[|x - Y_\tau|], \quad \forall x \in \mathbb{R}$*;*

(iii) There exists a sequence of standard stopping times $(\tau_l)_{l\to\infty}$ *such that* $\lim_{l\to\infty} \tau_l = \tau$ *a.s., and* $\lim_{l\to\infty} \mathbb{E}[|x - Y_{\tau_l}|] = \mathbb{E}[|x - Y_\tau|], \quad \forall x \in \mathbb{R}$*;*

(iv) $\mathbb{E}[|Y_\tau|] < +\infty$ *and* $\liminf_{l\to\infty} \mathbb{E}[|Y_l|\mathbb{1}_{\{\tau>l\}}] = 0$*.*

3.3 Almost surely finite Solutions

Suppose μ is a Borel measure on the Euclidean space $\mathbb{R}^n$ and (Z, κ) a Walsh Brownian motion. We want to provide a necessary and sufficient condition for μ to be a stopping distribution of (Z, κ). Here we say μ is a stopping distribution if and only if there exists a stopping time $\tau < +\infty$ such that $Z_\tau \sim \mu$.

In the case of that $\tilde{\mu}^\sigma = \sum_{i=1}^{l} c_i \delta_{\gamma_i}$ is sum of finitely many atoms, μ can be embedded if κ charges all directions γ_i, i.e.,

$$\kappa(\gamma_i) > 0, \quad i = 1, \dots, l.$$

Since $\kappa(\gamma_i) > 0$, we have $\mathbb{P}[\inf\{t \geq 0 : Z_t = (\gamma_i, r)\} < \infty] = 1$ for any $r > 0$. We enlarge $\mathcal{F}$ such that $\mathcal{F}_0$ is rich enough to support an independent variable (only in this section). Define the stopping time,

$$\tau = \inf\{t > 0 : \ Z_t = X\},$$

where X is $\mathcal{F}_0$-measurable and of distribution μ. It is clear that $Z_\tau \sim \mu$.

However, if $\tilde{\mu}^\sigma$ is continuous, it can be seen that $\kappa(\{\gamma\}) = 0$, $\tilde{\mu}^\sigma$-a.s. for any (Z, κ). Therefore, for any $(\gamma_0, r_0) \neq \mathbf{0}$ such that $\kappa(\{\gamma_0\}) = 0$, we have $\mathbb{P}[\inf\{t \geq 0 : Z_t = (\gamma_0, r_0)\} = +\infty] = 1$. Hence, the above construction does not provide us an almost surely finite stopping time. In [173], Rost answers a general question about the existence of embedding for an arbitrary Markov process. Suppose $(X_t)_{t\geq 0}$ is a transient Markov process, U^X is its potential operator, and $\nu_0 U^X, \nu_1 U^X$ are σ-finite. Then, for the initial distribution $X_0 \sim \nu_0$, there exists a stopping time τ such that $X_\tau \sim \nu_1$ if and only if $\nu_1 U^X \leq \nu_0 U^X$. If $(X_t)_{t\geq 0}$ is not transient, it would be killed at an independent time with exponential distribution (with parameter α), which results in $(X_t^\alpha)_{t\geq 0}$. Rost proved that ν_1 is a stopping distribution of $(X_t)_{t\geq 0}$ starting with ν_0, if and only if $\lim_{\alpha\to 0}(\nu_1 U^{X^\alpha} - \nu_0 U^{X^\alpha}) \leq 0$. The function $U^X(f)$ is excessive for any measurable f, so the result can be reformulated as the following (see [173, Theorem 4]).

Lemma 3.3.1. *Suppose $(X_t)_{t\geq 0}$ is a Markov process with such state space that is locally compact and completely separable. A necessary and sufficient condition for μ to be a stopping distribution of $(X_t)_{t\geq 0}$ starting with $\delta_{\mathbf{0}}$ is*

$$\downarrow \lim_{\alpha\to 0} \sup_{1\geq g\in \mathfrak{S}^\alpha} \langle \mu - \delta_{\mathbf{0}}, g\rangle = 0, \tag{3.7}$$

where $\mathfrak{S}^\alpha$ is the set of α-excessive functions (see Definition 3.2.6) of $(X_t)_{t\geq 0}$, and $\langle \nu, f\rangle := \int f d\nu$ for any measure ν and measurable function f.

Note that if $\kappa(A) = 0$ for some set $A \subset \mathcal{B}(\mathcal{S}^{n-1})$, then we have $\mathbb{P}[\inf\{t > 0 : \Gamma_t \in A\} = +\infty] = 1$, i.e., the Walsh Brownian motion does not visit the region $\{(\gamma, r) : r > 0, \gamma \in A\}$ a.s. Therefore, if $\tilde{\mu}^\sigma(A) > 0$, we must have $\kappa(A) > 0$ in order to make μ a stopping distribution, that is, $\tilde{\mu}^\sigma \ll \kappa$ is a necessary condition. We will show that it is also sufficient by checking (3.7) in Lemma 3.3.1.

Define $\tilde{\mu}_\gamma^\alpha$ to be the pushforward measure of $\tilde{\mu}_\gamma$ under the mapping $r \mapsto e^{2\sqrt{2\alpha}r}$. Let $g \in \mathfrak{S}^\alpha$ and $(W_\gamma)_{\gamma\in\mathcal{S}^{n-1}}$ be the characterization of g as in Proposition 3.2.8. We make two observations: (1) As $\alpha \to 0$, the measure $\tilde{\mu}_\gamma^\alpha$ weakly converges to δ_1; (2) The condition $g \leq 1$ is equivalent to that $W_\gamma(x) \leq \sqrt{x}$ for any $\gamma \in \mathcal{S}^{n-1}$.

Theorem 3.3.2. *If $\tilde{\mu}^\sigma$ is absolutely continuous with respect to κ, then the equation* (3.7) *holds for Walsh Brownian motion $(Z_t)_{t\geq 0}$. As a result, μ is a stopping distribu-*

tion of (Z,κ) if and only if $\tilde{\mu}^\sigma \ll \kappa$.

Proof. Recall that we have the characterization of α-excessive functions by Proposition 3.2.8. Taking $g \equiv 1$, we see that $\sup_{1 \geq g \in \mathfrak{S}^\alpha} \langle \mu - \delta_{\mathbf{0}}, g \rangle \geq 0$ for any $\alpha > 0$. It is sufficient to show for any $\epsilon > 0$, there exists $\alpha_0 > 0$ such that $\langle \mu - \delta_{\mathbf{0}}, g \rangle < \epsilon$ for any $\alpha < \alpha_0$ and $1 \geq g \in \mathfrak{S}^\alpha$.

Since $\tilde{\mu}^\sigma \ll \kappa$, we can find $K > \frac{8}{\epsilon}$ such that for any $A \subset \mathcal{B}(\mathcal{S}^{n-1})$, $\kappa(A) < \frac{1}{K}$ implies $\tilde{\mu}^\sigma(A) < \frac{\epsilon}{8}$. Take $\delta = \frac{\epsilon}{4K}$, and α_0 such that for any $\alpha < \alpha_0$

$$\int\limits_{\gamma \in \mathcal{S}^{n-1}} \tilde{\mu}^\alpha_\gamma([1, 1+\delta])\ \tilde{\mu}^\sigma(d\gamma) > 1 - \frac{\epsilon}{4}.$$

If $C := g(\mathbf{0}) > 1 - \epsilon$, we automatically have $\langle \mu - \delta_{\mathbf{0}}, g \rangle < 1 - C < \epsilon$. So without loss of generality, we assume $C \leq 1 - \epsilon$.

Since W_γ is concave, it is upper bounded on the interval $[1, 1+\delta]$ by the linear function

$$C + \partial_+ W_\gamma(1)(x-1).$$

In order to have

$$C + \partial_+ W_\gamma(1)(x-1) \leq \sqrt{x},\ x \in [1, 1+\delta],$$

the derivative $\partial_+ W_\gamma(1)$ cannot be greater than $\frac{\sqrt{1+\delta}-C}{\delta}$. Denote by H the collection of γ such that $\partial_+ W_\gamma(1) > \frac{\sqrt{1+\delta}-C}{\delta}$, i.e.,

$$H := \left\{ \gamma \in \mathcal{S}^{n-1} : \partial_+ W_\gamma(1) > \frac{\sqrt{1+\delta}-C}{\delta} \right\}.$$

By part *(iii)* of Proposition 3.2.8 and Markov's inequality, we have

$$\kappa(H) \leq \frac{C\delta}{2\sqrt{1+\delta} - 2C} \leq \frac{\delta}{2\epsilon} = \frac{1}{8K} < \frac{\epsilon}{64},$$

and Therefore,

$$\tilde{\mu}^\sigma(H) < \frac{\epsilon}{8}.$$

Denote

$$F := \{\gamma \in \mathcal{S}^{n-1} : \frac{d\tilde{\mu}^\sigma}{d\kappa} \le K\},$$

$$G := \mathcal{S}^{n-1} \setminus (H \cup F) \subset \{\gamma \in \mathcal{S}^{n-1} : \partial_+ W_\gamma(1) \le \frac{\sqrt{1+\delta} - C}{\delta}\}.$$

Note that $\int_{\gamma\in\mathcal{S}^{n-1}} \frac{d\tilde{\mu}^\sigma}{d\kappa}\ \kappa(d\gamma) = 1$. Therefore, by Markov's inequality, we have $\kappa(G) < \frac{1}{K}$ and thus $\tilde{\mu}^\sigma(G) < \frac{\epsilon}{8}$. According to our choice of δ, it can be seen that

$$\begin{aligned}
\mu(g) &= \int\limits_{\gamma\in\mathcal{S}^{n-1}} \tilde{\mu}^\sigma(d\gamma) \int\limits_0^{+\infty} g_\gamma(r)\ \tilde{\mu}_\gamma(dr) \\
&= \int\limits_{\gamma\in\mathcal{S}^{n-1}} \tilde{\mu}^\sigma(d\gamma) \int\limits_1^{+\infty} \frac{W_\gamma(x)}{\sqrt{x}}\ \tilde{\mu}^\alpha_\gamma(dx) \qquad (3.8) \\
&\le \int\limits_{\gamma\in\mathcal{S}^{n-1}} \tilde{\mu}^\sigma(d\gamma) \int\limits_1^{1+\delta} \frac{W_\gamma(x)}{\sqrt{x}}\ \tilde{\mu}^\alpha_\gamma(dx) + \frac{\epsilon}{4}.
\end{aligned}$$

We estimate the term $\frac{W_\gamma(x)}{\sqrt{x}}$ in (3.8). For $\gamma \in H$, we have inequalities

$$\int\limits_1^{1+\delta} \frac{W_\gamma(x)}{\sqrt{x}} \tilde{\mu}^\alpha_\gamma(dx) \le \tilde{\mu}^\alpha_\gamma([1, 1+\delta]) \le 1,$$

and for $\gamma \notin H$,

$$\int\limits_1^{1+\delta} \frac{W_\gamma(x)}{\sqrt{x}}\ \tilde{\mu}^\alpha_\gamma(dx) \le \int\limits_1^{1+\delta} \frac{C + \partial_+ W_\gamma(1)(x-1)}{\sqrt{x}}\ \tilde{\mu}^\alpha_\gamma(dx) \le C + \delta\partial_+ W_\gamma(1).$$

Therefore, we obtain the upper bound,

$$\mu(g) \leq C + \int\limits_{\gamma \in \mathcal{S}^{n-1} \setminus H} \delta \partial_+ W_\gamma(1) \; \tilde{\mu}^\sigma(d\gamma) + \tilde{\mu}^\sigma(H) + \frac{\epsilon}{4}.$$

For $\gamma \in F$, we have $\tilde{\mu}^\sigma(d\gamma) \leq K\kappa(d\gamma)$, and for $\gamma \in G$,

$$\partial_+ W_\gamma(1) \leq \frac{(\sqrt{1+\delta} - C)}{\delta}.$$

In conjunction with part (iii) of Proposition 3.2.8, we get

$$\begin{aligned}
&\int\limits_{\gamma \in \mathcal{S}^{n-1} \setminus H} \delta \partial_+ W_\gamma(1) \; \tilde{\mu}^\sigma(d\gamma) \\
&\qquad \leq \int\limits_{\gamma \in F} \delta \partial_+ W_\gamma(1) K \; \kappa(d\gamma) + \int\limits_{\gamma \in G} \frac{\delta(\sqrt{1+\delta} - C)}{\delta} \; \tilde{\mu}^\sigma(d\gamma) \\
&\qquad \leq \frac{\delta K C}{2} + 2\tilde{\mu}^\sigma(G).
\end{aligned}$$

Now we can conclude the result,

$$\mu(g) \leq C + \frac{\delta K C}{2} + 2\tilde{\mu}^\sigma(G) + \tilde{\mu}^\sigma(H) + \frac{\epsilon}{4} < \delta_{\mathbf{0}}(g) + \epsilon.$$

□

3.4 Minimality and Uniform Integrability

For Skorokhod embedding problem for Brownian motion, it was proved that a stopping time is minimal if and only if the stopped Brownian motion is uniformly integrable (see [71] and [150]). In this section, we prove the analogue in the case of Walsh Brownian motion. First we present a necessary condition for the stopped process $(Z_{\tau \wedge t})_{t \geq 0}$ to be uniformly integrable. Then we show that the uniform integrability of $(Z_{\tau \wedge t})_{t \geq 0}$ implies the minimality of τ. To show the other direction, we adopt the potential theory method of [70] and [71]. In the rest of the chapter, we assume that κ is not an atom, i.e., (Z,κ) does not degenerate to a reflected Brownian motion. We would like to remark that with careful modifications by reproving [70, Lemma 2.5, 3.2] for submartingales and redefining $W_t = R_t^{\mathcal{S}^{n-1}}$ in (3.9), our results still hold when κ is an atom.

Proposition 3.4.1. *For any stopping time τ such that $Z_\tau \sim \mu$ and $(Z_{\tau\wedge t})_{t\geq 0}$ is uniformly integrable, we have $m^\mu = \int_{\mathbb{R}^n} |z| \ \mu(dz)$ is finite, and*

$$\kappa(d\gamma) = \frac{m^\mu_\gamma}{m^\mu}\tilde{\mu}^\sigma(d\gamma). \tag{$*$}$$

Proof. Suppose τ is a stopping time such that $Z_\tau \sim \mu$ and $(Z_{\tau\wedge t})_{t\geq 0}$ is uniformly integrable. For any disjoint Borel subsets $A, B \subset \mathcal{S}^{n-1}$, recall the scalar process

$$h_{A,B}(Z_t) = \kappa(A)R_t^B - \kappa(B)R_t^A.$$

Due to the uniform integrability of $(Z_{\tau\wedge t})_{t\geq 0}$ and Lemma 3.2.4, $h_{A,B}(Z_{\tau\wedge t})_{t\geq 0}$ is a uniformly integrable martingale. Therefore, $h_{A,B}(Z_{\tau\wedge t})$ converges to $h_{A,B}(Z_\tau)$ in $\mathcal{L}^1$. In particular, when $0 < \kappa(A) < 1, B = A^c$, we have

$$\mathbb{E}[|Z_\tau|] \leq \max\left\{1/\kappa(A), 1/\kappa(A^c)\right\}\mathbb{E}[|h_{A,B}(Z_\tau)|] < +\infty.$$

Also by the optional sampling theorem we have that

$$\begin{aligned}
0 &= \mathbb{E}[h_{A,B}(Z_0)] = \mathbb{E}[h_{A,B}(Z_\tau)] \\
&= \kappa(A)\int\limits_B \tilde{\mu}^\sigma(d\gamma)\int\limits_{\mathbb{R}_+} r\ \tilde{\mu}_\gamma(dr) - \kappa(B)\int\limits_A \tilde{\mu}^\sigma(d\gamma)\int\limits_{\mathbb{R}_+} r\ \tilde{\mu}_\gamma(dr) \\
&= \kappa(A)\int\limits_B m^\mu_\gamma\ \tilde{\mu}^\sigma(d\gamma) - \kappa(B)\int\limits_A m^\mu_\gamma\ \tilde{\mu}^\sigma(d\gamma).
\end{aligned}$$

Since the choice of disjoint pair (A, B) is arbitrary, there exists a constant m such that for any $\gamma \in \mathcal{S}^{n-1}$,

$$m^{\mu}_{\gamma}\tilde{\mu}^{\sigma}(d\gamma) = m\kappa(d\gamma).$$

Integrating both sides of the above equation over $\mathcal{S}^{n-1}$, we get

$$m = \int_{\mathcal{S}^{n-1}} m\ \kappa(d\gamma) = \int_{\mathcal{S}^{n-1}} m^{\mu}_{\gamma}\ \tilde{\mu}^{\sigma}(d\gamma) = \int_{\mathbb{R}^n} |z|\ \ \mu(dz) = m^{\mu} < +\infty.$$

□

In the case of $n = 1$, S^0 consists of two directions $\{-, +\}$, and the process $(Z_t)_{t\geq 0}$ becomes a skew Brownian motion. Usually in the Skorokhod embedding problem for Brownian motion, we say a target distribution μ is centered if $\int_{-\infty}^{+\infty} x\ \mu(dx) = 0$. Since the spinning measure of Brownian motion is $\kappa(+) = \kappa(-) = \frac{1}{2}$, it can be seen that μ is centered if and only if $\tilde{\mu}^{\sigma}(+)m_+ = \tilde{\mu}^{\sigma}(-)m_-$, which is equivalent to $(*)$. We generalize the concept of being centered to the case of Walsh Brownian motion.

Definition 3.4.2. A pair (μ, κ) is balanced if $\int_{\mathbb{R}^n} |z|\ \mu(dz) < \infty$ and (μ, κ) satisfies $(*)$.

Let us also recall the definition of minimality for stopping times.

Definition 3.4.3. A stopping time τ is said to be minimal if for any stopping time $\upsilon \leq \tau$, $\mathcal{L}(Z_{\upsilon}) = \mathcal{L}(Z_{\tau})$ implies $\upsilon = \tau$ a.s.

By the same argument as in [150, Proposition 2], it can be seen that for any stopping time τ there exists a minimal stopping time υ such that $\mathcal{L}(Z_{\upsilon}) = \mathcal{L}(Z_{\tau})$. In the next proposition, by modifying the argument in the first paragraph of [150, Theorem 3], we show that for any stopping time τ, the uniform integrability of $(Z_{\tau\wedge t})_{t\geq 0}$ implies the minimality of τ.

Proposition 3.4.4. *If τ is a stopping time such that $(Z_{\tau\wedge t})_{t\geq 0}$ is uniformly integrable, then τ is minimal.*

Proof. Choose a measurable subset $A \subset \mathcal{S}^{n-1}$ such that $0 < \kappa(A) < 1$, and recall

$$h_{A,A^c}(Z_t) = \kappa(A)R_t^{A^c} - \kappa(A^c)R_t^A.$$

As in the proof of Proposition 3.2.5, $h_{A,A^c}(Z_t)$ is a martingale. Since $|h_{A,A^c}(Z_{\tau\wedge t})| \leq |Z_{\tau\wedge t}|$, the stopped process $h_{A,A^c}(Z_{\tau\wedge t})$ is also uniformly integrable. Let υ be a stopping time such that $\upsilon \leq \tau$ and $\mathcal{L}(Z_\upsilon) = \mathcal{L}(Z_\tau)$. Then it can be easily seen that $\mathcal{L}(h_{A,A^c}(Z_\upsilon)) = \mathcal{L}(h_{A,A^c}(Z_\tau))$. Define

$$\bar{a} := \sup_{a\in\mathbb{R}}\{a : a \in \mathit{supp}(\mathcal{L}(h_{A,A^c}(Z_\tau)))\}.$$

Using the equality of laws and the optional sampling theorem, for any $a \leq \bar{a}$ we have that

$$\begin{aligned}\mathbb{E}[h_{A,A^c}(Z_\tau)|h_{A,A^c}(Z_\tau) \geq a] &= \mathbb{E}[h_{A,A^c}(Z_\upsilon)|h_{A,A^c}(Z_\upsilon) \geq a] \\ &= \mathbb{E}[h_{A,A^c}(Z_\tau)|h_{A,A^c}(Z_\upsilon) \geq a],\end{aligned}$$

which implies that

$$\frac{\mathbb{E}[h_{A,A^c}(Z_\tau)\mathbb{1}_{\{h_{A,A^c}(Z_\tau)\geq a\}}]}{\mathbb{P}[h_{A,A^c}(Z_\tau) \geq a]} = \frac{\mathbb{E}[h_{A,A^c}(Z_\tau)\mathbb{1}_{\{h_{A,A^c}(Z_\upsilon)\geq a\}}]}{\mathbb{P}[h_{A,A^c}(Z_\upsilon) \geq a]}.$$

Since $h_{A,A^c}(Z_\tau)$ and $h_{A,A^c}(Z_\upsilon)$ have the same law, the above equality becomes

$$\mathbb{E}[h_{A,A^c}(Z_\tau)\mathbb{1}_{\{h_{A,A^c}(Z_\tau)\geq a\}}] = \mathbb{E}[h_{A,A^c}(Z_\tau)\mathbb{1}_{\{h_{A,A^c}(Z_\upsilon)\geq a\}}].$$

For $a > \bar{a}$, due to $\mathbb{1}_{\{h_{A,A^c}(Z_\tau)\geq a\}} = \mathbb{1}_{\{h_{A,A^c}(Z_\upsilon)\geq a\}} = 0$ a.s., the above equation still holds.

Suppose X is an integrable random variable and $a \in \mathbb{R}$. Then for any $\Omega' \subset \Omega$

such that $\mathbb{P}[\Omega'] = \mathbb{P}[X \geq a]$, we have

$$\begin{aligned} X &\geq a \quad \text{ on } \{X \geq a\} \setminus \Omega', \\ X &< a \quad \text{ on } \Omega' \setminus \{X \geq a\}, \end{aligned}$$

and $\mathbb{P}[\{X \geq a\} \setminus \Omega'] = \mathbb{P}[\Omega' \setminus \{X \geq a\}]$. Therefore,

$$\mathbb{E}[X \mathbb{1}_{\{X \geq a\}}] - \mathbb{E}[X \mathbb{1}_{\Omega'}] = \mathbb{E}[X \mathbb{1}_{\{X \geq a\} \setminus \Omega'}] - \mathbb{E}[X \mathbb{1}_{\Omega' \setminus \{X \geq a\}}] \geq 0.$$

And $\mathbb{E}[X \mathbb{1}_{\{X \geq a\}}] = \mathbb{E}[X \mathbb{1}_{\Omega'}]$ if and only if $\mathbb{P}[\{X \geq a\} \setminus \Omega'] = \mathbb{P}[\Omega' \setminus \{X \geq a\}] = 0$.

Let $X = h_{A,A^c}(Z_\tau), \Omega' = \{h_{A,A^c}(Z_\upsilon) \geq a\}$, we get that

$$\mathbb{P}[h_{A,A^c}(Z_\tau) \geq a,\ h_{A,A^c}(Z_\upsilon) < a] = \mathbb{P}[h_{A,A^c}(Z_\tau) < a,\ h_{A,A^c}(Z_\upsilon) \geq a] = 0.$$

It implies that $h_{A,A^c}(Z_\upsilon) = h_{A,A^c}(Z_\tau)$ a.s. If ξ is any stopping time such that $\upsilon \leq \xi \leq \tau$, then

$$h_{A,A^c}(Z_\xi) = \mathbb{E}[h_{A,A^c}(Z_\tau) | \mathcal{F}_\xi] = \mathbb{E}[h_{A,A^c}(Z_\upsilon) | \mathcal{F}_\xi] = h_{A,A^c}(Z_\upsilon) \quad \text{a.s.}$$

Therefore, $h_{A,A^c}(Z_t)$ is constant on the interval $\upsilon \leq t \leq \tau$, which is impossible unless $\upsilon = \tau$ a.s. □

Fix an $A_0 \subset \mathcal{S}^{n-1}$ such that $0 < \kappa(A_0) < 1$. Define $s := \frac{\kappa(A_0^c)}{\kappa(A_0)}$, and a scalar process

$$W_t = s R_t^{A_0} - R_t^{A_0^c}, \quad t \geq 0. \tag{3.9}$$

According to Lemma 3.2.4, it can be seen that $(W_t)_{t \geq 0}$ is a martingale.

Lemma 3.4.5. *For any stopping time τ such that $\mu(\tau)$ is balanced, we have for any*

$A \subset \mathcal{S}^{n-1}$

$$2\mathbb{E}[(R_\tau^A - x)^+] = 2x^- + 2 \int_{\gamma \in A} (c_\gamma^\tau(m^\tau x/m_\gamma^\tau) - (m^\tau x/m_\gamma^\tau)^+)\ \kappa(d\gamma),$$

and the potential function of W_τ

$$\begin{aligned}\mathbb{E}[|x - W_\tau|] =& |x| - 2\kappa(A_0^c)m^\tau + 2s \int_{\gamma \in A_0} (c_\gamma^\tau(m^\tau x/(m_\gamma^\tau s)) - (m^\tau x/(m_\gamma^\tau s))^+)\ \kappa(d\gamma) \\ &+ 2 \int_{\gamma \in A_0^c} (c_\gamma^\tau(-m^\tau x/m_\gamma^\tau) - (-m^\tau x/m_\gamma^\tau)^+)\ \kappa(d\gamma).\end{aligned}$$

Proof. Since $\mu(\tau)$ is balanced, we have $\kappa(d\gamma) = \frac{m_\gamma^\tau}{m^\tau}\widetilde{\mu(\tau)}^\sigma(d\gamma)$. According to the definition of c_γ^τ in (3.3), it can be easily verifying that

$$\int_{\mathbb{R}} |y - x|\ \widetilde{\mu(\tau)}_\gamma(dy) = \frac{2m_\gamma^\tau}{m^\tau} c_\gamma^\tau(m^\tau x/m_\gamma^\tau) - x - m_\gamma^\tau.$$

Therefore, by direct computation

$$\begin{aligned}&2\mathbb{E}[(R_\tau^A - x)^+] = \mathbb{E}[|R_\tau^A - x|] + \mathbb{E}[R_\tau^A] - x \\ &= \widetilde{\mu(\tau)}^\sigma(A^c)\ |x| + \int_{\gamma \in A} \int_{\mathbb{R}} |y - x|\ \widetilde{\mu(\tau)}_\gamma(dy)\ \widetilde{\mu(\tau)}^\sigma(d\gamma) + \int_{\gamma \in A} m_\gamma^\tau\ \widetilde{\mu(\tau)}^\sigma(d\gamma) - x \\ &= |x| - x + 2 \int_{\gamma \in A} \frac{m_\gamma^\tau}{m^\tau}(c_\gamma^\tau(m^\tau x/m_\gamma^\tau) - (m^\tau x/m_\gamma^\tau)^+)\ \widetilde{\mu(\tau)}^\sigma(d\gamma) \\ &= 2x^- + 2 \int_{\gamma \in A} (c_\gamma^\tau(m^\tau x/m_\gamma^\tau) - (m^\tau x/m_\gamma^\tau)^+)\ \kappa(d\gamma),\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}[|x-W_\tau|] =& s \int\limits_{\gamma\in A_0}\int\limits_{\mathbb{R}} |y-x/s| \ \widetilde{\mu(\tau)}_\gamma(dy)\ \widetilde{\mu(\tau)}^\sigma(d\gamma) \\
&+ \int\limits_{\gamma\in A_0^c}\int\limits_{\mathbb{R}} |y-(-x)| \ \widetilde{\mu(\tau)}_\gamma(dy)\ \widetilde{\mu(\tau)}^\sigma(d\gamma) \\
=& |x| + s \int\limits_{\gamma\in A_0} \left(\frac{2m^\tau_\gamma}{m^\tau} c^\tau_\gamma(m^\tau x/(m^\tau_\gamma s)) - x/s - m^\tau_\gamma - |x/s|\right)\ \widetilde{\mu(\tau)}^\sigma(d\gamma) \\
&+ \int\limits_{\gamma\in A_0^c} \left(\frac{2m^\tau_\gamma}{m^\tau} c^\tau_\gamma(-m^\tau x/m^\tau_\gamma) - (-x) - m^\tau_\gamma - |x|\right)\ \widetilde{\mu(\tau)}^\sigma(d\gamma) \\
=& |x| - 2\kappa(A_0^c)m^\tau + 2s \int\limits_{\gamma\in A_0} \left(c^\tau_\gamma(m^\tau x/(m^\tau_\gamma s)) - (m^\tau x/(m^\tau_\gamma s))^+\right)\ \kappa(d\gamma) \\
&+ 2 \int\limits_{\gamma\in A_0^c} \left(c^\tau_\gamma(-m^\tau x/m^\tau_\gamma) - (-m^\tau x/m^\tau_\gamma)^+\right)\ \kappa(d\gamma).
\end{aligned}
$$

□

For any two functions $f, g : \mathbb{R} \to \mathbb{R}$, we write $f(\cdot) = g(\cdot)$ if $f(x) = g(x), \forall x \in \mathbb{R}$, and $f(\cdot) \le g(\cdot)$ if $f(x) \le g(x), \forall x \in \mathbb{R}$.

Lemma 3.4.6. *A stopping time τ is standard (see Definition 3.2.10)) for $(W_t)_{t\ge 0}$, defined in* (3.9)*, if and only if $\mu(\tau)$ is balanced and*

$$
\lim_{l\to\infty} c^{\tau\wedge l}_\gamma(m^{\tau\wedge l}\cdot/m^{\tau\wedge l}_\gamma) - (m^{\tau\wedge l}\cdot/m^{\tau\wedge l}_\gamma)^+ = c^\tau_\gamma(m^\tau\cdot/m^\tau_\gamma) - (m^\tau\cdot/m^\tau_\gamma)^+, \quad \kappa\text{-a.s.}, \tag{3.10}
$$

where the limit is understood in the sense of pointwise convergence of functions.

Proof. Note that for any A, the process $(R^A_t)_{t\ge 0}$ is submartingale, and hence by Jensen's inequality $\mathbb{E}[(R^A_{\tau\wedge l} - x)^+]$ is increasing with respect to l. Since $\tau \wedge l$ is a bounded stopping time, the stopped process $(Z_{\tau\wedge l\wedge t})_{t\ge 0}$ is uniformly integrable. Hence by Proposition 3.4.1, $\mu(\tau\wedge l)$ is balanced. Therefore, according to Lemma 3.4.5, the following functional series is increasing with respect to l for any $A \subset \mathcal{S}^{n-1}$,

$$\int_{\gamma \in A} (c_\gamma^{\tau\wedge l}(m^{\tau\wedge l}x/m_\gamma^{\tau\wedge l}) - (m^{\tau\wedge l}x/m_\gamma^{\tau\wedge l})^+)\ \kappa(d\gamma).$$

Since the integral is increasing for an arbitrary A, it can be easily verified that

$$(c_\gamma^{\tau\wedge l}(m^{\tau\wedge l}x/m_\gamma^{\tau\wedge l}) - (m^{\tau\wedge l}x/m_\gamma^{\tau\wedge l})^+) \text{ is increasing with respect to } l\ \kappa\text{-a.s.}$$

According to Lemma 3.4.5, we also have

$$\begin{aligned}\mathbb{E}[|x - W_{\tau\wedge l}|] =& |x| - 2\kappa(A_0^c)m^{\tau\wedge l} \\ &+ 2s \int_{\gamma \in A_0} (c_\gamma^{\tau\wedge l}(m^{\tau\wedge l}x/(m_\gamma^{\tau\wedge l}s)) - (m^{\tau\wedge l}x/(m_\gamma^{\tau\wedge l}s))^+)\ \kappa(d\gamma) \\ &+ 2 \int_{\gamma \in A_0^c} (c_\gamma^{\tau}(-m^{\tau}x/m_\gamma^{\tau}) - (-m^{\tau}x/m_\gamma^{\tau})^+)\ \kappa(d\gamma).\end{aligned}$$

Since $(c_\gamma^{\tau\wedge l}(m^{\tau\wedge l}x/m_\gamma^{\tau\wedge l}) - (m^{\tau\wedge l}x/m_\gamma^{\tau\wedge l})^+) = m^{\tau\wedge l}$ when $x \leq 0$, the above equation becomes

$$\begin{aligned}&\mathbb{E}[|x - W_{\tau\wedge l}|] \\ &= \begin{cases} |x| + 2s\int_{\gamma \in A_0} (c_\gamma^{\tau\wedge l}(m^{\tau\wedge l}x/(m_\gamma^{\tau\wedge l}s)) - (m^{\tau\wedge l}x/(m_\gamma^{\tau\wedge l}s))^+)\ \kappa(d\gamma) & \text{if } x \geq 0, \\ |x| + 2\int_{\gamma \in A_0^c} (c_\gamma^{\tau\wedge l}(-m^{\tau\wedge l}x/m_\gamma^{\tau\wedge l}) - (-m^{\tau\wedge l}x/m_\gamma^{\tau})^+)\ \kappa(d\gamma) & \text{if } x \leq 0.\end{cases}\end{aligned} \tag{3.11}$$

Now suppose $\mu(\tau)$ is balanced and satisfies (3.10). Since $c_\gamma^{\tau\wedge l}(0) = m^{\tau\wedge l}$, we obtain that

$$\lim_{l\to+\infty} m^{\tau\wedge l} = m^\tau.$$

By the monotone convergence theorem, we conclude that

$$\lim_{l\to\infty} \mathbb{E}[|x - W_{\tau\wedge l}|] = |x| - 2\kappa(A_0^c)m^\tau + 2s \int\limits_{\gamma\in A_0} \left(c_\gamma^\tau(m^\tau x/(m_\gamma^\tau s)) - (m^\tau x/(m_\gamma^\tau s))^+\right) \kappa(d\gamma)$$
$$+2 \int\limits_{\gamma\in A_0^c} \left(c_\gamma^\tau(-m^\tau x/m_\gamma^\tau) - (-m^\tau x/m_\gamma^\tau)^+\right) \kappa(d\gamma) = \mathbb{E}[|x - W_\tau|],$$

and hence τ is standard for $(W_t)_{t\geq 0}$.

If τ is standard for the martingale $(W_t)_{t\geq 0}$, according to [70, Section 5], it is equivalent to that $(W_{\tau\wedge t})_{t\geq 0}$ is uniformly integrable. Therefore, the stopped processes $(Z_{\tau\wedge t})_{t\geq 0}$ and $(R^A_{\tau\wedge t})_{t\geq 0}$ are also uniformly integrable. Therefore, $\mu(\tau)$ is balanced, and $\mathbb{E}[(R^A_{\tau\wedge l} - x)^+] \leq \mathbb{E}[(R^A_\tau - x)^+]$ for any $x \in \mathbb{R}$. Hence

$$\int\limits_{\gamma\in A} \left(c_\gamma^{\tau\wedge l}(m^{\tau\wedge l}x/m_\gamma^{\tau\wedge l}) - (m^{\tau\wedge l}x/m_\gamma^{\tau\wedge l})^+\right) \kappa(d\gamma)$$
$$\leq \int\limits_{\gamma\in A} \left(c_\gamma^\tau(m^\tau x/m_\gamma^\tau) - (m^\tau x/m_\gamma^\tau)^+\right) \kappa(d\gamma).$$

The above inequality is true for any $A \subset \mathcal{S}^{n-1}$. Then it can be deduced that

$$c_\gamma^{\tau\wedge l}(m^{\tau\wedge l}\cdot/m_\gamma^{\tau\wedge l}) - (m^{\tau\wedge l}\cdot/m_\gamma^{\tau\wedge l})^+ \leq c_\gamma^\tau(m^\tau\cdot/m_\gamma^\tau) - (m^\tau\cdot/m_\gamma^\tau)^+ \quad \kappa\text{-}a.s.$$

Combining the equalities $\lim_{l\to 0}\mathbb{E}[|x - W_{\tau\wedge l}|] = \mathbb{E}[|x - W_\tau|]$, $\lim_{l\to+\infty} m^{\tau\wedge l} = m^\tau$ and the monotone convergence theorem, we conclude that

$$\lim_{l\to\infty} c_\gamma^{\tau\wedge l}(m^{\tau\wedge l}\cdot/m_\gamma^{\tau\wedge l}) - (m^{\tau\wedge l}\cdot/m_\gamma^{\tau\wedge l})^+ = c_\gamma^\tau(m^\tau\cdot/m_\gamma^\tau) - (m^\tau\cdot/m_\gamma^\tau)^+, \quad \kappa\text{-}a.s.$$

□

Corollary 3.4.7. *A stopping time τ is standard if and only if $(Z_{\tau\wedge t})_{t\geq 0}$ is uniformly integrable.*

Proof. According to [70, Section 5], τ is standard for the martingale $(W_t)_{t\geq 0}$ is equivalent to that $(W_{\tau\wedge t})_{t\geq 0}$ is uniformly integrable. Then the result follows from the fact that uniform integrability of $(W_{\tau\wedge t})_{t\geq 0}$ and $(Z_{\tau\wedge t})_{t\geq 0}$ are equivalent. $\square$

Definition 3.4.8. For two balanced measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^n)$, we say $\mu \leq_{wcx} \nu$ if

$$c^{\mu}_{\gamma}(m^{\mu} \cdot /m^{\mu}_{\gamma}) - (m^{\mu} \cdot /m^{\mu}_{\gamma})^{+} \leq c^{\nu}_{\gamma}(m^{\nu} \cdot /m^{\nu}_{\gamma}) - (m^{\nu} \cdot /m^{\nu}_{\gamma})^{+}, \quad \kappa\text{-a.s.}$$

Remark 3.4.9. As in the proof of Lemma 3.4.6, it can be seen that if $\upsilon \leq \tau$ are two stopping times with τ standard, then we have $\mu(\upsilon) \leq_{wcx} \mu(\tau)$.

Lemma 3.4.10. *Suppose υ, τ are two stopping times such that $\mu(\upsilon), \mu(\tau)$ are balanced. Then $\mu(\upsilon) \leq_{wcx} \mu(\tau)$ and $\mu(\tau) \leq_{wcx} \mu(\upsilon)$ imply that $\mathcal{L}(Z_\upsilon) = \mathcal{L}(Z_\tau)$.*

Proof. According to Lemma 3.4.5 and the fact that $\mu(\upsilon)$ $\mu(\tau)$ are balanced, we have for any $x \in \mathbb{R}$

$$\mathbb{E}[(R^A_\upsilon - x)^+] = \mathbb{E}[(R^A_\tau - x)^+].$$

It implies that $\mathcal{L}(R^A_\upsilon) = \mathcal{L}(R^A_\tau)$ for any $A \subset \mathcal{S}^{n-1}$. Regarding $\mu(\upsilon), \mu(\tau)$ as measures on the product space $\mathcal{S}^{n-1} \times R_+$, we have that

$$\mu(\upsilon)(A \times (a,b)) = \mu(\tau)(A \times (a,b)), \quad \forall (a,b) \subset R_+,$$

and hence $\mu(\upsilon) = \mu(\tau)$. $\square$

Lemma 3.4.11. *Suppose τ is a stopping time such that $\mu(\tau)$ is balanced. Let $\mathcal{D}$ be the collection of standard stopping times υ such that $\upsilon \leq \tau, \mu(\upsilon) \leq_{wcx} \mu(\tau)$. With the partial order of stopping times, any totally ordered subset $\mathcal{C} \subset \mathcal{D}$ has an upper bound in $\mathcal{D}$.*

Proof. Since $\mathcal{C}$ is totally ordered, any maximal element of $\mathcal{C}$ is an upper bound of $\mathcal{C}$. Therefore, without loss of generality, we assume that there does not exist a maximal element in $\mathcal{C}$. For each $\upsilon \in \mathcal{C}$,denote its survival function by $D_\upsilon : [0, +\infty) \to [0, 1]$, i.e.,

$$D_\upsilon(t) = \mathbb{P}[\upsilon > t].$$

Note that $\xi \leq \upsilon$ is equivalent to $D_\xi \leq D_\upsilon$. Let $(t_l)_{l\in\mathbb{N}}$ be a countable dense subset of $[0, +\infty)$. Define a sequence of stopping times by induction,

(i) For t_1, choose a stopping time $\upsilon_1 \in \mathcal{C}$ such that $D_{\upsilon_1}(t_1) \geq \frac{1}{2} \sup\limits_{\upsilon\in\mathcal{C}}\{D_\upsilon(t_1)\}$.

(ii) Suppose $\upsilon_1, \dots, \upsilon_{l-1}$ are well defined. For t_l, choose a stopping time $\upsilon_l \in \mathcal{C}$ such that $\upsilon_{l-1} \leq \upsilon_l$ and $D_{\upsilon_l}(t_k) \geq \frac{l}{l+1} \sup\limits_{\upsilon\in\mathcal{C}}\{D_\upsilon(t_k)\},\ k = 1, \dots, l$.

Since $\mathcal{C}$ is totally ordered and there does not exist a maximal element, it can be easily verified that the construction works. We define

$$\upsilon_0 = \lim_{l\to\infty} \upsilon_l.$$

For any $\xi \in \mathcal{C}$, there exists some t_k such that $D_\xi(t_k) < \sup\limits_{\upsilon\in\mathcal{C}}\{D_\upsilon(x_k)\}$. Otherwise, due to the right-continuity of D_ξ, it is the actually the maximal element of $\mathcal{C}$. Therefore, there exists some $l \in \mathbb{N}$ such that $D_\xi(t_k) < \frac{l}{l+1} \sup\limits_{\upsilon\in\mathcal{C}}\{D_\upsilon(t_k)\}$, and hence $\xi \leq \upsilon_l \leq \upsilon_0$. It remains to show that υ_0 is standard and $\mu(\upsilon_0) \leq_{wcx} \mu(\tau)$.

We claim that $\lim\limits_{l\to\infty} \mathbb{E}[|W_{\upsilon_l}|] = \mathbb{E}[|W_{\upsilon_0}|]$: Since paths of Walsh Brownian motion are continuous, $sR^{A_0}_{\upsilon_l}$ converges to $sR^{A_0}_{\upsilon_0}$ a.s., and hence $sR^{A_0}_{\upsilon_l}$ weakly converges to $sR^{A_0}_{\upsilon_0}$. Define $g^k : \mathbb{R} \to \mathbb{R}, k \in \mathbb{N}$ as follows

$$g^k(x) = |x| \wedge k.$$

Therefore, we have

$$\mathbb{E}[sR_{\upsilon_l}^{A_0}] = \mathbb{E}[g^k(sR_{\upsilon_l}^{A_0})] + s\mathbb{E}[(R_{\upsilon_l}^{A_0} - k/s)^+].$$

Since $\mu(\upsilon_l) \leq_{wcx} \mu(\tau)$, by Lemma 3.4.5 we obtain that

$$\begin{aligned}\mathbb{E}[(R_{\upsilon_l}^{A_0} - k/s)^+] &= \int\limits_{\gamma\in A_0} (c_\gamma^{\upsilon_l}(m^{\upsilon_l}k/(m_\gamma^{\upsilon_l}s)) - (m^{\upsilon_l}k/(m_\gamma^{\upsilon_l}s))^+)\ \kappa(d\gamma)\\ &\leq \int\limits_{\gamma\in A_0} (c_\gamma^{\tau}(m^{\tau}k/(m_\gamma^{\tau}s)) - (m^{\tau}k/(m_\gamma^{\tau}s))^+)\ \kappa(d\gamma).\end{aligned}$$

Therefore, $\mathbb{E}[g^k(sR_{\upsilon_l}^{A_0})]$ uniformly converges to $\mathbb{E}[sR_{\upsilon_l}^{A_0}]$ as $k \to \infty$. In conjunction with the fact that

$$\lim_{l\to\infty} \mathbb{E}[g^k(sR_{\upsilon_l}^{A_0})] = \mathbb{E}[g^k(sR_{\upsilon_0}^{A_0})],$$

we conclude that $\lim\limits_{l\to\infty} \mathbb{E}[sR_{\upsilon_l}^{A_0}] = \mathbb{E}[sR_{\upsilon_0}^{A_0}]$. Similarly, we have $\lim\limits_{l\to\infty} \mathbb{E}[R_{\upsilon_l}^{A_0^c}] = \mathbb{E}[R_{\upsilon_0}^{A_0^c}]$, and Therefore,

$$\lim_{l\to\infty} \mathbb{E}[|W_{\upsilon_l}|] = \lim_{l\to\infty} \mathbb{E}[sR_{\upsilon_l}^{A_0}] + \lim_{l\to\infty} \mathbb{E}[R_{\upsilon_l}^{A_0^c}] = \mathbb{E}[sR_{\upsilon_0}^{A_0}] + \mathbb{E}[R_{\upsilon_0}^{A_0^c}] = \mathbb{E}[|W_{\upsilon_0}|].$$

We show that υ_0 is standard for $(W_t)_{t\geq 0}$. The almost sure convergence of $W_{\upsilon_l} \to W_{\upsilon_0}$ implies the weak convergence. According to our assumption $\mu(\upsilon_l) \leq_{wcx} \mu(\tau)$, we have for any $x \in \mathbb{R}, l \in \mathbb{N}$,

$$\mathbb{E}[|W_{\upsilon_l} - x|] \leq \mathbb{E}[|W_\tau - x|].$$

In conjunction with the fact that $\lim\limits_{l\to\infty} \mathbb{E}[|W_{\upsilon_l}|] = \mathbb{E}[|W_{\upsilon_0}|]$ and Lemma 3.2.9, we have that $\lim\limits_{l\to\infty} \mathbb{E}[|W_{\upsilon_l} - x|] = \mathbb{E}[|W_{\upsilon_0} - x|]$ for any $x \in \mathbb{R}$. We can conclude that υ_0 is standard for W from Lemma 3.2.11.

To close the argument, we show that $\mu(\upsilon_0) \leq_{wcx} \mu(\tau)$. Since υ_0 is standard, according to Remark 3.4.9, we have that $\mu(\upsilon_l) \leq_{wcx} \mu(\upsilon_0)$. The equation (3.11) holds for any standard stopping time. Together with $\lim_{l\to\infty} \mathbb{E}[|W_{\upsilon_l} - x|] = \mathbb{E}[|W_{\upsilon_0} - x|]$ and the monotone convergence theorem, we conclude that

$$\lim_{l\to\infty} c_\gamma^{\upsilon_l}(m^{\upsilon_l}\cdot/m_\gamma^{\upsilon_l}) - (m^{\upsilon_l}\cdot/m_\gamma^{\upsilon_l})^+ = c_\gamma^{\upsilon_0}(m^{\upsilon_0}\cdot/m_\gamma^{\upsilon_0}) - (m^{\upsilon_0}\cdot/m_\gamma^{\upsilon_0})^+, \quad \kappa\text{-}a.s.$$

Now since $\mu(\upsilon_l) \leq_{wcx} \mu(\tau)$, we have that $\mu(\upsilon_0) \leq_{wcx} \mu(\tau)$. □

Lemma 3.4.12. *If $\mu \in \mathcal{P}(\mathbb{R}^n)$ is balanced, then the stopping time $\tau := \inf\{t \geq 0 : R_t = m^\mu_{\Gamma_t}\}$ is standard for $(W_t)_{t\geq 0}$.*

Proof. Define $\tau_k := \inf\{t \geq 0 : R_t = m^\mu_{\Gamma_t} \wedge k\}$. According to Proposition 3.2.5, we have that $\mathbb{E}[\tau_n] < +\infty$ and

$$\mathbb{P}[\Gamma_{\tau_k} \in d\gamma] = \frac{\frac{1}{m^\mu_\gamma \wedge k}\ \kappa(d\gamma)}{\int_{\beta\in\mathcal{S}^{n-1}} \frac{1}{m^\mu_\beta \wedge k}\ \kappa(d\beta)}.$$

It is clear that $\widetilde{\mu(\tau_k)}_\gamma = \delta_{m^\mu_\gamma \wedge k}$, $m^{\tau_k}_\gamma = m^\mu_\gamma \wedge k$, and $m^{\tau_k} = 1/\int_{\beta\in\mathcal{S}^{n-1}} \frac{1}{m^\mu_\beta \wedge k}\ \kappa(d\beta)$. Therefore, we obtain that

$$(c^{\tau_k}_\gamma(m^{\tau_k}x/m^{\tau_k}_\gamma) - (m^{\tau_k}x/m^{\tau_k}_\gamma)^+) = \begin{cases} m^{\tau_k}, & \text{if } x \in (-\infty, 0], \\ m^{\tau_k} - \frac{m^{\tau_k}x}{m^{\tau_k}_\gamma}, & \text{if } x \in [0, m^{\tau_k}_\gamma], \\ 0, & \text{if } x \in [m^{\tau_k}_\gamma, +\infty)\ . \end{cases}$$

Since Γ_{τ_k} converges to Γ_τ a.s., it can be seen that

$$\mathbb{P}[\Gamma_\tau \in d\gamma] = \lim_{k\to\infty} \frac{\frac{1}{m^\mu_\gamma \wedge k}\ \kappa(d\gamma)}{\int_{\beta\in\mathcal{S}^{n-1}} \frac{1}{m^\mu_\beta \wedge k}\ \kappa(d\beta)} = \frac{\frac{1}{m^\mu_\gamma}\ \kappa(d\gamma)}{\int_{\beta\in\mathcal{S}^{n-1}} \frac{1}{m^\mu_\beta}\ \kappa(d\beta)},$$

and also $m_\gamma^\tau = m_\gamma^\mu, m^\tau = m^\mu, \widetilde{\mu(\tau)}^\sigma(d\gamma) = \frac{m^\tau}{m_\gamma^\tau}\kappa(d\gamma)$. Therefore, τ is balanced and

$$(c_\gamma^\tau(m^\tau x/m_\gamma^\tau) - (m^\tau x/m_\gamma^\tau)^+) = \begin{cases} m^\tau, & \text{if } x \in (-\infty, 0], \\ m^\tau - \frac{m^\tau x}{m_\gamma^\tau}, & \text{if } x \in [0, m_\gamma^\tau], \\ 0, & \text{if } x \in [m_\gamma^\tau, +\infty)\ . \end{cases}$$

Since $\lim_{k\to\infty} m_\gamma^{\tau_k} = \lim_{k\to\infty} m_\gamma^\mu \wedge k = m_\gamma^\tau$ and $\lim_{k\to\infty} m^{\tau_k} = m^\tau$, it can be seen that

$$\lim_{k\to\infty} \mu(\tau_k) \geq_{wcx} \mu(\tau).$$

By monotone convergence theorem, we have $\lim_{k\to\infty} \mathbb{E}[|W_{\tau_k} - x|] = \mathbb{E}[|W_\tau - x|]$. Therefore, according to Lemma 3.2.11, τ is standard for $(W_t)_{t\geq 0}$. $\square$

Suppose ξ is a standard stopping time, and $(I_\gamma)_{\gamma\in\mathcal{S}^{n-1}}$ is a collection of intervals on rays. In the next lemma, we prove that the stopping time $\tau := \inf\{t \geq \xi : R_t \in I_{\Gamma_t}^c\}$ is still standard. The result for Brownian motion was proved in [70, Lemma 10.1]. Here we give another proof based on Lemma 3.2.11.

Lemma 3.4.13. *Given $A \subset \mathcal{S}^{n-1}$, suppose $(I_\gamma)_{\gamma\in A}$ is a collection of intervals such that $I_\gamma = (a_\gamma, b_\gamma) \subset (0, +\infty)$ and $\sup_{\gamma\in A} b_\gamma < +\infty$. For $\gamma \in A^c$, we define $I_\gamma = \emptyset$ by default. Let ξ be a standard stopping time, then stopping time given by $\tau = \inf\{t \geq \xi : R_t \in I_{\Gamma_t}^c\}$ is a standard stopping time as well.*

Proof. According to Lemma 3.2.11, it is equivalent to $\mathbb{E}[|W_\tau|] < \infty$ and $\liminf_{k\to\infty} \mathbb{E}[|W_k|\mathbb{1}_{\{\tau>k\}}] = 0$. The first condition follows from

$$\mathbb{E}[|W_\tau|] \leq \mathbb{E}[|W_\xi|] + \max\{s, 1\} \sup_{\gamma\in A} b_\gamma < +\infty,$$

Since paths of W are unbounded, it follows that $\tau < +\infty$ a.s. and hence

$\lim_{k\to\infty} \mathbb{P}[\tau > k \geq \xi] = 0$. The second condition follows from

$$\begin{aligned}\liminf_{k\to\infty} \mathbb{E}[|W_k|\mathbb{1}_{\{\tau>k\}}] &\leq \liminf_{k\to\infty} \mathbb{E}[|W_k|\mathbb{1}_{\{\xi>k\}}] + \lim_{k\to\infty} \mathbb{E}[|W_k|\mathbb{1}_{\{\tau>k\geq\xi\}}] \\ &\leq \liminf_{k\to\infty} \mathbb{E}[|W_k|\mathbb{1}_{\{\xi>k\}}] + \max\{s,1\} \sup_{\gamma\in A} b_\gamma \lim_{k\to\infty} \mathbb{P}[\tau > k \geq \xi] = 0.\end{aligned}$$

□

The proof of the following proposition is based on the idea of [71, Theorem 1].

Proposition 3.4.14. *For any stopping time τ such that $\mu(\tau)$ is balanced, there exists a standard stopping time $\upsilon \leq \tau$ such that $\mathcal{L}(Z_\upsilon) = \mathcal{L}(Z_\tau)$.*

Proof. Let $\mathcal{D}$ be the collection of standard stopping times ξ such that $\xi \leq \tau, \mu(\xi) \leq_{wcx} \mu(\tau)$. $\mathcal{D}$ is not empty since $\xi = 0$ is in $\mathcal{D}$. According to Lemma 3.4.11 and Zorn's lemma, there exists a maximal element in $\mathcal{D}$. Let denote it by υ. Let us show that $\mathcal{L}(Z_\upsilon) = \mathcal{L}(Z_\tau)$.

The equation (3.11) holds for any balanced stopping time ξ, and hence

$$\mathbb{E}[|x - W_\xi|] = \begin{cases} |x| + 2s\int_{\gamma\in A_0}(c^\xi_\gamma(m^\xi x/(m^\xi_\gamma s)) - (m^\xi x/(m^\xi_\gamma s))^+)\ \kappa(d\gamma) & \text{if } x \geq 0\ , \\ |x| + 2\int_{\gamma\in A_0^c}(c^\xi_\gamma(-m^\xi x/m^\xi_\gamma) - (-m^\xi x/m^\xi_\gamma)^+)\ \kappa(d\gamma) & \text{if } x \leq 0\ . \end{cases}$$

In conjunction with the fact that $\mu(\upsilon) \leq_{wcx} \mu(\tau)$ and Lemma 3.4.10, $\mathcal{L}(Z_\upsilon) = \mathcal{L}(Z_\tau)$ is equivalent to that $\mathbb{E}[|x - W_\upsilon|] = \mathbb{E}[|x - W_\tau|]$ for any $x \in \mathbb{R}$. Suppose not. Consider the open set $\mathcal{O} = \{x : \mathbb{E}[|x - W_\upsilon|] < \mathbb{E}[|x - W_\tau|]\}$. As in the proof of [71, Theorem 1], it can be shown that $\mathbb{P}[\{\upsilon < \tau\} \cap \{W_\upsilon \in \mathcal{O}\}] > 0$.

If $0 \in \mathcal{O}$ and $\mathbb{P}[\{\upsilon < \tau\} \cap \{W_\upsilon = 0\}] > 0$, it can be easily seen that $m^\upsilon < m^\tau$. Construct a new stopping time as follows

$$\upsilon'(\omega) = \upsilon(\omega) + \inf\{t \geq 0 : R_{\Gamma_t} = rm^\tau_{\Gamma_t}\}\mathbb{1}_{\{R_\upsilon(\omega)=0\}},$$

where $r > 0$ will be determined later. If Z_υ is not at the origin, we have $\upsilon' = \upsilon$, and if $Z_\upsilon = \mathbf{0}$, we transport the mass from origin to $(\gamma, rm^\tau_\gamma)_{\gamma\in\mathcal{S}^{n-1}}$ on rays. Denoting $p := \mathbb{P}[R_\upsilon = 0]$, for any $A \subset \mathcal{S}^{n-1}$, we have that

$$\mathbb{E}[|R^A_{\upsilon'} - x|\mathbb{1}_{\{R_\upsilon>0\}}] + \mathbb{E}[R^A_{\upsilon'}\mathbb{1}_{\{R_\upsilon>0\}}] = \mathbb{E}[|R^A_\upsilon - x|] + \mathbb{E}[R^A_\upsilon] - p|x|.$$

As in the proof of Lemma 3.4.12, it can be easily verified that

$$\begin{aligned}\mathbb{E}[|R^A_{\upsilon'} - x|\mathbb{1}_{\{R_\upsilon=0\}}] + \mathbb{E}[R^A_{\upsilon'}\mathbb{1}_{\{R_\upsilon=0\}}] =& p\widetilde{\mu(\tau)}^\sigma(A^c)\ |x| + p\int\limits_{\gamma\in A} |x - rm^\tau_\gamma|\widetilde{\mu(\tau)}^\sigma d(\gamma) \\ &+ p\int\limits_{\gamma\in A} rm^\tau_\gamma\widetilde{\mu(\tau)}^\sigma d(\gamma).\end{aligned}$$

Combining these two equations, we obtain that

$$\begin{aligned}2\mathbb{E}[(R^A_{\upsilon'} - x)^+] =& \mathbb{E}[|R^A_{\upsilon'} - x|\mathbb{1}_{\{R_\upsilon>0\}}] + \mathbb{E}[|R^A_{\upsilon'} - x|\mathbb{1}_{\{R_\upsilon=0\}}] \\ &+ \mathbb{E}[R^A_{\upsilon'}\mathbb{1}_{\{R_\upsilon>0\}}] + \mathbb{E}[R^A_{\upsilon'}\mathbb{1}_{\{R_\upsilon=0\}}] - x \\ =& |x| - x + 2\int\limits_{\gamma\in A} \left(c^\upsilon_\gamma(m^\upsilon x/m^\upsilon_\gamma) - (m^\upsilon x/m^\upsilon_\gamma)^+\right)\ \kappa(d\gamma) \\ &+ p\int\limits_{\gamma\in A} \frac{m^\tau}{m^\tau_\gamma}(|x - rm^\tau_\gamma| - |x| + rm^\tau_\gamma)\ \kappa(d\gamma).\end{aligned}$$

We want to guarantee that for any $\gamma \in \mathcal{S}^{n-1}$,

$$\begin{aligned}&2(c^\upsilon_\gamma(m^\upsilon x/m^\upsilon_\gamma) - (m^\upsilon x/m^\upsilon_\gamma)^+) + p\frac{m^\tau}{m^\tau_\gamma}(|x - rm^\tau_\gamma| - |x| + rm^\tau_\gamma) \\ &\leq 2(c^\tau_\gamma(m^\tau x/m^\tau_\gamma) - (m^\tau x/m^\tau_\gamma)^+).\end{aligned}$$

The left hand side evaluated at $x = 0$ is equal to $2m^\upsilon + 2prm^\tau$, and right hand side evaluated at $x = rm^\tau_\gamma$ is greater than $2m^\tau - 2rm^\tau$. Therefore, it is enough to take $r \leq \frac{m^\tau - m^\upsilon}{(1+p)m^\tau}$. As in Lemma 3.4.12, the stopping time υ' is also standard. Noting that

$\mathbb{P}[\{\upsilon < \tau\} \cap \{\upsilon < \upsilon'\}] > 0$, and $\mu(\upsilon') \leq_{wcx} \mu(\tau)$, the stopping time $\upsilon' \wedge \tau \in \mathcal{D}$ is strictly larger than υ, which violates our assumption that υ is maximal.

If $\mathbb{P}[\{\upsilon < \tau\} \cap \{W_\upsilon \in \mathcal{O} \setminus \{0\}\}] > 0$, there exist a subset $A \subset \mathcal{S}^{n-1}$, a collection of open intervals $(I_\gamma)_{\gamma \in A}$ and a collection of real numbers $(d_\gamma)_{\gamma \in A}$ such that

(i) $\kappa(A) > 0$, $I_\gamma = (a_\gamma, b_\gamma)$ and $\sup_{\gamma \in A} b_\gamma < +\infty$.

(ii) For any $\gamma \in A$, $\mathbb{P}[\{\upsilon < \tau\} \cap \{R_\upsilon \in I_\gamma\} | \Gamma_\upsilon = \gamma] > 0$;

(iii) For any $x \in I_\gamma$, $(c_\gamma^\upsilon(m^\upsilon x / m_\gamma^\upsilon) - (m^\upsilon x / m_\gamma^\upsilon)^+) < d_\gamma < (c_\gamma^\tau(m^\tau x / m_\gamma^\tau) - (m^\tau x / m_\gamma^\tau)^+)$.

Then we define stopping time $\upsilon'(\omega) := \inf\{t \geq \upsilon(\omega) : R_t \in I_{\Gamma_t}^c\}$. According Lemma 3.4.13, υ' is still standard. Note that our construction keeps $m_\gamma^\upsilon = m_\gamma^{\upsilon'}$. The new potential function $c_\gamma^{\upsilon'}$ is linear over the interval I_γ such that $c_\gamma^{\upsilon'}(x) = c_\gamma^\upsilon(x)$ for $x \in \{a_\gamma, b_\gamma\}$. Therefore, $\mu(\upsilon') \leq_{wcx} \mu(\tau)$, and $\mathbb{P}[\{\upsilon < \tau\} \cap \{\upsilon < \upsilon'\}] > 0$. Then the stopping time $\upsilon' \wedge \tau \in \mathcal{D}$ is strictly larger than υ, which violates the maximality of υ. $\square$

Theorem 3.4.15. *If τ is a stopping time such that $\mu(\tau)$ is balanced, then τ is minimal, in the sense of Definition 3.4.3, if and only if $(Z_{\tau \wedge t})_{t \geq 0}$ is uniformly integrable.*

Proof. The proof of "if" part is given by Proposition 3.4.4. Now consider the case when τ is balanced and minimal. First, according to Proposition 3.4.14, there exists a standard stopping time $\upsilon \leq \tau$ such that $\mathcal{L}(Z_\upsilon) = \mathcal{L}(Z_\tau)$. Now using the minimality of τ we obtain that $\upsilon = \tau$. Finally, according to Corollary 3.4.7, the process $(Z_{\tau \wedge t})_{t \geq 0}$ is uniformly integrable. $\square$

3.5 A generalization of Vallois' Skorokhod embedding

Chacon and Walsh [72] gave a general construction of the Skorokhod embedding based on one-dimensional potential theory. Later in [83], Cox and Hobson showed that the construction of both Azéma-Yor [10] and Vallois [180] can be interpreted in the framework of [72]. For a strictly convex function Ψ, Vallois also proved that his solution minimizes $\mathbb{E}[\Psi(L_t^0)]$ among all the minimal solutions, where $(L_t^0)_{t\geq 0}$ is the local time of a Brownian motion. Now we generalize this result to the Walsh Brownian motion by using the method established in [72] and [83].

Suppose that the target distribution $\mu \in \mathcal{P}(\mathbb{R}^n)$ is balanced (see Definition 3.4.2). Let Ψ be a strictly convex function defined on $\mathbb{R}$ such that $\Psi'(+\infty) \leq K$ for some positive constant K. Let $\mathcal{T}$ be the collection of stopping times τ such that the stopped process $(Z_{t\wedge\tau})_{t\geq 0}$ is uniform integrable and Z_τ is of distribution μ. We consider the optimization problem (3.1)

$$\inf_{\tau\in\mathcal{T}} \mathbb{E}[\Psi(L_\tau^Z)].$$

First we present a sufficient condition for the uniform integrability mentioned above. Choose any $A \in \mathcal{B}(\mathcal{S}^{n-1})$ such that $0 < \kappa(A) < 1$, and recall $h_{A,A^c}(Z_t)$ in (3.2). Define the hitting time for $x \in \mathbb{R}$,

$$H_x := \inf\{t \geq 0 : h_{A,A^c}(Z_t) = x\}. \tag{3.12}$$

Lemma 3.5.1. *If $x\mathbb{P}[\tau > H_x] \to 0$ as $x \to \pm\infty$, then the stopped process $(Z_{t\wedge\tau})_{t\geq 0}$ is uniformly integrable.*

Proof. The argument is part of [82, Theorem 5] and we repeat here for readers' convenience. Note that the uniform integrability of stopped processes $Z^\tau, R^\tau, h_{A,A^c}(Z^\tau)$ are equivalent, and the process $(h_{A,A^c}(Z_t))_{t\geq 0}$ is a martingale, so it is sufficient to show that for any stopping time $\upsilon \leq \tau$,

$$\mathbb{E}[h_{A,A^c}(Z_\tau)|\mathcal{F}_\upsilon] = h_{A,A^c}(Z_\upsilon).$$

Suppose $x < 0$, $F \in \mathcal{F}_\upsilon$, and set $F_x = F \cap \{\upsilon < H_x\}$. Since $h_{A,A^c}(Z_{t\wedge H_x})$ is a supermartingale, we have

$$\mathbb{E}[h_{A,A^c}(Z_{\upsilon\wedge H_x})\mathbb{1}_{F_x}] \geq \mathbb{E}[h_{A,A^c}(Z_{\tau\wedge H_x})\mathbb{1}_{F_x}].$$

By replacing B_t with $h_{A,A^c}(Z_t)$ in [82, Lemma 9], we know $h_{A,A^c}(Z_\upsilon)$ is integrable. As a result of the dominated convergence theorem, the left-hand side converges to $\mathbb{E}[h_{A,A^c}(Z_\upsilon)\mathbb{1}_F]$ as $x \to -\infty$. It is noted that the term on the right is equal to

$$\mathbb{E}[h_{A,A^c}(Z_\tau)\mathbb{1}_{F_x\cap\{\tau<H_x\}}] + x\mathbb{P}[F \cap \{\upsilon < H_x < \tau\}],$$

which converges to $\mathbb{E}[h_{A,A^c}(Z_\tau)\mathbb{1}_F]$ according to our assumption. We conclude $\mathbb{E}[h_{A,A^c}(Z_\upsilon)\mathbb{1}_F] \geq \mathbb{E}[h_{A,A^c}(Z_\tau)\mathbb{1}_F]$, and thus $\mathbb{E}[h_{A,A^c}(Z_\upsilon)|\mathcal{F}_\upsilon] \geq h_{A,A^c}(Z_\tau)$. By a same argument for $x > 0$, we obtain the result. □

3.5.1 Construction of the stopping time

For each $\gamma \in \mathcal{S}^{n-1}$, recall functions

$$c_\gamma(r) = \frac{\int_0^{+\infty} \left|mu/m_\gamma - r\right| \ \tilde{\mu}_\gamma(du) + r + m}{2}.$$

Here c_γ is our potential function on rays. It has the following properties (see e.g. [70] for proofs).

Lemma 3.5.2. *c_γ is a positive convex function such that*

(i) $c_\gamma(0) = m$ *and* $c_\gamma(r) \geq r$;

(ii) $\partial_+ c_\gamma(r) = \tilde{\mu}_\gamma([0, \frac{m_\gamma r}{m}])$, $\partial_- c_\gamma(r) = \tilde{\mu}_\gamma([0, \frac{m_\gamma r}{m}))$;

(iii) $\lim_{r\to+\infty} c_\gamma(r) - r = 0$.

Let us define

$$\zeta_\gamma(s) := \sup \operatorname*{argmin}_{r>0} \left\{ \frac{c_\gamma(r) - s}{r} \right\}.$$

It can be seen that $\zeta_\gamma(s)$ is the r-coordinate of the point on c_γ where the tangent line passes through $(0, s)$. Since such a point may not be unique, we choose the one with maximum r-coordinate. We also take

$$\begin{aligned} \phi_\gamma(s) &= \frac{c_\gamma(\zeta_\gamma(s)) - s}{\zeta_\gamma(s)}, \quad \Lambda(s) = \int\limits_{\gamma \in \mathcal{S}^{n-1}} \phi_\gamma(s)\ \tilde{\mu}^\sigma(d\gamma), \\ H(s) &= \int\limits_0^s \frac{1}{\Lambda(u)}\ du, \qquad a_\gamma(l) = \frac{m_\gamma}{m} \zeta_\gamma(H^{-1}(l)), \end{aligned} \tag{3.13}$$

where $H^{-1} : [0, +\infty) \to [0, +\infty)$ is the inverse function of H. We are now ready to define the stopping time,

$$\tau := \inf\{t \geq 0 : R_t \geq a_{\Gamma_t}(L_t^Z)\}, \tag{3.14}$$

that is we stop the process if its excursion travels beyond the hypersurface $\gamma \mapsto a_\gamma(L_t^Z)$.

We say a stopping time τ is of barrier type if there exists some some closed subset $B \subset [0, +\infty) \times \mathbb{R}^n$ such that τ is equal to the hitting time $\inf\{t \geq 0 : (L_t^Z, Z_t) \in B\}$. Since a_γ is non-increasing for any $\gamma \in \mathcal{S}^{n-1}$, stopping time τ is of barrier-type: taking

$$B := \underset{r \geq a_\gamma(l)}{\cup} \{[l, +\infty) \times r\gamma\} \subset [0, +\infty) \times \mathbb{R}^n,$$

it can be easily seen that $\tau = \inf\{t \geq 0 : (L_t^Z, Z_t) \in B\}$. Before verifying that $Z_\tau \sim \mu$, we need a technical lemma.

Lemma 3.5.3. *ϕ_γ is absolutely continuous on closed subsets $[0, m)$ for each $\gamma \in \mathcal{S}^{n-1}$ and*

$$\phi_\gamma(s) = 1 - \int\limits_0^s \frac{1}{\zeta_\gamma(u)}\ du.$$

Proof. The proof is from [83, Lemma 2], and we record here for the sake of completeness. The function ϕ_γ is the gradient of the tangent to c_γ that passes through $(0,s)$. By the convexity of c_γ, we easily see that ϕ_γ is non-increasing on $[0,m)$. In addition, note that c_γ is non-decreasing and ζ_γ is non-increasing. We estimate $\phi_\gamma(s-\delta)-\phi_\gamma(s)$ for small positive δ,

$$\begin{aligned}\phi_\gamma(s-\delta) =& \frac{c_\gamma(\zeta_\gamma(s-\delta))-(s-\delta)}{\zeta_\gamma(s-\delta)} \\ \leq& \frac{c_\gamma(\zeta_\gamma(s))-s+\delta}{\zeta_\gamma(s)} = \phi_\gamma(s)+\frac{\delta}{\zeta_\gamma(s)}.\end{aligned}$$

Therefore, ϕ_γ is $\frac{1}{\zeta_\gamma(s)}$-Lipschitz on closed intervals $[0,s]\subset[0,m)$ for any $s<m$. As a result, ϕ_γ is differentiable almost everywhere on $[0,m)$ and $\phi_\gamma(s)=\int_0^s \phi'_\gamma(u)\,du+1$. Since ζ_γ is left-continuous, we can calculate the left derivative of ϕ_γ,

$$\begin{aligned}\partial_-\phi_\gamma(s) =& \partial_-\frac{c_\gamma(\zeta_\gamma(s))-s}{\zeta_\gamma(s)} \\ =& \frac{\partial_+c_\gamma(\zeta_\gamma(s))\partial_-\zeta_\gamma(s)-1}{\zeta_\gamma(s)} - \frac{\partial_-\zeta_\gamma(s)(c_\gamma(\zeta_\gamma(s))-s)}{\zeta_\gamma^2(s)} \\ =& -\frac{1}{\zeta_\gamma(s)}+\frac{\partial_-\zeta_\gamma(s)}{\zeta_\gamma(s)}\left[\partial_+c_\gamma(\zeta_\gamma(s))-\frac{c_\gamma(\zeta_\gamma(s))-s}{\zeta_\gamma(s)}\right].\end{aligned}$$

If $\tilde{\mu}_\gamma$ has no atom at $\zeta_\gamma(s)$, c_γ is then differentiable at $\zeta_\gamma(s)$ and $\partial_+c_\gamma(\zeta_\gamma(s))$ is just the gradient of the tangent $\frac{c_\gamma(\zeta_\gamma(s))-s}{\zeta_\gamma(s)}$. If $\tilde{\mu}_\gamma$ has an atom at $\zeta_\gamma(s)$, we know $\partial_-\zeta_\gamma(s)$ is zero. In both of these two cases, the second term of the above equation vanishes and we obtain the result. □

Theorem 3.5.4. *The stopped process $(Z_{t\wedge\tau})_{t\geq 0}$ is uniformly integrable and Z_τ is of distribution μ, where τ is defined in* (3.14).

Before we prove this result we will give a corollary of this theorem and Proposition 3.4.1.

Corollary 3.5.5. *There exists a stopping time υ such that $Z_\upsilon\sim\mu$ and $(Z_{\upsilon\wedge t})_{t\geq 0}$ is uniformly integrable if and only if μ is balanced.*

Proof of Theorem 3.5.4. Our proof relies on the excursion theory (see e.g. [169], [171]). It is noted that L_τ^Z is no less than $H(s)$ is equivalent to excursions at local time l has maximum modulus less than $a_\gamma(l)$ for any $l \le H(s)$, where γ is the direction of excursions. Denote by $\mathcal{U}_R$ the excursion space of reflected Brownian motion $(R_t)_{t\ge 0}$ (see Subsection 3.2.1 for the discussion). Take a subset $\mathcal{V}$ of $\Pi := [0,+\infty)\times\mathcal{S}^{n-1}\times\mathcal{U}_R$,

$$\mathcal{V} := \{(l,\gamma,e):\ l < H(s),\ \gamma\in\mathcal{S}^{n-1},\ \sup_{t\ge 0} e(t) \ge a_\gamma(l)\}.$$

According to Lemma 3.2.2, the random variable $N^{\mathcal{V}} = \sum_{l>0} \mathbb{1}_{\mathcal{V}}(l,e_l)$ is Poisson with parameter

$$\int_0^{H(s)} dl \int_{\gamma\in\mathcal{S}^{n-1}} \frac{\kappa(d\gamma)}{a_\gamma(l)}.$$

Since $L_\tau^Z \ge H(s)$ if and only if $N^{\mathcal{V}} = 0$, we obtain

$$\mathbb{P}[L_\tau^Z \ge H(s)] = exp\left\{ -\int_0^{H(s)} dl \int_{\gamma\in\mathcal{S}^{n-1}} \frac{\kappa(d\gamma)}{a_\gamma(l)} \right\}.$$

By Lemma 3.5.3, we have

$$-\int_{\gamma\in\mathcal{S}^{n-1}} \frac{m\kappa(d\gamma)}{m_\gamma\zeta_\gamma(u)} = -\int_{\gamma\in\mathcal{S}^{n-1}} \frac{\tilde\mu^\sigma(d\gamma)}{\zeta_\gamma(u)} = \int_{\gamma\in\mathcal{S}^{n-1}} \phi_\gamma'(u)\ \tilde\mu^\sigma(d\gamma) = \Lambda'(u).$$

In conjunction with $H'(u) = \frac{1}{\Lambda(u)}$, we get

$$\begin{aligned}
\mathbb{P}[L_\tau^Z \geq H(s)] =& exp\left\{ - \int_0^{H(s)} dl \int_{\gamma\in\mathcal{S}^{n-1}} \frac{\kappa(d\gamma)}{a_\gamma(l)} \right\} \\
=& exp\left\{ - \int_0^{s} H'(u)\ du \int_{\gamma\in\mathcal{S}^{n-1}} \frac{m\kappa(d\gamma)}{m_\gamma\zeta_\gamma(u)} \right\} \\
=& exp\left\{ - \int_0^{s} \frac{\Lambda'(u)}{\Lambda(u)} du \right\} = \Lambda(s).
\end{aligned}$$

Recall the definition of τ: we will stop in the region $d\gamma \times [r, +\infty)$ at local time l if and only if $(Z_t)_{t\geq 0}$ does dot hit stopping region B until an excursion travels beyond $a_\gamma(l) \geq r$ at local time l. Take a subset of Π,

$$\mathcal{V}' := \left\{ (u, \gamma, e_u) :\ u \in (l, l + dl],\ \sup_{t\geq 0} e(t) \geq a_\gamma(l) \right\}.$$

Hence by Lemma 3.2.2, we obtain

$$\mathbb{P}[N^{\mathcal{V}'} \geq 1] = \frac{\kappa(d\gamma)}{a_\gamma(l)} dl.$$

Since $Z_\tau \in d\gamma \times [r, +\infty), L_\tau^Z \in dl$ if and only if $L_\tau^Z \geq l, N^{\mathcal{V}'} \geq 1, a_\gamma(l) \geq r$, we conclude

$$\mathbb{P}[Z_\tau \in d\gamma \times [r, +\infty), L_\tau^Z \in dl] = \mathbb{1}_{\{a_\gamma(l)\geq r\}} \mathbb{P}[L_\tau^Z \geq l] \frac{\kappa(d\gamma)}{a_\gamma(l)} dl,$$

and

$$\mathbb{P}[Z_\tau \in d\gamma \times [r,+\infty)] = \int\limits_{\{l:a_\gamma(l)\geq r\}} \mathbb{P}[L^Z_\tau \geq l]\frac{\kappa(d\gamma)}{a_\gamma(l)}\ dl$$

$$= \int\limits_{\{u:\zeta_\gamma(u)\geq \frac{mr}{m_\gamma}\}} H'(u)\Lambda(u)\frac{m\kappa(d\gamma)}{m_\gamma\zeta_\gamma(u)}\ du$$

$$= -\tilde{\mu}^\sigma(d\gamma) \int\limits_{\{u:\zeta_\gamma(u)\geq \frac{mr}{m_\gamma}\}} \phi'_\gamma(u)\ du.$$

Since $\phi_\gamma(s) = \partial_- c_\gamma(\zeta_\gamma(s)) = \tilde{\mu}_\gamma([0, \frac{m_\gamma\zeta_\gamma(s)}{m}))$, we obtain

$$\int\limits_{\{u:\zeta_\gamma(u)\geq \frac{mr}{m_\gamma}\}} \phi'_\gamma(u)\ du = \tilde{\mu}_\gamma([0,r)) - 1,$$

and therefore

$$\mathbb{P}[Z_\tau \in d\gamma \times [r,+\infty)] = \tilde{\mu}^\sigma(d\gamma) \times \tilde{\mu}_\gamma([r,+\infty)).$$

To finish the argument, we show that $(Z_{t\wedge\tau})_{t\geq 0}$ is uniform integrable by verifying Lemma 3.5.1. Recall our notation from (3.12), and consider the case $x > 0$. Due to the construction of τ, we have

$$\mathbb{P}[\tau > H_x] = \int\limits_{\gamma\in A^c} \kappa(d\gamma) \int\limits_{\{l:a_\gamma(l)\geq x\}} \frac{\mathbb{P}[L^Z_\tau \geq l]}{x} dl$$

$$= \int\limits_{\gamma\in A^c} \kappa(d\gamma) \int\limits_{\{u:\zeta_\gamma(u)\geq \frac{mx}{m_\gamma}\}} \frac{H'(u)\Lambda(u)}{x}\ du.$$

Therefore, we have

$$x\mathbb{P}[\tau > H_x] = \int\limits_{\gamma\in A^c} \mathrm{Leb}(\{u : \zeta_\gamma(u) \geq \frac{mx}{m_\gamma}\})\ \kappa(d\gamma).$$

The function $\gamma \mapsto \mathrm{Leb}(\{u : \zeta_\gamma(u) \geq \frac{mx}{m_\gamma}\})$ is bounded above by m, and decreases to 0 as $x \to +\infty$, so by the dominated convergence theorem, we have $\lim_{x\to+\infty} x\mathbb{P}[\tau >$

$H_x] = 0$. $\square$

To close this subsection, we show that there exists an integrable stopping time τ such that $Z_\tau \sim \mu$ if and only if (μ, κ) is balanced and μ has finite second moment. The proof of the "only if" part is a simple application of Lemma 3.2.4 and Doob's optional sampling theorem. To show the "if" part, we prove that when μ has finite second moment, the stopping time τ in Theorem 3.5.4 is actually integrable.

Corollary 3.5.6. *There exists a stopping time τ such that $Z_\tau \sim \mu$ and $\mathbb{E}[\tau] < +\infty$, if and only if (μ, κ) is balanced and the second moment of μ is finite, i.e.,*

$$\int_{\mathbb{R}^n} |z|^2 \ \mu(dz) < +\infty. \tag{3.15}$$

Proof. Suppose τ is a stopping time such that $\mathbb{E}[\tau] < +\infty$ and $Z_\tau \sim \mu$. The condition $\mathbb{E}[\tau] < +\infty$ implies that $(Z_{\tau\wedge t})_{t\geq 0}$ is uniformly integrable. Then due to Proposition 3.4.1, μ is balanced. Take a measurable function on $\mathcal{S}^{n-1} \times \mathbb{R}_+$, $g(\gamma, r) = r^2$. Applying Lemma 3.2.4, it can be seen that $g(Z_t) - t$ is a martingale. Since $\mathbb{E}[\tau] < +\infty$, we can employ Doob's optional sampling theorem and get

$$\mathbb{E}[\tau] = \mathbb{E}[g(Z_\tau)] = \int_{\mathbb{R}^n} |z|^2 \ \mu(dz) < +\infty.$$

For the converse, let $W_t := \frac{\kappa(A^c)}{\kappa(A)} R_t^A - R_t^{A^c}$ for some $\kappa(A) \in (0, 1)$, and τ be the stopping time constructed in (3.14). Due to Lemma 3.2.4, $(W_t)_{t\geq 0}$ is a martingale. According to Theorem 3.5.4, $(Z_{\tau\wedge t})_{t\geq 0}$ is uniformly integrable and $Z_\tau \sim \mu$, and hence $(W_{\tau\wedge t})_{t\geq 0}$ is also uniformly integrable. Applying [156, Proposition 2.1], we obtain that $\mathbb{E}[\tau] \leq C\mathbb{E}[|W_\tau|^2]$ for some constant $C > 0$. Due to the construction of $(W_t)_{t\geq 0}$, we conclude that

$$\mathbb{E}[\tau] \leq C\mathbb{E}[|W_\tau|^2] \leq C \max\left\{\left(\frac{\kappa(A^c)}{\kappa(A)}\right)^2, 1\right\} \int_{\mathbb{R}^n} |z|^2 \, \mu(dz) < +\infty.$$

$\square$

3.5.2 Verification of Optimality.

Beiglböck et al. have developed a new approach to the optimal Skorokhod embedding problem based on the ideas of optimal transport in [35] and [38], where the duality result and the monotonicity principle are presented. Most of their arguments are abstract and can carry over to the embedding problem for continuous Feller processes. By a similar argument as [35, Theorem 6.14], we know that the optimizer of problem (3.1) must be of barrier type. Since barrier type solutions are in general essentially unique (see [143]), our stopping time τ should solve the optimization problem (3.1).

Applying the method of pathwise inequalities established in [78] and [84], we verify the optimality of τ by constructing the dual optimizer (G, M). We define

$$\Delta(l) := \int_0^l dm \int_{\gamma \in \mathcal{S}^{n-1}} \frac{1}{a_\gamma(m)} \, \kappa(d\gamma),$$

$$A_\gamma(l) := \Psi'(+\infty) - \int_l^{+\infty} \frac{dm}{a_\gamma(m)} e^{\Delta(m)} \int_m^{+\infty} e^{-\Delta(n)} \Psi''(dn),$$

where a_γ is defined in (3.13). We now construct a function $G : \mathbb{R}^n \to \mathbb{R}$ and a local martingale $(M_t)_{t \geq 0}$ such that $M_t + G(Z_t) \leq \Psi(L_t^Z)$, and equality is obtained when

$Z_t = (\Gamma_t, a_{\Gamma_t}(L_t^Z))$. Define G to be concave on each ray,

$$G(\gamma, r) := \begin{cases} \Psi(0) - \int_0^{+\infty} dm\ e^{\Delta(m)} \int_m^{+\infty} e^{-\Delta(n)} \Psi''(dn) & \text{if } r = 0, \\ \inf_{l>0} \left\{ rA_\gamma(l) + \Psi(0) - \int_0^l dm\ e^{\Delta(m)} \int_m^{+\infty} e^{-\Delta(n)} \Psi''(dn) \right\} & \text{if } r > 0. \end{cases}$$

Denote by b_γ the right-continuous inverse of a_γ. Since $a_\gamma(r)$ is non-increasing with respect to r, it is easily seen that the infimum above is obtained at $l = b_\gamma(r)$, and hence

$$G'_\gamma(r+) \leq A_\gamma(b_\gamma(r)) \leq G'_\gamma(r-),$$

$$G(\gamma, r) = rA_\gamma(b_\gamma(r)) + \Psi(0) - \int_0^{b_\gamma(r)} dm\ e^{\Delta(m)} \int_m^{+\infty} e^{-\Delta(n)} \Psi''(dn).$$

Take

$$M_t := \int_0^{L_t^Z} dm \int_{\gamma \in \mathcal{S}^{n-1}} A_\gamma(m)\ \kappa(d\gamma) - A_{\Gamma_t}(L_t^Z) R_t.$$

Theorem 3.5.7. *The random process $(M_t)_{t \geq 0}$ is a local martingale. We have the pathwise inequality*

$$M_t + G(Z_t) \leq \Psi(L_t^Z),$$

where equality is obtained for those paths such that $Z_t = (\Gamma_t, a_{\Gamma_t}(L_t^Z))$. Therefore, the stopping time τ constructed in Section 3.5.1 solves the optimization problem

$$\inf_{\tau \in \mathcal{T}} \mathbb{E}[\Psi(L_\tau^Z)].$$

Proof. By Lemma 3.2.4, we have

$$
\begin{aligned}
-dM_t &= A'_{\Gamma_t}(L_t^Z)R_t \; dL_t^Z + \mathbb{1}_{\{R_t \neq 0\}} A_{\Gamma_t}(L_t^Z) \; dB_t^Z \\
&\quad + \int\limits_{\gamma \in \mathcal{S}^{n-1}} A_\gamma(L_t^Z) \; \kappa(d\gamma) \; dL_t^Z - \int\limits_{\gamma \in \mathcal{S}^{n-1}} A_\gamma(L_t^Z) \; \kappa(d\gamma) \; dL_t^Z \\
&= A'_{\Gamma_t}(L_t^Z)R_t \; dL_t^Z + \mathbb{1}_{\{R_t \neq 0\}} A_{\Gamma_t}(L_t^Z) \; dB_t^Z.
\end{aligned}
$$

Since $(L_t^Z)_{t\geq 0}$ is flat off $\{R_t = 0\}$, the first term vanishes. Therefore, $(M_t)_{t\geq 0}$ is a local martingale.

Note that

$$
\begin{aligned}
\int\limits_{\gamma \in \mathcal{S}^{n-1}} A_\gamma(l) \; \kappa(d\gamma) &= \Psi'(+\infty) - \int\limits_l^{+\infty} \Delta'(m) e^{\Delta(m)} \; dm \int\limits_m^{+\infty} e^{-\Delta(n)} \Psi''(dn) \\
&= \Psi'(+\infty) - \int\limits_l^{+\infty} e^{-\Delta(n)} \Psi''(dn) \int\limits_l^{n} \Delta'(m) e^{\Delta(m)} \; dm \\
&= \Psi'(+\infty) - \int\limits_l^{+\infty} e^{-\Delta(n)} \Psi''(dn) (e^{\Delta(n)} - e^{\Delta(l)}) \\
&= \Psi'(l) + e^{\Delta(l)} \int\limits_l^{+\infty} e^{-\Delta(n)} \Psi''(dn).
\end{aligned}
$$

Therefore, by the definition of G, we obtain

$$
\begin{aligned}
G(Z_t) &\leq A_{\Gamma_t}(L_t) R_t + \Psi(0) - \int\limits_0^{L_t^Z} dm \; e^{\Delta(m)} \int\limits_m^{+\infty} e^{-\Delta(n)} \Psi''(dn) \\
&= -M_t + \Psi(L_t^Z),
\end{aligned}
$$

where the inequality is strict unless $Z_t = (\Gamma_t, a_{\Gamma_t}(L_t^Z))$.

Suppose υ is a stopping time such that $Z_\upsilon \sim \mu$ and $(Z_{t\wedge\upsilon})_{t\geq 0}$ is uniformly integrable. Then by a similar argument as [84, Lemma 2.1], we know that the stopped process $(M_{t\wedge\upsilon})_{t\geq 0}$ is uniformly integrable, and hence $\mathbb{E}[M_{t\wedge\upsilon}] = 0$. So we have that

$$\int_{z\in\mathbb{R}^n} G(z)\ \mu(dz) \leq \mathbb{E}[\Psi(L_\upsilon^Z)],$$

where the equality is obtained when $\upsilon = \tau$. Therefore, the stopping time τ solves the optimization problem (3.1).

$\square$

CHAPTER IV

k-core in Percolated Dense Graph Sequences

4.1 Introduction

For an integer $k \geq 2$, the k-core of a graph G is the largest induced subgraph of G with minimum degree at least k. It was first introduced by Bollobás in [44] to find large k-connected subgraphs, and since then several studies have been devoted to investigate the existence and size of k-core. Apart from the theoretical interest, k-core has been applied to the study of social networks [42, 118], graph visualizing [6, 59], biology [183]. See also [132] for an extensive discussion on its applications. In the seminal paper [165], Pittel, Spencer and Wormald determined the threshold for the appearance of a non-empty k-core in Bernoulli random graphs and uniform random graphs. The size of k-core have been studied in different random graph ensembles such as Bernoulli random graphs [144], uniformly chosen random graphs and hypergraphs with specified degree sequence [80, 97, 121, 122, 149], Poisson cloning model [128] and the pairing-allocation model [54]. While almost all the previous work focused on k-core of homogeneous random graphs, Riordan [170] determined the asymptotic size of k-core for a sequence of inhomogeneous random graphs sampled from a graphon.

In this chapter we study the asymptotic size of k-core in random subgraphs of convergent dense graph sequences. Let G_n be a sequence of undirected weighted graphs on n vertices with edge weights $\{a^n_{i,j}\}$ that converges to a graphon W. For some

$c > 0$, we keep an edge (i, j) of G_n with probability $\min\{ca^n_{i,j}/n, 1\}$ independently, and denote the resulting random graph by $G_n(\frac{c}{n})$. For any kernel W, we can associate it with a branching process X^W, i.e., the number of children of a particle with type x has Poisson distribution with parameter $\int W(x, y)dy$ (see Section 4.2 for precise definition). Under some mild conditions, we show that

$$\text{size of } k\text{-core of } G_n\left(\frac{c}{n}\right) = n\mathbb{P}_{X^{cW}}(\mathcal{A}) + o_p(n), \tag{4.1}$$

where $\mathcal{A}$ is the event that the initial particle has at least k children, each of which has at least $k-1$ children, each of which has at least $k-1$ children, and so on.

Our contribution is two-fold. First, recall from [142] that every dense graph sequence has a convergent subsequence, and hence our result applies to a large class of dense graph sequences. In particular, our result together with [47, Lemma 1.6] recover [170] for bounded graphons. An important application of our result is quasi-random graph (see e.g. [76, 135]), which corresponds to dense graph sequences that converges to a constant limit, such as Paley graphs (see [120, 135]). As far as we know, other than the present work no result is known about the size of the k-core in random subgraphs of quasi-random graphs. Also, there are aplenty examples of dense random graph models (which are not quasi-random) that are known to converge to a positive limit (see [19, 41, 73, 74]). Second, as a byproduct of our proof of the main result, for any sequence of kernels W_n satisfying some mild assumptions that converges to W we have that

$$\mathbb{P}_{X^{W_n}}(\mathcal{A}) \to \mathbb{P}_{X^W}(\mathcal{A}),$$

a new continuity result concerning branching processes, which we believe is of independent interest. Even though the theory of graph limits received enormous attention in the last two decades, the only result alike that we are able to find is [47, Theorem

1.9], which concerns with the survival probability of a branching process.

Let us describe the main idea of our argument. The proof of upper bound of size of k-core is based on carefully computing the probability of the event $\mathcal{A}$, and the estimation of this probability heavily involves homomorphism density; see e.g. [45, 142]. The proof of lower bound is more delicate. First, we approximate W by a sequence of finitary kernels F_m as in [46]. Then, we show that for each fixed m, the branching process X^n associated with G_n contains $X^{(1-\varepsilon_m)F_m}$ as a subset for some ε_m with $0 < \varepsilon_m < \frac{1}{m}$ when n is large enough. To conclude the lower bound, we prove a continuity property and invoke a result (minor variant) from [170].

The rest of the chapter is organized as follows. In Section 4.2, we present our main results with some discussions. In Section 4.3 and Section 4.4, we prove the upper bound and lower bound of size of k-core respectively.

4.2 Main results and discussions

We now recall few definitions to state our results. A graphon (or kernel) is defined to be a symmetric measurable function $W : I \times I \to [0, \infty)$, where $I := [0, 1]$. Take $\mathcal{W}$ to be the space of graphons. The cut norm of $W : I \times I \to \mathbb{R}$ (signed graphon) is defined by

$$\|W\|_\square := \sup_{S,T \in \mathcal{B}(I)} \left| \int\limits_{S \times T} W(u, v)\, du\, dv \right|,$$

and the cut metric between two graphons W_1 and W_2 is defined by

$$d_\square(W_1, W_2) := \|W_1 - W_2\|_\square.$$

An undirected finite graph G_n with adjacency matrix $(a^n_{i,j})^n_{i,j=1}$ can be embedded into a symmetric kernel in a natural way

$$W_{G_n}(x,y) = \sum_{i,j=1}^{n} a^n_{i,j} \mathbb{1}_{J^n_i}(x) \mathbb{1}_{J^n_j}(y), \tag{4.2}$$

where $J^n_1 = [0, \frac{1}{n}]$ and for $i = 2, 3, \ldots, n$, $J^n_i = \left(\frac{i-1}{n}, \frac{i}{n}\right]$.

Let G_n be a sequence of simple graphs on n vertices with edge weights $\{a^n_{i,j}\}$ that converges to a kernel W. For some $c > 0$, we keep an edge (i, j) of G_n with probability $\min\{ca^n_{i,j}/n, 1\}$ independently, and denote the resulting random graph by $G_n(\frac{c}{n})$. Here and throughout the chapter we assume that edge weights $a^n_{i,j}$ are uniformly bounded by $a_M > 0$, and therefore for sufficiently large n we will have $\min\{ca^n_{i,j}/n, 1\} = ca^n_{i,j}/n$. Since retaining every edge independently is nothing but the bond-percolation on the graph , we call $G_n\left(\frac{c}{n}\right)$ percolated graph sequence (bond-percolation on arbitrary dense graph sequences was first studied in [45]). Our aim is to study the size of the k-core of the random graph sequence $G_n\left(\frac{c}{n}\right)$.

We will heavily use the branching process X^W associated with the kernel W. The process starts with a single particle with type x_0, which is chosen uniformly from $[0, 1]$. Conditional on generation t, each member in generation t has offspring in next generation independent of each other, and everything else. The number of children with types in a set $A \subset [0, 1]$ is Poisson with parameter $\int_A W(x, y)\, dy$, and these numbers are independent for disjoint sets.

Let $\mathcal{A}_d$ be the event that the root has at least k children, each of these k children has at least $k - 1$ children, each of those second generation of children has another $k - 1$ children and so on until the d-th generation. Define $\mathcal{A} = \cap_{d=1}^{\infty} \mathcal{A}_d$. Let $C_k(G)$ denote the size of the k-core of a graph G. We are now ready to discuss our main result, which provides asymptotic size of the k-core in random subgraphs of dense graph sequences or percolated dense graph sequences. First let us make the following assumption.

Assumption 4.2.1. *(i) There exists some positive constant δ such that*

$$\inf_{x,y} W(x,y) \geq \delta.$$

(ii) $\lambda \to \mathbb{P}_{X^{\lambda W}}(\mathcal{A})$ is continuous at $\lambda = c$ for some c positive.

Theorem 4.2.2. *Let G_n be a sequence of graphs with non-negative edge weights which are bounded above by a constant $a_M > 0$. Suppose that G_n converges to a graphon W as $n \to \infty$ and that the Assumption 4.2.1 holds. Then we have that*

$$C_k\left(G_n\left(\frac{c}{n}\right)\right) = n\mathbb{P}_{X^{cW}}(\mathcal{A}) + o_p(n). \tag{4.3}$$

It suffices to prove the case $c = 1$ in Theorem 4.2.2. To see this, let G_n be a graph with edge weights $\{a^n_{i,j}\}$ and consider another graph G'_n with edge weights $\{ca^n_{i,j}\}$. Therefore the random subgraphs $G_n\left(\frac{c}{n}\right)$ and $G'_n\left(\frac{1}{n}\right)$ are equal in distribution. Finally by our assumption G_n converges to W and this gives G'_n converges to cW. The result (4.3) then follows from the result with $c = 1$.

Our proof of (4.3) is divided into two parts, which will be given in the next two sections. We should remark that for the proof of $\leq$, we only need the assumption that the edge weights of G_n are uniformly bounded above by a_M and $G_n \to W$. Assumption 4.2.1 is used only in the proof of the $\geq$ direction in Section 4.4.

Remark 4.2.3. In Theorem 4.2.2, note that $\mathbb{P}_{X^{cW}}(\mathcal{A})$ could be zero and in that case we will only be able to say that there is no 'giant' k-core (as usual by 'giant' we mean 'of size order of n'). From Theorem 4.2.2 one can also obtain the emergence threshold for the giant k-core from the function $c \to \mathbb{P}_{X^{cW}}(\mathcal{A})$. More precisely, if there is a point $c_0 > 0$ such that for $0 \leq c < c_0$, $\mathbb{P}_{X^{cW}}(\mathcal{A}) = 0$ and for $c > c_0$, $\mathbb{P}_{X^{cW}}(\mathcal{A}) > 0$, then c_0 will be the threshold for the appearance of giant k-core. The other discontinuity points could be studied from this function as well.

Remark 4.2.4. In Theorem 4.2.2, it is not possible to remove the assumption that $\lambda \to \mathbb{P}_{X^{\lambda W}}(\mathcal{A})$ is continuous at $\lambda = c$, and the reason is explained at the end of Section 3.1 in [170]. It can be easily seen that $\inf_{x,y} W(x,y) \geq \delta$ implies the irreducibility of W (see e.g. [45] for the definition of irreducibility). It might be possible to replace our Assumption 4.2.1 (i) by the irreducibility assumption of W, and we defer it to a future work.

As a byproduct in the proof of Theorem 4.2.2, we also obtain a result regarding branching processes that might be of independent interest.

Proposition 4.2.5. *Let W_n be a sequence of graphons such that $d_\square(W_n, W) \to 0$. Also suppose there exists some positive constant δ such that $\inf_{x,y} W(x,y) \geq \delta$ and $\lambda \to \mathbb{P}_{X^{\lambda W}}(\mathcal{A})$ is continuous from below at $\lambda = 1$. Then it holds that*

$$\mathbb{P}_{X^{W_n}}(\mathcal{A}) \to \mathbb{P}_{X^W}(\mathcal{A}), \tag{4.4}$$

as $n \to \infty$.

Proof. It is proved in Propositions 4.3.7, 4.4.6. □

Let us point out that Proposition 4.2.5 has the following important consequence. Note that the function $\lambda \to \mathbb{P}_{X^{\lambda W}}(\mathcal{A})$ is non-decreasing, and therefore it can have at most countably many discontinuity points. Hence in many cases the next corollary provides a way to approximate the size of k-core using only G_n.

Corollary 4.2.6. *Let G_n be a sequence of graphs with non-negative edge weights which are bounded above by a constant $a_M > 0$. Suppose that G_n converges to a graphon W as $n \to \infty$, where $\inf_{x,y} W(x,y) \geq \delta$ for some $\delta > 0$, and $\lambda \to \mathbb{P}_{X^{\lambda W}}(\mathcal{A})$ is continuous at $\lambda = c$, then*

$$C_k\left(G_n\left(\frac{c}{n}\right)\right) = n\mathbb{P}_{X^{cW_{G_n}}}(\mathcal{A}) + o_p(n). \tag{4.5}$$

Proof of Corollary 4.2.6. The proof is immediate using Theorem 4.2.2 and Proposition 4.2.5. □

4.3 Proof of the upper bound in Theorem 4.2.2

We will first prove the upper bound, i.e.,

$$C_k\left(G_n\left(\frac{1}{n}\right)\right) \leq n\mathbb{P}_{X^W}(\mathcal{A}) + o_p(n).$$

The idea is as follows: if a vertex v of a graph is in the k-core, then for any $d > 0$ either v has property $\mathcal{A}_d$ or v is contained in a cycle of length smaller than $2d$. Since the probability of occurrence of short cycles is small for large enough n, the probability that v is in the k-core is bounded above by the probability of having property $\mathcal{A}_d$. Therefore to prove the upper bound, we explicitly calculate the probability of event $\mathcal{A}_d$ using homomorphism density, and a tightness argument. Finally, by letting $d \to \infty$, we obtain that $C_k\left(G_n\left(\frac{1}{n}\right)\right) \leq n\mathbb{P}_{X^W}(\mathcal{A}) + o_p(n)$. Note that we do not need the limit W to be bounded below by a constant or the continuity assumption for the upper bound.

Let us construct a branching process ${}^*X^n$ associated with the random graph $G_n\left(\frac{1}{n}\right)$. ${}^*X^n$ has n-types of offsprings $1, 2, \ldots, n$. It starts with a single particle whose type is chosen uniformly from $1, 2, \ldots, n$. Conditioning on generation t, each member of generation t has offsprings in the next generation independent of each other, and everything else. The number of j-offspring of a particle of type i is Bernoulli$(a_{i,j}^n/n)$.

We will also use another branching process where number of j-offsprings of a particle of type i is Poisson$(a_{i,j}^n\rho_n)$, where $\rho_n \geq \frac{1}{n}$ is to be determined. We denote this process by X^{n,ρ_n} (simply by X^n if $\rho_n = \frac{1}{n}$). By taking $\rho_n = \frac{1}{n-a_M}$, the Poisson branching process X^{n,ρ_n} stochastically dominates, in the first order, ${}^*X^n$ for $n > 3a_M$. To see this, it is sufficient to show the following inequality for any $i,j \in [n]$

$$\mathbb{P}\left(\text{Poisson}\left(a^n_{i,j}\rho_n\right) > t\right) \geq \mathbb{P}\left(\text{Bernoulli}\left(\frac{a^n_{i,j}}{n}\right) > t\right)$$

It is trivial for $t \geq 1$ and $t < 0$. We need to check only for $t = 0$. It can be easily verified that the above inequality is equivalent to

$$\frac{n\rho_n\left(1 - e^{-a^n_{i,j}\rho_n}\right)}{a^n_{i,j}\rho_n} \geq 1.$$

For $n > 3a_M$, we have that $a^n_{i,j}\rho_n = \frac{a^n_{i,j}}{n-a_M} < 1/2$, and hence according to the Taylor expansion of $e^{-a^n_{i,j}\rho_n}$,

$$\frac{n\rho_n\left(1 - e^{-a^n_{i,j}\rho_n}\right)}{a^n_{i,j}\rho_n} > n\rho_n(1 - a^n_{i,j}\rho_n/2) \geq (1 + a_M\rho_n)(1 - a_M\rho_n/2) \geq 1.$$

Note that we can write

$$C_k\left(G_n\left(\frac{1}{n}\right)\right) = \sum_{v\in[n]} \mathbb{1}\left\{v \in k\text{-core of } G_n\left(\frac{1}{n}\right)\right\}$$

If a vertex v is in the k-core, then one of the two things must be true:

(i) v is in a cycle within d-neighborhood (this implies v is in a cycle of length at most $2d$);

(ii) Starting from v there is a tree such that v has k neighbors, each of these k neighbors has at least $k-1$ neighbors and this happens up to generation d. In this case we will call vertex v has property $\mathcal{A}_d$.

Therefore

$$
\begin{aligned}
C_k\left(G_n\left(\frac{1}{n}\right)\right) \leq & \sum_{v\in[n]} \mathbb{1}\left\{v \text{ is in a cycle of length at most } 2d\right\} \\
& + \sum_{v\in[n]} \mathbb{1}\left\{v \text{ has property } \mathcal{A}_d\right\} \\
= & \text{Term I} + \text{Term II}. \qquad (4.6)
\end{aligned}
$$

Let V_n be an uniform random variable on $\{1, 2, \ldots, n\}$ independent of everything else. Then according to our construction,

$$
\begin{aligned}
\mathbb{E}(\text{Term II}) &\leq n\mathbb{P}\left({}^*X^n \text{ with root } V_n \text{ has property } \mathcal{A}_d\right) \\
&\leq n\mathbb{P}\left(X^{n,\rho_n} \text{ with root } V_n \text{ has property } \mathcal{A}_d\right). \qquad (4.7)
\end{aligned}
$$

Before presenting our first proposition, we state an auxiliary result, BKR inequality (see e.g. [48]). Consider a product space Ω of finite sets $\Omega_1, \ldots, \Omega_k$,

$$\Omega = \Omega_1 \times \ldots \times \Omega_k.$$

Let $\mathcal{F} = 2^\Omega$, and μ be a product of k probability measures $\mu_1, \ldots, \mu_k$. For any configuration $\omega = (\omega_1, \ldots, \omega_k) \in \Omega$, and any subset S of $[k]$, we define the cylinder $[\omega]_S$ by

$$[\omega]_S := \{\hat{\omega} : \hat{\omega}_i = \omega_i, \forall i \in S\}.$$

For any two subsets $A, B \subset \Omega$, define

$$A \circ B := \{\omega : \text{there exists some } S = S(\omega) \subset [k] \text{ such that } [\omega]_S \subset A,\ [\omega]_{S^c} \subset B\}.$$

Lemma 4.3.1. *For any product space Ω of finite sets, product probability measure μ on Ω and $A, B \subset \Omega$, we have the inequality*

$$\mu(A \circ B) \leq \mu(A)\mu(B).$$

In this chapter, to apply BKR inequality, we always take $\Omega^n_{i,j} = \{0,1\}$, $i \neq j \in \{1, \ldots, n\}$, and $\Omega = \prod_{i \neq j \in [n]} \Omega^n_{i,j}$. Then $\omega^n_{i,j} = 1$ represents that the node i and j are linked in the random graph $G_n\left(\frac{1}{n}\right)$. According to our construction, we also have $\mu_i(\{1\}) = \min\{a^n_{i,j}/n, 1\}$.

Proposition 4.3.2. *Let G_n be a sequence of graphs with non-negative edge weights which are bounded above by a constant $a_M > 0$. Then for any fixed d, it holds that*

$$C_k\left(G_n\left(\frac{1}{n}\right)\right) \leq n\mathbb{P}\left(X^{n,\rho_n} \textit{ with root } V_n \textit{ has property } \mathcal{A}_d\right) + o_p(n).$$

Proof. According to (4.6) and (4.7), it suffices to show that

$$\text{Term II} = \mathbb{E}(\text{Term II}) + o_p(n), \quad \text{and} \quad \text{Term I} = o_p(n).$$

In the first two steps, we show the concentration of Term II by computing its variance, and in the last step prove that Term I is small.

Step I: For any two independently and uniformly chosen vertices U and V of $G_n\left(\frac{1}{n}\right)$,

$$\mathbb{P}(d(U,V) \leq 2d) = \frac{1}{n^2} \sum_{u,v \in [n]} \mathbb{P}(d(u,v) \leq 2d) = o(1),$$

where d is the graph distance. To see this, note that $d(U,V) \leq 2d$ implies there is a path from U to V of length at most $2d$. Thus

$$\mathbb{P}(d(U,V) \leq 2d) \leq \sum_{i=1}^{2d} \mathbb{P}(\#\{\text{paths of length } i \text{ from } U \text{ to } V\} \geq 1)$$

Using Markov's inequality we get

$$\mathbb{P}(d(U,V) \leq 2d) \leq \frac{1}{n^2} \sum_{u,v \in [n]} \sum_{i=1}^{2d} \mathbb{E}(\#\{\text{paths of length } i \text{ from } u \text{ to } v\})$$

We can get a crude upper bound as

$$\mathbb{P}(d(U,V) \leq 2d) \leq \frac{1}{n^2} \sum_{u,v \in [n]} \sum_{i=1}^{2d} n^{i-1} \left(\frac{a_M}{n}\right)^i = o(1).$$

Step II: Let $G_n^d[v]$ be the subgraph of $G_n\left(\frac{1}{n}\right)$ formed by the vertices within distance d of $v \in [n]$, and define $B_v = \{\text{root } v \text{ has property } \mathcal{A}_d \text{ in } G_n^d[v]\}$. It can be easily verified that

$$\begin{aligned}\mathbb{E}(\text{Term II}^2) &= \sum_{v,v' \in [n]} \mathbb{P}\left(\text{root } v \text{ and } v' \text{ has property } \mathcal{A}_d\right) \\ &= \sum_{v \in [n]} \mathbb{P}(B_v) + \sum_{v \neq v'} \mathbb{P}(B_v \cap B_{v'}) \qquad (4.8)\end{aligned}$$

For two different vertices v and v', we break the probability in two parts,

$$\mathbb{P}(B_v \cap B_{v'}) = \mathbb{P}(B_v \cap B_{v'}, d(v,v') \leq 2d) + \mathbb{P}(B_v \cap B_{v'}, d(v,v') > 2d). \qquad (4.9)$$

For the second term on the right of (4.9), it can be easily seen that

$$\{d(v,v') > 2d\} \cap B_v \cap B_{v'} \subset \{d(v,v') > 2d\} \cap B_v \circ B_{v'}.$$

Therefore we get that

$$\begin{aligned}\mathbb{P}(B_v \cap B_{v'}) &= \mathbb{P}\left(d(v,v') \leq 2d\right) + \mathbb{P}\left(B_v \circ B_{v'}, d(v,v') > 2d\right) \\ &\leq \mathbb{P}\left(d(v,v') \leq 2d\right) + \mathbb{P}\left(B_v \circ B_{v'}\right).\end{aligned}$$

Now since B_v and $B_{v'}$ are increasing events, according to Lemma 4.3.1 we obtain that

$$\mathbb{P}\left(B_v \cap B_{v'}\right) \leq \mathbb{P}\left(d(v,v') \leq 2d\right) + \mathbb{P}(B_v)\mathbb{P}(B_{v'}). \tag{4.10}$$

Combining (4.10) and (4.8) we get

$$\mathbb{E}(\text{Term II}^2) \leq n^2\mathbb{P}\left(d(U,V) \leq 2d\right) + \left(\mathbb{E}\left(\text{Term II}\right)\right)^2 + \sum_{v\in[n]} \left(\mathbb{P}(B_v) - \mathbb{P}(B_v)^2\right). \tag{4.11}$$

Therefore using **Step I** we get $\mathbb{V}(\text{Term II}) = o(n^2)$. Now using Markov's inequality we conclude that Term II $= \mathbb{E}(\text{Term II}) + o_p(n)$.

Step III: Let us denote $C_v := \{v \text{ is in a cycle of length at most } 2d\}$. The first moment of the Term I is given by

$$\sum_{v\in[n]} \mathbb{P}\left(C_v\right) \leq n \sum_{l=3}^{2d} \frac{(n-1)!}{(n-l)!} \left(\frac{a_M}{n}\right)^l \leq \sum_{l=3}^{2d} a_M^l = o(n). \tag{4.12}$$

For the second moment, note that

$$\mathbb{E}(\text{Term I}^2) = \sum_{v\in[n]} \mathbb{P}(C_v) + \sum_{v\neq v'} \mathbb{P}(C_v \cap C_{v'}).$$

For two different vertices v, v', the probability can be written as

$$\mathbb{P}(C_v \cap C_{v'}) = \mathbb{P}(C_v \cap C_{v'}, d(v,v') > 2d) + \mathbb{P}(C_v \cap C_{v'}, d(v,v') \leq 2d),$$

and therefore

$$\mathbb{P}(C_v \cap C_{v'}) \leq \mathbb{P}(C_v \cap C_{v'}, d(v,v') > 2d) + \mathbb{P}(d(v,v') \leq 2d).$$

Note that

$$\{d(v,v') > 2d\} \cap C_v \cap C_{v'} \subset \{d(v,v') > 2d\} \cap C_v \circ C_{v'}.$$

Therefore according to Lemma 4.3.1, we obtain that

$$\begin{aligned}
\mathbb{P}(C_v \cap C_{v'}) &\le \mathbb{P}(C_v \circ C_{v'}, d(v,v') > 2d) + \mathbb{P}(d(v,v') \le 2d) \\
&\le \mathbb{P}(C_v \circ C_{v'}) + \mathbb{P}(d(v,v') \le 2d) \\
&\le \mathbb{P}(C_v)\mathbb{P}(C_{v'}) + \mathbb{P}(d(v,v') \le 2d).
\end{aligned}$$

Now summing over all $v, v' \in [n]$ and using **Step I**, we get

$$\mathbb{E}(\text{Term I}^2) = \mathbb{E}(\text{Term I})^2 + o(n^2)$$

We can conclude our result by using Markov's inequality. □

4.3.1 Recursive formula

Let us first introduce some notation. For any graphon W, we denote the initial particle of its associated branching process X^W by X_0^W, and the first generation by $X_{\{1\}}^W, \dots, X_{\{N(W)_0\}}^W$, where $N(W)_0$ is the number of offsprings of X_0^W. For each element in the d-th generation, we denote it by $X_{\{i_1|i_2|\dots|i_d\}}^W$ if he is the i_d-th child of $X_{\{i_1|i_2|\dots|i_{d-1}\}}^W$. Denote the number of offsprings of $X_{\{i_1|i_2|\dots|i_d\}}^W$ by $N(W)_{\{i_1|i_2|\dots|i_d\}}$, and the type of $X_{\{i_1|i_2|\dots|i_d\}}^W$ by $T(W)_{\{i_1|i_2|\dots|i_d\}}$. Define the collection of offspring numbers in the first d generations by

$$\mathbf{N}(W)^d := \{N(W)_0\} \cup \ldots \cup \{N(W)_{\{i_1|i_2|\dots|i_d\}} : i_j \le N(W)_{\{i_1|i_2|\dots|i_{j-1}\}}, j = 1, \dots, d\},$$

and the collection of offspring numbers of $X^W_{\{i\}}$, $1 \le i \le N(W)_0$ by

$$\mathbf{N}(W)^d_{\{i\}} := \{N(W)_{\{i\}}\} \cup \ldots \cup \{N(W)_{\{i|i_2|\ldots|i_d\}} : i_j \le N(W)_{\{i_1|i_2|\ldots|i_{j-1}\}}, j = 2, \ldots, d\}.$$

Denote the realizations of random variables $\mathbf{N}(W)^d$ and $\mathbf{N}(W)^d_{\{i\}}$ by $\mathbf{K}^d$ and $\mathbf{K}^d_{\{i\}}$ respectively, and especially denote the realization of $N(W)_0$ by k_0. Define functions

$$g(x, \mathbf{K}^d) := \mathbb{P}(\mathbf{N}(W)^d = \mathbf{K}^d \,|\, T(W)_0 = x).$$

It is clear that

$$\mathbb{P}(\mathbf{N}(W)^d = \mathbf{K}^d) = \int g(x, \mathbf{K}^d)\, dx.$$

Proposition 4.3.3. *We have that*

$$g(x, \mathbf{K}^d) = \frac{e^{-\int W(x,y)dy}}{k_0!} \prod_{j=1}^{k_0} \left(\int W(x,y) g(y, \mathbf{K}^d_{\{j\}})\, dy \right). \tag{4.13}$$

Proof. It can be easily seen that $g(x, k_0) = \frac{1}{k_0!} e^{-\int W(x,y)dy} \left(\int W(x,y)dy\right)^{k_0}$. For $d \ge 1$, we get that

$$\begin{aligned} g(x, \mathbf{K}^d) =& \mathbb{P}(\mathbf{N}(W)^d = \mathbf{K}^d \,|\, T(W)_0 = x) \\ =& \mathbb{P}(N(W)_0 = k_0 \,|\, T(W)_0 = x) \\ & \times \mathbb{P}(\mathbf{N}(W)^d_{\{j\}} = \mathbf{K}^d_{\{j\}}, j = 1, \ldots, k_0 \,|\, N(W)_0 = k_0, T(W)_0 = x) \\ =& g(x, k_0) \int_{y_1} \cdots \int_{y_{k_0}} \prod_{j=1}^{k_0} g(y_j, \mathbf{K}^d_{\{j\}})\, \mathbb{P}(T(W)_{\{j\}} \in dy_j \,|\, N(W)_0 = k_0, T(W)_0 = x). \end{aligned}$$

In conjunction with the equation

$$\prod_{j=1}^{k_0} \mathbb{P}(T(W)_{\{j\}} \in dy_j \mid N(W)_0 = k_0, T(W)_0 = i) = \frac{\prod_{j=1}^{k_0} W(x, y_j)\, dy_j}{(\int_y W(x,y)\, dy)^{k_0}},$$

we obtain the recursive formula

$$g(x, \mathbf{K}^d) = \frac{e^{-\int W(x,y)dy}}{k_0!} \prod_{j=1}^{k_0} \left(\int W(x,y) g(y, \mathbf{K}^d_{\{j\}})\, dy \right).$$

□

4.3.2 Convergence

Let W_n be a sequence of graphons such that $d_\square(W_n, W) \to 0$ and

$$\sup_{n,x,y} W_n(x,y) \le a_M$$

for some positive constant a_M. Let X^n be the associated branching process of W_n, and

$$g_n(x, \mathbf{K}^d) = \mathbb{P}(\mathbf{N}(W_n)^d = \mathbf{K}^d \mid T(W_n)_0 = x).$$

We want to show that as $n \to \infty$

$$\int g_n(x, \mathbf{K}^d)\, dx \to \int g(x, \mathbf{K}^d)\, dx.$$

To see this, for any graphon W, any finite tree T with root 0, any $x \in [0,1]$, we define the vertex prescribed homomorphism density

$$t^x(T, W) = \int_{[0,1]^{|V(T)|-1}} \prod_{0i \in E(T)} W(x, x_i) \prod_{ij \in E(T), i,j \ge 1} W(x_i, x_j)\, dx_1 \ldots dx_{|V(T)|-1},$$

and the homomorphism density

$$t(T,W) = \int\limits_{[0,1]} t^x(T,W)\,dx.$$

It is well-known that for finite T, $t(T,W_n) \to t(T,W)$ as long as $d_\square(W_n,W) \to 0$; see e.g. [49, 50, 142]. We will rewrite $\int g_n(x,\mathbf{K}^d)\,dx$ and $\int g(x,\mathbf{K}^d)\,dx)$ as $\sum_{m\geq 0}\lambda_m t(T_m,W_n)$ and $\sum_{m\geq 0}\lambda_m t(T_m,W)$ respectively for a sequence of trees T_m.

Proposition 4.3.4. *Suppose W is a graphon such that $\sup_{x,y} W(x,y) \leq a_M$. Then for any $d \in \mathbb{N}$ and any configuration $\mathbf{K}^d$, there exists a sequence of finite trees $(T_m)_{m\geq 0}$, and a sequence of real numbers $(\lambda_m)_{m\geq 0}$ such that*

(i) $\sum_{m\geq 0} |\lambda_m|\, a_M^{|E(T_m)|} < +\infty$;

(ii) $g(x,\mathbf{K}^d) = \sum_{m\geq 0}\lambda_m\, t^x(T_m,W)$.

Proof. Let us prove by induction. For $d=0$, we have that

$$g(x,k) = \frac{1}{k!}e^{-\int W(x,y)dy}\left(\int W(x,y)dy\right)^k = \frac{1}{k!}\sum_{m=0}\frac{(-1)^m}{m!}\left(\int W(x,y)dy\right)^{m+k}.$$

For any $m \in \mathbb{N}$, take T_m to be an $(m+k)$-star, i.e., a tree of height 1 with $(m+k)$ leaves. Define $\lambda_m := \frac{(-1)^m}{k!m!}$. Then it can be easily seen that

$$\sum_{m\geq 0}|\lambda_m|\,a_M^{|E(T_m)|} = \sum_{m\geq 0}\frac{a_M^{t+k}}{k!m!} < +\infty,$$

and

$$g(x,k) = \sum_{m\geq 0}\lambda_m\, t^x(T_m,W).$$

Now suppose that our claim is true for any configuration $\mathbf{K}^{d-1}$. According to our

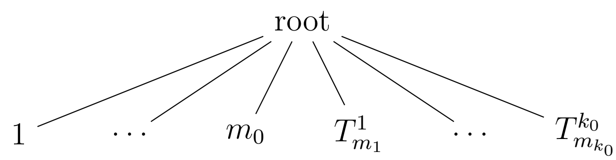


Figure 4.1: Tree T_m

recursive formulas (4.13), we expand the exponential term and obtain that

$$g(x,\mathbf{K}^d) = \frac{1}{k_0!}\sum_{m\geq 0}\frac{(-1)^m}{m!}\left(\int W(x,y)dy\right)^m \prod_{j=1}^{k_0}\left(\int W(x,y)\, g(y,\mathbf{K}^d_{\{j\}})\, dy\right).$$

For each $\mathbf{K}^d_{\{j\}}, j = 1,\ldots,k_0$, we have sequences $(\lambda^j_m)_{m\geq 0}, (T^j_m)_{m\geq 0}$ such that our claim is satisfied. For each $m = (m_0, m_1, \ldots, m_{k_0}) \in \mathbb{N}^{k_0+1}$, we define $\lambda_m = \frac{(-1)^{m_0}}{k_0! m_0!}\prod_{j=1}^{k_0}\lambda^j_{m_j}$, and tree T_m as in Figure 4.1. It is then clear that

$$\sum_{m\in\mathbb{N}^{k_0+1}} |\lambda_m|\, a_M^{|E(T_m)|} \leq \sum_{m_0\in\mathbb{N}}\frac{a_M^{k_0+m_0}}{k_0! m_0!}\prod_{j=1}^{k_0}\left(\sum_{m_j\in\mathbb{N}}|\lambda^j_{m_j}|\, a_M^{|E(T^j_{m_j})|}\right) < +\infty.$$

According to our induction, we have that

$$g(y,\mathbf{K}^d_{\{j\}}) = \sum_{m_j\geq 0}\lambda^j_{m_j} t^y(T^j_{m_j}, W).$$

Therefore, we obtain that

$$g(x,\mathbf{K}^d) = \sum_{m\in\mathbb{N}^{k_0+1}}\lambda_m\left(\int W(x,y)dy\right)^{m_0}\prod_{j=1}^{k_0}\left(\int W(x,y)\, t^y(T^j_{m_j}, W)\, dy\right).$$

It can be easily verified that for each $m \in \mathbb{N}^{k_0+1}$,

$$t^x(T_m,\mathbf{K}^d) = \left(\int W(x,y)dy\right)^{m_0}\prod_{j=1}^{k_0}\left(\int W(x,y)\, t^y(T^j_{m_j}, W)\, dy\right).$$

Thus, we conclude that

$$g(x, \mathbf{K}^d) = \sum_{m \in \mathbb{N}^{k_0+1}} \lambda_m \, t^x(T_m, W).$$

$\square$

Proposition 4.3.5. *Suppose W_n is a sequence of graphons such that $d_\square(W_n, W) \to 0$, and satisfying $\sup_{n,x,y} W_n(x, y) \leq a_M$ for some positive constant a_M. Then it holds that*

$$\lim_{n\to\infty} \mathbb{P}(\mathbf{N}(W_n)^d = \mathbf{K}^d) = \mathbb{P}(\mathbf{N}(W)^d = \mathbf{K}^d).$$

Proof. According to Proposition 4.3.4, we get that

$$\mathbb{P}(\mathbf{N}(W_n)^d = \mathbf{K}^d) = \int g_n(x, \mathbf{K}^d)\, dx = \sum_{m\geq 1} \lambda_m \, t(T_m, W_n),$$
$$\mathbb{P}(\mathbf{N}(W)^d = \mathbf{K}^d) = \int g(x, \mathbf{K}^d)\, dx = \sum_{m\geq 1} \lambda_m \, t(T_m, W).$$

Since W_n converges to W in that cut norm, we have that $t(T_m, W_n) \to t(T_m, W)$ as $n \to \infty$. Due to the uniform bound

$$\sum_{m\geq 1} \lambda_m \, t(T_m, W_n) \leq \sum_{m\geq 1} \lambda_m \, a_M^{|E(T_m)|} < +\infty,$$

we apply the dominated convergence theorem, and conclude that $\mathbb{P}(\mathbf{N}(W_n)^d = \mathbf{K}^d)$ converges to $\mathbb{P}(\mathbf{N}(W)^d = \mathbf{K}^d)$ as $n \to \infty$.

$\square$

4.3.3 Tightness

Notice that $X^W \in \mathcal{A}_d$ is equivalent to that $\mathbf{N}(W)^d \in \mathcal{A}_d$. To make our computation clear, we will sometimes adopt the latter notation. Recall we want to show that

$$\mathbb{P}(\mathbf{N}(W)^d \in \mathcal{A}_d) = \lim_{n\to\infty} \mathbb{P}(\mathbf{N}(W_n)^d \in \mathcal{A}_d). \tag{4.14}$$

To apply Proposition 4.3.5, we need a tightness result.

Lemma 4.3.6. *For $K \in \mathbb{N}$, we define $\mathbf{N}(W)^d \le K$ if $N(W)_{\{i_1|i_2|\dots|i_j\}} \le K$ for any $X^W_{\{i_1|i_2|\dots|i_j\}}$ in the first d generations. Suppose $\sup_{x,y} W(x,y) \le a_M$ for some positive constant a_M. Then for any $\alpha > 0$, $d \in \mathbb{N}$, there exists a large enough $K_0 \in \mathbb{N}$ uniformly for $x \in [0,1]$ such that $K \ge K_0$ implies*

$$\mathbb{P}(\mathbf{N}(W)^d \le K \,|\, T(W)_0 = x) > 1 - (1/K)^\alpha. \tag{4.15}$$

Here, the choice of K_0 only depends on α, d and a_M.

Proof. Let us prove (4.15) by induction. Recall for the initial generation we have that

$$g(x,k) = \frac{1}{k!} e^{-\int W(x,y)dy} \left(\int W(x,y)dy \right)^k.$$

For any $k \in \mathbb{N}$, we define for $c \in \mathbb{R}_+$

$$\psi_k(c) := \sum_{l=k+1}^{\infty} \frac{1}{l!} e^{-c} c^l.$$

Thus we have that

$$\mathbb{P}(N(W)_0 \le k \,|\, T(W)_0 = x) = 1 - \psi_k \left(\int W(x,y)dy \right).$$

It can be easily verified that $\psi_k'(c) = \frac{e^{-c}c^k}{k!} \geq 0$, and hence $\psi_k\left(\int W(x,y)dy\right) \leq \psi_k(a_M)$. Take K large enough that $\psi_K(a_M) < (1/K)^\alpha$. Then it is clear that

$$\mathbb{P}(N(W)_0 \leq K \,|\, T(X^n)_0 = x) = 1 - \psi_K\left(\int W(x,y)dy\right) > 1 - (1/K)^\alpha.$$

Assume our claim is true for $d-1$. Then for any $\beta > 0$, there exists a K such that

$$\mathbb{P}(\mathbf{N}(W)^d_{\{j\}} \leq K \,|\, T(W)^d_{\{j\}} = y) \geq 1 - (1/K)^\beta.$$

Note that

$$\begin{aligned}
&\mathbb{P}(\mathbf{N}(W)^d \leq K \,|\, T(W)_0 = x) \\
&= \sum_{k=0}^{K} \Big(\mathbb{P}(\mathbf{N}(W)^d_{\{j\}} \leq K, j = 1, \ldots, k \,|\, N(W)_0 = k, T(W)_0 = x) \\
&\qquad \times \mathbb{P}(N(W)_0 = k \,|\, T(W)_0 = x)\Big).
\end{aligned}$$

As in the proof of Proposition 4.3.3, we have that

$$\begin{aligned}
&\mathbb{P}(\mathbf{N}(W)^d \leq K \,|\, T(W)_0 = x) \\
&= \sum_{k=0}^{K} \left(\frac{e^{-\int W(x,y)dy}}{k!} \prod_{j=1}^{k} \left(\int_y W(x,y)\mathbb{P}(\mathbf{N}(W)^d_{\{j\}} \leq K \,|\, T(W)_{\{j\}} = y)\, dy \right)\right) \\
&> \sum_{k=0}^{K} \left(\frac{e^{-\int W(x,y)dy}}{k!} \prod_{j=1}^{k} \left(\int_y W(x,y)(1 - (1/K)^\beta\, dy) \right)\right) \\
&= \sum_{k=0}^{K} \left(\frac{e^{-\int W(x,y)dy}}{k!} \left(\int W(x,y)dy\right)^k (1 - (1/K)^\beta)^k \right).
\end{aligned}$$

Since $(1-(1/K)^{\beta})^K > 1-(1/K)^{\beta-2}$ for large K, we have that

$$\begin{aligned}\mathbb{P}(\mathbf{N}(W)^d \leq K \,|\, T(W)_0 = x) > & \sum_{k=0}^{K} \left(\frac{e^{-\int W(x,y)dy}}{k!} \left(\int W(x,y)dy \right)^k \right) (1-(1/K)^{\beta-2}) \\ > & (1-\psi_K(a_M))(1-(1/K)^{\beta-2}).\end{aligned}$$

Therefore by taking $\beta = \alpha + 3$, and large K such that $\psi_K(a_M) < (1/K)^{\alpha+1}$, we conclude that

$$\mathbb{P}(\mathbf{N}(W)^d \leq K \,|\, T(W)_0 = x) > 1-(1/K)^{\alpha}.$$

$\square$

Proposition 4.3.7. *Suppose W_n is a sequence of graphons such that $d_{\square}(W_n, W) \to 0$, and satisfying $\sup_{n,x,y} W_n(x,y) \leq a_M$ for some positive constant a_M. Then for any fixed d, we have that*

$$\lim_{n\to\infty} \mathbb{P}(\mathbf{N}(W_n)^d \in \mathcal{A}_d) = \mathbb{P}(\mathbf{N}(W)^d \in \mathcal{A}_d),$$

from which we conclude that

$$\limsup_{n\to\infty} \mathbb{P}_{X^{W_n}}(\mathcal{A}) \leq \mathbb{P}_{X^W}(\mathcal{A}).$$

Proof. Due to Proposition 4.3.5, it can be seen that for fixed d, K

$$\lim_{n\to\infty} \mathbb{P}(\mathbf{N}(W_n)^d \in \mathcal{A}_d, \mathbf{N}(W_n)^d \leq K) = \mathbb{P}(\mathbf{N}(W)^d \in \mathcal{A}_d, \mathbf{N}(W)^d \leq K).$$

Applying Lemma 4.3.6, we let $K \to \infty$, and obtain that

$$\lim_{n\to\infty} \mathbb{P}(\mathbf{N}(W_n)^d \in \mathcal{A}_d) = \mathbb{P}(\mathbf{N}(W)^d \in \mathcal{A}_d).$$

For any $\epsilon > 0$, there exists a d such that

$$\mathbb{P}(\mathbf{N}(W)^d \in \mathcal{A}_d) = \mathbb{P}(X^W \in \mathcal{A}_d) \leq \mathbb{P}_{X^W}(\mathcal{A}) + \epsilon.$$

Then, it can be easily verified that

$$\limsup_{n\to\infty} \mathbb{P}_{X^{W_n}}(\mathcal{A}) \leq \limsup_{n\to\infty} \mathbb{P}(\mathbf{N}(W_n)^d \in \mathcal{A}_d) = \mathbb{P}(\mathbf{N}(W)^d \in \mathcal{A}_d) \leq \mathbb{P}_{X^W}(\mathcal{A}) + \epsilon.$$

Therefore we obtain that

$$\limsup_{n\to\infty} \mathbb{P}_{X^{W_n}}(\mathcal{A}) \leq \mathbb{P}_{X^W}(\mathcal{A}).$$

□

4.3.4 Completing the proof of the upper bound

Recalling Proposition 4.3.2, we have that

$$C_k\left(G_n\left(\frac{1}{n}\right)\right) \leq n\mathbb{P}(X^{n,\rho_n} \in \mathcal{A}_d) + o_p(n).$$

Note that X^{n,ρ_n} is the branching process associated with the graphon $\rho_n W_{G_n}$, and $d_\square(\rho_n W_{G_n}, W) \to 0$. Applying Proposition 4.3.7 with $W_n = \rho_n W_{G_n}$, we obtain that

$$C_k\left(G_n\left(\frac{1}{n}\right)\right) \leq n\mathbb{P}_{X^W}(\mathcal{A}_d) + o_p(n).$$

Letting $d \to \infty$ in the above inequality, we conclude our result.

□

4.4 The proof of the lower bound in Theorem 4.2.2

We say a graphon F is finitary if there exist finitely many disjoint intervals $I_{t_i}, i = 1, \ldots, M$ such that $\cup_{i=1}^M I_{t_i} = [0,1]$ and the restriction of F on $I_{t_i} \times I_{t_j}$ is a constant for any $1 \leq i, j \leq M$. According to [46, Lemma 7.3], the graphon W can be approximated pointwise from below by finitary graphons. More precisely, we have that

Lemma 4.4.1. *There exists a sequence of finitary graphons $(F_m)_{m\in\mathbb{N}}$ such that $F_m \leq W$ and $\lim\limits_{m\to\infty} F_m(x,y) = W(x,y)$ a.s.*

Taking a sequence of finitary graphons $(F_m)_{m\in\mathbb{N}}$ as in Lemma 4.4.2, without loss of generality we can also assume that $\inf_{x,y} F_m(x,y) \geq \delta$ and $F_m(x,y)$ is increasing in m for any $x, y \in [0,1]^2$. Keep in mind that $\inf_{x,y} F_m(x,y) \geq \delta$ implies the irreducibility of F_m. We will prove in Subsection 4.4.1 that for any $\varepsilon > 0, m \in \mathbb{N}$,

$$C_k\left(G_n\left(\frac{1}{n}\right)\right) \geq (1-2\varepsilon)n\mathbb{P}_{X^{(1-2\varepsilon)F_m}}(\mathcal{A}) + o_p(n). \tag{4.16}$$

Then in Subsection 4.4.2, we will show the continuity property

$$\liminf_{\varepsilon\to 0, m\to\infty} \mathbb{P}_{X^{(1-2\varepsilon)F_m}}(\mathcal{A}) \geq \mathbb{P}_{X^W}(\mathcal{A}). \tag{4.17}$$

It is clear that (4.16) and (4.17) together prove the lower bound part of Theorem 4.2.2, i.e.,

$$C_k\left(G_n\left(\frac{1}{n}\right)\right) \geq n\mathbb{P}_{X^W}(\mathcal{A}) + o_p(n).$$

4.4.1 Proof of (4.16)

Fixing $m \in \mathbb{N}$, and $\varepsilon \in (0, \frac{1}{m})$ such that $\lambda \to \mathbb{P}_{X^{\lambda(1-\varepsilon)F_m}}(\mathcal{A})$ is continuous at $\lambda = 1$. Suppose $[0,1]$ is a disjoint union of intervals $I_{t_j}, j = 1, \ldots, M$, and there exists a collection $\{F_m(t_i, t_j) : 1 \leq i, j \leq M\}$ such that $F_m(x,y) = F_m(t_j, t_k)$ for

$x \in I_{t_j}, y \in I_{t_k}$. Here we say $t_h, h = 1, \ldots, M$ labels to distinguish types in the definition of branching process X^n.

Before proceeding to the rigorous proof, let us first give main ideas of our argument. We divide vertices of G_n into M groups $\mathbf{Good}_{n,t_1}, \ldots, \mathbf{Good}_{n,t_M}$ with the property that for any vertex $i \in \mathbf{Good}_{n,t_h}$ and $k = 1, \ldots, M$

$$\frac{\widetilde{d}^n_{i,t_k}}{n} := \sum_{j \in \mathbf{Good}_{n,t_k}} \frac{a^n_{i,j}}{n} \geq (1-\varepsilon) F_m(t_h, t_k) |I_k|.$$

Therefore, we can heuristically consider G_n as a 'finitary' graph by labelling vertices in $\mathbf{Good}_{n,t_h}$ by t_h, $h = 1, \ldots, M$. Due to the above inequality, the branching process X^n associated with G_n stochastically dominates, in the first order, the branching process $X^{(1-\varepsilon)F_m}$ associated with $(1-\varepsilon)F_m$. Take $F^{\varepsilon}_m\left(\frac{1}{n}\right)$ to be an n-vertex random graph sampled from $(1-\varepsilon)F_m$, i.e., independently uniformly select vertices $v_i \in [0,1]$ and then connect v_i, v_j independently with probability $(1-\varepsilon)F_m(v_i, v_j)/n$. By the standard exploration argument (see e.g. [46, Section 9]), locally the random graph $G_n\left(\frac{1}{n}\right)$ ($F^{\varepsilon}_m\left(\frac{1}{n}\right)$ resp.) is almost the branching process X^{G_n} ($X^{(1-\varepsilon)F_m}$ resp.). Thus, heuristically the random graph $G_n\left(\frac{1}{n}\right)$ is more connected than the random graph $F^{\varepsilon}_m\left(\frac{1}{n}\right)$, and thus has larger size of k-core. Therefore, the inequality (4.16) follows from [170, Theorem 3.1], which says

$$C_k\left(F^{\varepsilon}_m\left(\frac{1}{n}\right)\right) = n\mathbb{P}_{X^{(1-\varepsilon)F_m}}(\mathcal{A}).$$

The following simple lemma will be used to label vertices of G_n.

Lemma 4.4.2. *Let Assumption 4.2.1(i) hold and $\varepsilon \in (0, \frac{1}{m})$ be a fixed constant. Suppose that $\|W_{G_n} - W\|_\square \to 0$, and $\eta = \min\{|I_{t_1}|, \ldots, |I_{t_M}|\} > 0$. Let $c > 0$. For large n such that $\|W_{G_n} - W\|_\square \leq \frac{\eta\delta\varepsilon c}{2M}$, there exists a collection of disjoint subsets $\widetilde{\mathbf{Bad}}_{n,t_j} \subset I_{t_j}, j = 1, \ldots, M$ such that* *(i)* $|\widetilde{\mathbf{Bad}}_{n,t_j}| \leq c, j = 1, \ldots, M.$

(ii) For any $x \in I_{t_j} \setminus \widetilde{\mathbf{Bad}}_{n,t_j}$, *we have that*

$$\int\limits_{I_{t_k}} W_{G_n}(x,y)\; dy \geq (1-\varepsilon/2)F_m(t_j,t_k)|I_{t_k}|, \quad k = 1, \ldots, M. \tag{4.18}$$

Proof. First let us recall that one can also write

$$\|W\|_\square = \sup_{0\leq f,g\leq 1 \text{ measurable}} \left| \int f(x)g(y)W(x,y)\, dx\, dy \right|. \tag{4.19}$$

For any $1 \leq j, k \leq M$, define

$$\widetilde{\mathbf{Bad}}_{n,t_j,t_k} = \left\{ x \in I_{t_j} : \int\limits_{I_{t_k}} W_{G_n}(x,y)\; dy < (1-\varepsilon/2)F_m(t_j,t_k)|I_{t_k}| \right\}.$$

Taking $f(x) = \mathbb{1}_{\{x \in \widetilde{\mathbf{Bad}}_{n,t_j,t_k}\}}$ and $g(y) = \mathbb{1}_{\{y \in I_{t_k}\}}$ in (4.19) , we obtain that

$$\int\limits_{\widetilde{\mathbf{Bad}}_{n,t_j,t_k}} dx \int\limits_{I_{t_k}} (W_{G_n}(x,y) - W(x,y))\, dy \geq -\|W_{G_n} - W\|_\square = -\frac{\eta\delta\varepsilon c}{2M}. \tag{4.20}$$

In conjunction with the fact that $F_m \leq W$, it holds that

$$\int\limits_{\widetilde{\mathbf{Bad}}_{n,t_j,t_k}} dx \int\limits_{I_{t_k}} (W_{G_n}(x,y) - F_m(x,y))\, dy \geq -\frac{\eta\delta\varepsilon c}{2M}. \tag{4.21}$$

Since for any $x \in \widetilde{\mathbf{Bad}}_{n,t_j,t_k}$, $\int_{I_{t_k}} (W_{G_n}(x,y) - F_m(x,y))\, dy \leq -\eta\delta\varepsilon/2$, it follows that

$$|\widetilde{\mathbf{Bad}}_{n,t_j,t_k}| \leq c/M.$$

Let us take

$$\widetilde{\mathbf{Bad}}_{n,t_j} = \bigcup_{k=1}^{M} \widetilde{\mathbf{Bad}}_{n,t_j,t_k},$$

and it is clear that

$$|\widetilde{\mathbf{Bad}}_{n,t_j}| \le c.$$

$\square$

Before proving the main result in this subsection, we would like to point out that our main contribution here is the observation that one can label vertices of G_n so that heuristically it dominates the finitary graphon $(1-\varepsilon)F_m$. The remaining part of proof is just a modification of [170, Theorem 3.1]. We summarize it as the following lemma, and refer the reader to [170] for a detailed argument.

Lemma 4.4.3. *Suppose F_m is an irreducible finitary graphon with M labels $t_1, \dots, t_M$, and $\lambda \to \mathbb{P}_{X^{\lambda F_m}}(\mathcal{A})$ is continuous at $\lambda = 1$. Let G_n be a sequence of graphs such that $\sup\{a^n_{i,j}\} < +\infty$. Denote by $X_i^{G_n}$ ($X_{t_h}^{F_m}$ resp.) the branching process associated with G_n (F_m resp.) that has the initial particle with type i (label t_h resp.). If the vertices of G_n can be divided into M groups $\mathbf{G}_{n,t_h}, h = 1, \dots, M$ such that for some $\varepsilon \in (0,1)$*

(i) $\frac{|\mathbf{G}_{n,t_h}|}{n} \ge (1-\varepsilon)|I_{t_h}|, h = 1, \dots, M,$

(ii) For each vertex $i \in \mathbf{G}_{n,t_h}$, the branching process $X_i^{G_n}$ stochastically dominates, in the first order, the branching process $X_{t_h}^{F_m}$,

then it holds that

$$C_k\left(G_n\left(\frac{1}{n}\right)\right) \ge (1-\varepsilon)n\mathbb{P}_{X^{(1-\varepsilon)F_m}}(\mathcal{A}) + o_p(n).$$

Completing the proof of (4.16). Since $\lambda \to \mathbb{P}_{X^{\lambda F_m}}(\mathcal{A})$ is non-decreasing with respect to λ, it has only countably many discontinuity points. Therefore we can choose arbitrarily small ε such that $\lambda \to \mathbb{P}_{X^{\lambda(1-\varepsilon)F_m}}(\mathcal{A})$ is continuous at $\lambda = 1$. For concreteness we choose $0 < \varepsilon < \frac{1}{m}$. Take ε, c, η and $\widetilde{\mathbf{Bad}}_{n,t_h}$ as in Lemma 4.4.2. For $h = 1, \dots, M$, define

$$\mathbf{Good}_{n,t_h} := \left\{ i \in [n] : \left(\frac{i-1}{n}, \frac{i}{n} \right] \in I_{t_h} \setminus \widetilde{\mathbf{Bad}}_{n,t_h} \right\}.$$

Due to the construction of W_{G_n} in (4.2), for any $\left(\frac{i-1}{n}, \frac{i}{n}\right] \in I_{t_h}$, we have either $\left(\frac{i-1}{n}, \frac{i}{n}\right] \subset \widetilde{\mathbf{Bad}}_{n,t_h}$ or $\left(\frac{i-1}{n}, \frac{i}{n}\right] \cap \widetilde{\mathbf{Bad}}_{n,t_h} = \emptyset$. Therefore it can be easily verified that

$$\frac{|\mathbf{Good}_{n,t_h}|}{n} \geq |I_{t_h}| - c - \frac{2}{n}.$$

For any $i \in \mathbf{Good}_{n,t_h}$, define

$$\widetilde{d}^n_{i,t_k} := \sum_{j \in \mathbf{Good}_{n,t_k}} a^n_{i,j}, \quad k = 1, \dots, M.$$

As a result of (4.18), we obtain that

$$\begin{aligned} \frac{\widetilde{d}^n_{i,t_k}}{n} &\geq \int_{I_{t_k}} W_{G_n}\left(i/n, y\right)\, dy - a_M \left(|I_{t_k}| - \frac{|\mathbf{Good}_{n,t_k}|}{n} \right) \\ &\geq (1 - \varepsilon/2) F_m(t_h, t_k) |I_{t_k}| - (c + 2/n) a_M. \end{aligned}$$

Take $c \leq \min\left\{\frac{\varepsilon\delta\eta}{4a_M}, \frac{\varepsilon\eta}{4}\right\}$, $n \geq \max\left\{\frac{2}{\varepsilon\delta\eta}, \frac{2}{\varepsilon\eta}\right\}$ with $\|W_{G_n} - W\|_\square \leq \frac{\eta\delta\varepsilon c}{2M}$. We conclude that there exists a collection of disjoint $\mathbf{Good}_{n,t_h} \subset [n], h = 1, \dots, M$, which satisfies the following

(i) For all $h = 1, \dots, M$,

$$\frac{|\mathbf{Good}_{n,t_h}|}{n} \geq (1 - \varepsilon)|I_{t_h}|. \tag{4.22}$$

(ii) For any $i \in \mathbf{Good}_{n,t_h}$, it holds that

$$\frac{\widetilde{d}^n_{i,t_k}}{n} \geq (1-\varepsilon) F_m(t_h, t_k)|I_{t_k}|, \quad k = 1, \ldots, M. \tag{4.23}$$

For vertices in $\mathbf{Good}_{n,t_h}$, $h = 1, \ldots, M$, we label them with t_h. Let us define

$$\mathbf{Good}_n := \bigcup_{h=1}^{M} \mathbf{Good}_{n,t_h}, \quad \tilde{n} := |\mathbf{Good}_n|.$$

Let $\widetilde{G}_{\tilde{n}}$ be a graph with vertices $\mathbf{Good}_n$ such that $\widetilde{a}^n_{i,j} := \tilde{n} a^n_{i,j}/n$ for all $i, j \in \mathbf{Good}_n$. It is clear that

$$C_k\left(G_n\left(\frac{1}{n}\right)\right) \geq C_k\left(\widetilde{G}_{\tilde{n}}\left(\frac{1}{\tilde{n}}\right)\right). \tag{4.24}$$

Take $\widetilde{X}^{\tilde{n}}$ to be a branching process sampled from $\widetilde{G}_{\tilde{n}}$. For any $i \in \mathbf{Good}_{n,t_h}$, take $\widetilde{X}^{\tilde{n}}_i$ to be a branching process sampled from graph $\widetilde{G}_{\tilde{n}}$ with root i. For any t_h, take $X^{(1-\varepsilon)F_m}_{t_h}$ to be a branching process sampled from kernel $(1-\varepsilon)F_m$ with root of label t_h. Suppose a particle in generation t of $\widetilde{X}^{\tilde{n}}_i$ is of type j with label t_h, as a result of (4.23) the number of its t_k-labelled children has Poisson distribution with parameter $\widetilde{d}^n_{j,t_k}$ larger than $(1-\varepsilon)F_m(t_h, t_k)|I_{t_k}|$. Therefore, for any $i \in \mathbf{Good}_{n,t_h}$, we can consider $X^{(1-\varepsilon)F_m}_{t_h}$ as a subset of $\widetilde{X}^{\tilde{n}}_i$. Therefore for any increasing event $\mathcal{I}$, we have that $\mathbb{P}_{\widetilde{X}^{\tilde{n}}_i}(\mathcal{I}) \geq \mathbb{P}_{X^{(1-\varepsilon)F_m}_{t_h}}(\mathcal{I})$, and also

$$\begin{aligned} \mathbb{P}_{\widetilde{X}^{\tilde{n}}}(\mathcal{I}) &= \frac{1}{\tilde{n}} \sum_{i \in \mathbf{Good}_n} \mathbb{P}_{\widetilde{X}^{\tilde{n}}_i}(\mathcal{I}) = \frac{1}{\tilde{n}} \sum_{h=1}^{M} \sum_{i \in \mathbf{Good}_{n,t_h}} \mathbb{P}_{\widetilde{X}^{\tilde{n}}_i}(\mathcal{I}) \\ &\geq \frac{1}{\tilde{n}} \sum_{h=1}^{M} |\mathbf{Good}_{n,t_h}| \mathbb{P}_{X^{(1-\varepsilon)F_m}_{t_h}}(\mathcal{I}) \geq (1-\varepsilon) \sum_{h=1}^{M} |I_{t_h}| \mathbb{P}_{X^{(1-\varepsilon)F_m}_{t_h}}(\mathcal{I}) \\ &= (1-\varepsilon) \mathbb{P}_{X^{(1-\varepsilon)F_m}}(\mathcal{I}), \end{aligned} \tag{4.25}$$

where the second inequality follows from (4.22).

Now we apply Lemma 4.4.3 to $(1-\varepsilon)F_m$ and $\widetilde{G}_{\tilde{n}}$ to conclude that

$$C_k\left(G_n\left(\frac{1}{n}\right)\right) \geq C_k\left(\widetilde{G}_{\tilde{n}}\left(\frac{1}{\tilde{n}}\right)\right) \geq (1-\varepsilon)\tilde{n}\mathbb{P}_{X^{(1-2\varepsilon)F_m}}(\mathcal{A}) + o_p(n)$$
$$\geq (1-2\varepsilon)n\mathbb{P}_{X^{(1-2\varepsilon)F_m}}(\mathcal{A}) + o_p(n).$$

$\square$

4.4.2 Proof of (4.17)

Note that if F_m converges to W pointwise from below, by the dominated convergence theorem it can be easily seen that

$$\lim_{\varepsilon\to 0, m\to\infty} d_\square((1-2\varepsilon)F_m, W) = 0.$$

Therefore it is sufficient to show that

$$\lim_{n\to\infty} \mathbb{P}_{X^{W_n}}(\mathcal{A}) \geq \mathbb{P}_{X^W}(\mathcal{A}) \text{ if } \lim_{n\to\infty} d_\square(W_n, W) = 0,$$

which we will prove in Proposition 4.4.6.

We say a branching process has property $\mathcal{B}_d$ if the root has at least $k-1$ offsprings, each of these $k-1$ offsprings has at least $k-1$ offsprings, and this occurs up to generation d, and let $\mathcal{B} = \lim_{d\to\infty} \mathcal{B}_d$. Define functions

$$\Psi_k(\lambda) := \mathbb{P}(\text{Poi}(\lambda) \geq k).$$

For any graphon W, define

$$\beta_W(x,d) := \mathbb{P}(X^W \in \mathcal{B}_d \,|\, X_0 = x), \quad \beta_W(x) := \mathbb{P}(X^W \in \mathcal{B} \,|\, X_0 = x). \tag{4.26}$$

For $W = W_n$, we simply denote

$$\beta_n(x,d) := \beta_{W_n}(x,d), \quad \beta_n(x) := \beta_{W_n}(x).$$

Lemma 4.4.4. *Let $(W_n)_{n\in\mathbb{N}}$ be a sequence of graphons such that $\|W_n - W\|_\square \to 0$. Suppose that $\inf_{x,y} W(x,y) \geq \delta > 0$ for some constant $\delta > 0$, and $\alpha : [0,1] \to [0,1]$ is a measurable function such that $\inf_y \alpha(y) \geq \delta'$ for some constant $\delta' > 0$. Fix $\varepsilon > 0$. For any large n such that $\|W_n - W\|_\square \leq \frac{\varepsilon^3\delta\delta'}{2}$, there exists a subset $\mathbf{Bad} \subset [0,1]$ such that $Leb(\mathbf{Bad}) \leq \varepsilon^2$, and*

$$(1-\varepsilon/2)\int \alpha(y)W(x,y)\,dy \leq \int \alpha(y)W_n(x,y)\,dy \tag{4.27}$$

for all $x \in \mathbf{Bad}^c$. Note that the choice of $\mathbf{Bad}$ depends on $W_n, W, \delta, \delta', \varepsilon, \alpha$.

Proof. The proof is almost the same as Lemma 4.4.2. □

Lemma 4.4.5. *Let $k \in \mathbb{N}$, and W be a graphon with $W(x,y) \geq \delta$ for all $x, y \in [0,1]^2$ such that*

$$\alpha(x) = \Psi_k\left(\int W(x,y)\alpha(y)\,dy\right)$$

has a non-zero solution $\alpha(x)$. Then $\inf_x \alpha(x) \geq \delta' > 0$ for some $\delta' > 0$.

Proof. Let us write $\int_0^1 \alpha(x)\,dx = \Delta$. If $\alpha(x)$ is a non-zero solution then we have for any $x \in [0,1]$

$$\alpha(x) = \Psi_k\left(\int W(x,y)\alpha(y)\,dy\right) \geq \Psi_k\left(\delta\int \alpha(y)\,dy\right) = \Psi_k(\delta\Delta) := \delta'.$$

□

Proposition 4.4.6. *Let $(W_n)_{n\in\mathbb{N}}$ be a sequence of graphons such that $d_\square(W_n, W) \to 0$ as $n \to \infty$. Fix any $\varepsilon > 0$. Under Assumption 4.2.1 (i), we have that for large enough n*

$$\mathbb{P}_{X^{W_n}}(\mathcal{A}) \geq \mathbb{P}_{X^{(1-\epsilon)W}}(\mathcal{A}) - \varepsilon^2, \tag{4.28}$$

and moreover under Assumption 4.2.1 (ii)

$$\liminf_{n\to\infty} \mathbb{P}_{X^{W_n}}(\mathcal{A}) \geq \mathbb{P}_{X^W}(\mathcal{A}).$$

Proof. We will only prove (4.28), since the second statement follows from this directly. Due to the equality

$$\mathbb{P}_{X^{(1-\varepsilon)W}}(\mathcal{A}) = \int \Psi_k \left(\int (1-\varepsilon)W(x,y)\beta_{(1-\varepsilon)W}(y)\, dy \right) dx,$$

we assume that there exists an $\varepsilon_0 > 0$ such that $\mathrm{Leb}\{x : \beta_{(1-\varepsilon_0)W}(x) > 0\} > 0$. Otherwise there is nothing to prove. Since $\beta_{(1-\varepsilon_0)W}(x)$ is a non-zero solution of

$$\alpha(x) = \Psi_{k-1} \left(\int (1-\varepsilon_0)W(x,y)\alpha(y)\, dy \right),$$

according to Lemma 4.4.5 there exists a $\delta' > 0$ such that $\inf_x \beta_{(1-\varepsilon_0)W}(x) > \delta'$. Fix $\varepsilon \in (0, \min\{\varepsilon_0, \frac{\delta\delta'}{2a_M}\})$. We first prove the following statement: for any large n such that $\|W_n - W\|_\square \leq \frac{\varepsilon^3\delta\delta'}{2}$, there exists a subset $\mathbf{Bad}_d \subset [0,1]$ with $\mathrm{Leb}(\mathbf{Bad}_d) < \varepsilon^2$ for each $d \geq 1$ such that

$$\beta_n(x,d) \geq \beta_{(1-\varepsilon)W}(x,d), \quad \text{for any } x \in \mathbf{Bad}_d^c. \tag{4.29}$$

Applying Lemma 4.4.4 with $\alpha(y) = 1, \forall y \in [0,1]$, we obtain some $\mathbf{Bad}_1$ with $\mathrm{Leb}(\mathbf{Bad}_1) \leq \varepsilon^2$ such that $x \in \mathbf{Bad}_1^c$ implies $\int W_n(x,y)\, dy \geq (1-\varepsilon/2) \int W(x,y)\, dy$.

It follows that

$$\beta_n(x,1) = \Psi_{k-1}\left(\int W_n(x,y)\,dy\right) \geq \Psi_{k-1}\left((1-\varepsilon/2)\int W(x,y)\,dy\right)$$

$$\geq \Psi_{k-1}\left((1-\varepsilon)\int W(x,y)\,dy\right) = \beta_{(1-\varepsilon)W}(x,1).$$

Suppose there exists some $\mathbf{Bad}_{d-1}$ with $\mathrm{Leb}(\mathbf{Bad}_{d-1}) \leq \varepsilon^2$ such that $x \in \mathbf{Bad}_{d-1}^c$ implies $\beta_n(x,d-1) \geq \beta_{(1-\varepsilon)W}(x,d-1)$. Note that

$$\beta_n(x,d) = \Psi_{k-1}\left(\int W_n(x,y)\beta_n(y,d-1)\,dy\right), \tag{4.30}$$

$$\beta_{(1-\varepsilon)W}(x,d) = \Psi_{k-1}\left(\int (1-\varepsilon)W(x,y)\beta_{(1-\varepsilon)W}(y,d-1)\,dy\right). \tag{4.31}$$

Then applying Lemma 4.4.4 with $\alpha(y) = \beta_{(1-\varepsilon)W}(y,d-1) > \delta'$, we obtain some $\mathbf{Bad}_d$ with $\mathrm{Leb}(\mathbf{Bad}_d) \leq \varepsilon^2$ such that $x \in \mathbf{Bad}_d^c$ implies that

$$\int W_n(x,y)\beta_{(1-\varepsilon)W}(y,d-1)\,dy \geq (1-\varepsilon/2)\int W(x,y)\beta_{(1-\varepsilon)W}(y,d-1)\,dy.$$

By induction, it follows that for $x \in \mathbf{Bad}_d^c$

$$\begin{aligned}
\int W_n(x,y)\beta_n(y,d-1)\,dy &\geq \int\limits_{y\in\mathbf{Bad}_{d-1}^c} W_n(x,y)\beta_{(1-\varepsilon)W}(y,d-1)\,dy \\
&\geq \int W_n(x,y)\beta_{(1-\varepsilon)W}(y,d-1)\,dy - \mathrm{Leb}(\mathbf{Bad}_{d-1})a_M \\
&\geq (1-\varepsilon/2)\int W(x,y)\beta_{(1-\varepsilon)W}(y,d-1)\,dy - \mathrm{Leb}(\mathbf{Bad}_{d-1})a_M \\
&\geq (1-\varepsilon/2)\int W(x,y)\beta_{(1-\varepsilon)W}(y,d-1)\,dy - \frac{\varepsilon\delta\delta'}{2}.
\end{aligned}$$

Since $\inf_{x\in[0,1]}\int W(x,y)\beta_{(1-\varepsilon)W}(y,d-1)\,dy \geq \delta\delta'$, we get that

$$\int W_n(x,y)\beta_n(y,d-1)\,dy \geq (1-\varepsilon/2)\int W(x,y)\beta_{(1-\varepsilon)W}(y,d-1)\,dy - \frac{\varepsilon\delta\delta'}{2}$$
$$\geq (1-\varepsilon)\int W(x,y)\beta_{(1-\varepsilon)W}(y,d-1)\,dy. \tag{4.32}$$

In conjunction with (4.30) and (4.31), we obtain that for $x \in \mathbf{Bad}_d^c$, $\beta_n(x,d) \geq \beta_{(1-\varepsilon)W}(x,d)$. Therefore for all $d \geq 1$ there is a set $\mathbf{Bad}_d$ with $\mathrm{Leb}(\mathbf{Bad}_d) < \varepsilon^2$ such that for $x \in \mathbf{Bad}_d^c$

$$\beta_n(x,d) \geq \beta_{(1-\varepsilon)W}(x,d).$$

Now we prove that $\mathbb{P}_{X^{W_n}}(\mathcal{A}) \geq \mathbb{P}_{X^{(1-\epsilon)W}}(\mathcal{A}) - \varepsilon^2$. Note that

$$\mathbb{P}_{X^{W_n}}(\mathcal{A}_d) = \int \Psi_k\left(\int W_n(x,y)\beta_n(y,d-1)\,dy\right)\,dx,$$

and

$$\mathbb{P}_{X^{(1-\varepsilon)W}}(\mathcal{A}_d) = \int \Psi_k\left(\int (1-\varepsilon)W(x,y)\beta_{(1-\varepsilon)W}(y,d-1)\,dy\right)\,dx.$$

Due to (4.32), for $x \in \mathbf{Bad}_d^c$ we have that

$$\Psi_k\left(\int W_n(x,y)\beta_n(y,d-1)\,dy\right) \geq \Psi_k\left(\int (1-\varepsilon)W(x,y)\beta_{(1-\varepsilon)W}(y,d-1)\,dy\right).$$

Since $\mathrm{Leb}(\mathbf{Bad}_d) < \varepsilon^2$ and $\Psi_k(x) \leq 1, x \geq 0$, it can be easily verified that

$$\mathbb{P}_{X^{W_n}}(\mathcal{A}_d) \geq \mathbb{P}_{X^{(1-\varepsilon)W}}(\mathcal{A}_d) - \varepsilon^2.$$

Letting $d \to \infty$ in the above inequality, we conclude the result. $\square$

CHAPTER V

Finite-Time 4-Expert Prediction Problem

5.1 Introduction

In this chapter, we explicitly solve the degenerate nonlinear PDE with $N = 4$

$$\partial_t u^T(t,x) + \frac{1}{2} \sup_{J \in P(N)} e_J^\top \partial_{xx}^2 u^T(t,x) e_J = 0,$$
$$u^T(T,x) = \Phi(x) := \max_i x_i, \tag{5.1}$$

where $P(N)$ is the power set of $\{1,\ldots,N\}$ and $e_J := \sum_{j\in J} e_j$ with $\{e_j\}_{j\in\{1,\cdots,N\}}$ representing the standard basis of $\mathbb{R}^N$. Kohn and Drenska [91, 94] showed that this equation has a unique viscosity solution, which is the continuous limit of dynamic programming equation of the Expert Prediction Problem with finite stopping. The Expert Prediction Problem is a zero sum game between a player and an adversary (see e.g. [107]). Here we construct this unique viscosity solution explicitly

$$\begin{aligned} u^T(t,x) =& \frac{-1}{16\sqrt{2}} \int_{-\infty}^{\infty} \frac{e^{-(T-t)r^2}}{r^2} \left(\psi\left(r\theta \cdot x^o + \frac{\pi}{2}\right) \sum_{k=1}^{4} \cos\left(r\alpha_k \cdot x^o\right) - 4 \right. \\ & \left. -\psi\left(r\theta\cdot x^o\right) \sum_{k=1}^{4} \sin\left(r\alpha_k \cdot x^o\right)\right) dr + \frac{1}{4}\sum_{i=1}^{4} x_i + \frac{1}{2}\sqrt{\frac{(T-t)\pi}{2}}, \end{aligned} \tag{5.2}$$

where ψ is the 2π periodic square wave function, x^o is obtained from rearranging the coordinates of x in the increasing order, and $\alpha_k, \theta \in \mathbb{R}^4$ are defined by $\alpha_{k,j} = \frac{3}{\sqrt{2}}\mathbf{1}_{\{k=j\}} - \frac{1}{\sqrt{2}}\mathbf{1}_{\{k\neq j\}}$, $\theta = \frac{1}{\sqrt{2}}(1,1,-1,-1)$. We show that $u^T \in \mathcal{C}^2$, and due to this regularity, we are able to show that the balanced comb strategy and the probability matching algorithm proposed in [107] are the asymptotic saddle points for the game. As noted in [91], in particular for $x = 0, t = 0, T = 1$, the value $u^1(0,0)$ provides the expansion of the best regret as

$$V^M(0,0) = u^1(0,0)\sqrt{M} + o(\sqrt{M}) \text{ as } M \to \infty,$$

where V^M is the value function of expert prediction problem with time maturity M. According to our solution (5.2), we obtain the explicit value of the first order coefficient $u^1(0,0) = \frac{1}{2}\sqrt{\frac{\pi}{2}}$, which resolves the open problem in [106] for $N = 4$; see also [1].

Prediction problem with expert advice is classical and fundamental in the field of *machine learning*, and has been studied for decades. We refer the reader to [69] for a nice survey. It is a dynamic zero-sum game between a player and an adversary. At each of the M rounds, based on all the prior information, the player chooses one of the N experts to follow, and simultaneously the adversary chooses a set of winning experts. The increment of the gain for each expert is either 0 or 1 depending on whether the expert is chosen by the adversary, and the increment of the gain of the player is that of the expert the player follows. Given a fixed maturity M, the objective of the player is to minimize the regret $\max_i G^i_M - G_M$, while the adversary wants to maximize the regret, where G^i_M and G_M are the gain of the expert i and the player, respectively.

For the case of 2 experts, Cover [81] showed that the asymptotically optimal strategy for the adversary is the one that chooses an expert uniformly at random.

For the case of 3 experts with geometric stopping, Gravin, Peres and Sivan [107] showed that the comb strategy, which chooses the experts with the highest gain and the one with the lowest gain with probability $\frac{1}{2}$, and chooses the second leading expert with probability $\frac{1}{2}$ is asymptotically optimal for the adversary. They also showed that the probability matching algorithm, which consists of following an expert with the probability that under the comb strategy that that expert will be the leading one at the end of game, is the player's asymptotically optimal response. For the case of $N = 3$ experts with finite stopping, it has been shown in [1] that the comb strategy is asymptotically optimal. While both [1, 107] use the theory of random walk, [94] exploits the power of the PDE method. By considering a scaled game, they have shown that the value function of discrete games converges to the viscosity solution of a PDE. Following this setting, for the case of $N = 4$ experts in the geometric horizon setting, Bayraktar, Ekren and Zhang [28] showed that the comb strategy is asymptotically optimal by explicitly solving the corresponding nonlinear PDE. And very recently in [129], Kobzar, Kohn and Wang found lower and upper bounds for the optimal regret for finite stopping problem by constructing certain sub- and supersolutions of (5.1) following the method of [172]. Their results are only tight for $N = 3$ and improved those of [1]. Let us also mention the Multiplicative Weights Algorithm, which is asymptotically optimal as both $N, M \to \infty$ (see [68]).

In this chapter we construct an explicit solution to (5.1) for $N = 4$ with finite stopping. We build our candidate solution based on the conjecture of [107], which states that the comb strategy is asymptotically optimal for any number of experts in both finite and geometric horizon problem. Note that if the comb strategy is asymptotically optimal, the solution to (5.1) should also satisfy a linear PDE with comb strategy based coefficients (see (5.7)), which is shown to be true in the geometric horizon setting in [28]. The key observation is that the PDE of the finite horizon case can, at least heuristically, be obtained by applying the inverse Laplace transform to the solution of [28] extended to the complex plane. This is at a heuristic level because these linear PDEs, unlike (5.1), may not have unique solution and the analytic extension of our function to the complex plane is not well-behaved. In Section 5.5, we perform this formal inverse Laplace transform and obtain the explicit expression in (5.2). We show in Theorem 5.3.4 that (5.2) is the classical solution of (5.7). In Theorem 5.3.5, we show that it also satisfies (5.1) by verifying that the comb strategy is optimal for the limiting problem. In Theorem 5.3.10, we show that the probability matching strategy for the player and the comb strategy for the adversary form an asymptotic saddle point, resolving the conjecture of [107] for four experts. As a corollary, we resolve the Finite versus Geometric regret conjecture in [106] (see also [1]); see Corollary 5.3.11. Our work reveals that the ratio of the value of two problems (which was conjectured to be $\frac{2}{\sqrt{\pi}}$) actually comes from the inverse Laplace transform; see (5.17). We also apply our method to obtain an explicit expression for u^T in the 3 experts case, which was not known.

We now detail some of the difficulties in our proofs. The first main difficulty is showing that the boundary condition $u^T(T, x) = \Phi(x)$ is satisfied. We first write the function u^T in terms of sine and cosine integral functions (see [4]) and perform some intricate and long arguments from complex analysis relying on the properties of these functions. Second main difficulty is showing that the function u^T actually solves the nonlinear PDE. We perform this analysis through a verification type of argument, in which we show that certain inequalities are satisfied for all (t, x) and hence ruling out all the other alternative strategies for the adversary. This analysis is the most demanding part of the chapter in which we rely on the properties of the Jacobi-theta function (see [157]) and other properties of Fourier series. The third main difficulty is showing that the probability matching algorithm for the player and the comb strategy for the adversary form an asymptotic saddle point. Relying on some delicate estimates, we show that the value function of discrete game converges to u^T if either the player adopts the probability matching algorithm, or the adversary adopts the comb strategy.

The rest of the chapter is organized as follows. In Section 5.2, we introduce the problem and provide some of lemmata. In Section 5.3, we state the three main results of our chapter, namely Theorem 5.3.4, 5.3.5 and 5.3.10. Here we also state the Corollary 5.3.11 which resolves the "geometric versus finite horizon conjecture" for 4 experts. In Section 5.4, we provide all the proofs, and in Section 5.5, we provide a heuristic derivation of the value functions for $N = 3, 4$ via inverse Laplace transform.

In the rest of this section, we will provide some frequently used notation.

Notation. Denote the left hand side and the right hand side derivatives by ∂^-, ∂^+ respectively. Denote the number of experts by N, the time horizon of the discrete game by M, and the time horizon of the continuous time control problem by T (so M in our chapter represents the T in [1, 107]). Denote by U the set of probability measures on $\{1, \ldots, N\}$ and by V the set of probability measures on $P(N)$, the power set of $\{1, \ldots N\}$. We denote by $\{e_i\}_{i=\{1,\ldots,N\}}$ the canonical basis of $\mathbb{R}^N$, and for $J \in P(N)$, e_J is defined as $e_J := \sum_{j \in J} e_j$. For all $x \in \mathbb{R}^N$, we denote by x_i the i-th coordinate of x, by $\{x^{(i)}\}_{i=1,\ldots,N}$ the ranked coordinates of x with $x^{(1)} \leq x^{(2)} \leq \ldots \leq x^{(N)}$, by $\{i_1, \ldots, i_N\}$ the reordering of $\{1, \ldots, N\}$ such that $x_{i_1} \leq x_{i_2} \leq \ldots \leq x_{i_N}$ with the convention that if two components x_i and x_j are equal and $i < j$ then the ordering is defined to be $x_i \leq x_j$. We define $x^o := \left(x^{(1)}, \ldots, x^{(N)}\right)$.

5.2 Preliminaries

We assume that a player and an adversary are playing a zero-sum game, and they interact through the evolution of the gains of N experts. At step $m \in \mathbb{N}$, by $\{G_k^i\}_{k=1,\ldots,m-1}$, we denote the history of the gains of each expert $i = 1, \ldots N$, and by $\{G_k\}_{k=1,\ldots,m-1}$, the history of the gains of the player. After observing all the prior history $\mathcal{G}_{m-1} := \{(G_k^i, G_k) : 1 \leq i \leq N, 1 \leq k \leq m-1\}$, simultaneously, the adversary chooses some experts $J_m \in P(N)$, and the player chooses the expert $I_m \in \{1, \ldots, N\}$ to follow. For each $i = 1, \ldots, N$, the gain of expert i increases by 1 if he is chosen by the adversary, otherwise remains the same. The increment of the player's gain follows that of the expert I_m he chooses. Therefore we have

$$\begin{aligned} G_m^i &= G_{m-1}^i + \mathbb{1}_{\{i \in J_m\}}, \qquad i = 1, \ldots, N; \\ G_m &= G_{m-1} + \mathbb{1}_{\{I_m \in J_m\}}. \end{aligned}$$

In order to have a value for the game, we allow both the adversary and the player to adopt randomized strategies. At step $m \in \mathbb{N}$, the adversary decides on the distribution $\beta_m \in V$ to draw J_m from, and independently the player decides on the distribution $\alpha_m \in U$ of I_m. Then the dynamic of $\{(G_m^i, G_m : 1 \leq i \leq N\}$ is given by

$$\begin{aligned} \mathbb{E}^{\alpha_m, \beta_m}[G_m^i | \mathcal{G}_{m-1}] &= G_{m-1}^i + \sum_{J \in P(N)} \beta_m(J) \mathbb{1}_{\{i \in J\}}, \qquad i = 1, \ldots, N; \\ \mathbb{E}^{\alpha_m, \beta_m}[G_m | \mathcal{G}_{m-1}] &= G_{m-1} + \sum_{i=1}^{N} \sum_{J \in P(N)} \alpha_m(i) \beta_m(J) \mathbb{1}_{\{i \in J\}}. \end{aligned}$$

Denote by $\mathcal{U}$ the collection of sequences $\{\alpha_m\}_{m \in \mathbb{N}}$ such that α_m is a function of $\mathcal{G}_{m-1}$, by $\mathcal{V}$ the collection of such sequences $\{\beta_m\}_{m \in \mathbb{M}}$. We take

$$X_m := (X_m^1, \ldots, X_m^N) := (G_m^1 - G_m, \ldots, G_m^N - G_m), \tag{5.3}$$

the difference between the gain of the player and the experts. Define the function

$$\Phi : x \mapsto \max_{1 \leq i \leq N} x_i = x^{(N)},$$

and the regret of the player at step $m \in \mathbb{N}$,

$$\Phi(X_m) = \max_{i=1,\dots,N} G_m^i - G_m.$$

The objective of the player is to minimize his expected regret at maturity M while the objective of the adversary is to maximize the regret of the player. By the Minimax theorem, the game has a solution (see [91, 107]), i.e.,

$$\sup_{\beta\in\mathcal{V}} \inf_{\alpha\in\mathcal{U}} \mathbb{E}^{\alpha,\beta}\left[\Phi(X_M)|X_0 = x\right] = \inf_{\alpha\in\mathcal{U}} \sup_{\beta\in\mathcal{V}} \mathbb{E}^{\alpha,\beta}\left[\Phi(X_M)|X_0 = x\right], \tag{5.4}$$

where $\mathbb{E}^{\alpha,\beta}$ is the probability distribution under which we evaluate the regret given the controls $\alpha = \{\alpha_m\}$ and $\beta = \{\beta_m\}$. Therefore we can define the value function

$$V^M(m,x) := \sup_{\beta\in\mathcal{V}} \inf_{\alpha\in\mathcal{U}} \mathbb{E}^{\alpha,\beta}\left[\Phi(X_M)|X_m = x\right] = \inf_{\alpha\in\mathcal{U}} \sup_{\beta\in\mathcal{V}} \mathbb{E}^{\alpha,\beta}\left[\Phi(X_M)|X_m = x\right],$$

which satisfies the following dynamical programming principle

$$V^M(m,x) = \inf_{\alpha\in U} \sup_{\beta\in V} \sum_J \beta_J \left(V^M\left(m+1, x+e_J\right) - \alpha(J)\right).$$

Additionally, it was shown in [91] that for any sequence $m_M \in \mathbb{N}$ and $x_{m_M} \in \mathbb{R}^4$ such that $\frac{m_M T}{M} \to t$ and $\frac{x_{m_M}\sqrt{T}}{\sqrt{M}} \to x$ as $M \to \infty$, we have that

$$\lim_{M\to\infty} \frac{V^M(m_M, x_{m_M})\sqrt{T}}{\sqrt{M}} \to u^T(t,x),$$

where $u^T(t,x)$ is the unique viscosity solution to (5.1). Also, we have the Feynmann Kac representation of $u^T(t,x)$

$$u^T(t,x) = \sup_{\sigma} \mathbb{E}\left[\Phi(X_T^\sigma)|X_t = x\right], \tag{5.5}$$

where X^σ is defined by $X_u = X_t + \int_t^u \sigma_s dW_s$ with W a 1-dimensional Brownian motion and the progressively measurable process (σ_s) satisfying for all $s \in [t,u]$, $\sigma_s \in \{e_J : J \in P(N)\}$.

5.3 Main Results

5.3.1 Solution to PDE (5.1) with $N = 4$

Define $\alpha_k, \theta \in \mathbb{R}^4$ by $\alpha_{k,j} = \frac{3}{\sqrt{2}}\mathbf{1}_{\{k=j\}} - \frac{1}{\sqrt{2}}\mathbf{1}_{\{k\neq j\}}$ and $\theta = \frac{1}{\sqrt{2}}(1,1,-1,-1)$. Denote the 2π periodic square wave function by

$$\psi(r) := sign\left(\tan\left(\frac{r}{2}\right)\right) = sign\left(\sin\left(r\right)\right).$$

Define the auxiliary function

$$\Lambda(r,x) := \left(\psi\left(r\theta\cdot x + \frac{\pi}{2}\right)\sum_{k=1}^{4}\cos\left(r\alpha_k\cdot x\right) - 4 - \psi\left(r\theta\cdot x\right)\sum_{k=1}^{4}\sin\left(r\alpha_k\cdot x\right)\right),$$

and our conjectured solution to (5.1)

$$u^T(t,x) := \frac{-1}{16\sqrt{2}}\int_{-\infty}^{\infty}\frac{e^{-(T-t)r^2}}{r^2}\Lambda(r,x^o)dr + \frac{1}{4}\sum_{i=1}^{4}x_i + \frac{1}{2}\sqrt{\frac{(T-t)\pi}{2}}. \tag{5.6}$$

Remark 5.3.1. Due to the presence of r^{-2}, there is a possible integrability issue of

$$\int_{-\infty}^{\infty}\frac{e^{-(T-t)r^2}}{r^2}\Lambda(r,x^o)dr.$$

However as a result of the fact that $\sum_{k=1}^4 \alpha_k\cdot x^o = 0$, we have the Taylor expansion around 0

$$\Lambda(r,x^o) = \sum_{k=1}^{4}|r|\alpha_k\cdot x^o - \sum_{k=1}^{4}\frac{(r\alpha_k\cdot x^o)^2}{2} + o(r^2) = O(r^2).$$

Thus, $u^T(t,x)$ is well-defined.

Remark 5.3.2. Since the function $\Lambda(r,x)$ is even with respect to r, we sometimes use the expression

$$u^T(t,x) = \frac{-1}{8\sqrt{2}} \int_0^\infty \frac{e^{-(T-t)r^2}}{r^2} \Lambda(r,x^o)dr + \frac{1}{4}\sum_{i=1}^4 x_i + \frac{1}{2}\sqrt{\frac{(T-t)\pi}{2}}.$$

Definition 5.3.3. For all $x \in \mathbb{R}^4$ with $x_{i_1} \le x_{i_2} \le x_{i_3} \le x_{i_4}$, we denote by $\mathcal{J}_\mathcal{C}(x) \in P(4)$ the comb strategy which chooses the experts i_4 and i_2. Denote $\sigma_\mathcal{C}(X_s) := e_{\mathcal{J}_\mathcal{C}(X_s)}$ to be the corresponding control of problem (5.5). We take the convention that if two components x_i and x_j of the points are equal for $i < j$ then the ordering of the point is taken with $x_i \le x_j$.

The following theorem assembles properties of u^T, and its proof is provided in Section 5.4.1.

Theorem 5.3.4. *The function u^T is symmetric in x, satisfies $u^T \in \mathcal{C}([0,T]\times\mathbb{R}^4) \cap \mathcal{C}^2([0,T)\times\mathbb{R}^4)$ and*

$$\begin{aligned} \partial_t u^T(t,x) + \frac{1}{2} e^\top_{\mathcal{J}_\mathcal{C}(x)} \partial^2_{xx} u^T(t,x) e_{\mathcal{J}_\mathcal{C}(x)} &= 0, \\ u(T,x) &= \max_{i=1,\dots,4} x_i. \end{aligned} \tag{5.7}$$

The first derivative of u^T on $\theta \cdot x^o < 0$ is

$$\begin{aligned} \partial_{x_i} u^T(t,x) = &\frac{1}{16\sqrt{2}} \int_{-\infty}^\infty \frac{e^{-(T-t)r^2}}{r} \sum_{k=1}^4 \alpha_{k,i} \left(\psi\left(r\theta\cdot x^o + \frac{\pi}{2}\right) \sin\left(r\alpha_k \cdot x\right) \right. \\ &\left. + \psi\left(r\theta\cdot x^o\right)\cos\left(r\alpha_k\cdot x\right)\right) dr + \frac{1}{4}, \end{aligned} \tag{5.8}$$

and if $\theta \cdot x^o = 0$, *it is*

$$\partial_{x_i} u^T(t,x) = \frac{1}{4}. \tag{5.9}$$

If $\theta \cdot x^o < 0$, *and* $x^{(2)} < x^{(3)}$, *we have*

$$\begin{aligned}
\partial^2_{x_i x_j} u^T(t,x) =& \frac{1}{16\sqrt{2}} \int\limits_{-\infty}^{\infty} e^{-(T-t)r^2} \sum_{k=1}^{4} \alpha_{k,i}\alpha_{k,j} \Bigg(\psi\left(r\theta \cdot x^o + \frac{\pi}{2}\right) \cos\left(r\alpha_k \cdot x\right) \\
& -\psi\left(r\theta \cdot x^o\right) \sin\left(r\alpha_k \cdot x\right)\Big) \, dr \\
& + \frac{\partial_{x_j}(\theta \cdot x^o)}{16\sqrt{2}} \sum_{l \in \mathbb{Z}} \frac{(-1)^l e^{-\frac{(T-t)(\pi(l+1/2))^2}{(\theta \cdot x^o)^2}}}{\theta \cdot x^o} \sum_{k=1}^{4} 2\alpha_{k,i} \sin\left(\frac{\alpha_k \cdot x\pi(l+1/2)}{\theta \cdot x^o}\right) \\
& - \frac{\partial_{x_j}(\theta \cdot x^o)}{16\sqrt{2}} \sum_{l \in \mathbb{Z}} \frac{(-1)^l e^{-\frac{(T-t)(\pi l)^2}{(\theta \cdot x^o)^2}}}{\theta \cdot x^o} \sum_{k=1}^{4} 2\alpha_{k,i} \cos\left(\frac{\alpha_k \cdot x\pi l}{\theta \cdot x^o}\right),
\end{aligned} \tag{5.10}$$

if $\theta \cdot x^o < 0$ *and* $x^{(2)} = x^{(3)}$,

$$\begin{aligned}
\partial^2_{x_i x_j} u^T(t,x) =& \frac{1}{16\sqrt{2}} \int\limits_{-\infty}^{\infty} e^{-(T-t)r^2} \sum_{k=1}^{4} \alpha_{k,i}\alpha_{k,j} \Bigg(\psi\left(r\theta \cdot x^o + \frac{\pi}{2}\right) \cos\left(r\alpha_k \cdot x\right) \\
& -\psi\left(r\theta \cdot x^o\right) \sin\left(r\alpha_k \cdot x\right)\Big) \, dr,
\end{aligned} \tag{5.11}$$

and if $\theta \cdot x^o = 0$,

$$\partial^2_{x_i x_j} u^T(t,x) = \frac{1}{16\sqrt{2}} \int\limits_{-\infty}^{\infty} e^{-(T-t)r^2} \sum_{k=1}^{4} \alpha_{k,i}\alpha_{k,j} dr. \tag{5.12}$$

The proof of the following theorem is in Section 5.4.2.

Theorem 5.3.5. *The function* u^T *defined in* (5.6) *is also a solution to* (5.1) *and the comb strategy* $e_{\mathcal{J}_\mathcal{C}}$ *is optimal for the problem* (5.1).

5.3.2 An asymptotical Nash equilibrium for the game (5.4) with $N = 4$

Given the value of u^T, we now describe a family of asymptotically optimal strategies for both the player and the adversary. Inspired by [107] we give the following definition.

Definition 5.3.6. (i)For $M \in \mathbb{N}$, we denote by $\mathcal{J}^b_{\mathcal{C}}(M) \in \mathcal{V}$, the balanced comb strategy, which at state $x \in \mathbb{R}^4$ and round $m \in \mathbb{N}$, chooses experts $\mathcal{J}_{\mathcal{C}}(x) \in P(4)$ with probability $\frac{1}{2}$ and $\mathcal{J}^c_{\mathcal{C}}(x) \in P(4)$ with probability $\frac{1}{2}$.

(ii)For $M \in \mathbb{N}$, we denote by $\alpha^*(M) \in \mathcal{U}$, the strategy that, at state $x \in \mathbb{R}^4$ and round $m \in \mathbb{N}$, chooses the expert i with probability $\partial_{x_i} u^T\left(\frac{mT}{M}, \frac{x\sqrt{T}}{\sqrt{M}}\right)$ for all $i = 1, \ldots, 4$.

Remark 5.3.7. Note that Definition 5.3.3 defines a control for the problem (5.5) while Definition 5.3.6 defines controls for the game (5.4). Hence the latter depends on M, T and x, and the control $\alpha^*(M)$ actually reflects the scaling between the two problems (see [94] for details).

Remark 5.3.8. According to the Feynmann Kac representation (5.5) and Theorem 5.3.4, we have

$$u^T(t,x) = \mathbb{E}\left[\Phi\left(x + \int_t^T \sigma_{\mathcal{C}}(X_s)dW_s\right)\right].$$

Then heuristically

$$\partial_{x_i} u^T(t,x) = \mathbb{E}\left[\partial_{x_i}\Phi\left(x + \int_t^T \sigma_{\mathcal{C}}(X_s)dW_s\right)\right] = \mathbb{P}\left[(X_T^{\sigma_{\mathcal{C}}})_i = (X_T^{\sigma_{\mathcal{C}}})^{(4)} | X_t = x\right],$$

which is just the probability matching algorithm proposed in [107].

Definition 5.3.9. Define the following two value functions

$$\underline{V}^M(m,.) : x \mapsto \inf_{\alpha \in \mathcal{U}} \mathbb{E}^{\alpha, \mathcal{J}^b_{\mathcal{C}}(M)} \left[\Phi(X_M) | X_m = x \right],$$
$$\overline{V}^M(m,.) : x \mapsto \sup_{\beta \in \mathcal{V}} \mathbb{E}^{\alpha^*(M), \beta} \left[\Phi(X_M) | X_m = x \right],$$

and their limits

$$\underline{u}^T(t,x) := \liminf_{(M, \frac{m_M T}{N}, \frac{x_{m_M}\sqrt{T}}{\sqrt{M}}) \to (\infty, t, x)} \frac{\underline{V}^M\left(m_M, x_{m_M}\right)\sqrt{T}}{\sqrt{M}},$$
$$\overline{u}^T(t,x) := \limsup_{(M, \frac{m_M T}{M}, \frac{x_{m_M}\sqrt{T}}{\sqrt{M}}) \to (\infty, t, x)} \frac{\overline{V}^M(m_M, x_{m_M})\sqrt{T}}{\sqrt{M}}.$$

The proof the following theorem can be found in Section 5.4.3.

Theorem 5.3.10. *The family of strategies* $(\alpha^*(M))_{M\in\mathbb{N}} \in \mathcal{U}^{\mathbb{N}}$ *and* $(\mathcal{J}^b_{\mathcal{C}}(M))_{M\in\mathbb{N}} \in \mathcal{V}^{\mathbb{N}}$ *are asymptotic saddle points for the player and the adversary, in the sense that for all* $(t,x) \in [0,T] \times \mathbb{R}^4$

$$\underline{u}^T(t,x) = \overline{u}^T(t,x) = u^T(t,x).$$

It can be easily seen that $\underline{u}^T(t,x) \le u^T(t,x) \le \overline{u}^T(t,x)$, and our main result states that they are actually equal, which implies that at the leading order it is optimal for both the player and the adversary to choose respectively the controls $\alpha^*(M)$ and $\mathcal{J}^b_{\mathcal{C}}(M)$, i.e., for any $\alpha_M \in \mathcal{U}$, $\beta_M \in \mathcal{V}$ and $T > 0$, we have that

$$\liminf_{M\to\infty} \sqrt{\frac{T}{M}} \left(\mathbb{E}^{\alpha_M, \mathcal{J}^b_{\mathcal{C}}(M)} \left[\Phi(X_M) \middle| X_0 = \frac{\sqrt{M}x}{\sqrt{T}} \right] \right.$$
$$\left. - \mathbb{E}^{\alpha^*(M), \mathcal{J}^b_{\mathcal{C}}(M)} \left[\Phi(X_M) \middle| X_0 = \frac{\sqrt{M}x}{\sqrt{T}} \right] \right) \ge 0,$$

$$\limsup_{M\to\infty}\sqrt{\frac{T}{M}}\left(\mathbb{E}^{\alpha^*(M),\beta_M}\left[\Phi(X_M)\middle|X_0=\frac{\sqrt{M}x}{\sqrt{T}}\right]\right.$$
$$\left.-\mathbb{E}^{\alpha^*(M),\mathcal{J}^b_{\mathcal{C}}(M)}\left[\Phi(X_M)\middle|X_0=\frac{\sqrt{M}x}{\sqrt{T}}\right]\right)\leq 0.$$

5.3.3 Relation between the finite and geometric stopping

We recall the following results from [28] and [94]. Let T^δ be a geometric random variable with parameter $\delta > 0$. Define

$$V^\delta(X_0) := \sup_{\beta\in\mathcal{V}}\inf_{\alpha\in\mathcal{U}}\mathbb{E}^{\alpha,\beta}\left[\Phi(X_{T^\delta})\right] = \inf_{\alpha\in\mathcal{U}}\sup_{\beta\in\mathcal{V}}\mathbb{E}^{\alpha,\beta}\left[\Phi(X_{T^\delta})\right],$$

and

$$u^\delta : x\in\mathbb{R}^N \mapsto V^\delta\left(\frac{x}{\sqrt{\delta}}\right)\sqrt{\delta}.$$

so that as $\delta\downarrow 0$, the function u^δ converges locally uniformly to $u:\mathbb{R}^N\mapsto\mathbb{R}$ which is the unique viscosity solution of the equation

$$u(x) - \frac{1}{2}\sup_{J\in P(N)} e_J^\top \partial^2 u(x) e_J = \Phi(x). \tag{5.13}$$

The main conjecture in [107] regarding the relation between the finite and geometric horizon control problems is that

$$V^M(0,0) \underset{M\to+\infty}{\sim} \frac{2}{\sqrt{\pi}} V^{\frac{1}{M}}(0).$$

The corollary below shows that this statement is true for $N = 3, 4$.

Corollary 5.3.11. *For $N = 3, 4$, we have the limit*

$$\lim_{M\to\infty} \frac{V^M(0,0)}{V^{\frac{1}{M}}(0)} = \frac{2}{\sqrt{\pi}}.$$

Proof. According to Theorem 5.3.5 and Proposition 5.5.1, (5.6) and (5.72) are solutions to (5.1) with $N = 4$ and $N = 3$, respectively. As a result of [28, Proposition 6.1] and [94, Theorem 8], (5.67) and (5.71) are the solutions to (5.13) with $N = 4$ and $N = 3$, respectively. Plugging in $T = 1, t = 0, x = 0$ into these equations, we obtain that for $N = 4$, $u^1(0,0) = \frac{1}{2}\sqrt{\frac{\pi}{2}}$, $u(0) = \frac{\pi}{4\sqrt{2}}$, and for $N = 3$, $u^1(0,0) = \frac{4}{3\sqrt{2\pi}}$, $u(0) = \frac{4}{6\sqrt{2}}$. Due to the equalities

$$\lim_{M\to\infty} \frac{1}{\sqrt{M}} V^M(0,0) = u^1(0,0), \quad \lim_{M\to\infty} \frac{1}{\sqrt{M}} V^{\frac{1}{M}}(0) = u(0),$$

we conclude that for both $N = 3$ and $N = 4$,

$$\lim_{M\to\infty} \frac{V^M(0,0)}{V^{\frac{1}{M}}(0)} = \frac{u^1(0,0)}{u(0)} = \frac{2}{\sqrt{\pi}}.$$

□

5.3.3.1 From "optimality of the comb strategy conjecture" to "Finite vs Geometric regret conjecture"

For any $T > 0$ and $(t, x) \in [0, T] \times \mathbb{R}^N$ consider a given weak solution of the equation

$$X_u^{t,x} = x + \int_t^u \sigma_{\mathcal{C}}(X_s^{t,x}) dW_s, \quad \text{for } u \in [t, T]. \tag{5.14}$$

Proposition 5.3.12. *Let $N \geq 2$ and assume that the comb strategies are optimal in the sense that the weak solution of* (5.14) *is an optimizer of* (5.5) *and u^T is*

$C^0([0,T]\times\mathbb{R}^N)\cap C^{1,2}([0,T)\times\mathbb{R}^N)$ *and satisfies for some* $\varepsilon>0$ *and for all* $x\in\mathbb{R}^N$

$$\int_0^\infty e^{-T}\sup_{|x-y|\le\varepsilon}|\partial^2_{xx}u^T(0,y)|dT<\infty. \tag{5.15}$$

Then, the comb strategy is optimal for the problem (5.13) *and the function* u *defined at* (5.13) *satisfies*

$$u(x)=\mathbb{E}\left[\int_0^\infty e^{-T}\Phi(X_T^{0,x})dT\right]=\int_0^\infty e^{-T}u^T(0,x)dT. \tag{5.16}$$

Remark 5.3.13. Given the results in Proposition 5.3.12, a simple change of variable formula allows us to claim that the function

$$u^\lambda(x)=\lambda^{-3/2}u(\sqrt{\lambda}x)$$

solves the equation

$$\lambda u^\lambda(x)-\frac{1}{2}\sup_{J\in P(N)}e_J^\top\partial^2u^\lambda(x)e_J=\Phi(x)$$

and satisfies

$$u^\lambda(x)=\mathbb{E}\left[\int_0^\infty e^{-\lambda T}\Phi(X_T^{0,x})dT\right]=\int_0^\infty e^{-\lambda T}u^T(0,x)dT$$

Therefore, a corollary of (5.16) is the following relationship due to the Inverse Laplace transform from $u^\lambda(x)$ to $u^T(0,x)$,

$$u^1(0,0)=\frac{u(0)}{2\pi i}\int_{1-i\infty}^{1+i\infty}e^\lambda\lambda^{-3/2}d\lambda=-\frac{\Gamma\left(-\frac{1}{2}\right)}{\pi}u(0)=\frac{2}{\sqrt{\pi}}u(0), \tag{5.17}$$

where Γ is the gamma function. Thus, under the assumption of the optimality of the comb strategies for the finite time problem and some technical assumption the Proposition 5.3.12 yields the constant in the "Finite versus Geometric" conjecture of [106] for all N; see also [1]. According to (5.10), we have

$$|\partial^2_{x_i x_j} u^T(0,x)| \leq C \left(\int_{-\infty}^{\infty} e^{-Tr^2} dr + \sum_{l \in \mathbb{Z}} \frac{e^{\frac{-T(\pi l)^2}{4(\theta \cdot x^o)^2}}}{\theta \cdot x^o} \right),$$

where C is a positive constant. Multiplying both sides by e^{-T} and integrating from 0 to ∞, we can easily check (5.15) for our expression (5.6). As a result, Proposition 5.3.12 in fact implies Theorems 3.1 and 3.2 of [28].

5.4 Proofs

5.4.1 Proof of Theorem 5.3.4

5.4.1.1 Continuity of $x \mapsto u^T(t,x)$

Proof. Using (5.6) and the continuity of $x \mapsto x^o$, it suffices to show that

$$\Lambda(r,x) = \left(\psi \left(r\theta \cdot x + \frac{\pi}{2} \right) \sum_{k=1}^{4} \cos\left(r\alpha_k \cdot x \right) - 4 - \psi\left(r\theta \cdot x \right) \sum_{k=1}^{4} \sin\left(r\alpha_k \cdot x \right) \right).$$

is continuous with respect to x. Due to the formula $\sin(x)+\sin(y) = 2\sin(\frac{x+y}{2})\cos(\frac{x-y}{2})$, and the fact $\sum\limits_{k=1}^{4} \alpha_k \cdot x = 0$, we obtain

$$\begin{aligned}\sum_{k=1}^{4} \sin(r\alpha_k \cdot x) = 2\sin\left(\frac{r(\alpha_1+\alpha_2)\cdot x}{2}\right)\cos\left(\frac{r(\alpha_1-\alpha_2)\cdot x}{2}\right) \\ + 2\sin\left(\frac{r(\alpha_3+\alpha_4)\cdot x}{2}\right)\cos\left(\frac{r(\alpha_3-\alpha_4)\cdot x}{2}\right) \\ =2\sin\left(\frac{r(\alpha_1+\alpha_2)\cdot x}{2}\right)\left(\cos\left(\frac{r(\alpha_1-\alpha_2)\cdot x}{2}\right) - \cos\left(\frac{r(\alpha_3-\alpha_4)\cdot x}{2}\right)\right) \\ = -2\sin(r\theta\cdot x)\left(\cos\left(\frac{r(\alpha_1-\alpha_2)\cdot x}{2}\right) - \cos\left(\frac{r(\alpha_3-\alpha_4)\cdot x}{2}\right)\right).\end{aligned}$$

The square wave function $\psi(r\theta\cdot x)$ changes its sign at $r\theta\cdot x = k\pi, k \in \mathbb{Z}$, when $\sin(r\theta\cdot x)$ is equal to zero. Therefore the function $x \mapsto \psi(r\theta\cdot x)\sin(r\theta\cdot x)$ is continuous, and so is the term $\psi(r\theta\cdot x)\sum\limits_{k=1}^{4}\sin(r\alpha_k\cdot x)$.

Similarly, using the formula $\cos(x)+\cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$, we obtain

$$\sum_{k=1}^{4} \cos(r\alpha_k \cdot x) = 2\cos(r\theta\cdot x)\left(\cos\left(\frac{r(\alpha_1-\alpha_2)\cdot x}{2}\right) - \cos\left(\frac{r(\alpha_3-\alpha_4)\cdot x}{2}\right)\right).$$

Then the continuity of $x \mapsto \psi\left(r\theta\cdot x + \frac{\pi}{2}\right)\sum_{k=1}^{4}\cos\left(r\alpha_k\cdot x\right)$ follows from the continuity of $x \mapsto \psi\left(r\theta\cdot x + \frac{\pi}{2}\right)\cos(r\theta\cdot x)$, and we finish the proof. □

5.4.1.2 Terminal condition

Proof. Due to the continuity of $x \mapsto u^T(t,x)$ and the symmetry of u^T, we only need to show the equality $u^T(T,x) = \Phi(x)$ for the case $x^{(1)} < x^{(2)} < x^{(3)} < x^{(4)}$. Recall the definition of sine integral function $si(x)$ and cosine integral function $Ci(x)$ (see e.g. [4]),

$$si(x) = -\int_x^\infty \frac{\sin(t)}{t}dt, \quad Ci(x) = -\int_x^\infty \frac{\cos(t)}{t}dt,$$

and denote

$$T_0 = -\frac{\pi}{2\theta \cdot x^o}, \quad A_k = \alpha_k \cdot x^o, \quad R_k = |A_k T_0|, \quad k = 1, \dots, 4.$$

Under the assumption $x^{(1)} < x^{(2)} < x^{(3)} < x^{(4)}$, it is easy to check the following inequalities

$$-\frac{3\pi}{2} < A_1 T_0 < -\frac{\pi}{2} < A_2 T_0 < A_3 T_0 < \frac{\pi}{2} < A_4 T_0 < \frac{3\pi}{2}. \tag{5.18}$$

According to (5.6), we have

$$2\sqrt{2}\sum_{k=1}^{4} x_k - 8\sqrt{2}u^T(T, x) = \int\limits_0^\infty \frac{\Lambda(r, x^o)}{r^2} dr.$$

Note that

$$\psi\left(r\theta \cdot x^o + \frac{\pi}{2}\right) = \begin{cases} -1, & r \in [(4n+1)T_0, (4n+3)T_0] \\ +1, & r \in [(4n-1)T_0, (4n+1)T_0], \end{cases}$$

$$\psi\left(r\theta \cdot x^o\right) = \begin{cases} -1, & r \in [4nT_0, (4n+2)T_0] \\ +1, & r \in [(4n+2)T_0, (4n+4)T_0]. \end{cases}$$

We can rewrite the integral as infinite sum of integrals

$$\begin{aligned}
\int_0^\infty \frac{\Lambda(r,x^o)}{r^2}dr =& \int_0^\infty \left(\psi\left(r\theta\cdot x^o+\frac{\pi}{2}\right)\sum_{k=1}^4 \frac{\cos(A_k r)}{r^2}-\frac{4}{r^2}\right)dr \\
&-\int_0^\infty \psi\left(r\theta\cdot x^o\right)\sum_{k=1}^4 \frac{\sin(A_k 4)}{r^2}dr \\
=&\sum_{n=0}^\infty (-1)^n \int_{2nT_0}^{(2n+2)T_0} \sum_{k=1}^4 \frac{\sin(A_k r)}{r^2}dr + \int_0^{T_0}\sum_{k=1}^4 \frac{\cos(A_k r)-1}{r^2}dr \\
&+\sum_{n=1}^\infty (-1)^n \int_{(2n-1)T_0}^{(2n+1)T_0} \sum_{k=1}^4 \frac{\cos(A_k r)}{r^2}dr - \int_{T_0}^\infty \frac{4}{r^2}dr. \qquad (5.19)
\end{aligned}$$

Our aim is to prove $\int_0^\infty \frac{\Lambda(r,x^o)}{r^2}dr = -4A_4$, which is equivalent to $u^T(T,x)=\Phi(x)$.

It is easy to check the following indefinite integral formulas,

$$\begin{aligned}
\int \frac{\sin(x)}{x^2}dx &= Ci(x)-\frac{\sin(x)}{x}+\text{Constant}, \\
\int \frac{\cos(x)}{x^2}dx &= -si(x)-\frac{\cos(x)}{x}+\text{Constant}.
\end{aligned}$$

Let us compute the integral

$$\begin{aligned}
\int_0^{2T_0} \frac{1}{r^2}\sum_{k=1}^4 \sin(A_k r)dr =& \sum_{k=1}^4 A_k\left(Ci(2|A_kT_0|)-\frac{\sin(2|A_kT_0|)}{2|A_kT_0|}\right) \\
&-\lim_{\epsilon\to 0}\sum_{k=1}^4 A_k\left(Ci(|A_k\epsilon|)-\frac{\sin(|A_k\epsilon|)}{|A_k\epsilon|}\right).
\end{aligned}$$

According to $\sum\limits_{k=1}^4 A_k = 0$ and $\lim\limits_{x\to 0}\frac{\sin(x)}{x}=1$, the term $\sum\limits_{k=1}^4 A_k\frac{\sin(|A_k\epsilon|)}{|A_k\epsilon|}$ vanishes. Since the expansion of $Ci(x)$ near $x=0$ is $\ln(x)+\gamma$, where γ is the Euler-Mascheroni constant (see e.g. [79]), we obtain that

$$\lim_{\epsilon\to 0}\sum_{k=1}^{4} A_k Ci(|A_k\epsilon|) = \lim_{\epsilon\to 0}\sum_{k=1}^{4} A_k(\ln(|A_k|) + \ln(\epsilon) + \gamma) = \sum_{k=1}^{4} A_k \ln(|A_k|).$$

Accordingly, we have

$$\int_0^{2T_0} \frac{1}{r^2}\sum_{k=1}^{4}\sin(A_k r)dr = \sum_{k=1}^{4} A_k \left(Ci(2|A_k T_0|) - \frac{\sin(2|A_k T_0|)}{2|A_k T_0|}) - \sum_{k=1}^{4} A_k \ln(|A_k| \right),$$

and similarly for each $n \in \mathbb{N}$,

$$\begin{aligned}\int_{2nT_0}^{(2n+2)T_0} \frac{1}{r^2}\sum_{k=1}^{4}\sin(A_k r)dr =& \sum_{k=1}^{4} A_k (Ci((2n+2)|A_k T_0|) - \frac{\sin((2n+2)|A_k T_0|)}{(2n+2)|A_k T_0|}) \\ &- \sum_{k=1}^{4} A_k (Ci(2n|A_k T_0|) - \frac{\sin(2n|A_k T_0|)}{2n|A_k T_0|}).\end{aligned}$$

Therefore, we get the equation

$$\begin{aligned}&\int_0^{+\infty} -\psi\left(r\theta\cdot x^o + \frac{\pi}{2}\right)\sum_{k=1}^{4}\frac{\sin(A_k r)}{r^2}dr \\ &= -\sum_{k=1}^{4} A_k \ln(|A_k|) + 2\sum_{k=1}^{4}\sum_{n=1}^{\infty} A_k(-1)^{n+1} Ci(2n|A_k T_0|) \\ &\quad - 2\sum_{k=1}^{4}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(2nA_k T_0)}{2nT_0}.\end{aligned} \tag{5.20}$$

Now we deal with the cosine term in (5.19). Due to the equality $si(0) = -\frac{\pi}{2}$, it can be seen that

$$\begin{aligned}
\int\limits_0^{T_0} \frac{1}{r^2}(\sum_{k=1}^4 \cos(A_k r) - 4)dr &= -\sum_{k=1}^4 |A_k| si(|A_k T_0|) - \sum_{k=1}^4 \frac{\cos(|A_k T_0|)}{T_0} + \frac{4}{T_0} \\
&\quad + \lim_{\epsilon \to 0^+} \left(|A_k| \sum_{k=1}^4 si(|A_k \epsilon|) + \sum_{k=1}^4 \frac{\cos(|A_k \epsilon|) - 1}{\epsilon} \right) \\
&= -\frac{\pi}{2} \sum_{k=1}^4 |A_k| - \sum_{k=1}^4 |A_k| si(|A_k T_0|) - \sum_{k=1}^4 \frac{\cos(|A_k T_0|)}{T_0} + \frac{4}{T_0},
\end{aligned}$$

and similarly

$$\begin{aligned}
\int\limits_{(2n+1)T_0}^{(2n+3)T_0} \frac{1}{r^2} \sum_{k=1}^4 \cos(A_k r) dr &= -\sum_{k=1}^4 |A_k| si((2n+3)|A_k T_0|) - \sum_{k=1}^4 \frac{\cos((2n+3)|A_k T_0|)}{(2n+3)T_0} \\
&\quad + \sum_{k=1}^4 |A_k| si((2n+1)|A_k T_0|) - \sum_{k=1}^4 \frac{\cos((2n+1)|A_k T_0|)}{(2n+1)T_0}.
\end{aligned}$$

Then, in conjunction with the equality $\int_{T_0}^{+\infty} \frac{-4}{r^2} dr = -\frac{4}{T_0}$, we obtain that

$$\begin{aligned}
&\int\limits_0^{+\infty} \left(\psi\left(r\theta \cdot x^o + \frac{\pi}{2} \right) \sum_{k=1}^4 \frac{\cos(A_k r)}{r^2} - \frac{4}{r^2} \right) dr \\
&\quad = -\frac{\pi}{2} \sum_{k=1}^4 |A_k| - 2 \sum_{k=1}^4 \sum_{n=1}^\infty (-1)^{n+1} |A_k| si(|(2n-1) A_k T_0|) \\
&\qquad - 2 \sum_{k=1}^4 \sum_{n=1}^\infty (-1)^{n+1} \frac{\cos((2n-1) A_k T_0)}{(2n-1) T_0}. \qquad (5.21)
\end{aligned}$$

Using the inverse Fourier transform, we have

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos((2n-1)A_kT_0)}{(2n-1)T_0} = \frac{\pi}{4T_0}sign\left(\tan\left(\frac{\pi}{4}+\frac{A_kT_0}{2}\right)\right),$$
$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(2nA_kT_0)}{2nT_0} = \frac{i\left(\log\left(1+e^{-i2A_kT_0}\right)-\log\left(1+e^{i2A_kT_0}\right)\right)}{4T_0}.$$

Recalling the inequalities (5.18), for $k = 1, 4$, we have $|A_kT_0| \in (\frac{\pi}{2}, \frac{3\pi}{2})$, and hence the term $sign\left(\tan\left(\frac{\pi}{4}+\frac{A_kT_0}{2}\right)\right) = -1$. For $k = 2, 3$, since $|A_kT_0| < \frac{\pi}{2}$, we get $sign\left(\tan\left(\frac{\pi}{4}+\frac{A_kT_0}{2}\right)\right) = 1$, and therefore

$$\sum_{k=1}^{4}\frac{\pi}{2T_0}sign\left(\tan\left(\frac{\pi}{4}+\frac{A_kT_0}{2}\right)\right) = 0. \tag{5.22}$$

It can be seen that the function

$$x \mapsto i\left(\log(1+e^{-ix})-\log(1+e^{ix})\right) \equiv i\ \log\left(\frac{1+e^{-ix}}{1+e^{ix}}\right) \equiv x \mod 2\pi$$

is 2π-periodic, and equals to x when restricted to $(-\pi, \pi)$. So that we obtain

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(2nA_kT_0)}{2nT_0} = \begin{cases} \frac{2A_kT_0+2\pi}{4T_0}, & \text{if } \ k=1, \\ \frac{2A_kT_0}{4T_0}, & \text{if } \ k=2,3, \\ \frac{2A_kT_0-2\pi}{4T_0}, & \text{if } \ k=4, \end{cases}$$

and hence

$$\sum_{k=1}^{4}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(2nA_kT_0)}{2nT_0} = 0. \tag{5.23}$$

Combining (5.19), (5.20), (5.21), (5.22) and (5.23), we simplify the expression,

$$\begin{aligned}
&\int_0^\infty \frac{\Lambda(r,x^o)dr}{r^2} \\
&= -\sum_{k=1}^4 A_k \ln(|A_k|) - \frac{\pi}{2}\sum_{k=1}^4 |A_k| + 2\sum_{k=1}^4\sum_{n=1}^\infty (-1)^{n+1}|A_k| si(|(2n-1)A_kT_0|) \\
&\quad - 2\sum_{k=1}^4\sum_{n=1}^\infty (-1)^{n+1}\frac{\cos((2n-1)A_kT_0)}{(2n-1)T_0} + 2\sum_{k=1}^4\sum_{n=1}^\infty A_k(-1)^{n+1}Ci(2n|A_kT_0|) \\
&\quad - 2\sum_{k=1}^4\sum_{n=1}^\infty (-1)^{n+1}\frac{\sin(2nA_kT_0)}{2nT_0} \\
&= -\frac{\pi}{2}\sum_{k=1}^4 |A_k| + 2\sum_{k=1}^4\sum_{n=1}^\infty (-1)^{n+1}|A_k| si(|(2n-1)A_kT_0|) \\
&\quad - \sum_{k=1}^4 A_k \ln(|A_k|) + 2\sum_{k=1}^4\sum_{n=1}^\infty A_k(-1)^{n+1}Ci(2n|A_kT_0|).
\end{aligned} \tag{5.24}$$

It remains to calculate the infinite sum including $Ci(x)$ and $si(x)$. Note that

$$-2\sum_{n=1}^\infty (-1)^{n+1} si((2n-1)R_k) = 2\sum_{n=1}^\infty \int_{(4n-3)R_k}^{(4n-1)R_k} \frac{\sin(r)}{r}dr.$$

and $\frac{\sin(z)}{z} = Im\frac{e^{iz}}{z}$for $z \in \mathbb{R}$. We apply contour integral to $\frac{e^{iz}}{z}$. Denoting the curves in the counterclockwise direction by

$$\gamma_n^k := \{nR_ke^{i\theta} : \theta \in [0,\pi]\},$$

we have equalities

$$2\int_{(4n-3)R_k}^{(4n-1)R_k} - \int_{\gamma_{4n-3}^k} + \int_{\gamma_{4n-1}^k} \frac{e^{iz}}{z}dz = 0, n \in \mathbb{N}.$$

Therefore, we obtain

$$
\begin{aligned}
-2\sum_{n=1}^{\infty}(-1)^{n+1}si((2n-1)R_k) &= \sum_{n=1}^{\infty}(-1)^{n+1}Im\int\limits_{\gamma_{2n-1}^k}\frac{e^{iz}}{z}dz \\
&= \sum_{n=1}^{\infty}(-1)^{n+1}Re\int\limits_{0}^{\pi}e^{i(2n-1)R_ke^{i\theta}}d\theta. \qquad (*)
\end{aligned}
$$

According to the inequalities (5.18), we have $2R_k \notin \{-3\pi, -\pi, \pi, 3\pi\}$, and hence can exchange the infinite sum and the integral and compute the geometric series to obtain

$$
-2\sum_{n=1}^{\infty}(-1)^{n+1}si((2n-1)R_k) = Re\int\limits_{0}^{\pi}\frac{e^{R_ke^{i\theta}}}{1+e^{2R_ke^{i\theta}}}d\theta. \tag{5.25}
$$

Similarly, we calculate

$$
2\sum_{n=1}^{\infty}(-1)^{n+1}Ci(2nR_k) = -2Re\left(\sum_{n=1}^{\infty}\int\limits_{(4n-2)R_k}^{4nR_k}\frac{e^{iz}}{z}dz\right).
$$

Denoting the quarter of circles in the counterclockwise derivation by

$$
\tilde{\gamma}_n^k := \{nR_ke^{i\theta} : \theta \in [0, \pi/2]\},
$$

we obtain that

$$
\begin{aligned}
0 &= \int\limits_{(4n-2)R_k}^{4nR_k} + \int\limits_{\tilde{\gamma}_{4n}^k} - \int\limits_{i(4n-2)R_k}^{i4nR_k} - \int\limits_{\tilde{\gamma}_{4n-2}^k}\frac{e^{iz}}{z}dz \\
&= \int\limits_{(4n-2)R_k}^{4nR_k}\frac{e^{iz}}{z}dz + i\int\limits_{\frac{\pi}{2}}^{0}e^{(4n-2)R_ke^{i\theta}}d\theta + i\int\limits_{0}^{\frac{\pi}{2}}e^{4nR_ke^{i\theta}}d\theta + \int\limits_{4nR_k}^{(4n-2)R_k}\frac{e^{-r}}{r}dr.
\end{aligned}
$$

Recalling the definition of integral exponential function for $x > 0$,

$$E_1(x) = \int_{x}^{+\infty} \frac{e^{-r}}{r} dr = \int_{0}^{+\infty} \exp(-xe^t) dt,$$

it can be seen that

$$\begin{aligned} -2Re \int_{(4n-2)R_k}^{4nR_k} \frac{e^{iz}}{z} dz =& 2Re \left(-i \left(\int_{0}^{\frac{\pi}{2}} e^{(4n-2)R_k e^{i\theta}} d\theta - \int_{0}^{\frac{\pi}{2}} e^{4nR_k e^{i\theta}} d\theta \right) \right) \\ &+ 2(E_1(4nR_k) - E_1((4n-2)R_k)) \\ =& 2Im \left(\int_{0}^{\frac{\pi}{2}} e^{(4n-2)R_k e^{i\theta}} d\theta - \int_{0}^{\frac{\pi}{2}} e^{4nR_k e^{i\theta}} d\theta \right) \\ &+ 2(E_1(4nR_k) - E_1((4n-2)R_k)). \end{aligned}$$

By direct computation, we have

$$\begin{aligned} 2\sum_{n=1}^{\infty}(-1)^{n+1} Ci(2nR_k) &= 2Im \left(\int_{0}^{\frac{\pi}{2}} \frac{e^{i2R_k e^{i\theta}}}{1 + e^{i2R_k e^{i\theta}}} d\theta \right) \\ &\quad + 2 \left(\sum_{n=1}^{\infty} (E_1(4nR_k) - E_1((4n-2)R_k)) \right) \\ &= 2Im \left(\int_{0}^{\frac{\pi}{2}} \frac{e^{i2R_k e^{i\theta}}}{1 + e^{i2R_k e^{i\theta}}} d\theta \right) - 2\int_{0}^{\infty} \frac{e^{-2R_k e^r}}{1 + e^{-2R_k e^r}} dr \\ &= 2Im \left(\int_{0}^{\frac{\pi}{2}} \frac{e^{i2R_k e^{i\theta}}}{1 + e^{i2R_k e^{i\theta}}} d\theta \right) - 2\int_{R_k}^{\infty} \frac{e^{-2t}}{t(1 + e^{-2t})} dt, \qquad (**) \end{aligned}$$

where the last equation follows from the change of variable $t = R_k e^r$.

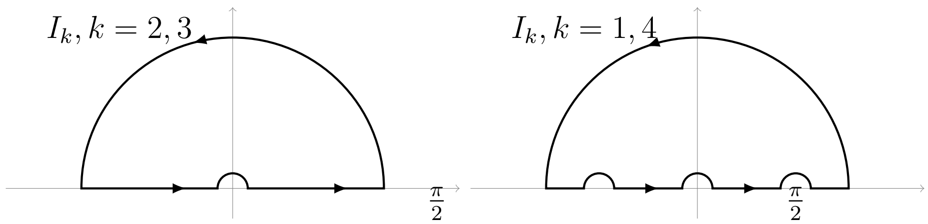


Figure 5.1: Contour of I_k

Then, we can further simplify the expression (5.24) using $(*)$ and $(**)$,

$$\begin{aligned}\int_0^\infty \frac{\Lambda(r, x^o)}{r^2}dr =& -\sum_{k=1}^4 A_k \ln(|A_k|) - \frac{\pi}{2}\sum_{k=1}^4 |A_k| + \sum_{k=1}^4 |A_k| Re \int_0^\pi \frac{e^{iR_k e^{i\theta}}}{1+e^{i2R_k e^{i\theta}}}d\theta \\ &+ \sum_{k=1}^4 2A_k Im \int_0^{\frac{\pi}{2}} \frac{e^{i2R_k e^{i\theta}}}{1+e^{i2R_k e^{i\theta}}}d\theta - 2A_k \int_{R_k}^\infty \frac{e^{-2t}}{t(1+e^{-2t})}dt. \end{aligned} \tag{5.26}$$

Let us denote

$$I_k = \int_0^\pi \frac{e^{iR_k e^{i\theta}}}{1+e^{i2R_k e^{i\theta}}}d\theta = -i\int_{\gamma_1^k} \frac{e^{iz}}{z(1+e^{i2z})}dz,$$

$$J_k = \int_0^{\frac{\pi}{2}} \frac{e^{i2R_k e^{i\theta}}}{1+e^{i2R_k e^{i\theta}}}d\theta = -i\int_{\tilde{\gamma}_1^k} \frac{e^{i2z}}{z(1+e^{i2z})}dz.$$

For $k = 2, 3$, we have $R_k < \frac{\pi}{2}$, and therefore 0 is the only pole of complex function $z \mapsto \frac{e^{iz}}{z(1+e^{i2z})}$ over the interval $[-R_k, R_k]$. According to the contour integral (see Figure 5.1), we have that

$$\begin{aligned} 0 =& \int_{\gamma_1^k} + \int_{-R_k}^{-\epsilon} - \int_{\{\epsilon e^{i\theta}:\theta\in[0,\pi]\}} + \int_\epsilon^{R_k} \frac{-ie^{iz}}{z(1+e^{i2z})}dz \\ =& I_k - i\int_{-R_k}^{-\epsilon} \frac{e^{it}}{t(1+e^{i2t})}dt - \int_0^\pi \frac{e^{i\epsilon e^{i\theta}}}{1+e^{i2\epsilon e^{i\theta}}}d\theta - i\int_\epsilon^{R_k} \frac{e^{it}}{t(1+e^{i2t})}dt. \end{aligned}$$

Since $\frac{e^{it}}{t(1+e^{i2t})} = \frac{e^{it}+e^{-it}}{t|1+e^{i2t}|^2}$ is real, and $\lim_{\epsilon\to 0} \frac{e^{i\epsilon e^{i\theta}}}{1+e^{i2\epsilon e^{i\theta}}} = \frac{1}{2}$, we obtain that $ReI_k = \frac{\pi}{2}$. For $k = 1, 4$, since $R_k \in (\frac{\pi}{2}, \frac{3\pi}{2})$, we have $-\frac{\pi}{2}, 0, \frac{\pi}{2}$ are three poles of the complex function

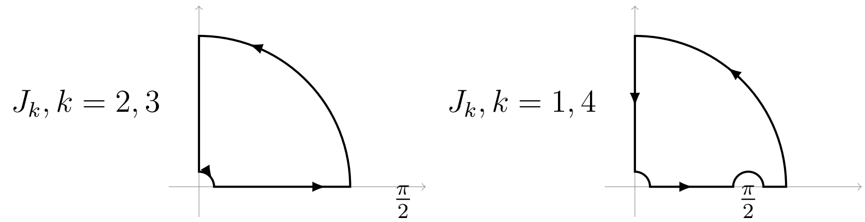


Figure 5.2: Contour of J_k

$z \mapsto \frac{e^{iz}}{z(1+e^{i2z})}$ over the interval $[-R_k, R_k]$. Again by contour integral (see Figure 5.1) the real part of I_k is equal to the integral around the three poles,

$$
\begin{aligned}
ReI_k &= \lim_{\epsilon\to 0} Re \int_0^{\pi} \frac{ie^{i\epsilon e^{i\theta}}\epsilon e^{i\theta}}{1-e^{i2\epsilon e^{i\theta}}}\left(\frac{1}{\frac{\pi}{2}+\epsilon e^{i\theta}} + \frac{1}{\frac{\pi}{2}-\epsilon e^{i\theta}}\right)d\theta + \frac{\pi}{2} \\
&= \int_0^{\pi} \lim_{\epsilon\to 0} Re \frac{ie^{i\epsilon e^{i\theta}}\epsilon e^{i\theta}}{1-e^{i2\epsilon e^{i\theta}}}\left(\frac{1}{\frac{\pi}{2}+\epsilon e^{i\theta}} + \frac{1}{\frac{\pi}{2}-\epsilon e^{i\theta}}\right)d\theta + \frac{\pi}{2} \\
&= \int_0^{\pi} -\frac{2}{\pi}d\theta + \frac{\pi}{2} = \frac{\pi}{2} - 2,
\end{aligned}
$$

and hence

$$
\sum_{k=1}^{4} |A_k| ReI_k = \frac{\pi}{2}\sum_{k=1}^{4}|A_k| - 2|A_1| - 2|A_4|. \tag{5.27}
$$

For $k = 2,3$, we apply contour integral to J_k (see Figure 5.2),

$$
\begin{aligned}
0 &= \int_{\tilde{\gamma}_1^k} + \int_{iR_k}^{i\epsilon} - \int_{\{\epsilon e^{i\theta}:\theta\in[0,\pi/2]\}} + \int_{\epsilon}^{R_k} \frac{-ie^{i2z}}{z(1+e^{i2z})}dz \\
&= J_k - i\int_{R_k}^{\epsilon} \frac{e^{-2t}}{t(1+e^{-2t})}dt + \int_{\frac{\pi}{2}}^{0} \frac{e^{i2\epsilon e^{i\theta}}}{1+e^{i2\epsilon e^{i\theta}}}d\theta - i\int_{\epsilon}^{R_k}\frac{e^{i2t}}{t(1+e^{i2t})}dt.
\end{aligned}
$$

Noting that imaginary part of $i\frac{e^{i2t}}{t(1+e^{i2t})}$ is just $\frac{1}{2t}$, we obtain that

$$ImJ_k = \int_{R_k}^{\epsilon} \frac{e^{-2t}}{t(1+e^{-2t})}dt + Im\int_0^{\frac{\pi}{2}} \frac{e^{i2\epsilon e^{i\theta}}}{1+e^{i2\epsilon e^{i\theta}}}d\theta + \int_{\epsilon}^{R_k} \frac{1}{2t}dt.$$

For $k = 1, 4$, $z = \frac{\pi}{2}$ is the other pole over the interval $[0, R_k]$ (see Figure 5.2), and we have

$$\begin{aligned} 0 =& J_k - i\int_{R_k}^{\epsilon} \frac{e^{-2t}}{t(1+e^{-2t})}dt + \int_{\frac{\pi}{2}}^{0} \frac{e^{i2\epsilon e^{i\theta}}}{1+e^{i2\epsilon e^{i\theta}}}d\theta - i\int_{\epsilon}^{\frac{\pi}{2}-\epsilon} \frac{e^{i2t}}{t(1+e^{i2t})}dt \\ &- i\int_{\frac{\pi}{2}+\epsilon}^{R_k} \frac{e^{i2t}}{t(1+e^{i2t})}dt + \int_0^{\pi} \frac{e^{i2\epsilon e^{i\theta}}\epsilon e^{i\theta}}{(\frac{\pi}{2}+\epsilon e^{i\theta})(1-e^{i2\epsilon e^{i\theta}})}d\theta, \end{aligned}$$

and therefore,

$$\begin{aligned} ImJ_k =& \int_{R_k}^{\epsilon} \frac{e^{-2t}}{t(1+e^{-2t})}dt + Im\int_0^{\frac{\pi}{2}} \frac{e^{i2\epsilon e^{i\theta}}}{1+e^{i2\epsilon e^{i\theta}}}d\theta + \int_{\epsilon}^{R_k} \frac{1}{2t}dt \\ &- \int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}+\epsilon} \frac{1}{2t}dt - Im\int_0^{\pi} \frac{e^{i2\epsilon e^{i\theta}}\epsilon e^{i\theta}}{(\frac{\pi}{2}+\epsilon e^{i\theta})(1-e^{i2\epsilon e^{i\theta}})}d\theta. \end{aligned}$$

Take

$$P_{\epsilon}^1 = Im\int_0^{\frac{\pi}{2}} \frac{e^{i2\epsilon e^{i\theta}}}{1+e^{i2\epsilon e^{i\theta}}}d\theta, \qquad P_{\epsilon}^2 = Im\int_0^{\pi} \frac{e^{i2\epsilon e^{i\theta}}\epsilon e^{i\theta}}{(\frac{\pi}{2}+\epsilon e^{i\theta})(1-e^{i2\epsilon e^{i\theta}})}d\theta.$$

Then we compute

$$
\begin{aligned}
&2ImJ_k - 2\int\limits_{R_k}^{\infty} \frac{e^{-2t}}{t(1+e^{-2t})}dt - \ln(|A_k|) \\
&= -2\int\limits_{\epsilon}^{\infty} \frac{e^{-2t}}{t(1+e^{-2t})}dt + (\ln(R_k) - \ln(|A_k|) - \ln(\epsilon)) \\
&+ 2P^1_\epsilon - \mathbb{1}_{\{k=1,4\}}\left(\ln\left(\frac{\pi/2+\epsilon}{\pi/2-\epsilon}\right) + 2P^2_\epsilon\right).
\end{aligned}
$$

As a result of $\sum\limits_{k=1}^{4} A_k = 0$, $\ln(R_k) - \ln(|A_k|) = \ln(T_0)$ and $\lim\limits_{\epsilon\to 0} P^2_\epsilon = 1$, it can be seen that

$$
\begin{aligned}
\sum_{k=1}^{4} A_k\Bigg(2ImJ_k - \ln(|A_k|) - 2\int\limits_{R_k}^{\infty} \frac{e^{-2t}}{t(1+e^{-2t})}dt\Bigg) \\
&= -2\sum_{k=1}^{4} A_k \int\limits_{\epsilon}^{\infty} \frac{e^{-2t}}{t(1+e^{-2t})}dt + \sum_{k=1}^{4} A_k(\ln(T_0) - \ln(\epsilon)) \\
&\quad + \sum_{k=1}^{4} A_k P^1_\epsilon - (A_1 + A_4)\left(\ln\left(\frac{\pi/2+\epsilon}{\pi/2-\epsilon}\right) + 2P^2_\epsilon\right) \\
&= -\lim_{\epsilon\to 0}(A_1 + A_4)\left(\ln\left(\frac{\pi/2+\epsilon}{\pi/2-\epsilon}\right) + 2P^2_\epsilon\right) \\
&= -2A_1 - 2A_4. \qquad (5.28)
\end{aligned}
$$

Combining (5.26), (5.27), (5.28), we obtain

$$
\int\limits_{0}^{\infty} \frac{\Lambda(r, x^o)}{r^2}dr = -2|A_1| - 2|A_4| - 2A_1 - 2A_4 = -4A_4, \qquad (5.29)
$$

which concludes the result.

□

5.4.1.3 Smoothness

Proof. Step 1: Equation (5.8). As a result of

$$\sum_{k=1}^{4}\sin(r\alpha_k\cdot x^o)=\sum_{k=1}^{4}\sin(r\alpha_k\cdot x),\sum_{k=1}^{4}\cos(r\alpha_k\cdot x^o)=\sum_{k=1}^{4}\cos(r\alpha_k\cdot x),$$

we obtain that

$$\begin{aligned}u^T(t,x)=&\frac{-1}{16\sqrt{2}}\int\limits_{-\infty}^{\infty}\frac{e^{-(T-t)r^2}}{r^2}\left(\psi\left(r\theta\cdot x^o+\frac{\pi}{2}\right)\sum_{k=1}^{4}\cos(r\alpha_k\cdot x)-4\right.\\&\left.-\psi\left(r\theta\cdot x^o\right)\sum_{k=1}^{4}\sin(r\alpha_k\cdot x)\right)dr+\frac{1}{4}\sum_{k=1}^{4}x_k+\frac{1}{2}\sqrt{\frac{(T-t)\pi}{2}}.\end{aligned}$$

To stress the dependence of T_0 on x, we denote it as $T_0(x):=-\frac{\pi}{2\theta\cdot x^o}$. Since $\theta\cdot x^o<0$, define two functional series $F_l(x),G_l(x),l\in\mathbb{Z}$ by

$$F_l(x):=\int\limits_{(2l-1)T_0(x)}^{(2l+1)T_0(x)}\frac{e^{-(T-t)r^2}}{r^2}\left(\psi\left(r\theta\cdot x^o+\frac{\pi}{2}\right)\sum_{k=1}^{4}\cos(r\alpha_k\cdot x)-4\right)dr,$$

$$G_l(x):=-\int\limits_{2lT_0(x)}^{(2l+2)T_0(x)}\frac{e^{-(T-t)r^2}}{r^2}\left(\psi\left(r\theta\cdot x^o\right)\sum_{k=1}^{4}\sin(r\alpha_k\cdot x)\right)dr.$$

Then we have

$$u^T(t,x)=-\frac{1}{16\sqrt{2}}\sum_{l\in\mathbb{Z}}(F_l(x)+G_l(x))+\frac{1}{4}\sum_{k=1}^{4}x_k+\frac{1}{2}\sqrt{\frac{(T-t)\pi}{2}}.$$

Noting that $\sum_{k=1}^{4} \cos(r\alpha_k \cdot x) = 0$ at endpoints $r = (2l-1)T_0(x), l \in \mathbb{Z}$, the partial derivative of $F_l(x)$ is given by

$$
\begin{aligned}
\partial_{x_i}^+ F_l(x) = &-\int_{(2l-1)T_0(x)}^{(2l+1)T_0(x)} \frac{e^{-(T-t)r^2}}{r} \sum_{k=1}^{4} \alpha_{k,i} \psi\left(r\theta \cdot x^o + \frac{\pi}{2}\right) \sin(r\alpha_k \cdot x) dr \\
&- 4(2l+1)\partial_{x_i}^+ T_0(x) \frac{e^{-(T-t)(2l+1)^2 T_0^2(x)}}{(2l+1)^2 T_0^2(x)} + 4(2l-1)\partial_{x_i}^+ T_0(x) \frac{e^{-(T-t)(2l-1)^2 T_0^2(x)}}{(2l-1)^2 T_0^2(x)},
\end{aligned}
$$

and similarly

$$
\partial_{x_i}^+ G_l(x) = -\int_{2lT_0(x)}^{(2l+2)T_0(x)} \frac{e^{-(T-t)r^2}}{r} \sum_{k=1}^{4} \alpha_{k,i} \psi\left(r\theta \cdot x^o\right) \cos(r\alpha_k \cdot x) dr.
$$

It is well-known that summation and differentiation are interchangeable if the partial sum of derivatives converges uniformly. Since $\sum_{l\in\mathbb{Z}} \partial_{x_i}^+ F_l(x)$ and $\sum_{l\in\mathbb{Z}} \partial_{x_i}^+ G_l(x)$ converge uniformly in any bounded region of x, we conclude that,

$$
\begin{aligned}
\partial_{x_i}^+ u^T(t,x) = &-\frac{1}{16\sqrt{2}} \sum_{l\in\mathbb{Z}} \big(\partial_{x_i} F_l(x) + \partial_{x_i} G_l(x)\big) + \frac{1}{4} \\
= &\frac{1}{16\sqrt{2}} \int_{-\infty}^{\infty} \frac{e^{-(T-t)r^2}}{r} \sum_{k=1}^{4} \alpha_{k,i} \left(\psi\left(r\theta \cdot x^o + \frac{\pi}{2}\right) \sin\left(r\alpha_k \cdot x\right)\right. \\
&\left. + \psi\left(r\theta \cdot x^o\right) \cos\left(r\alpha_k \cdot x\right)\right) dr + \frac{1}{4}.
\end{aligned}
$$

We can calculate $\partial_{x_i}^- u^T(t,x)$ in the exactly same way, and find that it has the same expression with $\partial_{x_i}^+ u^T(t,x)$. Therefore we proved the result (5.8).

Step 2: Equation (5.9). If $\theta \cdot x^o = 0$, then all the coordinates of x are equal, i.e., $x = (x_1, x_1, x_1, x_1)$. Let us compute the derivative of $u^T(t,x)$ by definition. Take

$\epsilon > 0$ and denote $(x_1 + \epsilon, x_1, x_1, x_1)$ simply by $x + \epsilon$. Then we have

$$
\begin{aligned}
u^T(t, x+\epsilon) = & \frac{-1}{8\sqrt{2}} \int_0^\infty \frac{e^{-(T-t)r^2}}{r^2} \left(\psi\left(\frac{\epsilon r}{\sqrt{2}} + \frac{\pi}{2}\right) \left(\cos\left(\frac{3\epsilon r}{\sqrt{2}}\right) + 3\cos\left(\frac{3\epsilon r}{\sqrt{2}}\right)\right) - 4 \right. \\
& \left. - \psi\left(\frac{\epsilon r}{\sqrt{2}}\right) \left(\sin\left(\frac{3\epsilon r}{\sqrt{2}}\right) - 3\sin\left(\frac{\epsilon r}{\sqrt{2}}\right)\right) \right) dr \\
& + x_1 + \frac{1}{4}\epsilon + \frac{1}{2}\sqrt{\frac{(T-t)\pi}{2}}.
\end{aligned}
$$

In order to conclude our result, it remains to show that

$$
\begin{aligned}
0 = & \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^\infty \frac{e^{-(T-t)r^2}}{r^2} \left(\psi\left(\frac{\epsilon r}{\sqrt{2}} + \frac{\pi}{2}\right) \left(\cos\left(\frac{3\epsilon r}{\sqrt{2}}\right) + 3\cos\left(\frac{\epsilon r}{\sqrt{2}}\right)\right) - 4 \right. \\
& \left. - \psi\left(\frac{\epsilon r}{\sqrt{2}}\right) \left(\sin\left(\frac{3\epsilon r}{\sqrt{2}}\right) - 3\sin\left(\frac{\epsilon r}{\sqrt{2}}\right)\right) \right) dr.
\end{aligned}
$$

According to the estimation

$$
\int_{\frac{1}{\sqrt{\epsilon}}}^\infty \frac{e^{-(T-t)r^2}}{r^2} dr \le \epsilon \int_{\frac{1}{\sqrt{\epsilon}}}^\infty e^{-(T-t)r^2} dr,
$$

it can be seen that

$$
\begin{aligned}
0 = & \lim_{\epsilon \to 0} \int_{\frac{1}{\sqrt{\epsilon}}}^\infty 12 e^{-(T-t)r^2} dr \ge \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\frac{1}{\sqrt{\epsilon}}}^\infty \frac{12 e^{-(T-t)r^2}}{r^2} dr \\
\ge & \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\frac{1}{\sqrt{\epsilon}}}^\infty \frac{e^{-(T-t)r^2}}{r^2} \left| \left(\psi\left(\frac{\epsilon r}{\sqrt{2}} + \frac{\pi}{2}\right) \left(\cos\left(\frac{3\epsilon r}{\sqrt{2}}\right) + 3\cos\left(\frac{\epsilon r}{\sqrt{2}}\right)\right) - 4 \right. \right. \\
& \left. \left. - \psi\left(\frac{\epsilon r}{\sqrt{2}}\right) \left(\sin\left(\frac{3\epsilon r}{\sqrt{2}}\right) - 3\sin\left(\frac{\epsilon r}{\sqrt{2}}\right)\right) \right) \right| dr.
\end{aligned}
$$

Now, both $\psi\left(\frac{\epsilon r}{\sqrt{2}}+\frac{\pi}{2}\right)$ and $\psi\left(\frac{\epsilon r}{\sqrt{2}}\right)$ are positive over the interval $\left[0, \frac{1}{\sqrt{\epsilon}}\right]$. In conjunction with two equalities

$$\cos\left(\frac{3\epsilon r}{\sqrt{2}}\right)+3\cos\left(\frac{\epsilon r}{\sqrt{2}}\right)=4\cos^3\left(\frac{\epsilon r}{\sqrt{2}}\right),$$

$$\sin\left(\frac{3\epsilon r}{\sqrt{2}}\right)-3\sin\left(\frac{\epsilon r}{\sqrt{2}}\right)=-4\sin^3\left(\frac{\epsilon r}{\sqrt{2}}\right),$$

we make the estimation

$$\begin{aligned}\left|\frac{1}{\epsilon}\int_0^{\frac{1}{\sqrt{\epsilon}}}\frac{e^{-(T-t)r^2}}{r^2}\Lambda(t,x+\epsilon)dr\right| &\leq \left|\frac{1}{\epsilon}\int_0^{\frac{1}{\sqrt{\epsilon}}}\frac{e^{-(T-t)r^2}}{r^2}\left(4\cos^3\left(\frac{\epsilon r}{\sqrt{2}}\right)-4+4\sin^3\left(\frac{\epsilon r}{\sqrt{2}}\right)\right)dr\right|\\ &\leq \epsilon\int_0^{\frac{1}{\sqrt{\epsilon}}}e^{-(T-t)r^2}\left(\frac{4-4\cos^3\left(\frac{\epsilon r}{\sqrt{2}}\right)}{(\epsilon r)^2}+\frac{4\sin^3\left(\frac{\epsilon r}{\sqrt{2}}\right)}{(\epsilon r)^2}\right)dr.\end{aligned}$$

Since the integral $\int_0^{\frac{1}{\sqrt{\epsilon}}}e^{-(T-t)r^2}\left(\frac{4-4\cos^3\left(\frac{\epsilon r}{\sqrt{2}}\right)}{(\epsilon r)^2}+\frac{4\sin^3\left(\frac{\epsilon r}{\sqrt{2}}\right)}{(\epsilon r)^2}\right)dr$ is bounded as $\epsilon\to 0$, we conclude that

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^{\infty}\frac{e^{-(T-t)r^2}}{r^2}\Lambda(t,x+\epsilon)dr=0,$$

and hence $\partial_{x_1}u^T(t,x)=\frac{1}{4}$. Similarly, we can prove that $\partial_{x_i}u^T(t,x)=\frac{1}{4}, i=2,3,4$.

Step 3: Equation (5.10). Define two functional series $H_l(x), K_l(x), l\in\mathbb{Z}$ by

$$H_l(x):=\int_{(2l-1)T_0(x)}^{(2l+1)T_0(x)}\frac{e^{-(T-t)r^2}}{r}\sum_{k=1}^{4}\alpha_{k,i}\psi\left(r\theta\cdot x^o+\frac{\pi}{2}\right)\sin\left(r\alpha_k\cdot x\right)dr,$$

$$K_l(x):=\int_{2lT_0(x)}^{(2l+2)T_0(x)}\frac{e^{-(T-t)r^2}}{r}\sum_{k=1}^{4}\alpha_{k,i}\psi\left(r\theta\cdot x^o\right)\cos\left(r\alpha_k\cdot x\right)dr.$$

Then we have $\partial_{x_i} u^T(t,x) = \frac{1}{4} + \frac{1}{16\sqrt{2}} \sum_{l\in\mathbb{Z}} (H_l(x) + K_l(x))$. We compute the right-hand derivatives of $H_l(x), K_l(x)$,

$$
\begin{aligned}
\partial^+_{x_j} H_l(x) = &\int_{(2l-1)T_0(x)}^{(2l+1)T_0(x)} e^{-(T-t)r^2} \sum_{k=1}^{4} \alpha_{k,i}\alpha_{k,j}\psi\left(r\theta\cdot x^o + \frac{\pi}{2}\right)\cos\left(r\alpha_k\cdot x\right)dr \\
&+ \partial^+_{x_j}(\theta\cdot x^o)\frac{(-1)^l e^{-\frac{(T-t)(\pi(l+1/2))^2}{(\theta\cdot x^o)^2}}}{\theta\cdot x^o}\sum_{k=1}^{4} 2\alpha_{k,i}\sin\left(\frac{\alpha_k\cdot x\pi(l+1/2)}{\theta\cdot x^o}\right) \\
&- \partial^+_{x_j}(\theta\cdot x^o)\frac{(-1)^l e^{-\frac{(T-t)(\pi(l-1/2))^2}{(\theta\cdot x^o)^2}}}{\theta\cdot x^o}\sum_{k=1}^{4} 2\alpha_{k,i}\sin\left(\frac{\alpha_k\cdot x\pi(l-1/2)}{\theta\cdot x^o}\right),
\end{aligned}
\tag{5.30}
$$

$$
\begin{aligned}
\partial^+_{x_j} K_l(x) = -&\int_{2lT_0(x)}^{(2l+2)T_0(x)} e^{-(T-t)r^2} \sum_{k=1}^{4} \alpha_{k,i}\alpha_{k,j}\psi\left(r\theta\cdot x^o\right)\sin\left(r\alpha_k\cdot x\right)dr \\
&+ \partial^+_{x_j}(\theta\cdot x^o)\frac{(-1)^l e^{-\frac{(T-t)(\pi(l+1))^2}{(\theta\cdot x^o)^2}}}{\theta\cdot x^o}\sum_{k=1}^{4} 2\alpha_{k,i}\cos\left(\frac{\alpha_k\cdot x\pi(l+1)}{\theta\cdot x^o}\right) \\
&- \partial^+_{x_j}(\theta\cdot x^o)\frac{(-1)^l e^{-\frac{(T-t)(\pi l)^2}{(\theta\cdot x^o)^2}}}{\theta\cdot x^o}\sum_{k=1}^{4} 2\alpha_{k,i}\cos\left(\frac{\alpha_k\cdot x\pi l}{\theta\cdot x^o}\right).
\end{aligned}
\tag{5.31}
$$

Replacing all the $\partial^+_{x_j}$ with $\partial^-_{x_j}$, we obtain the left hand side derivatives of $H_l(x)$ and $K_l(x)$. It can be easily checked that if $\theta\cdot x^o < 0, x^{(2)} < x^{(3)}$, the function $x \mapsto \theta\cdot x^o$ is differentiable, and hence $\partial^+_{x_j} H_l(x) = \partial^-_{x_j} H_l(x)$, $\partial^+_{x_j} K_l(x) = \partial^-_{x_j} K_l(x)$. Since $\sum_{l\in\mathbb{Z}} \partial_{x_j} H_j(x)$ and $\sum_{l\in\mathbb{Z}} \partial_{x_j} H_j(x)$ converge uniformly in any bounded region of x, we can interchange summation and differentiation and obtain (5.10).

Step 4: Equation (5.11). If $\theta\cdot x^o < 0, x^{(2)} = x^{(3)}$, the right derivative $\partial^+_{x_j}(\theta\cdot x^o)$ may not equal to the left derivative $\partial^-_{x_j}(\theta\cdot x^o)$. However, by showing that for each

$i = 1, 2, 3, 4,$

$$\sum_{k=1}^{4} \alpha_{k,i} \sin\left(\frac{\alpha_k \cdot x\pi(l+1/2)}{\theta \cdot x^o}\right) = \sum_{k=1}^{4} \alpha_{k,i} \cos\left(\frac{\alpha_k \cdot x\pi l}{\theta \cdot x^o}\right) = 0, \quad l \in \mathbb{Z},$$

functions $H_l(x)$, $K_l(x)$ are still differentiable, and hence we can conclude (5.11). Since we need to show the equality for any $i = 1, 2, 3, 4$, we can simply assume $x = x^o$ without loss of generality. It can be easily checked that

$$\begin{aligned} \frac{\alpha_1 \cdot x^o}{\theta \cdot x^o} &= \frac{\alpha_4 \cdot x^o}{\theta \cdot x^o} - 4 = \frac{3x^{(1)} - 2x^{(2)} - x^{(4)}}{x^{(1)} - x^{(4)}}, \\ \frac{\alpha_2 \cdot x^o}{\theta \cdot x^o} &= \frac{\alpha_3 \cdot x^o}{\theta \cdot x^o} = \frac{2x^{(2)} - x^{(1)} - x^{(4)}}{x^{(1)} - x^{(4)}}, \end{aligned}$$

and hence

$$\begin{aligned} \sin\left(\frac{\alpha_1 \cdot x\pi(l+1/2)}{\theta \cdot x^o}\right) &= \sin\left(\frac{\alpha_4 \cdot x\pi(l+1/2)}{\theta \cdot x^o}\right), \\ \sin\left(\frac{\alpha_2 \cdot x\pi(l+1/2)}{\theta \cdot x^o}\right) &= \sin\left(\frac{\alpha_3 \cdot x\pi(l+1/2)}{\theta \cdot x^o}\right), \\ \cos\left(\frac{\alpha_1 \cdot x\pi l}{\theta \cdot x^o}\right) &= \cos\left(\frac{\alpha_4 \cdot x\pi l}{\theta \cdot x^o}\right), \\ \cos\left(\frac{\alpha_2 \cdot x\pi l}{\theta \cdot x^o}\right) &= \cos\left(\frac{\alpha_3 \cdot x\pi l}{\theta \cdot x^o}\right). \end{aligned}$$

We finish the proof of (5.11) by the following computation

$$\begin{aligned} \sum_{k=1}^{4} \alpha_{k,i} \sin\left(\frac{\alpha_k \cdot x\pi(l+1/2)}{\theta \cdot x^o}\right) &= \pm 2\left(\sin\left(\frac{\alpha_1 \cdot x\pi(l+1/2)}{\theta \cdot x^o}\right) - \sin\left(\frac{\alpha_2 \cdot x\pi(l+1/2)}{\theta \cdot x^o}\right)\right) \\ &= \pm 4 \sin\left(\frac{2x^{(1)} - 2x^{(2)}}{x^{(1)} - x^{(4)}}\pi(l+1/2)\right) \cos\left(\pi(l+1/2)\right) = 0, \\ \sum_{k=1}^{4} \alpha_{k,i} \cos\left(\frac{\alpha_k \cdot x\pi l}{\theta \cdot x^o}\right) &= \pm 2\left(\cos\left(\frac{\alpha_1 \cdot x\pi l}{\theta \cdot x^o}\right) - \cos\left(\frac{\alpha_2 \cdot x\pi l}{\theta \cdot x^o}\right)\right) \\ &= \pm 4 \sin\left(\frac{2x^{(2)} - 2x^{(1)}}{x^{(1)} - x^{(4)}}\pi l\right) \sin\left(\pi l\right) = 0. \end{aligned}$$

Step 5: Equation (5.12). Finally, supposing $x = (x_1, x_1, x_1, x_1)$ and $x + \epsilon_j = (x_1 + \epsilon)\mathbb{1}_{\{k=j\}} + x_1 \mathbb{1}_{\{k \neq j\}}$, we calculate $\partial^2_{x_i x_j} u^T(t,x)$. According to (5.8), we have

$$\begin{aligned}\partial_{x_i} u^T(t, x+\epsilon_j) =& \frac{1}{8\sqrt{2}} \int_0^\infty \frac{e^{-(T-t)r^2}}{r} \sum_{k=1}^4 \alpha_{k,i} \left(\psi\left(\frac{\epsilon r}{\sqrt{2}} + \frac{\pi}{2}\right) \sin\big(r\alpha_k \cdot (x+\epsilon_j)\big) \right. \\ & \left. + \psi\left(\frac{\epsilon r}{\sqrt{2}}\right) \cos\big(r\alpha_k \cdot (x+\epsilon_j)\big)\right) dr + \frac{1}{4}.\end{aligned}$$

As a result of the equalities

$$\int_{\frac{1}{\sqrt{\epsilon}}}^\infty \frac{e^{-(T-t)r^2}}{r} dr = -\int_{\frac{1}{\sqrt{\epsilon}}}^\infty \frac{1}{2(T-t)r^2} de^{-(T-t)r^2} = \frac{\epsilon e^{-\frac{(T-t)}{\epsilon}}}{2(T-t)} + \int_{\frac{1}{\sqrt{\epsilon}}}^\infty \frac{e^{-(T-t)r^2}}{4(T-t)r^3} dr,$$

we deduce that

$$\begin{aligned} 0 =& \lim_{\epsilon\to 0} \left(\frac{e^{-\frac{(T-t)}{\epsilon}}}{2(T-t)} + \int_{\frac{1}{\sqrt{\epsilon}}}^\infty \frac{e^{-(T-t)r^2}}{4(T-t)r} \right) 6\sqrt{2} \geq \lim_{\epsilon\to 0} \frac{6\sqrt{2}}{\epsilon} \left(\frac{\epsilon e^{-\frac{(T-t)}{\epsilon}}}{2(T-t)} + \int_{\frac{1}{\sqrt{\epsilon}}}^\infty \frac{e^{-(T-t)r^2}}{4(T-t)r^3} \right) \\ \geq & \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_{\frac{1}{\sqrt{\epsilon}}}^\infty \frac{e^{-(T-t)r^2}}{r} \left| \sum_{k=1}^4 \alpha_{k,i} \left(\psi\left(\frac{\epsilon r}{\sqrt{2}} + \frac{\pi}{2}\right) \sin\big(r\alpha_k \cdot (x+\epsilon_j)\big) \right.\right. \\ & \left.\left. + \psi\left(\frac{\epsilon r}{\sqrt{2}}\right) \cos\big(r\alpha_k \cdot (x+\epsilon_j)\big)\right)\right| dr. \end{aligned} \tag{5.32}$$

From the equality

$$\begin{aligned}\sum_{k=1}^4 \alpha_{k,i} \cos(r\alpha_k \cdot (x+\epsilon_j)) =& \mathbb{1}_{\{i=j\}} \left(3\cos\left(\frac{3\epsilon r}{\sqrt{2}}\right) - 3\cos\left(\frac{\epsilon r}{\sqrt{2}}\right)\right) \\ & + \mathbb{1}_{\{i\neq j\}} \left(\cos\left(\frac{\epsilon r}{\sqrt{2}}\right) - \cos\left(\frac{3\epsilon r}{\sqrt{2}}\right)\right),\end{aligned}$$

it can be easily seen that

$$\lim_{\epsilon\to 0}\frac{1}{r\epsilon}\sum_{k=1}^{4}\alpha_{k,i}\cos(r\alpha_k\cdot(x+\epsilon_j))=0. \tag{5.33}$$

Combining (5.32) and (5.33), we conclude that

$$\begin{aligned}
&\partial^2_{x_ix_j}u^T(t,x)\\
&=\lim_{\epsilon\to 0}\frac{1}{8\sqrt{2}\epsilon}\int_0^{\frac{1}{\sqrt{\epsilon}}}\frac{e^{-(T-t)r^2}}{r}\sum_{k=1}^{4}\alpha_{k,i}\Big(\sin\big(r\alpha_k\cdot(x+\epsilon_j)\big)+\cos\big(r\alpha_k\cdot(x+\epsilon_j)\big)\Big)\,dr\\
&=\frac{1}{8\sqrt{2}}\lim_{\epsilon\to 0}\int_0^{\frac{1}{\sqrt{\epsilon}}}e^{-(T-t)r^2}\left(\sum_{k=1}^{4}\alpha_{k,i}\frac{\sin\big(r\alpha_k\cdot(x+\epsilon_j)\big)}{r\epsilon}+\frac{1}{r\epsilon}\sum_{k=1}^{4}\alpha_{k,i}\cos(r\alpha_k\cdot(x+\epsilon_j))\right)dr\\
&=\frac{1}{8\sqrt{2}}\int_0^{\infty}e^{-(T-t)r^2}\left(\sum_{k=1}^{4}\alpha_{k,i}\lim_{\epsilon\to 0}\frac{\sin\big(r\alpha_k\cdot(x+\epsilon_j)\big)}{r\epsilon}\right.\\
&\left.+\lim_{\epsilon\to 0}\frac{1}{r\epsilon}\sum_{k=1}^{4}\alpha_{k,i}\cos(r\alpha_k\cdot(x+\epsilon_j))\right)dr\\
&=\frac{1}{8\sqrt{2}}\int_0^{\infty}e^{-(T-t)r^2}\left(\sum_{k=1}^{4}\alpha_{k,i}\alpha_{k,j}\right)dr.
\end{aligned}$$

Since $\partial_{x_j}H_l(x),\partial_{x_j}K_l(x)$ defined in *Step 3* are continuous with respect to x, and both series $\sum\limits_{l\in\mathbb{Z}}\partial_{x_j}H_l(x),\sum\limits_{l\in\mathbb{Z}}\partial_{x_j}K_l(x)$ converge uniformly in any bounded region of x, the second derivative $\partial^2_{x_ix_j}u^T(t,x)$ is also continuous, and hence we have proved that $u^T(t,x)$ is in $C^2([0,T)\times\mathbb{R}^4)$. □

5.4.1.4 Solution Property

Proof. Supposing that $\{i_1,i_2,i_3,i_4\}=\{1,2,3,4\}$ are subscripts such that $x_{i_1}\le x_{i_2}\le x_{i_3}\le x_{i_4}$, we prove the equation

$$\partial_t u^T(t,x)+\frac{1}{2}\left(\partial_{x_{i_2}}+\partial_{x_{i_4}}\right)^2u^T(t,x)=0.$$

Taking derivative with respect to t, we obtain that

$$\begin{aligned}
\partial_t u^T(t,x) =& \frac{-1}{16\sqrt{2}} \int\limits_{-\infty}^{\infty} e^{-(T-t)r^2} \left(\psi \left(r\theta \cdot x^o + \frac{\pi}{2} \right) \sum_{k=1}^{4} \cos\left(r\alpha_k \cdot x \right) \right. \\
& \left. -\psi\left(r\theta \cdot x^o \right) \sum_{k=1}^{4} \sin\left(r\alpha_k \cdot x \right) \right) dr + \frac{1}{4\sqrt{2}} \int\limits_{-\infty}^{\infty} e^{-(T-t)r^2} dr - \frac{1}{4}\sqrt{\frac{\pi}{2(T-t)}} \\
=& \frac{-1}{16\sqrt{2}} \int\limits_{-\infty}^{\infty} e^{-(T-t)r^2} \left(\psi \left(r\theta \cdot x^o + \frac{\pi}{2} \right) \sum_{k=1}^{4} \cos\left(r\alpha_k \cdot x \right) \right. \\
& \left. -\psi\left(r\theta \cdot x^o \right) \sum_{k=1}^{4} \sin\left(r\alpha_k \cdot x \right) \right) dr. \qquad (5.34)
\end{aligned}$$

According to (5.10) and the equality $\partial_{x_{i_2}}\left(\theta \cdot x^o\right) + \partial_{x_{i_4}}\left(\theta \cdot x^o\right) = 0$, the series part cancels out and we have

$$\begin{aligned}
& \left(\partial_{x_{i_2}} + \partial_{x_{i_4}} \right)^2 u^T(t,x) \\
&= \frac{1}{16\sqrt{2}} \int\limits_{-\infty}^{\infty} e^{-(T-t)r^2} \sum_{k=1}^{4} (\alpha_{k,i_2} + \alpha_{k,i_4})^2 \left(\psi \left(r\theta \cdot x^o + \frac{\pi}{2} \right) \cos\left(r\alpha_k \cdot x \right) \right. \\
& \left. -\psi\left(r\theta \cdot x^o \right) \sin\left(r\alpha_k \cdot x \right) \right) dr.
\end{aligned}$$

Since $(\alpha_{k,i_2} + \alpha_{k,i_4})^2 = 2$ for every $k = 1, 2, 3, 4$, we conclude that

$$\begin{aligned}
\frac{1}{2} \left(\partial_{x_{i_2}} + \partial_{x_{i_4}} \right)^2 u^T(t,x) =& \frac{1}{16\sqrt{2}} \int\limits_{-\infty}^{\infty} e^{-tr^2} \sum_{k=1}^{4} \left(\psi \left(r\theta \cdot x^o + \frac{\pi}{2} \right) \cos\left(r\alpha_k \cdot x \right) \right. \\
& \left. -\psi\left(r\theta \cdot x^o \right) \sin\left(r\alpha_k \cdot x \right) \right) dr = -\partial_t u^T(t,x). \qquad (5.35)
\end{aligned}$$

$\square$

5.4.2 Proof of Theorem 5.3.5

Proof. By using arguments similar to the proof of (5.35), we have for $(j,k) = (1,3),(1,4),(2,3)$,

$$\partial_t u^T(t,x) + \frac{1}{2}\left(\partial_{x_{i_j}} + \partial_{x_{i_k}}\right)^2 u^T(t,x) = 0.$$

From (5.8), we obtain that $\sum_{k=1}^{4} \partial_{x_k} u^T(t,x) = 1$, which implies

$$\partial^2_{xx} u^T(t,x)\,(e_1 + e_2 + e_3 + e_4) = 0.$$

Subsequently, for all $J \in P(4)$, we have that

$$e_J^\top \partial^2_{xx} u^T(t,x) e_J - e_{J^c}^\top \partial^2_{xx} u^T(t,x) e_{J^c} = \left(e_J^\top - e_{J^c}^\top\right) \partial^2_{xx} u^T(t,x)\,(e_J + e_{J^c}) = 0.$$

Therefore, it remains to show that the strategies $J \in \{\emptyset, \{i_1, i_2\}, \{i_1\}, \{i_2\}, \{i_3\}, \{i_4\}\}$ are suboptimal, i.e.,

$$\partial_t u^T(t,x) + \frac{1}{2} \sup_{J \in P(N)} e_J^\top \partial^2_{xx} u^T(t,x) e_J \leq 0.$$

Since the second derivatives of $u^T(t,x)$ are continuous, we assume that $\theta \cdot x^o < 0, x^{(2)} < x^{(3)}$ without loss of generality. First we introduce some notations, and simplify the expressions for $\partial_t u^T(t,x), \partial^2_{xx} u^T(t,x)$. Define

$$
\begin{aligned}
S_k :=& \sqrt{T-t} \int_{-\infty}^{\infty} e^{-(T-t)r^2} \left(\psi\left(r\theta \cdot x^o + \frac{\pi}{2}\right) \cos\left(r\alpha_k \cdot x^o\right) - \psi\left(r\theta \cdot x^o\right) \sin\left(r\alpha_k \cdot x^o\right) \right) dr, \\
L_k :=& \sqrt{T-t} \left(\sum_{l\in\mathbb{Z}} \frac{(-1)^l e^{-\frac{(T-t)(\pi(l+1/2))^2}{(\theta\cdot x^o)^2}}}{\theta \cdot x^o} \sin\left(\frac{\alpha_k \cdot x^o \pi (l+1/2)}{\theta\cdot x^o} \right) \right. \\
& \left. - \sum_{l\in\mathbb{Z}} \frac{(-1)^l e^{-\frac{(T-t)(\pi l)^2}{(\theta\cdot x^o)^2}}}{\theta \cdot x^o} \cos\left(\frac{\alpha_k \cdot x^o \pi l}{\theta\cdot x^o} \right) \right).
\end{aligned}
$$

According to (5.34) and (5.10), it can be checked that

$$
\partial_t u^T(t,x) = \frac{-1}{16\sqrt{2(T-t)}} \sum_{k=1}^{4} S_k, \tag{5.36}
$$

$$
\partial^2_{x_{i_h} x_{i_j}} u^T(t,x) = \frac{1}{16\sqrt{2(T-t)}} \left(\sum_{k=1}^{4} \alpha_{k,h}\alpha_{k,j} S_k + 2\sum_{k=1}^{4} \partial_{x_{i_j}}(\theta\cdot x^o)\alpha_{k,h} L_k \right), \tag{5.37}
$$

where we use the fact that the i_h-th coordinate of x is the h-th coordinate of x^o.

Define $\tilde{T} := -\frac{\sqrt{T-t}\pi}{2\theta\cdot x^o}$, $\beta_k := \frac{\alpha_k \cdot x^o}{2\pi\sqrt{T-t}}, k = 1,2,3,4$, and

$$
f(r) = e^{-r^2}, \quad F_k^1(r) = f(r)\cos(2\pi\beta_k r), \quad F_k^2(r) = f(r)\sin(2\pi\beta_k r).
$$

Their Fourier transforms are given respectively by

$$
\begin{aligned}
\hat{f}(v) &:= \int_{-\infty}^{\infty} f(x) e^{-2\pi i v x} dx = \sqrt{\pi} e^{-\pi v^2}, \\
\hat{F_k^1}(v) &:= \frac{\hat{f}\left(v-\beta_k\right) + \hat{f}\left(v+\beta_k\right)}{2}, \\
\hat{F_k^2}(v) &:= \frac{\hat{f}\left(v-\beta_k\right) - \hat{f}\left(v+\beta_k\right)}{2i}.
\end{aligned}
$$

By change of variables, we obtain

$$S_k = \int_{-\infty}^{\infty} e^{-r^2} \left(\psi \left(r \frac{\theta \cdot x^o}{\sqrt{T-t}} + \frac{\pi}{2} \right) \cos \left(r \frac{\alpha_k \cdot x}{\sqrt{T-t}} \right) - \psi \left(r \frac{\theta \cdot x^o}{\sqrt{T-t}} \right) \sin \left(r \frac{\alpha_k \cdot x}{\sqrt{T-t}} \right) \right) dr$$
$$= \sum_{l \in \mathbb{Z}} (-1)^l \int_{(2l-1)\tilde{T}}^{(2l+1)\tilde{T}} F_k^1(r) dr + \sum_{l \in \mathbb{Z}} (-1)^l \int_{2l\tilde{T}}^{(2l+2)\tilde{T}} F_k^2(r) dr.$$

Since the functions F_k^1 are even and F_k^2 are odd, we obtain that

$$S_k = \int_{-\infty}^{\infty} F_k^1(r) dr - 2 \sum_{l \in \mathbb{Z}} \int_{\tilde{T}}^{3\tilde{T}} F_k^1(4l\tilde{T} + r) dr + 2 \sum_{l \in \mathbb{Z}} \int_{0}^{2\tilde{T}} F_k^2(4l\tilde{T} + r) dr. \tag{5.38}$$

Also it can be seen that

$$L_k = \frac{2\tilde{T}}{\pi} \left(\sum_{l \in \mathbb{Z}} (-1)^l F_k^2((2l+1)\tilde{T}) + \sum_{l \in \mathbb{Z}} (-1)^l F_k^1(2l\tilde{T}) \right). \tag{5.39}$$

Step 1: $J = \{i_1, i_2\}$. We prove the inequality

$$\partial_t u^T(t,x) + \frac{1}{2} \left(\partial_{x_{i_1}} + \partial_{x_{i_2}} \right)^2 u(t,x) \leq 0. \tag{5.40}$$

According to trigonometric formulas, we have the following equalities

$$\sin \left(\frac{\alpha_1 \cdot x^o \pi (l + 1/2)}{\theta \cdot x^o} \right) = \sin \left(\frac{\alpha_2 \cdot x^o \pi (l + 1/2)}{\theta \cdot x^o} \right),$$

$$\sin \left(\frac{\alpha_3 \cdot x^o \pi (l + 1/2)}{\theta \cdot x^o} \right) = \sin \left(\frac{\alpha_4 \cdot x^o \pi (l + 1/2)}{\theta \cdot x^o} \right),$$

$$\cos \left(\frac{\alpha_1 \cdot x^o \pi l}{\theta \cdot x^o} \right) = \cos \left(\frac{\alpha_2 \cdot x^o \pi l}{\theta \cdot x^o} \right),$$

$$\cos\left(\frac{\alpha_3 \cdot x^o \pi l}{\theta \cdot x^o}\right) = \cos\left(\frac{\alpha_4 \cdot x^o \pi l}{\theta \cdot x^o}\right).$$

Therefore we have $L_1 = L_2, L_3 = L_4$. Plugging in (5.36), (5.37) and noting that $\partial_{x_{i_1}}(\theta \cdot x^o) = \partial_{x_{i_2}}(\theta \cdot x^o) = \frac{1}{\sqrt{2}}$, it can be checked that

$$\begin{aligned}
\partial_t u^T(t,x) + \frac{1}{2}\left(\partial_{x_{i_1}} + \partial_{x_{i_2}}\right)^2 u(t,x) =& \frac{1}{16\sqrt{2(T-t)}}\left(\sum_{k=1}^{4}\left(\frac{1}{2}(\alpha_{k,1}+\alpha_{k,2})^2 - 1\right) S_k\right) \\
&+ \frac{1}{16\sqrt{2(T-t)}}(4L_1 - 4L_4) \\
=& \frac{1}{4\sqrt{2(T-t)}}(L_1 - L_4).
\end{aligned}$$

Let us introduce

$$\mu := \frac{\pi\alpha_1 \cdot x^o}{4\theta \cdot x^o} + \frac{\pi}{4}, \quad \nu := \frac{\pi\alpha_4 \cdot x^o}{4\theta \cdot x^o} + \frac{\pi}{4}, \quad \hat{\tau} := \frac{(T-t)i\pi}{4\left(\theta \cdot x^o\right)^2},$$

and the Jacobi-theta function

$$\theta_3\left(z, \tau\right) := \sum_{l=-\infty}^{\infty} \exp\left(\pi i l^2 \tau + 2ilz\right).$$

We rewrite sine and cosine terms as

$$(-1)^l \sin\left(\frac{\alpha_1 \cdot x^o \pi(l+1/2)}{\theta \cdot x^o}\right) = -Re\ e^{2i(2l+1)\mu},$$

$$-(-1)^l \cos\left(\frac{\alpha_1 \cdot x \pi l}{\theta \cdot x^o}\right) = -Re\ e^{4il\mu}.$$

Note that $\exp(\pi i l^2 \hat{\tau}) = e^{-\frac{(T-t)(\pi l)^2}{4(\theta \cdot x^o)^2}}$. Then according to the definition of L_1, we obtain that

$$\frac{L_1}{\sqrt{T-t}} = -\frac{Re\ \theta_3(\mu, \hat{\tau})}{\theta \cdot x^o} = -\frac{\theta_3(\mu, \hat{\tau})}{\theta \cdot x^o},$$

and

$$\frac{L_4}{\sqrt{T-t}} = -\frac{\theta_3(\nu, \hat{\tau})}{\theta \cdot x^o}.$$

Therefore, we obtain

$$\partial_t u^T(t,x) + \frac{1}{2}\left(\partial_{x_{i_1}} + \partial_{x_{i_2}}\right)^2 u(t,x) = -\frac{1}{4\sqrt{2}} \frac{\theta_3(\mu, \hat{\tau}) - \theta_3(\nu, \hat{\tau})}{\theta \cdot x^o}, \tag{5.41}$$

and (5.40) is equivalent to

$$\theta_3\left(\mu, \hat{\tau}\right) - \theta_3\left(\nu, \hat{\tau}\right) \leq 0. \tag{5.42}$$

Taking $q = e^{i\pi\hat{\tau}}$, we have the infinite product representation for the Jacobi-theta function (see e.g. [157])

$$\theta_3\left(z, \hat{\tau}\right) = \prod_{l=1}^{\infty}\left(1 - q^{2l}\right)\left(1 + 2q^{2l-1}\cos(2z) + q^{4l-2}\right). \tag{5.43}$$

By the definition of μ, ν, it can be easily checked that

$$\begin{aligned}
\mu - \nu &= \frac{\pi\left(x^{(1)} - x^{(4)}\right)}{x^{(1)} + x^{(2)} - x^{(3)} - x^{(4)}} \in (0, \pi), \\
\mu + \nu &= \frac{\pi\left(x^{(1)} - x^{(2)} - x^{(3)} + x^{(4)}\right)}{2\left(x^{(1)} + x^{(2)} - x^{(3)} - x^{(4)}\right)} + \frac{\pi}{2} \in (0, \pi),
\end{aligned}$$

and hence $dist(\mu, \mathbb{Z}\pi) > dist(\nu, \mathbb{Z}\pi)$. Subsequently, we have $\cos(2\mu) \leq \cos(2\nu)$, and therefore conclude (5.42) by (5.43).

Step 2: $J = \emptyset$. According to the Poisson summation formula for Fourier transform (see e.g. [178]), it can be seen that

$$\sum_{l\in\mathbb{Z}} F_k^1(4l\tilde{T}+r) = \sum_{l\in\mathbb{Z}} \frac{1}{4\tilde{T}}\hat{F_k^1}\left(\frac{l}{4\tilde{T}}\right)e^{i2\pi\frac{l}{4\tilde{T}}r},$$
$$\sum_{l\in\mathbb{Z}} F_k^2(4l\tilde{T}+r) = \sum_{l\in\mathbb{Z}} \frac{1}{4\tilde{T}}\hat{F_k^2}\left(\frac{l}{4\tilde{T}}\right)e^{i2\pi\frac{l}{4\tilde{T}}r}\ .$$

Then according to (5.38),

$$\begin{aligned}
S_k =&\hat{F_k^1}(0) - \frac{1}{2\tilde{T}}\sum_{l\in\mathbb{Z}}\hat{F_k^1}\left(\frac{l}{4T}\right)\int_{\tilde{T}}^{3\tilde{T}} e^{i2\pi\frac{l}{4\tilde{T}}r}dr + \frac{1}{2\tilde{T}}\sum_{l\in\mathbb{Z}}\hat{F_k^2}\left(\frac{l}{4\tilde{T}}\right)\int_{0}^{2\tilde{T}} e^{i2\pi\frac{l}{4\tilde{T}}r}dr\\
=&\sum_{l\in\mathbb{Z}}(-1)^l\hat{F_k^1}\left(\frac{2l+1}{4\tilde{T}}\right)\frac{2}{(2l+1)\pi} - \sum_{l\in\mathbb{Z}}\hat{F_k^2}\left(\frac{2l+1}{4\tilde{T}}\right)\frac{2}{i(2l+1)\pi}\\
=&\sum_{l\in\mathbb{Z}}(-1)^l\hat{f}\left(\frac{2l+1}{4\tilde{T}}+(-1)^{l+1}\beta_k\right)\frac{2}{(2l+1)\pi}\ .
\end{aligned}$$

By direct computation,

$$-\frac{1}{2\tilde{T}}\hat{F_k^1}\left(\frac{l}{4T}\right)\int_{\tilde{T}}^{3\tilde{T}} e^{i2\pi\frac{l}{4\tilde{T}}r}dr = \begin{cases}\hat{F_k^1}(0), & \text{if } l=0,\\ (-1)^{(l-1)/2}\hat{F_k^1}\left(\frac{l}{4\tilde{T}}\right)\frac{2}{l\pi}, & \text{if } l \text{ is odd},\\ 0, & \text{if } l \text{ is even},\end{cases}$$

$$\frac{1}{2\tilde{T}}\hat{F_k^2}\left(\frac{l}{4\tilde{T}}\right)\int_{0}^{2\tilde{T}} e^{i2\pi\frac{l}{4\tilde{T}}r}dr = \begin{cases}-\hat{F_k^2}\left(\frac{l}{4\tilde{T}}\right)\frac{2}{il\pi}, & \text{if } l \text{ is odd},\\ 0, & \text{if } l \text{ is even}.\end{cases}$$

Therefore, we obtain that

$$\begin{aligned}
S_k =& \sum_{l\in\mathbb{Z}}(-1)^l \hat{F}_k^1\left(\frac{2l+1}{4\tilde{T}}\right)\frac{2}{(2l+1)\pi} - \sum_{l\in\mathbb{Z}}\hat{F}_k^2\left(\frac{2l+1}{4\tilde{T}}\right)\frac{2}{i(2l+1)\pi}\\
=& \sum_{l\in\mathbb{Z}}(-1)^l \hat{f}\left(\frac{2l+1+(-1)^{l+1}4\tilde{T}\beta_k}{4\tilde{T}}\right)\frac{2}{(2l+1)\pi}\\
=& 2\sum_{l\geq 0}(-1)^l \hat{f}\left(\frac{2l+1+(-1)^{l+1}4\tilde{T}\beta_k}{4\tilde{T}}\right)\frac{2}{(2l+1)\pi},
\end{aligned} \tag{5.44}$$

where the last equation follows from the identity

$$\begin{aligned}
&(-1)^l \hat{f}\left(\frac{2l+1+(-1)^{l+1}4\tilde{T}\beta_k}{4\tilde{T}}\right)\frac{2}{(2l+1)\pi}\\
&= (-1)^{-l-1}\hat{f}\left(\frac{-2l-1+(-1)^{-l}4\tilde{T}\beta_k}{4\tilde{T}}\right)\frac{2}{(-2l-1)\pi}.
\end{aligned}$$

Denoting

$$\begin{aligned}
\eta_1 &:= 4\tilde{T}\beta_1 = -\frac{\alpha_1\cdot x^o}{\theta\cdot x^o} = \frac{3x^{(1)}-x^{(2)}-x^{(3)}-x^{(4)}}{x^{(4)}+x^{(3)}-x^{(2)}-x^{(1)}},\\
\eta_4 &:= 4\tilde{T}\beta_4 = -\frac{\alpha_4\cdot x^o}{\theta\cdot x^o} = \frac{-x^{(1)}-x^{(2)}-x^{(3)}+3x^{(4)}}{x^{(4)}+x^{(3)}-x^{(2)}-x^{(1)}},
\end{aligned}$$

it can be easily checked that they satisfy the constraints

$$\eta_1 \in [-3,-1], \quad \eta_4 \in [1,3], \quad \eta_4 - \eta_1 \leq 4. \tag{5.45}$$

Since $\partial_t u^T(t,x) = \frac{-1}{16\sqrt{2(T-t)}}\sum_{k=1}^{4} S_k$, the inequality

$$\partial_t u^T(t,x) \leq 0. \tag{5.46}$$

is equivalent to $\sum_{k=1}^{4} S_k \geq 0$. Due to definitions of $\tilde{T}$ and β_k, we have that $4\tilde{T}\beta_1+4\tilde{T}\beta_2 =$

$-2, 4\tilde{T}\beta_3 + 4\tilde{T}\beta_4 = 2$. Therefore we obtain the equations

$$\begin{aligned} S_1 + S_2 =& 2\sum_{l\geq 0}(-1)^l \hat{f}\left(\frac{2l+1+(-1)^{l+1}\eta_1}{4\tilde{T}}\right)\frac{2}{(2l+1)\pi} \\ &+ 2\sum_{l\geq 0}(-1)^l \hat{f}\left(\frac{2l+1+(-1)^{l+1}(-2-\eta_1)}{4\tilde{T}}\right)\frac{2}{(2l+1)\pi} \\ =& 2\sum_{l\geq 0}\left(\frac{2}{(4l+1)\pi} - \frac{2}{(4l+3)\pi}\right)\left(\hat{f}\left(\frac{4l+1-\eta_1}{4\tilde{T}}\right) + \hat{f}\left(\frac{4l+3+\eta_1}{4\tilde{T}}\right)\right), \end{aligned} \tag{5.47}$$

$$\begin{aligned} S_3 + S_4 =& 2\sum_{l\geq 0}(-1)^l \hat{f}\left(\frac{2l+1+(-1)^{l+1}\eta_4}{4\tilde{T}}\right)\frac{2}{(2l+1)\pi} \\ &+ 2\sum_{l\geq 0}(-1)^l \hat{f}\left(\frac{2l+1+(-1)^{l+1}(2-\eta_4)}{4\tilde{T}}\right)\frac{2}{(2l+1)\pi} \\ =& -2\sum_{l\geq 0}\left(\frac{2}{(4l-1)\pi} - \frac{2}{(4l+1)\pi}\right)\left(\hat{f}\left(\frac{4l-1+\eta_4}{4\tilde{T}}\right) + \hat{f}\left(\frac{4l+1-\eta_4}{4\tilde{T}}\right)\right). \end{aligned} \tag{5.48}$$

It is obvious that $S_1 + S_2 \geq 0$. As a result of (5.45), we obtain that $0 \leq -1 + \eta_4 \leq 5 - \eta_4 \leq 3 + \eta_4$, and hence the inequalities

$$\hat{f}\left(-1+\eta_4\right) \geq \hat{f}\left(5-\eta_4\right) \geq \hat{f}\left(3+\eta_4\right).$$

Noting that $2\sum_{l=1}^{\infty}\left(\frac{2}{(4l-1)\pi}-\frac{2}{(4l+1)\pi}\right)=\frac{4-\pi}{\pi}<\frac{4}{\pi}$, we get that

$$\begin{aligned}
S_3+S_4 =&\frac{8}{\pi}\hat{f}\left(\frac{-1+\eta_4}{4\tilde{T}}\right) \\
&-2\sum_{l=1}^{\infty}\left(\frac{2}{(4l-1)\pi}-\frac{2}{(4l+1)\pi}\right)\left(\hat{f}\left(\frac{4l-1+\eta_4}{4\tilde{T}}\right)+\hat{f}\left(\frac{4l+1-\eta_4}{4\tilde{T}}\right)\right) \\
\geq&\frac{8}{\pi}\hat{f}\left(\frac{-1+\eta_4}{4\tilde{T}}\right)-\frac{(4-\pi)}{\pi}\left(\hat{f}\left(\frac{3+\eta_4}{4\tilde{T}}\right)+\hat{f}\left(\frac{5-\eta_4}{4\tilde{T}}\right)\right)\geq 0,
\end{aligned}$$

and hence $\sum_{k=1}^{4}S_k\geq S_3+S_4\geq 0$.

Step 3: $J=\{i_h\}$. In the end, we prove the following inequality for each $h=1,2,3,4$,

$$-\partial_t u(t,x)+\frac{1}{2}\partial^2_{x_{i_h}x_{i_h}}u(t,x)\leq 0. \tag{5.49}$$

Recalling in (5.39), we have

$$L_k=\frac{2\tilde{T}}{\pi}\left(\sum_{l\in\mathbb{Z}}(-1)^l F_k^2((2l+1)\tilde{T})+\sum_{l\in\mathbb{Z}}(-1)^l F_k^1(2l\tilde{T})\right).$$

Applying Poisson summation formula, we obtain that

$$\begin{aligned}
\sum_{l\in\mathbb{Z}}(-1)^l F_k^2((2l+1)\tilde{T}) =&2\sum_{l\in\mathbb{Z}}F_k^2((4l+1)\tilde{T})=\sum_{l\in\mathbb{Z}}\frac{1}{2\tilde{T}}\hat{F_k^2}\left(\frac{l}{4\tilde{T}}\right)e^{i\frac{\pi l}{2}} \\
=&\sum_{l\in\mathbb{Z}}\frac{1}{2\tilde{T}}\hat{F_k^2}\left(\frac{2l+1}{4\tilde{T}}\right)e^{i\frac{\pi(2l+1)}{2}} \\
=&\frac{1}{4\tilde{T}}\sum_{l\in\mathbb{Z}}(-1)^l\left(\hat{f}\left(\frac{2l+1}{4\tilde{T}}-\beta_k\right)-\hat{f}\left(\frac{2l+1}{4\tilde{T}}+\beta_k\right)\right),
\end{aligned}$$

$$
\begin{aligned}
\sum_{l\in\mathbb{Z}}(-1)^l F_k^1(2l\tilde{T}) &= -\sum_{l\in\mathbb{Z}} F_k^1(2l\tilde{T}) + 2\sum_{l\in\mathbb{Z}} F_k^1(4l\tilde{T}) \\
&= -\sum_{l\in\mathbb{Z}} \frac{1}{2\tilde{T}}\hat{F}_k^1\left(\frac{l}{2\tilde{T}}\right) + \sum_{l\in\mathbb{Z}} \frac{1}{2\tilde{T}}\hat{F}_k^1\left(\frac{l}{4\tilde{T}}\right) \\
&= \frac{1}{4\tilde{T}}\sum_{l\in\mathbb{Z}}\left(\hat{f}\left(\frac{2l+1}{4\tilde{T}}-\beta_k\right)+\hat{f}\left(\frac{2l+1}{4\tilde{T}}+\beta_k\right)\right),
\end{aligned}
$$

and therefore

$$
L_k = \frac{1}{\pi}\sum_{l\in\mathbb{Z}}\hat{f}\left(\frac{2l+1+(-1)^{l+1}4\tilde{T}\beta_k}{4\tilde{T}}\right) = \frac{2}{\pi}\sum_{l\geq 0}\hat{f}\left(\frac{2l+1+(-1)^{l+1}4\tilde{T}\beta_k}{4\tilde{T}}\right). \quad (5.50)
$$

We first prove the following three inequalities by direct computation.

$$
S_1 \leq S_2, \quad S_3 \leq S_4, \quad S_2 \leq S_4.
$$

To prove the first inequality we write

$$
\begin{aligned}
&S_2 - S_1 \\
&= \sum_{l\geq 0}(-1)^l\frac{4}{(2l+1)\pi}\left(\hat{f}\left(\frac{2l+1+(-1)^{l+1}(-2-\eta_1)}{4\tilde{T}}\right)-\hat{f}\left(\frac{2l+1+(-1)^{l+1}\eta_1}{4\tilde{T}}\right)\right) \\
&= \sum_{l\geq 0}\left(\frac{4}{(4l+1)\pi}+\frac{4}{(4l+3)\pi}\right)\left(\hat{f}\left(\frac{4l+3+\eta_1}{4\tilde{T}}\right)-\hat{f}\left(\frac{4l+1-\eta_1}{4\tilde{T}}\right)\right).
\end{aligned}
$$

As a result of $0 \leq 4l+3+\eta_1 \leq 4l+1-\eta_1$, we have for every $l \geq 0$,

$$
\hat{f}\left(\frac{4l+3+\eta_1}{4\tilde{T}}\right)-\hat{f}\left(\frac{4l+1-\eta_1}{4\tilde{T}}\right) \geq 0,
$$

and hence we conclude the first inequality.

To show the second inequality we compute

$$
\begin{aligned}
&S_4 - S_3\\
&= \sum_{l\geq 0}(-1)^l \frac{4}{(2l+1)\pi}\left(\hat{f}\left(\frac{2l+1+(-1)^{l+1}\eta_4}{4\tilde{T}}\right) - \hat{f}\left(\frac{2l+1+(-1)^{l+1}(2-\eta_4)}{4\tilde{T}}\right)\right)\\
&= \sum_{l\geq 1}\left(\frac{4}{(4l-1)\pi} + \frac{4}{(4l+1)\pi}\right)\left(\hat{f}\left(\frac{4l+1-\eta_4}{4\tilde{T}}\right) - \hat{f}\left(\frac{4l-1+\eta_4}{4\tilde{T}}\right)\right).
\end{aligned}
$$

It can be easily seen that $\hat{f}\left(\frac{4l+1-\eta_4}{4\tilde{T}}\right) - \hat{f}\left(\frac{4l-1+\eta_4}{4\tilde{T}}\right) \geq 0$ for any $l \geq 1$, and therefore we have proved the second inequality.

Finally for the third inequality we have

$$
\begin{aligned}
&S_4 - S_2\\
&= \sum_{l\geq 0}(-1)^l \frac{4}{(2l+1)\pi}\left(\hat{f}\left(\frac{2l+1+(-1)^{l+1}\eta_4}{4\tilde{T}}\right) - \hat{f}\left(\frac{2l+1+(-1)^{l+1}(-2-\eta_1)}{4\tilde{T}}\right)\right).
\end{aligned}
$$

For even $l \geq 0$, we have $|2l+1-\eta_4| \leq |2l+3+\eta_1|$, and hence

$$
\hat{f}\left(\frac{2l+1-\eta_4}{4\tilde{T}}\right) - \hat{f}\left(\frac{2l+3+\eta_1}{4\tilde{T}}\right) \geq 0,
$$

while for odd $l \geq 0$, since $|2l+1+\eta_4| \geq |2l-1-\eta_1|$, we get that

$$
\hat{f}\left(\frac{2l+1+\eta_4}{4\tilde{T}}\right) - \hat{f}\left(\frac{2l-1-\eta_1}{4\tilde{T}}\right) \leq 0.
$$

Subsequently we conclude the third inequality.

Now we prove (5.49). According to (5.36)and (5.37), we have that

$$
\partial_t u^T(t,x) + \frac{1}{2}\partial_{x_{i_h}x_{i_h}} u^T(t,x) = \frac{1}{16\sqrt{2(T-t)}}\left(2S_h - \frac{3}{4}(S_1+S_2+S_3+S_4) + L_1 - L_4\right),
$$

and therefore the inequality is equivalent to

$$L_1 - \frac{3}{4}(S_1 + S_2) \leq L_4 + \frac{3}{4}(S_3 + S_4) - 2S_h. \tag{5.51}$$

We have shown that $S_1 \leq S_2 \leq S_4, S_3 \leq S_4$. Subsequently, it is enough for us to prove the inequality for the case $h = 4$. According to (5.44) and (5.50) can be checked that

$$\begin{aligned} L_4 + \frac{3}{4}S_3 - \frac{5}{4}S_4 = \sum_{l \geq 1} \Bigg(& \left(\frac{2}{\pi} + \frac{3}{(4l+1)\pi} + \frac{5}{(4l-1)\pi} \right) \hat{f}\left(\frac{4l - 1 + \eta_4}{4\tilde{T}} \right) \\ & + \left(\frac{2}{\pi} - \frac{3}{(4l-1)\pi} - \frac{5}{(4l+1)\pi} \right) \hat{f}\left(\frac{4l + 1 - \eta_4}{4\tilde{T}} \right) \Bigg), \end{aligned}$$

$$\begin{aligned} & L_1 - \frac{3}{4}(S_1 + S_2) \\ & = \sum_{l \geq 1} \left(\frac{2}{\pi} - \frac{3}{(4l+1)\pi} + \frac{3}{(4l+3)\pi} \right) \left(\hat{f}\left(\frac{4l + 1 - \eta_1}{4\tilde{T}} \right) + \hat{f}\left(\frac{4l + 3 + \eta_1}{4\tilde{T}} \right) \right). \end{aligned}$$

Note that $0 \leq 4l + 1 - \eta_4 \leq 4l - 1 + \eta_4 \leq 4l + 3 + \eta_1 \leq 4l + 1 - \eta_1$ for any $l \geq 1$. Subsequently we have that

$$\hat{f}\left(\frac{4l + 1 - \eta_4}{4\tilde{T}} \right) \geq \hat{f}\left(\frac{4l - 1 + \eta_4}{4\tilde{T}} \right) \geq \hat{f}\left(\frac{4l + 3 + \eta_1}{4\tilde{T}} \right) \geq \hat{f}\left(\frac{4l + 1 - \eta_1}{4\tilde{T}} \right),$$

and hence the inequalities

$$\begin{aligned}
\left(\frac{2}{\pi} - \frac{3}{(4l+1)\pi} + \frac{3}{(4l+3)\pi}\right) & \left(\hat{f}\left(\frac{4l+1-\eta_1}{4\tilde{T}}\right) + \hat{f}\left(\frac{4l+3+\eta_1}{4\tilde{T}}\right)\right) \\
& \leq \left(\frac{4}{\pi} - \frac{6}{(4l+1)\pi} + \frac{6}{(4l+3)\pi}\right) \hat{f}\left(\frac{4l+3+\eta_1}{4\tilde{T}}\right) \\
& \leq \frac{4}{\pi}\hat{f}\left(\frac{4l-1+\eta_4}{4\tilde{T}}\right) \\
& \leq \left(\left(\frac{2}{\pi} + \frac{3}{(4l+1)\pi} + \frac{5}{(4l-1)\pi}\right) \hat{f}\left(\frac{4l-1+\eta_4}{4\tilde{T}}\right)\right. \\
& \quad \left. + \left(\frac{2}{\pi} - \frac{3}{(4l-1)\pi} - \frac{5}{(4l+1)\pi}\right) \hat{f}\left(\frac{4l+1-\eta_4}{4\tilde{T}}\right)\right),
\end{aligned}$$

from which we conclude that $L_1 - \frac{3}{4}(S_1 + S_2) \leq L_4 + \frac{3}{4}S_3 - \frac{5}{4}S_4$ and also the inequality (5.51). $\square$

5.4.3 Proof of Theorem 5.3.10

Proof. The dynamics of state X_m is given by

$$X_m = X_{m-1} + e_{J_m} - \mathbb{1}_{\{I_m \in J_m\}}\mathbf{1}.$$

Take any sequence $m_M \in \mathbb{N}$ and $x_{m_M} \in \mathbb{R}^4$ such that $\frac{m_M T}{M} \to t$, $\frac{x_{m_M}\sqrt{T}}{\sqrt{M}} \to x$ as $M \to \infty$. Denote $t_m = \frac{mT}{M}$, $\Delta X_m = e_{J_m} - \mathbb{1}_{\{I_m \in J_m\}}\mathbf{1}$, and define the scaled state

$$\tilde{x}_{m_M} = \frac{x_{m_M}\sqrt{T}}{\sqrt{M}}, \quad \tilde{X}_m = \frac{X_m\sqrt{T}}{\sqrt{M}}, \quad \Delta\tilde{X}_m = \frac{\Delta X_m\sqrt{T}}{\sqrt{M}}.$$

Step 1: $\overline{u}^T(t,x) \leq u^T(t,x)$. To prove the inequality, we rewrite

$$\frac{\overline{V}^M(m_M, x_{m_M})\sqrt{T}}{\sqrt{M}} - u^T\left(t,x\right) = \frac{\sup_{\beta\in\mathcal{V}} \mathbb{E}^{\alpha^*(M),\beta}\left[\Phi(X_M)|X_{m_M} = x_{m_M}\right]\sqrt{T}}{\sqrt{M}} - u^T(t,x)$$

$$= \sup_{\beta\in\mathcal{V}} \mathbb{E}^{\alpha^*(M),\beta}\left[u^T(T,\tilde{X}_M) - u^T(t_{m_M},\tilde{X}_{m_M})|\tilde{X}_{m_M} = \tilde{x}_{m_M}\right]$$

$$+ u^T(t_{m_M},\tilde{x}_{m_M}) - u^T(t,x)$$

$$= \sum_{m=m_M+1}^{M} \sup_{\beta\in\mathcal{V}} \mathbb{E}^{\alpha^*(M),\beta}\left[\left(u^T\left(t_m,\tilde{X}_m\right) - u^T\left(t_{m-1},\tilde{X}_{m-1}\right)\right)|\tilde{X}_{m_M} = \tilde{x}_{m_M}\right]$$

$$+ u^T(t_{m_M},\tilde{x}_{m_M}) - u^T(t,x).$$

Note that

$$\mathbb{E}^{\alpha,\beta}\left[u^T\left(t_m,\tilde{X}_m\right) - u^T\left(t_{m-1},\tilde{X}_{m-1}\right)|\tilde{X}_{m-1} = \tilde{x}_{m-1}\right] \tag{$\star$}$$

$$= \mathbb{E}^{\alpha,\beta}\left[\partial_x u^T\left(t_{m-1},\tilde{x}_{m-1}\right)^\top \Delta\tilde{X}_m\right] \tag{5.52}$$

$$+ 2\mathbb{E}^{\alpha,\beta}\left[\int_0^{\sqrt{\frac{T}{M}}}\left(\sqrt{\frac{T}{M}} - s\right)\left(\partial_t u^T + \frac{1}{2}e_{J_m}^\top \partial_{xx}u^T e_{J_m}\right)(t_{m-1},\tilde{x}_{m-1} + s\Delta X_m)ds\right] \tag{5.53}$$

$$+ 2\mathbb{E}^{\alpha,\beta}\left[\int_0^{\sqrt{\frac{T}{M}}}\left(\sqrt{\frac{T}{M}} - s\right)\left(\partial_t u^T(t_{m-1},\tilde{X}_m) - \partial_t u^T(t_{m-1},\tilde{x}_{m-1} + s\Delta X_m)\right)ds\right] \tag{5.54}$$

$$+ \mathbb{E}^{\alpha,\beta}\left[\int_0^{\frac{T}{M}}\left(\partial_t u^T(t_{m-1}+s,\tilde{X}_m) - \partial_t u^T(t_{m-1},\tilde{X}_m)\right)ds\right]. \tag{5.55}$$

By the definition of $\alpha^*(M)$, the player chooses expert i with probability

$\partial_{x_i} u^T(t_{m-1}, \tilde{x}_{m-1})$ at round m for all $i = 1, 2, 3, 4$. Subsequently, we have

$$\begin{aligned}
&\mathbb{E}^{\alpha^*(M),\beta}\left[\partial_x u^T\left(t_{m-1}, \tilde{x}_{m-1}\right)^\top \Delta\tilde{X}_m\right] \\
&= \mathbb{E}^\beta\left[\sum_{i=1}^4 \partial_{x_i} u^T\left(e_{J_m}^\top \partial_x u^T - \mathbb{1}_{\{i\in J_m\}}\mathbf{1}^\top \partial_x u^T\right)\right]\sqrt{\frac{T}{M}} \\
&= \mathbb{E}^\beta\left[e_{J_m}^\top \partial_x u^T - \sum_{i=1}^4 \mathbb{1}_{\{i\in J_m\}}\partial_{x_i} u^T\right]\sqrt{\frac{T}{M}} = 0,
\end{aligned}$$

where all the partial derivatives of u^T are evaluated at $(t_{m-1}, \tilde{x}_{m-1})$.

As a result of the solution property of u, the term (5.53) is non-positive. Also, it is easy to find the partial derivatives $\partial_{tt}^2 u(t,x)$ and $\partial_{tx_i}^2 u(t,x)$

$$\begin{aligned}
\partial_{tt}^2 u^T(t,x) =& \frac{-1}{16\sqrt{2}}\int_{-\infty}^{\infty} r^2 e^{-(T-t)r^2}\left(\psi\left(r\theta\cdot x^o + \frac{\pi}{2}\right)\sum_{k=1}^4 \cos\left(r\alpha_k\cdot x\right)\right. \\
&\left.-\psi\left(r\theta\cdot x^o\right)\sum_{k=1}^4 \sin\left(r\alpha_k\cdot x\right)\right) dr,
\end{aligned}$$

$$\begin{aligned}
\partial_{tx_i}^2 u^T(t,x) =& \frac{1}{16\sqrt{2}}\int_{-\infty}^{\infty} r e^{-(T-t)r^2}\sum_{k=1}^4 \alpha_{k,i}\left(\psi\left(r\theta\cdot x^o + \frac{\pi}{2}\right)\sin\left(r\alpha_k\cdot x\right)\right. \\
&\left.+\psi\left(r\theta\cdot x^o\right)\cos\left(r\alpha_k\cdot x\right)\right) dr.
\end{aligned}$$

According to the boundedness of $\psi, \sin, \cos$, we obtain that

$$\left|\partial_{tt}^2 u^T(t,x)\right| \le \frac{1}{16\sqrt{2}}\int_{-\infty}^{\infty} 8r^2 e^{-(T-t)r^2} dr = \frac{1}{2\sqrt{2(T-t)^3}}\int_{-\infty}^{\infty} r^2 e^{-r^2} dr \le \frac{C}{\sqrt{(T-t)^3}},$$

$$\left|\partial^2_{tx_i} u^T(t,x)\right| \le \frac{1}{16\sqrt{2}} \int_{-\infty}^{\infty} 6\sqrt{2} r e^{-(T-t)r^2} dr \le \frac{6}{16(T-t)} \int_{-\infty}^{\infty} r e^{-r^2} dr \le \frac{C}{T-t},$$

where C is a positive constant independent of (t,x) and is allowed to change from line to line.

Noting that the above estimation is independent of x, we can therefore estimate the bound of (5.54) and (5.55).

$$\begin{aligned}
&\left|\mathbb{E}^{\alpha,\beta}\left[2\int_0^{\sqrt{\frac{T}{M}}}\left(\sqrt{\frac{T}{M}}-s\right)\left(\partial_t u^T(t_{m-1},\tilde{X}_m)-\partial_t u^T(t_{m-1},\tilde{x}_{n-1}+s\Delta X_m)\right)ds\right]\right| \\
&\quad\le C\mathbb{E}^{\alpha,\beta}\left[\int_0^{\sqrt{\frac{T}{M}}}\left(\sqrt{\frac{T}{M}}-s\right)ds\int_s^{\sqrt{\frac{T}{M}}}\left|\partial^2_{tx}u^T\left(t_{m-1},\tilde{x}_{m-1}+u\Delta X_m\right)\right|du\right] \\
&\quad\le \frac{C}{T-t_{m-1}}\int_0^{\sqrt{\frac{T}{M}}}\left(\sqrt{\frac{T}{M}}-s\right)^2 ds = \frac{C}{(T-t_{m-1})M^{\frac{3}{2}}},
\end{aligned} \tag{5.56}$$

$$\begin{aligned}
&\left|\mathbb{E}^{\alpha,\beta}\left[\int_0^{\frac{T}{M}}\left(\partial_t u^T(t_{m-1}+s,\tilde{X}_m)-\partial_t u^T(t_{m-1},\tilde{X}_m)\right)ds\right]\right| \\
&\quad\le C\mathbb{E}^{\alpha,\beta}\left[\int_0^{\frac{T}{M}}ds\int_0^s\left|\partial^2_{tt}u^T(t_{m-1}+u,\tilde{X}_m)\right|du\right] \\
&\quad= C\mathbb{E}^{\alpha,\beta}\left[\int_0^{\frac{T}{M}}\left(\frac{T}{M}-s\right)\left|\partial^2_{tt}u^T(t_{m-1}+s,\tilde{X}_m)\right|ds\right] \\
&\quad\le C\int_0^{\frac{T}{M}}\frac{\frac{T}{M}-s}{(T-t_{m-1}-s)^{\frac{3}{2}}}ds.
\end{aligned} \tag{5.57}$$

Therefore we obtain that

$$\sup_{\beta\in\mathcal{V}}\mathbb{E}^{\alpha^*(M),\beta}\left[u^T\left(t_m,\tilde{X}_m\right)-u^T\left(t_{m-1},\tilde{X}_{m-1}\right)|\tilde{X}_{m-1}=\tilde{x}_{m-1}\right]$$
$$\leq C\left(\frac{1}{(T-t_{m-1})M^{\frac{3}{2}}}+\int_0^{\frac{T}{M}}\frac{\frac{T}{M}-s}{(T-t_{m-1}-s)^{\frac{3}{2}}}ds\right).$$

Let us estimate

$$\sum_{m=1}^{M}\frac{1}{(T-t_{m-1})M^{\frac{3}{2}}}=\frac{1}{TM^{\frac{1}{2}}}\sum_{k=1}^{M}\frac{1}{k}\leq\frac{1}{TM^{\frac{1}{2}}}\int_{1/2}^{M+1/2}\frac{1}{\lambda}d\lambda=\frac{\ln(M+1/2)-\ln(1/2)}{TM^{\frac{1}{2}}},$$

and

$$\begin{aligned}\sum_{m=1}^{M}\int_0^{\frac{T}{M}}\frac{\frac{T}{M}-s}{(T-t_{m-1}-s)^{\frac{3}{2}}}ds&=\sum_{m=1}^{M-1}\int_0^{\frac{T}{M}}\frac{\frac{T}{M}-s}{(T-t_{m-1}-s)^{\frac{3}{2}}}ds+\int_0^{\frac{T}{M}}\frac{\frac{T}{M}-s}{(\frac{T}{M}-s)^{\frac{3}{2}}}ds\\&\leq\sum_{m=1}^{M-1}\frac{T}{M}\int_0^{\frac{T}{M}}\frac{1}{(T-t_{m-1}-s)^{\frac{3}{2}}}ds+\frac{1}{2}\sqrt{\frac{T}{M}}\\&=\frac{T}{M}\int_{\frac{T}{M}}^{T}\frac{1}{s^{\frac{3}{2}}}ds+\frac{1}{2}\sqrt{\frac{T}{M}}=\frac{T}{M}\left(\sqrt{\frac{M}{T}}-\frac{1}{\sqrt{T}}\right)+\frac{1}{2}\sqrt{\frac{T}{M}}.\end{aligned}$$

Thus we conclude that

$$\begin{aligned}&\lim_{M\to\infty}\sum_{m=1}^{M}\left(\frac{1}{(T-t_{m-1})M^{\frac{3}{2}}}+\int_0^{\frac{T}{M}}\frac{\frac{T}{M}-s}{(T-t_{m-1}-s)^{\frac{3}{2}}}ds\right)\\&\leq\lim_{M\to\infty}\left(\frac{T}{M}\left(\sqrt{\frac{M}{T}}-\frac{1}{\sqrt{T}}\right)+\frac{1}{2}\sqrt{\frac{T}{M}}+\frac{\ln(M+1/2)-\ln(1/2)}{TM^{\frac{1}{2}}}\right)=0,\end{aligned}\quad(5.58)$$

and furthermore

$$\overline{u}^T(t,x) - u^T(t,x) = \limsup_{\left(M, \frac{m_M T}{M}, \frac{x_{m_M}\sqrt{T}}{\sqrt{M}}\right) \to (\infty, t, x)} \left(\frac{\overline{V}^M(m_M, x_{m_M})\sqrt{T}}{\sqrt{M}} - u^T(t,x) \right)$$

$$\leq \limsup_{\left(M, \frac{m_M T}{M}, \frac{x_{m_M}\sqrt{T}}{\sqrt{M}}\right) \to (\infty, t, x)} \sum_{m=m_M+1}^{M} C \left(\frac{1}{(T - t_{m-1})M^{\frac{3}{2}}} + \int_0^{\frac{T}{M}} \frac{\frac{T}{M} - s}{(T - t_{m-1} - s)^{\frac{3}{2}}} ds \right)$$

$$+ \limsup_{\left(M, \frac{m_M T}{M}, \frac{x_{m_M}\sqrt{T}}{\sqrt{M}}\right) \to (\infty, t, x)} \left(u^T(t_{m_M}, \tilde{x}_{m_M}) - u^T(t,x) \right) = 0.$$

Step 2: $\underline{u}^T(t,x) \geq u^T(t,x)$. Similarly, we have

$$\frac{\underline{V}^M(m_M, x_{m_M})\sqrt{T}}{\sqrt{M}} - u^T(t,x)$$
$$= \sum_{m=m_M+1}^{M} \inf_{\alpha \in \mathcal{U}} \mathbb{E}^{\alpha, \mathcal{J}^b_{\mathcal{C}}(M)} \left[\left(u^T\left(t_m, \tilde{X}_m\right) - u^T\left(t_{m-1}, \tilde{X}_{m-1}\right) \right) | \tilde{X}_{m_M} = \tilde{x}_{m_M} \right]$$
$$+ u^T(t_{m_M}, \tilde{x}_{m_M}) - u^T(t,x),$$

and we need to estimate the conditional expectation $(\star)$. At round m, the adversary chooses experts $\mathcal{J}_{\mathcal{C}}(\tilde{x}_{m-1})$ with probability $\frac{1}{2}$, and $\mathcal{J}^c_{\mathcal{C}}(\tilde{x}_{m-1})$ with probability $\frac{1}{2}$. Therefore we compute

$$\mathbb{E}^{\alpha, \mathcal{J}^b_{\mathcal{C}}(M)} \left[\partial_x u^T\left(t_{m-1}, \tilde{x}_{m-1}\right)^\top \Delta \tilde{X}_m \right]$$
$$= \mathbb{E}^{\alpha} \left[\frac{1}{2} \left(e^\top_{\mathcal{J}_{\mathcal{C}}} \partial_x u^T - \mathbb{1}_{\{I_m \in \mathcal{J}_{\mathcal{C}}\}} \mathbf{1}^\top \partial_x u^T \right) + \frac{1}{2} \left(e^\top_{\mathcal{J}^c_{\mathcal{C}}} \partial_x u^T - \mathbb{1}_{\{I_m \in \mathcal{J}^c_{\mathcal{C}}\}} \mathbf{1}^\top \partial_x u^T \right) \right] \sqrt{\frac{T}{M}}$$
$$= \mathbb{E}^{\alpha} \left[\frac{1}{2} \left(\mathbf{1}^\top \partial_x u^T - \mathbf{1}^\top \partial_x u^T \right) \right] = 0. \tag{5.59}$$

Since the bounds of (5.54) and (5.55) are the same, it remains to find the lower bound of (5.53) when the adversary adopts the comb strategy. We show that if

$J_m = \mathcal{J}_{\mathcal{C}}(\tilde{x}_{m-1})$, then the following inequality holds

$$\left(\partial_t u^T + \frac{1}{2} e_{J_m}^\top \partial_{xx} u^T e_{J_m}\right)(t_{m-1}, \tilde{x}_{m-1} + s\Delta X_m) \geq -\frac{Cs}{T - t_{m-1}}, \tag{5.60}$$

where C is a positive constant independent of $\tilde{x}_{m-1}$ and is allowed to change from line to line. The proof for the case $J_m = \mathcal{J}_{\mathcal{C}}^c(\tilde{x}_{m-1})$ is the same. To simplify the notation, in the following argument, we denote $\mathcal{J} = \mathcal{J}_{\mathcal{C}}(\tilde{x}_{m-1})$ and $\tilde{x}_s = \tilde{x}_{m-1} + s\Delta X_m$.

Note that if $\tilde{x}_{m-1}^{(3)} \geq \tilde{x}_{m-1}^{(2)} + \sqrt{\frac{T}{M}}$, then according to Subsection 5.4.1.4, we have that for any $s \in \left[0, \sqrt{\frac{T}{M}}\right]$

$$\left(\partial_t u^T + \frac{1}{2} e_{\mathcal{J}}^\top \partial_{xx} u^T e_{\mathcal{J}}\right)(t_{m-1}, \tilde{x}_s) = 0, \tag{5.61}$$

which satisfies (5.60). Otherwise there exists a unique $s_0 \in \left[0, \sqrt{\frac{T}{M}}\right]$ such that $\tilde{x}_{s_0}^{(2)} = \tilde{x}_{s_0}^{(3)}$, i.e., $\tilde{x}_{m-1}^{(3)} = \tilde{x}_{m-1}^{(2)} + s_0$. Then for $s \in [0, s_0]$, we still have (5.61), but for $s \in \left[s_0, \sqrt{\frac{T}{M}}\right]$, according to the definition of $\mathcal{J}$, the adversary actually selects the first two leading experts. Recall (5.41),

$$\left(\partial_t u^T + \frac{1}{2} e_{\mathcal{J}}^\top \partial_{xx} u^T e_{\mathcal{J}}\right)(t_{m-1}, \tilde{x}_s) = \frac{\theta_3(\nu_s, \hat{\tau}_s) - \theta_3(\mu_s, \hat{\tau}_s)}{4\sqrt{2}\theta \cdot \tilde{x}_s^o}, \tag{5.62}$$

where

$$\mu_s := \frac{\pi\alpha_1 \cdot \tilde{x}_s^o}{4\theta \cdot \tilde{x}_s^o} + \frac{\pi}{4}, \quad \nu_s := \frac{\pi\alpha_4 \cdot \tilde{x}_s^o}{4\theta \cdot \tilde{x}_s^o} + \frac{\pi}{4}, \quad \hat{\tau}_s := \frac{i\pi(T - t_{m-1})}{4\left(\theta \cdot \tilde{x}_s^o\right)^2}.$$

Since $\tilde{x}_{s_0}^{(2)} = \tilde{x}_{s_0}^{(2)}$, it can be easily checked that $\mu_{s_0} - \nu_{s_0} = \pi$. According to the definition of Jacobi-theta function, $\theta_3(z + \pi, \tau) = \theta_3(z, \tau)$ and hence $\theta_3(\mu_{s_0}, \hat{\tau}_{s_0}) = \theta_3(\nu_{s_0}, \hat{\tau}_{s_0})$. Let us calculate $\mu_s - \nu_s$ for $s \geq s_0$,

$$\mu_s - \nu_s = \frac{\pi}{4}\left(\frac{\alpha_1 \cdot \tilde{x}_{s_0}^o - 2(s - s_0)}{\theta \cdot \tilde{x}_s^o} - \frac{\alpha_4 \cdot \tilde{x}_{s_0}^o + 2(s - s_0)}{\theta \cdot \tilde{x}_s^o}\right) = \pi - \frac{\pi(s - s_0)}{\theta \cdot \tilde{x}_s^o}.$$

Then we have the estimation

$$
\begin{aligned}
&\left|\theta_3(\nu_s,\hat{\tau}_s)-\theta_3(\mu_s,\hat{\tau}_s)\right| = \left|\theta_3\left(\mu_s+\frac{\pi(s-s_0)}{\theta\cdot\tilde{x}_s^o},\hat{\tau}_s\right)-\theta_3(\mu_s,\hat{\tau}_s)\right| \\
&= \left|\sum_{n=-\infty}^{\infty}\exp\left(i\pi\hat{\tau}_s n^2\right)\left(\cos\left(2n\left(\mu_s+\frac{\pi(s-s_0)}{\theta\cdot\tilde{x}_s^o}\right)\right)-\cos(2n\mu_s)\right)\right| \\
&\leq \left|\sum_{n=-\infty}^{\infty}\exp\left(i\pi\hat{\tau}_s n^2\right)\frac{2n\pi(s-s_0)}{\theta\cdot\tilde{x}_s^o}\right|. \qquad (5.63)
\end{aligned}
$$

To finish proofing (5.60), we need an auxiliary result

$$
\sup_{\lambda>0}\left(\sum_{n=1}^{\infty}n\lambda e^{-n^2\lambda}\right)=C<+\infty. \qquad (5.64)
$$

According to the inequality,

$$
\sum_{n=1}^{\infty}n\lambda e^{-n^2\lambda}\leq\sum_{n=1}^{\infty}n\lambda e^{-n\lambda}=\frac{\lambda e^{\lambda}}{(e^{\lambda}-1)^2},
$$

we conclude that $\lim\limits_{\lambda\to\infty}\sum\limits_{n=1}^{\infty}n\lambda e^{-n^2\lambda}=0$ and $\lambda\mapsto\sum\limits_{n=1}^{\infty}n\lambda e^{-n^2\lambda}$ is continuous over $\mathbb{R}_{>0}$. It remains to show that $\limsup\limits_{\lambda\to 0}\sum\limits_{n=1}^{\infty}n\lambda e^{-n^2\lambda}<\infty$. Fix $\lambda>0$, we can view $\sum\limits_{n=1}^{\infty}n\lambda e^{-n^2\lambda}$ as the Riemann sum of the integral $\int_1^{\infty}t\lambda e^{-t^2\lambda}dt$. It can be easily seen that $t\mapsto t\lambda e^{-t^2\lambda}$ is increasing over $\left[0,\frac{1}{\sqrt{2\lambda}}\right]$ and decreasing over $\left[\frac{1}{\sqrt{2\lambda}},\infty\right]$. Take $I(\lambda)$ to be largest integer that is smaller than or equal to $\frac{1}{\sqrt{2\lambda}}$. Then we obtain that

$$
\begin{aligned}
\sum_{n=1}^{I(\lambda)-1}n\lambda e^{-n^2\lambda}&\leq\int_0^{I(\lambda)}t\lambda e^{-t^2\lambda}dt,\\
\sum_{I(\lambda)+2}^{\infty}n\lambda e^{-n^2\lambda}&\leq\int_{I(\lambda)+1}^{\infty}t\lambda e^{-t^2\lambda}dt,
\end{aligned}
$$

and therefore

$$\sum_{n=1}^{\infty} n\lambda e^{-n^2\lambda} \leq I(\lambda)\lambda e^{-I(\lambda)^2\lambda} + (I(\lambda)+1)\lambda e^{-(I(\lambda)+1)^2\lambda} + \int_1^{\infty} t\lambda e^{-t^2\lambda} dt$$
$$\leq I(\lambda)\lambda e^{-I(\lambda)^2\lambda} + (I(\lambda)+1)\lambda e^{-(I(\lambda)+1)^2\lambda} + \frac{1}{2}e^{-\lambda}.$$

As a result of $I(x) = \left\lfloor \frac{1}{\sqrt{2\lambda}} \right\rfloor$, we conclude that

$$\limsup_{\lambda\to 0} \sum_{n=1}^{\infty} n\lambda e^{-n^2\lambda} \leq 2\lim_{\lambda\to 0} \frac{\lambda}{\sqrt{2\lambda}} e^{-\frac{1}{2}} + \lim_{\lambda\to 0} \frac{1}{2}e^{-\lambda} = \frac{1}{2}.$$

Taking $\lambda = \frac{\pi^2(T-t_{m-1})}{4(\theta\cdot\tilde{x}_s^o)^2}$ in (5.64), and combining (5.62),(5.63), (5.64), we obtain that

$$\left(\partial_t u^T + \frac{1}{2}e_{\mathcal{J}}^{\top}\partial_{xx}u^T e_{\mathcal{J}}\right)(t_{m-1},\tilde{x}_s) = \frac{\theta_3(\nu_s,\hat{\tau}_s) - \theta_3(\mu_s,\hat{\tau}_s)}{4\sqrt{2}\theta\cdot\tilde{x}_s^o}$$
$$\geq -\sum_{n=-\infty}^{\infty} \exp\left(i\pi\hat{\tau}_s n^2\right) \frac{2n\pi(s-s_0)}{(\theta\cdot\tilde{x}_s^o)^2} \geq -\frac{C(s-s_0)}{T-t_{m-1}} \sum_{n=1}^{\infty} n\lambda e^{-n^2\lambda} \geq -\frac{Cs}{T-t_{m-1}},$$

and hence

$$\mathbb{E}^{\alpha,\mathcal{J}}\left[\int_0^{\sqrt{\frac{T}{M}}} \left(\sqrt{\frac{T}{M}} - s\right)\left(\partial_t u^T + \frac{1}{2}e_{J_m}^{\top}\partial_{xx}u^T e_{J_m}\right)(t_{m-1},\tilde{x}_{m-1} + s\Delta X_m)ds\right]$$
$$\geq -\frac{C}{T-t_{m-1}} \int_0^{\sqrt{\frac{T}{M}}} \left(\sqrt{\frac{T}{M}} - s\right) sds \geq \frac{-C}{(T-t_{m-1})M^{3/2}}.$$

In conjunction with (5.56), (5.57), (5.58) and (5.59), we obtain that

$$\inf_{\alpha\in\mathcal{U}} \mathbb{E}^{\alpha,\mathcal{J}^b_{\mathcal{C}}(M)}\left[u^T\left(t_m,\tilde{X}_m\right)-u^T\left(t_{m-1},\tilde{X}_{m-1}\right)|\tilde{X}_{m-1}=\tilde{x}_{m-1}\right]$$
$$\geq -C\left(\frac{1}{(T-t_{m-1})M^{\frac{3}{2}}}+\int_0^{\frac{T}{M}}\frac{\frac{T}{M}-s}{(T-t_{m-1}-s)^{\frac{3}{2}}}ds\right),$$

and finally

$$\underline{u}^T(t,x)-u^T(t,x)=\liminf_{\left(M,\frac{m_MT}{M},\frac{x_{m_M}\sqrt{T}}{\sqrt{M}}\right)\to(\infty,t,x)}\left(\frac{\underline{V}^M(m_M,x_{m_M})\sqrt{T}}{\sqrt{M}}-u^T(t,x)\right)$$
$$\geq \liminf_{\left(M,\frac{m_MT}{M},\frac{x_{m_M}\sqrt{T}}{\sqrt{M}}\right)\to(\infty,t,x)}$$
$$-\sum_{m=m_M+1}^{M}C\left(\frac{1}{(T-t_{m-1})M^{\frac{3}{2}}}+\int_0^{\frac{T}{M}}\frac{\frac{T}{M}-s}{(T-t_{m-1}-s)^{\frac{3}{2}}}ds\right)$$
$$+\liminf_{\left(M,\frac{m_MT}{M},\frac{x_{m_M}\sqrt{T}}{\sqrt{M}}\right)\to(\infty,t,x)}\left(u^T(t_{m_M},\tilde{x}_{m_M})-u^T(t,x)\right)=0.$$

□

5.4.4 Proof of Proposition 5.3.12

Proof. The estimates in [94, Theorem 4] allows us to claim that there exists a constant $C>0$ so that for all $T>0$ and $(t,x)\in[0,T]\times\mathbb{R}^N$,

$$|u^T(t,x)|\leq C(T-t+1+|x|).$$

Thus, the function v defined by the expression (5.16)

$$v(x) := \mathbb{E}\left[\int_0^\infty e^{-T}\Phi(X_T^{0,x})dT\right] = \int_0^\infty e^{-T}u^T(0,x)dT.$$

has at most linear growth and due to (5.15) it is C^2. The optimality of comb strategies implies that for all $T > 0$ and $(t,x) \in [0,T) \times \mathbb{R}^N$,

$$\frac{1}{2}\sup_{J\in P(N)} e_J^\top \partial_{xx}^2 u^T(t,x)e_J = \frac{1}{2}e_{\mathcal{J}_\mathcal{C}(x)}^\top \partial_{xx}^2 u^T(t,x)e_{\mathcal{J}_\mathcal{C}(x)} = -\partial_t u^T(t,x).$$

Equation (5.15) and the optimality of comb strategies imply that

$$\frac{1}{2}\sup_{J\in P(N)} e_J^\top \partial_{xx}^2 v(x)e_J \geq \frac{1}{2}e_{\mathcal{J}_\mathcal{C}(x)}^\top \partial_{xx}^2 v e_{\mathcal{J}_\mathcal{C}(x)} \geq \int_0^\infty e^{-T}\frac{1}{2}e_{\mathcal{J}_\mathcal{C}(x)}^\top \partial_{xx}^2 u^T(0,x)e_{\mathcal{J}_\mathcal{C}(x)}dT$$
$$= \int_0^\infty e^{-T}\frac{1}{2}\sup_{J\in P(N)} e_J^\top \partial_{xx}^2 u^T(0,x)e_J dT \geq \frac{1}{2}\sup_{J\in P(N)} e_J^\top \partial_{xx}^2 v(x)e_J.$$

Thus, using the fact that for some function $u^\sharp$, $u^T(t,x) = u^\sharp(T-t,x)$ for all $T > 0$ and $(t,x) \in [0,T] \times \mathbb{R}^N$, all the inequalities above are equalities and

$$\frac{1}{2}\sup_{J\in P(N)} e_J^\top \partial_{xx}^2 v(x)e_J = -\int_0^\infty e^{-T}\partial_t u^T(0,x)dT = \int_0^\infty e^{-T}u^T(0,x)dT - u^0(0,x)$$
$$= v(x) - \Phi(x).$$

Given the uniqueness of viscosity solution with linear growth for (5.13) proven in [86, Theorem 5.1] $v = u$ and comb strategies are indeed optimal for the problem (5.13). □

5.5 Solutions of (5.1) from Inverse Laplace Transform

5.5.1 A heuristic derivation for $N = 4$

We derive the solution of (5.1) when there are 4 experts. From [28, Proposition 6.1], the solution of the linear PDE

$$u(x) - \frac{1}{2} e^{\top}_{\mathcal{J}_{\mathcal{C}}(x)} \partial^2 u(x) e_{\mathcal{J}_{\mathcal{C}}(x)} = \Phi(x), \tag{5.65}$$

is given by

$$\begin{aligned} u(x) =& x^{(4)} - \frac{\sqrt{2}}{4} \sinh(\sqrt{2}(x^{(4)} - x^{(3)})) + \frac{1}{4\sqrt{2}} \arctan\left(e^{\theta \cdot x^o}\right) \sum_{k=1}^{4} \cosh\left(\alpha_k \cdot x^o\right) \\ &+ \frac{1}{4\sqrt{2}} \operatorname{arctanh}\left(e^{\theta \cdot x^o}\right) \sum_{k=1}^{4} \sinh\left(\alpha_k \cdot x^o\right). \end{aligned}$$

It is well-known that an elliptic PDE can be solved by applying the Laplace transform to the corresponding parabolic one. Here, to obtain the solution to (5.7), we formally compute the inverse Laplace transform of (5.65). It can be easily checked that for $\lambda \in \mathbb{R}_+$

$$u^{\lambda}(x) = \lambda^{-3/2} u(\sqrt{\lambda} x)$$

solves the equation

$$\lambda u^{\lambda}(x) - \frac{1}{2} e^{\top}_{\mathcal{J}_{\mathcal{C}}(x)} \partial_{xx} u^{\lambda}(x) e_{\mathcal{J}_{\mathcal{C}}(x)} = \Phi(x).$$

We formally extend the function $\lambda \mapsto u^{\lambda}(x)$ to the complex plane with $\mathbb{R}_-$ as its branch cut. Applying the inverse Laplace transform for $t \in \mathbb{R}_+$,

$$u^{\#}(t, x) = \frac{1}{2\pi i} \int_{x_0 - i\infty}^{x_0 + i\infty} e^{t\lambda} u^{\lambda}(x) d\lambda, \tag{5.66}$$

should solve the PDE, at least heuristically,

$$\partial_t u^{\#}(t,x) - \frac{1}{2} e_{\mathcal{J}_{\mathcal{C}}(x)}^{\top} \partial_{xx}^2 u^{\#}(t,x) e_{\mathcal{J}_{\mathcal{C}}(x)} = 0,$$
$$u^{\#}(0,x) = \Phi(x),$$

where x_0 is chosen so that the function to integrate is analytic on the line of integration. Now the solution of (5.7) is given by

$$u^T(t,x) = u^{\#}(T-t,x).$$

Let us compute (5.66). Since the functions arctan, arctanh can be extended to the complex plane via the formulas,

$$\arctan(z) = \frac{1}{2i}\log\left(\frac{i-z}{i+z}\right), \quad \operatorname{arctanh}(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),$$

we obtain that

$$\begin{aligned} u(x) =& x^{(4)} - \frac{\sqrt{2}}{4}\sinh(\sqrt{2}(x^{(4)} - x^{(3)})) + \frac{1}{8i\sqrt{2}}\log\left(\frac{i - e^{\theta\cdot x^o}}{i + e^{\theta\cdot x^o}}\right)\sum_{k=1}^{4}\cosh\left(\alpha_k \cdot x^o\right) \\ &+ \frac{1}{8\sqrt{2}}\log\left(\frac{1+e^{\theta\cdot x^o}}{1-e^{\theta\cdot x^o}}\right)\sum_{k=1}^{4}\sinh\left(\alpha_k\cdot x^o\right). \end{aligned} \tag{5.67}$$

To cancel the singularity at $\lambda = 0$, we rewrite

$$\begin{aligned} u^{\#}(t,x) &= \frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty} e^{t\lambda}\frac{u(\sqrt{\lambda}x)-u(0)}{\lambda^{3/2}}d\lambda + \frac{u(0)}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty} e^{t\lambda}\frac{1}{\lambda^{3/2}}d\lambda \\ &= \frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty} e^{t\lambda}\frac{u(\sqrt{\lambda}x)-u(0)}{\lambda^{3/2}}d\lambda + \frac{1}{2}\sqrt{\frac{t\pi}{2}}, \end{aligned} \tag{5.68}$$

where we use the facts that $u(0) = \frac{\pi}{4\sqrt{2}}$, and the inverse Laplace transform of $\frac{1}{\lambda^{3/2}}$

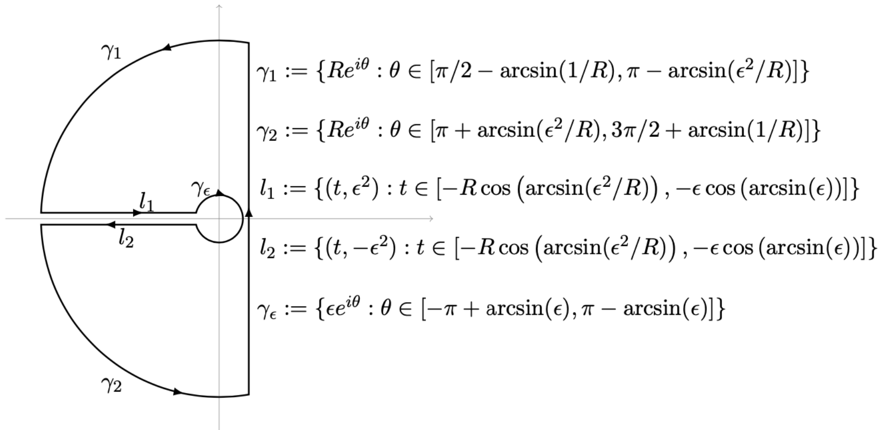


Figure 5.3: Contour of inverse Laplace transform

is $\frac{2\sqrt{t}}{\sqrt{\pi}}$. Take $x_0 = 1$, $\epsilon > 0$, $R > 0$, and the contour in Figure 5.3. The integral of $e^{t\lambda}(u(\sqrt{\lambda}x) - u(0))/\lambda^{\frac{3}{2}}$ along the contour is zero. Letting $R \to \infty$, $\epsilon \to 0$, and assuming that the limit of the integral along γ_1, γ_2 vanish, we obtain that

$$\frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty} e^{t\lambda}\frac{u(\sqrt{\lambda}x) - u(0)}{\lambda^{3/2}}d\lambda = -\lim_{(R,\epsilon)\to(\infty,0)}\frac{1}{2\pi i}\int_{\gamma_\epsilon+l_1+l_2} e^{t\lambda}\frac{u(\sqrt{\lambda}x) - u(0)}{\lambda^{3/2}}d\lambda.$$

It can be seen that

$$\begin{aligned}&\lim_{(R,\epsilon)\to(\infty,0)}\frac{1}{2\pi i}\int_{l_1+l_2} e^{t\lambda}\frac{u(\sqrt{\lambda}x) - u(0)}{\lambda^{3/2}}d\lambda \\ &\quad= \frac{1}{2\pi i}\int_{-\infty}^{0} e^{t\lambda}\frac{u(\sqrt{\lambda}x) - u(0)}{\lambda^{\frac{3}{2}}}d\lambda + \int_{0}^{-\infty} e^{t\lambda}\frac{u(\sqrt{\lambda}x) - u(0)}{\lambda^{\frac{3}{2}}}d\lambda, \end{aligned} \tag{5.69}$$

where the first integral is above the branch $\mathbb{R}_-$ and the second below. Thus the computation reduces to

$$\frac{1}{2\pi}\int_0^\infty e^{-tr}\frac{u(i\sqrt{r}x)+u(-i\sqrt{r}x)-2u(0)}{r^{\frac{3}{2}}}dr =$$

$$\frac{1}{16\sqrt{2}}\int_0^\infty \frac{e^{-tr}}{r^{3/2}}\left(\frac{1}{i\pi}\left(\log\left(\frac{i-e^{i\sqrt{r}\theta\cdot x^o}}{i+e^{i\sqrt{r}\theta\cdot x^o}}\right)+\log\left(\frac{i-e^{-i\sqrt{r}\theta\cdot x^o}}{i+e^{-i\sqrt{r}\theta\cdot x^o}}\right)\right)\sum_{k=1}^4\cos\left(\sqrt{r}\alpha_k\cdot x^o\right)-4\right.$$
$$\left.-\frac{1}{i\pi}\left(\log\left(\frac{1+e^{i\sqrt{r}\theta\cdot x^o}}{1-e^{i\sqrt{r}\theta\cdot x^o}}\right)-\log\left(\frac{1+e^{-i\sqrt{r}\theta\cdot x^o}}{1-e^{-i\sqrt{r}\theta\cdot x^o}}\right)\right)\sum_{k=1}^4\sin\left(\sqrt{r}\alpha_k\cdot x^o\right)\right)dr.$$

For some values of r depending on x, the first two log are respectively $\mp\infty$. But heuristically they cancel each other. Due to the factorizations

$$\frac{i-e^{i\sqrt{r}\theta\cdot x^o}}{i+e^{i\sqrt{r}\theta\cdot x^o}}=\frac{e^{i\left(\frac{\pi}{4}-\frac{\sqrt{r}\theta\cdot x^o}{2}\right)}-e^{i\left(-\frac{\pi}{4}+\frac{\sqrt{r}\theta\cdot x^o}{2}\right)}}{e^{i\left(\frac{\pi}{4}-\frac{\sqrt{r}\theta\cdot x^o}{2}\right)}+e^{i\left(-\frac{\pi}{4}+\frac{\sqrt{r}\theta\cdot x^o}{2}\right)}}=i\tan\left(\frac{\pi}{4}-\frac{\sqrt{r}\theta\cdot x^o}{2}\right),$$
$$\frac{1+e^{i\sqrt{r}\theta\cdot x^o}}{1-e^{i\sqrt{r}\theta\cdot x^o}}=\frac{e^{i\left(-\frac{\sqrt{r}\theta\cdot x^o}{2}\right)}+e^{i\left(\frac{\sqrt{r}\theta\cdot x^o}{2}\right)}}{e^{i\left(-\frac{\sqrt{r}\theta\cdot x^o}{2}\right)}-e^{i\left(\frac{\sqrt{r}\theta\cdot x^o}{2}\right)}}=\frac{1}{-i\tan\left(\frac{\sqrt{r}\theta\cdot x^o}{2}\right)},$$

and the identities

$$\log\left(\frac{i}{\tan\left(\frac{\sqrt{r}\theta\cdot x^o}{2}\right)}\right)-\log\left(\frac{-i}{\tan\left(\frac{\sqrt{r}\theta\cdot x^o}{2}\right)}\right)=i\pi sign\left(\tan\left(\frac{\sqrt{r}\theta\cdot x^o}{2}\right)\right),$$
$$\log\left(i\tan\left(\frac{\pi}{4}-\frac{\sqrt{r}\theta\cdot x^o}{2}\right)\right)+\log\left(i\tan\left(\frac{\pi}{4}+\frac{\sqrt{r}\theta\cdot x^o}{2}\right)\right)$$
$$=i\pi sign\left(\tan\left(\frac{\pi}{4}+\frac{\sqrt{r}\theta\cdot x^o}{2}\right)\right),$$

it can be checked that the integral (5.69) becomes

$$\frac{1}{16\sqrt{2}}\int_0^\infty \frac{e^{-tr}}{r^{3/2}}\left(sign\left(\tan\left(\frac{\pi}{4}+\frac{\sqrt{r}\theta\cdot x^o}{2}\right)\right)\sum_{k=1}^4\cos\left(\sqrt{r}\alpha_k\cdot x^o\right)-4\right.$$
$$\left.-sign\left(\tan\left(\frac{\sqrt{r}\theta\cdot x^o}{2}\right)\right)\sum_{k=1}^4\sin\left(\sqrt{r}\alpha_k\cdot x^o\right)\right)dr. \tag{5.70}$$

According to [28, Equation (3.4)], we have $\lim\limits_{\lambda\to 0}\frac{u(\lambda x)-u(0)}{\lambda}=\frac{1}{4}(x_1+x_2+x_3+x_4)$, and therefore

$$\begin{aligned}\frac{1}{2\pi i}\lim_{\epsilon\to 0}\int_{\gamma_\epsilon} e^{t\lambda}\frac{u(\sqrt{\lambda}x)-u(0)}{\lambda^{\frac{3}{2}}}d\lambda =&\frac{-1}{2\pi i}\lim_{\epsilon\to 0}\int_0^{2\pi} e^{t\epsilon e^{i\theta}}\frac{u(\sqrt{\epsilon e^{i\theta}}x)-u(0)}{\epsilon e^{i\theta\frac{1}{2}}}id\theta\\ =&-\frac{1}{4}(x_1+x_2+x_3+x_4).\end{aligned}$$

In conjunction with (5.68) and (5.70), we get that

$$\begin{aligned}u^{\#}(t,x)=&\frac{-1}{16\sqrt{2}}\int_0^\infty \frac{e^{-tr}}{r^{3/2}}\left(sign\left(\tan\left(\frac{\pi}{4}+\frac{\sqrt{r}\theta\cdot x^o}{2}\right)\right)\sum_{k=1}^4\cos\left(\sqrt{r}\alpha_k\cdot x^o\right)-4\right.\\ &\left.-sign\left(\tan\left(\frac{\sqrt{r}\theta\cdot x^o}{2}\right)\right)\sum_{k=1}^4\sin\left(\sqrt{r}\alpha_k\cdot x^o\right)\right)dr+\frac{1}{4}\sum_{i=1}^4 x_i+\frac{1}{2}\sqrt{\frac{t\pi}{2}}\\ =&\frac{-1}{8\sqrt{2}}\int_0^\infty \frac{e^{-tr^2}}{r^2}\left(sign\left(\tan\left(\frac{\pi}{4}+\frac{r\theta\cdot x^o}{2}\right)\right)\sum_{k=1}^4\cos\left(r\alpha_k\cdot x^o\right)-4\right.\\ &\left.-sign\left(\tan\left(\frac{r\theta\cdot x^o}{2}\right)\right)\sum_{k=1}^4\sin\left(r\alpha_k\cdot x^o\right)\right)dr+\frac{1}{4}\sum_{i=1}^4 x_i+\frac{1}{2}\sqrt{\frac{t\pi}{2}},\end{aligned}$$

where the last equality follows from the change of variable. Since $u^T(t,x)=u^{\#}(T-t,x)$, we obtain (5.2).

5.5.2 Explicit expressions for $N = 3$

According to [94, Theorem 8], the value function in the geometric stopping case is given by

$$u(x) = x^{(3)} + \frac{1}{2\sqrt{2}} e^{\sqrt{2}(x^{(2)} - x^{(3)})} + \frac{1}{6\sqrt{2}} e^{\sqrt{2}(2x^{(1)} - x^{(2)} - x^{(3)})}, \tag{5.71}$$

which solved (5.13) with $N = 3$. We compute the inverse Laplace transform

$$u^{\#}(t,x) = \frac{1}{2\pi i} \int_{x_0 - i\infty}^{x_0 + i\infty} e^{t\lambda} \lambda^{-3/2} u(\sqrt{\lambda} x) d\lambda,$$

where we extend the function $\lambda \mapsto u(\sqrt{\lambda}x)/\lambda^{\frac{3}{2}}$ naturally to $\mathbb{C} \setminus \mathbb{R}_-$. The inverse Laplace transform of $\frac{1}{s}$, $\frac{1}{\sqrt{s}}$ and $\frac{e^{-a\sqrt{s}}}{s}$ are 1, $\frac{1}{\sqrt{\pi t}}$ and $erfc\left(\frac{a}{2\sqrt{t}}\right)$ respectively, where $erfc$ is the complementary error function (see e.g. [4]). Subsequently according to the convolution theorem, it can be easily checked that

$$\begin{aligned} u^{\#}(t,x) = {} & x^{(2)} + \frac{1}{3}(2x^{(1)} - x^{(2)} - x^{(3)}) + \frac{\sqrt{t} e^{\frac{-(2x^{(1)} - x^{(2)} - x^{(3)})^2}{2t}}}{3\sqrt{2\pi}} + \frac{\sqrt{t} e^{\sqrt{t}\frac{-(x^{(2)} - x^{(3)})^2}{2t}}}{\sqrt{2\pi}} \\ & - \frac{1}{3\sqrt{\pi}}(2x^{(1)} - x^{(2)} - x^{(3)}) \int_{\frac{2x^{(1)} - x^{(2)} - x^{(3)}}{\sqrt{2t}}}^{\infty} e^{-y^2} dy - \frac{1}{\sqrt{\pi}}(x^{(2)} - x^{(3)}) \int_{\frac{x^{(2)} - x^{(3)}}{\sqrt{2t}}}^{\infty} e^{-y^2} dy. \end{aligned}$$

Then $u^T(t,x) := u^{\#}(T - t, x)$ is our conjectured solution to (5.1) with $N = 3$.

Proposition 5.5.1. *The explicit solution to Equation* (5.1) *with* $N = 3$ *is given by*

$$
\begin{aligned}
u^T(t,x) :=\ & x^{(2)} + \frac{1}{3}(2x^{(1)} - x^{(2)} - x^{(3)}) + \frac{\sqrt{T-t}e^{\frac{-(2x^{(1)}-x^{(2)}-x^{(3)})^2}{2(T-t)}}}{3\sqrt{2\pi}} \\
& + \frac{\sqrt{T-t}e^{\sqrt{T-t}\frac{-(x^{(2)}-x^{(3)})^2}{2(T-t)}}}{\sqrt{2\pi}} - \frac{1}{3\sqrt{\pi}}(2x^{(1)} - x^{(2)} - x^{(3)}) \int\limits_{\frac{2x^{(1)}-x^{(2)}-x^{(3)}}{\sqrt{2(T-t)}}}^{\infty} e^{-y^2} dy \\
& - \frac{1}{\sqrt{\pi}}(x^{(2)} - x^{(3)}) \int\limits_{\frac{x^{(2)}-x^{(3)}}{\sqrt{2(T-t)}}}^{\infty} e^{-y^2} dy. \qquad (5.72)
\end{aligned}
$$

Proof. The proof follows from straightforward computations and is left to the reader. $\square$

CHAPTER VI

Malicious Experts Versus the Multiplicative Weights Algorithm in Online Prediction

6.1 Introduction

Prediction with expert advice is classical and fundamental in the field of *online learning*, and we refer the reader to [69] for a nice survey. In this problem, a forecaster makes predictions based on advices of experts so as to minimize his loss, i.e., the cumulative difference between his predictions and true outcomes. A standard performance criterion is the regret: the difference between the loss of the forecaster and the minimum among losses of all experts. The prediction problem is often studied in the so-called adversarial setting and the stochastic setting. In the adversarial setting, the advice of experts is chosen by an adversary so as to maximize the regret of the forecaster, and therefore the problem can be viewed as a zero-sum game between the forecaster and the adversary (see e.g. [141] [107] [94] [28] [26]). In the stochastic setting, the losses of each expert are drawn independent and identically distributed (*i.i.d.*) over time from a fixed but unknown distribution, and smaller regrets can be achieved compared with the adversarial setting (see e.g. [87] [133] [151]).

In this chapter, we consider the model in [95] which considers a mix of adversarial and stochastic settings. It is a learning system with two experts and a forecaster.

One of the experts is honest, who at each round makes a correct prediction with probability μ. The other one is malicious, who knows the true outcome at each round and makes his predictions so as to maximize the loss of the forecaster. Here we assume that the forecaster adopts the classical multiplicative weights algorithm, and study its resistance to the corruption of the malicious expert. Denote by $V^\alpha(N,\rho)$ the expected cumulative loss for the forecaster, where α is the strategy chosen by the malicious expert, N is the fixed time horizon, and ρ is the initial weight of the malicious expert. Instead of regret, we analyze the asymptotic maximal loss $\lim_{N\to\infty} \max_\alpha \frac{V^\alpha(N,1/2)}{N}$.

It was proved in [95] that if the malicious expert is only allowed to adopt offline policies, i.e., to decide whether to tell the true outcome at each round at the beginning of the game, then we have $\lim_{N\to\infty} \max_\alpha \frac{V^\alpha(N,1/2)}{N} = 1-\mu$. It implies that the extra power of the malicious expert cannot incur extra losses to the forecaster.

Here we allow the malicious expert to adopt online policies, i.e., at each round, the malicious expert chooses whether to tell the truth based on all the prior histories. To find an upper bound on asymptotic losses, we rescale dynamic programming equations of the problem and obtain a partial differential equation (PDE). Then we prove that the unique solution of this PDE provides us an upper bound

$$\limsup_{N\to\infty} \max_\alpha \frac{V^\alpha(N,1/2)}{N} \leq 1-\mu^2.$$

For the lower bound, we design a simple strategy for the malicious expert and prove that

$$\liminf_{N\to\infty} \max_\alpha \frac{V^\alpha(N,1/2)}{N} > 1-\mu,$$

which implies that the malicious expert can incur extra losses to the forecaster when online policies are admissible. To make the forecaster more resistant to the malicious expert, we consider an adaptive multiplicative weights algorithm and prove that it is asymptotically optimal for the forecaster.

The rest of the chapter is organized as follows. In Section 6.2, we mathematically formulate this problem and develop its dynamic programming equations. In Section 6.3, we show the upper bound of asymptotic losses, and in Section 6.4 we find the lower bound. In Section 6.5, we consider the malicious expert versus the adaptive multiplicative weights algorithm. In Section 6.6, we summarize our results and their implications.

6.2 Problem Formulation

In this section, we introduce the mathematical model as in [95]. Consider a learning system with two experts and a forecaster. For each round $t \in \mathbb{N}_+$, denote the prediction of expert $i \in \{1,2\}$ by $x_t^i \in \{0,1\}$, and the true outcome by $y_t \in \{0,1\}$.

Suppose that the forecaster adopts the multiplicative weights algorithm. For each round $t \in \mathbb{N}_+$, denote by p_t^i the weight of expert $i \in \{1,2\}$, $p_t^1 + p_t^2 = 1$. Then the prediction of the forecaster is

$$\hat{y}_t := \sum_{i=1}^{2} p_t^i x_t^i.$$

Given $\epsilon \in (0,1)$, the weights evolve as follows

$$p_{t+1}^i = \frac{p_t^i \epsilon^{|x_t^i - y_t|}}{p_t^1 \epsilon^{|x_t^1 - y_t|} + p_t^2 \epsilon^{|x_t^2 - y_t|}}, \quad i = 1, 2.$$

Denote the entire history up to round $t-1$ by

$$\mathcal{G}_t := \{p_l^1, p_l^2, x_l^1, x_l^2, y_l : l = 1, \dots t-1\} \cup \{p_t^1, p_t^2\}.$$

Assume expert 2 is honest, and at each round $t \in \mathbb{N}_+$ make correct predictions with probability $\mu \in (0,1)$ independently of $\mathcal{G}_t$, i.e.,

$$x_t^2 = \begin{cases} y_t & \text{with probability } \mu, \\ 1 - y_t & \text{with probability } 1-\mu. \end{cases}$$

Expert 1 is malicious and knows the accuracy μ of expert 2 and the outcome y_t at each round. At each stage $t \in \mathbb{N}_+$, based on the information $\mathcal{G}_t$, the malicious expert can choose to lie, i.e., make $x_t^1 = 1 - y_t$, or to tell the truth, i.e., make $x_t^1 = y_t$. Denote by $\mathcal{A}_t$ the space of functions from $\mathcal{G}_t$ to $\{T, L\}$, where T (truth) and L (lie) represent $x_t^1 = y_t$ and $x_t^1 = 1 - y_t$ respectively.

At each round $t \in \mathbb{N}_+$, the loss of the forecaster is $l(\hat{y}_t, y_t) := |\hat{y}_t - y_t|$, which is also the gain of the malicious expert. It can be easily verified that

$$l(\hat{y}_t, y_t) = \begin{cases} p_t^1 & \text{if } \alpha_t = L, x_t^2 = y_t, \\ 1 & \text{if } \alpha_t = L, x_t^2 = 1 - y_t, \\ 0 & \text{if } \alpha_t = T, x_t^2 = y_t, \\ 1 - p_t^1 & \text{if } \alpha_t = T, x_t^2 = 1 - y_t. \end{cases} \tag{6.1}$$

And the evolution of p_t^1 is as follows:

$$p_{t+1}^1 = \begin{cases} g(p_t^1) & \text{if } \alpha_t = L, x_t^2 = y_t, \\ g^{(-1)}(p_t^1) & \text{if } \alpha_t = T, x_t^2 = 1 - y_t, \\ p_t^1 & \text{otherwise}, \end{cases} \tag{6.2}$$

where

$$g(p_t^1) = \frac{1}{1 + (1/p_t^1 - 1)/\epsilon},$$
$$g^{(-1)}(p_t^1) = \frac{1}{1 + (1/p_t^1 - 1)\epsilon}.$$

For a fixed time horizon N, the goal of the malicious expert is to maximize the cumulative loss of the forecaster by choosing a sequence of strategies $\alpha = \{(\alpha_1, \alpha_2, \dots) : \alpha_t \in \mathcal{A}_t, t \in \mathbb{N}_+\}$, i.e., solving the optimization problem

$$V(N, \rho) := \max_{\alpha} \mathbb{E}^{\alpha} \left[\sum_{t=1}^{N} l(\hat{y}_t, y_t) \,\middle|\, p_1^1 = \rho \right].$$

According to (6.1), we obtain the expected current loss

$$\mathbb{E}^{\alpha_t} \left[l(\hat{y}_t, y_t) | \mathcal{G}_t \right] = \begin{cases} (1 - \mu + \mu p_t^1) & \text{if } \alpha_t = L, \\ (1 - \mu)(1 - p_t^1) & \text{if } \alpha_t = T. \end{cases} \tag{6.3}$$

In combination with (6.2), we get dynamic programming equations

$$\begin{aligned} V(t+1, \rho) = \max\{&(1 - \mu + \mu\rho) + \mu V(t, g(\rho)) \\ &+ (1-\mu)V(t, \rho), \quad (1-\mu)(1-\rho) \\ &+ (1-\mu)V\left(t, g^{(-1)}(\rho)\right) + \mu V(t, \rho)\}, \end{aligned} \tag{6.4}$$

together with initial conditions $V(0, \rho) = 0$.

6.3 Upper bound on the Value function

In this section, we properly rescale the (6.4) and obtain a PDE (HJB). We explicitly solve this equation, and show that its solution (6.11) provides an upper bound

$$\limsup_{N\to\infty} \frac{V(N, 1/2)}{N} \leq 1 - \mu^2 \tag{6.5}$$

6.3.1 Limiting PDE

To appropriately rescale (6.4) and follow the formulation of [15], we change the variable

$$x = \frac{\ln(1/\rho - 1)}{\ln(1/\epsilon)}, \quad \rho = \frac{1}{1 + (1/\epsilon)^x},$$

and define

$$\tilde{V}(t, x) := -V\left(t, \frac{1}{1 + (1/\epsilon)^x}\right). \tag{6.6}$$

Then (6.4) becomes

$$\begin{aligned} \tilde{V}(t+1, x) = \min \Bigg\{ & -\left(1 - \mu + \frac{\mu}{1 + (1/\epsilon)^x}\right) \\ & + \mu\tilde{V}(t, x+1) + (1-\mu)\tilde{V}(t, x), \\ & -(1-\mu)\left(1 - \frac{1}{1 + (1/\epsilon)^x}\right) \\ & + (1-\mu)\tilde{V}\left(t, x-1\right) + \mu\tilde{V}(t, x) \Bigg\}. \end{aligned} \tag{6.7}$$

Define scaled value functions via the equation $\frac{\tilde{V}^{\delta}(\delta t, \delta x)}{\delta} = \tilde{V}(t, x)$. Substituting in

(6.7), we obtain that

$$\tilde{V}^\delta(t+\delta,x) = \min\left\{-\delta\left(1-\mu+\frac{\mu}{1+(1/\epsilon)^{x/\delta}}\right)\right.$$
$$+\mu\tilde{V}^\delta(t,x+\delta)+(1-\mu)\tilde{V}^\delta(t,x),$$
$$-\delta(1-\mu)\left(1-\frac{1}{1+(1/\epsilon)^{x/\delta}}\right)$$
$$\left.+(1-\mu)\tilde{V}^\delta\left(t,x-\delta\right)\right)+\mu\tilde{V}^\delta(t,x)\Big\}. \tag{6.8}$$

Taking δ to 0 in (6.8), we obtain a first order PDE

$$0 = v_t(t,x) + \max\left\{1-\mu+\mu s(x)-\mu v_x(t,x),\right.$$
$$\left.(1-\mu)(1-s(x))+(1-\mu)v_x(t,x)\right\}, \tag{6.9}$$

where $v(0,x)=0$, and

$$s(x) = \begin{cases} 0, & \text{if } x > 0, \\ 1, & \text{if } x < 0. \end{cases}$$

Define $\Omega_1 = \{x > 0\}, \Omega_2 = \{x < 0\}, \mathcal{H} = \{x = 0\}$. Note that such division corresponds to $\rho < 1/2$ and $\rho > 1/2$, i.e. whether the malicious expert is more credible than a benign one. Define Hamiltonians

$$H_1(x,p) = \max\{1-\mu-\mu p, 1-\mu+(1-\mu)p\}, x \in \bar{\Omega}_1,$$
$$H_2(x,p) = \max\{1-\mu p, (1-\mu)p\}, x \in \bar{\Omega}_2.$$

Then (6.9) becomes

$$v_t + H_i(x, v_x) = 0 \quad \text{for } x \in \Omega_i,\ i = 1,2. \tag{6.10}$$

Following Ishii's definition of viscosity solutions to discontinuous Hamiltonians, we complement (6.10) by

$$\min\{v_t + H_1(x, v_x), v_t + H_2(x, v_x)\} \le 0 \quad \text{for } x \in \mathcal{H},$$
$$\max\{v_t + H_1(x, v_x), v_t + H_2(x, v_x)\} \ge 0 \quad \text{for } x \in \mathcal{H},$$

where min and max should be understood in the sense of viscosity solutions.

Solving (6.10) by the method of characteristics and assuming that the value function is differentiable with respect to x on $\mathcal{H}$, we conjecture the solution

$$v(t, x) = \begin{cases} -(1-\mu)t, & \text{if } x \in [(1-\mu)t, \infty), \\ -(1-\mu^2)t + \mu x & \text{if } x \in [-\mu t, (1-\mu)t], \\ -t, & \text{if } x \in (-\infty, -\mu t]. \end{cases} \tag{6.11}$$

Proposition 6.3.1. *A viscosity solution of*

$$\begin{cases} v_t + H_i(x, v_x) = 0, \quad \textit{for } x \in \Omega_i, i = 1, 2, \\ \min\{v_t + H_1(x, v_x), v_t + H_2(x, v_x)\} \le 0 \quad \textit{for } x \in \mathcal{H}, \\ \max\{v_t + H_1(x, v_x), v_t + H_2(x, v_x)\} \ge 0 \quad \textit{for } x \in \mathcal{H}, \\ v(0, x) = 0. \end{cases} \tag{HJB}$$

is given by (6.11).

Proof. The initial condition $v(0, x) = 0$ is trivially satisfied. We show that v is a subsolution. Suppose $\phi : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is differentiable, and $v - \phi$ achieves a local maximum 0 at $(t_0, x_0) \in (0, \infty) \times \mathbb{R}$. Since v is differentiable in the domain $O := \{(t, x) : t > 0,\ x \neq (1-\mu)t,\ x \neq -\mu t\}$, we have $\phi_t(t_0, x_0) = v_t(t_0, x_0), \phi_x(t_0, x_0) = v_x(t_0, x_0)$ if $(t_0, x_0) \in O$. Then it is can be easily verified that $\phi_t + H_i(x, \phi_x) = 0$ at (t_0, x_0), where $i = 1$ if $x_0 \ge 0$, and $i = 2$ if $x_0 \le 0$.

Suppose (t_0, x_0) is on the line $\{(t,x) : t > 0,\ x = (1-\mu)t\}$. Note that

$$\begin{aligned}
&\partial_t^- v(t_0, x_0) = -(1-\mu),\ \partial_t^+(t_0, x_0) = -(1-\mu^2),\\
&\partial_x^- v(t_0, x_0) = \mu, \quad\ \ \partial_x^+ v(t_0, x_0) = 0.
\end{aligned}$$

Since (t_0, x_0) is a local maximum of $v - \phi$, we must have

$$\begin{aligned}
&(\phi_t(t_0, x_0), \phi_x(t_0, x_0))\\
&\quad \in \{(r,p) : r \in [-(1-\mu^2), -(1-\mu)],\ p \in [0, \mu]\}.
\end{aligned}$$

Take $\Delta x = (1-\mu)\Delta t$. As a result of

$$v(t_0 + \Delta t, x_0 + \Delta x) - \phi(t_0 + \Delta t, x_0 + \Delta x) \leq 0,$$

we obtain that

$$-(1-\mu)\Delta t - \phi_t \Delta t - \phi_x \Delta x + o(\Delta t) \leq 0.$$

Since we can choose Δt to be either positive or negative, it can be easily deduced that

$$-(1-\mu) - \phi_t - (1-\mu)\phi_x = 0.$$

Substituting into H_1, we obtain that

$$\begin{aligned}
&\phi_t(t_0, x_0) + H_1(x_0, \phi_x(t_0, x_0)).\\
&\quad = \phi_t(t_0, x_0) + (1-\mu) + (1-\mu)\phi_x(t_0, x_0) = 0.
\end{aligned}$$

If (t_0, x_0) is on the line $\{(t,x) : t > 0,\ x = -\mu t\}$, we have sub/super differentials of v,

$$\partial_t^- v(t_0, x_0) = -1,\ \partial_t^+(t_0, x_0) = -(1-\mu^2),$$
$$\partial_x^- v(t_0, x_0) = 0, \quad \partial_x^+ v(t_0, x_0) = \mu.$$

Therefore $v - \phi$ cannot achieve a local maximal on the line $\{(t, x) : t > 0,\ x = -\mu t\}$. Hence we have proved that v is a subsolution of (HJB), and similarly, we can show that v is a supersolution. □

6.3.2 Control problem

In this subsection, we show that there is a unique viscosity solution of (HJB) by applying results from [14] and [15]. Then in the next subsection, we will show that the unique solution given by (6.11) provides an upper bound of $\limsup_{N\to\infty} \frac{V(N,1/2)}{N}$ by using the comparison principle. First, we interpret (HJB) as a control problem.

In the domain $\Omega_i, i = 1, 2$, we take $A_i = [0, 1]$ as the space of controls, and

$$b_i(x, \alpha_i) = \alpha_i\mu - (1-\alpha_i)(1-\mu), \quad \alpha_i \in A_i,$$

as the controlled dynamics. For $x \in \mathcal{H}$, define the space of controls $A := A_1 \times A_2 \times [0, 1]$, and the dynamics

$$b_{\mathcal{H}}(x, (\alpha_1, \alpha_2, c)) := cb_1(x, \alpha_1) + (1-c)b_2(x, \alpha_2),$$

where $(\alpha_1, \alpha_2, c) \in A$. The running cost in the domain Ω_1 is given by $l_1(x, \alpha_1) = -(1-\mu)$, in the domain Ω_2 by $l_2(x, \alpha_2) = -\alpha_2$, and in $\mathcal{H}$ by

$$l_{\mathcal{H}}(x, (\alpha_1, \alpha_2, c)) = cl_1(x, \alpha_1) + (1-c)l_2(x, \alpha_2),$$

where $(\alpha_1, \alpha_2, c) \in A$.

In order to restrict the dynamics on the boundary $\mathcal{H}$, we require that $b_{\mathcal{H}}(x, (\alpha_1, \alpha_2, c)) = 0$ for $x \in \mathcal{H}$. We denote the collection of all such controls by

$$\begin{aligned} A_0(x) := \{a = (\alpha_1, \alpha_2, c) \in A : \\ b_{\mathcal{H}}(x, (\alpha_1, \alpha_2, c)) = 0\}. \end{aligned}$$

We say a control $a \in A_0(x)$ is regular if $b_1(x, \alpha_1) \leq 0, b_2(x, \alpha_2) \geq 0$, and denote

$$\begin{aligned} A_0^{reg}(x) := \{a = (\alpha_1, \alpha_2, c) \in A_0(x) : \\ (-1)^i b_i(x, \alpha_i) \geq 0\}. \end{aligned}$$

Define $\mathcal{A} := L^\infty([0, 1]; A)$. We say a Lipschitz function $X_x : [0, 1] \to \mathbb{R}, X_x(0) = x$, an admissible trajectory if there exists some control process $a(\cdot) \in \mathcal{A}$, such that for a.e. $t \in [0, 1]$

$$\begin{aligned} \dot{X}_x(t) =& b_1(X_x(t), \alpha_1(t))\mathbb{1}_{\{X_x(t)\in\Omega_1\}} \\ &+ b_2(X_x(t), \alpha_2(t))\mathbb{1}_{\{X_x(t)\in\Omega_2\}} \\ &+ b_{\mathcal{H}}(X_x(t), (\alpha_1(t), \alpha_2(t), c(t))\mathbb{1}_{\{X_x(t)\in\mathcal{H}\}}. \end{aligned} \tag{6.12}$$

According to [15, Theorem 2.1], we have $a(t) \in A_0(X_x(t))$ for a.e. $t \in \{s : X_x(s) \in \mathcal{H}\}$. Denote by $\mathcal{T}_x$ the set of admissible controlled trajectories starting from x, i.e.,

$$\begin{aligned} \mathcal{T}_x := \{(X_x(.), a(.)) \in \mathrm{Lip}([0, 1]; \mathbb{R}) \times \mathcal{A} \\ \text{such that (6.12) is satisfied and } X_x(0) = x\}. \end{aligned}$$

Let us also introduce the set of regular trajectories,

$$\mathcal{T}_x^{reg} := \{(X_x(.), a(.)) \in \mathcal{T}_x : a(t) \in A_0^{reg}(X_x(t))$$
$$\text{for a.e. } t \in \{s : X_x(s) \in \mathcal{H}\}\}.$$

For each $x \in \mathbb{R}, t \in [0, 1)$, we define two value functions

$$V^-(x,t) := \inf_{(X_x(.),a(.))\in\mathcal{T}_x} \int_0^t l(X_x(s), a(s))\, ds, \tag{6.13}$$

$$V^+(x,t) := \inf_{(X_x(.),a(.))\in\mathcal{T}_x^{reg}} \int_0^t l(X_x(s), a(s))\, ds, \tag{6.14}$$

where the cost function l is given by

$$l(X_x(s), a(s)) := \sum_{i=1,2} l_i(X_x(s), \alpha_i(s))\mathbb{1}_{\{X_x(s)\in\Omega_i\}}$$
$$+ l_{\mathcal{H}}(X_x(s), a(s))\mathbb{1}_{\{X_x(s)\in\mathcal{H}\}}.$$

Note that in $\Omega_i, i = 1, 2$, the associated Hamiltonian of (6.13) and (6.14)

$$(x, p) \mapsto \sup_{\alpha_i\in A_i} \{-b_i(x, \alpha_i)p - l_i(x, \alpha_i)\}$$

coincides with H_i in the last subsection. Then according to [15, Theorem 3.3], both V^- and V^+ are viscosity solutions of (HJB). We will show that they are actually equal and there is only one viscosity solution of (HJB).

Proposition 6.3.2. *$V^- = V^+$ is the unique viscosity solution of* (HJB)*, and V^- is the minimal supersolution of* (HJB)*.*

Proof. The argument is an application of results from [15]. Define the Hamiltonians on $\mathcal{H}$ via

$$H_T(x) := \sup_{A_0(x)} \{-l_{\mathcal{H}}(x,a)\},$$

$$H_T^{reg}(x) := \sup_{A_0^{reg}(x)} \{-l_{\mathcal{H}}(x,a)\},$$

Let us compute $H_T(x)$. Suppose $a = (\alpha_1, \alpha_2, c) \in A_0(x)$. Then it can be easily verified that maximizing $-l_{\mathcal{H}}(x,a)$ over $A_0(x)$ is equivalent to maximizing

$$c(1-\mu) + (1-c)\alpha_2, \tag{6.15}$$

subject to constraints,

$$\begin{aligned} &c(\alpha_1 + \mu - 1) + (1-c)(\alpha_2 + \mu - 1) = 0, \\ &c, \alpha_1, \alpha_2 \in [0,1]. \end{aligned} \tag{6.16}$$

We first fix α_2 and suppose $\alpha_2 > (1-\mu)$. Due to the equality

$$c(1-\mu) + (1-c)\alpha_2 = (1-\mu-\alpha_2)c + \alpha_2,$$

and the fact that the coefficient before c is negative, maximizing (6.15) is equivalent to minimizing c under the constraints. It can be easily seen that the minimum c can be obtained if and only if $\alpha_1 = 0$. Therefore the equation (6.16) becomes $1+\alpha_2 c = \alpha_2+\mu$, and hence (6.15) is equal to $(1+c)(1-\mu)$. Now fix $\alpha_1 = 0$. In order to obtain the maximum of c, we have to take $\alpha_2 = 1$. In that case $\alpha_1 = 0, \alpha_2 = 1, c = \mu$ and $c(1-\mu) + (1-c)\alpha_2 = 1-\mu^2$.

If $\alpha_2 \leq (1-\mu)$, we have $c(1-\mu) + (1-c)\alpha_2 \leq (1-\mu) < 1-\mu^2$. Since $(0,1,\mu)$ is a regular control, we conclude that

$$H_T(x) = H_T^{reg}(x) = 1 - \mu^2.$$

We say a continuous function v is viscosity solution of

$$\begin{aligned} v_t + \mathbb{H}^-(x, v_x) = 0 \text{ in } (0,1) \times \mathbb{R}, \\ \Big[resp., \quad v_t + \mathbb{H}^+(x, v_x) = 0 \text{ in } (0,1) \times \mathbb{R}\Big] \end{aligned} \tag{6.17}$$

if it satisfies (HJB) and

$$\begin{aligned} v_t + H_T(x) = 0 \text{ on } [0,1] \times \mathcal{H}, \\ [resp., \quad v_t + H_T^{reg}(x) = 0 \text{ on } [0,1] \times \mathcal{H}]. \end{aligned}$$

According to [15, Theorem 3.3], V^+ is a viscosity subsolution of $v_t + \mathbb{H}^+(x, v_x) = 0$, and hence also a viscosity subsolution of (6.17) since $H_T = H_T^{reg}$ in our case. As a result of [15, Theorem 4.2, 4.4], V^- is the viscosity solution of (6.17), and the comparison result holds for (6.17). Therefore we conclude that $V^+ \leq V^-$. Then according to their definitions (6.13) and (6.14), they must be equal.

Finally according to [15, Theorem 4.4], V^- is the minimal supersolution of (HJB) and V^+ is the maximal subsolution of (HJB). Then if v is a viscosity solution of (HJB), we must have $V^- \leq v \leq V^+$ and hence $v = V^- = V^+$. $\square$

6.3.3 Upper bound (6.5)

In this subsection, we show that

$$\underline{v}(t, x) := \liminf_{(s,y,\delta)\to(t,x,0)} \tilde{V}^\delta(s, y)$$

is a viscosity supersolution of (HJB). Then according to Proposition 6.3.2, we obtain that $\underline{v}(t,x) \geq v(t,x)$, and hence

$$\liminf_{N\to\infty} \frac{\tilde{V}(N, Nx)}{N} \geq v(1,x).$$

In particular, if we take $x = 0$, then due to (6.6) and (6.11) the above inequality becomes

$$\limsup_{N\to\infty} \frac{V(N, 1/2)}{N} \leq 1 - \mu^2.$$

Proposition 6.3.3. *$\underline{v}$ is a viscosity supersolution of* (HJB).

Proof. The proof is almost the same as [17, Theorem 2.1], and we record here for completeness. Fixing arbitrary $T > 0$, we show that $\underline{v}$ is a viscosity supersolution over $[0,T] \times \mathbb{R}$. Assume that (t_0, x_0) is a strict local minimum of $\underline{v} - \phi$ for some $\phi \in \mathcal{C}_b^\infty([0,T] \times \mathbb{R})$. As a result of (6.8), it can be easily seen that $\underline{v}(t,x) \in [-t, 0]$. Without loss of generality, we assume that $t_0 \in (0,T)$, $\underline{v}(t_0,x_0) = \phi(t_0,x_0)$, and there exists some $r > 0$ such that

(i) $\phi \leq -2T$ outside the ball $B((t_0,x_0),r) := \{(t,x) : (t-t_0)^2 + (x-x_0)^2 \leq r^2\}$,

(ii) $\underline{v} - \phi \geq 0 = \underline{(}t_0,x_0) - \phi(t_0,x_0)$ in the ball $B((t_0,x_0),r)$.

Then there exists a sequence of (t_n, x_n, δ_n) such that $(t_n, x_n, \delta_n) \to (t_0, x_0, 0)$ and (t_n, x_n) is a global minimum of $\tilde{V}^{\delta_n} - \phi$. Due to the definition of $\underline{v}$, we have that $\xi_n := \tilde{V}^{\delta_n}(t_n,x_n) - \phi(t_n,x_n) \to 0$ and $\tilde{V}^{\delta_n}(t,x) \leq \phi(t,x) + \xi_n$ for any $(t,x) \in [0,T] \times \mathbb{R}$.

According to (6.8), we obtain that

$$\begin{aligned}0 \leq & \phi(t_n, x_n) + \max\Bigg\{\delta_n\left(1-\mu+\frac{\mu}{1+(1/\epsilon)^{x_n/\delta_n}}\right) \\ & -\mu\phi(t_n-\delta_n, x_n+\delta_n) - (1-\mu)\phi(t_n-\delta_n, x_n), \\ & \delta_n(1-\mu)\left(1-\frac{1}{1+(1/\epsilon)^{x_n/\delta_n}}\right) \\ & -(1-\mu)\phi\left(t_n-\delta_n, x_n-\delta_n)\right) - \mu\phi(t_n-\delta_n, x_x)\Bigg\}. \end{aligned} \tag{6.18}$$

We prove for the case $x_0 = 0$, and the proof for $x \neq 0$ is the same. Since $\left\{\frac{1}{1+(1/\epsilon)^{x_n/\delta_n}}\right\}_{n\geq 0} \in [0,1]$, we can take a convergent subsequence. For simplicity, we still denote it by $\left\{\frac{1}{1+(1/\epsilon)^{x_n/\delta_n}}\right\}_{n\geq 0}$, and assume it converges to some $s \in [0,1]$. Letting $n \to \infty$ in (6.18), we obtain that

$$\begin{aligned}0 \leq \phi_t(t_0, x_0) + & \max\left\{1-\mu+\mu s - \mu\phi_x(t_0, x_0),\right. \\ & \left.(1-\mu)(1-s) + (1-\mu)\phi_x(t_0, x_0)\right\}.\end{aligned}$$

Note that if

$$1-\mu+\mu s-\mu\phi_x(t_0, x_0) \geq (1-\mu)(1-s) + (1-\mu)\phi_x(t_0, x_0),$$

then we have

$$\begin{aligned}H_2(x_0, \phi_x(t_0, x_0)) & \geq 1-\mu\phi_x(t_0, x_0) \\ & \geq 1-\mu+\mu s-\mu\phi_x(t_0, x_0),\end{aligned}$$

and hence

$$\phi_t(t_0, x_0) + H_2(x_0, \phi_x(t_0, x_0) \geq 0.$$

Similarly if

$$(1-\mu)(1-s)+(1-\mu)\phi_x(t_0,x_0) \geq 1-\mu+\mu s-\mu\phi_x(t_0,x_0),$$

then

$$\begin{aligned} H_1(x_0,\phi_x(t_0,x_0)) &\geq 1-\mu+(1-\mu)\phi_x(t_0,x_0) \\ &\geq (1-\mu)(1-s)+(1-\mu)\phi_x(t_0,x_0), \end{aligned}$$

and hence

$$\phi_t(t_0,x_0)+H_1(x_0,\phi_x(t_0,x_0) \geq 0.$$

Therefore, we have shown that

$$\begin{aligned} \max\{&\phi_t(t_0,x_0)+H_1(x_0,\phi_x(t_0,x_0)), \\ &\phi_t(t_0,x_0)+H_2(x_0,\phi_x(t_0,x_0))\} \geq 0. \end{aligned}$$

□

6.4 Lower Bound on the Value function

It was proved in [95] that the asymptotic average value is $(1-\mu)$ for any offline strategy of the malicious expert if starting with weight $p_1^1 = 1/2$. Recall that $g(\rho) = \frac{1}{1+(1/\rho-1)/\epsilon}$. Here we provide a lower bound on the value functions for the corresponding online problem

$$\begin{aligned} \liminf_{N\to\infty} \frac{V(N,\rho)}{N} &\geq 1-\mu+\mu(1-\mu)(\rho-g(\rho)) \\ &> 1-\mu, \end{aligned} \tag{6.19}$$

which shows that the malicious expert has more advantages when he adopts online policies.

This lower bound can be achieved if the malicious expert chooses to lie at state ρ and chooses to tell the truth at state $g(\rho)$. Since $g(\rho) < \rho$, intuitively the malicious experts lies when he is still credible, and tells the truth when its credibility has been lowered by the algorithm. For $p_1^1 = \rho$, define the corresponding strategies by

$$\alpha_t^\rho(\mathcal{G}_t) = \begin{cases} L & \text{if } p_t^1 = \rho, \\ T & \text{if } p_t^1 = g(\rho), \end{cases} \tag{6.20}$$

and $\alpha^\rho := (\alpha_1^\rho, \alpha_2^\rho, \dots)$. We denote the value function associated with α^ρ by

$$V^{\alpha^\rho}(N, \rho) = \mathbb{E}^{\alpha^\rho}\left[\sum_{t=1}^{N} l(\hat{y}_t, y_t) \,\Big|\, p_1^1 = \rho\right].$$

Proposition 6.4.1.

$$\lim_{N\to\infty} \frac{V^{\alpha^\rho}(N, \rho)}{N} = 1 - \mu + \mu(1-\mu)(\rho - g(\rho)).$$

Proof. Under strategy α^ρ, $\{p_t^1\}_{t\in\mathbb{N}}$ is a Markov chain with two states $\{\rho, g(\rho)\}$ starting with $p_0^1 = \rho$, and its transition probability is given by

$$\mathbb{P}\left[p_{t+1}^1 = \rho \mid p_t^1 = \rho\right] = 1 - \mu,$$

$$\mathbb{P}\left[p_{t+1}^1 = g(\rho) \mid p_t^1 = \rho\right] = \mu,$$

$$\mathbb{P}\left[p_{t+1}^1 = \rho \mid p_t^1 = g(\rho)\right] = 1 - \mu,$$

$$\mathbb{P}\left[p_{t+1}^1 = g(\rho) \mid p_t^1 = g(\rho)\right] = \mu.$$

Denote its distribution at time t by

$$\pi_t := \left(\mathbb{P}(p_t^1 = \rho), \mathbb{P}(p_t^1 = g(\rho))\right).$$

It can be easily seen that $(1-\mu, \mu)$ is the stationary distribution of $\{p_t^1\}_{t\in\mathbb{N}}$. According to [139, Theorem 4.9], the distribution π_t converges to $(1-\mu, \mu)$ as $t \to \infty$. Due to the equality

$$\begin{aligned}
&\mathbb{E}^{\alpha^\rho}\left[\sum_{t=0}^{N} l(\hat{y}_t, y_t) \,\middle|\, p_0^1 = \rho\right] \\
&= \sum_{t=0}^{N} \mathbb{P}(p_t^1 = \rho)(1 - \mu + \mu\rho) \\
&\quad + \sum_{t=0}^{N} \mathbb{P}(p_t^1 = g(\rho))(1 - \mu)(1 - g(\rho)),
\end{aligned}$$

it can be easily verified that

$$\begin{aligned}
\lim_{N\to\infty} \frac{V^{\alpha^\rho}(N, \rho)}{N} =& (1 - \mu)(1 - \mu + \mu\rho) \\
&+ \mu(1 - \mu)(1 - g(\rho)) \\
=& 1 - \mu + \mu(1 - \mu)(\rho - g(\rho)) \\
>& 1 - \mu.
\end{aligned}$$

□

6.5 asymptotically optimal strategy for the forecaster

In this section, we show that an adaptive multiplicative weights algorithm can resist corruptions of the malicious expert. Different from the multiplicative weights algorithm in Section 6.2, the adaptive multiplicative weights algorithm updates the weights $p_t^i, i = 1, 2$, as follows:

$$p_{t+1}^i = (p_t^i)^{\eta_{t+1}/\eta_t} e^{-\eta_{t+1}|x_t^i - y_t|},$$

where $\eta_t = \sqrt{8(\ln 2)/t}, t \in \mathbb{N}_+$ is time-varying. Hence the prediction of the forecaster is

$$\hat{y}_t = \frac{p_t^1 x_t^1 + p_t^2 x_t^2}{p_t^1 + p_t^2}.$$

Denote by $V^*(N, \tau_1, \tau_2)$ the value function for the malicious expert under the adaptive multiplicative weights algorithm with initial weights $p_1^1 = \tau_1, p_1^2 = \tau_2$. For any $t \in \mathbb{N}_+, p_t \in (0, \infty)$, define

$$h_t(p_t) := (p_t)^{\eta_{t+1}/\eta_t} e^{-\eta_{t+1}}.$$

It can be easily verified that $V^*(N, \tau_1, \tau_2)$ is the solution to dynamic programming equations

$$\begin{aligned}
V^*(t+1, \tau_1, \tau_2) = \max &\Bigg\{ \left(1 - \mu + \frac{\mu \tau_1}{\tau_1 + \tau_2}\right) \\
&+ \mu V^*(t, h_t(\tau_1), \tau_2) + (1-\mu) V^*(t, h_t(\tau_1), h_t(\tau_2)), \\
&\frac{(1-\mu)\tau_2}{\tau_1 + \tau_2} + (1-\mu) V^*\big(t, \tau_1, h_t(\tau_2)\big) + \mu V^*(t, \tau_1, \tau_2) \Bigg\},
\end{aligned}$$

together with initial conditions $V^*(0, \tau_1, \tau_2) = 0$.

Proposition 6.5.1.

$$\lim_{N \to \infty} \frac{V^*(N, 1, 1)}{N} = 1 - \mu, \tag{6.21}$$

which implies that this adaptive multiplicative weights algorithm is asymptotically optimal for the forecaster.

Proof. Suppose the malicious expert keeps lying, i.e. taking strategies $\alpha_t(\mathcal{G}_t) = L, t \in$

$\mathbb{N}_+$. Then according to (6.3), it can be easily seen that the cumulative loss under this strategy is greater than or equal to $(1-\mu)N$, and hence

$$\liminf_{N\to\infty} \frac{V^*(N,1,1)}{N} \geq 1-\mu.$$

To prove the other inequality, for any path $\mathcal{G}_{N+1}$ with $p_1^1 = p_1^2 = 1$, we define

$$\hat{L}_N := \sum_{t=1}^{N} l(\hat{y}_t, y_t), \quad L_N^i := \sum_{t=1}^{N} l(x_t^i, y_t), \;\; i=1,2.$$

Applying [69, Chapter 2, Theorem 2.3], we obtain that

$$\hat{L}_N - \min_{i=1,2} L_N^i \leq 2\sqrt{\frac{N}{2}\ln 2} + \sqrt{\frac{\ln 2}{8}},$$

and hence

$$\hat{L}_N \leq L_N^2 + 2\sqrt{\frac{N}{2}\ln 2} + \sqrt{\frac{\ln 2}{8}}.$$

Therefore for any strategy α, we obtain

$$\begin{aligned}\mathbb{E}^\alpha\left[\hat{L}_N \,\middle|\, p_1^1 = p_1^2 = 1\right] \leq & \mathbb{E}^\alpha\left[L_N^2 \,\middle|\, p_1^1 = p_1^2 = 1\right] \\ & + 2\sqrt{\frac{N}{2}\ln 2} + \sqrt{\frac{\ln 2}{8}} \\ =&(1-\mu)N + 2\sqrt{\frac{N}{2}\ln 2} \\ & + \sqrt{\frac{\ln 2}{8}},\end{aligned}$$

and also

$$\limsup_{N\to\infty} \frac{V^*(N,1,1)}{N} \leq 1-\mu.$$

$\square$

6.6 Conclusions

In this chapter, we have studied an online prediction problem with two experts of whom one is malicious. At each round, based on all the prior history, the malicious expert chooses to tell the true outcome or not so as to maximize the loss. We have shown that the multiplicative weights algorithm cannot resist the corruption of the malicious expert by explicitly finding upper and lower bounds on the value function; see (6.5) and (6.19). We have also proved that an adaptive multiplicative weights algorithm can resist the corruption; see Proposition 6.5.1.

CHAPTER VII

Prediction Against a Limited Adversary

7.1 Introduction

Prediction with expert advice is one of the fundamental problems in *online learning* and *sequential decision making*. In this problem, at each round a forecaster chooses between alternative actions based on his current and past observations with the objective of performing as well as the best constant strategy. We refer the reader to [69] for a survey. This problem is often studied in the adversarial setting where an adversary chooses the outcomes to maximize the regret of the forecaster. This interaction between the forecaster and the adversary can be seen as a zero-sum game (see e.g. [1, 2, 26, 28, 92, 94, 107, 168]). Using the minimax theorem, one can easily show that this zero-sum game admits a value under mild assumptions and the value function satisfies a discrete time dynamic programming principle. Then, the long-time behavior of the value function can be studied by showing that the discrete time dynamic programming equation "converges" to a differential operator and a scaled version of the value function converges to the solution of a partial differential equation associated to the differential operator. Viscosity solution theory provides formidable tools to rigorously show this convergence and study the properties of the long-time behavior of the value function.

One can also state the prediction problem in the stochastic setting where the actions of the adversary are drawn from a fixed distribution (unknown or known to the forecaster). Since the decisions of the adversary do not depend on the state, the forecaster has better performances and his regret is smaller.

Similar to [7, 30, 108, 123, 146], in this chapter, we bridge the adversarial and stochastic settings by considering an adversary who cannot freely choose the outcomes. In our framework, the gains of the experts are drawn from a fixed distribution. Then, without seeing the outcomes and each other's decisions, the adversary chooses to corrupt the gain of one of the experts and the forecaster chooses one of the experts. If the forecaster chose the corrupted expert, he obtains the corrupted gain. Otherwise he obtains the gain of the expert he chose. By studying the value function of this game between the adversary and the forecaster, we show that several important features of the fully adversarial setting do not extend to our framework and the assumptions on the data of the problem can lead to dramatic differences for the long-time behavior of regret.

First of all, we show that the existence of the value for the zero-sum game in the pre-limit regime is not guaranteed. Indeed, if one does not state the problem of the adversary properly, the strategies of the adversary might fail to range in a convex set. This point has crucial implications. Indeed, the minimax theorem fails and one cannot establish a dynamic programming equation and the analysis of the interaction becomes significantly more challenging. In our work, we identify a relevant set of strategies for the adversary that allows us to obtain the existence of the value and to use the viscosity machinery.

The second contribution of our chapter is to exhibit wildly different behavior of the regret in the long-time regime for different types of final conditions for the zero-sum game. In the classical statement of the prediction problem the gain of the forecaster is compared against the gain of the best expert. In this case, the payoff function at maturity of the zero-sum game is given by $\Phi_m(x) := \max_i x^i$, $\forall x \in \mathbb{R}^N$, where N is the number of experts in this game. One fundamental question is whether the long-time behavior of the prediction problem is robust with respect to the choice of this payoff function. Different choices of payoff functions are made in [112, Proposition 4.1], [92, 94], see also the distinction between internal and external regret in [100]. In particular, in [92], the authors assume that the payoff function satisfies a strict monotonicity condition which is for example not satisfied by the function Φ_m. Since the choice of the payoff function only impacts the final condition of the associated partial differential equation, the viscosity solution approach is a formidable tool to study the impact of the payoff function on the growth of the regret. Using these tools, we show that the long-time behavior of the regret have different regimes depending on whether we assume this strict monotonicity.

The third contribution of our chapter is to show that, although mathematically appealing, a comparison result for viscosity solutions of the limiting equation is not fundamental to obtain algorithms for the forecaster and the adversary and the growth of the regret. Indeed, similarly to [130, 129], algorithms for the adversary and a lower bound for the growth of regret can be found using a smooth subsolution of the limiting equation. Additionally, by considering a smooth supersolution of a relevant equation, one can construct an algorithm for the forecaster and an upper bound for the growth of the regret. As in Theorem 7.4.3, usually one can show that the infimum (supremum) limit of scaled value functions is a supersolution (subsolution) of the limiting equation. Therefore if a comparison result for viscosity solutions exists, one can conclude that the scaled value function converges and thus obtain the exact growth rate of regret. Note also that the Hamiltonian of the limiting equation we obtain has a discontinuous dependence on the first derivative and the equation is similar to the geometric equations studied in [75, 102, 176, 177].

Finally, unlike in [26, 55, 94] where the gradient (or simple transformation of the gradient) of the solution to the limiting equation yields an asymptotic optimal algorithm for the forecaster, we show that this gradient may not provide an asymptotic optimal algorithm for the forecaster in our problem. Unfortunately, this point shows that the solution to the limiting equation might fail to capture some feature of the prediction problem. There are other variants of the prediction problem, e.g., [145] considers online learning when the time horizon is unknown and similar to our case the controls of the adversary are limited, and [3] studies a repeated zero-sum game where an adversary plays on a budget.

The rest of the chapter is organized as follows. In Section 7.2, we formulate the problem of prediction against a limited adversary and state the relevant assumptions. In Section 7.3, we heuristically derive the limiting equation and in Section 7.4 state our main results. The Section 7.5 contains special cases where we can explicitly solve the limiting equation.

7.1.1 Notations

Let $N \geq 2$ and denote $\{e^i\}_{i=1}^N$ the canonical basis of $\mathbb{R}^N$. We define $\mathbf{1} = \sum_{i=1}^N e^i$, $\mathbb{R}_+^N = [0, \infty)^N$. We denote by $\mathbb{S}_N$ the set of symmetric matrices of dimension N.

7.2 Problem Formulation

Consider a learning system with $N \geq 2$ experts, an adversary and a forecaster. At each round m, each expert $i \in \{1, \ldots, N\}$ makes a prediction which yields a gain $g_m^i \in \{0, 1\}$. Here $g_m^i = 0$ (resp. $g_m^i = 1$) represents that the prediction is wrong (resp. correct) at this round. We assume that each expert is correct with probability μ^i, i.e., $\mathbb{E}[g_m^i] = \mu^i \in [0, 1]$. Knowing the values of $\{\mu^i : i \in \{1, \ldots, N\}\}$, the adversary and the forcaster play a zero-sum game. At each round, the adversary picks one expert $A_m \in \{1, \ldots, N\}$, and sets his gain to $h_m \in \{0, 1\}$. The adversary uses mixed type strategies and therefore, he chooses a distribution for $(A_m, h_m) \in \{1, \ldots, N\} \times \{0, 1\}$ that may depend on the past history of the game. The realized gain ΔG_m^i of the expert i at round m is

$$\Delta G_m^i = g_m^i \mathbf{1}_{\{i \neq A_m\}} + h_m \mathbf{1}_{\{i = A_m\}}$$

and the total gain of the expert i is

$$G_m^i = \sum_{k=1}^{m} \Delta G_k^i.$$

The fact that the adversary can only interfere on the outcome of the prediction of one expert is the main difference between our framework and the classical prediction with expert advice problems in [2, 68, 69, 94, 107], and also the bandit problems with corruption such as [108] and [146] where the regret bounds provided depend on the corruption. However, unlike [7] and [123], the adversary can optimally control the level of corruption at each round and therefore the level of corruption might be unbounded.

If the forecaster chooses to follow expert $F_m \in \{1, \ldots, N\}$ at each round, then his gain is given by

$$G_m := \sum_{k=1}^{m} \Delta G_k := \sum_{k=1}^{m} \Delta G_k^{F_k}.$$

The state of the zero-sum game between the adversary and the forecaster is

$$X_m = (X_m^1, \ldots, X_m^N) = (G_m^1 - G_m, \ldots G_m^N - G_m)$$

which evolves as

$$\Delta X_m = (\Delta G_m^1 - \Delta G_m, \ldots, \Delta G_m^N - \Delta G_m).$$

Given the state, the control of the adversary is $\alpha_m = \{(a_m^i, b_m^i)\}_{i=1,\dots N}$ where

$$a_m^i = \mathbb{P}(A_m = i, h_m = 0), \quad b_m^i = \mathbb{P}(A_m = i, h_m = 1),$$

and the control of the forecaster is $\phi_m = \{\phi_m^i\}_{i=1,\dots N}$ where

$$\phi_m^i = \mathbb{P}(F_m = i).$$

We assume that the random variables $\{g_m^i\} \cup \{(A_m, h_m)\} \cup \{F_m\}$ are mutually independent.

Remark 7.2.1. We do not assume that A_m and h_m are independent and this point is crucial. Indeed, in the definition of admissible strategies, if we require A_m and h_m to be independent, then the set of admissible distributions of ΔG_m might fail to be convex. Then, we would not be able to apply the minimax theorem to have a saddle point for the interaction between the adversary and the forecaster.

However, since we assume that A_m and h_m are not required to be independent, the set of distributions of ΔG_m is isomorphic to

$$\mathcal{A} := \left\{ ((a^i)_{i=1}^N, (b^i)_{i=1}^N) \in [0,1]^N \times [0,1]^N : \sum_{i=1}^N a^i + b^i = 1 \right\},$$

which is convex.

Simple computation yields that for all $j \in \{1, \dots, N\}$,

$$\mathbb{E}^{\alpha_m}[\Delta G_m^j] = (1 - a_m^j - b_m^j)\mu^j + b_m^j, \tag{7.1}$$

$$\mathbb{E}^{\phi_m, \alpha_m}[\Delta X_m^j] = (1 - a_m^j - b_m^j)\mu^j + b_m^j - \sum_{i=1}^N \phi_m^i((1 - a_m^i - b_m^i)\mu^i + b_m^i). \tag{7.2}$$

Suppose the maturity is $M > 0$ and let $\Phi : \mathbb{R}^N \mapsto \mathbb{R}$ be a given function. We define the regret of the forecaster via

$$\Phi(X_M) = \Phi(G_M^1 - G_M, \ldots, G_M^N - G_M).$$

We now state the following assumptions on Φ.

Assumption 7.2.2 (Assumptions on the final condition)**.** *(i) Φ is Lipschitz continuous and increasing in the sense that*

$$\Phi(x+y) \geq \Phi(x) \textit{ for all } x \in \mathbb{R}^N \textit{ and } y \in \mathbb{R}_+^N.$$

(ii)For all $x \in \mathbb{R}^N$ and $\lambda > 0$, $\Phi(\lambda x) = \lambda \Phi(x)$ and $\Phi(x + \lambda \mathbf{1}) = \Phi(x) + \lambda$.

(iii)There exists $\theta > 0$ so that

$$\Phi(x+y) \geq \Phi(x) + \frac{\theta}{N} y \cdot \mathbf{1} \textit{ for all } x \in \mathbb{R}^N \textit{ and } y \in \mathbb{R}_+^N.$$

Trivially, (i) and (ii) holds for classical examples of functions such as

$$\Phi_m(x) := \max_i x^i. \tag{7.3}$$

However, this choice of final value does not satisfy (iii). In order to satisfy all the assumption, one can perturb the function $\Phi_m(x)$ as

$$\Phi_{m,\theta}(x) := (1-\theta)\,\Phi_m(x) + \frac{\theta}{N} \sum_i x^i$$

for $\theta \in (0,1)$ by making the forecaster partially satisfied if he does better than the average. Our Theorems 7.4.3 and 7.4.7 below state that the leading order expansion of the regret crucially depends on whether Φ satisfies the Assumption 7.2.2 (iii) or not.

The objective of the forecaster is to minimize his expected regret at maturity M

while the objective of the adversary is to maximize the regret of the forecaster. Then, given the terminal condition Φ, for $x \in \mathbb{R}^N$ and $m \in \{0, \dots, M-1\}$, we can define the value function of interest via the iteration

$$V^M(M, x) := \Phi(x) \tag{7.4}$$

$$V^M(m, x) := \min_{\phi_m} \max_{\alpha_m} \mathbb{E}^{\phi_m, \alpha_m}[V^M(m+1, x + \Delta X_m)], \tag{7.5}$$

where $\mathbb{E}^{\phi_m, \alpha_m}$ is the expectation given the choices of ϕ_m and α_m. Since the space of strategies is the same for each m, we might suppress m from notation $\phi_m, \alpha_m, \Delta G_m, \Delta X_m$ in the dynamic programming equation (7.5).

We have the following result for the value function.

Lemma 7.2.3. *Under Assumption 7.2.2 (i) and (ii), for all $m \in \{0, \dots, M\}$ and $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N_+$, we have the following relations*

$$V^M(m, x) = \max_{\alpha} \min_{\phi} \mathbb{E}^{\phi, \alpha}[V^M(m+1, x + \Delta X)] \tag{7.6}$$

$$V^M(m, x + \lambda \mathbf{1}) = V^M(m, x) + \lambda, \quad \text{and } V^M(m, x + y) \geq V^M(m, x). \tag{7.7}$$

If we also make the Assumption 7.2.2 (iii), then

$$V^M(m, x + y) \geq V^M(m, x) + \frac{\theta}{N} y \cdot \mathbf{1}. \tag{7.8}$$

Proof. The integrability of the random variables are a direct consequence of the Lipschitz continuity of Φ that passes to V^M by induction. It is clear that for all ϕ the mapping $\alpha \mapsto \mathbb{E}^{\phi,\alpha}[V^M(m+1, x + \Delta X)]$ is linear therefore convex. Similarly, for all α the mapping $\phi \mapsto \mathbb{E}^{\phi,\alpha}[V^M(m+1, x + \Delta X)]$ is concave. Given the Remark 7.2.1, we can apply the classical minimax theorem to commute the min and the max. (7.7) is a simple consequence of the invariance of the final condition Φ in Assumption 7.2.2 (ii) and similarly (7.8) is a consequence of Assumption 7.2.2 (iii). $\square$

7.3 PDE describing the long-time regime

In order to study the behavior of V^M for large M, we define the scaled value function as

$$u^M(t,x) = \frac{1}{\sqrt{M}} V^M\left(\lceil Mt \rceil, \sqrt{M}x\right).$$

Thanks to Lemma 7.2.3, it can be easily seen that u^M satisfies equations

$$\begin{aligned} u^M(t,x) &= \min_{\phi} \max_{\alpha} \mathbb{E}^{\phi,\alpha}\left[u^M\left(t+\frac{1}{M}, x+\frac{1}{\sqrt{M}}\Delta X\right)\right] \\ &= \max_{\alpha} \min_{\phi} \mathbb{E}^{\phi,\alpha}\left[u^M\left(t+\frac{1}{M}, x+\frac{1}{\sqrt{M}}\Delta X\right)\right], \end{aligned} \tag{7.9}$$

where ϕ, α are the strategies of the forecaster and the adversary respectively.

Our objective is to study the value function V^M for large M via the limit of the scaled function u^M. In order to illustrate the underlying ideas of our main results and define the relevant quantities, we first assume that $u^M \to u$ as $M \to \infty$, and that u is regular enough. According to the Taylor expansion of the right-hand side of (7.9), we obtain that

$$0 = \min_{\phi} \max_{\alpha} \mathbb{E}^{\phi,\alpha}\left[\sqrt{M}\nabla u(t,x)\cdot \Delta X + \partial_t u(t,x) + \frac{1}{2}\sum_{i,j=1}^{N} \partial^2_{ij} u(t,x) \Delta X^i \Delta X^j\right] + o(1) \tag{7.10}$$

$$= \max_{\alpha} \min_{\phi} \mathbb{E}^{\phi,\alpha}\left[\sqrt{M}\nabla u(t,x)\cdot \Delta X + \partial_t u(t,x) + \frac{1}{2}\sum_{i,j=1}^{N} \partial^2_{ij} u(t,x) \Delta X^i \Delta X^j\right] + o(1). \tag{7.11}$$

For large enough M, in order to have the equality in this expansion, the following conditions have to hold for all $(t,x) \in [0,1) \times \mathbb{R}^N$,

$$0 = \min_{\phi} \max_{\alpha} \nabla u(t,x) \cdot \mathbb{E}^{\phi,\alpha}\left[\Delta X\right] = \max_{\alpha} \min_{\phi} \nabla u(t,x) \cdot \mathbb{E}^{\phi,\alpha}\left[\Delta X\right]. \tag{7.12}$$

Additionally, (7.7) yields that

$$\mathbf{1} \cdot \nabla u = 1.$$

Regarding the control of the adversary, assume that there exists $j_0, j_1 \in \{1, \ldots, N\}$ so that

$$(1 - a^{j_0} - b^{j_0})\mu^{j_0} + b^{j_0} < (1 - a^{j_1} - b^{j_1})\mu^{j_1} + b^{j_1}.$$

We can define the control ϕ by

$$\phi^j = \partial_j u(t,x) \text{ if } j \in \{1, \ldots, N\} \setminus \{j_0, j_1\}, \tag{7.13}$$

and

$$\phi^{j_1} = \partial_{x_{j_0}} u(t,x) + \partial_{x_{j_1}} u(t,x),\ \phi^{j_0} = 0.$$

Computing

$$\begin{aligned} 0 &= \max_{\alpha} \min_{\phi} \nabla u(t,x) \cdot \mathbb{E}^{\phi,\alpha}\left[\Delta X\right] \\ &= \max_{\alpha} \min_{\phi} \sum_{j=1}^{N} \left(\partial_j u(t,x) - \phi^j\right) \left((1 - a^j - b^j)\mu^j + b^j\right), \end{aligned} \tag{7.14}$$

this choice of ϕ leads to

$$\begin{aligned} &\min_{\phi} \sum_{j=1}^{N} \left(\partial_j u(t,x) - \phi^j\right) \left((1 - a^j - b^j)\mu^j + b^j\right) \\ &\leq \partial_{x_{j_0}} u(t,x) \left((1 - a^{j_0} - b^{j_0})\mu^{j_0} + b^{j_0} - (1 - a^{j_1} - b^{j_1})\mu^{j_1} - b^{j_1}\right). \end{aligned}$$

If $\partial_{x_{j_0}} u(t,x) > 0$, we obtain a contradiction with (7.14). Therefore, the first order condition (7.12) implies that for all j_0 so that $\partial_{x_{j_0}} u(t,x) > 0$, we have $\mathbb{E}^\alpha[\Delta G^{j_0}] = (1 - a^{j_0} - b^{j_0})\mu^{j_0} + b^{j_0} = \sup_j \big((1 - a^j - b^j)\mu^j + b^j\big) = \sup_j \mathbb{E}^\alpha[\Delta G^j]$. In order to describe the dynamics of u, we define

$$\mathcal{A}(p) := \left\{ \alpha \in \mathcal{A} : \mathbb{E}^\alpha[\Delta G^i] = \sup_j \mathbb{E}^\alpha[\Delta G^j] \text{ if } p_i > 0 \right\},$$

for any $p \in [0,\infty)^N$.

Therefore, by the heuristic expansion above, one expects that if there is a limit u of u^M, then u has to solve

$$0 = \partial_t u(t,x) + \frac{1}{2} \max_{\alpha \in \mathcal{A}(\partial u(t,x))} \sum_{i,j=1}^N \partial^2_{ij} u(t,x) \mathbb{E}^{\alpha,\partial u(t,x)} \left[\Delta X^i \Delta X^j \right].$$

Since $\mathbf{1} \cdot \nabla u = 1$ implies that $\mathbf{1} \cdot \nabla^2 u = 0$, the above equation is equivalent to that

$$0 = \partial_t u(t,x) + \frac{1}{2} \max_{\alpha \in \mathcal{A}(\partial u(t,x))} \sum_{i,j=1}^N \partial^2_{ij} u(t,x) \mathbb{E}^{\alpha} \left[\Delta G^i \Delta G^j \right]. \tag{7.15}$$

For notational simplicity, for all $p \in \mathbb{R}^N_+$ and $S \in \mathbb{S}_N$, we define

$$H(p,S) := \frac{1}{2} \max_{\alpha \in \mathcal{A}(p)} \sum_{i,j=1}^N S_{ij} \mathbb{E}^\alpha[\Delta G^i \Delta G^j], \tag{7.16}$$

so that (7.15) can be written as

$$0 = \partial_t u(t,x) + H(\nabla u(t,x), \nabla^2 u(t,x)). \tag{7.17}$$

Equations of type (7.17) are studied in [75, 102, 176, 177] in the context of geometric flows. In particular [176] provides a stochastic representation for geometric flow type equations. Note that our equation (7.17) is not geometric in the sense of [16, Equation (1.3)] and our problem can be seen as a deterministic game where the adversary and the forecaster chooses (deterministic controls) in $\mathcal{A}$ and the simplex of dimension N. Thus, in this regard, similar to [131], our main results can be seen as representations for the solutions to (7.17) as the limit of deterministic games (whenever wellposedness of (7.17) holds).

Similar equations also appear in [55, 92, 93] in the context of prediction. In particular, our Assumption 7.2.2 (iii) is inspired by [92] where the authors study the long-time behavior of a prediction problem where the experts are history-dependent and not controlled by the adversary. This point has a fundamental impact on the problem. Indeed, impressively, the limiting equation in [92] is geometric and can be solved by considering the evolution of its level sets. Similarly to [94, 107], in our framework the adversary has to solve a control problem in the long-time regime. Thus, the equation (7.17) is fully nonlinear and in general it is not solvable via geometric methods. However, in some particular cases, we find explicit solutions to (7.17) by finding an optimal control for the adversary; see Section 7.5.

Unlike the various cases in the literature where the generator is continuous on $\mathbb{R}^N - \{0\}$, depending on the specification of (μ_i), H might fail to be continuous in p on the set $\{(p, S) \in \mathbb{R}^N_+ \times \mathbb{S}_N : p_i = 0 \text{ for some } i\}$. This lack of continuity has a crucial impact on the wellposedness for viscosity solution of (7.17) and the comparison result for this PDE is not available in the literature.

Note also that under the Assumption 7.2.2 (iii), formally, we have the inequality $\partial_{x_j} u^M(t,x) \geq \frac{\theta}{N} > 0$ for all $j \in \{1, \dots, N\}$. Thus, in this case, one expects that

$$\mathcal{A}(\nabla u(t,x)) := \left\{ \alpha \in \mathcal{A} : \mathbb{E}^{\alpha}[\Delta G^i] = \sup_j \mathbb{E}^{\alpha}[\Delta G^j], \, \forall i \right\},$$

and the set of strategies for the adversary yields the balanced strategies defined in [107].

Definition 7.3.1. $\mathcal{A}_B$ denotes the set of "balanced" strategies α for the adversary, i.e., strategies $\alpha \in \mathcal{A}$ satisfying

$$\mathbb{E}^\alpha[\Delta G^{j_0}] = \mathbb{E}^\alpha[\Delta G^{j_1}] \tag{7.18}$$

for all $j_0, j_1 \in \{1, \ldots, N\}$. For any $\alpha \in \mathcal{A}_B$, we define

$$c_\alpha := \mathbb{E}^\alpha[\Delta G^i] = (1 - a^i - b^i)\mu^i + b^i, \text{ for any } i = 1, \ldots, N. \tag{7.19}$$

Note that for any $p \in (0, \infty)^N$, $\mathcal{A}(p) = \mathcal{A}_B$ no matter this set is empty or not. We provide a necessary and sufficient condition on (μ_i) for the existence of balanced strategies.

Proposition 7.3.2. *The set of balanced strategies $\mathcal{A}_B$ is not empty if and only if*

$$\inf_{c \in [0,1]} \sum_{i=1}^{N} \left(\frac{\mu^i - c}{\mu^i} \vee \frac{c - \mu^i}{1 - \mu^i} \right) \leq 1, \tag{7.20}$$

where we make the convention $\frac{0}{0} = 0$.

Proof. Suppose $\mathcal{A}_B \neq \emptyset$, and $\alpha \in \mathcal{A}_B$. Then according to (7.19), we have that $c_\alpha = (1 - a^j - b^j)\mu^j + b^j$, $\forall j$. If $c_\alpha \geq \mu^j$, we have that

$$c_\alpha - \mu^j = -(a^j + b^j)\mu^j + b^j \leq (1 - \mu^j)(a^j + b^j),$$

which is equivalent to that $\frac{c_\alpha - \mu^j}{1 - \mu^j} \leq a^j + b^j$. If $\mu^j \geq c_\alpha$, we obtain that

$$\mu^j - c_\alpha = (a^j + b^j)\mu^j - b^j \leq (a^j + b^j)\mu^j,$$

and hence $\frac{\mu^j - c_\alpha}{\mu_j} \leq a^j + b^j$. Since $\sum_{j=1}^N (a^j + b^j) = 1$, we get that

$$\inf_{c \in [0,1]} \sum_{i=1}^{N} \left(\frac{\mu^i - c}{\mu^i} \vee \frac{c - \mu^i}{1 - \mu^i} \right) \leq \sum_{j=1}^{N} \left(\frac{\mu^j - c_\alpha}{\mu^j} \vee \frac{c_\alpha - \mu^j}{1 - \mu^j} \right) \leq \sum_{j=1}^{N} (a^j + b^j) = 1.$$

For the converse, suppose there exists some $c \in [0,1]$ such that $\sum_{j=1}^N \left(\frac{\mu^j - c}{\mu^j} \vee \frac{c-\mu^j}{1-\mu^j} \right) \leq 1$. Denote $s := \sum_{j=1}^N \left(\frac{\mu^j - c}{\mu^j} \vee \frac{c-\mu^j}{1-\mu^j} \right)$. For each $j = 1, \ldots, N$, if $c \geq \mu^j$, we take

$$a^j = (c - \mu^j)\left(\frac{1}{s} - s\right), \quad b^j = (c - \mu^j)\left(1 + \frac{\mu^j}{s(1-\mu^j)}\right),$$

and if $\mu^j \geq c$,

$$a^j = (\mu^j - c)\left(\frac{1-\mu^j}{s\mu^j} + 1\right), \quad b^j = (\mu^j - c)\left(\frac{1}{s} - 1\right).$$

It can be easily verified that $a^j, b^j \in [0,1]$, $\sum_{j=1}^N (a^j + b^j) = 1$, and (a^j, b^j) satisfies (7.19). Therefore, it is a balanced strategies. $\square$

For notational simplicity, we also define the generator,

$$H_B(S) := \frac{1}{2} \max_{\alpha \in \mathcal{A}_B} \sum_{i,j=1}^N S_{ij} \mathbb{E}^\alpha[\Delta G^i \Delta G^j], \tag{7.21}$$

whose definition is motivated by the fact that

$$\begin{aligned}
H^*(p,S) &:= \limsup_{(q,R)\to(p,S)} H(q,R) = H(p,S) \text{ and} \\
H_*(p,S) &:= \liminf_{(q,R)\to(p,S)} H(q,R) = H_B(S) \text{ for all } p \in \mathbb{R}_+^N \text{ if } \mathcal{A}_B \neq \emptyset.
\end{aligned}$$

Given this discontinuity of the generator, we provide here the definition of viscosity solutions which is also available in [102].

Definition 7.3.3. An upper (resp. lower) semicontinuous function u is a viscosity subsolution (resp. supersolution) of (7.17) if for all $(t,x) \in [0,1) \times \mathbb{R}^N$ and smooth function ϕ so that $u - \phi$ has a local maximum (resp. minimum) at (t,x), we have that

$$-\partial_t \phi(t,x) - H^*(\nabla\phi(t,x), \nabla^2\phi(t,x)) \leq 0$$

$$(resp. -\partial_t\phi(t,x) - H_*(\nabla\phi(t,x), \nabla^2\phi(t,x)) \geq 0).$$

7.4 Main results

In this section, we provide the main results regarding the growth of regret and asymptotically optimal strategies of the forecaster and the adversary. The results fundemantally depend on whether $\mathcal{A}_B = \emptyset$ or not.

7.4.1 Growth of regret for the case $\mathcal{A}_B \neq \emptyset$

We assume in this subsection that $\mathcal{A}_B \neq \emptyset$. We prove the following priori bound for u^M.

Lemma 7.4.1. *Assume that Assumption 7.2.2 (i) holds. Then, there exists a constant C independent of M such that for all $(t,x) \in [0,1] \times \mathbb{R}^N$*

$$|u^M(t,x) - \Phi(x)| \leq C(2-t).$$

Proof. For any $\epsilon > 0$, there exists a mollifier η such that

$$|\eta * \Phi - \Phi|_\infty < \epsilon.$$

Define $\tilde{\Phi} := \eta * \Phi$, and it suffices for us to show that

$$|u^M(t,\cdot) - \tilde{\Phi}(\cdot)|_\infty \leq C(2-t). \tag{7.22}$$

According to the terminal condition of u^M, the inequality (7.22) holds for $t = 1$. Assume it is true for $t < 1$, we prove for $t - \frac{1}{M}$. Due to the dynamical programming equations and our induction, we get that

$$
\begin{aligned}
|u^M(t-1/M,x)-\tilde{\Phi}(x)| =& \left|\min_{\phi}\max_{\alpha}\mathbb{E}^{\phi,\alpha}\left[u^M(t,x+\Delta X/\sqrt{M})\right]-\tilde{\Phi}(x)\right| \\
\leq& \left|\min_{\phi}\max_{\alpha}\mathbb{E}^{\phi,\alpha}\left[u^M(t,x+\Delta X/\sqrt{M})-\tilde{\Phi}(x+\Delta X/\sqrt{M})\right.\right. \\
&\left.\left. +\ \tilde{\Phi}(x+\Delta X/\sqrt{M})-\tilde{\Phi}(x)\right]\right| \\
\leq& C(2-t)+\left|\min_{\phi}\max_{\alpha}\mathbb{E}^{\phi,\alpha}\left[\tilde{\Phi}(x+\Delta X/\sqrt{M})-\tilde{\Phi}(x)\right]\right|
\end{aligned}
$$

Applying Taylor expansion to $\tilde{\Phi}(x)$, we obtain that

$$
\begin{aligned}
&\min_{\phi}\max_{\alpha}\mathbb{E}^{\phi,\alpha}\left[\tilde{\Phi}(x+\Delta X/\sqrt{M})-\tilde{\Phi}(x)\right] \\
&=\frac{1}{\sqrt{M}}\min_{\phi}\max_{\alpha}\mathbb{E}^{\phi,\alpha}\left[\nabla\tilde{\Phi}(x)\cdot\Delta X\right]+O(1/M) \\
&=\frac{1}{\sqrt{M}}\min_{\phi}\max_{\alpha}\sum_{i=1}^{N}[\partial_i\tilde{\Phi}-\phi^i]\mathbb{E}^{\alpha}[\Delta G_i]+O(1/M).
\end{aligned}
$$

Choosing $\phi_i=\partial_i\tilde{\Phi}$, and $\alpha\in\mathcal{A}_B$, it can be easily checked that the minimax is zero. Since the second derivative of $\tilde{\Phi}$ is upper bounded, there exists a constant $C>0$ such that

$$
\min_{\phi}\max_{\alpha}\mathbb{E}^{\phi,\alpha}\left[\tilde{\Phi}(x+\Delta X/\sqrt{M})-\tilde{\Phi}(x)\right]\leq C/M,
$$

and hence

$$
|u^M(t-1/M,x)-\tilde{\Phi}(x)|\leq C(2-(t-1/M)).
$$

$\square$

Given Lemma 7.4.1, we define the functions

$$\overline{u}(t,x) := \limsup_{(M,s,y)\to(\infty,t,x)} u^M(s,y) \tag{7.23}$$

$$\underline{u}(t,x) := \liminf_{(M,s,y)\to(\infty,t,x)} u^M(s,y). \tag{7.24}$$

The following comparison principle is a special case of [102, Theorem 2.1] backwards in time.

Lemma 7.4.2. *Under Assumption 7.2.2 (i)-(ii) and subject to the final condition $U(1,x) = \Phi(x)$, there exists a unique viscosity solution to*

$$0 = \partial_t U(t,x) + H_B(\nabla^2 U(t,x)) \tag{7.25}$$

that grows at most linearly and is uniformly continuous. We will denote this unique solution by U. Moreover, if u_1 is a subsolution, and u_2 is a supersolution, then comparison principle holds, i.e., $u_1 \le U \le u_2$ on $[0,1]\times\mathbb{R}^N$.

Proof. The result is a direct consequence of [102, Theorem 2.1]. □

Thanks to the identity $H_*(p,S) = H_B(S)$, any supersolution to (7.17) is also a supersolution to (7.25). In the following theorem, using this property, we show that U provides a lower bound for the scaled value function.

Theorem 7.4.3. *Assume that Assumption 7.2.2 (i) and (ii) holds. Then, $\underline{u}$(resp. $\overline{u}$) is a supersolution (resp subsolution) of (7.17) subject to the terminal condition $\underline{u}(1,x) = \overline{u}(1,x) = \Phi(x)$, and hence the solution of (7.25) provides a lower bound to the growth of regret as*

$$\liminf_{M\to\infty} \frac{1}{\sqrt{M}} V^M\left(\lceil Mt\rceil, \sqrt{M}x\right) \ge \underline{u}(t,x) \ge U(t,x), \tag{7.26}$$

where U is the unique viscosity solution to (7.25).

Proof. The proof is almost the same as [17, Theorem 2.1] and [94, Theorem 7], and we only indicate our modifications. We first show the supersolution property of $\underline{u}$. Let $(t_0, x_0) \in [0,1) \times \mathbb{R}^N$ and ψ smooth so that $\underline{u} - \psi$ has a strict local minimum at (t_0, x_0). Then, similarly to [17, Theorem 2.1], there exists $M_n \to 0$ and $(s_n, y_n) \to (t_0, x_0)$ satisfying

$$u^{M_n}(s_n, y_n) \to \underline{u}(t_0, x_0) \text{ and } u^{M_n} - \psi \text{ has a local minimum at } (s_n, y_n).$$

Denote $\xi_n = u^{M_n}(s_n, y_n) - \psi(s_n, y_n)$ that converges to 0. The dynamic programming principle (7.9) and the minimality condition for $u^{M_n} - \psi$ yields that

$$\begin{aligned}
\xi_n &= u^{M_n}(s_n, y_n) - \psi(s_n, y_n) \\
&= \min_{\phi} \max_{\alpha} \mathbb{E}^{\phi,\alpha} \left[u^M \left(s_n + \frac{1}{M_n}, y_n + \frac{1}{\sqrt{M_n}} \Delta X \right) - \psi(s_n, y_n) \right] \\
&\geq \min_{\phi} \max_{\alpha} \mathbb{E}^{\phi,\alpha} \left[\psi \left(s_n + \frac{1}{M_n}, y_n + \frac{1}{\sqrt{M_n}} \Delta X \right) - \psi(s_n, y_n) + \xi_n \right]
\end{aligned}$$

where we used the minimality of $u^{M_n} - \psi$ to obtain the inequality. Given that ψ is fixed, we can now proceed to expand as in (7.10) to have

$$\begin{aligned}
&o(1) \geq \\
&\min_{\phi} \max_{\alpha \in \mathcal{A}} \mathbb{E}^{\phi,\alpha} \left[\sqrt{M_n} \nabla \psi(s_n, y_n) \cdot \Delta X + \partial_t \psi(s_n, y_n) + \frac{1}{2} \sum_{i,j=1}^{N} \partial_{ij}^2 \psi(s_n, y_n) \Delta X^i \Delta X^j \right].
\end{aligned}$$

By restricting the choice of α this inequality in particular implies that

$$
\begin{aligned}
&o(1) \\
&\geq \min_{\phi} \max_{\alpha \in \mathcal{A}_B} \mathbb{E}^{\phi,\alpha}\left[\sqrt{M_n}\nabla\psi(s_n,y_n)\cdot\Delta X + \partial_t\psi(s_n,y_n) + \frac{1}{2}\sum_{i,j=1}^{N}\partial_{ij}^2\psi(s_n,y_n)\Delta X^i\Delta X^j\right] \\
&\geq \max_{\alpha \in \mathcal{A}_B} \mathbb{E}^{\phi,\alpha}\left[\partial_t\psi(s_n,y_n) + \frac{1}{2}\sum_{i,j=1}^{N}\partial_{ij}^2\psi(s_n,y_n)\Delta X^i\Delta X^j\right]
\end{aligned}
$$

where we used the fact that for a balanced strategy the regret does not depend on the strategy of the forecaster. u^M satisfies

$$
u^M(s, y + \lambda\mathbf{1}) = u^M(s,y) + \lambda \tag{7.27}
$$

for all $(s,y) \in [0,1] \times \mathbb{R}^N$ and $\lambda \in \mathbb{R}$, and $u^{M_n} - \psi$ has a local minimum at (s_n, y_n). Thus, ψ satisfies $\nabla\psi(s_n,y_n)\cdot\mathbf{1} = 1$. Therefore, similarly as in (7.15), we easily obtain that

$$
o(1) \geq \partial_t\psi(s_n,y_n) + H_B(\nabla^2\psi(s_n,y_n)) = \partial_t\psi(s_n,y_n) + H_*(\nabla\psi(s_n,y_n), \nabla^2\psi(s_n,y_n)).
$$

The convergence of (s_n, y_n), and the continuity of H_B concludes the proof of the super solution.

We now prove the subsolution property of $\overline{u}$. Similarly as above, for a given $(t_0, x_0) \in [0,1) \times \mathbb{R}^N$ and ψ smooth so that $\underline{u} - \psi$ has a strict local maximum at (t_0, x_0), we can establish that

$$
\begin{aligned}
o(1) \leq \max_{\alpha\in\mathcal{A}} \min_{\phi} \Bigg\{ &\sqrt{M_n}\sum_{j=1}^{N}\left(\partial_j\psi(s_n,y_n) - \phi^j\right)\left((1-a^j-b^j)\mu^j + b^j\right) \\
&+\partial_t\psi(s_n,y_n) + \frac{1}{2}\sum_{i,j=1}^{N}\partial_{ij}^2\psi(s_n,y_n)\mathbb{E}^{\phi,\alpha}\left[\Delta X^i\Delta X^j\right]\Bigg\}.
\end{aligned} \tag{7.28}
$$

Let $\alpha \notin \mathcal{A}(\nabla\psi(s_n, y_n))$, then there exists $i \in \{1, \ldots, N\}$ so that $\partial_i \psi(s_n, y_n) > 0$ and

$$(1 - a^i - b^i)\mu^i + b^i < \sup_j (1 - a^j - b^j)\mu^j + b^j.$$

Similarly as (7.13), for such a strategy one can find strategy ϕ for the forecaster so that

$$\sum_{j=1}^{N} \left(\partial_j \psi(s_n, y_n) - \phi^j\right) \left((1 - a^j - b^j)\mu^j + b^j\right) \leq -\epsilon < 0$$

for all n large enough. Thus, the maximum in (7.28) cannot be achieved at such a strategy for n large enough. Therefore, (7.28) yields

$$\begin{aligned}
o(1) \leq & \max_{\alpha \in \mathcal{A}(\nabla\psi(s_n, y_n))} \min_{\phi} \left\{ \sqrt{M_n} \sum_{j=1}^{N} \left(\partial_j \psi(s_n, y_n) - \phi^j\right) \left((1 - a^j - b^j)\mu^j + b^j\right) \right. \\
& \left. + \partial_t \psi(s_n, y_n) + \frac{1}{2} \sum_{i,j=1}^{N} \partial^2_{ij} \psi(s_n, y_n) \mathbb{E}^{\phi,\alpha} \left[\Delta X^i \Delta X^j\right] \right\} \\
\leq & \max_{\alpha \in \mathcal{A}(\nabla\psi(s_n, y_n))} \left\{ \sqrt{M_n} \sup_j \left((1 - a^j - b^j)\mu^j + b^j\right) \left(\sum_{j=1,\, \partial_j \psi(s_n, y_n) > 0}^{N} \partial_j \psi(s_n, y_n) - 1 \right) \right. \\
& \left. + \partial_t \psi(s_n, y_n) + \frac{1}{2} \sum_{i,j=1}^{N} \partial^2_{ij} \psi(s_n, y_n) \mathbb{E}^{\alpha} \left[\Delta G^i \Delta G^j\right] \right\} \\
\leq & \ \partial_t \psi(s_n, y_n) + \max_{\alpha \in \mathcal{A}(\nabla\psi(s_n, y_n))} \frac{1}{2} \sum_{i,j=1}^{N} \partial^2_{ij} \psi(s_n, y_n) \mathbb{E}^{\alpha} \left[\Delta G^i \Delta G^j\right]
\end{aligned}$$

where we use the fact that

$$\sum_{j=1,\, \partial_j \psi(s_n, y_n) > 0}^{N} \partial_j \psi(s_n, y_n) = 1,$$

and

$$\sum_{i,j=1}^{N} \partial^2_{ij} \psi(s_n, y_n) \mathbb{E}^{\phi,\alpha} \left[\Delta X^i \Delta X^j\right] = \sum_{i,j=1}^{N} \partial^2_{ij} \psi(s_n, y_n) \mathbb{E}^{\alpha} \left[\Delta G^i \Delta G^j\right]$$

due to $\nabla\psi(t_n, y_n)\cdot \mathbf{1} = 1$. Thus, we finally obtain that

$$o(1) \leq \partial_t\psi(s_n, y_n) + H(\nabla\psi(s_n, y_n), \nabla^2\psi(s_n, y_n)),$$

which leads to the subsolution property

$$0 \leq \partial_t\psi(t_0, x_0) + H^*(\nabla\psi(t_0, x_0), \nabla^2\psi(t_0, x_0)).$$

Given the supersolution property of $\underline{u}$, the identity $H_*(p, S) = H_B(S)$ and the comparison result in Lemma 7.4.2, we easily have that $\underline{u} \geq U$ which implies (7.26). □

Remark 7.4.4. Although, it is mathematically appealing to have a comparison result for the PDE (7.17), we do not need it for practical problems such as lower bound of growth of regret such as (7.26). The lower bound of regret is a consequence of supersolution property of $\underline{u}$ and a comparison result for the PDE (7.25) (which is a significantly simpler task than a comparison for (7.17)).

Remark 7.4.5. The classical online problem easily yields an upper bound. Indeed, in this problem the adversary decides on the distribution of experts' predictions at each round, i.e., the adversary chooses an element in the probability space over $\{0,1\}^N$, see [26, 28, 94], etc. Denote the value function of this game by $W^M(m, x)$. According to [94, Theorem 7], we have that

$$\lim_{M\to\infty} \frac{1}{\sqrt{M}} W^M\left(\lceil Mt\rceil, \sqrt{M}x\right) = w(t, x),$$

where $w(t, x)$ is the viscosity solution to

$$\partial_t w(t, x) + \frac{1}{2}\max_{v\in\{0,1\}^N} \langle \partial^2_{xx} w(t, x)\cdot v, v\rangle = 0, \tag{7.29}$$

$$w(1, x) = \Phi(x). \tag{7.30}$$

Since the adversary in this game fully controls the the prediction of experts, it can be easily seen that $V^M(m,x) \le W^M(m,x)$, and therefore we obtain that

$$V^M(\lceil Mt \rceil, \sqrt{M}x) \le w(t,x)\sqrt{M} + o(\sqrt{M}). \tag{7.31}$$

Both in the classical problem in Remark 7.4.5 and in the description of the lower bound function u, the set of strategies of the adversary are balanced. We now provide a counter example that shows that without Assumption 7.2.2 (iii) it might be optimal for the adversary to choose a non balanced strategy by exhibiting a case where (7.26) is strict. Thus, unlike in [107], with corruption, the optimal strategy of the adversary is not always balanced. This example also shows that, in general, $\overline{u}$ can not be a subsolution to (7.25), but has to be characterized as a subsolution to (7.17). We will show in Theorem 7.4.7 that Assumption 7.2.2 (iii) is in fact sufficient to obtain that $\underline{u} = \overline{u}$ and solves (7.25).

Example 7.4.6. For $N = 3$, $\mu^1 = 0, \mu^2 = \mu^3 = 1$, it can be easily verified that $\mathcal{A}_B = \{(a^i, b^i) : b^1 = 1\}$, i.e., the adversary always corrupts the first expert, and set his gain to 1. Then the viscosity solution of (7.25) is $U(t,x) = \Phi_m(x) = \max_i x^i$. However, if the adversary chooses the strategy ($a^2 = a^3 = 1/2$), then we have that $u^M(t,x) > \Phi_m(x)$ for any $t \in [0,1)$. Therefore $\limsup_{M\to\infty} u^M(t,x)$ cannot always be a subsolution of (7.25).

The following Theorem and Example 7.4.6 show the importance of Assumption 7.2.2 (iii), which allows us to obtain the exact growth rate of regret. With Assumption 7.2.2 (iii), formally we obtain that $\nabla u^M \in (\theta/N, +\infty)^N$, and thus the adversary is forced to use balanced strategies. Therefore, we can show that the scaled value function converges to the solution U of (7.25).

Theorem 7.4.7. *Assume that Assumption 7.2.2 (i), (ii) and (iii) hold. Then, $\underline{u}$(resp. $\overline{u}$) is a lower (resp. upper) semicontinuous viscosity supersolution (resp. subsolution)* *of* (7.25) *subject to the terminal condition* $\underline{u}(1,x) = \overline{u}(1,x) = \Phi(x)$. *Therefore,* $\overline{u} = \underline{u} = U$ *provides the growth rate of regret as*

$$V^M\left(\lceil Mt\rceil, \sqrt{M}x\right) = U(t,x)\sqrt{M} + o(\sqrt{M}).$$

Proof. By (7.8), we have that

$$\begin{aligned} u^M(t,x+y) &= \min_{\phi}\max_{\alpha}\mathbb{E}^{\phi,\alpha}\left[u^M\left(t+\frac{1}{M}, x+y+\frac{1}{\sqrt{M}}\Delta X\right)\right] \\ &\geq \min_{\phi}\max_{\alpha}\mathbb{E}^{\phi,\alpha}\left[u^M\left(t+\frac{1}{M}, x+\frac{1}{\sqrt{M}}\Delta X\right)\right] + \frac{\theta}{N}\langle y, \mathbf{1}\rangle \\ &= u^M(t,x) + \frac{\theta}{N}\langle y, \mathbf{1}\rangle. \end{aligned}$$

Therefore if $\underline{u} - \psi$ or $\bar{u} - \psi$ attains a local extreme at $(t_0, x_0) \in [0,1) \times \mathbb{R}^N$, it follows that $\nabla\psi(t_0, x_0) \in \left[\frac{\theta}{N}, +\infty\right)^N$. Then, following the same arguments as in the proof of Theorem 7.4.3, we obtain the sign of

$$\partial_t\psi(s_n, y_n) + H(\nabla\psi(s_n, y_n), \nabla^2\psi(s_n, y_n))$$

for $(s_n, y_n) \to (t_0, x_0)$. The conclusion $\nabla\psi(t_0, x_0) \in \left[\frac{\theta}{N}, +\infty\right)^N$ and the identity $H(p, S) = H_B(S)$ if $p_i > 0$ for all i allows us to obtain the sign of

$$\partial_t\psi(s_n, y_n) + H(\nabla\psi(s_n, y_n), \nabla^2\psi(s_n, y_n)) = \partial_t\psi(s_n, y_n) + H_B(\nabla^2\psi(s_n, y_n)).$$

Therefore we obtain the required viscosity property. The conclusion of the theorem follows by the Lemma 7.4.2. □

Given the solution U of (7.25), we design strategies for the adversary. For a fixed maturity M, denote $\tilde{x} := \frac{x}{\sqrt{M}}, t_m := \frac{m}{M}$. We define the strategy for the adversary

$$\boldsymbol{\alpha}^* = (\alpha_1^*, \dots, \alpha_M^*) \tag{7.32}$$

via

$$\alpha_m^*(x) = \operatorname*{argmax}_{\alpha \in \mathcal{A}_B} \sum_{i,j=1}^{N} \partial_{ij}^2 U(t_{m-1}, \tilde{x}) \mathbb{E}^{\alpha}[\Delta G^i \Delta G^j].$$

Define

$$\underline{V}^M(0, x) = \inf_{\boldsymbol{\phi}} \mathbb{E}^{\boldsymbol{\phi}, \boldsymbol{\alpha}^*}[\Phi(X_M) \mid X_0 = x],$$

where $\boldsymbol{\phi} = (\phi_1, \dots, \phi_M)$ is any strategy of the forecaster. In the next proposition, we will show that

$$\lim_{M \to \infty} \frac{1}{\sqrt{M}} \underline{V}(0, \sqrt{M}x) \geq U(0, x), \tag{7.33}$$

under assumptions on U. In Section 7.5, we will verify these assumptions for a special case.

Proposition 7.4.8. *Assume that Assumption 7.2.2 (i) and (ii) hold. Suppose the solution U to (7.25) is smooth and satisfies the derivative bounds*

$$|\partial_{tt}^2 U(1-t, x)| \leq \frac{C}{t^{\frac{3}{2}}}, \quad |\partial_{tx}^2 U(1-t, x)| \leq \frac{C}{t}, \quad |\partial_{xxx}^3 U(1-t, x)| \leq \frac{C}{t}, \qquad \forall x \in \mathbb{R}^N, \tag{7.34}$$

for some positive constant C. Then (7.33) holds. Therefore, according to Theorem 7.4.3 the asymptotic strategy $\boldsymbol{\alpha}^*$ in (7.32) for the adversary guarantees U as a lower bound of regret, i.e.,

$$\lim_{M \to \infty} \frac{1}{M} V^M(0, \sqrt{M}x) \geq \lim_{M \to \infty} \frac{1}{M} \underline{V}^M(0, \sqrt{M}x) \geq U(0, x).$$

Proof. It can be easily verified that

$$
\begin{aligned}
\frac{1}{\sqrt{M}}\underline{V}^M(0,\sqrt{M}x) - U(0,x) =& \frac{\inf_{\boldsymbol{\phi}} \mathbb{E}^{\boldsymbol{\phi},\boldsymbol{\alpha}^*}[\Phi(X_M) \mid X_0 = \sqrt{M}x]}{\sqrt{M}} - U(0,x) \\
=& \inf_{\boldsymbol{\phi}} \mathbb{E}^{\boldsymbol{\phi},\boldsymbol{\alpha}^*}[U(1,\tilde{X}_M) \mid \tilde{X}_0 = x] - U(0,x) \\
=& \inf_{\boldsymbol{\phi}} \left(\sum_{m=1}^{M} \mathbb{E}^{\phi,\alpha_m^*}[U(t_m,\tilde{X}_m) - U(t_{m-1},\tilde{X}_{m-1}) \mid \tilde{X}_0 = x] \right).
\end{aligned}
$$

Note that

$$
\mathbb{E}^{\phi,\alpha}\left[U\left(t_m,\tilde{X}_m\right) - U\left(t_{m-1},\tilde{X}_{m-1}\right) \mid \tilde{X}_{m-1} = \tilde{x}_{m-1}\right] \tag{7.35}
$$

$$
= \mathbb{E}^{\phi,\alpha}\left[\partial_x U\left(t_{m-1},\tilde{x}_{m-1}\right)^\top \Delta\tilde{X}_m\right] \tag{7.36}
$$

$$
+ 2\mathbb{E}^{\phi,\alpha}\left[\int_0^{\sqrt{\frac{1}{M}}} \left(\sqrt{\frac{1}{M}} - s\right)\left(\partial_t U + \frac{1}{2}\Delta X_m^\top \cdot \partial_{xx}^2 U \cdot \Delta X_m\right)(t_{m-1},\tilde{x}_{m-1} + s\Delta X_m)ds\right] \tag{7.37}
$$

$$
+ 2\mathbb{E}^{\phi,\alpha}\left[\int_0^{\sqrt{\frac{1}{M}}} \left(\sqrt{\frac{1}{M}} - s\right)\left(\partial_t U(t_{m-1},\tilde{X}_m) - \partial_t U(t_{m-1},\tilde{x}_{m-1} + s\Delta X_m)\right)ds\right] \tag{7.38}
$$

$$
+ \mathbb{E}^{\phi,\alpha}\left[\int_0^{\frac{1}{M}} \left(\partial_t U(t_{m-1}+s,\tilde{X}_m) - \partial_t U(t_{m-1},\tilde{X}_m)\right)ds\right]. \tag{7.39}
$$

Under the strategy α_m^*, the term (7.36) is zero. Since $\partial_{xx}^2 U \cdot \mathbf{1} = 0$, the term (7.37) is independent of ϕ. Due to our choice of α_m^*, we have that

$$
\mathbb{E}^{\phi,\alpha_m^*}\left[\left(\partial_t U + \frac{1}{2}\Delta X_m^\top \cdot \partial_{xx}^2 U \cdot \Delta X_m\right)(t_{m-1},\tilde{x}_{m-1})\right] = 0.
$$

According to the derivatives bounds (7.34), it can be easily seen that

$$\begin{aligned}
&\left(\partial_t U + \frac{1}{2}\Delta X_m^\top \cdot \partial_{xx}^2 U \cdot \Delta X_m\right)(t_{m-1}, \tilde{x}_{m-1} + s\Delta X_m) \\
&\geq \left(\partial_t U + \frac{1}{2}\Delta X_m^\top \cdot \partial_{xx}^2 U \cdot \Delta X_m\right)(t_{m-1}, \tilde{x}_{m-1}) - \frac{Cs}{1 - t_{m-1}}.
\end{aligned}$$

Therefore the term (7.37) is bounded below by

$$-2C \int_0^{\sqrt{\frac{1}{M}}} \frac{\left(\sqrt{\frac{1}{M}} - s\right) s}{1 - t_{m-1}} ds = -\frac{C}{(1 - t_{m-1})M^{\frac{3}{2}}},$$

where C is allowed to change from line to line. Similarly, it can be easily verified that the term (7.38) is bounded below by $-\frac{C}{(1-t_{m-1})M^{\frac{3}{2}}}$. As a result of (7.34), we have that

$$\partial_t U(t_{m-1} + s, \tilde{X}_m) - \partial_t U(t_{m-1}, \tilde{X}_m) \geq -C \int_0^s \frac{1}{(1 - t_{m-1} - w)^{\frac{3}{2}}} dw,$$

and therefore the term (7.39) is bounded below by

$$\begin{aligned}
-C \int_0^{\frac{1}{M}} \int_0^s \frac{1}{(1 - t_{m-1} - w)^{\frac{3}{2}}} dw ds &= -C \int_0^{\frac{1}{M}} \int_w^{\frac{1}{M}} \frac{1}{(1 - t_{m-1} - w)^{\frac{3}{2}}} ds dw \\
&= -C \int_0^{\frac{1}{M}} \frac{\frac{1}{M} - s}{(1 - t_{m-1} - s)^{\frac{3}{2}}} ds.
\end{aligned}$$

Putting together all the estimates for (7.36), (7.37), (7.38) and (7.39) above, we conclude that

$$\mathbb{E}^{\phi,\alpha_m^*}\left[U\left(t_m,\tilde{X}_m\right)-U\left(t_{m-1},\tilde{X}_{m-1}\right)\mid \tilde{X}_{m-1}=\tilde{x}_{m-1}\right]$$
$$\geq -C\left(\frac{1}{(1-t_{m-1})M^{\frac{3}{2}}}+\int_0^{\frac{1}{M}}\frac{\frac{1}{M}-s}{(1-t_{m-1}-s)^{\frac{3}{2}}}ds\right).$$

It can be easily verified that

$$\lim_{M\to\infty}\sum_{m=1}^{M}\left(\frac{1}{(1-t_{m-1})M^{\frac{3}{2}}}+\int_0^{\frac{1}{M}}\frac{\frac{1}{M}-s}{(1-t_{m-1}-s)^{\frac{3}{2}}}ds\right)=0,$$

and therefore

$$\lim_{M\to\infty}\frac{1}{\sqrt{M}}\underline{V}^M(0,\sqrt{M}x)-U(0,x)\geq 0.$$

$\square$

One might expect that the function U captures important features of the problem, and the algorithm of the forecaster given by

$$\phi_m^*=\{\partial_j U(t_{m-1},\tilde{X}_{m-1})\}_{j=1}^N$$

yields the best algorithm for the growth of the regret, i.e., an equality holds in (7.33). Such a conjecture holds in [26, 55, 130, 129]. Unfortunately, as proved by the following counter example, in our case ϕ_m^* does not provide an asymptotic optimal algorithm.

Counter Example: Consider the case $N=2$, $\mu^1=\frac{3}{4},\mu^2=\frac{1}{4}$ with final condition

$\Phi = \Phi_{m,\theta}$. Let U be the viscosity solution to (7.25). Then it holds that

$$\lim_{M\to\infty} \frac{1}{\sqrt{M}} \sup_{\boldsymbol{\alpha}} \mathbb{E}^{\boldsymbol{\phi}^*,\boldsymbol{\alpha}}[\Phi(X_M) \,|\, X_0 = \sqrt{M}x] > U(0,x),$$

i.e., ϕ_m^* is not asymptotically optimal. According to Proposition 7.5.3 in the next section, it can be easily verified that (7.25) becomes

$$\partial_t U(t,x) + \frac{3}{32}(\partial_{11}^2 U(t,x) + \partial_{22}^2 U(t,x)) = 0.$$

From $\partial_{11}^2 U + \partial_{12}^2 U = \partial_{12}^2 U + \partial_{22}^2 U = 0$, we deduce that

$$\partial_{11}^2 U = \partial_{22}^2 U = -\partial_{12}^2 U = -\partial_{21}^2 U,$$

and hence

$$\partial_t U(t,x) + \frac{3}{16}\partial_{11}^2 U(t,x) = 0.$$

By Feynman-Kac representation of U, it can be easily verified $\partial_t U \le 0, \partial_{11}^2 U \ge 0$. By choosing $b^1 = a^2 = \frac{1}{2}$, we obtain that

$$\begin{aligned}
&\partial_t U(t,x) + \frac{1}{2}\max_{\alpha} \sum_{i,j} \partial_{ij}^2 U(t,x) \mathbb{E}^{\alpha}[\Delta G^i \Delta G^j] \\
&= \partial_t U(t,x) + \frac{3}{8}\partial_{11}^2 U(t,x) = -\partial_t U(t,x).
\end{aligned}$$

Therefore we obtain that

$$\begin{aligned}
&\sup_{\alpha} \mathbb{E}^{\phi_m^*,\alpha}\left[U\left(t_m, \tilde{X}_m\right) - U\left(t_{m-1}, \tilde{X}_{m-1}\right) \,|\, \tilde{X}_{m-1} = \tilde{x}_{m-1}\right] \\
&= -\frac{\partial_t U(t_{m-1}, \tilde{x}_{m-1})}{M} + o(1/M).
\end{aligned}$$

Due to the explicit formula of U, we obtain that

$$-\partial_t U(1-t, x^1, x^2) \geq ct^{-1/2} e^{-\frac{(x^1-x^2)^2}{4dt}}$$

for some positive constants c, d. To close the argument, we need an estimate of $\tilde{x}^1_{m-1} - \tilde{x}^2_{m-1}$.

Define the strategy $\hat{\boldsymbol{\alpha}} = (\hat{\alpha}_1, \ldots, \hat{\alpha}_M)$ such that

$$\hat{\alpha}_m = \{b^1_m = a^2_m = \frac{1}{2}\}.$$

Under the strategy $\hat{\boldsymbol{\alpha}}$, $Z_m := X^1_m - X^2_m$ becomes a random walk with

$$\mathbb{E}^{\hat{\boldsymbol{\alpha}}}[Z_m] = \frac{3}{4}, \quad \mathrm{Var}^{\hat{\boldsymbol{\alpha}}}[Z_m] = \frac{3}{16}.$$

Therefore, the scaled random walk $(t_m, \tilde{Z}_m)$ converges to a drifted Brownian motion $(B_t)_{t\geq 0}$ such that

$$\mathbb{E}[B_t] = \frac{3t}{4}, \quad \mathrm{Var}[B_t] = \frac{3t}{16}.$$

Since $\frac{B_t - \frac{3t}{4}}{\sqrt{\frac{3t}{16}}}$ has standard normal distribution, we define

$$p := \mathbb{P}\left[\frac{|B_t - \frac{3t}{4}|}{\sqrt{\frac{3t}{16}}} \leq 1\right] = \mathbb{P}\left[\frac{3t}{4} - \sqrt{\frac{3t}{16}} \leq B_t \leq \frac{3t}{4} + \sqrt{\frac{3t}{16}}\right].$$

Therefore, we obtain the estimate

$$\lim_{M\to\infty}\frac{1}{\sqrt{M}}\mathbb{E}^{\boldsymbol{\phi}^*,\hat{\boldsymbol{\alpha}}}[\Phi(X_M)\,|\,X_0=\sqrt{M}x]-U(0,x)$$
$$=\lim_{M\to\infty}\sum_{m=1}^{M}\mathbb{E}^{\boldsymbol{\phi}^*,\hat{\boldsymbol{\alpha}}}[U(t_m,\tilde{X}_m)-U(t_{m-1},\tilde{X}_{m-1})\,|\,\tilde{X}_0=0]$$
$$=\mathbb{E}\left[\int_0^1 c(1-t)^{-1/2}e^{-\frac{B_t^2}{4d(1-t)}}dt\right]$$
$$\geq p\int_0^1 c(1-t)^{-1/2}e^{-\frac{(\frac{3t}{4}+\sqrt{\frac{3t}{16}})^2}{4d(1-t)}}dt>0.$$

Algorithm for the forecaster: The decomposition in (7.35) and the identity (7.14) show that the algorithm for the forecaster defined as $\phi_m^*=\{\partial_j U(t_{m-1},\tilde{X}_{m-1})\}_{j=1}^N$ would set to 0 the term (7.36). However, even if u solves (7.17) but not (7.29), we cannot control the sign of (7.37) and there exist strategies for the adversary that renders $u(t_{m-1},\tilde{X}_{m-1})$ a submartingale (instead of a supermartingale). Thus, $\boldsymbol{\phi}^*=(\phi_1^*,\dots,\phi_M^*)$ is not the best strategy for the learner and ∇u does not necessarily provides the best algorithm for the forecaster.

However, if we assume that ψ is a smooth supersolution to (7.29), it can be easily verified that $\phi_m^*=\{\partial_j\psi(t_{m-1},\tilde{X}_{m-1})\}_{j=1}^N$ provides an algorithm for the forecaster for which the growth of the regret can be bounded from above as in (7.31).

7.4.2 Growth of regret when $\mathcal{A}_B=\emptyset$

We now assume that $\mathcal{A}_B=\emptyset$. In this case, we cannot rely on the PDE (7.25) to obtain the growth of the regret and we have to introduce some auxiliary functions. Without the Assumption 7.2.2 (iii), we provide an example showing that the regret is also of order $\sqrt{M}$. The following result holds no matter $\mathcal{A}_B$ is empty or not.

Proposition 7.4.9. *Assume that $\Phi(x)=\Phi_m(x)=\max_i x^i$. Then, there exists a* *function $\hat{u} : [0,1] \times \mathbb{R}^2 \mapsto \mathbb{R}$ solving a linear parabolic non-degenerate PDE with constant coefficients and terminal condition $\hat{u}(1,x) = x^1 \vee x^2$ for $x \in \mathbb{R}^2$ so that*

$$\liminf_{M\to\infty} \frac{V^M(0,\sqrt{M}x)}{\sqrt{M}} \geq \hat{u}(0,x^1,x^2).$$

Proof. Denote $\tilde{\Phi}(x) = x^1 \vee x^2$ for all $x \in \mathbb{R}^N$ so that $\Phi(x) \geq \tilde{\Phi}(x)$. Now consider a second game with final condition $\tilde{\Phi}$. We denote its value function by $\tilde{V}^M$. It is then clear that $V^M(m,x) \geq \tilde{V}^M(m,x)$ for all $x \in \mathbb{R}^N$ and $0 \leq m \leq M$.

The final condition of the second game only depends on the first two components of the state. Thus, $\tilde{V}^M(m,x) = \hat{V}^M(m,x^1,x^2)$ where $\hat{V}^M$ is the value of an auxiliary two-expert game. Thanks to the Proposition 7.3.2, the game with two experts always admits balanced strategies.

Thanks to Theorem 7.4.3,

$$\liminf_{M\to\infty} \frac{\hat{V}^M(tM,\sqrt{M}x^1,\sqrt{M}x^2)}{\sqrt{M}} \geq \hat{u}(t,x^1,x^2).$$

where $\hat{u}$ solves

$$0 = \partial_t \hat{u}(t,x) + \hat{H}_B(\nabla^2 \hat{u}(t,x)). \tag{7.40}$$

with final condition $\hat{u}(1,x) = x^1 \vee x^2$ and the generator $\hat{H}_B$ is associated to the balanced strategies to the auxiliary two-expert game. Thanks to Proposition 7.5.3 (which will be proved independently of this Proposition), the optimizer in (7.40) is associated to a constant strategy so that $\hat{u}$ in fact solves a linear non-degenerate PDE which concludes the proof.

□

Remark 7.4.10. Note that due to the non-degeneracy of the PDE solved by $\hat{u}$, we easily have that $\partial_i \hat{u}(t,x) > 0$ for all $t \in [0,1)$. Therefore, $\hat{u}$ is a smooth solution of

(7.17) and for the auxiliary two-expert game, $\frac{1}{\sqrt{M}}\hat{V}^M\left(\lceil Mt\rceil, \sqrt{M}x\right)$ indeed converges to $\hat{u}(t,x)$.

The following Proposition shows the importance of Assumption 7.2.2 (iii). For any Φ satisfying Assumption 7.2.2 (iii), we will show that $\Phi(x) \leq \Phi(0) + \Phi_{m,\theta}(x)$. Therefore the forecaster is partially satisfied when he does better than the average of the experts. Since no balanced strategies exist, the forecaster can do better than the average by following the best performed expert at each round, and thus the scaled value function tends to $-\infty$.

Proposition 7.4.11. *Assume that the terminal condition Φ satisfies Assumption 7.2.2. Then, we obtain that*

$$\limsup_{M\to\infty} \frac{1}{\sqrt{M}} V^M(0, \sqrt{M}x) = -\infty.$$

Proof. Recall that $\Phi_m(x) = \max_i x^i$. For any $x \in \mathbb{R}^N$, due to Assumption 7.2.2 (i), (ii), it follows that

$$\Phi(x) \leq \Phi(\Phi_m(x)\mathbf{1}) = \Phi(0) + \Phi_m(x).$$

And by Assumption 7.2.2 (iii), we obtain that

$$\Phi(0) + \Phi_m(x) - \Phi(x) = \Phi(\Phi_m(x)\mathbf{1}) - \Phi(x) \geq \frac{\theta}{N}(\Phi_m(x)\mathbf{1} - x)\cdot \mathbf{1},$$

and therefore

$$\begin{aligned}\Phi(x) &\leq \Phi(0) + \Phi_m(x) - \frac{\theta}{N}(\Phi_m(x)\mathbf{1} - x)\cdot\mathbf{1} \\ &= \Phi(0) + (1-\theta)\Phi_m(x) + \frac{\theta}{N}x\cdot\mathbf{1} =: \Phi(0) + \Phi_{m,\theta}(x). \end{aligned} \tag{7.41}$$

Since $V^M(0, \sqrt{M}x) \le V^M(0,0) + \sqrt{M}\Phi(x)$, it suffices to prove that

$$\limsup_{M\to\infty} \frac{1}{\sqrt{M}} V^M(0,0) = -\infty.$$

Denote $\boldsymbol{\phi} := (\phi_1, \dots, \phi_M)$ the sequence of strategies of the forecaster, and $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_M)$ the sequence of strategies of the adversary. Due to (7.41), we obtain that

$$\begin{aligned} V^M(0,0) &= \inf_{\boldsymbol{\phi}} \sup_{\boldsymbol{\alpha}} \mathbb{E}^{\boldsymbol{\phi},\boldsymbol{\alpha}}[\Phi(X_M) \mid X_0 = 0] \\ &\le \Phi(0) + \inf_{\boldsymbol{\phi}} \sup_{\boldsymbol{\alpha}} \mathbb{E}^{\boldsymbol{\phi},\boldsymbol{\alpha}}[\Phi_{m,\theta}(X_M) \mid X_0 = 0]. \end{aligned} \tag{7.42}$$

For any $\alpha \in \mathcal{A}$, we define

$$M(\alpha) = \max_i \mathbb{E}^\alpha[\Delta G^i],$$

and

$$m(\alpha) = \max_i \{\mathbb{E}^\alpha[\Delta G^i] : \mathbb{E}^\alpha[\Delta G^i] < M(\alpha)\}.$$

Here $M(\alpha)$ is the largest expected expert gain under the policy α, and $m(\alpha)$ is the second largest expected gain. Since $\mathcal{A}_B = \emptyset$, for any $\alpha \in \mathcal{A}$ we have that $m(\alpha) \ge 0$ and $M(\alpha) - m(\alpha) > 0$. Define

$$\delta := \inf_{a\in\mathcal{A}} (M(\alpha) - m(\alpha)).$$

It can be easily seen that $\delta > 0$.

For any value function V (7.5) with terminal condition Φ (7.4) satisfying $\Phi(x + \lambda\mathbf{1}) = \Phi(x) + \lambda$, it holds that $V^M(m+1, x+\lambda\mathbf{1}) = V^M(t,x) + \lambda$. It can be easily verified that

$$\begin{aligned}
V^M(m-1,x) &= \min_{\phi_m}\max_{\alpha_m}\mathbb{E}^{\phi_m,\alpha_m}[V^M(m,x+\Delta X_m)] \\
&= \max_{\alpha_m}\mathbb{E}^{\alpha_m}[V^M(m,x+\Delta G_m)] - \max_{\phi_m}\sum_i \phi_m^i \mathbb{E}^{\alpha_m}[\Delta G_m^i].
\end{aligned}$$

Therefore for any fixed strategy $\hat{\boldsymbol{\alpha}}$ of the adversary, the optimal response of the forecaster is to follow the experts with maximal expected gain under policy $\hat{\boldsymbol{\alpha}}$ at each round, i.e., $\phi_m^i = 0$ if and only if $\mathbb{E}^{\hat{\alpha}_m}[\Delta G^i] < M(\hat{\alpha}_m)$. Denote one such optimal response of the forecaster by $\hat{\boldsymbol{\phi}}$. Therefore we obtain that

$$\begin{aligned}
&\inf_{\boldsymbol{\phi}}\mathbb{E}^{\boldsymbol{\phi},\hat{\boldsymbol{\alpha}}}[\Phi_{m,\theta}(X_M)\,|\,X_0=0] = \mathbb{E}^{\hat{\boldsymbol{\phi}},\hat{\boldsymbol{\alpha}}}[\Phi_{m,\theta}(X_M)\,|\,X_0=0] \\
&\quad = (1-\theta)\mathbb{E}^{\hat{\boldsymbol{\phi}},\hat{\boldsymbol{\alpha}}}[\Phi_m(X_M)\,|\,X_0=0] + \frac{\theta}{N}\mathbb{E}^{\hat{\boldsymbol{\phi}},\hat{\boldsymbol{\alpha}}}[X_M\cdot\mathbf{1}\,|\,X_0=0] \\
&\quad = (1-\theta)\inf_{\boldsymbol{\phi}}\mathbb{E}^{\boldsymbol{\phi},\hat{\boldsymbol{\alpha}}}[\Phi_m(X_M)\,|\,X_0=0] + \frac{\theta}{N}\inf_{\boldsymbol{\phi}}\mathbb{E}^{\boldsymbol{\phi},\hat{\boldsymbol{\alpha}}}[X_M\cdot\mathbf{1}\,|\,X_0=0].
\end{aligned} \tag{7.43}$$

According to Remark 7.4.5, there exists some positive C independent of choice of $\hat{\boldsymbol{\alpha}}$ such that

$$\limsup_{M\to\infty}\frac{1}{\sqrt{M}}\inf_{\boldsymbol{\phi}}\mathbb{E}^{\boldsymbol{\phi},\hat{\boldsymbol{\alpha}}}[\Phi(X_M)\,|\,X_0=0] \le C. \tag{7.44}$$

Due to our definition of δ, we obtain that for any $\hat{\boldsymbol{\alpha}}$

$$\inf_{\boldsymbol{\phi}}\mathbb{E}^{\boldsymbol{\phi},\hat{\boldsymbol{\alpha}}}[X_M\cdot\mathbf{1}/N\,|\,X_0=0] \le -\delta M/N. \tag{7.45}$$

In conjunction with (7.43),(7.44) and (7.45), we conclude that

$$\limsup_{M\to\infty}\frac{1}{\sqrt{M}}V^M(0,0) = -\infty.$$

□

7.5 Explicit solutions in some special cases

In this section we exhibit some cases where the value function and the strategies of the adversary can be explicitly computed. The results are valid for final condition $\Phi_{\theta,m}(x) = (1-\theta)\Phi(x) + \frac{\theta}{N}\sum_{i=1}^{N} x^i$ for any fixed $\theta \geq 0$.

Our methodology is to provide a stochastic representation for the solution of (7.25) that allows us to claim that this solution also solves (7.17). Then, we use this solution to obtain a strategy for the adversary.

Definition 7.5.1. Any $\bar{\alpha} \in \mathcal{A}_B$ satisfying

$$c_{\bar{\alpha}} = \sup_{\alpha \in \mathcal{A}_B} c_\alpha$$

is called a generous adversary and any $\underline{\alpha} \in \mathcal{A}_B$ satisfying

$$c_{\underline{\alpha}} = \inf_{\alpha \in \mathcal{A}_B} c_\alpha$$

is called a greedy adversary.

Note that for $\alpha \in \mathcal{A}_B$, one has

$$\sum_{i,j=1}^{N} S_{ij}\mathbb{E}^\alpha[\Delta G^i \Delta G^j] = c_\alpha Tr\left(\Sigma_1 S\right) - Tr\left(\Sigma_2 S\right) \tag{7.46}$$

where $\Sigma_1 = \{\mu^i + \mu^j\}_{i,j=1}^N + diag(1-2\mu^1, \ldots, 1-2\mu^N)$ and $\Sigma_2 = \{\mathbb{1}_{\{i\neq j\}}\mu^i\mu^j\}_{i,j=1}^N$.

The next lemma shows that the linear differential operator associated to each balanced strategy is non-degenerate.

Lemma 7.5.2. *If $N \geq 2$, and $0 < \mu^i < 1, i = 1, \ldots, N$, then for any $\alpha \in \mathcal{A}_B$ (if this set is not empty) the matrix $c_\alpha\Sigma_1 - \Sigma_2$ is positive definite.*

Proof. It suffices to show that for any vector $y = (y^1, \dots, y^N) \neq 0$,

$$y(c_\alpha \Sigma_1 - \Sigma_2)y^\top = \sum_{i,j=1}^{N} y^i y^j \mathbb{E}^\alpha[\Delta G^i \Delta G^j] = \mathbb{E}^\alpha \left[\left(y^\top \Delta G \right)^2 \right] > 0. \tag{7.47}$$

As a result of (7.47), $y(c_\alpha \Sigma_1 - \Sigma_2)y^\top = 0$ if and only if $y^\top \Delta G = 0$ $\mathbb{P}^\alpha$-a.s. Denote the collection of all the possible realizations of ΔG by

$$O = \{z \in \{0,1\}^N : \mathbb{P}^\alpha[\Delta G = z] > 0\}.$$

We will prove that the dimension of the linear expansion $\langle O \rangle$ is N. Then it follows that $y^\top \Delta G = 0$ $\mathbb{P}^\alpha$-a.s. is impossible, and hence $y(c_\alpha \Sigma_1 - \Sigma_2)y^\top > 0$.

Recall that b^i is the probability that the adversary corrupts expert i and sets his gain to 1. Suppose there exists some $b^i > 0$. For any $z \in \{0,1\}^N$, we define $\hat{z} = z - z^i e^i + e^i$, and $p_j = \mathbb{1}_{\{z^j = 0\}}(1 - \mu^j) + \mathbb{1}_{\{z^j = 1\}}\mu^j$ for any $j \neq i$. Note that the i-th coordinate of $\hat{z}$ is always 1. It can be easily seen that

$$\begin{aligned} \mathbb{P}(\Delta G = \hat{z}) &\geq \mathbb{P}(\Delta G = \hat{z}, \text{the adversary corrupts expert } i \text{ and sets his gain to 1}) \\ &\geq b^i \prod_{j=1, j\neq i}^{N} p_i > 0. \end{aligned}$$

Therefore $\hat{z} \in O$ for any z, and $O \supset \{z \in \{0,1\}^N : z^i = 1\}$. Hence the dimension of $\langle O \rangle$ is N.

Recall that a^i is the probability that the adversary corrupts expert i and sets his gain to 0. If $b^i = 0, i = 1, \dots, N$, there must exist some $a^i > 0$. For any $z \in \{0,1\}^N$, we define $\tilde{z} = z - z^i e^i$. Note that the i-coordinate of $\tilde{z}$ is always 0. Since $\mathbb{P}(\Delta G = \tilde{z}) \geq a^i \prod_{j=1, j\neq i}^{N} p_i > 0$, we obtain that $\tilde{z} \in O$ for any z, and hence $O \supset \{z \in \{0,1\}^N : z^i = 0\}$. Since $N c_\alpha = \sum_{j=1}^{N}((1 - a^j - b^j)\mu^j + b^j) > 0$ for $N \geq 2$, it must hold that $\mathbb{E}^\alpha[\Delta G^i] > 0$. There must exist some $z \in O$ such that $z^i = 1$. Hence

$\langle O \rangle$ is of dimension N. $\square$

The next proposition provides a condition on $\{\mu^i\}$ that allows us to characterize the optimal strategy of the adversary as a constant strategy. Therefore, in this case, the optimal strategy of the adversary is the maximizer of the Hamiltonian in (7.25).

Proposition 7.5.3. *If* (7.20) *holds,* $\mu^i \in (0,1)$ *for any* i *and* $\mu^i + \mu^j \leq 1$ *(resp.* $\mu^i + \mu^j \geq 1$*) for any* $i \neq j$*, then any generous adversary* $\overline{\alpha}$ *(resp. greedy adversary* $\underline{\alpha}$*) is the maximizer of the Hamiltonian in* (7.17) *for all* $(t,x) \in [0,1) \times \mathbb{R}^N$*.*

Additionally, the solution u *to* (7.17) *is equal to* U *(which is the solution to* (7.25)*), and satisfies* (7.34)*. Therefore according to Proposition 7.4.8, the asymptotic strategy* $\boldsymbol{\alpha}^*$ *in* (7.32) *for the adversary guarantees* U *as a lower bound of regret.*

Proof. Note that according to Proposition 7.3.2, $\mathcal{A}_B \neq \emptyset$ if and only if (7.20) holds. We only provide the proof for the case where $\mu^i + \mu^j \geq 1$ for any $i \neq j$. The other case can be proved similarly. Our methodology is to show that the solution of the linear PDE associated with the constant strategy $\underline{\alpha}$ also provides a solution to (7.25) and (7.17).

Step 1: Approximating the final condition: Using the definition of $\Phi_{m,\theta}$ and the assumption on $\{\mu^i\}$, we first find an approximation sequence Φ_ϵ such that Φ_ϵ converges to $\Phi_{m,\theta}$ in $\mathcal{L}^\infty$ as $\epsilon \to 0$ and $Tr(\Sigma_1 \partial^2 \Phi_\epsilon(x)) \leq 0$. Since

$$\Phi_{m,\theta}(x) = (1-\theta)\max\{x^1, \dots, x^N\} + \frac{\theta}{N}\sum_{i=1}^N x^i = (1-\theta)\Phi_m(x) + \frac{\theta}{N}\sum_{i=1}^N x^i$$

and the second derivative of the linear part is 0, it is sufficient to prove the claim for $\theta = 0$. We prove the existence of such Φ_ϵ by induction.

First we approximate the absolute value function on $\mathbb{R}^1$. For each $\epsilon > 0$, it can be easily verified that there exists some $f_\epsilon : \mathbb{R}^1 \to \mathbb{R}^1$ such that

(i) $f_\epsilon(x) = |x|$ if $|x| \geq \epsilon$;

(ii) f_ϵ is convex;

(iii) $|x| \le f_\epsilon \le |x| + \epsilon$, $\forall x \in \mathbb{R}^1$.

Then in the case of $N = 2$, we define

$$\Phi_\epsilon^2(x^1, x^2) := \frac{x^1 + x^2 + f_\epsilon(x^1 - x^2)}{2}.$$

It can be easily seen that Φ_ϵ^2 converges to Φ^2 in $\mathcal{L}^\infty$, and $\partial_1 \Phi_\epsilon^2 + \partial_2 \Phi_\epsilon^2 = 1$. We compute the second derivative of Φ_ϵ^2 and obtain that

$$\begin{aligned}
\partial_{11}^2 \Phi_\epsilon^2(x) = \partial_{22}^2 \Phi_\epsilon^2(x) &= \frac{1}{2} f_\epsilon''(x^1 - x^2), \\
\partial_{12}^2 \Phi_\epsilon^2(x) = \partial_{12}^2 \Phi_\epsilon^2(x) &= -\frac{1}{2} f_\epsilon''(x^1 - x^2).
\end{aligned}$$

Since f_ϵ is convex, we have $f_\epsilon''(x^1 - x^2) \ge 0$, and therefore

$$Tr(\Sigma_1 \partial^2 \Phi_\epsilon^2(x)) = (\mu^1 + \mu^2 - 1) f_\epsilon''(x^1 - x^2) \le 0.$$

Suppose our claim is correct for $N-1$ many experts, let us prove it for N. Without loss of generality, we assume that $\mu^N = \max\{\mu^1, \dots, \mu^N\}$. Denote by $\tilde{x}$ the first $N-1$ components of x, and by $\tilde{\Sigma}_1$ the principal submatrix of Σ_1 by removing its last row and column. By induction, we have Φ_ϵ^{N-1} such that

(i) $\sum_{i=1}^{N-1} \partial_i \Phi_\epsilon^{N-1} = 1$ and $\partial_i \Phi_\epsilon^{N-1} \ge 0$, $\forall i \le N-1$;

(ii) $\Phi_\epsilon^{N-1} \to \Phi^{N-1}$ in $\mathcal{L}^\infty$ as $\epsilon \to 0$;

(iii) $Tr(\tilde{\Sigma}_1 \partial^2 \Phi_\epsilon^{N-1}) \le 0$.

Define

$$\Phi_\epsilon(x) := \frac{\Phi_\epsilon^{N-1}(\tilde{x}) + x^N + f_\epsilon(\Phi_\epsilon^{N-1}(\tilde{x}) - x^N)}{2}.$$

It is then clear that $\Phi_\epsilon \to \Phi$ in $\mathcal{L}^\infty$. To simplify notation, we omit the arguments $x, \tilde{x}$ when it is clear from the context. Let us compute its first derivatives

$$\begin{aligned}
\partial_i \Phi_\epsilon &= \frac{1}{2}\partial_i \Phi_\epsilon^{N-1} + \frac{1}{2} f_\epsilon'(\Phi_\epsilon^{N-1} - x^N)\partial_i \Phi_\epsilon^{N-1}, \; i \le N-1, \\
\partial_N \Phi_\epsilon &= \frac{1}{2} - \frac{1}{2} f_\epsilon'(\Phi_\epsilon^{N-1} - x^N),
\end{aligned}$$

and second derivatives

$$\begin{aligned}
\partial_{ij}^2 \Phi_\epsilon &= \frac{1}{2}\left(1 + f_\epsilon'(\Phi_\epsilon^{N-1} - x^N)\right)\partial_{ij}^2 \Phi_\epsilon^{N-1} + \frac{1}{2} f_\epsilon''(\Phi_\epsilon^{N-1} - x^N)\partial_i \Phi_\epsilon^{N-1}\partial_j \Phi_\epsilon^{N-1}, \; i,j \le N-1, \\
\partial_{iN}^2 \Phi_\epsilon &= -\frac{1}{2} f_\epsilon''(\Phi_\epsilon^{N-1} - x^N)\partial_i \Phi_\epsilon^{N-1}, \; i \le N, \\
\partial_{NN}^2 \Phi_\epsilon &= \frac{1}{2} f_\epsilon''(\Phi_\epsilon^{N-1} - x^N).
\end{aligned}$$

Due to $\sum_{i=1}^{N-1} \partial_i \Phi_\epsilon^{N-1} = 1$ and $1 + f_\epsilon', 1 - f_\epsilon' \ge 0$, we obtain that $\partial_i \Phi_\epsilon \ge 0$ and

$$\sum_{i=1}^{N} \partial_i \Phi_\epsilon^N = 1. \tag{7.48}$$

Denote by $\widetilde{\partial^2 \Phi_\epsilon^N}$ the principal submatrix of $\partial^2 \Phi_\epsilon^N$ by removing the last row and column. We rewrite the trace as

$$\begin{aligned}
Tr(\Sigma_1 \partial^2 \Phi_\epsilon^N) =& Tr(\tilde{\Sigma}_1 \widetilde{\partial^2 \Phi_\epsilon^N}) + 2\sum_{i=1}^{N-1}(\mu^i + \mu^N)\partial_{iN}^2 \Phi_\epsilon + \partial_{NN}^2 \Phi_\epsilon \\
=& \frac{1}{2}\left(1 + f_\epsilon'(\Phi_\epsilon^{N-1} - x^N)\right) Tr(\tilde{\Sigma}_1 \partial^2 \Phi_\epsilon^{N-1}) \\
&+ \frac{1}{2} f_\epsilon''(\Phi_\epsilon^{N-1} - x^N)\left((\partial \Phi_\epsilon^{N-1})^\top \cdot \tilde{\Sigma}_1 \cdot \partial \Phi_\epsilon^{N-1} - 2\sum_{i=1}^{N-1}(\mu^i + \mu^N)\partial_i \Phi_\epsilon^{N-1} + 1\right).
\end{aligned}$$

According to our induction, we know that $Tr(\tilde{\Sigma}_1 \partial^2 \Phi_\epsilon^{N-1}) \le 0$. Since

$\big(1+f_\epsilon'(\Phi_\epsilon^{N-1}-x^N)\big)\geq 0,\ f_\epsilon''(\Phi_\epsilon^{N-1}-x^N)\geq 0$, it suffices to show that

$$\left((\partial\Phi_\epsilon^{N-1})^\top\cdot\tilde{\Sigma}_1\cdot\partial\Phi_\epsilon^{N-1}-2\sum_{i=1}^{N-1}(\mu^i+\mu^N)\partial_i\Phi_\epsilon^{N-1}+1\right)\leq 0. \tag{7.49}$$

Due to the equality $\sum_{i=1}^{N-1}\partial_i\Phi_\epsilon^{N-1}=1$, we obtain that

$$\begin{aligned}(\partial\Phi_\epsilon^{N-1})^\top\cdot\tilde{\Sigma}_1\cdot\partial\Phi_\epsilon^{N-1}=&(\sum_{i=1}^{N-1}\partial_i\Phi_\epsilon^{N-1})^2+\sum_{i\neq j\leq N-1}(\mu^i+\mu^j-1)\partial_i\Phi_\epsilon^{N-1}\partial_j\Phi_\epsilon^{N-1}\\ =&1+\sum_{i\neq j\leq N-1}(\mu^i+\mu^j-1)\partial_i\Phi_\epsilon^{N-1}\partial_j\Phi_\epsilon^{N-1}.\end{aligned}$$

Similarly, we have that

$$2\sum_{i=1}^{N-1}(\mu^i+\mu^N)\partial_i\Phi_\epsilon^{N-1}=2+\sum_{i=1}^{N-1}(\mu^i+\mu^N-1)\partial_i\Phi_\epsilon^{N-1}.$$

Therefore (7.49) is equivalent to that

$$\sum_{i\neq j\leq N-1}(\mu^i+\mu^j-1)\partial_i\Phi_\epsilon^{N-1}\partial_j\Phi_\epsilon^{N-1}\leq\sum_{i=1}^{N-1}(\mu^i+\mu^N-1)\partial_i\Phi_\epsilon^{N-1}. \tag{7.50}$$

For fixed $i\leq N-1$, according to our assumption $\mu^N=\max\{\mu^1,\dots,\mu^N\}$ we have that

$$\begin{aligned}\sum_{j\leq N-1,j\neq i}(\mu^i+\mu^j-1)\partial_i\Phi_\epsilon^{N-1}\partial_j\Phi_\epsilon^{N-1}&\leq(\mu^i+\mu^N-1)\partial_i\Phi_\epsilon^{N-1}\sum_{j\leq N-1,j\neq i}\partial_j\Phi_\epsilon^{N-1}\\ &\leq(\mu^i+\mu^N-1)\partial_i\Phi_\epsilon^{N-1}.\end{aligned}$$

Summing from $i=1$ to $i=N-1$, we obtain the inequality (7.50), and hence (7.49). In conjunction with (7.48), we finish proving the induction.

Step 2: Solving the nonlinear PDE with the linear PDE: Denote by u the solution of

$$0 = \partial_t u(t,x) + \frac{1}{2} Tr \left(\underline{\Sigma} \partial^2_{xx} u(t,x) \right)$$

with terminal condition $\Phi_{\theta,m}$, and by u^ϵ the solution of

$$0 = \partial_t u^\epsilon(t,x) + \frac{1}{2} Tr \left(\underline{\Sigma} \partial^2_{xx} u^\epsilon(t,x) \right) \tag{7.51}$$

with terminal condition Φ_ϵ(smooth approximation of $\Phi_{\theta,m}$ satisfying $Tr(\Sigma_1 \partial^2 \Phi_\epsilon(x)) \leq 0$). In order to show that $\underline{\alpha}$ is optimal, it suffices to prove that u solves PDE (7.17). To show this solution property, it is sufficient to show that u solves (7.25), $\partial_i u(t,x) > 0$ for all $(t,x) \in [0,1) \times \mathbb{R}^N$, and $Tr(\Sigma_1 \partial^2_{xx} u(t,x)) \leq 0$.

We can differentiate (7.51) in x twice to obtain that

$$w^\epsilon(t,x) = Tr(\Sigma_1 \partial^2_{xx} u^\epsilon(t,x))$$

solves the same PDE

$$0 = \partial_t w^\epsilon(t,x) + \frac{1}{2} Tr \left(\underline{\Sigma} \partial^2_{xx} w^\epsilon(t,x) \right)$$

with final condition

$$w^\epsilon(1,x) = Tr(\Sigma_1 \partial^2_{xx} \Phi_\epsilon(x)).$$

Due to the choice of Φ_ϵ we have that that $w^\epsilon(1,x) \leq 0$.

Thus, by the maximum principle, $w^\epsilon(t,x) \leq 0$ for all $(\epsilon,t,x) \in (0,1) \times [0,1] \times \mathbb{R}^N$. Fix $t \in [0,1)$. Due the Malliavin calculus representation of $\partial^2_{xx} v^\epsilon$,

$$\partial^2_{xx} u^\epsilon(t,x) = \mathbb{E} \left[\Phi_\epsilon(x + \sqrt{\underline{\Sigma}}(W_{1-t})) \sqrt{\underline{\Sigma}}^{-1} \frac{W_{1-t} W_{1-t}^\top - (1-t) I_N}{(1-t)^2} \sqrt{\underline{\Sigma}}^{-1} \right]$$

and we have that $\partial^2_{xx} u^\epsilon(t,x) \to \partial^2_{xx} u(t,x)$. This implies that for all $t \in [0,1)$ and

$x \in \mathbb{R}^N$,

$$Tr(\Sigma_1 \partial^2_{xx} u(t,x)) \leq 0.$$

Thus, thanks to (7.46)

$$\begin{aligned}
2H_B(\partial^2_{xx} u(t,x)) &= \sup_{\alpha \in \mathcal{A}_B} \sum_{i,j=1}^{N} \partial^2_{ij} u(t,x) \mathbb{E}^\alpha [\Delta G^i \Delta G^j] \\
&= \sup_{\alpha \in \mathcal{A}_B} c_\alpha Tr\left(\Sigma_1 \partial^2_{xx} u(t,x)\right) - Tr\left(\Sigma_2 \partial^2_{xx} u(t,x)\right) \\
&= c_{\underline{\alpha}} Tr\left(\Sigma_1 \partial^2_{xx} u(t,x)\right) - Tr\left(\Sigma_2 \partial^2_{xx} u(t,x)\right),
\end{aligned}$$

and therefore $\underline{\alpha}$ is optimal among balanced strategies and u solves (7.25) and is therefore equal to U.

Note also that using the the density of the Brownian motion one can show that

$$\partial_i u\,(t,x) = (1-\theta)\mathbb{P}\left(i\text{th coordinate of } (x + \sqrt{\underline{\Sigma}} W_{1-t}) \text{ is maximal}\right) + \frac{\theta}{N} > 0 \tag{7.52}$$

for all $(t,x) \in [0,1) \times \mathbb{R}^N$ and $\theta \geq 0$. Thus, u also solves (7.17).

Step 3: The derivatives of u satisfies (7.34). According to Lemma 7.5.2 and Proposition 7.5.3, the coefficient matrix Σ of the optimal adversary is positive definite. Then there exists some matrix $P = (P_1, \dots, P_N)$ with $\det P > 0$ such that $\underline{\Sigma} = P^\top P$. It can be easily verified that the solution is given by

$$u(1-t,x) = \frac{\det P}{(2\pi t)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{|Py|^2}{2t}} \Phi_{m,\theta}(x-y)\, dy.$$

Note that $\Phi_{m,\theta}(x) = (1-\theta)\max\{x^1, \dots, x^N\} + \frac{\theta}{N}\sum x^i$ is differentiable almost everywhere and

$$\partial_i \Phi_{m,\theta}(x-y) = (1-\theta)\mathbb{1}_{\{x^i - y^i \geq \Phi_m(x-y)\}} + \frac{\theta}{N} \qquad \text{a.e.}$$

Therefore we obtain that

$$\begin{aligned}
\partial_i u(1-t,x) &= \frac{\theta}{N} + \frac{(1-\theta)\det P}{(2\pi t)^{N/2}} \int\limits_{\mathbb{R}^N} e^{-\frac{|Py|^2}{2t}} \mathbb{1}_{\{x^i - y^i \geq \Phi_m(x-y)\}}\, dy \\
&= \frac{\theta}{N} + \frac{(1-\theta)\det P}{(2\pi t)^{N/2}} \int\limits_{\mathbb{R}^N} e^{-\frac{|Px-Py|^2}{2t}} \mathbb{1}_{\{y^i \geq \Phi_m(y)\}}\, dy. \qquad (7.53)
\end{aligned}$$

Differentiating the above equation with respect to x^j, it follows that

$$\partial^2_{ij} u(1-t,x) = -\frac{(1-\theta)\det P}{(2\pi t)^{N/2}} \int\limits_{\mathbb{R}^N} e^{-\frac{|Px-Py|^2}{2t}} \left(\frac{P_j^\top P(x-y)}{t} \right) \mathbb{1}_{\{y^i \geq \Phi_m(y)\}}\, dy. \quad (7.54)$$

Similarly, it can be easily seen that

$$\begin{aligned}
&\partial^3_{ijk} u(1-t,x) \\
&= -\frac{(1-\theta)\det P}{(2\pi t)^{N/2}} \int\limits_{\mathbb{R}^N} e^{-\frac{|Px-Py|^2}{2t}} \left(\frac{P_j^\top P_k}{t} \right) \mathbb{1}_{\{y^i \geq \Phi_m(y)\}}\, dy \\
&\quad + \frac{(1-\theta)\det P}{(2\pi t)^{N/2}} \int\limits_{\mathbb{R}^N} e^{-\frac{|Px-Py|^2}{2t}} \left(\frac{P_j^\top P(x-y)}{t} \right) \left(\frac{P_k^\top P(x-y)}{t} \right) \mathbb{1}_{\{y^i \geq \Phi_m(y)\}}\, dy,
\end{aligned}$$

and also

$$
\begin{aligned}
&\partial^4_{ijkl} u(1-t,x)\\
&=\frac{(1-\theta)\det P}{(2\pi t)^{N/2}} \int\limits_{\mathbb{R}^N} e^{-\frac{|Px-Py|^2}{2t}} \left(\frac{P_j^\top P_k}{t}\right)\left(\frac{P_l^\top P(x-y)}{t}\right) \mathbb{1}_{\{y^i \geq \Phi_m(y)\}}\, dy\\
&\quad + \frac{(1-\theta)\det P}{(2\pi t)^{N/2}} \int\limits_{\mathbb{R}^N} e^{-\frac{|Px-Py|^2}{2t}} \left(\frac{P_j^\top P(x-y)}{t}\right)\left(\frac{P_k^\top P_l}{t}\right) \mathbb{1}_{\{y^i \geq \Phi_m(y)\}}\, dy\\
&\quad + \frac{(1-\theta)\det P}{(2\pi t)^{N/2}} \int\limits_{\mathbb{R}^N} e^{-\frac{|Px-Py|^2}{2t}} \left(\frac{P_j^\top P_l}{t}\right)\left(\frac{P_k^\top P(x-y)}{t}\right) \mathbb{1}_{\{y^i \geq \Phi_m(y)\}}\, dy\\
&\quad - \frac{(1-\theta)\det P}{(2\pi t)^{N/2}} \times\\
&\quad \int\limits_{\mathbb{R}^N} e^{-\frac{|Px-Py|^2}{2t}} \left(\frac{P_j^\top P(x-y)}{t}\right)\left(\frac{P_k^\top P(x-y)}{t}\right)\left(\frac{P_l^\top P(x-y)}{t}\right) \mathbb{1}_{\{y^i \geq \Phi_m(y)\}}\, dy.
\end{aligned}
$$

Let us show that there exists a constant C such that for any $x \in \mathbb{R}^N$

$$
|\partial^2_{ij} u(1-t,x)| \leq \frac{C}{\sqrt{t}}.
$$

It can be easily seen that

$$
\begin{aligned}
|\partial^2_{ij} u(1-t,x)| &= \left| -\frac{(1-\theta)\det P}{(2\pi t)^{N/2}} \int\limits_{\mathbb{R}^N} e^{-\frac{|Py|^2}{2t}} \left(\frac{P_j^\top P y}{t}\right) \mathbb{1}_{\{x^i - y^i \geq \Phi_m(x-y)\}}\, dy \right|\\
&= \left| -\frac{(1-\theta)\det P}{(2\pi)^{N/2}} \int\limits_{\mathbb{R}^N} e^{-\frac{|Py|^2}{2}} \left(\frac{P_j^\top P y}{\sqrt{t}}\right) \mathbb{1}_{\left\{\frac{x^i}{\sqrt{t}} - y^i \geq \Phi_m\left(\frac{x}{\sqrt{t}} - y\right)\right\}}\, dy \right|\\
&\leq \frac{(1-\theta)\det P}{(2\pi)^{N/2}\sqrt{t}} \int\limits_{\mathbb{R}^N} e^{-\frac{|Py|^2}{2}} |P_j^\top P y|\, dy = \frac{C}{\sqrt{t}}.
\end{aligned}
$$

Similarly, it can be proved that

$$
|\partial^3_{ijk} u(1-t,x)| \leq \frac{C}{t},
$$

and

$$|\partial^4_{ijkl}u(1-t,x)| \leq \frac{C}{t^{\frac{3}{2}}}.$$

Now let us estimate $\partial^2_{tt}u$ and $\partial^2_{tx}u$. Since $\partial_t u = -\frac{1}{2}Tr(\Sigma\partial^2_{xx}u)$, we obtain that

$$\begin{aligned}
|\partial^2_{tt}u(1-t,x)| &= \left|\frac{1}{2}\partial_t Tr(\Sigma\partial^2_{xx}u(1-t,x))\right| = \left|\frac{1}{2}Tr(\Sigma\partial^2_{xx}\partial_t u(1-t,x))\right| \\
&= \left|\frac{1}{2}Tr\left(\Sigma\partial^2_{xx}\left(\frac{1}{2}Tr(\Sigma\partial^2_{xx}u(1-t,x))\right)\right)\right|.
\end{aligned}$$

The right hand of the above equation is a linear combination of $\partial^4_{ijkl}u(1-t,x)$, and hence

$$|\partial^2_{tt}u(1-t,x)| \leq \frac{C}{t^{\frac{3}{2}}}, \qquad \forall x \in \mathbb{R}^N.$$

Similarly, it can be easily verified that

$$|\partial^3_{xxx}u(1-t,x)| \leq \frac{C}{t}, \quad |\partial^2_{tx}u(1-t,x)| \leq \frac{C}{t}, \qquad \forall x \in \mathbb{R}^N,$$

which concludes the proof of (7.34).

□

Example 7.5.4. In the special case of $\mu^i = \mu, i = 1, \ldots, N$, the equation becomes

$$0 = \partial_t U(t,x) + \frac{1}{2}\max_{\alpha\in\mathcal{A}_B}\left(\sum_{i,j=1}^N(2\mu c_\alpha - \mu^2)\partial^2_{ij}U(t,x) + \sum_{i=1}^N(c_\alpha - 2\mu c_\alpha + \mu^2)\partial^2_{ii}U(t,x)\right).$$

Since $\mathbf{1}\cdot\nabla U = 1$, we have $\sum_{i,j=1}^N(2\mu c_\alpha - \mu^2)\partial^2_{ij}U(t,x) = 0$. Hence the equation can be simplified as

$$0 = \partial_t U(t,x) + \frac{1}{2} \max_{\alpha \in \mathcal{A}_B} \left((c_\alpha - 2\mu c_\alpha + \mu^2) \Delta U(t,x) \right). \tag{7.55}$$

Note that

$$C := \frac{1}{2} \max_{\alpha \in \mathcal{A}_B} (c_\alpha - 2\mu c_\alpha + \mu^2) = \begin{cases} \frac{1}{2}(1-2\mu)(\mu + \frac{1}{N}(1-\mu)) + \frac{1}{2}\mu^2 & \text{if } \mu \leq 1/2 \ , \\ \frac{1}{2}(1-2\mu)(\mu - \frac{1}{N}\mu) + \frac{1}{2}\mu^2 & \text{if } \mu \geq 1/2 \ . \end{cases}$$

It can be easily verified that $C \geq 0$. We claim that the solution of (7.55) is the solution of the following equation,

$$0 = \partial_t U(t,x) + C \Delta U(t,x), \tag{7.56}$$

additionally the asymptotic strategy $\boldsymbol{\alpha}^*$ in (7.32) for the adversary guarantees U as a lower bound of regret.

7.6 Conclusion

In this chapter, we study an expert prediction problem, where an adversary only corrupts one expert at each round and a forecaster makes predictions based on experts' past gains. The forecaster aims at minimizing his regret, while the adversary wants to maximize it. Therefore this problem can be interpreted as a zero-sum game between the adversary and the forecaster. Using viscosity theory tools in the field of partial differential equation, we provided the growth rate of regret for the forecaster. A strategy of the adversary is called balanced if the expected gain of each expert is the same under this strategy. We showed that the growth rate of regret fundamentally depends on whether balanced strategies exist and whether the final condition Φ satisfies the strictly monotone condition Assumption 7.2.2 (iii),

(i) *Balanced strategies exist,* Φ *does not satisfy Assumption 7.2.2 (iii):* the growth rate of regret is bounded below by the solution of (7.25); see Theorem 7.4.3.

(ii) *Balanced strategies exist,* Φ *satisfies Assumption 7.2.2 (iii):* the growth rate of regret is given by the solution of (7.25); see Theorem 7.4.7.

(iii) *Balanced strategies do not exist,* Φ *does not satisfy Assumption 7.2.2 (iii):* the asymptotic regret is of order $\sqrt{M}$; see Proposition 7.4.9.

(iv) *Balanced strategies do not exist,* Φ *satisfies Assumption 7.2.2 (iii):* the asymptotic regret is $-\infty$; see Proposition 7.4.11.

Also, we designed asymptotic optimal strategies for the adversary in Proposition 7.4.8, and solved (7.17), (7.25) in some special cases; see Proposition 7.5.3.

CHAPTER VIII

On Non-uniqueness in Mean Field Games

8.1 Introduction

The theory of mean field games (MFGs) was introduced recently (2006-2007) independently by Lasry, Lions (see [136], [137], [138]) and Caines, Huang, Malhamé (see [116], [117]). It is an analysis of limit models for symmetric weakly interacting $N+1$-player differential games (see e.g. [61], [62]). The solution of MFGs provides an approximated Nash Equilibrium. It also under some conditions follows that MFGs are limit points of $N+1$-player Nash equilibria.

The influential work [56] by Cardaliaguet, Delarue, Lasry, and Lions established the convergence of closed loop equilibria using the the so-called master equation, which is a partial differential equation with terminal conditions whose variable are time, state and measure. It is known that under the monotonicity condition, the master equation possess a unique solution, which is used to show the above convergence. A similar analysis was carried in finite state mean field games by Bayraktar and Cohen [24] and Cecchin and Pelino [66] independently obtain the above convergence result (as well as the the analysis of its fluctuations).

In this chapter, we consider a case when the monotonicity assumption is not satisfied and resolve a conjecture of [109], in which a two-state mean field game with Markov feedback strategies is analyzed. In this game the transition rate of each player is the sum of his control and a background jump rate $\eta \geq 0$. Supposing an anti-monotone running cost (follow the crowd game), [109] poses a conjecture on the nature of the limits of $N+1$-player Nash equilibrium. We proceed by using similar techniques to [67], which considers an anti-monotone terminal condition. In particular, we again rely on the entropy solution of the master equation to prove the convergence and show that the limit of $N+1$-player Nash equilibrium selects the unique mean field equilibrium induced by this entropy solution. In [67], they showed that the mean field game equation has at most three equations, while in our model if $\eta < \frac{1}{2}$, the number of solutions is increasing with time horizon and can be arbitrarily large. Also, the entropy solution in our case cannot be written down explicitly, and so we need to construct using the characteristics and check that it is entropic. For numerical methods towards the convergence of $N+1$ player games to entropy solution, we refer readers to the work of Gomes et al. [103]. Let us mention the recent work by [88], where they study linear-quadratic mean field games in the diffusion setting. To re-establish the uniqueness of MFG solutions, they add a common noise and prove that the limit of MFG solutions as noise tends to zero is just the solution induced by the entropy solution of the master equation without common noise.

The chapter is organized as follows. In Section 8.2, we introduce the $N+1$-player game we are considering, and introduce the equations characterizing the mean field equilibria. In Section 8.3, we show that the forward backward equation characterizing the mean field game possesses a unique solution if $\eta \geq \frac{1}{2}$, may have multiple solutions if $\eta < \frac{1}{2}$. Furthermore, we also determine the number of solutions. In Section 8.4, we explicitly find the entropy solution of the master equation. In Section 8.5, we show that if $\eta = 0$ each player in the $N+1$-player game will follow the majority and briefly present that the optimal trajectories of $N+1$-player game converges to the optimal trajectory induced by the entropy solution of the master equation.

8.2 Two states mean field games

We consider the $N+1$-players game with state space $\Sigma = \{0,1\}$, and denote the state of players by $\mathbf{Z}(t) := (Z_j(t))_{j=1}^{N+1}$, which evolves as controlled Markov processes. The jump rate of $Z_j(t)$ is given by $\alpha^j(t,\mathbf{Z}(t))+\eta$, where $\alpha^j : [0,T]\times\Sigma^{N+1} \to [0,+\infty)$ is the control of player j and $\eta \geq 0$ is the minimum jump rate, i.e.,

$$\mathbb{P}[Z_j(t+h) = 1-i | Z_j(t) = i] = (\alpha^j(t,\mathbf{Z}(t)) + \eta)h + o(h).$$

Denote by $\mathcal{A}$ the collection of all the measurable and locally integrable functions $[0,T]\times\Sigma^{N+1} \to [0,+\infty)$, and by $\boldsymbol{\alpha}^{N+1} = (\alpha^1,\ldots,\alpha^{N+1}) \in \mathcal{A}^{N+1}$ the control of all players. It is can be easily seen that the law of Markov process is determined by the control vector $\boldsymbol{\alpha}^{N+1}$.

Let the empirical measure of player j at time t to be

$$\theta^{N+1,j}(t) = \frac{1}{N}\sum_{k=1,k\neq j}^{N+1} \delta_{Z_k(t)=0}.$$

Then given the running cost function

$$f(i,\theta) = |1-\theta-i| = \begin{cases} 1-\theta & i=0 \\ \theta & i=1, \end{cases} \tag{8.1}$$

the control vector $\boldsymbol{\alpha}^{N+1} \in \mathcal{A}^{N+1}$ and it is associated Markov process $(\mathbf{Z}(t))_{0\leq t\leq T}$, the objective function of the k-th player is defined by

$$J_k^{N+1}(\boldsymbol{\alpha}^{N+1}) = \mathbb{E}\Big[\int_0^T f(Z_k(t),\theta^{N+1,k}(t)) + \frac{\alpha^k(t,\mathbf{Z}(t))}{2}dt\Big]$$

For a control vector $\boldsymbol{\alpha}^{N+1} \in \mathcal{A}^{N+1}$ and $\beta \in \mathcal{A}$, define the perturbed control vector by

$$[\boldsymbol{\alpha}^{N+1,-j};\beta]_k := \begin{cases} \alpha_k, & k \neq j \\ \beta, & k = j. \end{cases}$$

Definition 8.2.1. A control vector $\boldsymbol{\alpha}^{N+1} \in \mathcal{A}^{N+1}$ is a Nash Equilibrium if for any $k = 1, \dots, N+1$

$$J_k^{N+1}(\boldsymbol{\alpha}^{N+1}) = \inf_{\beta \in \mathcal{A}} J_k^{N+1}([\boldsymbol{\alpha}^{N+1,-};\beta]).$$

To find the Nash equilibrium, it is standard to solve its corresponding Hamilton-Jacobi equations for value functions $V^{N+1}(t,i,\theta), i = 0,1$ (see e.g. [104]).

$$\begin{cases} -\frac{d}{dt}V^{N+1}(t,i,\theta) = f(i,\theta) - \frac{(\alpha_*^{N+1}(t,i,\theta))^2}{2} \\ \qquad\qquad +\eta(V^{N+1}(t,1-i,\theta) - V^{N+1}(t,i,\theta)) \\ \qquad\qquad +N(1-\theta)\left(\alpha_*^{N+1}(t,1,\theta+\frac{1-i}{N}) + \eta\right)(V^{N+1}(t,1,\theta+\frac{1}{N}) - V^{N+1}(t,1,\theta)) \\ \qquad\qquad +N\theta\left(\alpha_*^{N+1}(t,0,\theta-\frac{i}{N}) + \eta\right)(V^{N+1}(t,1,\theta-\frac{1}{N}) - V^{N+1}(t,1,\theta)), \\ V^{N+1}(T,i,\theta) = 0, \end{cases} \tag{HJB}$$

where the optimal control is given by

$$a_*^{N+1}(t,i,\theta) = (V^{N+1}(t,i,\theta) - V^{N+1}(t,1-i,\theta))_+.$$

It is also easy to write down the corresponding mean field game equation,

$$\begin{cases} \frac{d}{dt}\theta(t) = (1-\theta(t))((u(t,1)-u(t,0))_+ + \eta) - \theta(t)((u(t,0)-u(t,1))_+ + \eta), \\ -\frac{d}{dt}u(t,i) = f(i,\theta) - \eta(u(t,i)-u(t,1-i)) - \frac{((u(t,i)-u(t,1-i))_+)^2}{2}, \\ \theta(0) = \bar{\theta}, \\ u(T,i) = 0, \end{cases} \tag{MFG}$$

and see e.g. [104] and the corresponding master equation, the corresponding master equation,

$$\begin{cases} -\frac{\partial}{\partial t}U(t,i,\theta) = f(i,\theta) - \frac{[(U(t,i,\theta)-U(t,1-i,\theta)_+]^2}{2} + \eta(U(t,1-i,\theta) - U(t,i,\theta)) \\ \qquad\qquad + \frac{\partial}{\partial\theta}U(t,i,\theta)((U(t,1,\theta)-U(t,0,\theta)_+ + \eta)(1-\theta) \\ \qquad\qquad - \frac{\partial}{\partial\theta}U(t,i,\theta)((U(t,0,\theta)-U(t,1,\theta)_+ + \eta)\theta, \\ U(T,i,\theta) = 0, \end{cases} \tag{ME}$$

see Bayraktar, Cohen [24] and Cecchin, Pelino [66]. Recall from the latter two references that the uniqueness of (MFG) and (ME) is guaranteed by the so-called monotonicity condition, i.e., for every $\theta, \theta' \in [0,1]$,

$$\sum_{i=0,1} (-1)^i (f(i,\theta) - f(i,\theta'))(\theta - \theta') \geq 0,$$

which does not hold true with our choice of running cost.

8.3 non-uniqueness

We show that the mean field equations (MFG) may have multiple solutions. Taking

$$y(t) = u(t,1) - u(t,0), \quad x(t) = 2\theta(t) - 1,$$

then (MFG) becomes

$$\begin{cases} \frac{d}{dt}x = y - x|y| - 2\eta x \\ -\frac{d}{dt}y = x - \frac{1}{2}y|y| - 2\eta y \\ y(T) = 0, x(0) = 2\bar{\theta} - 1. \end{cases} \tag{8.2}$$

The second one of (8.2) is equivalent to

$$x = \frac{1}{2}y|y| + 2\eta y - \frac{d}{dt}y. \tag{8.3}$$

Taking derivative with respect to t in (8.3) and in conjunction with (8.2), we obtain

$$\frac{d^2}{dt^2}y + y - \frac{1}{2}y^3 - 3\eta|y|y - 4\eta^2 y = 0. \tag{8.4}$$

For simplicity, we time reverse the system and try to solve

$$\begin{cases} \frac{d^2}{dt^2}y + y - \frac{1}{2}y^3 - 3\eta|y|y - 4\eta^2 y = 0 \\ \frac{1}{2}y(T)|y(T)| + 2\eta y(T) + \frac{d}{dt}y(T) = x(T) = 2\bar{\theta} - 1 \\ y(0) = 0. \end{cases} \tag{8.5}$$

Since (8.5) contains only the y variable, it can be uniquely solved if imposing the initial conditions $y(0) = 0, \frac{d}{dt}y(0) = v$, and we denote its $\mathcal{C}^1$ solution as $y_v(.)$. Therefore the number of solutions to (8.5) is just the number of initial velocity v such that $2\bar{\theta} - 1 = x_v(T)$, where for any $t \geq 0$

$$x_v(t) := \frac{1}{2}y_v(t)|y_v(t)| + 2\eta y_v(T) + \frac{d}{dt}y_v(t) \tag{8.6}$$

We rewrite the differential equation as a derivative with respect to y instead of t, i.e.,

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{1}{2}(\frac{dy}{dt})^2\right)\frac{dt}{dy} = \frac{d}{dy}\left(\frac{1}{2}(\frac{dt}{dy})^{-2}\right).$$

We can therefore get an implicit solution

$$\frac{dt}{dy} = \pm\frac{1}{\sqrt{G(y)+v^2}}, \tag{8.7}$$

where $G(y) = \frac{1}{4}y^4 + 2\eta|y|^3 + 4\eta^2y^2 - y^2$.

When $y \geq 0$, the first order derivative of G is

$$G'(y) = y^3 + 6\eta y^2 + 8\eta^2 y - 2y = y(y + 3\eta - \sqrt{\eta^2+2})(y + 3\eta + \sqrt{\eta^2+2}).$$

It is then easy to conclude the following results

- If $\eta \geq \frac{1}{2}$, the function $G(y)$ is strictly increasing for $y \geq 0$;
- If $0 \leq \eta < \frac{1}{2}$, the function $G(y)$ decreases on the interval $[0, \sqrt{\eta^2+2} - 3\eta]$ and increases on the interval $[\sqrt{\eta^2+2} - 3\eta, +\infty)$;
- If $\eta < \frac{1}{2}, |v| < v_0$, the function $G(y) + v^2$ maybe negative for some $y \in \mathbb{R}$. Let us denote by $y(v)$ the smallest positive root of $G(y) + v^2 = 0$. Since the function $y \mapsto G(y)$ first decreases to $-v_0^2$ over the interval $[0, \sqrt{\eta^2+2} - 3\eta]$, and then increasing to $+\infty$ over the interval $[\sqrt{\eta^2+2} - 3\eta, +\infty)$, we know that the function $y \mapsto G(y) + v^2$ decreases over $[0, y(v))$ and crosses 0 at $y(v)$, which implies that $y(v)$ is a simple root.

Let $v_0 := \sqrt{-G(\sqrt{\eta^2+2} - 3\eta)}$ if $\eta < \frac{1}{2}$. and

$$T(v) := \int_0^{y(v)} \frac{dz}{\sqrt{G(z)+v^2}}, \quad v \in (0, v_0), \tag{8.8}$$

whose role will be clear in the next result.

Lemma 8.3.1. *The following properties hold for solutions $y_v(.)$,*

- *$y_v(.)$ is strictly increasing if $v > 0$, strictly decreasing if $v < 0$, identically 0 if $v = 0$;*

- *If either $\eta \geq \frac{1}{2}, v \in \mathbb{R}$ or $\eta < \frac{1}{2}, |v| \geq v_0$, then the solution $y_v(t) < +\infty$ if and only if $t < \int_0^{+\infty} \frac{dz}{\sqrt{G(z)+v^2}}$. Furthermore, $y_v(.)$ is strictly increasing if $v > 0$, strictly decreasing if $v < 0$;*

- *If $\eta < \frac{1}{2}, |v| \in (0, v_0)$, the solution $y_v(.)$ is a periodic function.*

Proof. The first statement is clear. We prove the rest by writing down the unique $\mathcal{C}^1$ solution explicitly.

If either $\eta \geq \frac{1}{2}, v \in \mathbb{R}$ or $\eta < \frac{1}{2}, |v| \geq v_0$, then $G(z) + v^2 \geq 0$ for any $z \in \mathbb{R}$ and thus we obtain from (8.7) that

$$t = \operatorname{sign}(v) \int_0^y \frac{dz}{\sqrt{G(z) + v^2}}.$$

Since the function $y \mapsto \int_0^y \frac{dz}{\sqrt{G(z)+v^2}}$ is strictly increasing, for any $t < \int_0^{+\infty} \frac{dz}{\sqrt{G(z)+v^2}}$, we can find a unique $y_v(t)$ such that

$$t = \int_0^{y_v(t)} \frac{dz}{\sqrt{G(z) + v^2}}.$$

It can be seen that the function $t \mapsto y_v(t)$ is $\mathcal{C}^1$, and therefore is the unique solution to (8.5).

Since $G(y_v(t)) + v^2$ is always nonnegative, the solution $y_v(t)$ must oscillate between $[-y(v), y(v)]$. For any $0 \leq t \leq T(v)$, there exists a unique $y_v(t)$ such that

$$t = \int_0^{y_v(t)} \frac{dz}{\sqrt{G(z) + v^2}}.$$

Define a periodic function, still denoted by $y_v(.)$,

$$y_v(t) = \begin{cases} y_v(t - 4kT(v)) & t \in [4kT(v), (4k+1)T(v)), \\ y_v((4k+2)T(v) - t) & t \in [(4k+1)T(v), (4k+2)T(v)), \\ -y_v(t - (4k+2)T(v)) & t \in [(4k+2)T(v), (4k+3)T(v)), \\ -y_v((4k+4)T(v) - t) & t \in [(4k+3)T(v), (4k+4)T(v)). \end{cases}$$

It can be easily seen that $y_v(t)$ is the unique $\mathcal{C}^1$ solution to (8.5). □

Proposition 8.3.2. *If $\eta \geq \frac{1}{2}$, then $x_v(T)$ is strictly increasing with respect to v and therefore* (8.5) *has unique solution.*

Proof. It can be seen that both of the equation (8.5) and the function $v \mapsto x_v(T)$ are odd. Therefore $y_{-v}(.) = -y_v(.)$, $x_{-v}(T) = -x_v(T)$, and we only need to prove the proposition for $v \geq 0$.

The strictly decreasing function $v \mapsto \int_0^{+\infty} \frac{dz}{\sqrt{G(z)+v^2}}$ approaches $+\infty$ as $v \to 0$, approaches 0 as $v \to +\infty$. Therefore any positive T there exists a unique $u > 0$ such that

$$\int_0^{+\infty} \frac{dz}{\sqrt{G(z) + u^2}} = T.$$

As a result of Lemma 8.3.1, the solution $y_v(.)$ is finite at T if and only if $v < u$, and there exists a unique $y_v(T) > 0$ such that

$$T = \int_0^{y_v(T)} \frac{dz}{\sqrt{G(z) + v^2}},$$

and also $\frac{dy_v}{dt}|_T = \sqrt{G(y_v(T)) + v^2}$. Suppose $0 \leq v_1 < v_2 < u$. Due to the fact that $G(z) + v_1^2 < G(z) + v_2^2, \forall z \in \mathbb{R}$, we obtain

$$y_{v_1}(T) < y_{v_2}(T), \frac{d}{dt} y_{v_1}(T) < \frac{d}{dt} y_{v_2}(T),$$

from which we can conclude $x_{v_1}(T) < x_{v_2}(T)$. As a result of $\lim\limits_{v\to u} y_v(T) = +\infty$, we obtain $\lim\limits_{v\to u} x_v(T) = +\infty$, and thus there exists a unique solution to (8.5) for any $2\bar{\theta} - 1 \in \mathbb{R}$.

□

As a result of the above proposition, the mean field equation (8.2) may have multiple solutions only if $\eta < \frac{1}{2}$. To find the number of solutions, we study the period of $y_v(.)$ in the following lemma. Note that since $y_{-v}(t) = -y_v(t)$ and $y_0(t) = 0$, it suffices for us to consider the period of $y_v(.)$ for $v \in (0, v_0)$.

Lemma 8.3.3. *Suppose $0 \le \eta < \frac{1}{2}$, $v \in (0, v_0)$, and $y(v)$ is the smallest postive root of $z \mapsto G(z) + v^2$. Recall (8.8) and define*

$$H(v) := \int_v^{y(v)} \frac{dz}{\sqrt{G(z) + v^2}}.$$

Take $T(v) = T(-v), H(v) = H(-v)$ if $v \in (-v_0, 0)$. Then both $T(.)$ and $H(.)$ are increasing with respect to v over the interval $(0, v_0)$, and $\lim\limits_{v\to v_0} T(v) = +\infty$.

Proof. By the definition, we have $G(y) + v^2 = (\frac{y^2}{2} + 2\eta|y|)^2 + v^2 - y^2$, from which we can conclude that $y(v) \ge v$, and therefore $H(v)$ is positive.

By change of variable $p = \frac{z}{y(v)}$, we obtain

$$T(v) = \int_0^1 \frac{dp}{\sqrt{\frac{G(y(v)p)}{y(v)^2} + \frac{v^2}{y(v)^2}}} = \int_0^1 \frac{dp}{\sqrt{\frac{1}{4}y(v)^2p^4 + 2\eta y(v)p^3 + (4\eta^2 - 1)p^2 + \frac{v^2}{y(v)^2}}}.$$

Denote the square of the bottom of the integrand by $P(v, p)$, i.e.,

$$P(v, p) := \frac{1}{4}y(v)^2p^4 + 2\eta y(v)p^3 + (4\eta^2 - 1)p^2 + \frac{v^2}{y(v)^2}.$$

To prove $T(v)$ is increasing, it suffices to show that $P(v, p)$ is decreasing with respect to v for any fixed $p \in [0, 1]$.

Since $y(v)$ is an increasing function of v, the derivative $\frac{dP}{dv}(v, p)$ is no larger than $\frac{dP}{dv}(v, 1)$, which is equal to 0 according to the definition of $y(v)$,

$$\frac{dP}{dv}(v, 1) = \frac{d(G(y(v)) + v^2)}{dv} = 0.$$

Therefore $P(v_1, p) \geq P(v_2, p)$ for any $p \in [0, 1], 0 < v_1 < v_2 < v_0$.

We can also rewrite $H(v)$ as

$$H(v) = \int_{\frac{v}{y(v)}}^{1} \frac{dp}{\sqrt{P(v, p)}},$$

and it is enough to show that $v \mapsto \frac{v}{y(v)}$ is decreasing. Taking derivative of the following equation with respect to v,

$$G(y(v)) + v^2 = 0,$$

we get $\frac{dy(v)}{dv} = -\frac{2v}{G'(y(v))}$, and thus

$$\frac{d}{dv}(\frac{v}{y(v)}) = \frac{y(v) - v\frac{dy(v)}{dv}}{y(v)^2} = \frac{y(v) + \frac{2v^2}{G'(y(v))}}{y(v)^2}.$$

As a result of $\frac{dy(v)}{dv} \geq 0$, we obtain that $G'(y(v)) < 0$ and $\frac{d}{dv}(\frac{v}{y(v)}) \leq 0$ is equivalent to $G'(y(v))y(v) + 2v^2 \geq 0$. We conclude our claim by the following computation,

$$\begin{aligned} G'(y(v))y(v) + 2v^2 &= G'(y(v))y(v) + 2v^2 - 2(G(y(v)) + v^2) \\ &= \frac{1}{2}y(v)^4 + 2\eta y(v)^3 > 0 \end{aligned}$$

In the end, it can be seen that the function $z \mapsto G(z) + v_0^2$ is always positive over the interval $[0, +\infty)$ and only attains 0 at $z = \sqrt{\eta^2 + 2} - 3\eta$. Since $G(z) + v_0^2$ is a polynomial, we obtain that $y(v_0) = \sqrt{\eta^2+2} - 3\eta$, $(z - \sqrt{\eta^2+2} + 3\eta)^2$ is a factor of $G(z) + v_0^2$, and hence

$$\lim_{v \to v_0} T(v) = \int_0^{\sqrt{\eta^2+2}-3\eta} \frac{dz}{\sqrt{G(z) + v_0^2}} = +\infty.$$

$\square$

For each $k \in \mathbb{N}$, define $T_k(v) := (2k-1)T(v) + H(v)$ if $|v| \in (0, v_0)$, and $T_k(v) := +\infty$ if $|v| > v_0$. Now we show that for $v \neq 0$, $\{T_k(v) : k \in \mathbb{N}\}$ is the set of times T such that $x_v(T)$ attains 0 ($T_k(v) = +\infty$ for $|v| \geq v_0$ simply implies that $x_v(t)$ never reaches 0 for those v). As a result of Lemma 8.3.1, the function $x_v(T)$ can equal to 0 only if $\eta < \frac{1}{2}$, $|v| \in (0, v_0)$ or $v = 0$. Setting $x_v(T) = 0$, by (8.6) we get

$$\begin{aligned} 0 = x_v(T) &= \frac{1}{2} y_v(T)|y_v(T)| + 2\eta y_v(T) + \frac{d}{dt} y_v(T) \\ &= \frac{1}{2} y_v(T)|y_v(T)| + 2\eta y_v(T) + \operatorname{sign}(\frac{d}{dt} y_v(T)) \sqrt{G(y_v(T)) + v^2}. \end{aligned}$$

Moving the last term to the left, taking square of both sides and plugging in the formula of $G(y)$, it becomes

$$(\frac{1}{2} y_v(T)|y_v(T)| + 2\eta y_v(T))^2 + v^2 - (y_v(T))^2 = (\frac{1}{2} y_v(T)|y_v(T)| + 2\eta y_v(T))^2,$$

which is equivalent to $v^2 - (y_v(T))^2 = 0$. Therefore we obtain that $|y_v(T)| = v, \operatorname{sign}(y_v(T)) = -\operatorname{sign}(\frac{d}{dt} y_v(T))$, from which we conclude that $x_v(T) = 0$ if and only if $T = T_k(v)$ or $v = 0$.

Therefore $T_1(v)$ is the first time $x_v(t)$ reaches 0. Taking $T_k(0+) := \lim_{v \downarrow 0} T_k(v)$, it can be seen that for $t \leq T_1(0+), v \neq 0$, we have $x_v(t) \neq 0$. Before computing the number of solutions, we still need one more result, which is also important for us to construct the entropy solution of the master equation in the next section.

Lemma 8.3.4. *Suppose $\eta < \frac{1}{2}$. Then for any $(x,t) \in \mathbb{R} \times \mathbb{R}_+ \setminus \{0\} \times \mathbb{R}_+$, there exists a unique $v(x,t) \in \mathbb{R}_+$ such that $x_v(t) = x, t < T_1(v)$ (simply take $v(x,t) = 0$ if $x = 0$).*

Proof. **Step 1.** For any $0 < v_1 < v_2 \leq v_0$, we prove that $y_{v_1}(t) < y_{v_2}(t), \forall t \in (0, T_1(v_1)]$. Otherwise suppose $y_{v_1}(t) = y_{v_2}(t)$ for some $t \in (0, T_1(v_1)]$. If $t \leq T(v_1)$, as in the proof of Lemma 8.3.1 we have

$$t = \int_0^{y_{v_1}(t)} \frac{dz}{\sqrt{G(z) + v_1^2}} = \int_0^{y_{v_2}(t)} \frac{dz}{\sqrt{G(z) + v_2^2}}, \tag{8.9}$$

which is impossible since $G(z) + v_1^2 < G(z) + v_2^2$. If $t \in (T(v_1), T(v_2)]$, then $y_{v_2}(t) > y_{v_2}(T(v_1)) > y_{v_1}(T(v_1)) > y_{v_1}(t)$, which is contradictory to our assumption. If $t \in (T(v_2), T_1(v_1)]$, we have

$$2T(v_1) - t = \int_0^{y_{v_1}(t)} \frac{dz}{\sqrt{G(z) + v_1^2}} > \int_0^{y_{v_2}(t)} \frac{dz}{\sqrt{G(z) + v_2^2}} = 2T(v_2) - t,$$

which contradicts to Lemma 8.3.3.

Step 2. For any $v_0 \leq v_1 < v_2, t \in \big(0, \int_0^{+\infty} \frac{dz}{\sqrt{G(z)+v^2}}\big]$, we have $y_{v_1}(t) < y_{v_2}(t)$, which can be proved as in **Step 1**.

Step 3. For any $0 < v_1 < v_2 \leq v_0$, we prove that $x_{v_1}(t) < x_{v_2}(t), \forall t \in [0, T_1(v_1))]$. Otherwise suppose $t = \sup\{t : x_{v_1}(t) = x_{v_2}(t), t \leq T_1(v_1)\}$, where supreme is attained by the continuity of $x_{v_1}(.)$ and $x_{v_2}(.)$. To show the contradiction, we prove that $\frac{d}{dt}(x_{v_2}(t) - x_{v_1}(t)) < 0$, in which case these two curves have to intersect after time t since x_{v_2} decreases to 0 at time $T_1(v_2) > T_1(v_1)$.

If $t \geq T(v_1)$, we have

$$\begin{aligned} x_{v_1}(t) &= \frac{1}{2} y_{v_1}(t)^2 + 2\eta y_{v_1}(t) - \sqrt{G(y_{v_1}(t)) + v_1^2} \\ &= \frac{1}{2} y_{v_2}(t)^2 + 2\eta y_{v_2}(t) + \operatorname{sign}(\frac{d}{dt} y_{v_2}(t)) \sqrt{G(y_{v_2}(t)) + v_2^2} = x_{v_2}(t). \end{aligned}$$

Since we proved $y_{v_1}(t) < y_{v_2}(t)$, the derivative $\frac{d}{dt}y_{v_2}(t)$ must be negative, and hence

$$\frac{1}{2}y_{v_1}(t)^2+2\eta y_{v_1}(t)-\sqrt{G(y_{v_1}(t))+v_1^2}=\frac{1}{2}y_{v_2}(t)^2+2\eta y_{v_2}(t)-\sqrt{G(y_{v_2}(t))+v_2^2}. \quad (8.10)$$

Combining (8.10) and $\frac{d}{dt}y_{v_i}(t) = -\sqrt{G(y_{v_i}(t)) + v_i^2}, i = 1, 2$, we obtain

$$\begin{aligned}\frac{d}{dt}(x_{v_2}(t) - x_{v_1}(t)) =&y_{v_1}(t)\Big(\sqrt{G(y_{v_1}(t)) + v_1^2} - \frac{1}{2}y_{v_1}(t)^2 - 2\eta y_{v_1}(t) + 1)\Big)\\ &- y_{v_2}(t)\Big(\sqrt{G(y_{v_2}(t)) + v_2^2} - \frac{1}{2}y_{v_2}(t)^2 - 2\eta y_{v_2}(t) + 1)\Big).\end{aligned}$$

Because of (8.10) and the fact that $y_{v_2}(t) > y_{v_1}(t)$, we deduce that $\frac{d}{dt}(x_{v_2}(t)-x_{v_1}(t)) < 0$ is equivalent to $\sqrt{G(y_{v_2}(t)) + v_2^2} - \frac{1}{2}y_{v_2}(t)^2 - 2\eta y_{v_2}(t) + 1 > 0$, which is true since

$$\begin{aligned}&\sqrt{G(y_{v_2}(t)) + v_2^2} - \frac{1}{2}y_{v_2}(t)^2 - 2\eta y_{v_2}(t) + 1 > -\frac{1}{2}y_{v_2}(t)^2 - 2\eta y_{v_2}(t) + 1\\ &> -\frac{1}{2}(\sqrt{\eta^2+2} - 3\eta)^2 - 2\eta(\sqrt{\eta^2}+2-3\eta) + 1 > 0.\end{aligned}$$

If $t < T(v_1)$, by the same reasoning we have

$$\frac{1}{2}y_{v_1}(t)^2 + 2\eta y_{v_1}(t) + \sqrt{G(y_{v_1}(t)) + v_1^2} = \frac{1}{2}y_{v_2}(t)^2 + 2\eta y_{v_2}(t) + \sqrt{G(y_{v_2}(t)) + v_2^2},$$

and also

$$\begin{aligned}\frac{d}{dt}(x_{v_2}(t) - x_{v_1}(t)) =&y_{v_2}(t)\Big(\sqrt{G(y_{v_2}(t)) + v_2^2} + \frac{1}{2}y_{v_2}(t)^2 + 2\eta y_{v_2}(t) - 1\Big)\\ &- y_{v_1}(t)\Big(\sqrt{G(y_{v_1}(t)) + v_1^2} + \frac{1}{2}y_{v_1}(t)^2 + 2\eta y_{v_1}(t) - 1\Big).\end{aligned}$$

Accordingly, it suffices to show that $\Big(\sqrt{G(y_{v_2}(t)) + v_2^2} + \frac{1}{2}y_{v_2}(t)^2 + 2\eta y_{v_2}(t) - 1\Big) < 0$, which is equivalent to

$$\sqrt{G(y_{v_2}(t)) + v_2^2} < 1 - \frac{1}{2}y_{v_2}(t)^2 - 2\eta y_{v_2}(t). \quad (8.11)$$

Taking square of (8.11) , we obtain the equivalent inequality $v_2^2 + 4\eta y_{v_2}(t) - 1 < 0$. Since $y_{v_2}(t) \leq y(v_2)$, we conclude our claim by the following computation

$$\begin{aligned} v_2^2 + 4\eta y_{v_2}(t) - 1 \leq & v_2^2 + 4\eta y(v_2) - 1 = -G(y(v_2)) + 4\eta y(v_2) - 1 \\ = & -(\frac{1}{2}y(v_2)^2 + 2\eta y(v_2) - 1)^2 < 0. \end{aligned}$$

Step 4. For any $v_0 \leq v_1 < v_2, t \in \left(0, \int_0^{+\infty} \frac{dz}{\sqrt{G(z)+v^2}}\right]$, we have $x_{v_1}(t) < x_{v_2}(t)$, which can be proved as in **Step 3**.

Step 5. Until now we have shown that the stopped curves $\{x_v(t) : 0 \leq t < T_1(v)\}$ do not intersect, and it remains to prove that for any $(x,t) \in \mathbb{R}_+ \times \mathbb{R}_+$, there exists a $v(x,t) \in \mathbb{R}_+$ such that $x_v(t) = x, t < T_1(v)$. Note that according to (8.5), for any fixed t, the couple $(y_v(t), \frac{d}{dt}y_v(t))$ is continuous with respect to the initial velocity v, and thus the mapping $v \mapsto x_v(t)$ is also continuous.

First suppose $x < x_{v_0}(t)$ and $t \leq T_1(0+)$. As a result of $\lim\limits_{v\to 0} x_v(t) = 0$, $\lim\limits_{v\to v_0} x_v(t) = x_{v_0}(t)$ and the continuity of $v \mapsto x_v(t)$, we know that there must exist some $v \in (0, v_0)$ such that $x_v(t) = x$. The equality $t < T_1(v)$ simply follows from the inequality $t \leq T_1(0+) < T_1(v)$.

Suppose $x < x_{v_0}(t)$ and $t > T_1(0+)$. Since $T_1(v)$ increases to $+\infty$ as v increases to v_0, we know that there exists a unique $v' \in (0, v_0)$ such that $t = T_1(v')$, which also implies $x_{v'}(t) = 0$. According to the continuity of $v' \mapsto x_{v'}(t)$, and the fact that $\lim\limits_{v\to v_0} x_v(t) = x_{v_0}(t)$, we know there must exist a $v > v'$ such that $x_v(t) = x$, and $t = T_1(v') < T_1(v)$.

In the end suppose $x > x_{v_0}(t)$. Because the mapping $v \mapsto \int_0^{+\infty} \frac{dz}{\sqrt{G(z)+v^2}}$ is decreasing from $+\infty$ to 0 over the interval $(v_0, +\infty)$, there exists a unique $v' > v_0$ such that $\int_0^{+\infty} \frac{dz}{\sqrt{G(z)+v'^2}} = t$, which also implies $x_{v'}(t) = +\infty$. Again by the continuity of $v \mapsto x_v(t)$ and the fact that $\lim\limits_{v\to v_0} x_v(t) = x_{v_0}(t) < x$, there exists a $v > v_0$ such that $x_v(t) = x$. $\square$

Proposition 8.3.5. *Suppose $\eta < \frac{1}{2}$. Then there exists a unique solution to* (8.5) *for any $T > 0$ if $|2\bar{\theta} - 1| \geq 1 - \eta^2 - \eta\sqrt{\eta^2 + 2}$, and the number of solutions to* (8.5) *can be arbitrarily large if $|2\bar{\theta} - 1| < 1 - \eta^2 - \eta\sqrt{\eta^2 + 2}$ and T is large enough. In particular, the number of solutions with boundary condition $2\bar{\theta} - 1 = 0$ is given by*

$$1 + 2 \sup_{k \in \mathbb{N}} \{k : T_k(0+) < T\}.$$

Proof. Recalling $v_0 = \sqrt{-G(\sqrt{\eta^2 + 2} - 3\eta)}$, we first prove that $x_{v_0}(t)$ is increasing with respect to t and $\lim_{t \to +\infty} x_{v_0}(t) = 1 - \eta^2 - \eta\sqrt{\eta^2 + 2}$.

Taking derivative of the following equation,

$$x_{v_0}(t) = \frac{1}{2} y_{v_0}(t) y_{v_0}(t) + 2\eta y_{v_0}(t) + \frac{d}{dt} y_{v_0}(t),$$

we get $\frac{d}{dt} x_{v_0}(t) = (y_{v_0}(t) + 2\eta) \frac{d}{dt} y_{v_0}(t) + \frac{1}{2} G^{'}(y_{v_0}(t))$. Therefore $x_{v_0}(t)$ is increasing is equivalent to

$$(y_{v_0}(t) + 2\eta) \frac{d}{dt} y_{v_0}(t) \geq -\frac{1}{2} G^{'}(y_{v_0}(t)). \tag{8.12}$$

Since both sides of (8.12) are positive, it is enough to show that

$$(y_{v_0}(t) + 2\eta)^2 (\frac{d}{dt} y_{v_0}(t))^2 - \frac{1}{4} (G^{'}(y_{v_0}(t)))^2 > 0.$$

Plugging in the equality $\frac{d}{dt} y_{v_0}(t) = \sqrt{G(y_{v_0}(t)) + v_0^2}$ and the formula of G, the inequality becomes

$$2\eta (y_{v_0}(t))^3 + (4\eta^2 - 1 + v_0^2)(y_{v_0}(t))^2 + 4\eta v_0^2 y_{v_0}(t) + 4\eta^2 v_0^2 \geq 0.$$

Now we finish proving $x_{v_0}(t)$ is increasing by the following equality,

$$\begin{aligned}&2\eta(y_{v_0}(t))^3 + (4\eta^2 - 1 + v_0^2)(y_{v_0}(t))^2 + 4\eta v_0^2 y_{v_0}(t) + 4\eta^2 v_0^2\\&= (y_{v_0}(t) - \sqrt{\eta^2+2} + 3\eta)^2\left(2\eta y + \frac{4\eta^2 v_0^2}{(\sqrt{\eta^2+2} - 3\eta)^2}\right)\end{aligned}$$

Recall Lemma 8.3.1, $y_{v_0}(t)$ is given by the equation

$$t = \int_0^{y_{v_0}(t)} \frac{dz}{\sqrt{G(z) + v_0^2}}.$$

Combining the equality proved in Lemma 8.3.3 that $\int_0^{\sqrt{\eta^2+2}-3\eta} \frac{dz}{\sqrt{G(z)+v_0^2}} = +\infty$, we conclude that $\lim_{t\to+\infty} y_{v_0}(t) = \sqrt{\eta^2+2} - 3\eta$. Also, according to (8.7), we get that

$$\lim_{t\to+\infty} \frac{d}{dt} y_{v_0}(t) = \sqrt{G(\sqrt{\eta^2+2} - 3\eta) + v_0^2} = 0.$$

Therefore by (8.6), we conclude the second claim

$$\lim_{t\to+\infty} x_{v_0}(t) = \frac{1}{2}(\sqrt{\eta^2+2} - 3\eta)^2 + 2\eta(\sqrt{\eta^2+2} - 3\eta) = 1 - \eta^2 - \eta\sqrt{\eta^2+2}.$$

It can be seen that the curves $\{x_v(t) : t \geq 0, v \geq v_0\}$ never cross each other, and that $x_v(t) < 1 - \eta^2 - \eta\sqrt{\eta^2+2}$ for any $t > 0$ if $v < v_0$. Therefore according to Lemma 8.3.4, if $|2\bar{\theta} - 1| \geq 1 - \eta^2 - \eta\sqrt{\eta^2+2}$, there exists only one $v \geq v_0$ such that $x_v(T) = 2\bar{\theta} - 1$.

Now suppose that $0 < 2\bar{\theta} - 1 < 1 - \eta^2 - \eta\sqrt{\eta^2+2}$. For each $v \in (0, v_0)$, define

$$M(v) := \max_{t\geq 0} x_v(t).$$

As a result of Lemma 8.3.4, $M(v)$ is actually an increasing function, and there exists a unique $\bar{v} \in (0, v_0)$ such that $M(\bar{v}) = 2\bar{\theta} - 1$. Also for any $v \in [\bar{v}, v_0)$, we can define

$t(v)$ as the unique t satisfying $x_v(t) = 2\bar{\theta} - 1, t < T_1(v)$, which is also an increasing function of v. Then $(x_v(.), y_v(.))$ is a solution of (8.4) with time horizon $T = t(v)$. Since the period of $x_v(.)$ is $4T(v)$, and $\lim\limits_{v \to v_0} t(v) = +\infty$, for each $k \in \mathbb{N}$ we know that if $T > t(\bar{v}) + 4kT(\bar{v})$, there must exist some $v' \in [\bar{v}, v_0)$ such that $T = t(v') + 4kT(v')$. Therefore we conclude that the number of solutions to (8.4) with time horizon T is greater than

$$\sup_{k \in \mathbb{N}} \{k : T \geq t(\bar{v}) + 4kT(\bar{v})\},$$

which can be arbitrarily large if T is large enough.

In the end, we consider the number of solutions for the terminal condition $2\bar{\theta} - 1 = 0$. We have already shown that $T_k(v)$ is the time when $x_v(t)$ attains zero. According to Lemma 8.3.3, the functions $T_k(v)$ are increasing with respect to v for each $k \in \mathbb{N}$ and $\lim\limits_{v \to v_0} T_k(v) = +\infty$. Since $x_{-v}(t) = -x_v(t)$, and $v = 0$ is always a solution, the number of solutions is just

$$1 + 2\{(k, v) : T_k(v) = T, k \in \mathbb{N}, v \in (0, v_0)\} = 1 + 2 \sup_{k \in \mathbb{N}} \{k : T_k(0+) < T\}.$$

$\square$

8.4 The Master Equation

Letting $Y(t, \theta) = U(t, 1, \theta) - U(t, 0, \theta)$, $x = 2\theta - 1$, and time reverse the master equation (ME), we obtain the equation

$$\frac{\partial Y}{\partial t} + \frac{\partial}{\partial x}\left(2\eta x Y + \frac{xY|Y|}{2} - \frac{Y^2}{2} - \frac{x^2}{2}\right) = 0, \tag{8.13}$$

with the boundary condition $Y(0, x) = 0, \forall x \in [-1, 1]$.

Since the equation has the form of a scalar conservation law, there exists a unique entropy solution. By the method of characteristics, we directly construct a piecewise

$\mathcal{C}^1$ solution to (8.13) and then check it is entropic.

Rewriting (8.13) as

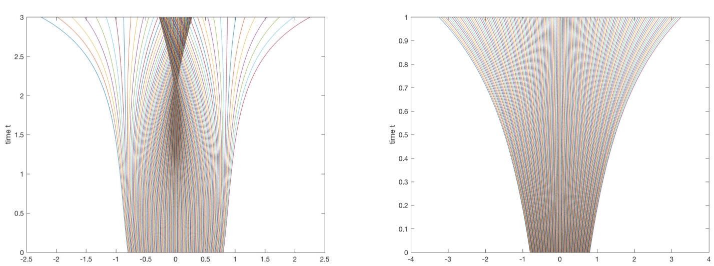
$$\frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial x}(2\eta x - Y + x|Y|) = -2\eta Y - \frac{Y|Y|}{2} + x,$$

and letting $y(t) = Y(t, x(t))$, $\frac{d}{dt}x = 2\eta x - y + x|y|$, we obtain the characteristic curve of (8.13)

$$\begin{cases} \frac{d}{dt}x = 2\eta x - y + x|y|, \\ \frac{d}{dt}y = -2\eta y - \frac{y|y|}{2} + x, \\ y(0) = 0, x(0) = \frac{dy}{dt}(0), \end{cases}$$

whose solution is given explicitly in Lemma 8.3.1. If $\eta \geq \frac{1}{2}$, the solution given by characteristic curves is smooth everywhere. If $\eta < \frac{1}{2}$, the shock curve is taken to be $\gamma(t) = 0, t \in \mathbb{R}_+$. See our illustration in Figure 8.1.

Figure 8.1: Characteristic curves, $\eta = 0.1, T = 3$ on the left; $\eta = 0.6, T = 1$ on the right.



Proposition 8.4.1. *The function $Y(x,t) := y_{v(x,t)}(t)$ is the entropy solution of* (8.13) *with shock curve $\gamma(t) = 0, t > T_1(0+)$, where $v(x,t) \in \mathbb{R}$ is defined in Lemma 8.3.4.*

Proof. It is clear that the function $Y(x,t)$ is $\mathcal{C}^1$ outside the shock curve, and we only need to check the *Rankine-Hugoniot condition* and the *Lax condition* (see [67, Proposition 3]). Define

$$Y_+(t) := \lim_{x \downarrow 0} Y(x,t), \quad Y_- := \lim_{x \uparrow 0} Y(x,t).$$

If $t > T_1(0+)$, there exists a $v > 0$ such that $t = T_1(v)$ since $v \mapsto T_1(v)$ is increasing to $+\infty$ as v increases to v_0. Also it can be seen that $\lim_{x \downarrow 0} v(x,t) = v$. According to the discussion above Lemma 8.3.4, we conclude that $Y_+(t) = y_v(t) = v = \lim_{x \downarrow 0} v(x,t)$, and similarly $Y_-(t) = -\lim_{x \downarrow 0} v(x,t)$. If $t \le T_1(0+)$, the mapping $v \mapsto x_v(t)$ is continuous and strictly increasing, which is zero at $v = 0$. Therefore $\lim_{x \downarrow 0} v(x,t) = 0$, and $Y_+(t) = Y_-(t) = 0$. In summary, we have

$$Y_+(t) = -Y_-(t) = \begin{cases} \lim_{x \downarrow 0} v(x,t) & \text{if } t > T_1(0+), \\ 0 & \text{if } t \le T_1(0+). \end{cases}$$

Taking $\mathfrak{g}(x,Y) = 2\eta x Y + \frac{xY|Y|}{2} - \frac{Y^2}{2} - \frac{x^2}{2}$, we have

$$\frac{d}{dt}\gamma(t) = 0 = \frac{-\frac{(Y_+(t))^2}{2} + \frac{(Y_-(t))^2}{2}}{Y_+(t) - Y_-(t)} = \frac{\mathfrak{g}(\gamma(t), Y_+(t)) - \mathfrak{g}(\gamma(t), Y_-(t))}{Y_+(t) - Y_-(t)},$$

which verifies the *Rankine-Hugoniot condition*.

For any c strictly between $Y_-(t)$ and $Y_+(t)$, $t > T_1(0+)$, we have

$$\frac{\mathfrak{g}(\gamma(t), c) - \mathfrak{g}(\gamma(t), Y_+(t))}{c - Y_+(t)} = \frac{\frac{(Y_+(t))^2}{2} - \frac{c^2}{2}}{c - Y_+(t)} = -\frac{c + Y_+(t)}{2},$$

$$\frac{\mathfrak{g}(\gamma(t), c) - \mathfrak{g}(\gamma(t), Y_-(t))}{c - Y_-(t)} == \frac{\frac{(Y_-(t))^2}{2} - \frac{c^2}{2}}{c - Y_-(t)} = -\frac{c + Y_-(t)}{2},$$

and therefore

$$\frac{\mathfrak{g}(\gamma(t), c) - \mathfrak{g}(\gamma(t), Y_+(t))}{c - Y_+(t)} < \frac{d}{dt}\gamma(t) = 0 < \frac{\mathfrak{g}(\gamma(t), c) - \mathfrak{g}(\gamma(t), Y_-(t))}{c - Y_-(t)},$$

which verifies the *Lax condition*. □

Remark 8.4.2. It is easily seen that the entropy solution of (8.13) corresponds to a solution of (ME).

Remark 8.4.3. By Lemma 8.3.4, we know that for any $\bar{\theta} \in [0,1]$, there exists a unique v' such that $x_{v'}(T) = 2\bar{\theta} - 1, T < T_1(v')$. Then $(x_{v'}(T-t), y_{v'}(T-t))$ solves (8.2), which is the mean field equilibrium induced the entropy solution.

8.5 $N+1$-player game and the selection of Equilibrium

In this section, we consider the $N+1$-player game and always assume $\eta = 0$. Since the model we are considering is invariant under permutation, it can be easily seen that

$$V^{N+1}(t, 0, 1-\theta) = V^{N+1}(t, 1, \theta),$$

and therefore we only need to consider the HJB systems for $V^{N+1}(t, 1, \theta)$:

$$\begin{cases} -\frac{d}{dt}V^{N+1}(t,1,\theta) = f(1,\theta) - \frac{(\alpha_*^{N+1}(t,1,\theta))^2}{2} \\ \qquad\qquad +N(1-\theta)\alpha_*^{N+1}(t,1,\theta)(V^{N+1}(t,1,\theta+\frac{1}{N}) - V^{N+1}(t,1,\theta)) \\ \qquad\qquad +N\theta\alpha_*^{N+1}(t,0,\theta-\frac{1}{N})(V^{N+1}(t,1,\theta-\frac{1}{N}) - V^{N+1}(t,1,\theta)) \\ V^{N+1}(T,1,\theta) = 0, \end{cases} \tag{8.14}$$

where the optimal control policy is

$$a_*^{N+1}(t, i, \theta) = (V^{N+1}(t, i, \theta) - V^{N+1}(t, 1-i, \theta))_+.$$

As a result of the local Lipschitz continuity of the HJB equation (8.14), the system can be uniquely solved with terminal condition $V^{N+1}(T, 0, \theta) = 0$, which provides us the unique Nash Equilibrium of the game. Supposing that the representative player is applying the zero control while the other players are taking the optimal policy, then by the definition of Nash Equilibrium we conclude that

$$V^{N+1}(t,1,\theta) \le \mathbb{E}\Big[\int_t^T f(i(t),\theta_t)dt\Big] \le T-t.$$

Now we prove that if the representative player agrees with the majority, then he will keep his state by taking the zero control.

Proposition 8.5.1. *Taking*

$$Y^{N+1}(t,\theta) = V^{N+1}(t,1,\theta) - V^{N+1}(t,0,\theta) = V^{N+1}(t,1,\theta) - V^{N+1}(t,1,1-\theta),$$

for any $\theta \in \{0, \frac{1}{N}, \dots, 1\}$ *we have*

$$\begin{aligned} &Y^{N+1}(t,\theta) \ge 0 \ (\alpha_*^{N+1}(t,0,\theta)=0) \quad \text{if } \theta \ge \frac{1}{2}, \\ &Y^{N+1}(t,\theta) \le 0 \ (\alpha_*^{N+1}(t,1,\theta)=0) \quad \text{if } \theta \le \frac{1}{2}. \end{aligned} \tag{8.15}$$

Proof. We only prove the first inequality of (8.15) for even N, and the rest can be proved similarly. As a result of $Y^{N+1}(t,\frac{1}{2}) = 0$, it is enough for us to show it for $\theta \ge \frac{1}{2} + \frac{1}{N}$. Take

$$W^{N+1}(t,\theta) = V^{N+1}(t,1,\theta) - V^{N+1}(t,1,\theta-\frac{1}{N}).$$

According to (8.14), we obtain

$$\begin{aligned} \frac{d}{dt}Y^{N+1}(t,\theta) = {} & 1 - 2\theta + \frac{|Y^{N+1}(t,\theta)|Y^{N+1}(t,\theta)}{2} \\ & + N\theta\Big(Y^{N+1}(t,\theta-\frac{1}{N})_- W^{N+1}(t,\theta) + Y^{N+1}(t,\theta)_- W^{N+1}(t,1-\theta+\frac{1}{N})\Big) \\ & - N(1-\theta)\Big(Y^{N+1}(t,\theta)_+ W^{N+1}(t,\theta+\frac{1}{N}) + Y^{N+1}(t,\theta+\frac{1}{N})_+ W^{N+1}(t,1-\theta)\Big), \end{aligned} \tag{8.16}$$

and

$$
\begin{aligned}
\frac{d}{dt}W^{N+1}(t,1-\theta) =& -\frac{1}{N}+\frac{Y^{N+1}(t,1-\theta)_+^2}{2}-\frac{Y^{N+1}(t,1-\theta-\frac{1}{N})_+^2}{2}\\
&-N\theta Y^{N+1}(t,1-\theta)_+W^{N+1}(t,1-\theta+\frac{1}{N})\\
&+N(1-\theta)Y^{N+1}(t,1-\theta-\frac{1}{N})_-W^{N+1}(t,1-\theta)\\
&+N(\theta+\frac{1}{N})Y^{N+1}(t,1-\theta-\frac{1}{N})_+W^{N+1}(t,1-\theta)\\
&-N(1-\theta-\frac{1}{N})Y^{N+1}(t,1-\theta-\frac{2}{N})_-W^{N+1}(t,1-\theta-\frac{1}{N}).
\end{aligned}
\tag{8.17}
$$

By our terminal condition $V^{N+1}(T,1,\theta)=0$, it is easy to see that $Y^{N+1}(T,\theta)=W^{N+1}(T,\theta)=0$, and both $\frac{d}{dt}Y^{N+1}(T,\theta)$, $\frac{d}{dt}W^{N+1}(T,1-\theta)$ are negative if $\theta>\frac{1}{2}$. And therefore by the continuity of $V^{N+1}(t,1,\theta)$, there exists a small positive $\epsilon>0$ such that $Y^{N+1}(t,\theta), W^{N+1}(t,1-\theta)$ are positive during the time interval $[T-\epsilon,T)$. Define

$$
s:=\sup_{\{t<T-\epsilon\}}\{t: W^{N+1}(t,1-\theta)=0 \text{ or } Y^{N+1}(t,\theta)=0 \text{ for some } \theta>\frac{1}{2}\}.
$$

We finish the argument by showing that $Y^{N+1}(t,\theta)$ and $W^{N+1}(t,1-\theta)$ are both positive for $t\in[s,T-\epsilon], \theta>\frac{1}{2}$, which implies s has to be $-\infty$. By the definition of s, we have $Y^{N+1}(t,\theta)=-Y^{N+1}(t,1-\theta)\geq 0, W^{N+1}(t,1-\theta)\geq 0$ if $t\in[s,T-\epsilon)$, $\theta>\frac{1}{2}$, and therefore we obtain the following inequality from (8.16),

$$
\frac{d}{dt}Y^{N+1}(t,\theta)\leq Y^{N+1}(t,\theta)\Big(\frac{Y^{N+1}(t,\theta)}{2}-N(1-\theta)W^{N+1}(t,\theta+\frac{1}{N})\Big).
$$

Since $V^{N+1}(t,1,\theta)\leq T$, we get that $|Y^{N+1}(t,\theta)|\leq 2T$, $|W^{N+1}(t,\theta)|\leq 2T$ for any $\theta\in\{0,\frac{1}{N},\dots,1\}$. Therefore $Y^{N+1}(t,\theta)$ is bounded below by the solution of

$$
\begin{cases}
\frac{d}{dt}l(t)=(T+2NT)l(t)\\
l(T-\epsilon)=Y^{N+1}(T-\epsilon,\theta),
\end{cases}
$$

which is always positive. Similarly, for $t \in [s, T-\epsilon], \theta > \frac{1}{2}$, we obtain the inequality from (8.17)

$$\begin{aligned}
&\frac{d}{dt}W^{N+1}(t, 1-\theta) \leq N(1-\theta)Y^{N+1}(t, 1-\theta-\frac{1}{N})_{-}W^{N+1}(t, 1-\theta) \\
&\leq 2NT(1-\theta)W^{N+1}(t, 1-\theta),
\end{aligned}$$

which implies $W^{N+1}(t, 1-\theta) > 0$ for $t \in [s, T-\epsilon]$. □

Remark 8.5.2. Recall that $\mathbf{Z}(t)$ is the state of the $N+1$ players at time t when agents play the Nash equilibrium given by (HJB). Denote by $\theta^{N+1}(t)$ the fraction of players at state 0, i.e.,

$$\theta^{N+1}(t) = \frac{1}{N+1}\sum_{j=1}^{N+1}\delta_{Z_j(t)=0}.$$

and let U be the solution of (ME) corresponding to the entropy solution of (8.13). According to Proposition 8.5.1, $\theta^{N+1}(t)$ will always stay on one side of $\frac{1}{2}$ if $\theta^{N+1}(0) \neq \frac{1}{2}$. In combination with the fact that $U(t, i, \theta)$ is smooth outside the curve $\bar{\gamma}(t) = \frac{1}{2}$, it can be easily seen that $V^{N+1}(t, 1, \theta)$ converges to $U(t, 1, \theta)$ if $\theta \neq \frac{1}{2}$ (see e.g. [67, Theorem 8]).

Let $(\xi_j)_{j\in\mathbb{N}}$ be the i.i.d initial datum of Z_j such that $\mathbb{P}[\xi_j = 0] = \bar{\theta} \neq \frac{1}{2}, \mathbb{P}[\xi_j = 1] = 1-\bar{\theta}$. Denote by $\tilde{Z}_j$ the i.i.d process in which players choose the optimal control $\tilde{\alpha}(t, i) := (U(t, i, \theta(t)) - U(t, 1-i, \theta(t)))_{+}$, where U is the corresponding entropy solution of (ME). Also, we can prove the propagation of chaos property by using the technique developed in [66] and [67].

8.6 Conclusion

When $\eta > 1/2$, the N-player game converges to the mean field game following the analysis of [24] and [66]. Here we considered the case when $\eta = 0$ and showed that the N-player game value functions converge to the entropic mean-field game solution and verified in this case the conjecture of [109].

When $\eta \in (0, \frac{1}{2})$, it is always possible for players to jump to the other state. Therefore $\theta^{N+1}(t)$ may not always stay on one side of $\frac{1}{2}$, and when we use Itô's formula to the entropy solution U, there would be extra jump terms. Subsequently our strategy does not work when $\eta \in (0, 1/2)$, and new techniques are needed. We leave this as an open problem.

When $\bar{\theta} = 1/2$, it is expected that the N player limit will charge the two solutions we obtain with equal probability (as in [88]), which is numerically justified by the Figure 3 of [109]. Hence in that case the N-player empirical distribution will not converge to the stable fixed points of the MFG map (in the language of [109]) unlike what is claimed in the conjecture.

CHAPTER IX

Solvability of Infinite Horizon McKean-Vlasov FBSDEs in Mean Field Control Problems and Games

9.1 Introduction

Motivated by infinite horizon mean field control and mean field game, in this chapter we establish existence and uniqueness of solutions to an infinite horizon McKean-Vlasov FBSDE

$$\begin{cases} dX_t = B(t, X_t, Y_t, \mathcal{L}(X_t, Y_t))\, dt + \sigma\, dW_t, \\ dY_t = -F(t, X_t, Y_t, \mathcal{L}(X_t, Y_t))\, dt + Z_t\, dW_t, \quad \forall t \geq 0, \\ X_0 = \xi, \end{cases} \tag{9.1}$$

where (W_t) is a Brownian motion on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, $B, F : \Omega \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathcal{P}_2(\mathbb{R}^2) \to \mathbb{R}$ are two progressively measurable functions, and ξ is an $\mathcal{F}_0$-measurable square integrable random variable. Compared with finite horizon FBSDEs, in (9.1) no terminal conditions are required. Instead, for the well-posedness we specify the solution space which determines asymptotic behavior of the processes. Due to our interest in infinite horizon discounted problems, we look for solutions

(X_t, Y_t, Z_t) to (9.1) in $L^2_K(0, \infty, \mathbb{R}^3)$, where $K \in \mathbb{R}$ and $L^2_K(0, \infty, \mathbb{R}^3)$ is the Hilbert space of all $\mathbb{R}^2$-valued adapted stochastic process (v_t) such that

$$\mathbb{E}\left[\int_0^\infty e^{Kt}|v_t|^2\,dt\right] < +\infty.$$

Using methods of [163] and [161, 182], we show that there exists a unique solution $(X_t, Y_t) \in L^2_K(0, \infty, \mathbb{R}^2)$ to (9.1) under two sets of assumptions. As applications, we solve the corresponding infinite horizon McKean-Vlasov FBSDEs of mean field type control and mean field game problems.

The study of mean field games was initiated independently by Lasry, Lions (see [136], [137], [138]) and Caines, Huang, Malhamé (see [116], [117]), which is an analysis of limit models for symmetric weakly interacting $N + 1$-player differential games. Since then, mean field game has been an active research area. We refer the readers to [21, 22, 24, 66] for the study of finite state mean field games, to [31, 58, 67, 88] for uniqueness of mean field game solutions, and to [61, 62] for a nice survey. Also, inspired by the surge of interest in optimal control, several works have been published for the analysis of mean field type control, which includes the distribution of controlled states in coefficients; see e.g. [8, 39, 63].

The investigation of BSDEs was pioneered by Pardoux and Peng [158, 159] in the early 90s, which is now a standard tool in stochastic optimization problems (see e.g. [60, 164]). Applying Pontryagin's maximum principle, both mean field game and mean field type control can be studied using McKean-Vlasov FBSDEs; see e.g. [64, 63]. For analysis of FBSDE, we refer to a common reference [148].

The linear quadratic model for infinite horizon mean field game and mean field type control have been studied in [57, 116, 166] using HJB-FP equations and in [20] using martingale method respectively. [166] provided the exact stationary solution to linear quadratic infinite horizon mean field games. We also refer to [57, 77] for the PDE analysis of long time behavior of mean field game. For the best of our knowledge, this chapter is the first to investigate infinite horizon mean field game and mean field type control problems using FBSDE techniques.

The result of the chapter is organized as follows. In Section 9.2, we prove the existence and uniqueness of solutions to (9.1) under two sets of assumptions; see Theorems 9.2.4 and 9.2.8. In Section 9.3, as an application, we solve the infinite horizon mean field type control problems and games. In Section 9.4, we analyze the particular case of linear quadratic models.

In this rest of this section we will list some frequently used notation.

Notation. Denote by $\mathcal{P}_2(\mathbb{R}^n)$ the space of random variables in $\mathbb{R}^n$ with finite second moment endowed with the Wasserstein 2-metric $\mathcal{W}_2$. For any $\mathbb{R}^n$, define δ_0 to be the Dirac measure at the origin, and for any random variable X, denote by $\mathcal{L}(X)$ the law of X.

9.2 Solutions to infinite horizon McKean-Vlasov FBSDEs

In this section, we establish the existence and uniqueness of the infinite horizon McKean-Vlasov FBSDE (9.1) under two sets of assumptions. For any $(v_t) \in L^2_K(0,\infty,\mathbb{R}^n)$, we define the exponentially weighted L^2 norm

$$||v||_K^2 := \mathbb{E}\left[\int_0^\infty e^{Kt}|v_t|^2\,dt\right].$$

For simplicity, we only solve (9.1) for one dimensional (X_t, Y_t, Z_t), but our results can be easily generalized to multidimensional case.

9.2.1 Continuity method

As in [163], we study the following family of infinite horizon FBSDEs parametrized by $\lambda \in [0,1]$,

$$\begin{cases} dX_t^\lambda = (\lambda B(t, X_t^\lambda, Y_t^\lambda, \mathcal{L}(X_t^\lambda, Y_t^\lambda)) - \kappa(1-\lambda)Y_t^\lambda + \phi(t))\,dt + \sigma\,dW_t, \\ dY_t^\lambda = -(\lambda F(t, X_t^\lambda, Y_t^\lambda, \mathcal{L}(X_t^\lambda, Y_t^\lambda)) + \kappa(1-\lambda)X_t^\lambda + \psi(t))\,dt + Z_t^\lambda\,dW_t, \\ X_0^\lambda = \xi, \quad (X_t^\lambda, Y_t^\lambda, Z_t^\lambda) \in L_K^2(0,\infty,\mathbb{R}^3), \end{cases} \tag{9.2}$$

where ϕ, ψ are two arbitrary processes in $L_K^2(0,\infty,\mathbb{R})$ and κ is a positive constant to be determined below in Assumption 9.2.2. Note that when $\lambda = 1$, $\phi \equiv 0$, $\psi \equiv 0$, (9.2) becomes (9.1), and when $\lambda = 0$, (9.2) becomes

$$\begin{cases} dX_t^0 = (-\kappa Y_t^0 + \phi(t))\,dt + \sigma\,dW_t, \\ dY_t^0 = -(\kappa X_t^0 + \psi(t))\,dt + Z_t^0\,dW_t, \\ X_0^0 = \xi. \end{cases} \tag{9.3}$$

Lemma 9.2.1. *Assume that $-2\kappa < K < 0$. For any $\phi, \psi \in L_K^2(0,\infty,\mathbb{R})$, there exists a unique solution $(X^0, Y^0, Z^0) \in L_K^2(0,\infty,\mathbb{R}^3)$ to (9.3).*

Proof. The argument is almost the same as [163, Lemma 2], and we repeat it here for readers' convenience. Let us consider the following infinite horizon BSDE,

$$dP_t = -(-\kappa P_t + \phi(t) + \psi(t))\,dt + (Q_t - \sigma)\,dW_t, \quad \forall t \geq 0.$$

Applying [163, Theorem 4] with the fact that $K + 2\kappa > 0$, the above equation has a unique solution $(P, Q) \in L_K^2(0,\infty,\mathbb{R})$. Then we consider the following SDE,

$$dX_t = (-\kappa X_t - \kappa P_t + \phi(t))\,dt + \sigma\,dW_t, \quad X_0 = \xi.$$

Since $P, Q, \phi \in L_K^2(0, \infty, \mathbb{R})$, it can be easily seen that the above equation has a unique solution over arbitrary finite horizon $[0, T]$. Therefore, it remains to show that $X \in L_K^2(0, \infty, \mathbb{R})$. Applying Itô's formula to $e^{Kt}|X_t|^2$, it follows that

$$
\begin{aligned}
&\mathbb{E}[e^{KT}|X_T|^2] - \mathbb{E}[\xi^2] \\
&= \mathbb{E}\left[\int_0^T (K - 2\kappa)e^{Kt}|X_t|^2 + 2e^{Kt}X_t \cdot (-\kappa P_t + \phi(t)))\, dt\right] + \mathbb{E}\left[\int_0^T e^{Kt}\sigma^2\, dt\right].
\end{aligned}
$$

Choose a positive ϵ such that $K - 2\kappa + \epsilon < 0$. Using the inequality

$$
2e^{Kt}X_t \cdot (-\kappa P_t + \phi(t)) \le \epsilon e^{Kt}|X_t|^2 + \frac{1}{\epsilon}(-\kappa P_t + \phi(t))^2,
$$

we easily obtain that

$$
\mathbb{E}[e^{KT}|X_T|^2] - \mathbb{E}[\xi^2] \le \mathbb{E}\left[\int_0^T (K - 2\kappa + \epsilon)e^{Kt}|X_t|^2\, dt\right] + C_\epsilon,
$$

where C_ϵ is a constant that only depends on ϵ and $\|P\|_K^2, \|Q\|_K^2, \|\phi\|_K^2$. Letting $T \to \infty$ in the above inequality, we conclude that $X \in L_K^2(0, \infty, \mathbb{R})$. It can be easily verified that $(X^0, Y^0, Z^0) = (X, X + P, Q) \in L_K^2(0, \infty, \mathbb{R}^3)$ is a solution to (9.3). The uniqueness can be proved in a similar way as in Theorem 9.2.4.

□

Assumption 9.2.2. *(i) There exists a positive constant l such that for any $x, x', y, y' \in \mathbb{R}$, $m, m' \in \mathcal{P}_2(\mathbb{R}^2)$*

$$
\begin{aligned}
|B(t, x, y, m) - &B(t, x', y', m')| + |F(t, x, y, m) - F(t, x', y', m')| \\
&\le l(|x - x'| + |y - y'| + \mathcal{W}_2(m, m')) \quad a.s.
\end{aligned}
$$

(ii) There exist constants $\kappa > -K/2 > 0$ such that for any $t \ge 0$ and any square

integrable random variables X, Y, X', Y'

$$\mathbb{E}\left[K\hat{X}\hat{Y} - \hat{X}(F(t,U) - F(t,U')) + \hat{Y}(B(t,U) - B(t,U'))\right] \leq -\kappa\mathbb{E}\left[\hat{X}^2 + \hat{Y}^2\right],$$

where $\hat{X} = X - X', \hat{Y} = Y - Y'$ *and* $U = (X, Y, \mathcal{L}(X,Y)), U' = (X', Y', \mathcal{L}(X', Y'))$.

Proposition 9.2.3. *Suppose* $\lambda_0 \in [0,1)$ *and for any* $\mathcal{F}_0$*-measurable square integrable random variable* ξ*,* $\phi, \psi \in L^2_K(0,\infty,\mathbb{R})$*,* (9.2) *has a unique solution* $(X^{\lambda_0}, Y^{\lambda_0}, Z^{\lambda_0})$ *in* $L^2_K(0,\infty,\mathbb{R}^3)$*. Then under Assumption 9.2.2 there exists a* δ_0 *independent of* λ_0 *such that for any* $\delta \in [0,\delta_0]$*,* $\phi, \psi \in L^2_K(0,\infty,\mathbb{R}^2)$*,* (9.2) *also has a unique solution* $(X^{\lambda_0+\delta}, Y^{\lambda_0+\delta}, Z^{\lambda_0+\delta})$ *in* $L^2_K(0,\infty,\mathbb{R}^3)$*.*

Proof. For any pair $(x,y) \in L^2_K(0,\infty,\mathbb{R}^2)$ such that $x_0 = \xi$, according to our hypothesis, there exists a unique solution (X,Y,Z) to the following equation

$$\begin{aligned}
&dX_t = (\lambda_0 B(t,X_t,Y_t,M_t) - \kappa(1-\lambda_0)Y_t + \delta(B(t,x_t,y_t,m_t) + \kappa y_t) + \phi(t))\,dt + \sigma\,dW_t,\\
&dY_t = -(\lambda_0 F(t,X_t,Y_t,M_t) + \kappa(1-\lambda_0)X_t + \delta(F(t,x_t,y_t,m_t) - \kappa x_t) + \psi(t))\,dt + Z_t\,dW_t,\\
&X_0 = \xi,
\end{aligned}$$

where $m_t := \mathcal{L}(x_t,y_t)$ and $M_t := \mathcal{L}(X_t,Y_t)$. We define a map Φ via

$$\Phi : (x,y) \mapsto (X,Y).$$

Then a fixed point of Φ is a solution to (9.2) with parameter $\lambda_0 + \delta$. Let us prove that Φ is actually a contraction.

Take another (x',y') and its image (X',Y') under Φ. Denote $u_t = (x_t,y_t,m_t), U_t = (X_t,Y_t,M_t)$, and $\hat{x}_t = x_t - x'_t, \hat{y}_t = y_t - y'_t$ and similarly $\hat{X}_t, \hat{Y}_t$. Since $\hat{X}, \hat{Y} \in$

$L_K^2(0,\infty,\mathbb{R})$, there exists an increasing sequence of T_i such that $\lim\limits_{i\to\infty} T_i = \infty$ and

$$\lim_{i\to\infty} \mathbb{E}\left[e^{KT_i}\hat{X}_{T_i}\hat{Y}_{T_i}\right] = 0.$$

By Itô's formula, it can be easily seen that

$$\begin{aligned}
&\mathbb{E}\left[e^{KT_i}\hat{X}_{T_i}\hat{Y}_{T_i}\right] \\
&= \lambda_0\mathbb{E}\left[\int_0^{T_i} e^{Kt}\left(K\hat{X}_t\hat{Y}_t - \hat{X}_t(F(t,U_t) - F(t,U_t')) + \hat{Y}_t(B(t,U_t) - B(t,U_t'))\right)dt\right] \\
&- \kappa(1-\lambda_0)\mathbb{E}\left[\int_0^{T_i} e^{Kt}\left(\hat{X}_t^2 + \hat{Y}_t^2\right)dt\right] + (K - \lambda_0 K)\mathbb{E}\left[\int_0^{T_i} e^{Kt}\hat{X}_t\hat{Y}_t\,dt\right] \\
&+ \kappa\delta\mathbb{E}\left[\int_0^{T_i} e^{Kt}\left(\hat{X}_t\hat{x}_t + \hat{Y}_t\hat{y}_t\right)dt\right] \\
&+ \delta\mathbb{E}\left[\int_0^{T_i} e^{Kt}\left(-\hat{X}_t(F(t,u_t) - F(t,u_t')) + \hat{Y}_t(B(t,u_t) - B(t,u_t'))\right)dt\right]. \qquad (9.4)
\end{aligned}$$

According to Assumption 9.2.2 (ii), it holds that

$$\mathbb{E}\left[K\hat{X}_t\hat{Y}_t - \hat{X}_t(F(t,U_t) - F(t,U_t')) + \hat{Y}_t(B(t,U_t) - B(t,U_t'))\right] \le -\kappa\mathbb{E}\left[\hat{X}_t^2 + \hat{Y}_t^2\right]. \qquad (9.5)$$

Therefore by Assumption 9.2.2 (i) and the fact that

$$\mathcal{W}_2(\mathcal{L}(x_t,y_t),\mathcal{L}(x_t',y_t')) \le \sqrt{\mathbb{E}[|x_t - x_t'|^2]} + \sqrt{\mathbb{E}[|y_t - y_t'|^2]},$$

it can be easily deduced from (9.4)

$$\mathbb{E}\left[e^{KT_i}\hat{X}_{T_i}\hat{Y}_{T_i}\right] \leq -\left(\kappa + K/2 - \frac{k\delta + 4l\delta}{2}\right)\mathbb{E}\left[\int\limits_0^{T_i} e^{Kt}\left(\hat{X}_t^2 + \hat{Y}_t^2\right)dt\right]$$
$$+\frac{\kappa\delta + 4l\delta}{2}\mathbb{E}\left[\int\limits_0^{T_i} e^{Kt}\left(\hat{x}_t^2 + \hat{y}_t^2\right)dt\right].$$

Letting $i \to \infty$ and choosing $\delta \leq \frac{2\kappa}{3\kappa+12l)}$, we actually obtain that

$$\mathbb{E}\left[\int\limits_0^{\infty} e^{Kt}\left(\hat{X}_t^2 + \hat{Y}_t^2\right)dt\right] \leq \frac{1}{2}\mathbb{E}\left[\int\limits_0^{\infty} e^{Kt}\left(\hat{x}_t^2 + \hat{y}_t^2\right)dt\right],$$

and therefore Φ is a contraction. $\square$

Theorem 9.2.4. *Under Assumption 9.2.2, for each $\mathcal{F}_0$-measurable square integrable random variable ξ,* (9.1) *has a unique solution in* $L_K^2(0,\infty,\mathbb{R}^3)$.

Proof. By Lemma 9.2.1, for any $\phi,\psi \in L_K^2(0,\infty,\mathbb{R})$, there exists a solution in $L_K^2(0,\infty,\mathbb{R})$ to (9.2) with $\lambda = 0$. Then according to Proposition 9.2.3, for any $\phi,\psi \in L_K^2(0,\infty,\mathbb{R})$ there exists a solution to (9.2) with $\lambda = \delta_0$. Repeating this process for $\lceil \frac{1}{\delta_0} \rceil$ many times, we conclude that there exists a solution to (9.2) with $\lambda = 1$. In particular, letting $\phi \equiv 0, \psi \equiv 0$, we get a solution to (9.1).

For the uniqueness, suppose there exist two solution $(X,Y,Z),(X',Y',Z') \in L_K^2(0,\infty,\mathbb{R}^3)$ to (9.1), and denote $\hat{X} = X - X', \hat{Y} = Y - Y', \hat{Z} = Z - Z'$. There exists a sequence of $T_i \to \infty$ such that $\mathbb{E}\left[e^{KT_i}\hat{X}_{T_i}\hat{Y}_{T_i}\right] \to 0$. By Itô's formula and

Assumption 9.2.2, we have that

$$
\begin{aligned}
&\mathbb{E}\left[e^{KT_i}\hat{X}_{T_i}\hat{Y}_{T_i}\right] \\
&= \mathbb{E}\left[\int_0^{T_i} e^{Kt}\left(K\hat{X}_t\hat{Y}_t - \hat{X}_t(F(t,U_t) - F(t,U_t')) + \hat{Y}_t(B(t,U_t) - B(t,U_t'))\right) dt\right] \\
&\leq -(\kappa + K/2)\mathbb{E}\left[\int_0^{T_i} e^{Kt}\left(\hat{X}_t^2 + \hat{Y}_t^2\right) dt\right].
\end{aligned}
$$

Letting $T_i \to \infty$, we conclude that $\|\hat{X}\|_K^2 = \|\hat{Y}\|_K^2 = 0$, and hence complete the proof. □

9.2.2 Fixed point argument

We prove the existence of solution to (9.1) under another monotonicity condition, which in the spirit of [161],[182]. The main idea is as follows. Take any process $(x_t) \in L_K^2(0,\infty,\mathbb{R})$ such that $x_0 = \xi$. Using [160, Theorem 4.1], there exists a unique solution $(\overline{y}_t, \overline{z}_t)$ to the following infinite horizon BSDE

$$
d\overline{y}_t = -F(t, x_t, \overline{y}_t, \mathcal{L}(x_t, \overline{y}_t))\, dt + \overline{z}_t\, dW_t, \quad \forall t \geq 0. \tag{9.6}
$$

And then we show that there exists a unique solution to the forward McKean Vlasov SDE

$$
\begin{cases}
dX_t = B(t, X_t, \overline{y}_t, \mathcal{L}(X_t, \overline{y}_t))\, dt + \sigma\, dW_t, \\
X_0 = \xi,
\end{cases} \tag{9.7}
$$

and hence we construct a mapping which sends (x_t) to (X_t). We will prove that this mapping is a contraction, and hence its unique fixed point is the unique solution to (9.1). First we present the main assumption of this subsection.

Assumption 9.2.5. *(i) There exists some constants κ_1, κ_2 such that for any $t \in \mathbb{R}_+$, $x, x', y, y' \in \mathbb{R}$, $m \in \mathcal{P}_2(\mathbb{R}^2)$*

$$(y - y')(F(t,x,y,m) - F(t,x,y',m) \leq -\kappa_1 |y - y'|^2 \quad a.s.,$$
$$(x - x')(B(t,x,y,m) - B(t,x',y,m)) \leq -\kappa_2 |x - x'|^2 \quad a.s.$$

(ii) $F(t,x,y,m), B(t,x,y,m)$ are Lipschitz in (x,y,m). There exist some positive constant l_1, l_2 such that for any $t \in \mathbb{R}_+$, $x, x', y, y' \in \mathbb{R}$, $m, m' \in \mathcal{P}_2(\mathbb{R}^2)$

$$|F(t,x,y,m) - F(t,x',y,m')| \leq l_1(|x - x'| + \mathcal{W}_2(m,m')) \quad a.s.,$$
$$|B(t,x,y,m) - B(t,x,y',m')| \leq l_2(|y - y'| + \mathcal{W}_2(m,m')) \quad a.s.$$

(iii) There exist some positive constants ϵ_1, ϵ_2 and negative constant K such that

$$-2\kappa_1 + 2l_1 + 2l_1\epsilon_1 < K < 2\kappa_2 - 2l_2 - 2l_2\epsilon_2,$$

and also

$$4l_1 l_2 \leq \epsilon_1\epsilon_2(K + 2\kappa_1 - 2l_1 - 2l_1\epsilon_1)(-K + 2\kappa_2 - 2l_2 - 2l_2\epsilon_2).$$

(iv) $\|F(\cdot,0,0,\delta_0)\|_K^2 + \|B(\cdot,0,0,\delta_0)\|_K^2 < +\infty$.

Lemma 9.2.6. *Under Assumption 9.2.5, for any $(x_t) \in L_K^2(0,\infty,\mathbb{R})$ there exists a unique solution $(\overline{y}, \overline{z})$ to (9.6) such that $(\overline{y}, \overline{z}) \in L_K^2(0,\infty,\mathbb{R}^2)$.*

Proof. According to [160, Theorem 4.1], for any $(y_t) \in L_K^2(0,\infty,\mathbb{R})$, there exists a unique solution $(\overline{y}_t, \overline{z}_t) \in L_K^2(0,\infty,\mathbb{R}^2)$ to the infinite horizon BSDE

$$d\overline{y}_t = -F(t, x_t, \overline{y}_t, \mathcal{L}(x_t, y_t))\, dt + \overline{z}_t\, dW_t, \quad \forall t \geq 0. \tag{9.8}$$

Therefore it suffices to show that $(y_t) \mapsto (\overline{y_t})$ is a contraction on $L^2_K(0,\infty,\mathbb{R})$. Take any $(y_t), (y'_t) \in L^2_K(0,\infty,\mathbb{R})$, and denote by $(\overline{y}_t), (\overline{y}'_t)$ their corresponding solutions to (9.8).

From Itô's formula, one can easily deduce that

$$Ke^{Kt}|\overline{y}_t - \overline{y}'_t|^2\,dt + e^{Kt}|\overline{z}_t - \overline{z}'_t|^2\,dt = de^{Kt}|\overline{y}_t - \overline{y}'_t|^2 - 2e^{Kt}(\overline{y}_t - \overline{y}'_t)\,d(\overline{y}_t - \overline{y}'_t) \tag{9.9}$$

Since $\overline{y}, \overline{y}' \in L^2_K(0,\infty,\mathbb{R})$, there exists a sequence of $T_i \to \infty$ such that $\mathbb{E}\left[e^{KT_i}|\overline{y}_{T_i} - \overline{y}'_{T_i}|^2\right] \to 0$. Integrating (9.9) over interval $[0, T_i]$, taking expectation, and letting $T_i \to \infty$, we obtain that

$$\begin{aligned}
\mathbb{E}\left[\int_0^\infty Ke^{Kt}|\overline{y}_t - \overline{y}'_t|^2 + e^{Kt}|\overline{z}_t - \overline{z}'_t|^2\,dt\right] &= -\mathbb{E}\left[|\overline{y}_0 - \overline{y}'_0|^2\right] \\
&\quad + \mathbb{E}\left[\int_0^\infty 2e^{Kt}(\overline{y}_t - \overline{y}'_t)\left(F(t, x_t, \overline{y}_t, \mathcal{L}(x_t, y_t)) - F(t, x_t, \overline{y}'_t, \mathcal{L}(x_t, y'_t))\right)dt\right].
\end{aligned}$$

For the second term on the right hand side, we have that

$$\begin{aligned}
&2e^{Kt}(\overline{y}_t - \overline{y}'_t)\left(F(t, x_t, \overline{y}_t, \mathcal{L}(x_t, y_t)) - F(t, x_t, \overline{y}'_t, \mathcal{L}(x_t, y'_t))\right) \\
&\quad \le 2e^{Kt}(\overline{y}_t - \overline{y}'_t)\left(F(t, x_t, \overline{y}_t, \mathcal{L}(x_t, y_t)) - F(t, x_t, \overline{y}'_t, \mathcal{L}(x_t, y_t))\right) \\
&\qquad + 2e^{Kt}(\overline{y}_t - \overline{y}'_t)\left(F(t, x_t, \overline{y}'_t, \mathcal{L}(x_t, y_t)) - F(t, x_t, \overline{y}'_t, \mathcal{L}(x_t, y'_t))\right) \\
&\quad \le -2\kappa_1 e^{Kt}|\overline{y}_t - \overline{y}'_t|^2 + 2e^{Kt}|\overline{y}_t - \overline{y}'_t|\left(\mathcal{W}_2(\mathcal{L}(x_t, y_t), \mathcal{L}(x_t, y'_t))\right)
\end{aligned}$$

Together with $\mathcal{W}_2^2(\mathcal{L}(x_t, y_t), \mathcal{L}(x_t, y'_t)) \le l_1 \mathbb{E}[|y_t - y'_t|^2]$, it holds that

$$(K + 2\kappa_1 - l_1)\|\overline{y} - \overline{y}'\|_K^2 + \|\overline{z} - \overline{z}'\|_K^2 \le l_1\|y - y'\|_K^2.$$

Since $K + 2\kappa_1 - l_1 > l_1$, the mapping $(y_t) \mapsto (\overline{y}_t)$ is indeed a contraction. □

Proposition 9.2.7. *Under Assumption 9.2.5, for any* $(\overline{y}_t) \in L^2_K(0,\infty,\mathbb{R})$ *there exists a unique solution* X *to* (9.7)*, and furthermore* $X \in L^2_K(0,+\infty,\mathbb{R})$*.*

Proof. The existence and uniqueness of solution to (9.7) is standard (see e.g. [60]). We only need to show that the unique solution X belongs to the space $L^2_K(0,+\infty,\mathbb{R})$.

Applying Itô's formula, it can be easily seen that

$$\begin{aligned}\mathbb{E}\left[e^{Kt}|X_t|^2\right] =&\mathbb{E}[\xi^2] + 2\,\mathbb{E}\left[\int_0^t e^{Ks}X_s\cdot B(s,X_s,\overline{y}_s,\mathcal{L}(X_s,\overline{y}_s))\,ds\right]\\ &+ K\mathbb{E}\left[\int_0^s e^{Ks}|X_s|^2\,ds\right] + \mathbb{E}\left[\int_0^t e^{Ks}\sigma^2\,ds\right].\end{aligned} \tag{9.10}$$

For the integrand of the second term on the right, we have that

$$\begin{aligned}&X_s\cdot B(s,X_s,\overline{y}_s,\mathcal{L}(X_s,\overline{y}_s))\\ &= X_s\cdot\big(B(s,X_s,\overline{y}_s,\mathcal{L}(X_s,\overline{y}_s)) - B(s,0,\overline{y}_s,\mathcal{L}(X_s,\overline{y}_s))\big)\\ &\qquad + X_s\cdot B(s,0,\overline{y}_s,\mathcal{L}(X_s,\overline{y}_s))\\ &\le -\kappa_2|X_s|^2 + |X_s|\cdot\big(|B(s,0,\overline{y}_s,\delta_0\otimes\mathcal{L}(\overline{y}_s))| + l_2\mathcal{W}_2(\delta_0\otimes\mathcal{L}(\overline{y}_s),\mathcal{L}(X_s,\overline{y}_s))\big)\,.\end{aligned}$$

With the fact that $\mathcal{W}_2(\delta_0\otimes\mathcal{L}(\overline{y}_s),\mathcal{L}(X_s,\overline{y}_s)) \le \sqrt{\mathbb{E}[|X_s|^2]}$, one can easily derive that

$$\mathbb{E}\left[X_s\cdot B(s,X_s,\overline{y}_s,\mathcal{L}(X_s))\right] \le (-\kappa_2 + l_2 + \epsilon_2)|X_s|^2 + \frac{1}{4\epsilon_2}\left(|B(s,0,\overline{y}_s,\delta_0\otimes\mathcal{L}(\overline{y}_s))|^2\right).$$

Therefore from (9.10), we obtain that

$$\mathbb{E}\left[e^{Kt}|X_t|^2\right] \le (-2\kappa_2 + 2l_2 + K + 2\epsilon_2)\int_0^t e^{Ks}|X_s|^2\,ds + C_{\epsilon_2},$$

where C_{ϵ_2} is a constant depends on $K,\sigma,\mathbb{E}[\xi^2],l_2,\|B(\cdot,0,0,\delta_0)\|^2_K,\|\overline{y}\|^2_K$. Due to Assumption 9.2.5 (iii), the coefficient before the integral on the right hand side is negative, and thus we conclude the $\|X\|_K^2 < +\infty$. $\square$

Theorem 9.2.8. *There exists a unique solution (X,Y,Z) to* (9.1) *in* $L_K^2(0,\infty,\mathbb{R}^3)$.

Proof. For any $x \in L_K^2(0,\infty,\mathbb{R})$ such that $x_0 = \xi$, define $\Psi(x) := (\overline{y},\overline{z}) \in L_K^2(0,\infty,\mathbb{R}^2)$ to be the unique solution to (9.6), and for any $(\overline{y},\overline{z}) \in L_K^2(0,\infty,\mathbb{R}^2)$, define $\Phi(\overline{y},\overline{z}) := X \in L_K^2(0,\infty,\mathbb{R})$ to be the unique solution to (9.7). We prove that the composition $\Phi\circ\Psi : L_K^2(0,\infty,\mathbb{R}) \to L_K^2(0,\infty,\mathbb{R})$ is a contraction, and hence the fixed point of $\Phi\circ\Psi$ provides the unique solution to (9.1). Take $x, x' \in L_K^2(0,\infty,\mathbb{R})$ such that $x_0 = x_0' = \xi$, $(\overline{y},\overline{z}) = \Psi(x)$, $(\overline{y}',\overline{z}') = \Psi(x')$, and $X = \Phi(\overline{y},\overline{z})$, $X' = \Phi(\overline{y}',\overline{z}')$.

From Itô's formula, one can easily deduce that

$$Ke^{Kt}|\overline{y}_t - \overline{y}_t'|^2\,dt + e^{Kt}|\overline{z}_t - \overline{z}_t'|^2\,dt = de^{Kt}|\overline{y}_t - \overline{y}_t'|^2 - 2e^{Kt}(\overline{y}_t - \overline{y}_t')\,d(\overline{y}_t - \overline{y}_t'). \tag{9.11}$$

Since $\overline{y}, \overline{y}' \in L_K^2(0,\infty,\mathbb{R})$, there exists a sequence of $T_i \to \infty$ such that $\mathbb{E}\left[e^{KT_i}|\overline{y}_{T_i} - \overline{y}_{T_i}'|^2\right] \to 0$. Integrating (9.11) over interval $[0,T_i]$, taking expectation, and letting $T_i \to \infty$, we obtain that

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty Ke^{Kt}|\overline{y}_t - \overline{y}_t'|^2 + e^{Kt}|\overline{z}_t - \overline{z}_t'|^2\,dt\right] &= -\mathbb{E}\left[|\overline{y}_0 - \overline{y}_0'|^2\right] \\ &\quad + \mathbb{E}\left[\int_0^\infty 2e^{Kt}(\overline{y}_t - \overline{y}_t')\left(F(t,x_t,\overline{y}_t,\mathcal{L}(x_t,\overline{y}_t)) - F(t,x_t',\overline{y}_t',\mathcal{L}(x_t',\overline{y}_t'))\right)dt\right].\end{aligned} \tag{9.12}$$

For the second term on the right hand side, we have that

$$
\begin{aligned}
&2e^{Kt}(\overline{y}_t-\overline{y}'_t)\left(F(t,x_t,\overline{y}_t,\mathcal{L}(x_t,\overline{y}_t))-F(t,x'_t,\overline{y}'_t,\mathcal{L}(x'_t,\overline{y}'_t))\right)\\
&\le -2\kappa_1 e^{Kt}|\overline{y}_t-\overline{y}'_t|^2+2l_1e^{Kt}|\overline{y}_t-\overline{y}'_t|\left(|x_t-x'_t|+\mathcal{W}_2(\mathcal{L}(x_t,\overline{y}_t),\mathcal{L}(x'_t,\overline{y}'_t))\right)\\
&\le -2\kappa_1 e^{Kt}|\overline{y}_t-\overline{y}'_t|^2+2l_1e^{Kt}|\overline{y}_t-\overline{y}'_t|\left(|x_t-x'_t|+\sqrt{\mathbb{E}[|x_t-x'_t|^2]}+\sqrt{\mathbb{E}[|\overline{y}_t-\overline{y}'_t|^2]}\right).
\end{aligned}
$$

Therefore it holds that

$$
(K+2\kappa_1-2l_1-2l_1\epsilon_1)\|\overline{y}-\overline{y}'\|_K^2+\|\overline{z}-\overline{z}'\|_K^2\le\frac{2l_1}{\epsilon_1}\|x-x'\|_K^2. \tag{9.13}
$$

Applying Itô's formula to $de^{Kt}|X_t-X'_t|^2$, similarly we obtain that

$$
\begin{aligned}
&-\mathbb{E}\left[\int_0^\infty Ke^{Kt}|X_t-X'_t|^2\,dt\right]\\
&\quad=\mathbb{E}\left[\int_0^\infty 2e^{Kt}(X_t-X'_t)\left(B(t,X_t,\overline{y}_t,\mathcal{L}(X_t,\overline{y}_t))-B(t,X'_t,\overline{y}'_t,\mathcal{L}(X'_t,\overline{y}'_t))\right)dt\right]
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\mathbb{E}\left[2e^{Kt}(X_t-X'_t)\left(B(t,X_t,\overline{y}_t,\mathcal{L}(X_t,\overline{y}_t))-B(t,X'_t,\overline{y}'_t,\mathcal{L}(X'_t,\overline{y}'_t))\right)\right]\\
&\le\mathbb{E}\left[-2\kappa_2e^{Kt}|X_t-X'_t|^2+2l_2e^{Kt}|X_t-X'_t|\left(|\overline{y}_t-\overline{y}'_t|+\mathcal{W}_2(\mathcal{L}(X_t,\overline{y}_t),\mathcal{L}(X'_t,\overline{y}'_t))\right)\right]\\
&\le(-2\kappa_2+2l_2+2l_2\epsilon_2)\mathbb{E}\left[e^{Kt}|X_t-X'_t|^2\right]+\frac{2l_2}{\epsilon_2}\mathbb{E}\left[e^{Kt}|\overline{y}_t-\overline{y}'_t|^2\right],
\end{aligned}
$$

and therefore

$$
(-K+2\kappa_2-2l_2-2l_2\epsilon_2)\|X-X'\|_K^2\le\frac{2l_2}{\epsilon_2}\|\overline{y}-\overline{y}'\|_K^2. \tag{9.14}
$$

According to Assumption 9.2.5 (iii), (9.13), (9.14), it can be easily seen that

$$\begin{aligned}
&\|X - X'\|_K^2 \leq \frac{2l_2}{\epsilon_2(-K + 2\kappa_2 - 2l_2 - 2l_2\epsilon_2)} \|\overline{y} - \overline{y}'\|_K^2 \\
&\leq \frac{4l_1 l_2}{\epsilon_1\epsilon_2(K + 2\kappa_1 - 2l_1 - 2l_1\epsilon_1)(-K + 2\kappa_2 - 2l_2 - 2l_2\epsilon_2)} \|x - x'\|_K^2 < \|x - x'\|_K^2,
\end{aligned}$$

and therefore $\Phi \circ \Psi$ is a contraction.

□

9.3 Infinite horizon mean field game and mean field type control

In this section, we apply our main results to solve the infinite horizon mean field type control problem and the infinite horizon mean field game. First in Subsection 9.3.1, we derive the corresponding McKean-Vlasov FBSDEs (9.20) and (9.27) by Pontryagin's maximum principle, and solve the problems given solutions to (9.20) and (9.27). Then in Subsection 9.3.2, we provide sufficient conditions for the existence of solutions to (9.20) and (9.27). Let $r > 0$ be a discount factor and $A \subset \mathbb{R}$ be a convex control space. Suppose $b, f : \mathbb{R}_+ \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \times A \to \mathbb{R}$ are two measurable functions. We work under the following assumption.

Assumption 9.3.1. *(i) $b(t, x, \mu, a)$ is Lipschitz in (x, μ, a), and $f(t, x, \mu, a)$ is of at most quadratic growth in (x, μ, a). There exists a positive constant l such that for any $\mu, \mu' \in \mathcal{P}_2(\mathbb{R})$, $t \in \mathbb{R}_+$, $x \in R$, $a \in A$,*

$$|b(t, x, \mu, a) - b(t, x, \mu', a)| \leq l\mathcal{W}_2(\mu, \mu').$$

(ii) $||b(\cdot, 0, \delta_0, a)||_{-r}^2 < +\infty$, $\int_0^\infty e^{-rt}|f(\cdot, 0, \delta_0, a)|\, dt < +\infty$ for some (and thus any) $a \in A$.

(iii) There exists a constant $\kappa > l - \frac{r}{2}$ such that for any $t > 0$, $a \in A$, $\mu \in \mathcal{P}_2(\mathbb{R})$,

$x, x' \in \mathbb{R}$, it holds that

$$(x - x')(b(t, x, \mu, a) - b(t, x', \mu, a)) \leq -\kappa(x - x')^2.$$

9.3.1 Pontryagin's maximum principle

Define $\mathcal{A} := L^2_{-r}(0, \infty, A)$ to be the space of all admissible controls. For any control $\alpha \in \mathcal{A}$, let (X_t) be a strong solution to the following controlled McKean-Vlasov SDE

$$\begin{cases} dX_t = b(t, X_t, \mathcal{L}(X_t), \alpha_t)\, dt + \sigma\, dW_t, \\ X_0 = \xi. \end{cases}$$

As in the proof of Proposition 9.2.7, it can be easily shown that under Assumption 9.3.1, we have that $(X_t) \in L^2_{-r}(0, \infty, \mathbb{R})$. The cost functional takes the form

$$J(\alpha) := \mathbb{E}\left[\int_0^\infty e^{-rt} f(t, X_t, \mathcal{L}(X_t), \alpha_t)\, dt\right],$$

which is finite for any $\alpha \in \mathcal{A}$ due to Assumption 9.3.1. We want to solve the minimization problem

$$\inf_{\alpha \in \mathcal{A}} J(\alpha). \tag{9.15}$$

Let us formally derive the maximum principle of the mean field type control problem. Suppose α is an optimal control. Choose another admissible control β, define $\alpha^\epsilon := \alpha + \epsilon\beta$, and denote by X^ϵ the state trajectory corresponding to the control α^ϵ. Let

$$V_t = \lim_{\epsilon \to 0} \frac{X_t^\epsilon - X_t}{\epsilon}$$

be the variation process. Introduce the short-hand notation

$$\theta_t := (X_t, \mathcal{L}(X_t), \alpha_t), \quad \theta_t^\epsilon = (X_t^\epsilon, \mathcal{L}(X_t^\epsilon), \alpha_t^\epsilon).$$

Then it can be shown that V satisfies

$$\begin{aligned} dV_t &= \left(\partial_x b(t,\theta_t) \cdot V_t + \widetilde{\mathbb{E}}\left[\partial_\mu b(t,\theta_t)(\widetilde{X}_t) \cdot \widetilde{V}_t\right] + \partial_a b(t,\theta_t) \cdot \beta_t\right) dt, \\ V_0 &= 0, \end{aligned}$$

where $(\widetilde{X}, \widetilde{V})$ is an independent copy of (X, V) defined on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ and $\widetilde{\mathbb{E}}\left[\partial_\mu b(t,\theta_t)(\widetilde{X}_t) \cdot \widetilde{V}_t\right]$ is the derivative on the probability measure space when the state variable and the control are fixed, i.e.,

$$\widetilde{\mathbb{E}}\left[\partial_\mu b(t,x,\mathcal{L}(X_t),a)(\widetilde{X}_t) \cdot \widetilde{V}_t\right]\Big|_{x=X_t, a=\alpha_t}. \tag{9.16}$$

To make (9.16) clear, in the following remark we briefly introduce how to differentiate functions of probability measures. We refer readers to [61, Chapter 5] for a nice survey on this topic.

Remark 9.3.2. Let $\overline{\Omega}$ be a polish space and $(\overline{\mathbb{P}}, \overline{\mathcal{F}})$ be an atomless probability measure over $\overline{\Omega}$. For any function $u : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$, we define its lift to the Hilbert space $L^2(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}; \mathbb{R})$ by $\overline{u}(X) := u(\mathcal{L}(X))$. Then u is said to differentiable at $\mu_0 = \mathcal{L}(X_0)$ if $\overline{u}$ is Fréchet differentiable at X_0. By identifying $L^2(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}; \mathbb{R})$ with its dual, the Fréchet derivative of $\overline{u}$ at X_0, denoted by $D\overline{u}(X_0)$, is an element in $L^2(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}; \mathbb{R})$. It can be shown that there exists a measurable function $\partial_\mu u(\mu_0) : \mathbb{R} \to \mathbb{R}$ such that $\partial_\mu u(\mu_0)(X_0) = D\overline{u}(X_0)$ $\mathbb{P}$-a.s. Therefore we define the derivative of u at μ_0 as the measurable function $\partial_\mu u(\mu_0)$, which satisfies

$$u(\mu) = u(\mu_0) + \mathbb{E}\left[\partial_\mu u(\mu_0)(X_0) \cdot (X - X_0)\right] + o(||X - X_0||_2),$$

where $\mathcal{L}(X) = \mu, \mathcal{L}(X_0) = \mu_0$.

The function $\alpha \to J(\alpha)$ is Gâteaux differentiable in the direction β and its derivative is given by

$$\begin{aligned}
&\frac{d}{d\epsilon} J(\alpha + \epsilon\beta)\Big|_{\epsilon=0} \\
&= \mathbb{E}\left[\int_0^\infty e^{-rt}\left(\partial_x f(t,\theta_t)\cdot V_t + \widetilde{\mathbb{E}}\left[\partial_\mu f(t,\theta_t)(\widetilde{X}_t)\cdot \widetilde{V}_t\right] + \partial_a f(t,\theta_t)\cdot\beta_t\right) dt\right].
\end{aligned}$$

Define the generalized Hamiltonian

$$\mathcal{H}(t,x,\mu,a,y) := b(t,x,\mu,a)\cdot y + f(t,x,\mu,a) - rxy. \tag{9.17}$$

We consider the following infinite horizon BSDE

$$dY_t = -\left(\partial_x \mathcal{H}(t,\Theta_t) + \widetilde{\mathbb{E}}\left[\partial_\mu \mathcal{H}(t,\widetilde{\Theta}_t)(X_t)\right]\right) dt + Z_t\, dW_t, \tag{9.18}$$

where $\Theta_t := (\theta_t, Y_t) = (X_t, \mathcal{L}(X_t), \alpha_t, Y_t)$ and $(\widetilde{\Theta}, \widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ is an independent copy of $(\Theta, \Omega, \mathcal{F}, \mathbb{P})$.

According to Itô's formula, it can be easily seen that

$$\frac{d}{d\epsilon} J(\alpha+\epsilon\beta)\Big|_{\epsilon=0} = \mathbb{E}\left[\int_0^\infty e^{-rt}\partial_a \mathcal{H}(t,\Theta_t)\cdot\beta_t\, dt\right].$$

Thus when α is an optimal admissible control with the associated stochastic processes (X_t, Y_t, Z_t), it holds that

$$\mathcal{H}(t, X_t, \mathcal{L}(X_t), \alpha_t, Y_t) = \min_{a\in A} \mathcal{H}(t, X_t, \mathcal{L}(X_t), a, Y_t) \qquad \text{Leb}\otimes\mathbb{P} \quad a.e.$$

For any $x, y \in \mathbb{R}$, $m \in \mathcal{P}_2(\mathbb{R}^2)$ with first marginal $\mu \in \mathcal{P}_2(\mathbb{R})$, define

$$\hat{\alpha}_t(x, y, \mu) = \underset{a \in A}{\operatorname{argmin}}\, \mathcal{H}(t, x, \mu, a, y), \tag{9.19}$$

and

$$
\begin{aligned}
&B_c(t, x, y, m) := b(t, x, \mu, \hat{\alpha}_t(x, y, \mu)),\\
&F_c(t, x, y, m) := \partial_x \mathcal{H}(t, x, \mu, \hat{\alpha}_t(x, y, \mu), y) + \int_{x', y'} \partial_\mu \mathcal{H}(t, x', \mu, \hat{\alpha}_t(x', y', \mu), y')(x)\, dm(x', y').
\end{aligned}
$$

The above discussion connects the infinite horizon mean field control problem to the McKean-Vlasov FBSDE

$$
\begin{cases}
dX_t = B_c(t, X_t, Y_t, \mathcal{L}(X_t, Y_t))\, dt + \sigma\, dW_t,\\
dY_t = -F_c(t, X_t, Y_t, \mathcal{L}(X_t, Y_t))\, dt + Z_t\, dW_t,\\
X_0 = \xi.
\end{cases} \tag{9.20}
$$

Proposition 9.3.3. *Let (b, f) be differentiable in (x, μ, a), Assumption 9.3.1 hold and $\mathcal{H}$ be convex in (x, μ, a). Suppose $||\hat{\alpha}_\cdot(0, 0, \delta_0)||^2_{-r} < +\infty$, $\hat{\alpha}_t$ is Lipschitz and (B_c, F_c) satisfies either Assumption 9.2.2 or 9.2.5 with $K = -r$. Then we have that $J(\hat{\alpha}) = \min_\alpha J(\alpha)$.*

Proof. Due to Theorem 9.2.4, 9.2.8, there exists a uniques solution (X, Y, Z) to (9.20). Let us denote $\theta_t^\wedge := (X_t, \mathcal{L}(X_t), \hat{\alpha}_t(X_t, Y_t, \mathcal{L}(X_t)))$ and $\Theta_t^\wedge := (\theta_t^\wedge, Y_t)$. For an arbitrary admissible control α' and its associated process X', we have that

$$\begin{aligned}
J(\hat{\alpha}) - J(\alpha') =& \mathbb{E}\left[\int_0^\infty e^{-rt}\left(\mathcal{H}(t, X_t, \mathcal{L}(X_t), \hat{\alpha}_t, Y_t) - \mathcal{H}(t, X'_t, \mathcal{L}(X'_t), \alpha'_t, Y_t)\right)\, dt\right] \\
& - \mathbb{E}\left[\int_0^\infty e^{-rt}\left(b(t, X_t, \mathcal{L}(X_t), \hat{X}_t) - b(t, X'_t, \mathcal{L}(X'_t), \alpha'_t)\right)\cdot Y_t\, dt\right] \\
& + r\mathbb{E}\left[\int_0^\infty e^{-rt}(X_t - X'_t)\cdot Y_t\, dt\right]. \qquad (9.21)
\end{aligned}$$

It can be easily seen that there exists a sequence of $T_i \to \infty$ such that $\mathbb{E}\left[e^{-rT_i}(X_{T_i} - X_{T'_i})\cdot Y_{T_i}\right] \to 0$. Applying Itô's formula to $e^{-rT_i}(X_{T_i} - X_{T'_i})\cdot Y_{T_i}$ and letting $T_i \to \infty$, we obtain that

$$\begin{aligned}
& \mathbb{E}\left[\int_0^\infty e^{-rt}(X_t - X'_t)\left(\partial_x \mathcal{H}(t, \Theta_t^\wedge) + \widetilde{\mathbb{E}}\left[\partial_\mu \mathcal{H}(\widetilde{\Theta}_t^\wedge)(X_t)\right]\right)\, dt\right] \\
& = \mathbb{E}\left[\int_0^\infty e^{-rt}\left(-r(X_t - X'_t) + b(t, X_t, \mathcal{L}(X_t), \hat{X}_t) - b(t, X'_t, \mathcal{L}(X'_t), \alpha'_t)\right)\cdot Y_t\, dt\right].
\end{aligned} \qquad (9.22)$$

According to the convexity of $\mathcal{H}$ and the fact that $\hat{\alpha}_t = \operatorname{argmin}_{a\in A} \mathcal{H}(t, X_t, \mathcal{L}(X_t), a, Y_t)$, it holds that

$$\begin{aligned}
& \mathcal{H}(t, X'_t, \mathcal{L}(X'_t), \alpha'_t, Y_t) - \mathcal{H}(t, X_t, \mathcal{L}(X_t), \hat{\alpha}_t, Y_t) \\
& \geq (X'_t - X_t)\cdot \partial_x \mathcal{H}(t, \Theta_t^\wedge) + \widetilde{\mathbb{E}}\left[\partial_\mu \mathcal{H}(t, \Theta_t^\wedge)(\widetilde{X}_t)\cdot(\widetilde{X}'_t - \widetilde{X}_t)\right] + (\alpha'_t - \hat{\alpha}_t)\cdot \partial_a \mathcal{H}(t, \Theta_t^\wedge) \\
& \geq (X'_t - X_t)\cdot \partial_x \mathcal{H}(t, \Theta_t^\wedge) + \widetilde{\mathbb{E}}\left[\partial_\mu \mathcal{H}(t, \Theta_t^\wedge)(\widetilde{X}_t)\cdot(\widetilde{X}'_t - \widetilde{X}_t)\right]. \qquad (9.23)
\end{aligned}$$

By Fubini's theorem, we have that

$$\mathbb{E}\left[(X_t' - X_t) \cdot \widetilde{\mathbb{E}}\left[\partial_\mu \mathcal{H}(\widetilde{\Theta}_t^\wedge)(X_t)\right]\right] = \mathbb{E}\widetilde{\mathbb{E}}\left[\partial_\mu \mathcal{H}(t, \Theta_t^\wedge)(\widetilde{X}_t) \cdot (\widetilde{X}_t' - \widetilde{X}_t)\right].$$

In conjunction with (9.21), (9.22), (9.23), we conclude that

$$\begin{aligned}
J(\hat{\alpha}) - J(\alpha') &\leq \mathbb{E}\left[\int_0^\infty e^{-rt}\left(\mathcal{H}(t, X_t, \mathcal{L}(X_t), \hat{\alpha}_t, Y_t) - \mathcal{H}(t, X_t', \mathcal{L}(X_t'), \alpha_t', Y_t)\right) dt\right] \\
&- \mathbb{E}\left[\int_0^\infty e^{-rt}(X_t - X_t')\left(\partial_x \mathcal{H}(t, \Theta_t^\wedge) + \widetilde{\mathbb{E}}\left[\partial_\mu \mathcal{H}(\widetilde{\Theta}_t^\wedge)(X_t)\right]\right) dt\right] \\
&\leq \mathbb{E}\left[\int_0^\infty e^{-rt}\left(\mathcal{H}(t, X_t, \mathcal{L}(X_t), \hat{\alpha}_t, Y_t) - \mathcal{H}(t, X_t', \mathcal{L}(X_t'), \alpha_t', Y_t)\right) dt\right] \\
&- \mathbb{E}\left[\int_0^\infty e^{-rt}\left((X_t - X_t') \cdot \partial_x \mathcal{H}(t, \Theta_t^\wedge) + \widetilde{\mathbb{E}}\left[\partial_\mu \mathcal{H}(t, \Theta_t^\wedge)(\widetilde{X}_t) \cdot (\widetilde{X}_t - \widetilde{X}_t')\right]\right) dt\right] \leq 0.
\end{aligned}$$

□

Now we introduce an infinite horizon mean field game with discounted cost. Suppose there are N players, and each player i has state variable X_t^i at time t. Denote the empirical distribution of N players by $\overline{\mu}_t := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}$. Given admissible controls $\alpha^1, \ldots, \alpha^N \in \mathcal{A}$ and N independent Brownian motions $W^i, \ldots, W^N$, the players have dynamics

$$dX_t^i = b(t, X_t^i, \overline{\mu}_t, \alpha_t^i)\, dt + \sigma\, dW_t^i, \quad i = 1, \ldots, N. \tag{9.24}$$

The cost functional for player i is given by

$$J^i(\alpha^1, \dots, \alpha^N) := \mathbb{E}\left[\int_0^\infty e^{-rt} f(t, X_t^i, \overline{\mu}_t, \alpha_t^i)\, dt\right], \tag{9.25}$$

where $r > 0$ is the discount factor and $f : \mathbb{R}_+ \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \times A \to \mathbb{R}$ is the running cost. We want to study the Nash equilibrium as $N \to \infty$.

Suppose $\overline{\mu}_t$ converges to a measure flow μ_t in equilibrium as $N \to \infty$. Then a representative player wants to minimize

$$J^\mu(\alpha) := \mathbb{E}\left[\int_0^\infty e^{-rt} f(t, X_t, \mu_t, \alpha_t)\, dt\right],$$

under the constraint

$$dX_t = b(t, X_t, \mu_t, \alpha_t)\, dt + \sigma\, dW_t.$$

As the variational argument for the mean field type control problem, the optimal strategy of the representative should be given by $\hat{\alpha}(t, X_t, Y_t, \mu_t)$ where (X, Y, Z) is the solution to

$$\begin{cases} dX_t = b\left(t, X_t, \mu_t, \hat{\alpha}_t(X_t, Y_t, \mu_t)\right)\, dt + \sigma\, dW_t, \\ dY_t = -\partial_x \mathcal{H}\left(t, X_t, \mu_t, \hat{\alpha}_t(X_t, Y_t, \mu_t), Y_t\right)\, dt + Z_t\, dW_t, \quad \forall t \geq 0, \\ X_0 = \xi. \end{cases} \tag{9.26}$$

For any $m \in \mathcal{P}_2(\mathbb{R}^2)$ with first marginal $\mu \in \mathcal{P}_2(\mathbb{R})$, define

$$B_g(t, x, y, m) := b(t, x, \mu, \hat{\alpha}_t(x, y, \mu)),$$
$$F_g(t, x, y, m) := -\partial_x \mathcal{H}(t, x, \mu, \hat{\alpha}(x, y, \mu), y).$$

It also required that the law of X_t coincides with μ_t. Thus plugging $\mu_t = \mathcal{L}(X_t)$ in (9.26), we obtain the FBSDE of mean field game

$$\begin{cases} dX_t = B_g(t, X_t, Y_t, \mathcal{L}(X_t, Y_t))\, dt + \sigma\, dW_t, \\ dY_t = -F_g(t, X_t, Y_t, \mathcal{L}(X_t, Y_t))\, dt + Z_t\, dW_t, \quad \forall t \geq 0. \\ X_0 = \xi. \end{cases} \tag{9.27}$$

Proposition 9.3.4. *Let (b, f) be differentiable in (x, a), Assumption 9.3.1 hold and $\mathcal{H}$ be convex in (x, a). Suppose $||\hat{\alpha}.(0, 0, \delta_0)||^2_{-r} < +\infty$, $\hat{\alpha}_t$ is Lipschitz and (B_g, F_g) satisfies either Assumption 9.2.2 or 9.2.5 with $K = -r$. Then there exists a unique solution $(X, Y, Z) \in L^2_{-r}(0, \infty, \mathbb{R}^3)$ to (9.27) which provides an equilibrium to the infinite horizon mean field game, i.e.,*

$$J^{\mathcal{L}(X)}(\hat{\alpha}) \leq J^{\mathcal{L}(X)}(\alpha), \quad \forall \alpha \in \mathcal{A}.$$

Proof. Given the existence of solutions to (9.27), the proof is standard, see e.g. [61, Theorem 3.17]. □

Remark 9.3.5. In the mean field game, since there are large number of players, any change of a representative player doesn't impact the measure flow (μ_t). Therefore (μ_t) is fixed in the derivation of (9.26). That's the main difference from mean field control problem, where the law $\mathcal{L}(X_t)$ changes as the control changes. For more detailed discussions, see e.g. [63].

9.3.2 Solvability of Mean field type control and Mean field game FBSDEs

In this subsection, we find sufficient conditions on the given data for the existence and uniqueness of solutions to (9.20) and (9.27). For the mean field type control problem, we assume that $b(t, x, \mu, a) = b_0(t) + \bar{b}_1(t)\overline{\mu} + b_1(t)x + b_2(t)a$, where

$b_0(t), \overline{b}_t(t), b_1(t), b_2(t)$ are deterministic functions. For the mean field game problem, we assume that $b(t,x,\mu,a) = b_0(t,\mu) + b_1(t)x + b_2(t)a$, where by abuse of notation $b_0(t,\cdot)$ is a measurable function of $\mu \in \mathcal{P}_2(\mathbb{R})$ for any $t \in \mathbb{R}_+$. Let us compute (B_c, F_c),

$$\begin{aligned} B_c(t,x,y,m) =& b_0(t) + \overline{b}_1(t)\overline{\mu} + b_1(t)x + b_2(t)\hat{\alpha}_t(x,y,\mu), \\ F_c(t,x,y,m) =& b_1(t)y + \partial_x f(t,x,\mu,\hat{\alpha}_t(x,y,\mu)) - ry \\ &+ \overline{b}_1(t)\overline{\nu} + \int\limits_{x',y'} \partial_\mu f(t,x',\mu,\hat{\alpha}_t(x',y',\mu))(x)\, dm(x',y'), \end{aligned} \tag{9.28}$$

where μ is the first marginal of m.

Definition 9.3.6. A continuously differentiable function $\rho : \mathbb{R} \to \mathbb{R}$ is said to be η-convex for some $\eta > 0$ if

$$\rho(z') - \rho(z) - (z'-z)\cdot \partial_z \rho(z) \geq \eta (z'-z)^2, \quad \forall z, z' \in \mathbb{R}.$$

It can be easily seen that if $\partial_z \rho$ is ζ-Lipschitz, then

$$\begin{aligned} \rho(z') - \rho(z) - (z'-z)\cdot \partial_z \rho(z) &\leq \left| (z'-z)\cdot \int\limits_0^1 \partial_z \rho(t(z'-z)+z) - \partial_z\rho(z)\, dt \right| \\ &\leq \frac{\zeta}{2}(z'-z)^2. \end{aligned}$$

First, we show the Lipschitz and convex property of the minimizer $\hat{\alpha}_t$ (9.19).

Lemma 9.3.7. *Suppose $b(t,x,\mu,a) = b_0(t,\mu)+b_1(t)x+b_2(t)a$, f is once continuously differentiable in (x,a), η-convex in a, and $\partial_a f$ is l-Lipschitz in (μ,x). Then it holds that*

$$|\hat{\alpha}_t(x,y,\mu) - \hat{\alpha}_t(x',y',\mu')| \leq \frac{l}{2\eta}|x'-x| + \frac{|b_2(t)|}{2\eta}|y'-y| + \frac{l}{2\eta}\mathcal{W}_2(\mu,\mu'), \tag{9.29}$$

and for any $(t,x,y,\mu)\in\mathbb{R}_+\times\mathbb{R}^2\times\mathcal{P}_2(\mathbb{R})$,

$$|\hat{\alpha}_t(x,y,\mu)|\le\eta^{-1}(|\partial_a f(t,x,\mu,a_0)|+|b_2(t)y|)+|a_0|. \tag{9.30}$$

Furthermore, if $A=\mathbb{R}$ *and* $\partial_a f$ *is* ζ*-Lipchitz in* a*, it follows that*

$$b_2(t)(y'-y)\cdot\big(\hat{\alpha}_t(x,y',\mu)-\hat{\alpha}_t(x,y,\mu)\big)\le-\frac{2b_2(t)^2\eta}{\zeta^2}(y'-y)^2. \tag{9.31}$$

Proof. The proofs of (9.29) and (9.30) are from [61, Lemma 3.3, Lemma 6.18]. Denote $\hat{\alpha}_t=\hat{\alpha}_t(x,y,\mu)$ and $\hat{\alpha}'_t=\hat{\alpha}_t(x,y',\mu)$. In the case that $A=\mathbb{R}$, it is clear that $\partial_a\mathcal{H}(t,x,\mu,\hat{\alpha}_t,y)=\partial_a\mathcal{H}(t,x,\mu,\hat{\alpha}'_t,y')=0$, and thus

$$b_2(t)(y'-y)+\big(\partial_a f(t,x,\mu,\hat{\alpha}'_t)-\partial_a f(t,x,\mu,\hat{\alpha}_t)\big)=0. \tag{9.32}$$

Since f is η-convex in a and $\partial_a f$ in ζ-Lipschitz in a, we obtain that

$$\frac{\zeta}{2}(\hat{\alpha}'_t-\hat{\alpha}_t)^2\ge f(t,x,\mu,\hat{\alpha}'_t)-f(t,x,\mu,\hat{\alpha}_t)-(\hat{\alpha}'_t-\hat{\alpha}_t)\cdot\partial_a f(t,x,\mu,\hat{\alpha}_t)\ge\eta(\hat{\alpha}'_t-\hat{\alpha}_t)^2,$$
$$\frac{\zeta}{2}(\hat{\alpha}'_t-\hat{\alpha}_t)^2\ge f(t,x,\mu,\hat{\alpha}_t)-f(t,x,\mu,\hat{\alpha}'_t)-(\hat{\alpha}_t-\hat{\alpha}'_t)\cdot\partial_a f(t,x,\mu,\hat{\alpha}'_t)\ge\eta(\hat{\alpha}'_t-\hat{\alpha}_t)^2,$$

and therefore

$$\zeta(\hat{\alpha}'_t-\hat{\alpha}_t)^2\ge(\hat{\alpha}'_t-\hat{\alpha}_t)\cdot\big(\partial_a f(t,x,\mu,\hat{\alpha}'_t)-\partial_a f(t,x,\mu,\hat{\alpha}_t)\big)\ge 2\eta(\hat{\alpha}'_t-\hat{\alpha}_t)^2.$$

Multiplying (9.32) by $(\hat{\alpha}'_t-\hat{\alpha}_t)$ and using the above inequality, we get that

$$\zeta(\hat{\alpha}'_t-\hat{\alpha}_t)^2\ge-b_2(t)(y'-y)\cdot(\hat{\alpha}'_t-\hat{\alpha}_t)\ge 2\eta(\hat{\alpha}'_t-\hat{\alpha}_t)^2,$$

and also

$$|\hat{\alpha}_t' - \hat{\alpha}_t| \geq \frac{|b_2(t)|}{\zeta}|y' - y|.$$

Therefore we conclude that

$$b_2(t)(y' - y) \cdot (\hat{\alpha}_t' - \hat{\alpha}_t) \leq -2\eta(\hat{\alpha}_t' - \hat{\alpha}_t)^2 \leq -\frac{2b_2(t)^2\eta}{\zeta^2}(y' - y)^2.$$

$\square$

We show that the following function, as a part of F_c (9.28), is Lipschitz

$$\Psi : (t, x, m) \mapsto \overline{b}_1(t)\overline{\nu} + \Phi(t, x, m),$$

where

$$\Phi(t, x, m) = \int_{x',y'} \partial_\mu f(t, x', \mu, \hat{\alpha}_t(x', y', \mu))(x) \, dm(x', y').$$

Lemma 9.3.8. *Assume that f is once continuously differentiable in (x, μ, a), η-convex in a, $\partial_a f$ is l-Lipschitz in (x, μ), and $\partial_\mu f(t, x', \mu, a)(x)$ is l-Lipschitz in (x', μ, a, x). Then for any $x, \overline{x} \in \mathbb{R}$, $m, \overline{m} \in \mathcal{P}_2(\mathbb{R}^2)$ it holds that*

$$|\Psi(t, x, m) - \Psi(t, \overline{x}, \overline{m})| \leq \left(|\overline{b}_1(t)| + \frac{l(4\eta + 2l + |b_2(t)|)}{2\eta}\right) \mathcal{W}_2(m, \overline{m}) + l|x - \overline{x}|. \tag{9.33}$$

Proof. Together with Lemma 9.3.7, we have the Lipschitz property

$$\begin{aligned}
&\left|\partial_\mu f(t,x',\mu,\hat{\alpha}_t(x',y',\mu))(\overline{x}) - \partial_\mu f(t,\overline{x}',\mu,\hat{\alpha}_t(\overline{x}',\overline{y}',\mu))(\overline{x})\right| \\
&\leq l\left(|x'-\overline{x}'| + |\hat{\alpha}_t(x',y',\mu) - \hat{\alpha}_t(\overline{x}',\overline{y}',\mu)|\right) \\
&\leq l\left(|x'-\overline{x}'| + \frac{l}{2\eta}|x'-\overline{x}'| + \frac{|b_2(t)|}{2\eta}|y'-\overline{y}'|\right).
\end{aligned}$$

Therefore it holds that

$$\left|\int\limits_{x',y'} \partial_\mu f(t,x',\mu,\hat{\alpha}_t(x',y',\mu))(\overline{x})\, d(m-\overline{m})(x',y')\right| \leq \frac{l(2\eta+l+|b_2(t)|)}{2\eta}\mathcal{W}_2(m,\overline{m}),$$

and hence

$$\begin{aligned}
&|\Psi(t,x,m) - \Psi(t,\overline{x},\overline{m})| \\
&\leq |\bar{b}_1(t)|\mathcal{W}_1(\nu,\nu') + |\Phi(t,x,m) - \Phi(t,\overline{x},m)| + |\Phi(t,\overline{x},m) - \Phi(t,\overline{x},\overline{m})| \\
&\leq |\bar{b}_1(t)|\mathcal{W}_2(m,\overline{m}) + l|x-\overline{x}| + \left|\int\limits_{x',y'} \partial_\mu f(t,x',\mu,\hat{\alpha}_t(x',y',\mu))(\overline{x})\, d(m-\overline{m})(x',y')\right| \\
&\qquad + \int\limits_{x',y'} \left|\partial_\mu f(t,x',\mu,\hat{\alpha}_t(x',y',\mu))(\overline{x}) - \partial_\mu f(t,x',\overline{\mu},\hat{\alpha}_t(x',y',\overline{\mu}))(\overline{x})\right| d\overline{m}(x',y') \\
&\leq \left(|\bar{b}_1(t)| + \frac{l(4\eta+2l+|b_2(t)|)}{2\eta}\right)\mathcal{W}_2(m,\overline{m}) + l|x-\overline{x}|.
\end{aligned}$$

□

Remark 9.3.9. [61, Lemma 5.41] provides a sufficient condition for the Lipschitz property of

$$(x',\mu,a,x) \mapsto \partial_\mu f(t,x',\mu,a)(x).$$

Theorem 9.3.10. *Let $b(t,x,\mu,a) = b_0(t) + \overline{b}_1(t)\overline{\mu} + b_1(t)x + b_2(t)a$. The conclusion of Proposition 9.3.3 holds under either conditions $(i),(ii),(iii),(iv)$ or conditions*

$(i'), (ii'), (iii'), (iv')$ below, and thus $\hat{\alpha}$ solves the minimization problem (9.15).

(i) *$b_1(t), b_2(t)$ are uniformly bounded, and there exists a positive constant l such that $|\bar{b}_1(t)| \le l$ and $-\max_t b_1(t) \ge l - \frac{r}{2}$. f is once continuously differentiable in (x, μ, a), of at most quadratic growth in (x, μ, a), and it holds that $b_0(\cdot), |f(\cdot, 0, \delta_0, a)|^{1/2} \in L^2_{-r}(0, \infty, \mathbb{R})$ for some (any thus any) $a \in A$.*

(ii) *There exist some positive constants η, ι such that the following convexity condition holds*

$$\begin{aligned}&f(t, x', \mu', a') - f(t, x, \mu, a) - \partial_{(x,a)} f(t, x, \mu, a) \cdot (x' - x, a' - a)\\&- \mathbb{E}\left[\partial_\mu f(t, x, \mu, a)(X) \cdot (X' - X)\right] \ge \iota(x' - x)^2 + \eta(a' - a)^2,\end{aligned}$$

for any $t \in \mathbb{R}_+$ whenever X', X have distributions μ', μ respectively.

(iii) *$\partial_x f$ and $\partial_a f$ are l-Lipschitz in (μ, a) and (x, μ) respectively. $\partial_a f$ is ζ-Lipschitz in a, and $\partial_\mu f(t, x', \mu, a)(x)$ is l-Lipschitz in (x', μ, a, x).*

(iv) *$A = \mathbb{R}$, and it holds that*

$$\inf_t \min \left\{ 2\iota - \frac{13l}{2} - \frac{5l^2 + 3|b_2(t)|l}{2\eta}, \frac{2b_2(t)^2\eta}{\zeta^2} - \frac{3l}{2} - \frac{l^2 + 2|b_2(t)|l}{2\eta} \right\} > \frac{r}{2}. \tag{9.34}$$

(i') *$b_1(t), b_2(t)$ are uniformly bounded, and there exists a positive constant l such that $|\bar{b}_1(t)| \le l$. f is once continuously differentiable in (x, μ, a), of at most quadratic growth in (x, μ, a), and it holds that $b(\cdot), |f(\cdot, 0, \delta_0, a)|^{1/2} \in L^2_{-r}(0, \infty, \mathbb{R})$ for some (any thus any) $a \in A$.*

(ii') *There exists a positive constant η such that the following convexity condition*

holds

$$f(t,x',\mu',a') - f(t,x,\mu,a) - \partial_{(x,a)} f(t,x,\mu,a)\cdot(x'-x,a'-a)$$
$$- \mathbb{E}\left[\partial_\mu f(t,x,\mu,a)(X)\cdot(X'-X)\right] \geq \eta(a'-a)^2,$$

for any $t \in \mathbb{R}_+$ whenever X', X have distributions μ', μ respectively.

(iii') $\partial_x f$ and $\partial_a f$ are l-Lipschitz in (x,μ,a) and (x,μ) respectively. $\partial_\mu f(t,x',\mu,a)(x)$ is l-Lipschitz in (x',μ,a,x).

(iv') It holds that

$$\max_t b_1(t) \leq -\max\left\{9l - \frac{r}{2} + \max_t \frac{9l^2 + 4l|b_2(t)|}{2\eta},\right.$$
$$\left. 3l - \frac{r}{2} + \max_t \frac{4|b_2(t)|l + 3b_2(t)^2}{2\eta}\right\}. \tag{9.35}$$

Proof. Assume that conditions $(i), (ii), (iii), (iv)$ hold. It is clear that Assumption 9.3.1 is satisfied, and due to Lemma 9.3.7 $\hat{\alpha}_t$ is Lipschitz and $\hat{\alpha}_\cdot(0,0,\delta_0) \in L^2_{-r}(0,\infty,\mathbb{R})$. According to condition (ii), it can be easily seen that $\mathcal{H}$ is convex in (x,μ,a). By Lemma 9.3.8 and explicit formulas of (B_c, F_c) (9.28), Assumption 9.2.2 (i) can be easily verified. It remains to to check Assumption 9.2.2 (ii) with $K = -r$.

Take any square integrable random variables X, Y, X', Y', and denote $\mu = \mathcal{L}(X)$, $\mu' = \mathcal{L}(X'), m = \mathcal{L}(X,Y), m' = \mathcal{L}(X',Y')$. Define $\hat{X} = X - X', \hat{Y} = Y - Y'$ and $U = (X,Y,\mathcal{L}(X,Y)), U' = (X',Y',\mathcal{L}(X',Y'))$. Let us compute

$$\begin{aligned}
&- r\hat{X}\hat{Y} - \hat{X}(F_c(t,U) - F_c(t,U')) + \hat{Y}(B_c(t,U) - B_c(t,U')) \\
&= -\hat{X}\left(\partial_x f(t,X,\mu,\hat{\alpha}_t(X,Y,\mu)) - \partial_x f(t,X',\mu',\hat{\alpha}_t(X',Y',\mu')) + \Psi(X,m) - \Psi(X',m')\right) \\
&\quad + \hat{Y}\left(\bar{b}_1(t)\mathbb{E}[\hat{X}] + b_2(t)(\hat{\alpha}_t(X,Y,\mu) - \hat{\alpha}_t(X',Y',\mu'))\right).
\end{aligned} \tag{9.36}$$

Since f is ι-convex in x, we have that

$$
\begin{aligned}
&-\hat{X}\left(\partial_x f(t,X,\mu,\hat{\alpha}_t(X,Y,\mu)) - \partial_x f(t,X',\mu',\hat{\alpha}_t(X',Y',\mu'))\right) \\
&= -\hat{X}\left(\partial_x f(t,X,\mu,\hat{\alpha}_t(X,Y,\mu)) - \partial_x f(t,X',\mu,\hat{\alpha}_t(X,Y,\mu))\right) \\
&\quad - \hat{X}\left(\partial_x f(t,X',\mu,\hat{\alpha}_t(X,Y,\mu)) - \partial_x f(t,X',\mu',\hat{\alpha}_t(X',Y',\mu'))\right) \\
&\le -2\iota\hat{X}^2 + l|\hat{X}|\left(\mathcal{W}_2(\mu,\mu') + \frac{l}{2\eta}|\hat{X}| + \frac{|b_2(t)|}{2\eta}|\hat{Y}| + \frac{l}{2\eta}\mathcal{W}_2(\mu,\mu')\right).
\end{aligned} \tag{9.37}
$$

According to (9.31), it follows that

$$
\begin{aligned}
&\hat{Y}b_2(t)\left(\hat{\alpha}_t(X,Y,\mu) - \hat{\alpha}_t(X',Y',\mu')\right) \\
&\quad= \hat{Y}b_2(t)\left(\hat{\alpha}_t(X,Y,\mu) - \hat{\alpha}_t(X,Y',\mu)\right) + \hat{Y}b_2(t)\left(\hat{\alpha}_t(X,Y',\mu) - \hat{\alpha}_t(X',Y',\mu')\right) \\
&\quad\le -\frac{2b_2(t)^2\eta}{\zeta^2}\hat{Y}^2 + |\hat{Y}b_2(t)|\left(\frac{l}{2\eta}|\hat{X}| + \frac{l}{2\eta}\mathcal{W}_2(\mu,\mu')\right).
\end{aligned} \tag{9.38}
$$

Using Lemma 9.3.8, equations (9.36),(9.37),(9.38), condition (iv) and basic inequalities, Assumption 9.2.2 (ii) can be verified.

Assume that conditions (i'), (ii'), (iii'), (iv') hold. We only check Assumption 9.2.5, and the rest is very similar to the first part of proof. Recalling the formula (9.28), it can be easily verified that

$$
\begin{aligned}
&(y-y')\left(F_c(t,x,y,m) - F_c(t,x,y',m)\right) \le \left(b_1(t) - r + \frac{|b_2(t)|l}{2\eta}\right)(y-y')^2, \\
&(x-x')\left(B_c(t,x,y,m) - B_c(t,x',y,m)\right) \le \left(b_1(t) + \frac{|b_2(t)|l}{2\eta}\right)(x-x')^2, \\
&|F_c(t,x,y,m) - F_c(t,x',y,m')| \\
&\le \left(3l + \frac{3l^2 + |b_2(t)|l}{2\eta}\right)\mathcal{W}_2(m,m') + \left(2l + \frac{l^2}{2\eta}\right)|x-x'|, \\
&|B_c(t,x,y,m) - B_c(t,x,y',m')| \\
&\le \left(l + \frac{|b_2(t)|l}{2\eta}\right)\mathcal{W}_2(m,m') + \frac{b_2(t)^2}{2\eta}|y-y'|.
\end{aligned}
$$

Therefore we define

$$
\begin{aligned}
\kappa_1 &= -\max_t \left(b_1(t) - r + \frac{|b_2(t)| l}{2\eta} \right), \\
\kappa_2 &= -\max_t \left(b_1(t) + \frac{|b_2(t)| l}{2\eta} \right), \\
l_1 &= \max_t \left(3l + \frac{3l^2 + |b_2(t)| l}{2\eta} \right), \\
l_2 &= \max_t \left(l + \frac{|b_2(t)| l}{2\eta} + \frac{b_2(t)^2}{2\eta} \right).
\end{aligned}
$$

Due to condition (iv'), it can be easily verified that

$$-2\kappa_1 + 6l_1 < -r < 2\kappa_2 - 6l_2,$$

and hence Assumption 9.2.5 (iii) is satisfied. □

Now we provide sufficient conditions to solve (9.27). Assume that $b(t,x,\mu,a) = b_0(t,\mu) + b_1(t)x + b_2(t)a$. Then it is clear that

$$
\begin{aligned}
B_g(t,x,y,\mu) =& b_0(t,\mu) + b_1(t)x + b_2(t)\hat{\alpha}_t(x,y,\mu), \\
F_g(t,x,y,\mu) =& b_1(t)y + \partial_x f(t,x,\mu,\hat{\alpha}_t(x,y,\mu)) - ry.
\end{aligned}
$$

Theorem 9.3.11. *Let $b(t,x,\mu,a) = b_0(t,\mu) + b_1(t)x + b_2(t)a$. The conclusion of Proposition 9.3.4 holds under either conditions $(i),(ii),(iii),(iv)$ or conditions $(i'),(ii'),(iii'),(iv')$ below, and thus $(\mathcal{L}(X_t),\hat{\alpha}_t)$ solves the infinite horizon mean field game.*

(i) $b_1(t), b_2(t)$ are uniformly bounded, and $b_0(t,\mu)$ is l-Lipschitz in μ, such that $-\max_t b_1(t) \geq l - \frac{r}{2}$. f is once continuously differentiable in (x,a), of at most quadratic growth in (x,μ,a), and it holds that $b(\cdot,\delta_0), |f(\cdot,0,\delta_0,a)|^{1/2} \in$

$L^2_{-r}(0,\infty,\mathbb{R})$ *for some (any thus any)* $a \in A$.

(ii) f *is* ι*-convex in* x *and* η*-convex in* a.

(iii) $\partial_x f$ *and* $\partial_a f$ *are* l*-Lipschitz in* (μ, a) *and* (x, μ) *respectively.* $\partial_a f$ *is* ζ*-Lipschitz in* a.

(iv) $A = \mathbb{R}$ *and it holds that*

$$\inf_t \min\left\{2\iota - \frac{3l}{2} - \frac{l^2}{\eta} - \frac{3|b_2(t)|l}{4\eta}, \frac{2b_2(t)^2\eta}{\zeta^2} - \frac{l}{2} - \frac{3|b_2(t)|l}{4\eta}\right\} \geq \frac{r}{2}. \tag{9.39}$$

(i') $b_1(t), b_2(t)$ *are uniformly bounded, and* $b_0(t,\mu)$ *is* l*-Lipschitz in* μ. f *is once continuously differentiable in* (x, a)*, of at most quadratic growth in* (x, μ, a)*, and it holds that* $b(\cdot, \delta_0)$*,* $|f(\cdot, 0, \delta_0, a)|^{1/2} \in L^2_{-r}(0,\infty,\mathbb{R})$ *for some (any thus any)* $a \in A$.

(ii') f *is* η*-convex in* a*, convex* x.

(iii') $\partial_x f$ *is* l*-Lipschitz in* (x, μ, a)*, and* $\partial_a f$ *is* l*-Lipschitz in* (x, μ).

(iv') *It holds that*

$$\begin{aligned}\max_t b_1(t) \leq -\max\Bigg\{&3l - \frac{r}{2} + \max_t \frac{3l^2 + |b_2(t)|l}{2\eta}, \\ &3l - \frac{r}{2} + \max_t \frac{4|b_2(t)|l + 3b_2(t)^2}{2\eta}\Bigg\}.\end{aligned} \tag{9.40}$$

Proof. The proof is almost the same as that of Theorem 9.3.10. □

Remark 9.3.12. Using PDE tools, [57, 77] studied the long time behavior of mean fields games in the special case when $b(t, x, \mu, a) = a$, $f(t, x, \mu, a) = L(x, a) + F(x, \mu)$. Their main assumption, the convexity of $y \mapsto -\inf_a\{ay + L(x, a)\}$, is similar to (9.31) which is needed in Assumption 9.2.2 (ii). However in the case that $A \neq \mathbb{R}$, (9.31) may no longer hold. This is a case when Assumption 9.2.5 can prove to be less demanding since (9.31) is not needed.

[57, 77] proved that the vanishing discount limit for the infinite horizon problem is the solution to an ergodic mean field games [77, Theorem 6.4], and that the solution to the discounted mean field game converges to the unique stationary solution exponentially fast [57, Theorem 3.7]. It remains open to show the above convergence results for general models using FBSDE techniques, and we leave it for future research.

9.4 Linear quadratic models

In this section, we apply Theorem 9.3.10, 9.3.11 to linear quadratic models. For any $\mu \in \mathcal{P}_2(\mathbb{R})$, define $\overline{\mu} := \int x\, \mu(dx)$ as the mean of distribution μ. Let us suppose $A = \mathbb{R}$, and

$$\begin{aligned} b(t,x,\mu,a) :=& b_1(t)x + \bar{b}_1(t)\overline{\mu} + b_2(t)a, \\ f(t,x,\mu,a) :=& \frac{1}{2}\left(x^2 q(t) + (x-\overline{\mu})^2 \overline{q}(t) + a^2 p(t)\right), \end{aligned}$$

where $b_1(t), \bar{b}_1(t), b_2(t), q(t), \overline{q}(t), p(t)$ are deterministic functions.

In this simple case, we can explicitly compute (9.19)

$$\hat{\alpha}_t(x,y,\mu) = -\frac{b_2(t)}{p(t)} y.$$

Plugging in (9.20) and (9.27), we obtain that

$$\begin{aligned} B_c(t,x,y,m) &= B_g(t,x,y,m) = b_1(t)x - \frac{b_2(t)^2}{p(t)} y + \bar{b}_1(t)\overline{\mu}, \\ F_c(t,x,y,m) &= b_1(t)y + (q(t)+\overline{q}(t))x - \overline{q}(t)\overline{\mu} - ry + \bar{b}_1(t)\overline{\nu}, \\ F_g(t,x,y,m) &= b_1(t)y + (q(t)+\overline{q}(t))x - \overline{q}(t)\overline{\mu} - ry, \end{aligned}$$

where μ and ν are the first and second marginals of m respectively.

Applying Theorem 9.3.10, 9.3.11, we can easily obtain the following two corollaries.

Corollary 9.4.1. *Suppose $b_1(t), b_2(t), q(t), \overline{q}(t), p(t)$ are bounded. Let l, ι, η, ξ be some positive constants. Then $\hat{\alpha}_t$ solves the mean field type control problem under either of the following:*

(i) $|\overline{b}_1(t)| \leq l$, $-b_1(t) \geq l - \frac{r}{2}$, $\xi \geq p(t) \geq 2\eta$, $q(t) \geq 2\iota$, $\overline{q}(t) \geq 0$, $|\overline{q}(t)| \leq l$ *for all* t, *and* (9.34) *holds.*

(ii) $|\overline{b}_1(t)| \leq l$, $p(t) \geq 2\eta$, $q(t) \geq 0$, $\overline{q}(t) \geq 0$, $|q(t)| + |\overline{q}(t)| \leq l$ *for all* t, *and* (9.35) *holds.*

Corollary 9.4.2. *Suppose $b_1(t), b_2(t), q(t), \overline{q}(t), p(t)$ are bounded. Let l, ι, η, ξ be some positive constants. $\hat{\alpha}_t$ solves the mean field game under either of the following two conditions:*

(i) $|\overline{b}_1(t)| \leq l$, $-b_1(t) \geq l - \frac{r}{2}$, $\xi \geq p(t) \geq 2\eta$, $q(t) \geq 2\iota$, $|\overline{q}(t)| \leq l$ *for all* t, *and* (9.39) *holds.*

(ii) $|\overline{b}_1(t)| \leq l$, $p(t) \geq 2\eta$, $q(t) + \overline{q}(t) \geq 0$, $|q(t)| + |\overline{q}(t)| \leq l$ *for all* t, *and* (9.40) *holds.*

Remark 9.4.3. It is known that one can solve linear quadratic mean field games by Riccati equations, and thus the solution Y_t is a linear transformation of X_t. As in [61, Section 3.5], one may assume that $Y_t = \eta(t)X_t + \chi(t)$, $Z_t = \eta(t)\sigma$, and it can be shown that $(\eta(t), \chi(t))$ solves

$$\begin{cases} 0 = \dot{\eta}(t) - \eta(t)^2 \frac{b_2(t)^2}{p(t)} + \eta(t)\left(2b_1(t) - r\right) + q(t) + \overline{q}(t), \\ 0 = \dot{\chi}(t) + \chi(t)\left(-\eta(t)\frac{b_2(t)^2}{p(t)} + b_1(t) - r\right) - \overline{q}(t)\overline{x}(t) + \eta(t)\overline{b}_1(t)\overline{x}(t), \quad \forall t \geq 0, \end{cases} \tag{9.41}$$

where $\overline{x}(t) := \mathbb{E}[X_t]$ together with $\overline{\eta}(t)$ is the solution to

$$\begin{cases} 0 = \dot{\overline{\eta}}(t) + \overline{\eta}(t)\left(2b_1(t) + \overline{b}_1(t) - r\right) - \overline{\eta}(t)^2 \frac{b_2(t)^2}{p(t)} + q(t), \\ \dot{\overline{x}}(t) = \left(b_1(t) + \overline{b}_1(t) - \overline{\eta}(t)\frac{b_2(t)^2}{p(t)}\right)\overline{x}(t), \quad \forall t \geq 0, \\ \overline{x}(0) = \mathbb{E}[\xi]. \end{cases} \tag{9.42}$$

Both (9.41) and (9.42) are systems of infinite horizon ordinary differential equations, and we impose the growth condition $\int_0^\infty e^{-rt}\left(\overline{x}(t)^2 + \chi(t)^2\right)dt + \sup_t |\eta(t)| < \infty$, and that $\eta(t) \geq \frac{p(t)}{b_2(t)^2}(b_1(t) - r/2)$.

When there exists a solution $(\eta(t), \chi(t), \overline{x}(t))$ to (9.41)(9.42), it can be easily verified that $Y_t = \eta_t X_t + \chi_t$, $\mathbb{E}[X_t] = \overline{x}(t)$ solves (9.27) and that $(X_t, Y_t) \in L^2_{-r}(0, \infty, \mathbb{R}^2)$. Therefore by the uniqueness result of MFG FBSDE (9.27), the solution to (9.41)(9.42) is also unique. The solvability of (9.41) and (9.42) is strongly connected with an equivalent deterministic linear quadratic optimal control problem, which is beyond the scope of this chapter and we refer to [61, Section 3.5.1]. Similarly, one can also write down ordinary equations for solutions to infinite horizon linear quadratic mean field control problems.

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