# The $\mathcal{N} = 4$ SU($N$) Super-Yang-Mills Index and Dual AdS Black Holes

by

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# Abstract

The main subjects of this dissertation are indices of the $\mathcal{N} = 4$ SU($N$) Super-Yang-Mills (SYM) theory, namely the topologically twisted index and the superconformal index. These indices have received a lot of attention since they provide microscopic understanding of $\mathrm{AdS}_5$ black strings and black holes respectively through the AdS/CFT correspondence. In this dissertation, we focus on the field theory side and investigate these indices with a goal of improving the current microscopic understanding of $\mathrm{AdS}_5$ black strings and black holes. As a result, we unveil interesting physics of the 4d indices such as modular properties and a relation to the $S^3$ partition function of effective Chern-Simons theory, and also make suggestions in the gravity side based on the structure of indices through the AdS/CFT correspondence.

First, we study the topologically twisted index of the $\mathcal{N} = 4$ SU($N$) SYM theory on $T^2 \times S^2$. We introduce the Bethe Ansatz (BA) formula that gives the twisted index as a sum over solutions to the Bethe Ansatz Equations (BAE) and categorize various solutions into two groups: standard ones that compose the SL(2,$\mathbb{Z}$) family and non-standard ones that denote all the other BAE solutions including continuous families. Focusing on the contribution from standard BAE solutions, we confirm that it behaves as an elliptic genus with certain modular properties and further investigate its asymptotic behaviors in the Cardy-like limit. Lastly, we review how the twisted index counts the microstates associated with the dual $\mathrm{AdS}_5$ black string entropy in the Cardy-like limit, and discuss missing steps that should be taken care of to validate the microstate counting.

Next, we study the superconformal index of the $\mathcal{N} = 4$ SU($N$) SYM theory. We compute the superconformal index first by saddle point evaluation and then by the BA formula. In due process, we establish a direct relation between the 4d superconformal index and the $S^3$ partition function of Chern-Simons theory. Then we investigate the phase structure of the superconformal index in the large-$N$ after the Cardy-like limit, which contains partially-deconfined phases distinguished from the previously well-known fully-deconfined/confined

phases. Finally, we discuss implications of a partially-deconfined phase, based on the counting of microstates associated with the dual $AdS_5$ black hole entropy by the superconformal index.

# Chapter 1

# Introduction

## 1.1 Microscopic understanding of black hole thermodynamics

During the last few decades, quantum field theory (QFT) has been one of the most successful frameworks through which we understand our nature. It is a theoretical foundation of the Standard Model that decribes various quarks, leptons and their interactions consistently. Beyond theoretical consistency, its validity has also been strongly supported by numerous experiments.

QFT does not fully account for our nature, however, particularly when the longest known fundamental interaction of particles – gravity – comes into the picture. We may try to quantize gravity following a systematic formulation of perturbative quantum field theories that have successfully described other types of fundamental interactions such as strong, weak, and electromagnetic forces. Non-renormalizability of gravity, however, does not allow for a verifiable quantum field theory of gravity.

In typical laboratory conditions, we can circumvent the issue of lack of a consistent quantum theory of gravity by treating gravity classically. This is allowed since gravity is fairly suppressed compared to the other fundamental interactions. The aforementioned success of QFT and the Standard Model is also valid in this weak gravity regime. This limitation of QFT still implies, however, that QFT is not enough to describe our nature completely.

String theory has received a lot of attention since it provides a possible explanation of the quantum nature of gravity. Furthermore, due to its theoretical sophistication, string theory has been considered as one of the leading candidates for a consistent quantum theory of gravity so far. It is still unclear, however, if the theoretical validity of string theory as a quantum theory of gravity can be supported by experiments.

This situation has motivated research on black holes. First of all, a black hole is a perfect background to test if a given candidate is a valid quantum theory of gravity since conventional QFT does not work around black holes due to strong gravity; we need a quantum description of gravity to address quantum properties of black holes. If string theory successfully describes such properties, it could provide solid evidence for accepting string theory as a quantum theory of gravity. Furthermore, black holes are not just made out of our theoretical imagination, but are actual celestial objects that have been observed in our real world. This opens up a possibility that quantum theoretical explanation for black hole physics based on string theory could be tested by experiments in the future. Hence both theoretical and experimental understanding of black holes hold the key towards establishing a consistent quantum theory of gravity.

In this context, black hole thermodynamics has received particular interest among other properties of black holes. The microscopic origin of thermodynamic quantities of black holes, in particular, requires a quantum theoretical explanation and therefore becomes an important subject of a quantum theory of gravity. To be specific, since Bekenstein and Hawking introduced the macroscopic entropy of a black hole [11, 64] as

$$S_{\text{BH}} = \frac{A}{4G} \tag{1.1}$$

in terms of the horizon area $A$ and the Newton's constant $G$, the quantum origin of microstates associated with the black hole entropy (1.1) has been considered one of the most important questions in high energy theory.

For a certain class of supersymmetric (or BPS from Bogomolny-Prasad-Sommerfield) black holes in asymptotically flat backgrounds, Strominger and Vafa successfully resolved this problem by reproducing the black hole entropy as the logarithm of the number of BPS states in string theory [107]. To be specific, they have focused on asymptotically flat $\frac{1}{4}$-BPS black holes in 5d $\mathcal{N} = 4$ supergravity which arising in the low energy regime of type IIB string theory compactified on $K3 \times S^1$. The number of corresponding BPS states was then counted by investigating the relevant D-brane world volume theory in the small $K3$ limit. The logarithm of this number of BPS states was then matched with the black hole entropy. Applying the same technique to slightly different near-BPS black holes was also successful [33].

Even though the first successes of black hole microstate counting were made in asymptotically flat backgrounds, more systematic understanding of the microscopic origin of black hole entropy has been initiated in asymptotically Anti-de Sitter (AdS) backgrounds through the AdS/CFT correspondence. The AdS/CFT correspondence, which was first suggested by Maldacena in [86], tells us that a gravity theory in AdS is dual to a non-gravitational conformal field theory (CFT) on the conformal boundary of AdS. According to this duality, an asymptotically AdS solution of a gravity theory corresponds to an ensemble of quantum states of the dual CFT. Applying this relation to asymptotically AdS black hole solutions (AdS black holes), we can provide a quantum theoretical explanation of microstates associated with the AdS black hole entropy. Based on this idea, Strominger successfully accounted for the entropy of $\mathrm{AdS}_3$ black holes [106].

However, generalization to higher dimensional AdS black holes seemed non-trivial. In [106], the authors computed the entropy of a 3-dimensional Banados-Teitelboim-Zanelli (BTZ) black hole [9] with locally $\mathrm{AdS}_3$ near horizon geometry in terms of the central charge of a dual $\mathrm{CFT}_2$ using the Brown-Henneaux central charge relation [26]. The resulting entropy was then matched with the logarithm of the number of dual quantum states computed by the Cardy formula [34]. Compared to this $\mathrm{AdS}_3$ black hole case equipped with the Brown-Henneaux central charge relation and the Cardy formula, there is no apparent tool that we can use to count the number of quantum states dual to $\mathrm{AdS}_{d+1}$ black holes with $d \geq 3$. Hence AdS black hole microstate counting in higher dimensions had remained as an open question for a while.

While microscopic understanding of $\mathrm{AdS}_{d+1}$ ($d \geq 3$) black holes has remained unresolved, there have been important developments in supersymmetry. First, the idea of supersymmetric localization introduced in [97] has allowed for exact calculation of important physical quantities of various supersymmetric field theories such as partition functions and Witten indices. Based on this achievement, it is now possible to compute various exact quantities of superconformal field theories (SCFT) even in the strong coupling limit and to compare the results with their holographic duals in the supergravity limit through the AdS/CFT correspondence. Meanwhile, rigid supersymmetry has allowed for systematic construction of supersymmetric field theories on curved manifolds [53]. It has been done by fixing the metric to a curved background in the corresponding supergravity theory with appropriate topological twists to keep supersymmetry. Combining these two achievements, numerous exact results for supersymmetric field theories on various backgrounds have been obtained in the literature.

Based on these recent developments in supersymmetry, the microscopic origin of higher dimensional AdS black holes has been recently addressed for a supersymmetric magnetic $\mathrm{AdS}_4$ black hole in 4d $\mathcal{N} = 2$ gauged supergravity coupled to vector multiplets [16]. First, the authors computed the partition function of 3d Aharony-Bergman-Jafferis-Maldacena (ABJM) theory [2] on $S^2 \times S^1$ with topological twist over $S^2$ as a function of magnetic charges and chemical potentials associated with flavor symmetries. It is called the topologically twisted index [21][1]. The logarithm of the twisted index was then successfully matched with the $AdS_4$ black hole entropy upon extremization with respect to chemical potentials, which was dubbed as $I$-extremization [16]. Since then, numerous following works have been done in similar directions and have improved our microscopic understanding of $AdS_{d+1}$ ($d \geq 3$) black holes. For other examples in $AdS_4/CFT_3$, refer to the following references: generalization to dyonic cases [17]; indices on general 3-dimensional manifolds [29, 22, 39, 41, 40, 42]; $AdS_4$ black holes and their near-horizon geometries uplifted to 11-dimensional supergravity along with 7-dimensional Sasaki-Einstein internal manifolds and their dual indices [75, 72, 44, 56, 57, 76, 68]; $AdS_4$ black holes uplifted to Type IIA supergravity and their dual indices [18, 70]; a universal behavior of various examples [8]. Also refer to [74, 54, 108, 109, 46, 45] for examples in $AdS_6/CFT_5$.

In this dissertation, we are particularly interested in microscopic understanding of supersymmetric, magnetically charged $AdS_5$ black strings and supersymmetric, rotating, electrically charged $AdS_5$ black holes in 5d $\mathcal{N} = 2$ gauged supergravity. Since these $AdS_5$ black strings and black holes are dual to the corresponding ensembles of BPS states of the $\mathcal{N} = 4$ SU($N$) Super-Yang-Mills (SYM) theory, the main subjects would be appropriate indices that count such BPS states. They are *the topologically twisted index [43, 21, 66, 40] and the superconformal index [83, 101, 40] of the $\mathcal{N} = 4$ SU($N$) SYM theory* associated with $AdS_5$ black strings and black holes respectively. Compared to the topologically twisted index of ABJM theory on $S^2 \times S^1$, these 4d indices have richer structures such as modular properties and interesting physics related to lower-dimensional effective theories. On the other hand, compared to 5d indices of various SCFTs, we have more control of 4d indices and therefore may investigate their physics more precisely. These features partially explain our particular interest in the microstate counting in $AdS_5/CFT_4$ among other examples in different dimensions.

Hence, in this dissertation, we explore the aforementioned indices of the $\mathcal{N} = 4$ SU($N$) SYM theory in the context of microstate counting of dual $AdS_5$ black string/hole entropy. This work is based on

- Ref. [67]: Junho Hong, James T. Liu, *The topologically twisted index of $\mathcal{N} = 4$ super Yang-Mills on $T^2 \times S^2$ and the elliptic genus*, **JHEP 07 (2018) 018**, [arXiv:1804.04592],
- Ref. [5]: Arash Arabi Ardehali, Junho Hong, James T. Liu, *Asymptotic growth of the 4d $\mathcal{N} = 4$ index and partially deconfined phases*, **JHEP 07 (2020) 073**,

[1] The topologically twisted index of $\mathcal{N} = 2$ supersymmetric gauge theories was in fact studied in [96] first based on the Bethe/gauge correspondence [93, 94, 95]. The authors of [21] used supersymmetric localization instead.

[arXiv:1912.04169],

- Ref. [58]: Alfredo González Lezcano, Junho Hong, James T. Liu, Leopoldo A. Pando Zayas, *Sub-leading Structures in Superconformal Indices: Subdominant Saddles and Logarithmic Contributions*, **JHEP 01 (2021) 001**, [arXiv:2007.12604].

Before providing details of these indices, however, first we review recent developments in supersymmetry that have formed a foundation of this line of research in section 1.2 and the $\mathcal{N} = 2$ gauged supergravity theory that contains $\mathrm{AdS}_5$ black strings and black holes of our interest in section 1.3. In section 1.4, we summarize the main part of this dissertation.

## 1.2 Developments in supersymmetry

As briefly mentioned in the previous section 1.1, recent progress in the exact calculation of indices of various superconformal field theories are due in large part to two main developments in supersymmetry: rigid supersymmetry and supersymmetric localization. In this section we review these techniques schematically.

### 1.2.1 Supersymmetric field theory on a curved space

Supersymmetry is a very powerful tool which allows us to have more control of a given field theory. For example, based on supersymmetry, we can explore protected physical quantities such as the Witten index that gives information about ground states of the theory. In the dual gravity side, we can find numerous solutions of supergravity theories with the help of supersymmetry that reduces the 2nd order equations of motion to the 1st order BPS equations.

Hence it is natural to ask if we can construct supersymmetric field theories in general backgrounds beyond Minkowski space. This question has been answered by various works in different backgrounds. For example, the $\mathcal{N} = 4$ SYM theory was constructed on $S^4$ in [97, 98], and [79, 80, 81] studied the partition function of some $\mathcal{N} = 2$ theories on $S^3$. A systematic construction of supersymmetric field theories on curved backgrounds was then established in [53, 52]. Here we review this systematic procedure schematically.

First, based on a given supersymmetric field theory, one introduces a supergravity multiplet that contains metric $g_{\mu\nu}$ and auxiliary fields. Along with the original supermultiplets including chiral/vector multiplets, this gives a off-shell supergravity. If necessary, we may also add background vector multiplets. The next step is to fix the metric to a curved space where we want to construct a given supersymmetric field theory. Other auxiliary fields within the supergravity multiplets should also be frozen under supersymmetry transformation since we want a supersymmetric field theory on a curved background, and not a supergravity theory. In short, we impose

$$\delta_{\rm susy} g_{\mu\nu} = \delta_{\rm susy}(\text{auxiliary fields}) = 0. \tag{1.2}$$

To satisfy the condition (1.2), we need to turn off a gravitino field $\Psi_\mu$, which necessarily means

$$\delta_{\rm susy}\Psi_\mu = (\nabla_\mu + \cdots)\zeta = 0. \tag{1.3}$$

These constraints are called the Killing spinor equations. For some degree of supersymmetry to survive, the Killing spinor equations (1.3) must allow for a non-trivial Killing spinor $\zeta$, which imposes certain constraints on fixed values of auxiliary fields hidden in "$\cdots$" of (1.3). In a flat space, the Killing spinor equations (1.3) would be automatically satisfied with zero auxiliary fields and a constant Killing spinor $\zeta$. In a curved space, however, they typically assign particular values to auxiliary fields for the theory to remain supersymmetric.

For concreteness, we repeat the above procedure in a specific example of interest, namely the construction of $\mathcal{N}=1$ theory on $T^2\times S^2$ following [52, 66]. We start with $\mathcal{N}=1$ chiral/vector multiplets:

$$\text{chiral multiplet: } (\phi,\psi,F), \qquad \text{vector multiplet: } (A_\mu,\lambda,D). \tag{1.4}$$

These superfields make up $\mathcal{N}=1$ theories. To define the theory on $T^2\times S^2$, one introduces a new minimal off-shell supergravity multiplet [104, 103]

$$\text{supergravity multiplet : } (g_{\mu\nu},\Psi_\mu,\mathcal{A}_\mu,B_{\mu\nu}), \tag{1.5}$$

with auxiliary fields $\mathcal{A}_\mu$ and $B_{\mu\nu}$. Then, following the logic introduced above, one must impose the Killing spinor equations

$$\begin{aligned}\delta_{\rm susy}\Psi_\mu &= -2(\nabla_\mu - i\mathcal{A}_\mu)\zeta - 2iV_\mu - 2iV_\mu\zeta - 2iV^\nu\sigma_{\mu\nu}\zeta = 0,\\ \delta_{\rm susy}\tilde\Psi_\mu &= -2(\nabla_\mu + i\mathcal{A}_\mu)\tilde\zeta - 2iV_\mu + 2iV_\mu\tilde\zeta + 2iV^\nu\tilde\sigma_{\mu\nu}\tilde\zeta = 0,\end{aligned} \tag{1.6}$$

with non-trivial Killing spinors $(\zeta,\tilde\zeta)$, for the $\mathcal{N}=1$ theory to remain supersymmetric on $T^2\times S^2$. Here $V_\mu$ is defined in terms of the field strength of $B_{\mu\nu}$ as $V^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\partial_\nu B_{\rho\lambda}$. Since $\nabla_\mu = \partial_\mu$ on a flat space, it is obvious that the Killing spinor equations (1.6) allow for constant Killing spinors $(\zeta,\tilde\zeta)$ with zero auxiliary fields. On a curved space where $\nabla_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}$, however, one must investigate the Killing spinor equations (1.6) to figure out particular values of auxiliary fields $\mathcal{A}_\mu, V_\mu$ that allow for non-trivial Killing spinors $(\zeta, \tilde{\zeta})$. On $T^2 \times S^2$ whose metric is given explicitly as

$$ds^2 = d\omega d\bar{\omega} + \left(\frac{2}{1+z\bar{z}}\right)^2 dz d\bar{z}, \tag{1.7}$$

the auxiliary fields that allow for non-trivial Killing spinors $(\zeta, \tilde{\zeta})$ are determined as

$$\mathcal{A} = -\frac{i}{2}\frac{\bar{z}dz - zd\bar{z}}{1+z\bar{z}}, \qquad V = 0. \tag{1.8}$$

The non-trivial Killing spinors that satisfy the Killing spinor equations (1.6) along with auxiliary fields (1.8) are then given as

$$\zeta \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \tilde{\zeta} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{1.9}$$

up to multiplicative constants. Note that the background $\mathcal{A}$ field in (1.8) gives a non-trivial magnetic flux on $S^2$, which is called a topological twist on $S^2$. This is a quite general feature of supersymmetric field theories on curved backgrounds, and we call them topologically twisted theories. The topologically twisted index of $\mathcal{N} = 4$ SU($N$) SYM theory on $T^2 \times S^2$, which is of our interest in chapter 2, is basically the partition function of the topologcially twisted $\mathcal{N} = 4$ SU($N$) SYM theory on $T^2 \times S^2$ with a topological twist on $S^2$. In this example, there are additional background gauge fields associated with flavor symmetries so the fixed values of auxiliary fields would be more involved than (1.8). However, they are determined by the same logic we have reviewed above; refer to [66] for more details about the topologcially twisted $\mathcal{N} = 4$ SU($N$) SYM theory on $T^2 \times S^2$. In chapter 2, we focus on the calculation of the topologically twisted index.

### 1.2.2 Supersymmetric localization

The AdS/CFT correspondence [86] was first introduced as a strong-weak duality, which connects string theories on AdS backgrounds in the small curvature limit (the radius of curvature is much larger than the string length) and conformal field theories in the large 't-Hooft coupling limit. It had not been easy to test this duality explicitly, however, since the large 't-Hooft coupling limit does not allow for a perturbative calculation in the field theory side.

Supersymmetric localization, which was first introduced in [97], provides a way to overcome this issue for certain supersymmetric conformal field theories (SCFT). It allows for exact calculation of various physical quantities of SCFTs regardless of the size of the 't-Hooft coupling, so we can compare field theory results with gravity results in the small curvature limit (or equivalently in the supergravity limit). Here we review the mechanism of supersymmetric localization schematically.

Interesting physical quantities of SCFTs such as partition functions and supersymmetric indices can be written as a path integral as

$$Z = \int \mathcal{D}\phi \, \exp[-S[\phi]], \tag{1.10}$$

where $\phi$ denotes a set of fields in a given SCFT and $S[\phi]$ is the action. By assumption, the action is invariant under a supersymmetry transformation $Q$ as $QS[\phi] = 0$.

Now we introduce a potential $V[\phi]$ that satisfies $Q^2V[\phi] = 0$ and a deformed quantity $Z(t)$ as

$$Z(t) = \int \mathcal{D}\phi \, \exp[-S[\phi] + tQV[\phi]]. \tag{1.11}$$

Note that the physical quantity of interest (1.10) is equivalent to $Z = Z(0)$. In fact, provided the path integration measure $\mathcal{D}\phi$ is invariant under a supersymmetry transformation $Q$, we can conclude $Z = Z(t)$ for an arbitrary value of $t$ since

$$Z'(t) = \int \mathcal{D}\phi \, (QV[\phi]) \exp[-S[\phi] + tQV[\phi]] = \int \mathcal{D}\phi \, Q\big(\exp[-S[\phi] + tQV[\phi]]\big) = 0. \tag{1.12}$$

In the 2nd equation we have used $QS[\phi] = Q^2V[\phi] = 0$ and in the 3rd equation we have used that the integral of a total derivative vanishes.

Next we find a localization locus $\phi_0$ where the deformation term $QV[\phi]$ satisfies

$$QV[\phi_0] = 0, \qquad \frac{\delta(QV)}{\delta\phi(x)}[\phi_0] = 0, \qquad \frac{\delta^2(QV)}{\delta\phi(x)\delta\phi(y)}[\phi_0] \sim \delta^d(x-y). \tag{1.13}$$

Substituting the Taylor expansion around this locus, namely $\phi = \phi_0 + \frac{1}{\sqrt{t}}\phi_1$, into $Z = Z(t)$ (1.11) then gives

$$Z = Z(t) = \int d\phi_0 \mathcal{D}\phi_1 \, \exp\bigg[-S[\phi_0] + \frac{1}{2}\int d^dx \left(\frac{\delta^2(QV)}{\delta\phi(x)^2}[\phi_0]\right) \phi_1(x)^2 + \mathcal{O}(t^{-\frac{1}{2}})\bigg]. \tag{1.14}$$

Note that the $\phi_0$-integral is over the localization locus and not a path integral anymore.

Finally, taking the limit $t \to \infty$, we obtain the physical quantity of our interest (1.14) as

$$Z = Z(\infty) = \int d\phi_0 \underbrace{\exp[-S[\phi_0]]}_{Z_{\text{classical}}} \underbrace{\left[\det\left(\frac{\delta^2(QV)}{\delta\phi(x)^2}[\phi_0]\right)\right]^{\#}}_{Z_{\text{1-loop}}}, \tag{1.15}$$

where $\# = -\frac{1}{2}$ for bosonic fields and $\# = 1$ for fermionic fields. In the final formula (1.15), the original path integral expression (1.10) reduces to an ordinary integral of the classical contribution $Z_{\text{classical}}$ multiplied by 1-loop determinants $Z_{\text{1-loop}}$ over the localization locus $\phi_0$.

It is remarkable that the above procedure is independent of the size of coupling constants and therefore valid even in the strong coupling limit. Since the work of Pestun [97], other physical quantities of various SCFTs have been computed exactly following the above procedure. The $S^3$ partition function of Chern-Simons-Matter theory is a typical example [79]. Indices of $\mathcal{N} = 4$ SU($N$) SYM theory of our interest have also been studied by supersymmetric localization: see [43, 21, 66] for the topologically twised index and [27] for the superconformal index[2]. In the main part of this dissertation, we will focus on the results of supersymmetric localization and investigate various properties of those indices.

## 1.3 5-dimensional $\mathcal{N} = 2$ gauged STU model

In this section, we turn to the gravity side and investigate AdS$_5$ black strings/holes dual to $\mathcal{N} = 4$ SU($N$) SYM theory of our interest: to be specific, supersymmetric, magnetically charged AdS$_5$ black strings and supersymmetric, rotating, electrically charged AdS$_5$ black holes. They are solutions of the 5d $\mathcal{N} = 2$ gauged supergravity coupled to two vector multiplets, so we start from there.

Five-dimensional $\mathcal{N} = 2$ gauged supergravity coupled to vector multiplets was constructed in [59, 60]. The supergravity multiplet consists of a graviton $g_{\mu\nu}$, a gravitino $\psi_\mu^\alpha$, and a graviphoton $A_\mu$. The vector multiplet consists of a real scalar $\phi^x$, a dilatino $\lambda^{x\alpha}$, and a gauge field $A_\mu^x$ where $x$ take values in $x = 1, \cdots, n_V$ with $n_V$ the number of vector multiplets. All the fermionic fields $\psi_\mu^\alpha$ and $\lambda^{x\alpha}$ are symplectic-Majorana spinors with the corresponding SU(2) index $\alpha \in \{1, 2\}$.

We are particularly interested in the case with two vector multiplets ($n_V = 2$), namely the STU model. The 5d $\mathcal{N} = 2$ gauged STU model was known as a consistent truncation

[2]In chapter 3, we formulate the superconformal index following the Hamiltonian approach [83, 101] instead. The result is slightly different from the one from supersymmetric localization [27] by the factor of supersymmetric Casimir energy, which we will not discuss in this dissertation. Refer to [6, 23] for more details about the supersymmetric Casimir energy.

of 10d Type IIB supergravity on $\text{AdS}_5 \times S^5$ [47], where the three abelian gauge fields of the STU model including a graviphoton originate from the $\text{U}(1)^3$ Cartan subalgebra of the full gauge group SO(6) acting on the internal manifold $S^5$. This means that the equations of motion of the truncated Type IIB supergravity are equivalent to those of the $\mathcal{N}=2$ gauged STU model. Then, through the duality between the Type IIB supergravity on $\text{AdS}_5 \times S^5$ and the $\mathcal{N}=4$ SYM theory on the Minkowski space through the AdS/CFT correspondence [86], we can understand various solutions of the 5d $\mathcal{N}=2$ gauged STU model as holographic duals of the ensemble of quantum states in the $\mathcal{N}=4$ SYM theory. Based on this overall picture, first we study supersymmetric (or BPS) AdS black string/hole solutions of the 5d $\mathcal{N}=2$ gauged STU model in this section. We will then move on to the indices of the $\mathcal{N}=4$ SYM theory that count the number of BPS states dual to these black string/hole solutions in the main part of this dissertation.

To find a supersymmetric AdS black string/hole solution of the 5d $\mathcal{N}=2$ gauged STU model, first we need its bosonic action. Fermions will not matter since we can turn them off with appropriate consistency conditions called the BPS equations that will be specified later. Following the convention of [47, 87], the bosonic action is given as

$$S = \frac{1}{16\pi G_{(5)}} \int d^5x \sqrt{g} \left[ R + 4g^2 \sum_{i=1}^{3} \frac{1}{X^i} - \frac{1}{2} \sum_{x=1}^{2} \partial_\mu \phi^x \partial^\mu \phi^x - \frac{1}{4} \sum_{i=1}^{3} (X^i)^{-2} F^i_{\mu\nu} F^{i\mu\nu} \right. \\ \left. + \frac{1}{24} |\varepsilon_{ijk}| \varepsilon^{\mu\nu\rho\sigma\lambda} F^i_{\mu\nu} F^j_{\rho\sigma} A^k_\lambda \right], \tag{1.16}$$

where $x$ and $i,j,k$ take values in $x \in \{1,2\}$ and $i,j,k \in \{1,2,3\}$ respectively and the convention of the Levi-Civita symbol is given as

$$\varepsilon^{\mu\nu\rho\sigma\lambda} = \begin{cases} -|g|^{-1/2} & \text{(even permutation)} \\ +|g|^{-1/2} & \text{(odd permutation)} \end{cases}. \tag{1.17}$$

The physical scalars $\phi^x$ are parametrized by $X^i$ as

$$X^1 = e^{-\frac{1}{\sqrt{6}}\phi^1 - \frac{1}{\sqrt{2}}\phi^2}, \qquad X^2 = e^{-\frac{1}{\sqrt{6}}\phi^1 + \frac{1}{\sqrt{2}}\phi^2}, \qquad X^3 = e^{\frac{2}{\sqrt{6}}\phi^1}. \tag{1.18}$$

From the bosonic action (1.16), the Einstein equations are given as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R + 4g^2\sum_{i=1}^{3}(X^i)^{-1}) = \frac{1}{2}\sum_{x=1}^{2}\partial_\mu\phi^x\partial_\nu\phi^x - \frac{1}{4}g_{\mu\nu}\sum_{x=1}^{2}\partial_\rho\phi^x\partial^\rho\phi^x \\ + \frac{1}{2}\sum_{i=1}^{3}(X^i)^{-2}F^i_{\mu\rho}F^i{}_\nu{}^\rho - \frac{1}{8}g_{\mu\nu}\sum_{i=1}^{3}(X^i)^{-2}F^i_{\rho\sigma}F^{i\rho\sigma}. \tag{1.19}$$

The scalar equations of motion are given as

$$0 = \nabla_\mu\nabla^\mu\phi^x - \frac{1}{4}\sum_{i=1}^{3}\partial_{\phi^x}(X^i)^{-2}F^i_{\mu\nu}F^{i\mu\nu} + 4g^2\sum_{i=1}^{3}\partial_{\phi^x}(X^i)^{-1}. \tag{1.20}$$

The Bianchi identity and the vector equations of motion are given as

$$0 = \partial_{[\rho}F_{\lambda\sigma]}, \tag{1.21a}$$

$$0 = \nabla_\mu((X^i)^{-2}F^{i\mu\nu}) + \frac{1}{4}\sqrt{g}|\varepsilon_{ijk}|\varepsilon^{\mu\lambda\rho\sigma\nu}F^j_{\mu\lambda}F^k_{\rho\sigma}. \tag{1.21b}$$

For a supersymmetric bosonic solution, the supersymmetry transformations of fermionic fields $\delta\psi^\alpha_\mu$ and $\delta\lambda^{x\alpha}$ must vanish. This condition gives the aforementioned BPS equations

$$0 = \left[\partial_\mu + \frac{1}{4}\omega^{ab}_\mu\gamma_{ab} + \frac{i}{24}(\gamma_\mu{}^{\nu\rho} - 4\delta^\nu_\mu\gamma^\rho)\sum_{i=1}^{3}(X^i)^{-1}F^i_{\nu\rho} + \frac{g}{6}\sum_{i=1}^{3}X^i\gamma_\mu - \frac{ig}{2}\sum_{i=1}^{3}A^i_\mu\right]\epsilon, \tag{1.22a}$$

$$0 = \left[-\frac{i}{4}\partial_\mu\phi^x\gamma^\mu + \frac{1}{8}\sum_{i=1}^{3}(\partial_{\phi^x}(X^i)^{-1})F^i_{\mu\nu}\gamma^{\mu\nu} + \frac{ig}{2}\sum_{i=1}^{3}\partial_{\phi^x}X^i\right]\epsilon. \tag{1.22b}$$

To find a supersymmetric solution of the $\mathcal{N} = 2$ gauged STU model from the bosonic action (1.16), we must find a field configuration that satisfies all the above equations (1.19), (1.20), (1.21), and (1.22). In [55, 61], however, it has been found that the BPS equations (1.22) and the following integrability condition are in fact sufficient to find a supersymmetric solution with a time-like Killing vector. In the following subsections, based on this observation, we discuss supersymmetric $AdS_5$ black string/hole solutions that have been found by investigating the BPS equations (1.22).

### 1.3.1 $AdS_5$ black strings

In this subsection, we review a supersymmetric, magnetically charged $AdS_5$ black string and its entropy in 5d $\mathcal{N} = 2$ gauged STU model. First, a supersymmetric, magnetically charged $AdS_5$ black string solution of the BPS equations (1.22) with identical vector fields has been found [84]:

$$
\begin{aligned}
ds^2 &= r^{\frac{1}{2}}\left(r-\frac{1}{3r}\right)^{\frac{3}{2}}(-dt^2+dz^2)+\left(r-\frac{1}{3r}\right)^{-2}dr^2+r^2(d\theta^2+\sinh^2\theta d\phi^2), && (1.23a)\\
A^i &= -\frac{1}{3}\cosh\theta d\phi, && (1.23b)\\
X^i &= 1. && (1.23c)
\end{aligned}
$$

Note that we identify magnetic charges $p^i$ in (30,31) of [84] as $p^i = \frac{1}{3g}$ to restore a pure $AdS_5$ solution with vanishing physical scalar fields, namely $\phi^x \to 0$ or $X^i \to 1$, in the asymptotic region $r \to \infty$. Note $p^i = q_I^{\text{there}}$. Then we set $g = 1$ which fixes the $AdS_5$ radius. In the near-horizon limit $r \to \frac{1}{\sqrt{3}}$, the geometry becomes $AdS_3 \times H^2$. A magnetic string solution similar to (1.23) was found for the case with $S^2$ instead of $H^2$ but it has a naked singularity [35].

Since the work of [35, 84], there have been numerous works that try to generalize the $AdS_5$ black string solution (1.23) to one with general magnetic charges. In [32, 7], for example, the authors reduced the BPS equations (1.22) for a magnetic $AdS_5$ black string ansatz to a system of SO(2,1) spinning top equations. Then the $AdS_5$ black string solution (1.23) was partially generalized to the case with different configurations of magnetic charges. Despite the effort, an analytic $AdS_5$ black string solution with fully general magnetic charges has not yet been constructed.

We are mainly interested in the entropy of an $AdS_5$ black string, however, and therefore we do not need a full analytic solution. It suffices to know the near-horizon limit to compute the entropy. Fortunately, the near-horizon limit of an $AdS_5$ black string that satisfies the BPS equations (1.22) has been found for general magnetic charges. It is given as [12]

$$
\begin{aligned}
ds^2 &= \left(\frac{8p^1p^2p^3\Pi}{\Theta^3}\right)^{\frac{2}{3}}\left(\frac{1}{4}\left(-r^2dt^2+\frac{dr^2}{r^2}\right)+\rho^2\left(dx^5+\frac{r}{2\rho}dt\right)^2\right) && (1.24a)\\
&\quad+\left(\frac{(p^1p^2p^3)^2}{\Pi}\right)^{\frac{1}{3}}e^{2h_{\Sigma_\mathfrak{g}}(x,y)}(dx^2+dy^2), && (1.24b)\\
A^i &= -p^i\omega_{\Sigma_\mathfrak{g}}, && (1.24c)\\
X^i &= \frac{p^i(p^1+p^2+p^3-2p^i)}{(p^1p^2p^3\Pi)^{\frac{1}{3}}}, && (1.24d)\\
-\kappa &= p^1+p^2+p^3, && (1.24e)
\end{aligned}
$$

where we have introduced a function $h_{\Sigma_\mathfrak{g}}$ and a one-form $\omega_{\Sigma_\mathfrak{g}}$ that characterize the Riemann surface $\Sigma_\mathfrak{g}$ of genus $\mathfrak{g}$ as ($\kappa = 1$, $\kappa = 0$, and $\kappa = -1$ for $\mathfrak{g} = 0$, $\mathfrak{g} = 1$, and $\mathfrak{g} > 1$ respectively)

$$e^{h_{\Sigma_\mathfrak{g}}(x,y)} = \begin{cases} \frac{2}{1+x^2+y^2} & (\mathfrak{g}=0) \\ \sqrt{2\pi} & (\mathfrak{g}=1) \\ \frac{1}{y} & (\mathfrak{g}>1) \end{cases}, \qquad \omega_{\Sigma_\mathfrak{g}} = \begin{cases} \frac{2(-ydx+xdy)}{1+x^2+y^2} & (\mathfrak{g}=0) \\ \pi(-ydx+xdy) & (\mathfrak{g}=1) \\ \frac{dx}{y} & (\mathfrak{g}>1) \end{cases}. \tag{1.25}$$

The constants $\Pi$ and $\Theta$ are given in terms of magnetic charges as

$$\Pi = (p^1 + p^2 - p^3)(p^1 - p^2 + p^3)(-p^1 + p^2 + p^3), \tag{1.26a}$$

$$\Theta = -(p^1)^2 - (p^2)^2 - (p^3)^2 + 2(p^1p^2 + p^2p^3 + p^3p^1). \tag{1.26b}$$

In the metric (1.24b), we have replaced the Poincaré $AdS_3$ part in [12] with the extremal BTZ metric following [77]. Since they are locally equivalent, we still dub (1.24) as an $AdS_3 \times \Sigma_g$ near-horizon solution for simplicity. It is remarkable that an $AdS_3 \times \Sigma_g$ solution (1.24) of the BPS equations (1.22) is truly a near-horizon limit of an $AdS_5$ black string with general magnetic charges, which was confirmed by constructing numerical black string solutions whose near-horizon limits correspond to (1.24) [12].

Now the entropy of an $AdS_5$ black string whose near-horizon limit corresponds to (1.24) is given from the Bekenstein-Hawking entropy formula [11, 64] as

$$S = \frac{A_{(3)}}{4G_{(5)}} = \frac{\text{vol}[\Sigma_\mathfrak{g}]}{2G_{(5)}} \frac{p^1p^2p^3\rho}{\Theta} \Delta_{x^5}, \tag{1.27}$$

where $A_{(3)}$ denotes the volume of a 3-dimensional time-slice at the horizon $r \to 0$ and $\Delta_{x^5}$ is the period of the $x^5$ coordinate. $G_{(5)}$ is a 5-dimensional Newton's constant.

### Corresponding 4D black holes

Following the AdS/CFT picture described in section 1.1, we would like to relate the black string entropy (1.27) to the index of a dual superconformal field theory. For that purpose, first we should clarify the physical meaning of a free parameter $\rho$ in the entropy formula (1.27) and then find out its dual in the field theory side. This is because we need an exact dictionary between parameters in both sides of the AdS/CFT correspondence to match the entropy and the index precisely. In the 5d $\mathcal{N} = 2$ gauged STU model, however, the physical meaning of a free parameter $\rho$ is unclear.

This issue has been overcome in [77] by relating the near-horizon limit of an $AdS_5$ black string (1.24) to the near-horizon limit of a 4-dimensional black hole in the 4d $\mathcal{N} = 2$ STU model [31]. To elaborate this relation, we briefly introduce the 4d $\mathcal{N} = 2$ gauged STU model and the map that connects the 5d theory to the 4d theory.

The 4d $\mathcal{N}=2$ gauged STU model has a supergravity multiplet, which contains a graviton $g_{\mu\nu}$, a gravitino $\psi^\alpha_\mu$, and a graviphoton $A^0_\mu$, and three vector multiplets, each of which contains a complex scalar $z^i$, a gauigino $\lambda^{i\alpha}$, and a gauge field $A^i_\mu$. Here $i$ take values in $i\in\{1,2,3\}$ and $\alpha$ is the SU(2) index $\alpha\in\{1,2\}$. The bosonic action of the 4d $\mathcal{N}=2$ gauged STU model is then given as

$$\begin{aligned}S=\frac{1}{16\pi G_{(4)}}\int d^4x\sqrt{-g}\bigg[R+4ig^2\sum_{i=1}^{3}\frac{1}{z^i-\bar{z}^i}+2g^2\sum_{i=1}^{3}\frac{\partial_\mu z^i\partial^\mu\bar{z}^i}{(z^i-\bar{z}^i)^2}\\+\frac{1}{2}\mathcal{I}_{IJ}F^I_{\mu\nu}F^{J\mu\nu}-\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\mathcal{R}_{IJ}F^I_{\mu\nu}F^{J\mu\nu}\bigg],\end{aligned}\tag{1.28}$$

where $I,J$ take values in $I,J\in\{0,1,2,3\}$. The coefficients $\mathcal{I}_{IJ}$ and $\mathcal{R}_{IJ}$ are given in terms of the symplectic language as

$$\mathcal{R}_{IJ}+i\mathcal{I}_{IJ}=\bar{\mathcal{F}}_{IJ}+2i\frac{(\operatorname{Im}\mathcal{F}_{IK})\mathcal{X}^K(\operatorname{Im}\mathcal{F}_{JL})\mathcal{X}^L}{(\operatorname{Im}\mathcal{F}_{KL})\mathcal{X}^K\mathcal{X}^L},\tag{1.29}$$

where $\mathcal{F}_{IJ}\equiv\partial_{\mathcal{X}^I}\partial_{\mathcal{X}^J}\mathcal{F}$ and the prepotential $\mathcal{F}$ for the STU model is given as

$$\mathcal{F}=-\frac{\mathcal{X}^1\mathcal{X}^2\mathcal{X}^3}{\mathcal{X}^0}.\tag{1.30}$$

Here $\mathcal{X}^I$ parametrizes physical scalars $z^i$'s, and we use a particular parametrization

$$z^i=\frac{\mathcal{X}^i}{\mathcal{X}^0}.\tag{1.31}$$

In this parametrization, the coefficients $\mathcal{I}_{IJ}$ and $\mathcal{R}_{IJ}$ introduced in (1.29) are written explicitly as

$$
\begin{aligned}
\mathcal{R}_{IJ} &= \begin{pmatrix} -\frac{(z^1+\bar{z}^1)(z^2+\bar{z}^2)(z^3+\bar{z}^3)}{4} & \frac{(z^2+\bar{z}^2)(z^3+\bar{z}^3)}{4} & \frac{(z^3+\bar{z}^3)(z^1+\bar{z}^1)}{4} & \frac{(z^1+\bar{z}^1)(z^2+\bar{z}^2)}{4} \\ \frac{(z^2+\bar{z}^2)(z^3+\bar{z}^3)}{4} & 0 & -\frac{z^3+\bar{z}^3}{2} & -\frac{z^2+\bar{z}^2}{2} \\ \frac{(z^3+\bar{z}^3)(z^1+\bar{z}^1)}{4} & -\frac{z^3+\bar{z}^3}{2} & 0 & -\frac{z^1+\bar{z}^1}{2} \\ \frac{(z^1+\bar{z}^1)(z^2+\bar{z}^2)}{4} & -\frac{z^2+\bar{z}^2}{2} & -\frac{z^1+\bar{z}^1}{2} & 0 \end{pmatrix}, \\
\mathcal{I}_{00} &= \frac{(z^1-\bar{z}^1)(z^2-\bar{z}^2)(z^3-\bar{z}^3)}{8i}\left(1-\sum_{i=1}^{3}\frac{(z^i+\bar{z}^i)^2}{(z^i-\bar{z}^i)^2}\right), \\
\mathcal{I}_{0i} = \mathcal{I}_{i0} &= \frac{(z^i+\bar{z}^i)(z^1-\bar{z}^2)(z^2-\bar{z}^2)(z^3-\bar{z}^3)}{4i(z^\alpha-\bar{z}^\alpha)^2}, \\
\mathcal{I}_{ij} &= \text{diag}\left(\frac{i(z^2-\bar{z}^2)(z^2-\bar{z}^3)}{2(z^1-\bar{z}^1)}, \frac{i(z^3-\bar{z}^3)(z^1-\bar{z}^1)}{2(z^2-\bar{z}^2)}, \frac{i(z^1-\bar{z}^1)(z^2-\bar{z}^3)}{2(z^3-\bar{z}^3)}\right).
\end{aligned} \tag{1.32}
$$

Lastly, the 4d electromagnetic charges are given as

$$
\begin{aligned}
p^I &= \frac{1}{\text{vol}[\Sigma_\mathfrak{g}]}\int_{\Sigma_\mathfrak{g}} F^I, \\
q_I &= \frac{1}{\text{vol}[\Sigma_\mathfrak{g}]}\int_{\Sigma_\mathfrak{g}} (-\mathcal{I}_{IJ} *_4 F^J + \mathcal{R}_{IJ}F^J).
\end{aligned} \tag{1.33}
$$

It is straightforward to show that the 5d bosonic action (1.16) reduces to the 4d bosonic action (1.28) under the Kaluza-Klein (KK) reduction

$$
\begin{aligned}
ds^2_{(5)} &= e^{-\varphi}ds^2_{(4)} + e^{2\varphi}(dx^5 + \frac{1}{2}A^0_{(4)})^2, \\
A^i_{(5)} &= -A^i_{(4)} + (z^i+\bar{z}^i)(dx^5 + \frac{1}{2}A^0_{(4)}), \\
X^i &= -ie^{-\varphi}(z^i - \bar{z}^i), \\
e^{3\varphi} &= i(z^1-\bar{z}^1)(z^2-\bar{z}^2)(z^3-\bar{z}^3),
\end{aligned} \tag{1.34}
$$

where we have distinguished metric and gauge fields in different dimensions using the corresponding subscripts. In particular, the KK reduction (1.34) maps the near-horizon limit of a supersymmetric, magnetic $AdS_5$ black string (1.24) to the near-horizon limit of a 4d black hole, namely

$$ds^2_{(4)} = \frac{2p^1p^2p^3\rho\Pi}{\Theta^3}\left(-r^2dt^2 + \frac{dr^2}{r^2} + \frac{\Theta^2}{\Pi}e^{2h_{\Sigma_{\mathfrak{g}}}(x,y)}(dx^2+dy^2)\right), \tag{1.35a}$$

$$A^0_{(4)} = \frac{r}{\rho}dt, \tag{1.35b}$$

$$A^i_{(4)} = p^i\omega_{\Sigma_{\mathfrak{g}}}, \tag{1.35c}$$

$$z^i = \frac{i\rho p^i(p^1+p^2+p^3-2p^i)}{\Theta}, \tag{1.35d}$$

$$-\kappa = p^1+p^2+p^3. \tag{1.35e}$$

Note that this is not the near-horizon limit of an $\mathrm{AdS}_4$ black hole [31]. The entropy of a 4d black hole whose near-horizon limit corresponds to (1.35) can be computed by the Bekenstein-Hawking formula [11, 64] as

$$S = \frac{A_{(2)}}{4G_{(4)}} = \frac{\mathrm{vol}[\Sigma_{\mathfrak{g}}]}{2G_{(4)}}\frac{p^1p^2p^3\rho}{\Theta}, \tag{1.36}$$

where $A_{(2)}$ denotes the area of a 2-dimensional time-slice at the horizon $r \to 0$. $G_{(4)}$ is a 4-dimensional Newton's constant. This black hole entropy (1.36) is exactly the same as the original black string entropy (1.27) under the identification

$$G_{(5)} = \Delta_{x^5}G_{(4)}, \tag{1.37}$$

which is justified by the relations

$$\begin{aligned}\frac{1}{16\pi G_{(5)}}\int d^5x\sqrt{g_{(5)}}R_{(5)} &= \frac{1}{16\pi G_{(5)}}\int dx^5d^4x\sqrt{g_{(4)}}\left[R_{(4)}+\cdots\right]\\ &= \frac{1}{16\pi G_{(4)}}\int d^4x\sqrt{g_{(4)}}\left[R_{(4)}+\cdots\right]\end{aligned} \tag{1.38}$$

for the KK reduction (1.34).

Finally, we discuss the physical menaing of a free parameter $\rho$ in the context of the 4d $\mathcal{N}=2$ gauged STU model. Substituting (1.35) into (1.33) along with (1.32) gives an expression for $\rho$ in terms of 4d electromagnetic charges as

$$\begin{aligned} q_0 &= \frac{1}{\mathrm{vol}[\Sigma_{\mathfrak{g}}]}\int_{\Sigma_{\mathfrak{g}}}\left(\frac{\rho^3 p^1 p^2 p^3 \Pi}{\Theta^3} *_4 F^0\right) \\ &= \frac{1}{\mathrm{vol}[\Sigma_{\mathfrak{g}}]}\int_{\Sigma_{\mathfrak{g}}}\left(\frac{\rho^3 p^1 p^2 p^3 \Pi}{\Theta^3}\frac{\Theta^2}{\rho\Pi}e^{2h_{\Sigma_{\mathfrak{g}}}(x,y)}dx\wedge dy\right), \\ \rightarrow \quad \rho &= \sqrt{\frac{q_0\Theta}{p^1p^2p^3}} \end{aligned} \tag{1.39}$$

Hence the degree of freedom in $\rho$ is directly related to the Kaluza-Klein electric charge associated to the 4d graviphoton $A^0$. The $\mathrm{AdS}_5$ black string entropy (1.27), or equivalently the corresponding 4d black hole entropy (1.36), now can be rewritten in terms of 4d electromagnetic charges as

$$S = \frac{\mathrm{vol}[\Sigma_{\mathfrak{g}}]}{2G_{(4)}}\sqrt{\frac{q_0p^1p^2p^3}{\Theta}}. \tag{1.40}$$

We will discuss the microscopic origin of this black string entropy using the topologically twisted index of dual $\mathcal{N}=4$ SU($N$) SYM theory in Chapter 2.

### 1.3.2 $\mathrm{AdS}_5$ black holes

In this subsection, we review a supersymmetric, rotating, electrically charged $\mathrm{AdS}_5$ black hole and its entropy in 5d $\mathcal{N}=2$ gauged STU model. The first example of a supersymmetric, rotating, electrically charged $\mathrm{AdS}_5$ black hole in 5d $\mathcal{N}=2$ minimal gauged supergravity was found in [62]. It is then generalized to the theory coupled to vector multiplets [61]. Generalization to non-extremal black holes with two independent angular momenta has been done in [37, 38]. Here we focus on the most general supersymmetric, rotating, electrically charged $\mathrm{AdS}_5$ black hole in 5d $\mathcal{N}=2$ gauged STU model [85] that incorporates all the aforementioned black holes as its special case.

The supersymmetric, rotating, electrically charged $\mathrm{AdS}_5$ black hole solution to the BPS equations (1.22) is given as [85]

$$ds^2 = -(H_1H_2H_3)^{-\frac{2}{3}}(dt+\omega_\phi d\phi+\omega_\psi d\psi)^2+(H_1H_2H_3)^{\frac{1}{3}}h_{mn}dx^mdx^n, \tag{1.41a}$$

$$A^i = H_i^{-1}(dt+\omega_\phi d\phi+\omega_\psi d\psi)+U^i_\phi d\phi+U^i_\psi d\psi, \tag{1.41b}$$

$$X^i = H_i^{-1}(H_1H_2H_3)^{\frac{1}{3}}. \tag{1.41c}$$

Here we have used the $(t,r,\theta,\phi,\psi)$ coordinate, where $(\theta,\phi,\psi)$ is a typical $S^3$ coordinate with $0\leq\theta\leq\frac{\pi}{2}$, $\phi\sim\phi+2\pi$, and $\psi\sim\psi+2\pi$. The 4d base metric $h_{mn}dx^mdx^n$ in (1.41) is given as

$$
\begin{aligned}
h_{mn}dx^m dx^n = & \frac{r^2}{\Delta_r}dr^2 + \frac{r^2}{\Delta_\theta}d\theta^2 + \frac{r^2\sin^2\theta}{\Xi_a^2}\left[\Xi_a + \sin^2\theta(\rho^2 g^2 + 2(1+ag)(a+b)g)\right]d\phi^2 \\
& + \frac{r^2\cos^2\theta}{\Xi_b^2}\left[\Xi_b + \cos^2\theta(\rho^2 g^2 + 2(1+bg)(a+b)g)\right]d\psi^2 \\
& + \frac{2r^2\sin^2\theta\cos^2\theta}{\Xi_a\Xi_b}\left[\rho^2 g^2 + 2(a+b)g + (a+b)^2 g^2\right]d\phi d\psi,
\end{aligned}
\tag{1.42}
$$

and the other functions in (1.41) are given as

$$
H_i = 1 + \frac{\sqrt{\Xi_a\Xi_b}(1+g^2\mu_i) - \Xi_a\cos^2\theta - \Xi_b\sin^2\theta}{g^2 r^2}, \tag{1.43a}
$$

$$
\omega_\phi = -\frac{g\sin^2\theta}{r^2\Xi_a}\left[\rho^4 + (2r_m^2 + a^2)\rho^2 + \frac{1}{2}(\beta_2 - a^2b^2 - g^{-2}(a^2-b^2))\right], \tag{1.43b}
$$

$$
\omega_\psi = -\frac{g\cos^2\theta}{r^2\Xi_b}\left[\rho^4 + (2r_m^2 + b^2)\rho^2 + \frac{1}{2}(\beta_2 - a^2b^2 + g^{-2}(a^2-b^2))\right], \tag{1.43c}
$$

$$
U^i_\phi = \frac{g\sin^2\theta}{\Xi_a}\left[\rho^2 + 2r_m^2 + a^2 - \sqrt{\Xi_a\Xi_b}\mu_i + g^{-2}(1-\sqrt{\Xi_a\Xi_b})\right], \tag{1.43d}
$$

$$
U^i_\psi = \frac{g\cos^2\theta}{\Xi_b}\left[\rho^2 + 2r_m^2 + b^2 - \sqrt{\Xi_a\Xi_b}\mu_i + g^{-2}(1-\sqrt{\Xi_a\Xi_b})\right], \tag{1.43e}
$$

where we have used the following definitions:

$$
\begin{aligned}
\Delta_r &= r^2(g^2r^2 + (1+ag+bg)^2), \\
\Delta_\theta &= 1 - a^2g^2\cos^2\theta - b^2g^2\sin^2\theta, \\
\Xi_a &= 1 - a^2g^2, \\
\Xi_b &= 1 - b^2g^2, \\
\rho^2 &= r^2 + a^2\cos^2\theta + b^2\sin^2\theta, \\
r_m^2 &= g^{-1}(a+b) + ab, \\
\beta_2 &= \Xi_a\Xi_b(\mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1) - 2g^{-2}\sqrt{\Xi_a\Xi_b}(1-\sqrt{\Xi_a\Xi_b})(\mu_1+\mu_2+\mu_3) \\
&\quad + 3g^{-4}(1-\sqrt{\Xi_a\Xi_b})^2.
\end{aligned}
\tag{1.44}
$$

The five parameters $\mu_i, a, b$ $(i = 1, 2, 3)$ used above to describe the black hole solution are in fact constrained by

$$
\mu_1 + \mu_2 + \mu_3 = \frac{1}{\sqrt{\Xi_a\Xi_b}}\left[2r_m^2 + 3g^{-2}(1-\sqrt{\Xi_a\Xi_b})\right]. \tag{1.45}
$$

They are also supposed to satisfy

$$0 \leq a, b < g^{-1}, \qquad 0 \leq \sqrt{\Xi_b/\Xi_a} - 1 < g^2\mu_i, \tag{1.46}$$

where the last one is from the regularity of scalar fields $X^i$, for the case with $a \geq b$ without loss of generality.

The Bekenstein-Hawking entropy of a black hole (1.41) is then given as

$$\begin{aligned} S &= \frac{A_{(3)}}{4G_{(5)}} \\ &= \frac{\pi^2\sqrt{[1+g^2(\mu_1+\mu_2+\mu_3)]\mu_1\mu_2\mu_3 - \frac{1}{4}g^2(\mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1)^2 + \frac{(\sqrt{\Xi_a}-\sqrt{\Xi_b})^2}{g^6\sqrt{\Xi_a\Xi_b}}\mathcal{J}}}{2G_{(5)}}, \end{aligned} \tag{1.47}$$

where we have defined $\mathcal{J}$ as

$$\mathcal{J} = (1+g^2\mu_1)(1+g^2\mu_2)(1+g^2\mu_3). \tag{1.48}$$

In terms of electric charges and angular momenta following the convention of [20, 82],

$$Q_i = \frac{\pi}{4G_{(5)}}\left[\frac{\mu_i}{g} + \frac{g}{2}(\mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1 - 2\frac{\mu_1\mu_2\mu_3}{\mu_i})\right], \tag{1.49a}$$

$$J_\phi = \frac{\pi}{4G_{(5)}}\left[\frac{g}{2}(\mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1) + g^3\mu_1\mu_2\mu_3 + g^{-3}\mathcal{J}(\sqrt{\Xi_b/\Xi_a}-1)\right], \tag{1.49b}$$

$$J_\psi = \frac{\pi}{4G_{(5)}}\left[\frac{g}{2}(\mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1) + g^3\mu_1\mu_2\mu_3 + g^{-3}\mathcal{J}(\sqrt{\Xi_a/\Xi_b}-1)\right], \tag{1.49c}$$

we can rewrite the black hole entropy (1.47) as

$$S = 2\pi\sqrt{Q_1Q_2 + Q_2Q_3 + Q_3Q_1 - \frac{\pi(J_\phi + J_\psi)}{4G_{(5)}g^3}}. \tag{1.50}$$

We will discuss microscopic origin of this black hole entropy using the superconformal index of dual $\mathcal{N} = 4$ SU($N$) SYM theory in Chapter 3.

## 1.4 Overview of this dissertation

As explained in section 1.1, this dissertation focuses on appropriate indices of the $\mathcal{N} = 4$ SU($N$) SYM theory that count microstates associated with the AdS$_5$ black string entropy (1.27) and the AdS$_5$ black hole entropy (1.50) respectively. Here we summarize what we discuss about those indices in the next two chapters with more details.

In Chapter 2, we study the topologically twisted index that counts microstates associated with the $AdS_5$ black string entropy (1.27). First, we introduce the topologically twisted index of $\mathcal{N}=4$ SU($N$) SYM theory on $T^2 \times S^2$ [21] and show how to compute it through the Bethe Ansatz (BA) formula [73]. Then we show how the previous calculation in the Cardy-like limit [73] can be improved and obtain exact expressions of the SL(2,$\mathbb{Z}$) family of contributions to the twisted index through the BA formula. This SL(2,$\mathbb{Z}$) family of contributions turns out to behave as an elliptic genus of 2d SCFT and also have interesting Cardy-like asymptotics that plays an important role in the microstate counting. We will also discuss other contributions to the twisted index, which have been ignored in the conventional BA formula. Finally we revisit the relation between the twisted index and the black string entropy (1.27) through the $I$-extremization [73]. In this last step, we will emphasize subtle issues in the microstate counting by the topologically twisted index. This chapter is based on joint work in collaboration with Arash Arabi Ardehali and James T. Liu, previously published as [67, 5].

In Chapter 3, we investigate the superconformal index that counts microstates associated with the $AdS_5$ black hole entropy (1.50). First, we introduce the superconformal index of $\mathcal{N}=4$ SU($N$) SYM theory through the Hamiltonian formalism[3] and derive the elliptic hypergeometric integral representation. Then we explain how to compute it in two different ways [36, 20]: saddle-point evaluation and the BA formula.[4] To be specific, we improve the previous saddle-point evaluation in the Cardy-like limit and obtain an all-order result up to exponentially suppressed terms using the 3d effective Chern-Simons theory. The result is then confirmed by the BA formula. Next we investigate the phase structure of the superconformal index in the large-$N$ after the Cardy-like limit. As a result, we find a 'partially deconfined' phase of the superconformal index, which is distinguished from previously well-known fully-deconfined/confined phases in the large-$N$ limit. Finally we discuss the relation between the superconformal index and the black hole entropy (1.50). Here we will see how recent developments in the superconformal index, especially a 'partially deconfined' phase of the superconformal index, implies the existence of missing gravity dual solutions different from the known $AdS_5$ black hole in subsection 1.3.2. This chapter is based on joint work in collaboration with Arash Arabi Ardehali, Alfredo González Lezcano, James T. Liu, and Leopoldo A. Pando Zayas, previously published as [5, 58].

[3]You may use the superymmetric localization instead. See [27] for example.

[4]There is another approach using elliptic extension [28, 30] but we will not discuss this method in this dissertation.

# Chapter 2

# The Topologically Twisted Index

## 2.1 Introduction

In this section, we introduce the partition function of $\mathcal{N} = 4$ SU($N$) SYM theory on $T^2 \times S^2$ with a partial topological twist on $S^2$, which is introduced to preserve supersymmetry following the procedure in 1.2.1. It is also called the topologically twisted index based on its alternative interpretation as a supersymmetric index, namely [21, 73, 43, 66]

$$\mathcal{Z}(\tau, \Delta_a, \mathfrak{n}_a) = \mathrm{Tr}_{S^1 \times S^2} \Big[(-1)^F q^P \prod_{a=1}^{3} y_a^{J_a}\Big], \tag{2.1}$$

where $y_q = e^{2\pi i \Delta_a}$ are fugacities associated with flavor symmetry charges $J_a$ and $q = e^{2\pi i \tau}$ is given in terms of $\tau$, the modular parameter of $T^2$. $P$ is the momentum along the spatial direction of $T^2$ and $\mathfrak{n}_a$ are magnetic fluxes associated with flavor symmetries. Note that angular momenta and corresponding fugacities are omitted in (2.1). A general rotating case has also been considered in [71], but here we focus on the case without angular momenta.

The topologically twisted index (2.1) has been computed using supersymmetric localization described in subsection 1.2.2. Here we review the procedure schematically based on [14, 15, 21, 66]. The starting point is the path integral representation of the twisted index (2.1), namely

$$\mathcal{Z}(\tau, \Delta_a, \mathfrak{n}_a) = \int \mathcal{D}\phi \, \exp[-S[\phi]] = \int \mathcal{D}\phi \, \exp[-S_{\mathrm{FI}}[\phi] - S_{\mathrm{chiral}}[\phi] - S_{\mathrm{vector}}[\phi]]. \tag{2.2}$$

Here $\phi$ denotes a set of fields that compose the $\mathcal{N} = 4$ SU($N$) SYM theory on $T^2 \times S^2$. Since the action from chiral/vector multiplets are exact with respect to supersymmetry transformation [66], one can treat $S_{\text{chiral}}$ and $S_{\text{vector}}$ as deformation terms $QV[\phi]$ in 1.2.2[1]. The result is given as

$$\mathcal{Z}(\tau, \Delta_a, \mathfrak{n}_a) = \lim_{t_c, t_v \to \infty} \int \mathcal{D}\phi \, \exp[-S_{\text{FI}}[\phi] - t_c S_{\text{chiral}}[\phi] - t_v S_{\text{vector}}[\phi]]. \tag{2.3}$$

Then [21, 66] have determined localization locus $\phi_0$ that extremizes the deformation terms $S_{\text{chiral}}$ and $S_{\text{vector}}$. The resulting localization locus $\phi_0$ is parametrized by the holonomy of the gauge field $A_\mu$ along the two cycles of the torus $T^2$ with the modular parameter $\tau$

$$u = 2\pi \oint_{A\text{-cycle}} A - 2\pi\tau \oint_{B\text{-cycle}} A, \tag{2.4}$$

and integer gauge magnetic fluxes $\{\mathfrak{m}_i\}$. Refer to [21, 73] for more details about the localization locus. The path integral expresssion (2.3) then reduces to a saddle point evaluation around the localization locus $\phi_0$, which gives the integrand in terms of a classical action and 1-loop determinants as

$$\mathcal{Z}(\tau, \Delta_a, \mathfrak{n}_a) = \int d\phi_0 \, Z_{\text{classical}}[\phi_0] \, Z^{\text{chiral}}_{\text{1-loop}}[\phi_0] \, Z^{\text{vector}}_{\text{1-loop}}[\phi_0]. \tag{2.5}$$

Here the classical action can be computed by evaluating the on-shell action at the localization locus and 1-loop determinants can be computed by Gaussian integrals around it. It is remarkable that they are equivalent to the contributions from chiral/vector multiplets to the elliptic genera of 2d $\mathcal{N} = (0, 2)$ supersymmetric gauge theories [14, 15].

The result of supersymmetric localization (2.5) now contains an ordinary integral, which is much simpler than the original path integral in (2.2). It still has a subtle issue, however, that the 1-loop determinant from chiral multiplets $Z^{\text{chiral}}_{\text{1-loop}}[\phi_0]$ is a meromorphic function of $\phi_0$ with singularities and therefore the naive expression (2.5) does not converge. This issue has been treated carefully in [14, 15] for 2d elliptic genera and the same technique has been applied to the 4d case of our interest in [21, 66]. To be precise, the naive expression (2.5) should be written as a sum over a specific set of poles of $Z^{\text{chiral}}_{\text{1-loop}}[\phi_0]$, namely the sum of Jeffrey-Kirwan (JK) residues [78]. Then one can rewrite the sum of JK residues as appropriate contour integrals. Refer to [21, 66] for more details. The final integral representation of the topologically twisted index of $\mathcal{N} = 4$ SU($N$) SYM theory on $T^2 \times S^2$ (2.1) is given as a sum over gauge magnetic fluxes $\{\mathfrak{m}_i\}$ along with integrals over complex holonomies $u_i$'s, namely

[1]In fact, [21] and [66] have used slightly different deformation terms but the final results are consistent with each other as expected. See [66] for detailed explanation.

[73]

$$\mathcal{Z}(\tau,\Delta_a,\mathfrak{n}_a)=\frac{\mathcal{A}}{N!}\sum_{\{\mathfrak{m}_i\}\in\mathbb{Z}^N}^{\sum_i\mathfrak{m}_i=0}\int_{\mathcal{C}}\Big(\prod_{i=1}^{N-1}\frac{dz_i}{2\pi iz_i}\Big)\prod_{i,j=1\,(i\neq j)}^{N}\left[\frac{\theta_1(u_{ij};\tau)}{i\eta(\tau)}\prod_{a=1}^{3}\left(\frac{i\eta(\tau)}{\theta_1(u_{ij}+\Delta_a;\tau)}\right)^{\mathfrak{m}_{ij}-\mathfrak{n}_a+1}\right]. \tag{2.6}$$

Here $z_i=e^{2\pi iu_i}$, $u_{ij}\equiv u_i-u_j$, $\mathfrak{m}_{ij}\equiv\mathfrak{m}_i-\mathfrak{m}_j$, and we have defined the prefactor $\mathcal{A}$ as

$$\mathcal{A}=\eta(\tau)^{2(N-1)}\prod_{a=1}^{3}\left(\frac{i\eta(\tau)}{\theta_1(\Delta_a;\tau)}\right)^{(N-1)(1-\mathfrak{n}_a)}. \tag{2.7}$$

Special functions $\eta(\cdot)$ and $\theta_1(\cdot;\cdot)$ are defined in Appendix A. For the twisted index (2.6) to be well-defined, we impose the inequality $|q|<1$ or equivalently $0<\arg\tau<\pi$. The SU($N$) constraint is given as

$$\sum_{i=1}^{N}u_i\in\mathbb{Z}. \tag{2.8}$$

Chemical potentials $\Delta_a$ and flavor magnetic fluxes $\mathfrak{n}_a$ are constrained as

$$\sum_{a=1}^{3}\Delta_a\in\mathbb{Z},\qquad\sum_{a=1}^{3}\mathfrak{n}_a=2, \tag{2.9}$$

from the invariance of superpotential under flavor symmetries and supersymmetry respectively.

## 2.2 The Bethe Ansatz formula

In this section, we explain how to compute the integral representation of the topologically twisted index (2.6) using the Bethe Ansatz (BA) formula. First, using the trick introduced in [21, 16, 73], we can take the sum over gauge magnetic fluxes $\{\mathfrak{m}_i\}$ before evaluating the integral. The result is given as

$$\begin{aligned}\mathcal{Z}(\tau,\Delta_a,\mathfrak{n}_a)=\frac{\mathcal{A}}{N!}\int_{\mathcal{B}}\frac{dw}{2\pi iw}\int_{\mathcal{C}}\Big(\prod_{i=1}^{N-1}\frac{dz_i}{2\pi iz_i}\Big)\Big(\prod_{i=1}^{N}\frac{Q_i^M}{Q_i-1}\Big)&\\ \times\prod_{i,j=1\,(i\neq j)}^{N}\left[\frac{\theta_1(u_{ij};\tau)}{i\eta(\tau)}\prod_{a=1}^{3}\left(\frac{i\eta(\tau)}{\theta_1(u_{ij}+\Delta_a;\tau)}\right)^{1-\mathfrak{n}_a}\right]&\end{aligned} \tag{2.10}$$

with a large positive integer $M$ and the integration contour $\mathcal{B}$ that encloses the origin $w=0$. Here the Bethe Ansatz (BA) operator $Q_i$ is defined as

$$Q_i(\{u_j\};\tau,\Delta_a) = e^{2\pi i\lambda}\prod_{j=1}^{N}\prod_{a=1}^{3}\frac{\theta_1(u_{ji}+\Delta_a;\tau)}{\theta_1(u_{ij}+\Delta_a;\tau)} \tag{2.11}$$

where we have introduced $\lambda$ as $w=e^{2\pi i\lambda}$. Now we can evaluate the integral (2.10) by picking residues of the integrand determined by the Bethe Ansatz Equations (BAE), namely

$$Q_i(\{u_j\};\tau,\Delta_a) = 1. \tag{2.12}$$

The topologically twisted index (2.10) then reduces to a sum over BAE solutions as

$$\mathcal{Z}(\tau,\Delta_a,\mathfrak{n}_a) = \sum_{\{u_i\}\in\text{BAE}}\frac{\mathcal{A}}{\mathcal{H}(\{u_j\};\tau,\Delta_a)}\prod_{i,j=1\,(i\neq j)}^{N}\prod_{a=1}^{3}\left(\frac{\theta_1(u_{ij};\tau)}{\theta_1(u_{ij}+\Delta_a;\tau)}\right)^{1-\mathfrak{n}_a} \tag{2.13}$$

where we have used the constraint on the flavor magnetic fluxes (2.9) and introduced a Jacobian determinant

$$\mathcal{H}(\{u_j\};\tau,\Delta_a) = \det[H(\{u_j\};\tau,\Delta_a)] = \det\left[\frac{1}{2\pi i}\frac{\partial(Q_1,\cdots Q_N)}{\partial(u_1,\cdots,u_{N-1},\lambda)}\right]. \tag{2.14}$$

We call (2.13) the Bethe Ansatz (BA) formula of the toplogically twisted index of $\mathcal{N}=4$ SU($N$) SYM theory on $T^2\times S^2$. Note that the factor of $N!$ in the denominator of (2.10) is canceled by treating BAE solutions related by rearranging holonomies $\{u_i\}$ as equivalent.

The BA formula (2.13) is not completely rigorous, however, since it implicitly assumes that all the solutions of the BAE (2.12) are isolated. Only under this assumption, we can apply the Cauchy's integral formula to pick residues in (2.10) and derive the BA formula (2.13). If there are continuous family of solutions to the BAE (2.12), we should therefore modify the BA formula (2.13) accordingly. Schematically we may write it as

$$\begin{aligned}\mathcal{Z}(\tau,\Delta_a,\mathfrak{n}_a) = &\sum_{\{u_i\}\in\text{BAE}}\frac{\mathcal{A}}{\mathcal{H}(\{u_j\};\tau,\Delta_a)}\prod_{i,j=1\,(i\neq j)}^{N}\prod_{a=1}^{3}\left(\frac{\theta_1(u_{ij};\tau)}{\theta_1(u_{ij}+\Delta_a;\tau)}\right)^{1-\mathfrak{n}_a}\\ &+(\text{contribution from continuous family of BAE solutions}).\end{aligned} \tag{2.15}$$

Another issue is the contour $\mathcal{C}$ in (2.10), which is introduced to capture the JK residues. To compute the contribution from isolated BAE solutions correctly through the BA formula (2.15) derived from (2.10), one should sum over BAE solutions within the contour $\mathcal{C}$ only.

This has not yet been investigated thoroughly in the literature [73, 67]. We leave it for future research.

### 2.2.1 Solutions of the BAE

To compute the topologically twisted index through the BA formula (2.13) or the modified one (2.15), the first step is to solve the BAE (2.12) and find its most general solutions. Since the BAE is a system of highly involved non-linear algebraic equations, however, it is difficult to find the most general solutions. Hence an asymptotic solution of the BAE (2.12) in the Cardy-like limit, namely $|\tau| \to 0$ with fixed $0 < \arg \tau < \pi$, was found first in [73]:

$$u_i = \left( \frac{N+1}{2N} - \frac{i}{N} \right) \tau \quad (i = 1, \cdots, N), \qquad \lambda = \frac{N+1}{2}. \tag{2.16}$$

In [67], it has been observed that (2.16) in fact satisfies the BAE (2.12) for any $\tau$ in the upper half plane. Furthermore, a large class of BAE solutions that includes (2.16) as its special case was reported in the same reference. We study these BAE solutions in the following subsection 2.2.1 in detail. Later in [5], new BAE solutions including a continuous family have been found. We investigate these new BAE solutions and their implications in the next subsection 2.2.1.

#### Standard BAE solutions

The 'basic' BAE solution (2.16) is obtained by distributing $N$ holonomies $\{u_i\}$ along the thermal circle of the torus $T^2$ with the modular parameter $\tau$. This particular distribution of holonomies defines the torus $T^2/\mathbb{Z}_N$ with modular parameter $\tilde{\tau} = \tau/N$. Based on this observation and from the modular property of the twisted index that we will discuss in subsection 2.3.1, [67] proved that any set of holonomies $\{u_i\}$ evenly distributed over the torus satisfy the BAE (2.12). In this case the set of holonomies $\{u_i\}$ defines a freely acting orbifold $T^2/\mathbb{Z}_m \times \mathbb{Z}_n$ where $m$ takes all positive divisors of $N$ with $N = mn$. This BAE solution can be written explicitly as

$$u_{\hat{j}\hat{k}} = \bar{u} + \frac{\hat{j}}{m} + \frac{\hat{k}}{n}(\tau + \frac{r}{m}) = \bar{u} + \frac{\hat{j} + \hat{k}\tilde{\tau}}{m} \tag{2.17}$$

with $\tilde{\tau} = \frac{m\tau + r}{n}$, where we have introduced a double index notation ($u_N = u_0$)

$$u_{n\hat{j}+\hat{k}} \to u_{\hat{j}\hat{k}} \qquad (\hat{j} = 0, 1, \cdots, m-1, \ \hat{k} = 0, 1, \cdots, n-1). \tag{2.18}$$

The constant $\bar{u}$ in (2.17) is chosen to enforce the SU($N$) constraint (2.8). We call the BAE solutions (2.17) standard ones.

Here we prove that (2.17) truly satisfies the BAE (2.12) following [67]. Substituting (2.17) into (2.12), we can write the BAE explicitly as

$$e^{2\pi i\lambda} \stackrel{!}{=} \prod_{a=1}^{3}\prod_{\hat{j}=0}^{m-1}\prod_{\hat{k}=0}^{n-1}\frac{\theta_1(\Delta_a - \frac{(\hat{j}-\hat{j}_0)+(\hat{k}-\hat{k}_0)\tilde{\tau}}{m};\tau)}{\theta_1(\Delta_a + \frac{(\hat{j}-\hat{j}_0)+(\hat{k}-\hat{k}_0)\tilde{\tau}}{m};\tau)}. \tag{2.19}$$

To prove this equation, it suffices to show that the RHS of (2.19) is independent of the double index $(\hat{j}_0, \hat{k}_0)$. With this goal in mind, we derive the following identity using the double-periodicity of the elliptic theta function $\theta_1(u;\tau)$ (A.7b):

$$\begin{aligned}&\prod_{\hat{j}=-\hat{j}_0}^{m-\hat{j}_0-1}\prod_{\hat{k}=-\hat{k}_0}^{n-\hat{k}_0-1}\theta_1(\Delta_a \pm \frac{\hat{j}+\hat{k}\tilde{\tau}}{m};\tau)\\&= (-1)^{n\hat{j}_0+(r-1)\hat{k}_0}e^{\pm 2\pi i\hat{k}_0(m\Delta_a \pm \frac{2n-\hat{k}_0-1}{2}\tilde{\tau})}e^{-i\pi m\hat{k}_0\tau}\prod_{\hat{j}=0}^{m-1}\prod_{\hat{k}=0}^{n-1}\theta_1(\Delta_a \pm \frac{\hat{j}+\hat{k}\tilde{\tau}}{m};\tau).\end{aligned} \tag{2.20}$$

Substituting (2.20) into (2.19) and using the constraint $\sum_{a=1}^{3}\Delta_a \in \mathbb{Z}$ from (2.9) then gives

$$e^{2\pi i\lambda} \stackrel{!}{=} \prod_{a=1}^{3}\prod_{\hat{j}=0}^{m-1}\prod_{\hat{k}=0}^{n-1}\frac{\theta_1(\Delta_a - \frac{\hat{j}+\hat{k}\tilde{\tau}}{m};\tau)}{\theta_1(\Delta_a + \frac{\hat{j}+\hat{k}\tilde{\tau}}{m};\tau)}. \tag{2.21}$$

Now the RHS of (2.21) is manifestly independent of $(\hat{j}_0, \hat{k}_0)$, which demonstrates that the BAE reduces to a single algebraic equation that can be satisfied for some parameter $\lambda$. This proves that (2.17) truly satisfies the BAE (2.12).

While the proof has already been completed, we can determine the parameter $\lambda$ explicitly by simplifying the RHS of (2.21) further. First, choosing $(\hat{j}_0, \hat{k}_0) = (m-1, n-1)$ in (2.20) with the upper sign, we obtain

$$\prod_{\hat{j}=0}^{m-1}\prod_{\hat{k}=0}^{n-1}\frac{\theta_1(\Delta_a - \frac{\hat{j}+\hat{k}\tilde{\tau}}{m};\tau)}{\theta_1(\Delta_a + \frac{\hat{j}+\hat{k}\tilde{\tau}}{m};\tau)} = e^{2\pi i(\frac{N+1}{2}+(n-1)m\Delta_a)}. \tag{2.22}$$

Substituting this identity (2.22) back into (2.21) then gives

$$\lambda = \frac{N+1}{2}. \tag{2.23}$$

Note that this is consistent with the basic solution (2.16). The above calculation shows that this value of $\lambda$ (2.23) is in fact valid for all standard BAE solutions (2.17).

In summary, we have found multiple BAE solutions to (2.12), namely (2.17), denoted by three integers $\{m, n, r\}$ where $m, n$ are positive divisors of $N$ such that $N = mn$ and $r$ take values in $r \in \{0, 1, \cdots, n-1\}$. Recall that we call them standard BAE solutions.

### Non-standard BAE solutions

In the last subsection, we found a large set solutions of the BAE (2.12) dubbed as standard BAE solutions (2.17). It is important that they are not the most general solutions to the BAE (2.12). Since the BA formula of the topologically twisted index (2.15) requires a complete set of BAE solutions, we cannot compute the twisted index through the BA formula (2.15) relying solely on standard BAE solutions. In this subsection, following [5], we will therefore investigate other BAE solutions that we call non-standard ones. For simplicity, here we assume real chemical potentials $\Delta_a \in \mathbb{R}$. Then, since the BAE (2.12) is invariant under the integer shift $\Delta_a \to \Delta + \mathbb{Z}$ and the flip $(\lambda, \Delta_a) \to (-\lambda, -\Delta_a)$, we can set

$$0 < \Delta_a < 1, \qquad \sum_{a=1}^{3} \Delta_a = 1 \tag{2.24}$$

without loss of generality for real chemical potentials. Note that we exclude a pathological case with integer chemical potentials.

### $N = 2$ case

Since the BAE (2.12) is difficult to solve in general, we start with the simplest case with $N = 2$. For $N = 2$, the BAE (2.12) reduces to a single algebraic equation as

$$\pm 1 = e^{-2\pi i \lambda} = \prod_{a=1}^{3} \frac{\theta_1(\Delta_a + u_{21}; \tau)}{\theta_1(\Delta_a - u_{21}; \tau)}. \tag{2.25}$$

Due to the double periodicity (A.7b), for any given solution $u_{21}$ of the $N = 2$ BAE (2.25), $u_{21} + \mathbb{Z} + \mathbb{Z}\tau$ will also be BAE solutions. Then, since an elliptic function takes all complex values once within the fundamental domain, the $N = 2$ BAE (2.25) will have 3 solutions in the fundamental domain up to identification $u_{21} \sim u_{21} + \mathbb{Z} + \mathbb{Z}\tau$ for each choice of $e^{-2\pi i \lambda} = \pm 1$. To be explicit, we found [5]

$$e^{-2\pi i \lambda} = -1 \quad : \quad u_{21} \in \left\{ \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} \right\}, \tag{2.26a}$$

$$e^{-2\pi i \lambda} = 1 \quad : \quad u_{21} \in \{0, u_\Delta, -u_\Delta\}. \tag{2.26b}$$

The first three BAE solutions (2.26a) correspond to the standard solutions denoted by three integers $\{2,1,0\}$, $\{1,2,0\}$, and $\{1,2,1\}$ respectively, which we have already encountered in the previous subsection 2.2.1. Here we are interested in non-standard BAE solutions listed in (2.26b). The first one is a trivial solution and therefore we are interested in the last two BAE solutions in (2.26b), namely $u_{21} \in \{\pm u_\Delta\}$. The subscript "$\Delta$" represents that they are functions of chemical potentials $\Delta_a$ in general, which distinguishes these solutions from the standard ones. Even though the explicit form of $u_\Delta$ has not yet been known for a general $\tau$, their asymptotic behaviors have been studied in [5] and here we review them briefly.

In the 'low-temperature' limit ($|\tau| \to \infty$ with fixed $0 < \arg\tau < \pi$), we can approximate the $N = 2$ BAE (2.25) using the product form (A.3b) as

$$\pm 1 = e^{-2\pi i\lambda} = \prod_{a=1}^{3} \frac{1 - e^{2\pi i(u_{21}-\Delta_a)}}{1 - e^{2\pi i(u_{21}+\Delta_a)}} \frac{1 - e^{2\pi i(\tau-u_{21}+\Delta_a)}}{1 - e^{2\pi i(\tau-u_{21}-\Delta_a)}} \left(1 + \mathcal{O}(e^{-2\pi|\tau|\sin(\arg\tau)})\right), \tag{2.27}$$

where we locate $u_{21}$ within the fundamental domain without loss of generality. Solving (2.27) for $e^{-2\pi i\lambda} = 1$ then gives a trivial solution and the low-temperature asymptotic forms of the other two non-standard solutions $u_{21} \in \{\pm u_\Delta\}$, namely

$$\lim_{|\tau|\to\infty} u_\Delta = \frac{1}{2\pi i} \log\left[-\frac{1-\sum_a \cos 2\pi\Delta_a}{2} + \sqrt{\left(\frac{1-\sum_a \cos 2\pi\Delta_a}{2}\right)^2 - 1}\right]. \tag{2.28}$$

In the Cardy-like limit ($|\tau| \to 0$ with fixed $0 < \arg\tau < \pi$), we can approximate the $N = 2$ BAE (2.25) using the asymptotic form (A.23) as

$$\begin{aligned}\pm 1 = e^{-2\pi i\lambda} = \prod_{a=1}^{3} & e^{\frac{\pi i}{\tau}[\{\Delta_a+u_{21}\}_\tau(1-\{\Delta_a+u_{21}\}_\tau)-\{\Delta_a-u_{21}\}_\tau(1-\{\Delta_a-u_{21}\}_\tau)]+\pi i(\lfloor\Delta_a+\tilde{u}_{21}\rfloor-\lfloor\Delta_a-\tilde{u}_{21}\rfloor)} \\ & \times \frac{(1-e^{-\frac{2\pi i}{\tau}(1-\{\Delta_a+u_{21}\}_\tau)})(1-e^{-\frac{2\pi i}{\tau}\{\Delta_a+u_{21}\}_\tau})}{(1-e^{-\frac{2\pi i}{\tau}(1-\{\Delta_a-u_{21}\}_\tau)})(1-e^{-\frac{2\pi i}{\tau}\{\Delta_a-u_{21}\}_\tau})} \left(1 + \mathcal{O}(e^{-\frac{2\pi\sin(\arg\tau)}{|\tau|}})\right).\end{aligned} \tag{2.29}$$

Solving (2.29) for $e^{-2\pi i\lambda} = 1$ then gives a trivial solution and the Cardy-like asymptotic forms of the other two non-standard solutions $u_{21} \in \{\pm u_\Delta\}$. First, when $\max[\Delta_a] < \frac{1}{2}$, we found

$$\lim_{|\tau|\to 0} u_\Delta = \frac{1}{2} + \frac{\tau}{4} \qquad (\max[\Delta_a] < \frac{1}{2}). \tag{2.30}$$

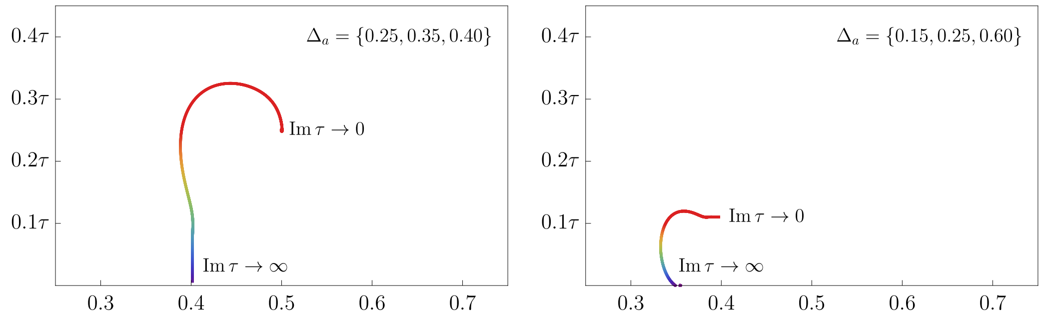


Figure 2.1: Numerical plots of the non-standard $N = 2$ solution $u_{21} = u_\Delta$ with $\arg\tau = \pi/4$. The figure on the left corresponds to the case with $\max[\Delta_a] < 1/2$ while that on the right corresponds to the case with $\max[\Delta_a] \geq 1/2$. Note that the vertical axis is given in units of $\tau$.

Otherwise, namely when $\max[\Delta_a] \geq \frac{1}{2}$, we found

$$\lim_{|\tau|\to 0} u_\Delta = 1 - \max[\Delta_a] + i\frac{\log 2}{2\pi}\tau \qquad (\max[\Delta_a] \geq \frac{1}{2}). \tag{2.31}$$

In [5], we have also found numerical BAE solutions that connect the low-temperature asymptotic solutions (2.28) to the Cardy-like asymptotic ones (2.30) and (2.31) precisely. See Figure 2.1. This confirms the existence of non-standard BAE solutions with a generic $\tau$.

### $N = 3$ case

For $N = 3$, the BAE (2.12) reduces to a couple of algebraic equations as

$$\begin{aligned}\{1, w, w^2\} \ni e^{-2\pi i\lambda} &= \prod_{a=1}^{3} \frac{\theta_1(\Delta_a + u_{21};\tau)\theta_1(\Delta_a + u_{31};\tau)}{\theta_1(\Delta_a - u_{21};\tau)\theta_1(\Delta_a - u_{31};\tau)} \\ &= \prod_{a=1}^{3} \frac{\theta_1(\Delta_a - u_{21};\tau)\theta_1(\Delta_a + u_{31} - u_{21};\tau)}{\theta_1(\Delta_a + u_{21};\tau)\theta_1(\Delta_a - u_{31} + u_{21};\tau)},\end{aligned} \tag{2.32}$$

where $w = e^{2\pi i/3}$ is a primitive cube root of unity. Even though it is the next simplest case, the $N = 3$ BAE (2.32) is much more complicated to solve compared to the $N = 2$ one (2.25) because there are more than one holonomy pairs, namely $(u_{21}, u_{31})$, we should solve for. In particular, this makes it difficult to classify the most general solutions to the $N = 3$ BAE (2.32) within the fundamental domain.

Even though a full classification of solutions to the $N = 3$ BAE (2.32) is highly involved and has not yet been done, [5] did find some non-standard solutions to the $N = 3$ BAE (2.32).

As we did in the $N = 2$ case, we investigated BAE solutions in both asymptotic regions first and confirmed that they are at least numerically connected for a generic $\tau$. It is particularly interesting that it is a complex 1-dimensional continuous family of BAE solutions. To be specific, the $N = 3$ BAE (2.32) imposes only one constraint on the holonomy pair $(u_{21}, u_{31})$ in this case and therefore the resulting BAE solution has a complex 1-dimensional degree of freedom. Its asymptotic forms can be found in [5] and here we focus on the exact solution within this continuous family of solution, namely

$$(u_{21}, u_{31}) = \left(\frac{1}{2}, \frac{\tau}{2}\right). \tag{2.33}$$

It is straightforward to check that (2.33) indeed satsifies the $N = 3$ BAE (2.32) using the double periodicity (A.7b) and the inversion (A.8b).

The BAE solution (2.33) is independent of chemical potentials $\Delta_a$, but distinguished from standard BAE solutions (2.17) in that it is a part of the continuous family and not isolated. Here we prove it by showing that the Jacobian matrix $H(\{u_j\}; \tau, \Delta_a)$ in (2.14) has at least one zero eigenvalue at the BAE solution (2.33). To begin with, applying the relation (2.48), which we will derive in the next section, to the present $N = 3$ case, we find

$$(2\pi i)^3 \mathcal{H}(\{u_j\}; \tau, \Delta_a)\big|_{(u_{21},u_{31})=(\frac{1}{2},\frac{\tau}{2})} = 36 \sum_{I=2}^{4} \frac{\prod_{J=2}^{4} \sum_{a=1}^{3} \theta_J'(\Delta_a;\tau)/\theta_J(\Delta_a;\tau)}{\sum_{a=1}^{3} \theta_I'(\Delta_a;\tau)/\theta_I(\Delta_a;\tau)}. \tag{2.34}$$

Here $\theta_{2,3,4}$ are the Jacobi theta functions given explicitly in (B.3). Next we prove the following lemma in Appendix B.1:

**Lemma 1.** *For any $\tau$ in the upper half plane and any complex $\Delta_a$ subject to the constraint $\sum_{a=1}^{3} \Delta_a \in \mathbb{Z}$, we have*

$$\sum_{I=2}^{4} \frac{1}{\sum_{a=1}^{3} \theta_I'(\Delta_a;\tau)/\theta_I(\Delta_a;\tau)} = 0. \tag{2.35}$$

Finally, substituting Lemma 1 into (2.34), we have

$$\mathcal{H}(\{u_j\}; \tau, \Delta_a)\big|_{(u_{21},u_{31})=(\frac{1}{2},\frac{\tau}{2})} = 0. \tag{2.36}$$

This establishes that the Jacobian matrix $H(\{u_j\}; \tau, \Delta_a)$ in (2.14) indeed has a vanishing eigenvalue at the BAE solution (2.33) and thereby proves that (2.33) is within a continuous family of BAE solutions.

### $N > 3$ case

Investigating continuous family of solutions to the BAE (2.12) for $N > 3$ is highly involved and it has not yet been studied analytically so far. The authors of [5] observed an interesting pattern, however, in continuous families of BAE solutions for $N = 4, 5, 6, 7, 8, 9, 10$ through numerical analysis. First, we found numerical BAE solutions where the Jacobian matrix $\mathcal{H}(\{u_j\}; \tau, \Delta_a)$ (2.14) has a single vanishing eigenvalue for $N = 4, 5$. Then for $N = 6, 7, 8, 9$, we found new BAE solutions where the Jacobian matrix has two zero eigenvalues. Lastly for $N = 10$, numerical analysis shows that there is a BAE solution where the Jacobian matrix has three zero eigenvalues. This implies that the maximum complex dimension of a continuous family of BAE solutions increases as $N$ increases with a certain pattern. The exact pattern in the dimensionality of the space of continuous family of BAE solutions has been conjectured in [5] as

**Conjecture 1.** *For $N \geq \frac{(l+1)(l+2)}{2}$, the BAE of $\mathcal{N} = 4$ SU(N) SYM theory (2.12) has l-complex dimensional continua of solutions at most.*

This conjecture is based on a duality between the space of BAE solutions of $\mathcal{N} = 4$ SU($N$) SYM theory and the vacua of $\mathcal{N} = 1^*$ theory proposed in [5]. To be specific, the authors of [5] related the standard BAE solutions (2.17) to the massive vacua of the $\mathcal{N} = 1^*$ theory studied in [50] and the continuous family of non-standard BAE solutions to the Coulomb vacua of the $\mathcal{N} = 1^*$ theory studied in [51]. Then, based on the analysis of the Coulomb vacua of the $\mathcal{N} = 1^*$ theory with SU($N$) gauge group in [99], we have conjectured the maximum dimension of continuous family of BAE solutions as 1. As we have confirmed above numerically, the conjecture 1 works well at least for small rank of the gauge group $N = 2, 3, \cdots, 10$. Furthermore, the exact non-standard BAE solution within a continuous family for the SU(3) case (2.33) turns out to have its explicit counterpart in dual $\mathcal{N} = 1^*$ theory [51]. Both observations strongly support the proposed duality [5] even though the exact dictionary has not yet been established.

Even though a full classification of non-standard BAE solutions beyond the conjecture 1 has not yet been done, there are particularly interesting class of such solutions. Note that the structure of standard BAE solutions (2.17) depends heavily on the factorization of the rank of a gauge group $N$. For example, if $N$ is a prime number, there are only $N+1$ number of standard BAE solutions denoted by $\{N, 1, 0\}$ and $\{1, N, r\}$ ($r = 0, \cdots, N-1$). On the contrary, if $N$ has many positive divisors, there are much more standard BAE solutions denoted by $\{m, n, r\}$. This brings the following question. Consider a large, odd number $N$ for example. Then for the case with the gauge group SU($N-1$), the BAE (2.12) has a standard solution (2.17) denoted by $\{2, \frac{N-1}{2}, 0\}$. But if you slightly change the rank of the gauge group by $N-1 \to N$, a similar standard solution like $\{2, \frac{N}{2}, 0\}$ does not exist anymore since 2 does not divide $N$. Since the change $N-1 \to N$ should be ignorable in the large-$N$ limit, such a drastic outcome must be explained somehow. This naturally suggests that there is a non-standard BAE solution for an odd $N$, which looks similar to the $\{2, \frac{N-1}{2}, 0\}$ standard BAE solution.

We did find such a non-standard solution, analytically in the Cardy-like limit and numerically for a generic $\tau$. First we discuss the asymptotic analysis following [5]. In the Cardy-like limit, the BAE (2.12) reduces to

$$\begin{aligned} e^{-2\pi i\lambda} = \prod_{a=1}^{3}\prod_{j=1}^{N} & e^{\frac{\pi i}{\tau}[\{\Delta_a+u_{ji}\}_\tau(1-\{\Delta_a+u_{ji}\}_\tau)-\{\Delta_a-u_{ji}\}_\tau(1-\{\Delta_a-u_{ji}\}_\tau)]+\pi i(\lfloor\Delta_a+\tilde{u}_{ji}\rfloor-\lfloor\Delta_a-\tilde{u}_{ji}\rfloor)} \\ & \times \frac{(1-e^{-\frac{2\pi i}{\tau}(1-\{\Delta_a+u_{ji}\}_\tau)})(1-e^{-\frac{2\pi i}{\tau}\{\Delta_a+u_{ji}\}_\tau})}{(1-e^{-\frac{2\pi i}{\tau}(1-\{\Delta_a-u_{ji}\}_\tau)})(1-e^{-\frac{2\pi i}{\tau}\{\Delta_a-u_{ji}\}_\tau})}\Big(1+\mathcal{O}(e^{-\frac{2\pi\sin(\arg\tau)}{|\tau|}})\Big) \end{aligned} \tag{2.37}$$

from the asymptotic expansion (A.23). Then we find asymptotic solutions to (2.37) for the two different cases, namely $\max[\Delta_a] \geq \frac{1}{2}$ and $\max[\Delta_a] < \frac{1}{2}$. They are given explicitly as

**CASE 1.** $\max[\Delta_a] \geq \frac{1}{2}$

$$\{u_j\} = \left\{\frac{j}{(N+1)/2}\tau \,\Big|\, j=0,\cdots,\frac{N-1}{2}\right\} \cup \left\{\frac{1}{2}+\frac{j-1/2}{(N-1)/2}\tau \,\Big|\, j=1,\cdots,\frac{N-1}{2}\right\}. \tag{2.38}$$

**CASE 2.** $\max[\Delta_a] < \frac{1}{2}$

$$\begin{aligned} \{u_j\} &= \left\{\pm\frac{\min[\Delta_a]-\epsilon}{2}\right\} \cup \left\{\frac{j}{(N-3)/2}\tau \,\Big|\, j=0,\cdots,\frac{N-5}{2}\right\} \\ &\quad \cup \left\{\frac{1}{2}+\frac{j-1/2}{(N-1)/2}\tau \,\Big|\, j=1,\cdots,\frac{N-1}{2}\right\}, \\ \epsilon &= \frac{\tau}{2\pi i}\exp\left[-\frac{\pi i}{\tau}\left(-\frac{N-1}{2}-\min[\Delta_a]+(N-1)(1-\max[\Delta_a])\right)\right]. \end{aligned} \tag{2.39}$$

Here, for notational convenience, we omitted a universal additive constant $\bar{u}$ that should be added to all holonomies $u_j$'s to satisfy the SU($N$) constraint $\sum_{i=1}^{N} u_i \in \mathbb{Z}$ (2.8) as in (2.17). Although (2.38) and (2.39) are BAE solutions valid only in the Cardy-like limit, we confirmed numerically that there are indeed exact BAE solutions corresponding to them. Furthermore, we checked that they can be continuously deformed to satisfy the BAE (2.12) for a generic $\tau$. See Figure 2.2.

While we have focused on non-standard solutions whose holonomies are packed into two groups, numerical investigations confirm that similar non-standard solutions whose holonomies are divided into more than two nearly equal packs do exist, at least for $N$

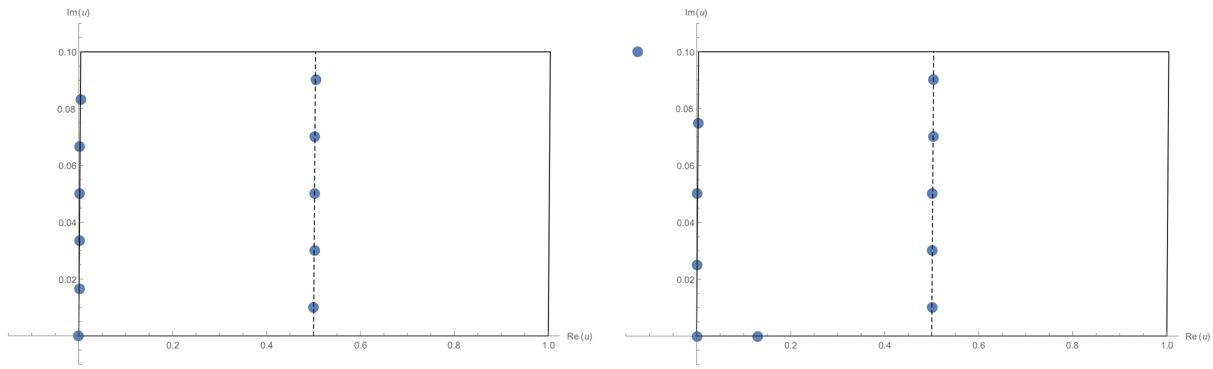


Figure 2.2: Numerical solutions to the BAE (2.12) with $N = 11$ and $\tau = \frac{1+23i}{230}$. Here (a) corresponds (2.38) with $(\Delta_1, \Delta_2) = (\frac{105}{517}, \frac{75}{287})$ and (b) corresponds to (2.39) with $(\Delta_1, \Delta_2) = (\frac{75}{287}, \frac{152}{517})$.

sufficiently large. These non-standard solutions are similar to the above discussed solutions (2.38) and (2.39) in that their explicit forms are sensitive to the configuration of chemical potentials $\Delta_a$. The simplest asymptotic BAE solution, corresponding to CASE 1 above, occurrs only when $\max[\Delta_a] \geq 1 - 1/C$ where $C$ denotes the number of packs that we divide holonomies into. In this case, the non-standard solution in the Cardy-like limit is given by

$$\begin{aligned}\{u_j\} = &\left\{\frac{J}{C} + \frac{j}{(N+C-D)/C}\tau \,\middle|\, J = 0, \cdots, D-1,\ j = 0, \cdots, \frac{N-D}{C}\right\} \\ &\cup \left\{\frac{J}{C} + \frac{j-1/2}{(N-D)/2}\tau \,\middle|\, J = D, \cdots, C-1,\ j = 1, \cdots, \frac{N-D}{C}\right\},\end{aligned} \tag{2.40}$$

where $N = C\lfloor N/C \rfloor + D$ $(D = 1, \cdots, C-1)$. This solution (2.40) satisfies the BAE in the Cardy-like limit (2.37) up to exponentially suppressed terms.

Although this extension of the CASE 1 solution (2.38) to arbitrary values of $C$ only holds for sufficiently large $\max[\Delta_a]$, we expect that generalizations of the CASE 2 solution (2.38), whose pairs of holonomies may be pulled away from the main packs, exist for other values of the chemical potentials. We thus conjecture that solutions to the BAE (2.12) exist for all values of $C$ and $N$ with $d \leq N/C < d+1$. Here, $d$ corresponds to the minimum number of holonomies in a single pack that allows the solution to be categorized as a set of packs instead of individually distributed holonomies. When $C$ divides $N$, the solution is standard but otherwise it is non-standard.

### 2.2.2 The topologically twisted index from the BA formula

In the previous subsection 2.2.1, we investigate various solutions to the BAE (2.12). Standard BAE solutions (2.17) were obtained explicitly and we have also found various non-standard

BAE solutions, which simply denote all the BAE solutions except standard ones. Non-standard BAE solutions, however, have not yet been classified completely. Hence, even though the existence of non-standard BAE solutions is clear from various examples, it is still difficult to compute the topologically twisted index through the (modified) BA formula (2.15).

There is a part that we can compute explicitly, however, namely the standard contribution from the standard BAE solutions (2.17) to the twisted index through the BA formula (2.15). To be more explicit, first we rewrite the BA formula (2.15) as

$$\mathcal{Z}(\tau,\Delta_a,\mathfrak{n}_a)=\mathcal{Z}_{\text{standard}}(\tau,\Delta_a,\mathfrak{n}_a)+\mathcal{Z}_{\text{non-standard}}(\tau,\Delta_a,\mathfrak{n}_a), \tag{2.41}$$

where we have defined the standard contribution $\mathcal{Z}_{\text{standard}}$ as

$$\begin{aligned}\mathcal{Z}_{\text{standard}}(\tau,\Delta_a,\mathfrak{n}_a)&=\sum_{n=1\,(n|N)}^{N}\sum_{r=0}^{n-1}Z_{\{m,n,r\}}(\tau,\Delta_a,\mathfrak{n}_a),\\ Z_{\{m,n,r\}}(\tau,\Delta_a,\mathfrak{n}_a)&=\frac{\mathcal{A}}{\mathcal{H}_{\{m,n,r\}}}\prod_{a=1}^{3}\prod_{(\hat{j}_1,\hat{k}_1)\neq(\hat{j}_2,\hat{k}_2)}\left(\frac{\theta_1(\frac{(\hat{j}_1-\hat{j}_2)+(\hat{k}_1-\hat{k}_2)\tilde{\tau}}{m};\tau)}{\theta_1(\Delta_a+\frac{(\hat{j}_1-\hat{j}_2)+(\hat{k}_1-\hat{k}_2)\tilde{\tau}}{m};\tau)}\right)^{1-\mathfrak{n}_a}\\ &=\frac{\mathcal{A}}{\mathcal{H}_{\{m,n,r\}}}\prod_{a=1}^{3}\prod_{\hat{j}_2=0}^{m-1}\prod_{\hat{k}_2=0}^{n-1}\prod_{\hat{j}_1=-\hat{j}_2}^{m-\hat{j}_2-1}{}'\prod_{\hat{k}_1=-\hat{k}_2}^{n-\hat{k}_2-1}{}'\left(\frac{\theta_1(\frac{\hat{j}_1+\hat{k}_1\tilde{\tau}}{m};\tau)}{\theta_1(\Delta_a+\frac{\hat{j}_1+\hat{k}_1\tilde{\tau}}{m};\tau)}\right)^{1-\mathfrak{n}_a}.\end{aligned} \tag{2.42}$$

Here $\mathcal{H}_{\{m,n,r\}}(\tau,\Delta_a)$ denotes the Jacobian determinant $\mathcal{H}(\{u_i\};\tau,\Delta_a)$ (2.14) evaluated at the standard BAE solution (2.17). The primed sums in (2.42) indicate that $(\hat{j}_1,\hat{k}_1)=(0,0)$ is to be omitted from the double product. It is striaghtforward to see that the standard contribution $\mathcal{Z}_{\text{standard}}$ (2.42) is obtained by summing over all contributions from the standard BAE solutions (2.17) to the twisted index through the BA formula (2.15). The non-standard contribution $\mathcal{Z}_{\text{non-standard}}$ is the remaining contribution which has not yet been known explicitly.

Here we focus on simplifying the standard contribution $\mathcal{Z}_{\text{standard}}$ (2.42). To begin with, we shift the product over $\hat{j}_1,\hat{k}_1$ in (2.42) using (2.20) as

$$\prod_{\hat{j}_1=-\hat{j}_2}^{m-\hat{j}_2-1}{}'\prod_{\hat{k}_1=-\hat{k}_2}^{n-\hat{k}_2-1}{}'\frac{\theta_1(\frac{\hat{j}_1+\hat{k}_1\tilde{\tau}}{m};\tau)}{\theta_1(\Delta_a+\frac{\hat{j}_1+\hat{k}_1\tilde{\tau}}{m};\tau)}=e^{-2\pi i m\hat{k}_2\Delta_a}\prod_{\hat{j}_1=0}^{m-1}{}'\prod_{\hat{k}_1=0}^{n-1}{}'\frac{\theta_1(\frac{\hat{j}_1+\hat{k}_1\tilde{\tau}}{m};\tau)}{\theta_1(\Delta_a+\frac{\hat{j}_1+\hat{k}_1\tilde{\tau}}{m};\tau)}. \tag{2.43}$$

Then $Z_{\{m,n,r\}}$ in (2.42) reads

$$Z_{\{m,n,r\}} = \frac{\mathcal{A}}{\mathcal{H}_{\{m,n,r\}}} \prod_{a=1}^{3} \left[ e^{-\pi i m(n-1)\Delta_a} \prod_{\hat{j}_1=0}^{m-1}{}' \prod_{\hat{k}_1=0}^{n-1}{}' \frac{\theta_1(\frac{\hat{j}_1+\hat{k}_1\tilde{\tau}}{m};\tau)}{\theta_1(\Delta_a + \frac{\hat{j}_1+\hat{k}_1\tilde{\tau}}{m};\tau)} \right]^{N(1-\mathfrak{n}_a)} . \tag{2.44}$$

Substituting the identity (A.4) and the prefactor (2.7) into (2.44) then gives

$$Z_{\{m,n,r\}} = \frac{i^{N-1}}{\mathcal{H}_{\{m,n,r\}}} \prod_{a=1}^{3} \left[ \left( \frac{\theta_1(\Delta_a;\tau)}{\eta(\tau)^3} \right) \left( \frac{m\eta(\tilde{\tau})^3}{\theta_1(m\Delta_a;\tilde{\tau})} \right)^N \right]^{1-\mathfrak{n}_a} . \tag{2.45}$$

To simplify the Jacobian determinant $\mathcal{H}_{\{m,n,r\}}$ in (2.45), first note that the element of the $N \times N$ Jacobian matrix $H$ is given from (2.14) as

$$H_{\mu,\nu} = \frac{1}{2\pi i}\frac{\partial Q_\mu}{\partial u_\nu} = \delta_{\mu\nu} \sum_{j=1}^{N} g(u_{\mu j};\tau,\Delta_a) - g(u_{\mu\nu};\tau,\Delta_a) + g(u_{\mu N};\tau,\Delta_a), \tag{2.46a}$$

$$H_{N,\nu} = \frac{1}{2\pi i}\frac{\partial Q_N}{\partial u_\nu} = -\sum_{j=1}^{N} g(u_{Nj};\tau,\Delta_a) - g(u_{N\nu};\tau,\Delta_a) + g(0;\tau,\Delta_a), \tag{2.46b}$$

$$H_{\mu,N} = \frac{1}{2\pi i}\frac{\partial Q_\mu}{\partial \lambda} = 1, \tag{2.46c}$$

$$H_{N,N} = \frac{1}{2\pi i}\frac{\partial Q_N}{\partial \lambda} = 1, \tag{2.46d}$$

where $\mu,\nu \in \{1,2,\ldots,N-1\}$ and we have defined

$$g(u;\tau,\Delta_a) \equiv \frac{i}{2\pi} \sum_{a=1}^{3} \frac{\partial}{\partial \Delta_a} \log\big[\theta_1(\Delta_a + u;\tau)\theta_1(\Delta_a - u;\tau)\big]. \tag{2.47}$$

From (2.46), we can derive

$$\mathcal{H} = \det H = N \det\left[\frac{1}{2\pi i}\frac{\partial(Q_1,\cdots,Q_{N-1})}{\partial(u_1,\cdots,u_{N-1})}\right]. \tag{2.48}$$

Hence, to determine the Jacobian determinant $\mathcal{H}$ or our interest, it suffices to study the determinant of the $(N-1)\times(N-1)$ square matrix whose elements are given as (2.46a).

Since we are particularly interested in $\mathcal{H}_{\{m,n,r\}}$, namely the Jacobian determinant (2.14) of a standard BAE solution (2.17), we turn to a double index notation as (2.18). We also introduce the $\mathcal{G}$-function as

$$\mathcal{G}_{\{m,n,r\}}(\hat{j},\hat{k};\tau,\Delta_a) \equiv \frac{i}{2\pi}\sum_{a=1}^{3}\frac{\partial}{\partial\Delta_a}\log\left[\theta_1(\Delta_a+\frac{\hat{j}+\hat{k}\tilde{\tau}}{m};\tau)\theta_1(\Delta_a-\frac{\hat{j}+\hat{k}\tilde{\tau}}{m};\tau)\right], \tag{2.49}$$

which is related to the $g$-function (2.47) as

$$\mathcal{G}_{\{m,n,r\}}(\hat{j}-\hat{j}_0,\hat{k}-\hat{k}_0;\tau,\Delta_a) = g(u_{\hat{j}\hat{k}}-u_{\hat{j}_0\hat{k}_0};\tau,\Delta_a). \tag{2.50}$$

Then the matrix element (2.46a) can be rewritten in terms of a double index notation as

$$\begin{aligned}[H_{\{m,n,r\}}]_{\mu,\nu} = \delta_{\mu\nu}\sum_{\hat{j}=0}^{m-1}\sum_{\hat{k}=0}^{n-1}\mathcal{G}_{\{m,n,r\}}(\hat{j}-\hat{j}_\mu,\hat{k}-\hat{k}_\mu;\tau,\Delta_a)+\mathcal{G}_{\{m,n,r\}}(\hat{j}_\mu,\hat{k}_\nu;\tau,\Delta_a)\\ -\mathcal{G}_{\{m,n,r\}}(\hat{j}_\mu-\hat{j}_\nu,\hat{k}_\mu-\hat{k}_\nu;\tau,\Delta_a)\end{aligned} \tag{2.51}$$

where $(\mu,\nu)=(n\hat{j}_\mu+\hat{k}_\mu, n\hat{j}_\nu+\hat{k}_\nu)$ following (2.18). The sum in (2.51) can be simplified as

$$\begin{aligned}\sum_{\hat{j}=0}^{m-1}\sum_{\hat{k}=0}^{n-1}\mathcal{G}_{\{m,n,r\}}(\hat{j}-\hat{j}_\mu,\hat{k}-\hat{k}_\mu;\tau,\Delta_a) &= \sum_{\hat{j}=0}^{m-1}\sum_{\hat{k}=0}^{n-1}\mathcal{G}_{\{m,n,r\}}(\hat{j},\hat{k};\tau,\Delta_a)\\ &= \frac{i}{\pi}\sum_{a=1}^{3}\frac{\partial}{\partial\Delta_a}\log\theta_1(m\Delta_a;\tilde{\tau}),\end{aligned} \tag{2.52}$$

where we have used (2.20) and (A.4) in the first and in the second equations respectively. Substituting (2.52) back into (2.51) then gives

$$\begin{aligned}[H_{\{m,n,r\}}]_{\mu,\nu} &= \left(\frac{i}{\pi}\sum_{a=1}^{3}\frac{\partial}{\partial\Delta_a}\log\theta_1(m\Delta_a;\tilde{\tau})\right)\left[I_{N-1}+\widetilde{H}_{\{m,n,r\}}\right]_{\mu,\nu},\\ [\widetilde{H}_{\{m,n,r\}}]_{\mu,\nu} &\equiv \frac{\mathcal{G}_{\{m,n,r\}}(\hat{j}_\mu,\hat{k}_\mu;\tau,\Delta_a)-\mathcal{G}_{\{m,n,r\}}(\hat{j}_\mu-\hat{j}_\nu,\hat{k}_\mu-\hat{k}_\nu;\tau,\Delta_a)}{\frac{i}{\pi}\sum_{a=1}^{3}\frac{\partial}{\partial\Delta_a}\log\theta_1(m\Delta_a;\tilde{\tau})}.\end{aligned} \tag{2.53}$$

The Jacobian determinant $\mathcal{H}_{\{m,n,r\}}$ is then given from (2.48) and (2.53) as

$$\mathcal{H}_{\{m,n,r\}} = \det H_{\{m,n,r\}} = N\left(\frac{i}{\pi}\sum_{a=1}^{3}\frac{\partial}{\partial\Delta_a}\log\theta_1(m\Delta_a;\tilde{\tau})\right)^{N-1}\det\Big(1+\widetilde{H}_{\{m,n,r\}}\Big). \tag{2.54}$$

Finally, the contribution from the standard BAE solution (2.17) denoted by three integers

$\{m,n,r\}$ is given by substituting (2.54) into (2.45) as

$$Z_{\{m,n,r\}}(\tau,\Delta_a,\mathfrak{n}_a)=\frac{\prod_{a=1}^{3}\left[\left(\frac{\theta_1(\Delta_a;\tau)}{\eta(\tau)^3}\right)\left(\frac{m\eta(\tilde{\tau})^3}{\theta_1(m\Delta_a;\tilde{\tau})}\right)^N\right]^{1-\mathfrak{n}_a}}{N\det\Big(1+\widetilde{H}_{\{m,n,r\}}\Big)\left[\frac{1}{\pi}\sum_{a=1}^{3}\frac{\partial}{\partial\Delta_a}\log\theta_1\left(m\Delta_a;\tilde{\tau}\right)\right]^{N-1}}. \tag{2.55}$$

The standard contribution $\mathcal{Z}_{\text{standard}}$ then naturally follows from (2.42).

## 2.3 Standard contribution to the topologically twisted index

In this section we investigate the standard contribution $\mathcal{Z}_{\text{standard}}$ to the topologically twisted index based on (2.42) and (2.55). Even though this is not enough to get the full topologically twisted index due to the non-standard contribution $\mathcal{Z}_{\text{non-standard}}$ in (2.41), we will see that the standard contribution itself has an interesting transformation property and an asymptotic behavior in the Cardy-like limit.

### 2.3.1 The index as an elliptic genus

In [43, 66], it has been observed that the partition function of a theory on $T^2\times S^2$ with $\mathcal{N}=1$ chiral/vector multiplets is equivalent to an elliptic genus of certain $\mathcal{N}=(0,2)$ theory on $T^2$. Since the topologically twisted index of $\mathcal{N}=4$ SU($N$) SYM theory means the partition function of a supersymmetric gauge theory on $T^2\times S^2$ that consists of three $\mathcal{N}=1$ chiral multiplets and one $\mathcal{N}=1$ vector multiplet, we should be able to interpret the twisted index of our interest as an elliptic genus too. A direct evidence for this would be that the topologically twisted index of $\mathcal{N}=4$ SU($N$) SYM theory has the same periodicity and modular property with an elliptic genus. To be specific, to show that the twisted index becomes an elliptic genus, we must prove that the twisted index transforms as a weak Jacobi form of weight zero as an elliptic genus does. Here it is worth recalling that, for a single chemical potential, a Jacobi form of weight $k$ and index $m$ transforms according to

$$\phi(u+\lambda\tau+\mu;\tau)=(-1)^{2m(\lambda+\mu)}q^{-m\lambda^2}e^{-4\pi im\lambda u}\phi(\tau,u), \tag{2.56a}$$

$$\phi\left(\frac{u}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^k e^{\frac{2\pi imcu^2}{c\tau+d}}\phi(\tau,u), \tag{2.56b}$$

for $a,b,c,d,\lambda,\mu\in\mathbb{Z}$ with $ad-bc=1$. It is then straightforward to generalize this to the case of three chemical potentials $\Delta_a$ ($a=1,2,3$) of our interest.

We do not have an explicit expression for the full twisted index, however, since the non-standard contribution $\mathcal{Z}_{\text{non-standard}}$ in (2.41) is unknown. Hence it is difficult to show that the full twisted index $\mathcal{Z} = \mathcal{Z}_{\text{standard}} + \mathcal{Z}_{\text{non-standard}}$ transforms as a weak Jacobi form of weight zero. We therefore focus on the standard contribution $\mathcal{Z}_{\text{standard}}$ and show that it transforms as an elliptic genus. More explicitly, we verify that the standard contribution $\mathcal{Z}_{\text{standard}}$ from (2.42) and (2.55) transforms as a weak Jacobi form of weight zero and indices

$$m_a = -\frac{N^2-1}{2}(1-\mathfrak{n}_a), \tag{2.57}$$

under the constraint $\sum_a \Delta_a = 0$. To do so, in subsection 2.3.1, we first consider the periodic shifts $\Delta_a \to \Delta_a + 1$ and $\Delta_a \to \Delta_a + \tau$ to confirm (2.56a). Then in the next subsection 2.3.1, we consider the modular transformations $T : \tau \to \tau + 1$ and $S : \tau \to -1/\tau$ to confirm (2.56b). Note that the index $m_a$ is a half-integer when both $N$ and $\mathfrak{n}_a$ are even, and an integer otherwise.

**Periodic shifts of chemical potentials**

We first consider the shift $\Delta_{\hat{a}} \to \Delta_{\hat{a}} + 1$ for a single $\Delta_{\hat{a}}$. From (A.7b), we find that $Z_{\{m,n,r\}}$ (2.45) transforms under this shift as

$$\begin{aligned} Z_{\{m,n,r\}} \to (-1)^{(1-mN)(1-\mathfrak{n}_{\hat{a}})} Z_{\{m,n,r\}} &= (-1)^{2m_{\hat{a}}} (-1)^{N(N-m)(1-\mathfrak{n}_{\hat{a}})} Z_{\{m,n,r\}} \\ &= (-1)^{2m_{\hat{a}}} Z_{\{m,n,r\}}, \end{aligned} \tag{2.58}$$

where we substituted in the index $m_{\hat{a}}$ from (2.57). In the last equation we have used that $N(N-m) = m^2 n(n-1)$ is an even integer. The result (2.58) is in agreement with (2.56a). Note that this result is valid even if we only shift a single $\Delta_{\hat{a}}$.

For the shift $\Delta_{\hat{a}} \to \Delta_{\hat{a}} + \tau$, we first consider how the numerator in $Z_{\{m,n,r\}}$ (2.55) transforms. Using (A.7b), we find

$$\theta_1(\Delta_{\hat{a}} + \tau; \tau) = -q^{-1/2} e^{-2\pi i \Delta_{\hat{a}}} \theta_1(\Delta_{\hat{a}}; \tau), \tag{2.59a}$$

$$\theta_1(m(\Delta_{\hat{a}} + \tau); \tilde{\tau}) = (-1)^{n+r(n+1)} q^{-N/2} e^{-2\pi N i \Delta_{\hat{a}}} \theta_1(m\Delta_{\hat{a}}; \tilde{\tau}). \tag{2.59b}$$

This demonstrates that the numerator picks up an overall factor as

$$Z^{\text{numer}}_{\{m,n,r\}} \to \left[(-1)^{1-N(n+r(n+1))} q^{(N^2-1)/2} y_{\hat{a}}^{N^2-1}\right]^{1-\mathfrak{n}_{\hat{a}}} Z^{\text{numer}}_{\{m,n,r\}} \tag{2.60}$$

under the shift $\Delta_{\hat{a}} \to \Delta_{\hat{a}} + \tau$. As above, the sign factor can be rewritten as

$$\begin{aligned}1 - N(n + r(n+1)) &= -(N^2 - 1) + N(n(m-1) - r(n+1)) \\ &= -(N^2 - 1) + n^2 m(m-1) - rmn(n+1).\end{aligned} \tag{2.61}$$

Since the last two terms in the final expression are even, they do not contribute to the overall sign, and we are left with

$$Z^{\text{numer}}_{\{m,n,r\}} \to (-1)^{2m_{\hat{a}}} q^{-m_{\hat{a}}} e^{-4\pi i m_{\hat{a}} \Delta_{\hat{a}}} Z^{\text{numer}}_{\{m,n,r\}}, \tag{2.62}$$

where we have substituted in the index $m_{\hat{a}}$ from (2.57). Since the numerator by itself transforms properly under the shift $\Delta_{\hat{a}} \to \Delta_{\hat{a}} + \tau$ as a weak Jacobi form of index $m_{\hat{a}}$ (2.57) following (2.56a), we see that the denominator must be invariant under this shift. This is not entirely obvious, however, as the logarithmic derivatives of $\theta_1$ transform as

$$\begin{aligned}\partial_{\Delta_{\hat{a}}} \log \theta_1(\Delta_{\hat{a}} + \tau, \tau) &= \partial_{\Delta_{\hat{a}}} \log \theta_1(\Delta_{\hat{a}}, \tau) - 2\pi i, \\ \partial_{\Delta_{\hat{a}}} \log \theta_1(m(\Delta_{\hat{a}} + \tau), \tilde{\tau}) &= \partial_{\Delta_{\hat{a}}} \log \theta_1(m\Delta_{\hat{a}}, \tilde{\tau}) - 2\pi N i,\end{aligned} \tag{2.63}$$

as can be seen directly from (2.59a) and (2.59b). The sum of logarithmic derivatives, however, is invariant so long as we simultaneously shift another chemical potential, say $\Delta_{\hat{b}}$, by $-\tau$, since then these additional factors will cancel. Therefore the denominator is invariant under this combined shift, and hence (2.62) leads to

$$Z_{\{m,n,r\}} \to (-1)^{2m_{\hat{a}}} q^{-m_{\hat{a}}} e^{-4\pi i m_{\hat{a}} \Delta_{\hat{a}}} Z_{\{m,n,r\}}, \tag{2.64}$$

which is in agreement with (2.56a). Note that this simultaneous shift is in fact required to maintain the constraint $\sum_{a=1}^{3} \Delta_a \in \mathbb{Z}$ (2.9).

Finally, since the standard contribution $\mathcal{Z}_{\text{standard}}$ is a sum of all $Z_{\{m,n,r\}}$ (2.42) and each $Z_{\{m,n,r\}}$ transforms under periodic shifts of chemical potentials as (2.58) and (2.64) consistent with (2.56a), we conclude that $\mathcal{Z}_{\text{standard}}$ transforms under periodic shifts of chemical potentials as a weak Jacobi form of weight zero and indices $m_a$ (2.57).

**Modular transformations**

This time we check the properties of the standard contribution $\mathcal{Z}_{\text{standard}}$ (2.42) under modular transformations (2.56b). Since a general transformation can be generated by a combination of $T$ and $S$ transformations, it is sufficient for us to demonstrate the following properties:

$$T: \quad \mathcal{Z}_{\text{standard}}(\tau+1, \Delta_a, \mathfrak{n}_a) = \mathcal{Z}_{\text{standard}}(\tau, \Delta_a, \mathfrak{n}_a), \tag{2.65a}$$

$$S: \quad \mathcal{Z}_{\text{standard}}(-1/\tau, \Delta_a/\tau, \mathfrak{n}_a) = e^{\frac{2\pi i}{\tau}\sum_{a=1}^{3} m_a \Delta_a^2} \mathcal{Z}_{\text{standard}}(\tau, \Delta_a, \mathfrak{n}_a). \tag{2.65b}$$

These follow from the modular transformation of a weak Jacobi form of weight zero and indices $m_a$ (2.57) with chemical potentials $\Delta_a$ based on (2.56b).

### $T$-transformation

We begin with the $T$-transformation $T: \tau \to \tau+1$. Under this transformation, an individual contribution $Z_{\{m,n,r\}}$ will get permuted even though their sum $\mathcal{Z}_{\text{standard}}$ will be invariant. We thus work each contribution at a time, namely the $T$-transformation of $Z_{\{m,n,r\}}$.

To proceed, we consider the expression (2.45), and observe that the numerator is built from the combination

$$\psi(u;\tau) \equiv \frac{\theta_1(u;\tau)}{\eta(\tau)^3}, \tag{2.66}$$

which transforms as a weak Jacobi form of weight $-1$ and index $1/2$, as can be seen from (A.12). For $\psi(\Delta_a;\tau)$, we have simply

$$\psi(\Delta_a;\tau+1) = \psi(\Delta_a;\tau). \tag{2.67}$$

However, the transformation is not as direct for $\psi(m\Delta_a;\tilde{\tau})$, since $T: \tilde{\tau} \to \tilde{\tau} + m/n$ is not a $\mathrm{SL}(2,\mathbb{Z})$ transformation on $\tilde{\tau}$. In this case, it is more useful to note that

$$T: \frac{m\tau+r}{n} \to \frac{m\tau+(r+m)}{n} = \frac{m\tau+r'}{n} + \left\lfloor \frac{r+m}{n} \right\rfloor, \tag{2.68}$$

where $r' = r+m \pmod{n}$. Since $\psi$ (2.66) is invariant under integer shifts of the modular parameter, we end up with

$$\psi(m\Delta_a; \frac{m(\tau+1)+r}{n}) = \psi(m'\Delta_a; \frac{m'\tau+r'}{n'}), \tag{2.69}$$

where

$$\tilde{\tau}' \equiv \frac{m'\tau+r'}{n'}, \qquad \{m',n',r'\} = \{m,n,r+m \pmod{n}\}. \tag{2.70}$$

Substituting (2.67) and (2.69) into the numerator of (2.45) then gives a simple transformation

$$Z^{\text{numer}}_{\{m,n,r\}}(\tau+1,\Delta_a,\mathfrak{n}_a) = Z^{\text{numer}}_{\{m',n',r'\}}(\tau,\Delta_a,\mathfrak{n}_a), \tag{2.71}$$

where $\{m', n', r'\}$ is from (2.70).

To be complete, we must also investigate the $T$-transformation on the denominator of (2.45), namely the Jacobian determinant $\mathcal{H}_{\{m,n,r\}}$ (2.54). Here we use the double periodicity (A.7b) and the modular property (A.11), to obtain the map

$$\mathcal{G}_{\{m,n,r\}}(\hat{j}, \hat{k}; \tau, \Delta_a + 1) = \mathcal{G}_{\{m',n',r'\}}(\hat{j}', \hat{k}'; \tau, \Delta_a), \tag{2.72}$$

with $\{m', n', r'\}$ from (2.70) and

$$\hat{j}' = \hat{j} + \hat{k} \left\lfloor \frac{r+m}{n} \right\rfloor \pmod{m}, \qquad \hat{k}' = \hat{k}. \tag{2.73}$$

Then since the above $(\hat{j}, \hat{k}) \to (\hat{j}', \hat{k}')$ is a bijective map from $\mathbb{Z}_m \times \mathbb{Z}_n$ to $\mathbb{Z}_{m'} \times \mathbb{Z}_{n'}$, we get (see Appendix B.3)

$$\mathcal{H}_{\{m,n,r\}}(\tau + 1, \Delta_a) = \mathcal{H}_{\{m',n',r'\}}(\tau, \Delta_a), \tag{2.74}$$

and hence the denominator transforms in the expected manner as well.

Substituting (2.71) and (2.74) into (2.45) then gives

$$Z_{\{m,n,r\}}(\tau + 1, \Delta_a, \mathfrak{n}_a) = Z_{\{m',n',r'\}}(\tau, \Delta_a, \mathfrak{n}_a) \tag{2.75}$$

where $\{m', n', r'\}$ is from (2.70). Finally, since $\{m, n, r\} \to \{m', n', r'\}$ in (2.70) is bijective, it is clear that the standard contribution $\mathcal{Z}_{\text{standard}}$ (2.42) is indeed invariant under $T$-transformation as (2.65a).

#### $S$-transformation

We now turn to the $S$-transformation, which takes $\Delta_a \to \Delta_a/\tau$ along with $\tau \to -1/\tau$. Once again, we start with the numerator. Since $\psi(u;\tau)$ defined in (2.66) is a weak Jacobi form of weight $-1$ and index $1/2$, we immediately have

$$\psi(\Delta_a/\tau; -1/\tau) = \frac{1}{\tau} e^{\frac{\pi i \Delta_a^2}{\tau}} \psi(\Delta_a, \tau) \tag{2.76}$$

from (2.56b). For $\psi(m\Delta_a; \tilde{\tau})$, it is important to realize that $S$ does not simply take $\tilde{\tau}$ to $-1/\tilde{\tau}$. Instead, we want to map $\tilde{\tau}$ into a new $\tilde{\tau}'$, at least up to a SL(2,$\mathbb{Z}$) transformation. In particular, we demand

$$S : \frac{m\tau + r}{n} \to \frac{r\tau - m}{n\tau} = \frac{a\tilde{\tau}' + b}{c\tilde{\tau}' + d}, \tag{2.77}$$

where $\tilde{\tau}' = (m'\tau + r')/n'$. The resulting SL(2,$\mathbb{Z}$) transformation is given by

$$a = \frac{r}{g}, \qquad c = \frac{n}{g}, \qquad ad - bc = 1, \qquad g \equiv \gcd(n,r), \tag{2.78}$$

and $\tilde{\tau}'$ takes the form

$$\tilde{\tau}' = \frac{m'\tau + r'}{n'}, \qquad \{m', n', r'\} = \{g, N/g, -dm\}. \tag{2.79}$$

Here $b$ and $d$ are uniquely determined as the solution to (2.78) is under the constraint for $r'$, $0 \le r' < n'$. Also note that we can make use of the simple relation $c\tilde{\tau}' + d = m'\tau/m$, which can be derived without explicit knowledge of $b$ and $d$. From (2.77) and (2.56b), we then find

$$\psi\left(\frac{m\Delta_a}{\tau}; \frac{r\tau - m}{n\tau}\right) = \psi\left(\frac{m'\Delta_a}{c\tilde{\tau}' + d}; \frac{a\tilde{\tau}' + b}{c\tilde{\tau}' + d}\right) = \frac{m}{m'\tau} e^{\frac{iN\Delta_a^2}{4\pi\tau}} \psi(m'\Delta_a, \tilde{\tau}'). \tag{2.80}$$

Substituting this expression along with (2.76) into the numerator of (2.45) then gives

$$Z^{\rm numer}_{\{m,n,r\}}(-1/\tau, \Delta_a/\tau, \mathfrak{n}_a) = \tau^{N-1} e^{\frac{2\pi i}{\tau}\sum_a m_a \Delta_a^2} Z^{\rm numer}_{\{m',n',r'\}}(\tau, \Delta_a, \mathfrak{n}_a), \tag{2.81}$$

where $m_a$ is from (2.57) and $\{m', n', r'\}$ is from (2.79).

The extra factor of $\tau^{N-1}$ is expected to be canceled by a similar factor arising from the denominator of (2.45), namely the Jacobian determinant $\mathcal{H}_{\{m,n,r\}}$. For this determinant, we use the double periodicity (A.7b) and the modular property (A.11), along with the requirement $\sum_a \Delta_a = 0$ to obtain the map

$$\mathcal{G}_{\{m,n,r\}}(\hat{j}, \hat{k}; -1/\tau, \Delta_a/\tau) = \tau \mathcal{G}_{\{m',n',r'\}}(\hat{j}', \hat{k}'; \tau, \Delta_a) \tag{2.82}$$

with $\{m', n', r'\}$ from (2.79) and

$$\hat{j}' = -\frac{g}{n}(\hat{k} + d\hat{k}') \mod g, \tag{2.83a}$$

$$\hat{k}' = \frac{n}{g}\hat{j} + \frac{r}{g}\hat{k} \mod \frac{N}{g}. \tag{2.83b}$$

In Appendix B.2, we show that the above $(\hat{j}, \hat{k}) \to (\hat{j}', \hat{k}')$ is a bijective map from $\mathbb{Z}_m \times \mathbb{Z}_n$ to $\mathbb{Z}_{m'} \times \mathbb{Z}_{n'}$. Then from Appendix B.3 we get

$$\mathcal{H}_{\{m,n,r\}}(-1/\tau, \Delta_a/\tau) = \tau^{N-1} \det \mathcal{H}_{\{m',n',r'\}}(-1/\tau, \Delta_a/\tau), \tag{2.84}$$

which will cancel the extra factor of $\tau^{N-1}$ from the transformation of the numerator (2.81).

Substituting (2.81) and (2.84) into (2.45) then gives

$$Z_{\{m,n,r\}}(-1/\tau,\Delta_a/\tau,\mathfrak{n}_a)=e^{\frac{2\pi i}{\tau}\sum_a m_a\Delta_a^2}Z_{\{m',n',r'\}}(\tau,\Delta_a,\mathfrak{n}_a) \tag{2.85}$$

where $\{m',n',r'\}$ is from (2.79). Finally, since $\{m,n,r\}\to\{m',n',r'\}$ in (2.79) is self-inverse and therefore bijective, it is clear that the standard contribution $\mathcal{Z}_{\text{standard}}$ (2.42) transforms under the $S$-transformation as (2.65b).

Here we wish to explain why the chemical potentials must sum to zero in order for the standard contribution $\mathcal{Z}_{\text{standard}}$ (2.42) to be a proper modular form, in particular under the $S$-transformation. Since $S$-transformation takes $\Delta_a$ to $\Delta_a/\tau$, we must demand the simultaneous conditions

$$\sum_{a=1}^{3}\Delta_a\in\mathbb{Z}\quad\text{and}\quad\sum_{a=1}^{3}\Delta_a\in\tau\mathbb{Z} \tag{2.86}$$

to satisfy the constraint given in (2.9) for both $\mathcal{Z}(\tau,\Delta_a,\mathfrak{n}_a)$ and $\mathcal{Z}(-1/\tau,\Delta_a/\tau,\mathfrak{n}_a)$. This leads to $\sum_{a=1}^{3}\Delta_a=0$. Of course, we can always shift chemical potentials by integers as $\Delta_a\to\Delta_a+\mathbb{Z}$ to set $\sum_{a=1}^{3}\Delta_a=0$, so this is not a physically sensible restriction on chemical potentials.

### 2.3.2 Cardy-like asymptotics

In this subsection we investigate the Cardy-like limit of the standard contribution $\mathcal{Z}_{\text{standard}}$ given from (2.42) and (2.55) that we repeat here for convenience

$$\mathcal{Z}_{\text{standard}}(\tau,\Delta_a,\mathfrak{n}_a)=\sum_{n=1\,(n|N)}^{N}\sum_{r=0}^{n-1}Z_{\{m,n,r\}}(\tau,\Delta_a,\mathfrak{n}_a), \tag{2.87a}$$

$$Z_{\{m,n,r\}}(\tau,\Delta_a,\mathfrak{n}_a)=\frac{\prod_{a=1}^{3}\left[\psi(\Delta_a;\tau)\psi(m\Delta_a;\tilde{\tau})^{-N}\right]^{1-\mathfrak{n}_a}}{n\det\Big(1+\widetilde{H}_{\{m,n,r\}}\Big)\left[\frac{1}{\pi}\sum_{a=1}^{3}\frac{\psi'(m\Delta_a;\tilde{\tau})}{\psi(m\Delta_a;\tilde{\tau})}\right]^{N-1}}. \tag{2.87b}$$

Note that we simplify (2.55) in terms of the $\psi$-function (2.66) to get (2.87b). Here the Cardy-like limit means

$$|\tau|\to 0+\quad\text{with fixed}\quad 0<\arg\tau<\pi. \tag{2.88}$$

The Cardy-like limit of the twisted index has been first investigated in [73]. It was restricted to the Cardy-like limit of $Z_{\{1,N,0\}}$, however, and here we extends a similar analysis to generic $Z_{\{m,n,r\}}$ and therefore the total standard contribution $\mathcal{Z}_{\text{standard}}$ following [67], but with generic complex parameters $\tau$ and $\Delta_a$.

To begin with, using the Cardy-like asymptotics of elliptic functions (A.16) and (A.23), we derive

$$\begin{aligned}\log\psi(\Delta_a;\tau)=&\frac{\pi i}{\tau}\{\Delta_a\}_\tau(1-\{\Delta_a\}_\tau)+\log\tau-\frac{\pi i}{2}+p_a\pi i\\&+\mathcal{O}(e^{-\frac{2\pi\sin(\arg\tau)}{|\tau|}\min(\{\tilde{\Delta}_a\},1-\{\tilde{\Delta}_a\})}),\end{aligned}\tag{2.89}$$

where we have defined an integer $p_a$ as

$$p_a\equiv\lfloor\mathrm{Re}\,\Delta_a-\cot(\arg\tau)\,\mathrm{Im}\,\Delta_a\rfloor.\tag{2.90}$$

See (A.17) and (A.19) for the definitions of the $\tau$-modded value $\{\cdot\}_\tau$ and the 'tilde' compoment of chemical potentials $\tilde{\Delta}_a$. Next, for $\psi(m\Delta_a;\tilde{\tau})$, first we take the SL(2,$\mathbb{Z}$) transformation with

$$c=\frac{n}{g},\qquad d=-\frac{r}{g},\qquad ad-bc=1,\qquad g\equiv\gcd(n,r),\tag{2.91}$$

in (A.12). The result is given as

$$\psi(m\Delta_a;\tilde{\tau})=\psi(m\Delta_a;\frac{m\tau+r}{n})=\frac{m\tau}{g}e^{-\frac{N\pi i}{\tau}\Delta_a^2}\psi(\frac{g\Delta_a}{\tau};-\frac{g^2}{N\tau}+\frac{a}{c}).\tag{2.92}$$

Applying the product forms (A.2) and (A.3b) to the RHS of (2.92) then gives

$$\begin{aligned}\log\psi(m\Delta_a;\tilde{\tau})=&\frac{g^2\pi i}{N\tau}\{\frac{N\Delta_a}{g}\}_\tau(1-\{\frac{N\Delta_a}{g}\}_\tau)+\log\left(\frac{m\tau}{ig}\right)+q_a\pi i\left(1+(q_a+1)\frac{a}{c}\right)\\&+\mathcal{O}(e^{-\frac{2\pi\sin(\arg\tau)}{|\tau|}\min(\{\frac{N}{g}\tilde{\Delta}_a\},1-\{\frac{N}{g}\tilde{\Delta}_a\})}),\end{aligned}\tag{2.93}$$

where we have defined an integer $q_a$ as

$$q_a\equiv\left\lfloor\mathrm{Re}\,\frac{N\Delta_a}{g}-\cot(\arg\tau)\,\mathrm{Im}\,\frac{N\Delta_a}{g}\right\rfloor.\tag{2.94}$$

Lastly, the diagonal term from the Jacobian matrix in the denominator of (2.87b) can be expanded using (2.93) as

$$\frac{1}{\pi}\sum_{a=1}^{3}\frac{\psi'(m\Delta_a;\tilde{\tau})}{\psi(m\Delta_a;\tilde{\tau})}=-\frac{ig}{m\tau}\eta_{N/g},\tag{2.95}$$

where we have assumed $C\tilde{\Delta}_a\notin\mathbb{Z}$ and introduced $\eta_C\in\{\pm1\}$ for a positive integer $C$ as

$$\sum_{a=1}^{3}\{C\tilde{\Delta}_a\}=\frac{3+\eta_C}{2}.\tag{2.96}$$

Here we have used the constraint $\sum_{a=1}^3 \Delta_a \in \mathbb{Z}$ (2.9) too.

Finally, substituting (2.89), (2.93), and (2.95) into (2.87b) gives

$$
\begin{aligned}
&\log Z_{\{m,n,r\}}(\tau,\Delta_a,\mathfrak{n}_a) \\
&= \frac{\pi i}{\tau}\sum_{a=1}^{3}(1-\mathfrak{n}_a)\left[\{\Delta_a\}_\tau(1-\{\Delta_a\}_\tau)-g^2\{\frac{N\Delta_a}{g}\}_\tau(1-\{\frac{N}{g}\Delta_a\}_\tau)\right]-\log\frac{N}{g} \\
&\quad+\pi i\sum_{a=1}^{3}(1-\mathfrak{n}_a)\left[p_a-Nq_a(1+(q_a+1)\frac{a}{c})\right]+\frac{(1+\eta_{N/g})(N-1)\pi i}{2} \\
&\quad-\log\det\Big(1+\widetilde{H}_{\{m,n,r\}}\Big)+\mathcal{O}(e^{-\frac{2\pi\sin(\arg\tau)}{|\tau|}\min(\{\frac{N}{g}\tilde{\Delta}_a\},1-\{\frac{N}{g}\tilde{\Delta}_a\},\{\tilde{\Delta}_a\},1-\{\tilde{\Delta}_a\})}).
\end{aligned}
\tag{2.97}
$$

Note that the 2nd line is pure imaginary, which is meaningful up to $2\pi i\mathbb{Z}$ only. Substituting the asymptotic expansion of each sector (2.97) into the sum (2.87a) then determines the Cardy-like limit of the standard contribution $\mathcal{Z}_{\text{standard}}$, where the contribution from the determinant in the last line of (2.97) is left implicit.

Since the contribution from determinant, namely $-\log\det(1+\widetilde{H}_{\{m,n,r\}})$ in (2.97), is rather intricate, its Cardy-like limit has not yet been known explicitly except for some special cases. Here we summarize the key observations of [67] but in a slightly different way with general complex $\tau$ and $\Delta_a$. First, recall that elements of an $(N-1)\times(N-1)$ square matrix $\widetilde{H}_{\{m,n,r\}}$ are given from (2.52) and (2.53) as

$$
[\widetilde{H}_{\{m,n,r\}}]_{\mu,\nu}=\frac{\mathcal{G}_{\{m,n,r\}}(\hat{j}_\mu,\hat{k}_\nu;\tau,\Delta_a)-\mathcal{G}_{\{m,n,r\}}(\hat{j}_\mu-\hat{j}_\nu,\hat{k}_\mu-\hat{k}_\nu;\tau,\Delta_a)}{\sum_{\hat{j}=0}^{m-1}\sum_{\hat{k}=0}^{n-1}\mathcal{G}_{\{m,n,r\}}(\hat{j},\hat{k};\tau,\Delta_a)},
\tag{2.98}
$$

where $(\mu,\nu)=(n\hat{j}_\mu+\hat{k}_\mu,n\hat{k}_\nu+\hat{k}_\nu)\in\mathbb{Z}_m\times\mathbb{Z}_n\setminus\{(0,0)\}$ following (2.18). To compute the determinant of the matrix with elements (2.98) in the Cardy-like limit, first we need to expand the $\mathcal{G}$-function in the Cardy-like limit. By substituting the Cardy-like expansion (A.23) into the definition (2.49), we obtain

$$
\begin{aligned}
\mathcal{G}_{\{m,n,r\}}(\hat{j},\hat{k};\tau,\Delta_a)&=\frac{1}{\tau}\sum_{a=1}^{3}\left(B_1(\{\Delta_a+\frac{n\hat{j}+r\hat{k}}{N}\}_\tau)+B_1(\{\Delta_a-\frac{n\hat{j}+r\hat{k}}{N}\}_\tau)\right) \\
&\quad+\frac{1}{\tau}\sum_{a=1}^{3}\sum_{\sigma=\pm}\left(\frac{e^{-\frac{2\pi i}{\tau}(1-\{\Delta_a+\sigma\frac{\hat{j}+\hat{k}\tilde{\tau}}{m}\}_\tau)}}{1-e^{-\frac{2\pi i}{\tau}(1-\{\Delta_a+\sigma\frac{\hat{j}+\hat{k}\tilde{\tau}}{m}\}_\tau)}}-\frac{e^{-\frac{2\pi i}{\tau}\{\Delta_a+\sigma\frac{\hat{j}+\hat{k}\tilde{\tau}}{m}\}_\tau}}{1-e^{-\frac{2\pi i}{\tau}\{\Delta_a+\sigma\frac{\hat{j}+\hat{k}\tilde{\tau}}{m}\}_\tau}}\right) \\
&\quad+\mathcal{O}(e^{-\frac{2\pi\sin(\arg\tau)}{|\tau|}}),
\end{aligned}
\tag{2.99}
$$

where $B_n(x)$ denotes the $n$-th Bernoulli polynomial.

In the simplest case with $\{m,n,r\}=\{1,N,0\}$, (2.99) shows that the $\mathcal{G}$-function becomes independent of an entry $(\hat{j},\hat{k})$ in the Cardy-like limit as

$$\mathcal{G}_{\{1,N,0\}}(\hat{j},\hat{k};\tau,\Delta_a)\sim\frac{2}{\tau}\sum_{a=1}^{3}B_1(\{\Delta_a\}_\tau),\tag{2.100}$$

where "$\sim$" means that an equation is valid up to exponentially suppressed terms. Substituting (2.100) back into (2.98) then gives

$$[\widetilde{H}_{\{1,N,0\}}]_{\mu,\nu}\sim 0\qquad\to\qquad\log\det\Big(1+\widetilde{H}_{\{1,N,0\}}\Big)\sim 1.\tag{2.101}$$

Substituting (2.101) into (2.97) then gives the Cardy-like asymptotics of $Z_{\{1,N,0\}}$ explicitly as

$$\begin{aligned}\log Z_{\{1,N,0\}}(\tau,\Delta_a,\mathfrak{n}_a)=&-\frac{(N^2-1)\pi i}{\tau}\sum_{a=1}^{3}(1-\mathfrak{n}_a)\{\Delta_a\}_\tau(1-\{\Delta_a\}_\tau)\\&-(N-1)\pi i\sum_{a=1}^{3}(1-\mathfrak{n}_a)p_a+\frac{(1+\eta_1)(N-1)\pi i}{2}\\&+\mathcal{O}(e^{-\frac{2\pi\sin(\arg\tau)}{|\tau|}\min(\{\tilde{\Delta}_a\},1-\{\tilde{\Delta}_a\})}).\end{aligned}\tag{2.102}$$

Note that the 2nd line is pure imaginary which is physically meaningful up to $2\pi i\mathbb{Z}$ only.

The problem is, as observed in [67], the determinant may vanish in general sectors $\{m,n,r\}\neq\{1,N,0\}$ for some chemical potentials up to exponentially suppressed terms, namely

$$\det\Big(1+\widetilde{H}_{\{m,n,r\}}\Big)\sim 0\quad(\text{for some }\Delta_a).\tag{2.103}$$

This is not always the case, and the determinant may yield a finite non-zero value for other configurations of chemical potentials. In that case, the determinant contribution in (2.97) is of order $\mathcal{O}(|\tau|^0)$ so can be ignored compared to the leading $\mathcal{O}(|\tau|^{-1})$ order. When the determinant vanishes as (2.103), however, we must keep track of the first exponentially suppressed terms of the $\mathcal{G}$-function, namely the 2nd line of (2.99), to improve the estimate of the determinant contribution in (2.97) as

$$-\log\det\Big(1+\widetilde{H}_{\{m,n,r\}}\Big)=\infty\quad\overset{improve}{\to}\quad-\log\det\Big(1+\widetilde{H}_{\{m,n,r\}}\Big)=\mathcal{O}(|\tau|^{-1}).\tag{2.104}$$

This makes the evaluation of the determinant contribution highly involved, which cannot be ignored since it affects the $\frac{1}{|\tau|}$-leading order in the Cardy-like limit. Refer to [67] for this calculation with $N=2,3$. We leave a systematic understanding of the determinant contribution in (2.97) for future research, which is required to evaluate the Cardy-like limit of the standard contribution $\mathcal{Z}_{\text{standard}}$ (2.87a) completely.

## 2.4 Microscopic understanding of an AdS$_5$ black string

In this section, we come back to the original motivation of studying the topologically twisted index of $\mathcal{N}=4$ SU($N$) SYM theory on $T^2 \times S^2$. First we review how the twisted index has been used to count the microstates associated with the AdS$_5$ black string entropy (1.27) [73, 77, 69, 111, 71]. Then we discuss how recent developments in the twisted index summarized in sections 2.2 and 2.3 affects the previous microstate counting.

### 2.4.1 Microstate counting by the topologically twisted index

In [73], the authors related the topologically twisted index to the central charge associated with the AdS$_3$ geometry in the near-horizon limit of an AdS$_5$ black string (1.24) in the Cardy-like limit. To be precise, they claimed

$$\lim_{|\tau|\to 0} \log \mathcal{Z}(\tau, \Delta_a^\star, \mathfrak{n}_a) = \frac{\pi i}{12\tau} c_r(\mathfrak{n}_a) + o(N^2) \tag{2.105}$$

where $\Delta_a^\star$ stands for an extremum of the twisted index and the right-moving central charge $c_r(\mathfrak{n}_a)$ is computed by applying the Brown-Henneaux central charge formula ([26]) to the AdS$_3$ part of (1.24) as

$$\begin{aligned} c(\mathfrak{n}_a) &= \frac{3R_{\text{AdS}_3}}{2G_{(3)}} \\ &= 3\left(\frac{8p^1p^2p^3\Pi}{\Theta^3}\right)^{\frac{1}{3}} \times \frac{1}{2}\left(\frac{2N^2}{\pi^4}4\pi\left(\frac{(p^1p^2p^3)^2}{\Pi}\right)^{\frac{1}{3}}\pi^3\right) \\ &= 3N^2\frac{\mathfrak{n}_1\mathfrak{n}_2\mathfrak{n}_3}{1-(\mathfrak{n}_1\mathfrak{n}_2+\mathfrak{n}_2\mathfrak{n}_3+\mathfrak{n}_3\mathfrak{n}_1)}. \end{aligned} \tag{2.106}$$

Here we have identified $p^a = -\frac{1}{2}\mathfrak{n}_a$ and used

$$\begin{aligned} \frac{1}{G_{(3)}} &= \frac{\text{vol}_7}{G_{(10)}} = \frac{2N^2}{\pi^4}\text{vol}_7, \\ \text{vol}_7 &= 4\pi|\mathfrak{g}-1|\left(\frac{(p^1p^2p^3)^2}{\Pi}\right)^{\frac{1}{3}} \times \pi^3, \end{aligned} \tag{2.107}$$

where $\mathfrak{g} = 0$ is for $\Sigma_{\mathfrak{g}} = S^2$ in (1.24) and $\pi^3$ is for the volume of an internal manifold $S^5$ with unit radius. The result is not surprising since (2.105) is the expected behavior in the Cardy-like limit of the SCFT.

Since we want to understand the microscopic origin of the $AdS_5$ black string entropy (1.27) using the topologically twisted index, however, we take a different route. Following [77, 69, 111, 71] instead, we will match the twisted index directly to the $AdS_5$ black string entropy (1.27) upon appropriate extremization.

It is also important to note that the aforementioned literature [73, 77, 69, 111, 71] have investigated the topologically twisted index implicitly assuming

$$\mathcal{Z}(\tau, \Delta_a^\star, \mathfrak{n}_a) \sim Z_{\{1,N,0\}}(\tau, \Delta_a^\star, \mathfrak{n}_a) \tag{2.108}$$

in the Cardy-like limit. Given (2.41) and (2.42), this is not a trivial statement and we will discuss this issue in the following subsection 2.4.2. For now, we assume (2.108) and how the resulting twisted index counts the microstates of a dual $AdS_5$ black string entropy (1.27).

To begin with, recall that a partition function can be written schematically as

$$\mathcal{Z}(\mu) = \sum_Q \Omega(Q) e^{2\pi i \mu Q}, \tag{2.109}$$

where $Q$ denotes a set of charges that specify microstates of degeneracy $d(Q)$ and $\mu$ is a set of chemical potentials associated with $Q$. The number of microstates $d(Q)$ is then obtained by the Cauchy's integral formula as

$$\Omega(Q) = \int_0^1 d\mu\, \mathcal{Z}(\mu) e^{-2\pi i \mu Q} = \int_0^1 d\mu\, e^{\mathcal{S}(\mu;Q)}. \tag{2.110}$$

where we have introduced

$$\mathcal{S}(\mu;Q) \equiv \log \mathcal{Z}(\mu) - 2\pi i \mu Q. \tag{2.111}$$

If $\text{Re}[\mathcal{S}(\mu;Q)]$ becomes a large positive number around a saddle point, we can compute the integral (2.110) using a saddle point approximation as

$$\log \Omega(Q) \sim \mathcal{S}(\mu^\star;Q), \tag{2.112}$$

where $\mu^\star$ is an extremum of $\mathcal{S}(\mu;Q)$.

Following the above picture, we can count the number of BPS states of $\mathcal{N} = 4$ $SU(N)$ SYM theory on $T^2 \times S^2$ specified by electric charges $q_a$ ($a = 1, 2, 3$) and a Kaluza-Klein momentum along the spatial direction of $T^2$, namely $q$, as [111, 71]

$$\Omega(q_a, q, \mathfrak{n}_a) = \int_0^1 d\Delta_1 d\Delta_2 d\tau\, \mathcal{Z}(\tau, \Delta_a, \mathfrak{n}_a) e^{-2\pi i \sum_a \Delta_a q_a - 2\pi i \tau q}\Big|_{\sum_{a=1}^3 \Delta_a \in \mathbb{Z}} \tag{2.113}$$

from the relation

$$\mathcal{Z}(\tau, \Delta_a, \mathfrak{n}_a) = \sum_{q_a, q} \Omega(q_a, q, \mathfrak{n}_a)\, e^{2\pi i \sum_a \Delta_a q_a + 2\pi i \tau q}. \tag{2.114}$$

Then the saddle point approximation (2.112) simplifies (2.113)

$$\log \Omega(q_a, q, \mathfrak{n}_a) \sim \log \mathcal{Z}(\tau^\star, \Delta_a^\star, \mathfrak{n}_a) - 2\pi i \sum_{a=1}^3 \Delta_a^\star q_a - 2\pi i \tau^\star q, \tag{2.115}$$

where an extremum $(\tau^\star, \Delta_a^\star)$ is determined by the saddle point equations

$$0 = \partial_{\Delta_a} \log \mathcal{Z}(\tau, \Delta_a, \mathfrak{n}_a) - 2\pi i q^a - 2\pi i \lambda, \tag{2.116a}$$

$$0 = \partial_\tau \log \mathcal{Z}(\tau, \Delta_a, \mathfrak{n}_a) - 2\pi i q. \tag{2.116b}$$

The parameter $\lambda$ in (2.116a) is a Lagrange muliplier for the constraint $\sum_{a=1}^3 \Delta_a \in \mathbb{Z}$.

The saddle point approximation (2.115) is of course valid only if $\text{Re}[\log \mathcal{Z}(\tau, \Delta_a, \mathfrak{n}_a)]$ becomes a large positive number around a saddle point. Assuming (2.108), we will check this in the large-$N$ after the Cardy-like limit *posteriori*. Before that, we solve the saddle point equations (2.116b) by substituting (2.102) and (2.108) and therefore making them explicit in the Cardy-like limit as

$$0 \sim \partial_{\Delta_a} \log Z_{\{1,N,0\}}(\tau, \Delta_a, \mathfrak{n}_a) - 2\pi i q^a - 2\pi i \lambda, \tag{2.117a}$$

$$0 \sim \partial_\tau \log Z_{\{1,N,0\}}(\tau, \Delta_a, \mathfrak{n}_a) - 2\pi i q, \tag{2.117b}$$

where ($i\varphi$ is a phase independent of $\tau, \Delta_a$)

$$\log Z_{\{1,N,0\}}(\tau, \Delta_a, \mathfrak{n}_a) \sim -\frac{(N^2-1)\pi i}{\tau} \sum_{a=1}^3 (1-\mathfrak{n}_a)\{\Delta_a\}_\tau (1 - \{\Delta_a\}_\tau) + i\varphi. \tag{2.118}$$

Now we solve the saddle point equations in the Cardy-like limit (2.117). To begin with, the first saddle point equation (2.117a) with $q_a = 0$ (the case of our interest) gives

$$0 \sim -\frac{(N^2-1)\pi i}{\tau^\star}(1-\mathfrak{n}_a)(1 - 2\{\Delta_a^\star\}_{\tau^\star}) - 2\pi i \lambda. \tag{2.119}$$

assuming $\tilde{\Delta}_a \notin \mathbb{Z}$ around a saddle point $\Delta_a = \Delta_a^\star$. Summing (2.119) over $a = 1, 2, 3$, we have

$$\lambda=\frac{\frac{\pi i(N^2-1)}{\tau^\star}\eta_1}{2\pi i\sum_{a=1}^3\frac{1}{1-\mathfrak{n}_a}}. \tag{2.120}$$

where $\eta_C$ is defined in (2.96) for $C\in\mathbb{Z}$. Substituting this $\lambda$ back into (2.119) then gives an extremum

$$\{\Delta_a^\star\}_{\tau^\star}=\frac{1}{2}\left(1+\frac{\frac{\eta_1}{1-\mathfrak{n}_a}}{\sum_{b=1}^3\frac{1}{1-\mathfrak{n}_b}}\right)=\frac{1}{2}\left(1+\eta_1\left(1-\frac{\mathfrak{n}_a(\mathfrak{n}_a-1)}{1-\mathfrak{n}_1\mathfrak{n}_2-\mathfrak{n}_2\mathfrak{n}_3-\mathfrak{n}_3\mathfrak{n}_1}\right)\right) \tag{2.121}$$

under the constraint $\sum_{a=1}^3\mathfrak{n}_a=2$ from (2.9). Note that $\Delta_a^\star\in\mathbb{R}$ from (2.121). The corresponding value of $\log\Omega(0,q,\mathfrak{n}_a)$ is obtained by substituting the extremum (2.121) into (2.118) and then into (2.115) under the assumption (2.108). The result is given for $q_a=0$ as

$$\log\Omega(0,q,\mathfrak{n}_a)\sim\frac{\pi i(N^2-1)}{4\tau^\star}\frac{\mathfrak{n}_1\mathfrak{n}_2\mathfrak{n}_3}{1-\mathfrak{n}_1\mathfrak{n}_2-\mathfrak{n}_2\mathfrak{n}_3-\mathfrak{n}_3\mathfrak{n}_1}-2\pi i\tau^\star q+i\varphi. \tag{2.122}$$

Next, the 2nd saddle point equation (2.117b) gives

$$\begin{aligned}0&=\frac{(N^2-1)\pi i}{\tau^{\star2}}\sum_{a=1}^3(1-\mathfrak{n}_a)\{\Delta_a^\star\}_{\tau^\star}(1-\{\Delta_a^\star\}_{\tau^\star})-2\pi iq\\&=-\frac{(N^2-1)\pi i}{4\tau^{\star2}}\frac{\mathfrak{n}_1\mathfrak{n}_2\mathfrak{n}_3}{1-\mathfrak{n}_1\mathfrak{n}_2-\mathfrak{n}_2\mathfrak{n}_3-\mathfrak{n}_3\mathfrak{n}_1}-2\pi iq\\\to\quad\tau^\star&=i\sqrt{\frac{N^2-1}{8q}\frac{\mathfrak{n}_1\mathfrak{n}_2\mathfrak{n}_3}{1-\mathfrak{n}_1\mathfrak{n}_2-\mathfrak{n}_2\mathfrak{n}_3-\mathfrak{n}_3\mathfrak{n}_1}},\end{aligned} \tag{2.123}$$

where we have assumed $\tilde{\Delta}_a\notin\mathbb{Z}$ around a saddle point $\Delta_a=\Delta_a^\star$ again and used the above result (2.121). Substituting the value of $\tau^\star$ from (2.123) into (2.122) then gives

$$\log\Omega(0,q,\mathfrak{n}_a)\sim\pi\sqrt{\frac{2(N^2-1)q\mathfrak{n}_1\mathfrak{n}_2\mathfrak{n}_3}{1-\mathfrak{n}_1\mathfrak{n}_2-\mathfrak{n}_2\mathfrak{n}_3-\mathfrak{n}_3\mathfrak{n}_1}}+i\varphi. \tag{2.124}$$

The real part of (2.124) is a large positive number in the large-$N$ limit, if the two of $\mathfrak{n}_a$'s are negative. This condition on flavor magnetic charges is in fact required for the 2d $\mathcal{N}=(0,2)$ SCFT arising from the KK compactification of 4d $\mathcal{N}=4$ SU($N$) SYM theory over $S^2$ to have a positive central charge (2.106) [12, 73]. Therefore the saddle point approximation (2.115) is indeed valid in the large-$N$ after the Cardy-like limit.

Now we want to relate the field theory result (2.124) to the dual black string entropy

(1.27). Under the AdS/CFT dictionary (see (5.12) of [71] for example)

$$\mathfrak{n}_a = -2p^a, \qquad q = \frac{\Delta_{x^5}}{G_{(5)}} q_0 = \frac{1}{G_{(4)}} q_0, \qquad N^2 = \frac{\Delta_{x^5}}{4G_{(5)}} = \frac{1}{4G_{(4)}}, \tag{2.125}$$

the logarithm of the number of BPS states (2.124) can be rewritten as

$$\begin{aligned} \log \Omega(0, q, \mathfrak{n}_a) &\sim \frac{2\pi}{G_{(4)}} \sqrt{\frac{-q_0 p^1 p^2 p^3}{1 - 4(p^1 p^2 + p^2 p^3 + p^3 p^1)}} + o(N^2) \\ &= \frac{2\pi}{G_{(4)}} \sqrt{\frac{q_0 p^1 p^2 p^3}{-(p^1)^2 - (p^2)^2 - (p^3)^2 + 2(p^1 p^2 + p^2 p^3 + p^3 p^1)}} + o(N^2). \end{aligned} \tag{2.126}$$

Here we have also used the constraint $\sum_{a=1}^3 \mathfrak{n}_a = 2$ from (2.9) or equivalently $\sum_{a=1}^3 p^a = -1$. This result (2.126) matches the $AdS_5$ black string entropy (1.27), or equivalently the corresponding $AdS_4$ black hole entropy (1.40), in the large-$N$ limit where we set the Riemann surface $\Sigma_{\mathfrak{g}} = S^2$. Hence we conclude that the ensemble of BPS states of $\mathcal{N} = 4$ SU($N$) SYM theory on $T^2 \times S^2$ counted by the topologically twisted index provides quantum origin of microstates associated with the dual $AdS_5$ black string entropy (1.27).

### 2.4.2 Implication of new BAE solutions

The conclusion made at the very end of the last subsection is of course valid only if the assumption (2.108) is true in the Cardy-like limit. Here we discuss its validity. To check (2.108) based on (2.41) and (2.42), we must show

$$\text{Re}[\log Z_{\{1,N,0\}}(\tau, \Delta_a, \mathfrak{n}_a)] > \text{Re}[\log Z_{\{m,n,r\}}(\tau, \Delta_a, \mathfrak{n}_a)] \quad (\{m,n,r\} \neq \{1,N,0\}), \tag{2.127a}$$

$$\text{Re}[\log Z_{\{1,N,0\}}(\tau, \Delta_a, \mathfrak{n}_a)] > \text{Re}[\log \mathcal{Z}_{\text{non-standard}}(\tau, \Delta_a, \mathfrak{n}_a)], \tag{2.127b}$$

in the Cardy-like limit, around a saddle point $(\tau^\star, \Delta_a^\star)$ given by (2.121) and (2.123). These conditions (2.127) have been implicitly assumed in the literature. Since we do not have explicit expression for the non-standard contribution $\mathcal{Z}_{\text{non-standard}}$, however, we leave (2.127b) for future research.

To prove (2.127a), we substitute (2.97) and (2.102) into (2.127a). Then the inequality

(2.127a) reduces to

$$\begin{aligned}
&\mathrm{Re}\Bigg[\frac{N^2\pi i}{\tau^\star}\sum_{a=1}^{3}(1-\mathfrak{n}_a)\left(\frac{\{\frac{N\Delta_a^\star}{g}\}_{\tau^\star}(1-\{\frac{N\Delta_a^\star}{g}\}_{\tau^\star})}{N^2/g^2}-\{\Delta_a^\star\}_{\tau^\star}(1-\{\Delta_a^\star\}_{\tau^\star})\right)\\
&\qquad+\log\det\Big(1+\widetilde{H}_{\{m,n,r\}}(\tau^\star,\Delta_a^\star)\Big)\Bigg]>0
\end{aligned}\tag{2.128}$$

for any positive integer $g=\gcd(n,r)$ that divides $N$, where a saddle point $(\tau^\star,\Delta_a^\star)$ is given by (2.121) and (2.123). As we have discussed in subsection 2.3.2, the determinant contribution in (2.128) is difficult to analyze in general. In particular, when $\det(1+\widetilde{H}_{\{m,n,r\}}(\tau^\star,\Delta_a^\star))$ vanishes in the Cardy-like limit, we must keep track of the leading exponentially suppressed terms of the form $\mathcal{O}(e^{-1/|\tau|})$. The determinant contribution in (2.128) then becomes a negative number of order $\mathcal{O}(|\tau|^{-1})$, which makes the proof of the inequality (2.128) more complicated. Hence we focus on the case where $\det(1+\widetilde{H}_{\{m,n,r\}}(\tau^\star,\Delta_a^\star))$ does not vanish in the Cardy-like limit. The determinant contribution in (2.128) then becomes a finite number of order $\mathcal{O}(|\tau|^0)$, which is sub-leading to the 1st line of (2.128). The inequality then reduces to

$$\mathrm{Re}\left[\frac{i}{\tau^\star}\sum_{a=1}^{3}(1-\mathfrak{n}_a)\left(\frac{\{\frac{N\Delta_a^\star}{g}\}_{\tau^\star}(1-\{\frac{N\Delta_a^\star}{g}\}_{\tau^\star})}{N^2/g^2}-\{\Delta_a^\star\}_{\tau^\star}(1-\{\Delta_a^\star\}_{\tau^\star})\right)\right]>0.\tag{2.129}$$

Since chemical potentials at the saddle point $\Delta_a^\star$ given in (2.121) are real and $\arg\tau^\star=\frac{\pi}{2}$ from (2.123), we can simplify the inequality (2.129) further as

$$\sum_{a=1}^{3}(1-\mathfrak{n}_a)\left(\frac{\{\frac{N\Delta_a^\star}{g}\}(1-\{\frac{N\Delta_a^\star}{g}\})}{N^2/g^2}-\{\Delta_a^\star\}(1-\{\Delta_a^\star\})\right)>0.\tag{2.130}$$

Note that $\{\Delta_a^\star\}_{\tau^\star}=\{\Delta_a^\star\}$ since $\Delta_a^\star$ is real from (2.121). Here the curly bracket $\{\cdot\}$ denotes a normal modded value on the real line as (A.20).

Now we review the proof of the reduced claim (2.130) in [67] for flavor magnetic charges $\mathfrak{n}_a$ satisfying the constraint $\sum_{a=1}^{2}\mathfrak{n}_a=2$ with two of them being negative. See the explanation following (2.124) for why. To begin with, note that the saddle point (2.121) is in fact an invertible map between

$$\begin{aligned}
&\{\mathfrak{n}_a\,|\,\sum_a\mathfrak{n}_a=2,\ \text{two of them are negative}\}\\
\leftrightarrow\quad&\left\{\{\Delta_a^\star\}\,|\,(1+\frac{\eta_1}{2}-\{\Delta_1^\star\}-\{\Delta_2^\star\})^2>(\{\Delta_1^\star\}+\frac{1+\eta_1}{2})(\{\Delta_2^\star\}+\frac{1+\eta_1}{2})\right\},
\end{aligned}\tag{2.131}$$

where the inverse is given as

$$\mathfrak{n}_a = \frac{2\{\Delta_a^\star\}(2\{\Delta_a^\star\}-1)}{1-4(\{\Delta_1^\star\}\{\Delta_2^\star\}+\{\Delta_2^\star\}\{\Delta_3^\star\}+\{\Delta_3^\star\}\{\Delta_1^\star\})}. \tag{2.132}$$

Recall $\eta_1 \in \{\pm 1\}$ from (2.96). From here on, we take $\eta_1 = -1$ and the other case can be studied in a similar way. Then, using

$$(1-\mathfrak{n}_a)(1-2\{\Delta_a^\star\}) = -\frac{(1-\mathfrak{n}_1)(1-\mathfrak{n}_2)(1-\mathfrak{n}_3)}{1-\mathfrak{n}_1\mathfrak{n}_2-\mathfrak{n}_2\mathfrak{n}_3-\mathfrak{n}_3\mathfrak{n}_1} > 0 \tag{2.133}$$

derived from (2.121) and the constraint $\sum_{a=1}^{2}\mathfrak{n}_a = 2$ with two of them being negative, the inequality (2.130) can be rewritten equivalently as

$$f(d_1,d_2) \equiv \sum_{a=1}^{3}\frac{1}{1-2\{\Delta_a^\star\}}\left(\frac{\{n'\Delta_a^\star\}(1-\{n'\Delta_a^\star\})}{n'^2} - \{\Delta_a^\star\}(1-\{\Delta_a^\star\})\right) > 0, \tag{2.134}$$

for any $\{\Delta_a^\star\}$ within the domain (2.131) and a positive integer $n'$. Here we define a function $f(d_1,d_2)$ as the LHS of the above inequality where we have introduced abbreviations

$$d_a \equiv \{\Delta_a^\star\}, \qquad x_a \equiv \{n'\Delta_a^\star\} = \{n'd_a\}. \tag{2.135}$$

Then within the subdomain of fixed integers $\lfloor n'\Delta_a^\star\rfloor$, where $\partial_{d_a} f$ is well defined, we can consider an extremum of $f$ under the constraint $\sum_{a=1}^{3} d_a = 1$ which satisfies

$$\begin{aligned}&\partial_{d_1} f = \partial_{d_2} f = 0\\ \Rightarrow\ &\frac{2}{(1-d_a)^2}\left(\frac{x_a(1-x_a)}{n'^2} - d_a(1-d_a)\right) + \frac{1}{1-2d_a}\left(\frac{1-2x_a}{n'} - (1-2d_a)\right) = k,\end{aligned} \tag{2.136}$$

where $k$ is some constant independent of an index $a = 1,2,3$. At this extremum, the determinant of the Hessian is given by

$$\begin{vmatrix} \partial_{d_1}^2 f & \partial_{d_1}\partial_{d_2} f \\ \partial_{d_2}\partial_{d_1} f & \partial_{d_2}^2 f \end{vmatrix} = \frac{16k^2}{(1-2d_1)(1-2d_2)(1-2d_3)} < 0, \tag{2.137}$$

where the last inequality is valid within the domain (2.131) with $\eta_1 = -1$. So the extremum is in fact a saddle point, not a minimum or maximum. This implies that the minimum of $f$ within the subdomain of fixed integers $\lfloor n'\Delta_a^\star\rfloor$ must stays on its boundary. If one investigates the values of $f$ on the boundary, it is straightforward though tedious to check that $f$ is minimized under

$$x_1 \to 0^+, x_2 \to 0^+, x_3 \to 1^- \quad \text{for} \quad \sum_{a=1}^{3} \lfloor n'\Delta_a^\star \rfloor = n' - 1, \tag{2.138}$$

$$x_1 \to 0^+, x_2 \to 1^-, x_3 \to 1^- \quad \text{for} \quad \sum_{a=1}^{3} \lfloor n'\Delta_a^\star \rfloor = n' - 2, \tag{2.139}$$

where we have ordered $d_a$ as $d_1 \leq d_2 < 1/2 < d_3$ without loss of generality (recall we set $\eta_1 = -1$ so $\sum_a d_a = 1$). For both cases, we have

$$f(d_1, d_2) \to -\sum_{a=1}^{3} \frac{d_a(1-d_a)}{1-2d_a} = -\frac{\mathfrak{n}_1\mathfrak{n}_2\mathfrak{n}_3}{4(1-\mathfrak{n}_1)(1-\mathfrak{n}_2)(1-\mathfrak{n}_3)} > 0. \tag{2.140}$$

This means that the LHS of (2.134) has a positive minimum value in every subdomain of fixed integers $\lfloor n'\Delta_a^\star \rfloor$, and thereby proves the claim (2.134) for any $\{\Delta_a^\star\}$ within the domain (2.131) and a positive integer $n'$.

Hence we conclude that the inequality (2.127a) is valid in the Cardy-like limit, provided the determinant contribution in (2.128) is of order $\mathcal{O}(|\tau|^0)$. This partially supports the validity of the microstate counting by the topologically twisted index in subsection 2.4.1 based on the assumption (2.108). It is still incomplete, however, and to fully justify the statement of 2.4.1 we must figure out how to investigate both inequalities in (2.127). We leave it for future research.

## 2.5 Concluding remarks

In this section, we summarize the main results of this chapter about the topologically twisted index of $\mathcal{N} = 4$ SU($N$) SYM theory on $T^2 \times S^2$ and discuss future directions.

First, in section 2.2, we have reviewed the Bethe Ansatz (BA) formula that gives the topologically twisted index of $\mathcal{N} = 4$ SU($N$) SYM theory on $T^2 \times S^2$ as a sum over solutions to the Bethe Ansatz Equations (BAE). We have also investigated various BAE solutions including the SL$(2, \mathbb{Z})$ family of standard ones and the other non-standard ones. Compared to standard BAE solutions, however, a full classification of all possible non-standard BAE solutions has not yet been known. Since the BA formula gives the exact twisted index only if the most general solutions to the BAE are known, more systematic understanding of non-standard BAE solutions would be the first step towards a complete understanding of the twisted index. Furthermore, the existence of continuous family of non-standard BAE solutions requires the conventional BA formula to be modified since it has been derived assuming all BAE solutions are isolated. This motivates an improvement of the BA formula that takes the contribution from generic BAE solutions to the twisted index into account.

Second, in section 2.3, we have focused on the contribution from standard BAE solutions to the twisted index through the BA formula and investigated its properties. To be specific, we confirm that it behaves as an elliptic genus of 2d $\mathcal{N} = (0, 2)$ SCFT and then explore the Cardy-like asymptotics. First of all, it is surprising that the standard contribution itself satisfies periodicites and modular properties of an elliptic genus, considering that it is not exactly the twisted index due to the non-standard contribution. This implies that the total non-standard contribution must vanish or at least satisfy the properties of an elliptic genus by itself. For a complete understanding of the twisted index, this would be the first and foremost statement we need to check in the future. When it comes to the Cardy-like asymptotics, even the standard contribution has not yet been fully digested. This is mainly because of a complicated structure of the Jacobian in the BA formula, a systematic understanding of which may lead to an interesting future work.

Lastly, in section 2.4, we have reviewed how the twisted index counts the microstates associated with the dual $\text{AdS}_5$ black string entropy in the large-$N$ after the Cardy-like limit. Then we clarified a hidden assumption for the microstate counting. The proof of this hidden assumption, however, is not complete mainly because the Cardy-like asymptotics of various contributions to the twisted index through the BA formula have not been computed explicitly. The first step towards a complete microstate counting of $\text{AdS}_5$ black holes would therefore be figuring out the Cardy-like asymptotics of such contributions. Taking one step further, since the AdS/CFT correspondence does not require the Cardy-like limit, we may even try to compute the twisted index in the pure large-$N$ limit (without the Cardy-like limit) and count the dual microstates based on the resulting twisted index. Then we can count the microstates associated with holographic dual solutions by the large-$N$ twisted index that preserves a certain modular property, which was broken in the Cardy-like limit. This could be a very interesting future direction that possibly unravels new holographic dual solutions different from the $\text{AdS}_5$ black string discussed in this dissertation.

# Chapter 3

# The Superconformal Index

## 3.1 Introduction

The Witten index of a 4-dimensional superconformal field theory (SCFT) that contains information about protected short multiplets, the ones that cannot pair up into long multiplets, has been introduced in [83, 101]. By construction, this superconformal index (SCI) remains the same under continuous deformations of the theory. Upon a radial quantization where the spacetime now reads $S^3 \times \mathbb{R}$, the SCI can be understood as an index that receives contributions only from the BPS states that do not combine into long representations. In this chapter, we are particularly interested in the SCI of $\mathcal{N} = 4$ SU($N$) SYM theory over $\frac{1}{16}$-BPS states.

In this section, we provide a trace formula for the SCI of our interest and then derive an elliptic hypergeometric integral formula from it for further calculation in the next section 3.2. To begin with, the SCI of $\mathcal{N} = 4$ SYM theory on $S^3 \times \mathbb{R}$ for $\frac{1}{16}$-BPS states can be written as a trace formula from (3.1), (3.3), (3.4), (4.2) of [83] as

$$\mathcal{I}(t, y, v, w) = \mathrm{Tr}[(-1)^F e^{-2\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}} t^{2(H+J_1^z)} y^{2J_2^z} v^{R_2} w^{R_3}] \tag{3.1}$$

where $2\{\mathcal{Q}, \mathcal{Q}^\dagger\} \equiv \Delta = H - 2J_1^z - \frac{1}{2}(3R_1 + 2R_2 + R_3)$. Here $H$ is Hamiltonian, $J_{1,2}^z$ are angular momenta associated with $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \cong \mathrm{SO}(4)$ acting on $S^3$ respectively, $R_a$ $(a = 1, 2, 3)$ are Cartan generators of the SU(4) $R$-symmetry. Out of these charges, $H + J_1^z$, $J_2^z$, $R_2$, $R_3$ are the conserved charges that commute with the supercharges $\mathcal{Q}, \mathcal{Q}^\dagger$ and $t, y, v, w$ are fugacities associated with them respectively.

Here we introduce a symmetric form of the SCI (3.1). First we introduce new fugacities

$p, q, y_a$ following (5) of [105] as

$$y_1 = t^2 v, \quad y_2 = t^2 w^{-1}, \quad y_3 = t^2 w v^{-1}, \quad p = t^3 y, \quad q = t^2 y^{-1}. \tag{3.2}$$

Note that they are constrained as $pq = y_1 y_2 y_3$ by construction, so there are still 4 independent fugacities. Then we take the change of basis for $R$-charges given in Appendix C of [83] as

$$Q_1 = \frac{R_1}{2} + R_2 + \frac{R_3}{2}, \quad Q_2 = \frac{R_1 - R_3}{2}, \quad Q_3 = \frac{R_1 + R_3}{2}, \tag{3.3}$$

where $Q_a$'s are Cartan generators of the SO(6)≅SU(4). Substituting (3.2) and (3.3) into the SCI (3.1) then gives a symmetric representation

$$\begin{aligned} \mathcal{I}(t, y, v, w) = \mathcal{I}(p, q, y_a) &= \mathrm{Tr}\Big[(-1)^F e^{-2\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}} t^{2\Delta} p^{J_1^z + J_2^z} q^{J_1^z - J_2^z} y_1^{Q_1} y_2^{Q_2} y_3^{Q_3}\Big] \\ &= \mathrm{Tr}_{\Delta=0}[(-1)^F p^{J_1^z + J_2^z} q^{J_1^z - J_2^z} y_1^{Q_1} y_2^{Q_2} y_3^{Q_3}]. \end{aligned} \tag{3.4}$$

In the second line we have used that the SCI (3.1) receives contributions from the $\frac{1}{16}$-BPS states satisfying the constraint $\Delta = 0$ only.

Now we review the derivation of an elliptic hypergeometric integral formula of the SCI of $\mathcal{N} = 4$ SU($N$) SYM theory [49, 105], which is a foundation for explicit calculation in the next section 3.2. To begin with, we rewrite the SCI in terms of a path integral as

$$\mathcal{I}(p, q, y_a) = \int \mathcal{D}U \, \exp\left[\sum_{m=1}^{\infty} \frac{1}{m} f(t^m, y^m, v^m, w^m) \,\mathrm{Tr}\, U^m \,\mathrm{Tr}\, U^{\dagger m}\right], \tag{3.5}$$

where the single particle index $f(t, y, v, w)$ is given as [83]

$$f(t, y, v, w) = \frac{t^2(v + \frac{1}{w} + \frac{w}{v}) - t^3(y + \frac{1}{y}) - t^4(w + \frac{1}{v} + \frac{v}{w}) + 2t^6}{(1 - t^3 y)(1 - \frac{t^3}{y})}. \tag{3.6}$$

In terms of symmetric fugacities $p, q, y_a$ given in (3.2), the single particle index (3.6) can be rewritten as

$$f(t, y, v, w) = \tilde{f}(p, q, y_a) = \frac{\sum_{a=1}^3 y_a - pq \sum_{a=1}^3 \frac{1}{y_a} - p - q + 2pq}{(1-p)(1-q)}. \tag{3.7}$$

This can also be derived from single particle indices for a chiral multiplet and a vector multiplet given in [102], using that the $\mathcal{N} = 4$ SYM theory consists of three chiral multiplets and a vector multiplet. Thanks to the identity $\tilde{f}(t^m, y^m, v^m, w^m) = f(p^m, q^m, y_a^m)$, the path integral form of the SCI (3.5) now reads

$$\mathcal{I}(p,q,y_a)=\int\mathcal{D}U\,\exp\Bigg[\sum_{m=1}^{\infty}\frac{1}{m}\tilde{f}(p^m,q^m,y_a^m)\,\mathrm{Tr}\,U^m\,\mathrm{Tr}\,U^{\dagger m}\Bigg]. \tag{3.8}$$

Let us first consider the case where the gauge group is U($N$). In this case we have $U=\mathrm{diag}(z_1,\cdots,z_N)$ with $z_i^*=z_i^{-1}$ and the integration measure of (3.8) is given as

$$\mathcal{D}U=\frac{1}{N!}\prod_{i=1}^{N}\frac{dz_i}{2\pi iz_i}\prod_{i<j}^{N}|z_i-z_j|^2. \tag{3.9}$$

Substituting (3.9), (A.27), and (A.28) into (3.8) then gives the SCI of $\mathcal{N}=4$ U($N$) SYM theory as

$$\mathcal{I}(p,q,y_a)=\frac{((p;p)_\infty(q;q)_\infty)^N}{N!}\prod_{a=1}^{3}\Gamma(y_a;p,q)^N\oint\prod_{j=1}^{N-1}\frac{dz_j}{2\pi iz_j}\prod_{1\leq i,j\leq N}^{i\neq j}\frac{\prod_{a=1}^3\Gamma(y_a\frac{z_i}{z_j};p,q)}{\Gamma(\frac{z_i}{z_j};p,q)}, \tag{3.10}$$

where the contour integral is over a unit circle at the origin. Refer to (A.1) and (A.5a) for the definitions of a pochhammer symbol and an ellptic gamma function respectively. The SCI of $\mathcal{N}=4$ SU($N$) SYM theory can be obtained just by replacing the trace of unitary matrices in (3.8) as [3]

$$\mathrm{Tr}\,U^m\,\mathrm{Tr}\,U^{\dagger m}\quad\to\quad\mathrm{Tr}\,U^m\,\mathrm{Tr}\,U^{\dagger m}-1. \tag{3.11}$$

From (A.27), it is straightforward to show that the replacement (3.11) simply changes the prefactor in (3.10) as

$$\left((p;p)_\infty(q;q)_\infty\prod_{a=1}^{3}\Gamma(y_a;p,q)\right)^N\quad\to\quad\left((p;p)_\infty(q;q)_\infty\prod_{a=1}^{3}\Gamma(y_a;p,q)\right)^{N-1}. \tag{3.12}$$

Finally, the elliptic hypergeometric integral formula of the SCI of $\mathcal{N}=4$ SU($N$) SYM theory is given by applying (3.12) to (3.10) as

$$\mathcal{I}(p,q,y_a)=\frac{((p;p)_\infty(q;q)_\infty)^{N-1}}{N!}\prod_{a=1}^{3}\Gamma(y_a;p,q)^{N-1}\oint\prod_{j=1}^{N-1}\frac{dz_j}{2\pi iz_j}\prod_{1\leq i,j\leq N}^{i\neq j}\frac{\prod_{a=1}^3\Gamma(y_a\frac{z_i}{z_j};p,q)}{\Gamma(\frac{z_i}{z_j};p,q)}. \tag{3.13}$$

For later purpose, we introduce holonomies $u_i$'s as $z_i=e^{2\pi iu_i}$ and chemical potentials $\sigma,\tau,\Delta_a$ as $p=e^{2\pi i\sigma}$, $q=e^{2\pi i\tau}$, and $y_a=e^{2\pi i\Delta_a}$. The SU($N$) constraint and the constraint for chemical potentials are given as

$$\sum_{i=1}^{N} u_i \in \mathbb{Z}, \tag{3.14}$$

$$\sum_{a=1}^{3} \Delta_a - \tau - \sigma \in \mathbb{Z}, \tag{3.15}$$

respectively. Note that $\Delta_a$ satisfy a constraint (3.15), which is different from the one in (2.9) we encountered when computing the topologically twisted index.

## 3.2 The calculation of the superconformal index

In this section, we compute the SCI (3.13) in two different ways: a saddle point evaluation and the Bethe Ansatz (BA) approach. For simplicity, we identify $p = q$ ($\sigma = \tau$) in (3.13) and replace the argument of the SCI (3.13) with chemical potentials as

$$\mathcal{I}(p, q, y_a) \quad \rightarrow \quad \mathcal{I}(\tau, \Delta). \tag{3.16}$$

For a saddle point evaluation in subsection 3.2.1, we take the Cardy-like limit ($|\tau| \to 0$ with fixed $0 < \arg\tau < \pi$) following [36, 27, 65, 4], but keep track of terms up to exponentially suppressed ones of the form $\mathcal{O}(e^{-1/|\tau|})$. For the BA approach in subsection 3.2.2, we take the large-$N$ limit but keep track of terms up to $\mathcal{O}(N^0)$. We will also take the Cardy-limit in the BA approach and confirms that the result is consistent with the one from a saddle point evaluation.

### 3.2.1 The saddle point evaluation

When you compute a complicated matrix integral, a saddle point evaluation is one of the most straightforward approach. To apply this technique to the elliptic hypergeometric integral formula, first we rewrite (3.13) with $p = q = e^{2\pi i\tau}$ as

$$\mathcal{I}(\tau, \Delta) = \frac{1}{N!} \int_{-\frac{1}{2N}}^{1-\frac{1}{2N}} \prod_{\mu=1}^{N-1} du_\mu \, \exp\big[N^2 S_{\text{eff}}(\{u_i\}; \tau, \Delta)\big], \tag{3.17}$$

where the integration range was chosen for later convenience and we have introduced an effective action $S_{\text{eff}}$ as

$$\begin{aligned} N^2 S_{\text{eff}}(\{u_i\};\tau,\Delta) = &\sum_{i\neq j}\left(\sum_{a=1}^{3}\log\widetilde{\Gamma}(u_{ij}+\Delta_a;\tau)+\log\theta_0(u_{ij};\tau)\right) \\ &+(N-1)\sum_{a=1}^{3}\log\widetilde{\Gamma}(\Delta_a;\tau)+2(N-1)\log(q;q)_\infty. \end{aligned} \tag{3.18}$$

Refer to (A.5b) for the definition of 'tilde' elliptic gamma function and $\{u_i\}$ is a shorthand notation for a set of holonomies $\{u_i\,|\,i=1,\cdots,N\}$. To obtain the expression (3.17), we have also replaced $-\sum_{i\neq j}\log\widetilde{\Gamma}(u_{ij};\tau)$ with $\sum_{i\neq j}\log\theta_0(u_{ij};\tau)$, using the quasi-double-periodicity (A.7a), (A.9) and the inversion formula (A.8a), (A.10) of elliptic functions.

Given the effective action (3.18) and the integral form of the index (3.17), we can now apply the saddle-point approach. First, we find solutions $\{u_i\}=\{u_i^\star\}$ to the saddle point equations

$$0=\frac{\partial}{\partial u_\mu}S_{\text{eff}}(\{u_i\};\tau,\Delta)\bigg|_{\{u_i\}=\{u_i^\star\}}\qquad(\mu=1,\cdots,N-1). \tag{3.19}$$

Then the index (3.17) can be approximated around these saddle points as

$$\mathcal{I}(\tau,\Delta)\sim\sum_{\{u_i^\star\}\in\mathcal{C}'}\frac{1}{N!}\int_{D_{\{u_i^\star\}}}\prod_{\mu=1}^{N-1}du_\mu\,\exp\big[N^2S_{\text{eff}}(\{u_i\};\tau,\Delta)\big], \tag{3.20}$$

where the integration is along the steepest descent contour $\mathcal{C}'$ passing through one or more saddle points. For each saddle point $\{u_i\}=\{u_i^\star\}$, $D_{\{u_i^\star\}}$ is a neighborhood of the corresponding saddle point solution $\{u_i^\star\}$. If all the saddle points $\{u_i^\star\}$ that contribute to the SCI through (3.20) are real and on the original contour $\mathcal{C}$ of (3.17), namely

$$\mathcal{C}=\bigcup_{\mu=1}^{N-1}[-\frac{1}{2N},1-\frac{1}{2N}], \tag{3.21}$$

we have $\mathcal{C}'=\mathcal{C}$ and

$$\{u_i^\star\}\in D_{\{u_i^\star\}}\subseteq\mathcal{C}. \tag{3.22}$$

However, in general, we may expect some saddle points to be complex, in which case the original contour $\mathcal{C}$ has to be deformed to $\mathcal{C}'\,(\neq\mathcal{C})$ passing through those complex saddle points. Here we assume this to be the case, and comment on the contour deformation further in subsection 3.2.1.

Note that (assuming contour deformation is possible) if we did not restrict the integral in (3.20) to the neighborhoods of the saddle points and kept the full integration contour $\mathcal{C}'$, we would still have an exact expression for the index. The approximation comes from integrating only near the saddle points, and this needs to be controlled by a large parameter. Such a parameter would naturally be $N^2$ in the 't Hooft expansion. But in the Cardy-like limit, we will see that $1/|\tau|$ can also play the role of a large parameter. In either case, the saddle point evaluation (3.20) is valid up to exponentially suppressed terms with respect to the large parameter. From here on, we take the Cardy-like limit ($|\tau| \to 0$ with fixed $0 < \arg\tau < \pi$) and use $1/|\tau|$ as a large control parameter.

In subsection 3.2.1, we revisit the leading term in the Cardy-like limit $|\tau| \to 0$ [36, 27, 65, 4]. In subsection 3.2.1, we keep track of sub-leading corrections in the finite Cardy-like expansion with $|\tau| \ll 1$. In both sections, our goal is to obtain an explicit expression for the SCI using the saddle point evaluation (3.20).

**Leading term in the Cardy-like limit**

In the Cardy-like limit, $|\tau| \to 0$, we substitute the asymptotic expansions (A.15), (A.21), and (A.25) into the effective action (3.18). The leading order terms then reads

$$N^2 S_{\text{eff}}(\{u_i\};\tau,\Delta) = -\frac{\pi i}{3\tau^2}\sum_{a=1}^{3}\left(\sum_{i\neq j} B_3(\{u_{ij}+\Delta_a\}_\tau) + (N-1)B_3(\{\Delta_a\}_\tau)\right) + \mathcal{O}(|\tau|^{-1}), \tag{3.23}$$

where $B_n(x)$ is the $n$-th Bernoulli polynomial. Refer to (A.17) for the definition of a $\tau$-modded value $\{\cdot\}_\tau$. Here we assumed

$$\tilde{u}_{ij} + \tilde{\Delta}_a \not\to \mathbb{Z} \tag{3.24}$$

for any $u_i$'s and $\Delta_a$'s, to use the asymptotic expansion of an elliptic gamma function (A.25). The 'tilde' values $\tilde{u}_i$ and $\tilde{\Delta}_a$ are defined following (A.19) and the curly bracket $\{\cdot\}$ stands for a modded value of a real number as (A.20).

The saddle point equation (3.19) is then given from the effective action (3.23) as

$$\begin{aligned} 0 = -\frac{\pi i}{\tau^2}\sum_{a=1}^{3}\sum_{j=1}^{N}\Big(&B_2(\{u_{\mu j}+\Delta_a\}_\tau) - B_2(\{u_{Nj}+\Delta_a\}_\tau) \\ &- B_2(\{-u_{\mu j}+\Delta_a\}_\tau) + B_2(\{-u_{Nj}+\Delta_a\}_\tau)\Big) + \mathcal{O}(|\tau|^{-1}), \end{aligned} \tag{3.25}$$

under the assumption (3.24). The leading order saddle point equation (3.25) has a rich set of solutions and we expect that one or a handful of solutions yields a dominant contribution to the SCI through the saddle point approximation (3.20). One of the most well known solutions is the one with all identical holonomies, namely $u_i = u_j$ for all $i, j \in \{1, \cdots, N\}$ [36, 65, 4]. The effective action at this saddle point successfully counted the dual $AdS_5$ black hole microstates [36] as we will review in subsection 3.4.1. Here we focus on the case where this particular saddle point with identical holonomies is dominant over the other saddle points and therefore this black hole microstate counting is valid. We put off the discussion on other saddle points, in particular the ones dubbed as $C$-center saddles[1] in [5], to Appendix C.1.

On the integration contour (3.22), there are $N$ distinct sets of identical holonomies satisfying the SU($N$) constraint $\sum_{i=1}^{N} u_i \in \mathbb{Z}$, namely

$$\{u_j^{(m)}\} = \left\{u_j^{(m)} = \frac{m}{N} \,\middle|\, j = 1, \cdots, N\right\} \quad (m = 0, 1, \cdots, N-1). \tag{3.26}$$

We can compute the effective action (3.23) at this saddle point (3.26) as

$$N^2 S_{\text{eff}}(\{u_i^{(m)}\}; \tau, \Delta) = -\frac{\pi i(N^2-1)}{\tau^2} \prod_{a=1}^{3} \left(\{\tilde{\Delta}_a\} - \frac{1+\eta_1}{2}\right) + \mathcal{O}(|\tau|^{-1}), \tag{3.27}$$

where $\eta_1 \in \{\pm 1\}$ is defined from

$$\sum_{a=1}^{3} \{C\Delta_a\}_\tau = 2C\tau + \frac{3+\eta_C}{2} \quad \Leftrightarrow \quad \sum_{a=1}^{3} \{C\tilde{\Delta}_a\} = \frac{3+\eta_C}{2} \tag{3.28}$$

under the assumption

$$C\tilde{\Delta}_a \notin \mathbb{Z}. \tag{3.29}$$

Refer to (A.19) for the definition of a 'tilde' component of chemical potentials. The SCI is then given by substituting (3.27) into the saddle point evaluation (3.20) as

$$\begin{aligned} \mathcal{I}(\tau, \Delta) = N \exp\left[-\frac{\pi i(N^2-1)}{\tau^2} \prod_{a=1}^{3} \left(\{\tilde{\Delta}_a\} - \frac{1+\eta_1}{2}\right) + o\left(|\tau|^{-2}\right)\right] \\ + \text{(contribution from other saddles)}. \end{aligned} \tag{3.30}$$

This reproduces the result of [36, 27, 65, 4] provided that the other contributions are subdominant. The factor of $N!$ in the denominator of (3.20) is removed by the degeneracy from permuting $N$ holonomies within the saddle point (3.26). The prefactor $N$ in (3.20) comes from the number of distinct sets of identical holonomies on the contour (3.22).

[1]The $C$-center solution is related to the $\{C, N/C, 0\}$ BAE solution in [67] and the $(C, N/C)$ saddle in [30].

### Sub-leading terms in the Cardy-like expansion

The fact that the $|\tau|^{-2}$-leading term in the Cardy-like limit (3.30) also captures the $N^2$-leading term in the large-$N$ limit is not clear *a priori*, since (3.30) could have terms of order $N^2$ but sub-leading in the Cardy-like expansion such as $\mathcal{O}(N^2|\tau|^{-1})$. In this subsection we clarify that such a correction does *not* show up in fact and therefore (3.30) captures the $N^2$-leading term in the large-$N$ limit correctly except the ones of the form $\mathcal{O}(N^2 e^{-1/|\tau|})$, by keeping track of all the sub-leading terms up to exponentially suppressed ones in the Cardy-like expansion.

To go beyond the leading term in the Cardy-like limit, we have to expand various elliptic functions to higher order. In particular, we substitute the asymptotic expansions (A.15), (A.21), and (A.25) into (3.18) and keep track of sub-leading terms in the finite Cardy-like expansion $|\tau| \ll 1$. The result is given in terms of Bernoulli polynomials as

$$
\begin{aligned}
&N^2 S_{\text{eff}}(\{u_i\};\tau,\Delta) \\
&= -\frac{\pi i}{3\tau^2}\sum_{a=1}^{3}\left(\sum_{i\neq j} B_3(\{u_{ij}+\Delta_a\}_\tau) + (N-1)B_3(\{\Delta_a\}_\tau)\right) \\
&\quad + \frac{\pi i}{\tau}\left(\sum_{a=1}^{3}\sum_{i\neq j} B_2(\{u_{ij}+\Delta_a\}_\tau) + (N-1)\sum_{a=1}^{3} B_2(\{\Delta_a\}_\tau) + \sum_{i\neq j}\{u_{ij}\}_\tau(1-\{u_{ij}\}_\tau)\right) \\
&\quad - \frac{5\pi i}{6}\sum_{a=1}^{3}\left(\sum_{i\neq j} B_1(\{u_{ij}+\Delta_a\}_\tau) + (N-1)B_1(\{\Delta_a\}_\tau)\right) \\
&\quad + \pi i\sum_{i\neq j}\{u_{ij}\}_\tau + \frac{\pi i(2\tau^2-3\tau-1)N^2}{6\tau} + \pi i N - \frac{\pi i(2\tau^2+3\tau-1)}{6\tau} \\
&\quad - (N-1)\log\tau + \sum_{i\neq j}\log\Big(1-e^{-\frac{2\pi i}{\tau}(1-\{u_{ij}\}_\tau)}\Big)\left(1-e^{-\frac{2\pi i}{\tau}\{u_{ij}\}_\tau}\right) \\
&\quad + \mathcal{O}\left(|\tau|^{-1}e^{-\frac{2\pi\sin(\arg\tau)}{|\tau|}X}\right),
\end{aligned}
\tag{3.31}
$$

where the first line above is just the leading order term (3.23). As in the previous subsection, we follow the conventions in (A.17), (A.19), (A.20) and the assumption (3.24). The higher order terms are of order $\mathcal{O}(|\tau|^{-1}e^{-\frac{2\pi\sin(\arg\tau)}{|\tau|}X})$ where $X$ is defined as

$$
X = \min(\{\tilde{u}_{ij}+\tilde{\Delta}_a\},\ 1-\{\tilde{u}_{ij}+\tilde{\Delta}_a\} : a=1,2,3,\ i,j=1,\cdots,N). \tag{3.32}
$$

This is exponentially suppressed under the assumption (3.24). Thus, we are treating the SCI in all powers of $\tau$ up to exponentially suppressed terms.

Using this finite Cardy-like expansion of the effective action (3.31), we would like to evaluate sub-leading corrections to the leading order saddle point solution (3.26) and the SCI (3.30) obtained in the infinite Cardy-like limit. For that purpose, it suffices to focus on the effective action (3.31) *near* the leading order saddle point solution (3.26). To be specific, we make an ansatz for a saddle point solution in the finite Cardy-like expansion,

$$\{u_j^{(m)}\} = \left\{u_j^{(m)} = \frac{m}{N} + v_j\tau \,\middle|\, v_j \sim \mathcal{O}(|\tau|^0),\ \sum_{j=1}^{N} v_j = 0\right\} \quad (m = 0, 1, \cdots, N-1), \tag{3.33}$$

and investigate the effective action (3.31) around this ansatz. This ansatz (3.33) is natural as it is equivalent to the leading order solution (3.26) up to sub-leading corrections. We call this ansatz a 'basic' ansatz, and the corresponding solution a 'basic' saddle point. Note that $\sum_{j=1}^{N} v_j = 0$ is required to satisfy the $\mathrm{SU}(N)$ constraint.

The effective action (3.31) *near* the basic ansatz (3.33) can be simplified using

$$\begin{aligned} \{u_{ij} + \Delta_a\}_\tau &= u_{ij} + \{\Delta_a\}_\tau, \\ \{u_{ij}\}_\tau &= \begin{cases} u_{ij} & (\tilde{u}_i \geq \tilde{u}_j) \\ 1 + u_{ij} & (\tilde{u}_i < \tilde{u}_j). \end{cases} \end{aligned} \tag{3.34}$$

Here (3.34) was derived by factoring out $u_{ij}$ from modded values carefully based on that $u_{ij} = v_{ij}\tau$ is at most of order $\mathcal{O}(|\tau|)$. The resulting simplified effective action is given as

$$\begin{aligned} N^2 S_{\text{eff}}(\{u_i\}; \tau, \Delta) = &-\frac{\eta\pi i}{\tau^2} N \sum_{j=1}^{N} \left(u_j - \frac{\sum_{k=1}^{N} u_k}{N}\right)^2 + \sum_{j\neq k} \log\Big(2\sin\frac{\pi u_{jk}}{\tau}\Big) \\ &- \frac{\pi i}{\tau^2}(N^2-1)\prod_{a=1}^{3}\left(\{\Delta_a\}_\tau - \frac{1+\eta_1}{2}\right) + \frac{\pi i(6-5\eta_1)(N^2-1)}{12} \\ &- \frac{\pi i N(N-1)}{2} - (N-1)\log\tau + \mathcal{O}(|\tau|^{-1} e^{-\frac{2\pi\sin(\arg\tau)}{|\tau|}X}), \end{aligned} \tag{3.35}$$

in terms of $\eta_1 \in \{\pm 1\}$ introduced in (3.28).

The saddle point equation (3.19) is then given from the effective action (3.35) and the basic ansatz (3.33) as

$$i\eta\, v_j = \frac{1}{N} \sum_{k=1\,(\neq j)}^{N} \cot \pi v_{jk} \quad (i = 1, \cdots, N), \tag{3.36}$$

which is valid up to exponentially suppressed terms. Note that the system of equations is

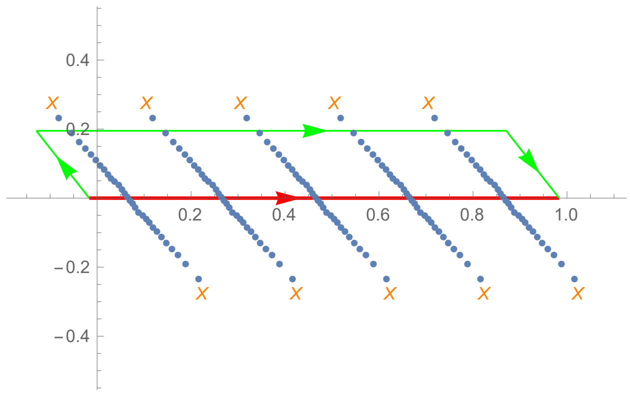


Figure 3.1: Numerical leading saddle points (blue dots) discussed in Appendix C.2.3 with $N = 30$ and $\tau = \frac{ie^{\pi i/6}}{\pi}$. There must be $N = 30$ distinct sets of holonomies in the above figure but here only 5 copies of them are shown for presentation. Orange crosses denote $\pm\tau + \frac{m}{N}$ $(m = 2, 8, 14, 20, 26)$ and therefore it is straightforward to see that each set of holonomies collapses to $\frac{m}{N}$ as $|\tau| \to 0$.

$\tau$-independent, thus justifying our assumption $v_j \sim \mathcal{O}(|\tau|^0)$ in the basic ansatz (3.33). In addition, the log term in the first line of (3.35) leads to a repulsion between pairs of eigenvalues. It is this term that shows up away from the strict Cardy-like limit that pushes the eigenvalues apart and modifies the leading order saddle point, (3.26), of condensed eigenvalues. In fact, as will be highlighted below, this set of equations closely resemble those of an SU($N$) Chern-Simons model.

**The steepest descent contour.** At leading order in the Cardy-like limit, we found $N$ distinct real saddle points (3.26). However, at sub-leading order, while there are still $N$ distinct basic saddle points, each one is now complex, as the solutions to (3.36) are complex. As a result, we seek to deform the original contour (3.22) to a new contour $\mathcal{C}'$ that passes through these $N$ basic saddle points.

To be more specific, we show a typical complex basic saddle points in Figure 3.1. The original contour integrates all eigenvalues along the real line, as shown by the red path. The first step is then to deform the contour so that the integration path of each holonomy $u_\mu$ passes through the corresponding saddle point as indicated by the green path in the figure (the one in Figure 3.1 is particularly for the second holonomy $u_2$ from the above). Since the contributions from the left and the right ends of green contours cancel each other, the deformed contour can be written simply as

$$\mathcal{C}' = \bigcup_{\mu=1}^{N-1} (v_\mu\tau - \frac{1}{2N}, v_\mu\tau + 1 - \frac{1}{2N}], \tag{3.37}$$

where $\{v_j\,|\,j=1,\cdots,N\}$ is a solution to the saddle point equation (3.36) under the SU($N$) constraint $\sum_{j=1}^N v_j=0$. Note that we are implicitly assuming that the effective action is analytic in this region so that the deformation is valid.

The saddle-point evaluation of the SCI (3.20) around basic saddle points $\{u_i^{(m)}\}$, obtained from the ansatz (3.33) satisfying the saddle point equation (3.36), is then given from the effective action (3.35) as

$$\begin{aligned}\mathcal{I}(\tau,\Delta)\sim\sum_{m=0}^{N-1}\frac{\mathcal{A}}{N!}\int_{D_{\{u_i^{(m)}\}}}\prod_{\mu=1}^{N-1}du_\mu\,\exp\left[-\frac{\eta_1\pi i}{\tau^2}N\sum_{j=1}^N\Big(u_j-\frac{\sum_{k=1}^N u_k}{N}\Big)^2+\sum_{j\neq k}\log\Big(2\sin\frac{\pi u_{jk}}{\tau}\Big)\right]\\+\text{(contribution from other saddles)}\end{aligned}\tag{3.38}$$

up to exponentially suppressed terms. Here $D_{\{u_i^{(m)}\}}$ denote small neighborhoods of basic saddle points $\{u_i^{(m)}\}$ on the deformed contour (3.37), namely

$$D_{\{u_i^{(m)}\}}=\bigcup_{\mu=1}^{N-1}(v_\mu\tau+\frac{m}{N}-\epsilon,v_\mu\tau+\frac{m}{N}+\epsilon]\subseteq\mathcal{C}'\tag{3.39}$$

for some small positive number $\epsilon$. The prefactor $\mathcal{A}$ in (3.38) is defined as

$$\begin{aligned}\mathcal{A}=\exp\Bigg[-\frac{\pi i}{\tau^2}(N^2-1)\prod_{a=1}^3\left(\{\Delta_a\}_\tau-\frac{1+\eta_1}{2}\right)+\frac{\pi i(6-5\eta_1)(N^2-1)}{12}\\-\frac{\pi iN(N-1)}{2}-(N-1)\log\tau+\mathcal{O}(|\tau|^{-1}e^{-\frac{2\pi\sin(\arg\tau)}{|\tau|}X})\Bigg].\end{aligned}\tag{3.40}$$

It is convenient to introduce new integration variables $\lambda_j$ with the constraint $\sum_{j=1}^N\lambda_j=0$ as

$$u_j=u_j^{(m)}-(i\lambda_j+v_j)\tau=\frac{m}{N}-i\lambda_j\tau.\tag{3.41}$$

This allows us to rewrite (3.38) as

$$\begin{aligned}\mathcal{I}(\tau,\Delta)\sim N\tau^{N-1}e^{-\frac{\pi i(N^2-1)}{2}}\frac{\mathcal{A}}{N!}\int_{D_{\{\lambda_i\}}}\prod_{\mu=1}^{N-1}d\lambda_\mu\,\exp\left[\eta_1\pi iN\sum_{j=1}^N\lambda_j^2+\sum_{j\neq k}\log(2\sinh\pi\lambda_{jk})\right]\\+\text{(contribution from other saddles)},\end{aligned}\tag{3.42}$$

where the integration contour $D_{\{\lambda_i\}}$ is given from the original one (3.39) and the change of variables (3.41) as

$$D_{\{\lambda_i\}} = \bigcup_{\mu=1}^{N-1} (iv_\mu - \frac{i\epsilon}{\tau}, iv_\mu + \frac{i\epsilon}{\tau}]. \tag{3.43}$$

Remarkably, the steepest descent integral in (3.42) is identical to that used to evaluate the $S^3$ partition function of supersymmetric SU$(N)_k$ Chern-Simons theory

$$Z^{CS}_{\text{SU}(N)_k} = \frac{1}{N!}\int_{-\infty}^{\infty}\prod_{\mu=1}^{N-1} d\lambda_\mu \, \exp\left[-\pi ik\sum_{j=1}^{N}\lambda_j^2 + \sum_{j\neq k}\log(2\sinh\pi\lambda_{jk})\right], \tag{3.44}$$

provided we make the identification $k = -\eta N$. This does depend on the ability to deform the contour of the Chern-Simons theory in (3.44) to pass through the $D_{\{\lambda_i\}}$ contour in (3.42), which we assume to be the case. We briefly comment on this issue in the beginning of Appendix C.2. The SCI (3.42) then reads in terms of the $S^3$ partition function (3.44) as

$$\begin{aligned}\mathcal{I}(\tau,\Delta) \sim N\tau^{N-1}e^{-\frac{\pi i(N^2-1)}{2}}\mathcal{A}\, Z^{CS}_{\text{SU}(N)_{k=-\eta N}} \\ + \text{(contribution from other saddles)}.\end{aligned} \tag{3.45}$$

We have computed this SU($N$) partition function in Appendix C.3 based on the U($N$) partition function from [79]. Substituting the result (C.60) into (3.45), we get

$$\begin{aligned}\mathcal{I}(\tau,\Delta) \sim N\exp\left[-\frac{\pi i(N^2-1)}{\tau^2}\prod_{a=1}^{3}\left(\{\Delta_a\}_\tau - \frac{1+\eta_1}{2}\right) + \mathcal{O}(e^{-1/|\tau|})\right] \\ + \text{(contribution from other saddles)}.\end{aligned} \tag{3.46}$$

Here (3.46) shows that there are in fact no sub-leading corrections besides exponentially suppressed ones. We also obtain a $\log N$ contribution to the logarithm of the SCI, which comes directly from the degeneracy of $N$ different saddle points contributing equally to the SCI. This is in fact an important lesson we learn, and the universality of the logarithmic correction has been confirmed for a large class of $\mathcal{N} = 1$ 4d SCFT's in [58].

### 3.2.2 The Bethe Ansatz formula

In this section, we compute the SCI (3.13) using the Bethe Ansatz (BA) approach first in the large-$N$ limit and then in the Cardy-like limit. The BA approach was first applied to the SCI of a generic 4d $\mathcal{N} = 1$ quiver gauge theories in [19] and then specialized to that of $\mathcal{N} = 4$ SU($N$) SYM theory with $p = q$ in [20]. The latter is then generalized to the case where $(p, q) = (h^a, h^b)$ with $a, b \in \mathbb{N}$ ($\gcd(a, b) = 1$) in [13]. Since we are interested in the

SCI of $\mathcal{N} = 4$ SU($N$) SYM theory with $p = q$, we mainly follow [20] but go beyond the leading order.

According to the BA formula [19], the elliptic hypergeometric integral formula of the SCI of $\mathcal{N} = 4$ SU($N$) SYM theory (3.13) can be rewritten in terms of a sum over Bethe vacua as

$$\mathcal{I}(\tau,\Delta) = \kappa_N \sum_{\{u_i\}\in\mathcal{M}_{\text{BAE}}} \mathcal{Z}(\{u_i\};\tau,\Delta)\mathcal{H}(\{u_i\};\tau,\Delta)^{-1}, \tag{3.47}$$

where the building blocks are given as

$$\kappa_N = \frac{1}{N!}\left((q;q)_\infty^2 \prod_{a=1}^3 \widetilde{\Gamma}(\Delta_a;\tau)\right)^{N-1} \tag{3.48a}$$

$$\mathcal{Z}\left(\{u_i\};\tau,\Delta\right) = \prod_{i\neq j}^N \frac{\prod_{a=1}^3 \widetilde{\Gamma}(u_{ij}+\Delta_a;\tau)}{\widetilde{\Gamma}\left(u_{ij};\tau\right)} \tag{3.48b}$$

$$\mathcal{H}\left(\{u_i\};\tau,\Delta\right) = \det\left[\frac{1}{2\pi i}\frac{\partial\left(Q_1,\cdots,Q_N\right)}{\partial\left(u_1,\cdots,u_{N-1},\lambda\right)}\right]. \tag{3.48c}$$

Here the BA operator $Q_i(\{u_i\};\tau,\Delta)$ is defined as

$$Q_i(\{u_i\};\tau,\Delta) \equiv e^{2\pi i\lambda}\prod_{\Delta}\prod_{j=1}^N \frac{\theta_1(u_{ji}+\Delta;\tau)}{\theta_1(u_{ij}+\Delta;\tau)}, \tag{3.49}$$

where $\Delta$ take values in $\Delta \in \{\Delta_1, \Delta_2, -\Delta_1-\Delta_2\}$. Note a notational difference from the one we used to compute the topologically twisted index (2.11) due to different constraints for chemical potentials, (2.9) and (3.15): we must use $-\Delta_1-\Delta_2$ instead of $\Delta_3$ in (3.49). The Bethe Ansatz Equations (BAE) are then given as

$$Q_i(\{u_i\};\tau,\Delta) = 1, \tag{3.50}$$

which have the same form as (2.12). The sum over Bethe vacua in the BA formula (3.47) must be taken for solutions to the BAE (3.50) whose first $N-1$ holonomies are within a particular domain, namely

$$\{u_i\}\in\mathcal{M}_{\text{BAE}} \quad \text{iff} \quad \begin{cases} i)\ Q_i(\{u_i\};\tau,\Delta) = 1 \quad (i=1,\cdots,N), \\ ii)\ u_i = x_i + y_i\tau \text{ with } -1\leq y_i<0 \quad (i=1,\cdots,N-1). \end{cases} \tag{3.51}$$

The BA operator (3.49) has a double-periodicity, namely

$$Q_i(\{u_i\};\tau,\Delta) = Q_i(\{u_i+m_i+n_i\tau\};\tau,\Delta) \qquad (m_i,n_i\in\mathbb{Z}). \tag{3.52}$$

Hence, if we find one solution $\{u_i\}$ to the BAEs (3.50), we can generate countably many solutions $\{u_i+m_i+n_i\tau\}$ with different sets of integers $m_i$'s and $n_i$'s. To compare the contributions of these BAE solutions through the BA formula (3.47), we need to understand how the building blocks $\mathcal{H}(\{u_i\};\tau,\Delta)$ and $\mathcal{Z}(\{u_i\};\tau,\Delta)$ transform under the shifting $\{u_i\}\to\{u_i+m_i+n_i\tau\}$. First, it is straightforward to show

$$\mathcal{H}(\{u_i+m_i+n_i\tau\};\tau,\Delta) = \mathcal{H}(\{u_i\};\tau,\Delta) \tag{3.53}$$

from (3.49) and (A.7b). The other building block $\mathcal{Z}(\{u_i\};\tau,\Delta)$ transforms as

$$\mathcal{Z}(\{u_i+m_i\};\tau,\Delta) = \mathcal{Z}(\{u_i\};\tau,\Delta), \tag{3.54a}$$

$$\mathcal{Z}(\{u_i-\delta_{ik}\tau\};\tau,\Delta) = (-1)^{N-1}e^{-2\pi i\lambda}Q_k(\{u_i\};\tau,\Delta)\mathcal{Z}(\{u_i\};\tau,\Delta) \tag{3.54b}$$

which are derived from (3.49) and (A.9). This implies that for a standard BAE solution (2.17) satisfying (3.50) with $\lambda=\frac{N+1}{2}$ (2.23), both building blocks $\mathcal{H}(\{u_i\};\tau,\Delta)$ and $\mathcal{Z}(\{u_i\};\tau,\Delta)$ are invariant under $\{u_i\}\to\{u_i+m_i+n_i\tau\}$ from (3.53) and (3.54). The contributions from standard BAE solutions to the SCI through the BA formula (3.47) are therefore invariant under shiting holonomies as $\{u_i\}\to\{u_i+m_i+n_i\tau\}$. Note that this argument is not valid for non-standard BAE solutions in general since $\lambda$ may take different values. In any case, we don't have to consider countably many contributions in the BA formula (3.47) due to the holonomy shifting $\{u_i\}\to\{u_i+m_i+n_i\tau\}$ in fact, since the sum over Bethe vacua in the BA formula (3.47) is taken only for BAE solutions satisfying the 2nd constraint in (3.51).

One of the simplest solution to the BAE (3.50) is a so-called 'basic' standard solution, namely

$$\{u_i\}_{\text{basic}} = \left\{u_i=\bar{u}+\frac{i}{N}\tau\,\middle|\, i=1,2,\cdots,N-1\right\}\cup\{u_N=\bar{u}\} \tag{3.55}$$

where $\bar{u}$ is supposed to satisfy the SU($N$) constraint $\sum_{i=1}^N u_i\in\mathbb{Z}$ (3.14). Following the notation of [67], this is a $\{1,N,0\}$ BAE solution. Due to the double-periodicity of the BA operator (3.52), there are countably many basic solutions as

$$\{u_i\}_{\text{basic}} = \left\{u_i=\bar{u}+\frac{i}{N}\tau+m_i+n_i\tau\,\middle|\, i=1,2,\cdots,N-1\right\}\cup\{u_N=\bar{u}+m_N+n_N\tau\} \tag{3.56}$$

with arbitrary integers $m_i$'s and $n_i$'s. Note that $m_i$'s are redundant since $u_i$'s are introduced modulo integers in the first place. As we mentioned above, only some of the basic BAE solutions (3.56) that satisfy the 2nd constraint in (3.51), will contribute to the SCI through the BA formula (3.47). From here on, we will compute the contribution from such basic BAE solutions to the SCI through the BA formula (3.47) in the large-$N$ limit and then in the Cardy-like limit.

**Degeneracy**

To determine the contribution from basic BAE solutions (3.56) to the SCI through the BA formula (3.47), first we must figure out how many of them satisfy the 2nd constraint in (3.51) and therefore contribute to the SCI. In short, we need the relevant degeneracy of basic BAE solutions. To begin with, we classify all possible basic BAE solutions (3.56) satisfying the 2nd constraint in (3.51) into two cases.

**Case 1:**

$$\{u_i\}_{\text{basic}} = \left\{u_i = \bar{u} + \frac{i}{N}\tau \,\middle|\, i = 1, 2, \cdots, N-1\right\} \cup \{u_N = \bar{u} + n_N\tau\} . \tag{3.57}$$

In this case, the value of $\bar{u}$ is determined by the SU($N$) constraint $\sum_{i=1}^{N} u_i \in \mathbb{Z}$ (3.14) and the 2nd constraint in (3.51) as

$$\bar{u} \in \left\{\frac{i}{N} - \frac{\frac{N-1}{2} + n_N}{N}\tau \,\middle|\, i = 0, 1, \cdots, N-1,\ n_N = \lfloor\frac{N+1}{2}\rfloor, \lfloor\frac{N+3}{2}\rfloor\right\} . \tag{3.58}$$

Hence the degeneracy of CASE 1 is $2N \times (N-1)!$, where $(N-1)!$ is from permuting $\{u_1, \cdots, u_{N-1}\}$.

**Case 2:**

$$\begin{aligned}\{u_i\}_{\text{basic}} = &\left\{u_i = \bar{u} + \frac{i}{N}\tau \,\middle|\, i = 0, \cdots, j-1, j+1, \cdots, N-1\right\} \\ &\cup \left\{u_N = \bar{u} + \frac{j}{N}\tau + n_N\tau\right\} .\end{aligned} \tag{3.59}$$

Here $j \in \{1, \cdots, N-2\}$. As in Case 1, the value of $\bar{u}$ is determined by the SU($N$) constraint $\sum_{i=1}^{N} u_i \in \mathbb{Z}$ (3.14) and the 2nd constraint in (3.51) as

$$\bar{u} \in \left\{\frac{i}{N} - \frac{\frac{N-1}{2} + n_N}{N}\tau \,\middle|\, i = 0, 1, \cdots, N-1,\ n_N = \lfloor\frac{N+1}{2}\rfloor\right\} . \tag{3.60}$$

Hence the degeneracy of Case 2 is $(N-2) \times N \times (N-1)!$, where $(N-2)$ is from choosing different $j \in \{1, \cdots, N-2\}$ and $(N-1)!$ is from permuting $\{u_1, \cdots, u_{N-1}\}$.

The total relevant degeneracy for the basic BAE solutions (3.56) is obtained by summing over degeneracies of the above two cases, which yields $N \times N!$. Consequently, the BA formula of the SCI (3.47) now reads

$$\begin{aligned}\mathcal{I}(\tau, \Delta) = N \times N! \times \kappa_N \mathcal{Z}(\{u_i\}_{\text{basic}}; \tau, \Delta) \mathcal{H}(\{u_i\}_{\text{basic}}; \tau, \Delta)^{-1} \\ + (\text{from other BAE solutions}).\end{aligned} \tag{3.61}$$

Focusing on the logarithm of the basic contribution, namely the first line of (3.61), we get

$$\begin{aligned}\log \mathcal{I}(\tau, \Delta)\big|_{\text{basic}} &= \log N! + \log N + \log \kappa_N \\ &\quad + \log \mathcal{Z}(\{u_i\}_{\text{basic}}; \tau, \Delta) - \log \mathcal{H}(\{u_i\}_{\text{basic}}; \tau, \Delta).\end{aligned} \tag{3.62}$$

The contribution $\log \kappa_N$ can be written explicitly from the definition (3.48a) as

$$\log \kappa_N = -\log N! + (N-1) \left( \sum_{a=1}^{3} \log \widetilde{\Gamma}(\Delta_a; \tau) + 2 \log(q;q)_\infty \right). \tag{3.63}$$

In the remaining part of this subsection, we compute the remaining two contributions in the second line of (3.62) in order mainly following the results of [20], but kepp track of sub-leading corrections. We omit the subscript 'basic' of $\{u_i\}_{\text{basic}}$ for notational convenience from here on.

**The contribution from** $\log \mathcal{Z}(\{u_i\}; \tau, \Delta)$

The contribution $\log \mathcal{Z}(\{u_i\}; \tau, \Delta)$ to the SCI (3.62) can be written explicitly as

$$\log \mathcal{Z}(\{u_i\}; \tau, \Delta) = \sum_{i \neq j}^{N} \left( \sum_{a=1}^{3} \log \widetilde{\Gamma}(\frac{i-j}{N}\tau + \Delta_a; \tau) - \log \widetilde{\Gamma}(\frac{i-j}{N}\tau; \tau) \right). \tag{3.64}$$

To simplify the expression (3.64) further, recall that section 4 of [20] yields

$$\begin{aligned}\sum_{i\neq j}^{N}\log\widetilde{\Gamma}(\frac{i-j}{N}\tau+\Delta_a;\tau)&=2\pi i\sum_{i\neq j}^{N}Q(\frac{i-j}{N}\tau+\{\Delta_a\}_\tau;\tau)-N\log\frac{\theta_0(\frac{N(\{\Delta_a\}_\tau-1)}{\tau};-\frac{N}{\tau})}{\theta_0(\frac{\{\Delta_a\}_\tau-1}{\tau};-\frac{1}{\tau})}\\&\quad+\sum_{k=0}^{\infty}\left(\log\frac{\psi(\frac{N(k+\{\Delta_a\}_\tau)}{\tau})}{\psi(\frac{N(k+1-\{\Delta_a\}_\tau)}{\tau})}-N\log\frac{\psi(\frac{k+\{\Delta_a\}_\tau}{\tau})}{\psi(\frac{k+1-\{\Delta_a\}_\tau}{\tau})}\right),\qquad(3.65\text{a})\\\sum_{i\neq j}^{N}\log\widetilde{\Gamma}(\frac{i-j}{N}\tau;\tau)&=2\pi i\sum_{i\neq j}^{N}Q(\frac{i-j}{N}\tau+1;\tau)-N\log N\\&\quad-2N\log\frac{(\tilde{q}^N;\tilde{q}^N)_\infty}{(\tilde{q};\tilde{q})_\infty}+\frac{\pi i}{12}(N-1),\qquad(3.65\text{b})\end{aligned}$$

where we have followed conventions in Appendix A. We have also defined $\tilde{q}\equiv e^{-\frac{2\pi i}{\tau}}$ and

$$Q(u;\tau)\equiv-\frac{B_3(u)}{6\tau^2}+\frac{B_2(u)}{2\tau}-\frac{5}{12}B_1(u)+\frac{\tau}{12}\qquad(3.66)$$

in terms of the $n$-th Bernoulli polynomials $B_n(x)$. Then, using the asymptotic expansions in Appendix A, we can show that some of the contributions in (3.65) are exponentially suppressed in the large-$N$ limit as

$$\begin{aligned}\log\big(\tilde{q}^N;\tilde{q}^N\big)_\infty&=\mathcal{O}(e^{-\frac{2\pi N\sin(\arg\tau)}{|\tau|}}),\qquad(3.67\text{a})\\\theta_0(\frac{N(\{\Delta_a\}_\tau-1)}{\tau};-\frac{N}{\tau})&=\mathcal{O}(e^{-\frac{2\pi N\sin(\arg\tau)}{|\tau|}\min(\{\tilde{\Delta}_a\},1-\{\tilde{\Delta}_a\})}),\qquad(3.67\text{b})\\\sum_{k=0}^{\infty}\log\psi(\frac{N(k+1-\{\Delta_a\}_\tau)}{\tau})&=\mathcal{O}(Ne^{-\frac{2\pi N\sin(\arg\tau)}{|\tau|}(1-\{\tilde{\Delta}_a\})}),\qquad(3.67\text{c})\\\sum_{k=0}^{\infty}\log\psi(\frac{N(k+\{\Delta_a\}_\tau)}{\tau})&=\mathcal{O}(Ne^{-\frac{2\pi N\sin(\arg\tau)}{|\tau|}\{\tilde{\Delta}_a\}}),\qquad(3.67\text{d})\end{aligned}$$

where we have assumed

$$\tilde{\Delta}_a\not\to\mathbb{Z}.\qquad(3.68)$$

Ignoring the above exponentially suppressed terms and using the identity (A.13), we can simplify (3.65) as

$$\begin{aligned}\sum_{i\neq j}^{N}\log\widetilde{\Gamma}(\frac{i-j}{N}\tau+\Delta_a;\tau)&=-\frac{\pi i N^2(\{\Delta_a\}_\tau-\tau)(\{\Delta_a\}_\tau-\tau-\frac{1}{2})(\{\Delta_a\}_\tau-\tau-1)}{3\tau^2}\\&\quad+\frac{\pi i(\{\Delta_a\}_\tau-\tau-\frac{1}{2})}{6}-N\log\widetilde{\Gamma}(\Delta_a;\tau)\\&\quad+\mathcal{O}(Ne^{-\frac{2\pi N\sin(\arg\tau)}{|\tau|}\min(\{\tilde{\Delta}_a\},1-\{\tilde{\Delta}_a\})}),\end{aligned}\tag{3.69a}$$

$$\begin{aligned}\sum_{i\neq j}^{N}\log\widetilde{\Gamma}(\frac{i-j}{N}\tau;\tau)&=\frac{\pi i N^2(\tau-\frac{1}{2})(\tau-1)}{3\tau}-\frac{\pi i N(\tau^2-3\tau+1)}{6\tau}-\frac{\pi i\tau}{6}\\&\quad-N\log N+2N\log(\tilde{q};\tilde{q})_\infty+\mathcal{O}(e^{-\frac{2\pi N\sin(\arg\tau)}{|\tau|}}).\end{aligned}\tag{3.69b}$$

Finally, substituting (3.69) into (3.64) and introducing $\eta_1\in\{\pm1\}$ as (3.28), we obtain

$$\begin{aligned}\log\mathcal{Z}(\{u_i\};\tau,\Delta)&=-\frac{\pi i N^2}{\tau^2}\prod_{a=1}^{3}\left(\{\Delta_a\}_\tau-\frac{1+\eta_1}{2}\right)+\frac{(1-\eta_1)\pi i}{2}N^2+\frac{\eta_1\pi i}{12}\\&\quad+N\log N-N\sum_{a=1}^{3}\log\widetilde{\Gamma}(\Delta_a;\tau;\tau)-2N\log(\tilde{q};\tilde{q})_\infty\\&\quad+\frac{\pi i N(\tau^2-3\tau+1)}{6\tau}+\mathcal{O}(Ne^{-\frac{2\pi N\sin(\arg\tau)}{|\tau|}\min(\{\tilde{\Delta}_a\},1-\{\tilde{\Delta}_a\})}).\end{aligned}\tag{3.70}$$

**The contribution from $-\log\mathcal{H}(\{u_i\};\tau,\Delta)$**

Next we consider the Jacobian contribution (3.48c) to the SCI (3.47), which has been already introduced in (2.14) and studied in subsection 2.2.2. The result (2.54) for the basic BAE solution $\{m,n,r\}=\{1,N,0\}$ reads

$$\begin{aligned}-\log\mathcal{H}(\{u_i\};\tau,\Delta)&=-\log N-(N-1)\log\left(\frac{i}{\pi}\sum_{\Delta}\partial_\Delta\log\theta_1(\Delta;\frac{\tau}{N})\right)\\&\quad+\log\det\Big(I_{N-1}+\widetilde{H}(\{u_i\};\tau,\Delta)\Big),\end{aligned}\tag{3.71}$$

where $\Delta$ take values in $\{\Delta_1, \Delta_2, -\Delta_1-\Delta_2\}$ and we have slightly redefined an $(N-1)\times(N-1)$ square matrix $\widetilde{H}$ (2.53) for the basic BAE solution as

$$\Big[\widetilde{H}(\{u_i\};\tau,\Delta)\Big]_{\mu,\nu} \equiv \frac{g(\mu;\tau,\Delta)-g(\mu-\nu;\tau,\Delta)}{\sum_{k=1}^{N} g(k;\tau,\Delta)}, \tag{3.72a}$$

$$g(j;\tau,\Delta) \equiv \frac{i}{2\pi}\sum_{\Delta}\partial_\Delta \log\Big[\theta_1(\frac{j}{N}\tau+\Delta;\tau)\theta_1(-\frac{j}{N}\tau+\Delta;\tau)\Big]. \tag{3.72b}$$

The second term of (3.71) can be computed explicitly in the large-$N$ limit using the asymptotic expansion (A.23) as

$$\frac{i}{\pi}\sum_{\Delta}\partial_\Delta \log\theta_1(\Delta;\frac{\tau}{N}) = \eta_1\frac{N}{\tau} + \mathcal{O}(e^{-\frac{2\pi N\sin(\arg\tau)}{|\tau|}\min(\{\tilde{\Delta}_a\},1-\{\tilde{\Delta}_a\})}) \tag{3.73}$$

in terms of $\eta_1 \in \{\pm 1\}$ in (3.28) and under the assumption (3.68). Substituting (3.73) into (3.71) then gives

$$\begin{aligned}-\log\mathcal{H}(\{u_i\};\tau,\Delta) = &-N\log N + (N-1)\log\frac{\tau}{\eta} - \log\det\Big(I_{N-1}+\widetilde{H}(\{u_i\};\tau,\Delta)\Big)\\ &+\mathcal{O}(e^{-\frac{2\pi N\sin(\arg\tau)}{|\tau|}\min(\{\tilde{\Delta}_a\},1-\{\tilde{\Delta}_a\})}).\end{aligned} \tag{3.74}$$

The final step would be therefore estimating $-\log\det(I_{N-1}+\widetilde{H})$.

Since it is difficult to estimate $-\log\det(I_{N-1}+\widetilde{H})$ in general, first we take the Cardy-like limit $|\tau| \ll 1$. Using the asymptotic expansion (A.23), we can obtain the Cardy-like expansion of the $g$-function (3.72b) under the assumption (3.68) as

$$g(j;\tau,\Delta) = \frac{\eta_1}{\tau} + \mathcal{O}(e^{-\frac{2\pi\sin(\arg\tau)}{|\tau|}\min(\{\tilde{\Delta}_a\},1-\{\tilde{\Delta}_a\})}). \tag{3.75}$$

Substituting (3.75) back into $[\widetilde{H}(\{u_i\};\tau,\Delta)]_{\mu,\nu}$ (3.72a) then gives

$$[\widetilde{H}]_{\mu,\nu} = \mathcal{O}(N^{-1}e^{-\frac{2\pi\sin(\arg\tau)}{|\tau|}\min(\{\tilde{\Delta}_a\},1-\{\tilde{\Delta}_a\})}). \tag{3.76}$$

The Jacobian contribution (3.74) is then simplified as

$$-\log\mathcal{H}(\{u_i\};\tau,\Delta) = -N\log N + (N-1)\log\frac{\tau}{\eta} + \mathcal{O}(e^{-\frac{2\pi\sin(\arg\tau)}{|\tau|}\min(\{\tilde{\Delta}_a\},1-\{\tilde{\Delta}_a\})}) \tag{3.77}$$

in the Cardy-like limit.

We want to estimate $-\log\det(I_{N-1}+\widetilde{H})$ in the large-$N$ limit, however, not in the Cardy-like limit. To do that, we use the Gershgorin Circle Theorem: every eigenvalue of $\widetilde{H}$ lies within at least one of the $N-1$ Gershgorin discs ($\mu = 1, 2, \cdots, N-1$)

$$D([\widetilde{H}]_{\mu,\mu}, \sum_{\nu=1\,(\neq\mu)}^{N-1} |[\widetilde{H}]_{\mu,\nu}|), \tag{3.78}$$

where the first and the second argument of $D(\cdot,\cdot)$ denotes the center and the radius of a disk respectively. Due to (3.76), every Gershgorin disc can be located within the unit disk at the origin for a small enough but finite $|\tau|$, and therefore every eigenvalue of the matrix $\tilde{H}$ has modulus less than 1 in that regime. Hence we can estimate $-\log\det(I_{N-1}+\widetilde{H})$ for a small enough $|\tau|$ as

$$-\log\det(I_{N-1}+\widetilde{H}) = -\operatorname{tr}\log(I_{N-1}+\widetilde{H}) = \operatorname{tr}\Big(\sum_{n=1}^{\infty}\frac{1}{n}(-\widetilde{H})^n\Big) = \mathcal{O}(N^0). \tag{3.79}$$

Here we have used that every eigenvalue of $[\widetilde{H}]_{\mu,\nu}$ has modulus less than 1 for the taylor expansion of a logarithm in the 2nd equation. The Jacobian contribution (3.74) is then estimated as

$$-\log\mathcal{H}(\{u_i\};\tau,\Delta) = -N\log N + (N-1)\log\frac{\tau}{\eta} + \mathcal{O}(N^0). \tag{3.80}$$

for a small enough but finite $|\tau|$.

We have not been able to estimate $-\log\det(I_{N-1}+\widetilde{H})$ analytically for a generic finite $\tau$, where some eigenvalues of $\widetilde{H}$ can be be greater than equal to 1. Hence we move on to a numerical analysis. We investigated $-\log\det(I_{N-1}+\widetilde{H})$ with $(\Delta_1,\Delta_2) = (\frac{1}{\pi},\frac{1}{e})$ and $\tau = 2+i$ for $N = 30, 35, \cdots, 200$ numerically. In this case the corresponding matrix $\widetilde{H}$ (3.72a) does have eigenvalues greater than 1 so we cannot rely on the analytic argument (3.79). As in Figure 3.2, however, $-\log\det(I_{N-1}+\widetilde{H})$ still seems to be of order $\mathcal{O}(N^0)$. We obtained similar results with other chemical potentials $\Delta_a$'s and $\tau$. Based on these numerical results, we believe that (3.79) and (3.80) are valid for a generic finite $\tau$ in fact.

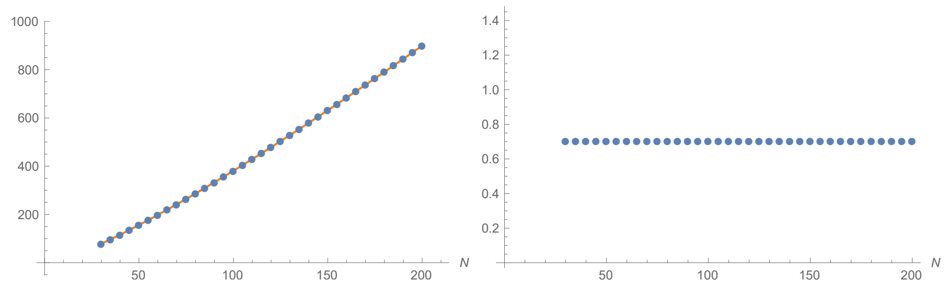


Figure 3.2: In the left hand side, blue dots represent numerical values of the real part of the Jacobian contribution $\mathrm{Re}\log\mathcal{H}(\{u_i\};\tau,\Delta)$ and an orange line shows the first two leading terms in (3.74), namely $N\log N-(N-1)\log|\tau|$. The figure in the right hand side shows numerical values of $\mathrm{Re}\log(I_{N-1}+\widetilde{H})$, obtained by subtracting an orange line from blue dots in the left hand side. It converges to a certain finite value and therefore we can conclude it is of order $\mathcal{O}(N^0)$.

### The sum of all contributions

Substituting (3.63), (3.70), (3.74) into (3.62) and using the identity (A.11a), we have

$$\begin{aligned}\log\mathcal{I}(\tau,\Delta)\big|_{\text{basic}}=&-\frac{\pi iN^2}{\tau^2}\prod_{a=1}^{3}\left(\{\Delta_a\}_\tau-\frac{1+\eta_1}{2}\right)+\log N-\sum_{a=1}^{3}\log\widetilde{\Gamma}(\Delta_a;\tau)\\&-2\log(q;q)_\infty+\frac{(1-\eta_1)\pi iN(N-1)}{2}+\frac{\pi i(6-5\eta_1)}{12}\\&-\log\tau-\log\det(I_{N-1}+\widetilde{H}(\{u_i\};\tau,\Delta))\\&+\mathcal{O}(e^{-\frac{2\pi N\sin(\arg\tau)}{|\tau|}\min(\{\tilde{\Delta}_a\},1-\{\tilde{\Delta}_a\})}).\end{aligned}\tag{3.81}$$

Recall that $\log\det(I_{N-1}+\widetilde{H})$ is of order $\mathcal{O}(N^0)$ as (3.80) so (3.81) can be simplified as

$$\log\mathcal{I}(\tau,\Delta)\big|_{\text{basic}}=-\frac{\pi iN^2}{\tau^2}\prod_{a=1}^{3}\left(\{\Delta_a\}_\tau-\frac{1+\eta_1}{2}\right)+\log N+\mathcal{O}(N^0)\tag{3.82}$$

in the large-$N$ limit. Note that the pure imaginary term $\frac{(1-\eta_1)\pi iN(N-1)}{2}\in 2\pi i\mathbb{Z}$ is removed due to $\eta_1\in\{\pm1\}$.

Even though (3.81) has been derived in the large-$N$ limit, the result is also valid in the Cardy-like limit up to exponentially suppressed terms in the last line. Furthermore, we can simplify (3.81) further using the Jacobian contribution in the Cardy-like limit (3.77), the asymptotic expansion of a Pochhammer symbol (A.15), and the following expansion

$$\begin{aligned}\sum_{a=1}^{3}\log\widetilde{\Gamma}(\Delta_a;\tau)=&-\frac{\pi i}{\tau^2}\prod_{a=1}^{3}\left(\{\Delta_a\}_\tau-\frac{1+\eta_1}{2}\right)+\frac{\pi i(\tau-2\eta_1)(2\tau-\eta_1)}{12\tau}\\&+\mathcal{O}(e^{-\frac{2\pi\sin(\arg\tau)}{|\tau|}\min(\{\tilde{\Delta}_a\},1-\{\tilde{\Delta}_a\})})\end{aligned}\tag{3.83}$$

derived from (A.25). The result is given as

$$\begin{aligned}\log\mathcal{I}(\tau,\Delta)\big|_{\text{basic}}=&-\frac{\pi i(N^2-1)}{\tau^2}\prod_{a=1}^{3}\left(\{\Delta_a\}_\tau-\frac{1+\eta_1}{2}\right)+\log N\\&+\mathcal{O}(e^{-\frac{2\pi\sin(\arg\tau)}{|\tau|}\min(\{\tilde{\Delta}_a\},1-\{\tilde{\Delta}_a\})}),\end{aligned}\tag{3.84}$$

which matches the result from a saddle point evaluation (3.46) precisely. In the BA approach, the origin of the $\log N$ term can be found in the relevant degeneracy of the basic BAE solutions (3.56) and a similar result has been confirmed for more generic $\mathcal{N}=1$ SCFT's in [58].

## 3.3 The phase structure of the superconformal index

In this section, based on the results from the previous section 3.2 and the Appendix C.1, we investigate the phase structure of the SCI of $\mathcal{N}=4$ SU($N$) SYM theory. As a final result, following [5], we will conjecture the leading asymptotics of the SCI in the large-$N$ after the Cardy-like limit. In due process, we will show that the leading asymptotics displays "infinite temperature" Roberge-Weiss-type first order phase transitions [100] between the fully-deconfined phase and confined or partially deconfined phases under a variation of chemical potentials.

In (C.13) that generalizes (3.46), we computed contributions from $C$-center saddles to the SCI of $\mathcal{N}=4$ SU($N$) SYM theory through the saddle point evaluation (3.20) in the Cardy-like limit. The leading order from (C.13) reads

$$\begin{aligned}\mathcal{I}(\tau,\Delta)=&\sum_{C=1}^{C|N}\exp\left[-\frac{\pi i}{6\tau^2}\sum_{a=1}^{3}\left(\frac{N^2}{C^3}\kappa(C\tilde{\Delta}_a)-\kappa(\tilde{\Delta}_a)\right)+o(|\tau|^{-2})\right]\\&+\text{(contribution from other saddles)}.\end{aligned}\tag{3.85}$$

Here we have introduced the $\kappa$-function as

$$\kappa(x)\equiv\{x\}(1-\{x\})(1-2\{x\}),\tag{3.86}$$

which satisfies the following identity under the constraint (3.28):

$$\sum_{a=1}^{3}\kappa(C\tilde{\Delta}_a)=6\prod_{a=1}^{3}\left(\{C\tilde{\Delta}_a\}-\frac{1+\eta_C}{2}\right). \tag{3.87}$$

The Cardy-like asymptotics (3.85) is consistent with the results from the BA approach given in (C.18) that generalizes (3.84).

Since contributions from other saddles (or BAE solutions in the BA approach) in the second line of (3.85) may affect the SCI, we can only provide the lower-bound for the Cardy-like asymptotics of the SCI from (3.85) as

$$\lim_{|\tau|\to 0}\mathcal{I}(\tau,\Delta)\geq\exp\left[-\frac{\pi i}{6\tau^2}\sum_{a=1}^{3}\left(\frac{N^2}{C_m^3}\kappa(C_m\tilde{\Delta}_a)-\kappa(\tilde{\Delta}_a)\right)\right]. \tag{3.88}$$

Here $C_m$ is a positive divisor of $N$ that maximizes

$$\mathrm{Re}\left[-\frac{\pi i}{6\tau^2}\sum_{a=1}^{3}\left(\frac{N^2}{C_m^3}\kappa(C_m\tilde{\Delta}_a)-\kappa(\tilde{\Delta}_a)\right)\right]. \tag{3.89}$$

The authors of [5] have investigated numerically if the lower-bound (3.88) is optimal for small values $N$. For $N=3,4$, it has been observed that the lower-bound for the Cardy-like asymptotics of the SCI (3.88) is indeed optimal. That means, the inequality in (3.88) turns out to hold as an exact equation in this case. For $N=5,6$, however, some configurations of chemical potentials satisfied (3.88) as a strict inequality only. For $N=6$ the lower-bound (3.88) is almost optimal though, which is not surprising since 6 has more positive divisors $\{1,2,3,6\}$ than 5 does $\{1,5\}$. In any case, this situation motivates an improvement of the lower-bound of the Cardy-like asymptotics of the SCI (3.88).

A conceptually straightforward way to improve the lower-bound of the Cardy-like asymptotics of the SCI (3.88) is to consider contributions from different saddles in the 2nd line of (3.85). Recall that we have only considered $C$-center saddles where $N$ holonomies are evenly distributed into $C$ packs, which are then evenly spaced along the domain $[0,1)$ with 0 and 1 being identified. See (C.1) for their explicit configurations. It is remarkable that the number of different $C$-center saddles considered in (3.88) heavily depends on the factorization of the rank of the gauge group $N$. For example, if $N$ is a prime number, there are not many $C$-center saddles even though $N$ is large. As a result, it is unlikely for the lower-bound (3.88) to be optimal for a prime $N$. To improve the lower-bound (3.88), this issue needs to be taken care of.

In [5], we therefore considered new saddles where the packs are nearly uniform by first distributing $\lfloor N/C \rfloor$ holonomies into each of the $C$ packs. This leaves $N \bmod C$ holonomies left over, which can then be distributed in some prescribed manner in the $C$ packs. Note that such saddles are closely related to the non-standard BAE solutions discussed at the end of subsection 2.2.1. We can take these almost $C$-center saddles into account regardless of the factorization of the rank of the gauge group $N$, and consequently improve the lower-bound (3.88). However, the resulting lower-bound would be sensitive to the particular distribution of the left over $N \bmod C$ holonomies, and can no longer be expressed as compact as (3.88).

This problem can be treated by taking the large-$N$ limit (after the Cardy-like limit we have already taken). To be specific, although the refined lower-bound that is obtained by splitting the holonomies into $C$ packs for all integers $C$ does not admit a simple expression for finite $N$, it nevertheless simplifies in the large-$N$ limit, at least for the leading order of the SCI. The idea here is that, instead of taking all $C = 1, 2, \ldots, N$, we cut off the set of almost $C$-center saddles that we will consider to improve the lower-bound (3.88) at some large but finite $C_{\max}$ that is independent of $N$. Then, for a given $C$, we start with $C$ packs of $\lfloor N/C \rfloor$ holonomies and compute $S_{\text{eff}}$ (3.31) for this subset of $C\lfloor N/C \rfloor$ holonomies. This is of course incomplete, but the missing contributions turns out to be of order $\mathcal{O}(N)$ at most. Hence the contribution from almost $C$-center saddles to the SCI is equivalent to the contribution from 'exact' $C$-center saddles in (3.85) up to $\mathcal{O}(N)$ order in the large-$N$ after the Cardy-like limit.

As a result, the lower-bound for the SCI in the large-$N$ after the Cardy-like limit is improved from (3.88) as

$$\lim_{N\to\infty}\lim_{|\tau|\to 0}\mathcal{I}(\tau,\Delta) \geq \exp\left[-\frac{\pi i N^2}{6C_m^3\tau^2}\sum_{a=1}^{3}\kappa(C_m\tilde{\Delta}_a) + o(N^2|\tau|^{-2})\right], \tag{3.90}$$

where $C_m \leq C_{\max}$ is a natural number that maximizes

$$\mathrm{Re}\left[-\frac{i}{C_m^3\tau^2}\sum_{a=1}^{3}\kappa(C_m\tilde{\Delta}_a)\right]. \tag{3.91}$$

Here, because the lower-bound applies for any finite $C_{\max} \in \mathbb{N}$ in the large-$N$ limit, we can in fact remove the cutoff $C_{\max}$ and instead choose $C_m$ from any natural number. Note that the improved large-$N$ lower-bound (3.90) confirms that the finite $N$ lower-bound (3.88) is not optimal in general, and therefore explains why the lower-bound (3.88) is not optimal for $N = 5, 6$ as we mentioned above.

Now we are ready to discuss the phase structure of the SCI in the large-$N$ after the Cardy-like limit. As long as the RHS of (3.90) has a positive real part, the SCI will exhibit

$\mathcal{O}(N^2)$ growth in the large-$N$ after the Cardy-like limit. This corresponds to either a full deconfinement when $C_m = 1$, or a partial deconfinement when $C_m \neq 1$. In the $M$-wing where

$$M\text{-wing:}\quad \text{Re}\left[-\frac{i}{\tau^2}\sum_{a=1}^{3}\kappa(\tilde{\Delta}_a)\right] > 0, \tag{3.92}$$

we always have $C_m = 1$ in (3.90) even at finite $N$ [5] and therefore the SCI is fully deconfined. On the other hand, the situation is more elaborate in the $W$-wing where

$$W\text{-wing:}\quad \text{Re}\left[-\frac{i}{\tau^2}\sum_{a=1}^{3}\kappa(\tilde{\Delta}_a)\right] < 0, \tag{3.93}$$

and therefore $C_m \neq 1$ in (3.90). In the $W$-wing, the question therefore becomes whether for any given chemical potentials we can find an integer $C > 1$ such that

$$\text{Re}\left[-\frac{i}{C^3\tau^2}\sum_{a=1}^{3}\kappa(C\tilde{\Delta}_a)\right] > 0. \tag{3.94}$$

If this is the case, we can conclude that the asymptotics of the SCI in the large-$N$ after the Cardy-like limit does have a partially deconfined phase in the $W$-wing.

Following [5], here we prove the existence of such $C \in \mathbb{N}$. Let us fix $0 < \arg\tau < \frac{\pi}{2}$ for concreteness. Then the $W$-wing (3.93) consists of all $\tilde{\Delta}_{1,2}$ subject to $0 < \{\tilde{\Delta}_1\}, \{\tilde{\Delta}_2\}, 1 - \{\tilde{\Delta}_1\} - \{\tilde{\Delta}_2\} < 1$. Hence it suffices to show that for any such $\{\tilde{\Delta}_{1,2}\}$ we can find an integer $C > 1$ satisfying the condition (3.94). Since (3.94) is periodic under $\tilde{\Delta}_{1,2} \to \tilde{\Delta}_{1,2} + 1/C$, we can simply focus on the square $0 < \tilde{\Delta}_{1,2} < 1/C$. Now, it follows from the scaling $\tilde{\Delta}_a \to \tilde{\Delta}_a/C$ that on this square the sign of $\sum_a \kappa(C\tilde{\Delta}_a)$ is positive (resp. negative) if the representatives $\{C\tilde{\Delta}_{1,2}\}/C$ of $\tilde{\Delta}_{1,2}$ on the square $0 < \tilde{\Delta}_{1,2} < 1/C$ lie on the lower triangle with vertices $(0,0), (0,1/C), (1/C,0)$ (resp. the upper triangle with vertices $(0,1/C), (1/C,0), (1/C,1/C)$). Hence the question boils down to *whether we can find a natural number $C$ such that the representatives are on the upper triangle* where $\{C\tilde{\Delta}_1\}/C + \{C\tilde{\Delta}_2\}/C > 1/C$. The following Lemma answers this question in the positive. An elementary proof can be found in Appendix C.4.[2]

**Lemma 2.** *For every pair of real numbers $x, y$ subject to $0 < x, y, 1-x-y < 1$, there exists a natural number $C > 1$ such that $\{Cx\} + \{Cy\} > 1$.*

Similar arguments apply when $\frac{\pi}{2} < \arg\tau < \pi$. We thus conclude that for all chemical potentials strictly inside the $W$-wing (3.93), there exists an integer $C > 1$ that satisfies the condition (3.94). Hence the SCI is partially deconfined in the $W$-wing.

[2]We learned the proof from David E Speyer, a mathematician at University of Michigan.

A "non-deconfined" behavior (i.e. $o(N^2)/\tau\sigma$ growth for $\log\mathcal{I}$ as $N \to \infty$ after the Cardy-like limit) might appear in non-generic situations where $\arg\tau = \frac{\pi}{2}$ (c.f. section 3 of [4]), or $\tilde{\Delta}_a \in \mathbb{Z}$. In such cases, subdominant terms of $O(N)$ or smaller may be important in order to fully pin down the behavior of the SCI. As a special example of such non-deconfined behaviors, recall that the SCI will show a confined behavior of order $\mathcal{O}(N^0)$ for real fugacities, or equivalently for pure imaginary chemical potentials $\arg\tau = \arg\Delta_a = \frac{\pi}{2}$. [83].

Finally, with some optimism, we conjecture that the improved lower-bound (3.90) with the cut-off $C_{\max}$ removed gives not just a lower-bound but the *actual leading asymptotics* of the SCI [5]. This is formally written in the following conjecture.

**Conjecture 2.** *The leading asymptotics of the SCI of the 4d $\mathcal{N} = 4$ SU(N) SYM theory in the large-N after the Cardy-like limit is given as*

$$\lim_{N\to\infty}\lim_{|\tau|\to 0}\mathcal{I}(\tau,\Delta) \sim \exp\left(-\frac{i\pi N^2}{6C_m^3\tau^2}\sum_{a=1}^{3}\kappa(C_m\tilde{\Delta}_a)\right), \tag{3.95}$$

*with the error such that logarithms of the two sides differ by $o(N^2|\tau|^{-2})$. Here $C_m$ is a natural number that maximizes*

$$\mathrm{Re}\left[-\frac{i}{C_m^3\tau^2}\sum_{a=1}^{3}\kappa(C_m\tilde{\Delta}_a)\right]. \tag{3.96}$$

*Recall that Lemma 2 guarantees that (3.96) is positive.*

This conjecture is motivated in part by the following two observations: *i*) in the $N \to \infty$ limit, (3.95) takes infinitely many almost $C$-center saddles into account and hence increases chance of their sufficiency; *ii*) already for $N$ as small as 6, as witnessed by Figure 4 of [5], exact $C$-center saddles considered in (3.88) provide a good estimate for Cardy-like asymptotics of the SCI.

## 3.4 Microscopic understanding of an $AdS_5$ black hol

In this section, we discuss how the SCI of $\mathcal{N} = 4$ SU($N$) SYM theory counts the microstates associated with the dual $AdS_5$ black hole entropy (1.50). First we review the literature that have succefully counted the microstates using the SCI. Then we consider a further development in the SCI studied in previous sections 3.2 and 3.3 and its implication to the previous black hole microstate counting.

### 3.4.1 Microstate counting by the superconformal index

In [83], the SCI of $\mathcal{N}=4$ SU($N$) SYM theory (3.13) was computed for real fugacities $p, q, y_a$. It does not show the $\mathcal{O}(N^2)$ order deconfined behavior in the large-$N$ limit though, and they explained it as a large cancelation between bosonic and fermionic BPS states. Since then the dual black hole microstates counting had remained unresolved for a while.

Recently, by three different groups [36, 27, 20], it has been shown that the SCI can be used to count the dual black hole microstates by allowing for complex fugacities, which is natural in Euclidean theory. Even though they computed the SCI in different ways, final results turned out to be consistent with each other. Hence the microstates counting steps are almost identical in all three references. In this subsection, we review that process.

Before getting into details, it is important to note that the aforementioned literature have investigated the SCI implicitly assuming

$$\begin{aligned}\mathcal{I}(\tau,\Delta) &\sim \exp\left[-\frac{\pi i(N^2-1)}{\tau^2}\prod_{a=1}^{3}\left(\{\tilde{\Delta}_a\}-\frac{1+\eta_1}{2}\right)+\mathcal{O}(|\tau|^{-1})\right]\\ &\sim \exp\left[-\frac{\pi i N^2}{\tau^2}\prod_{a=1}^{3}\left(\{\Delta_a\}_\tau-\frac{1+\eta_1}{2}\right)+\mathcal{O}(N)\right]\end{aligned} \tag{3.97}$$

in the Cardy-like limit [36, 27] or in the large-$N$ limit [20]. Note that (3.97) is consistent with the result from a saddle point evaluation (3.46) and that from the BA approach (3.82) only if the 'basic' contribution dominates the other. Recall that we have already shown in the previous section 3.3 that this is not always the case: $C$-center contribution with $C\neq 1$ dominates the 'basic' one in (3.95) where chemical potentials are within the $W$-wing (3.93). We will discuss how this affects the microstate counting in the next subsection 3.4.2. For now, we consider the case where the assumption (3.97) is valid, or equivalently where chemical potentials are within the $M$-wing (3.92).

Following the subsection 2.4.1, we can count the number of $\frac{1}{16}$-BPS states of $\mathcal{N}=4$ SU($N$) SYM theory specified by electric charges $Q_a$ ($a=1,2,3$) and identical angular momenta $J=J_1=J_2$ as[3]

$$\Omega(Q_a,J)=\int_0^1 d\Delta_1 d\Delta_2 d\sigma d\tau\,\mathcal{I}(\tau,\Delta)e^{-2\pi i(2\tau J+\sum_a\Delta_a Q_a)}\Big|_{\sum_{a=1}^3\Delta_a\in\mathbb{Z}}. \tag{3.98}$$

Substituting the large-$N$ asymptotics in (3.97) and using the saddle point approximation

[3] To be precise, the number of bosonic BPS states minus the fermionic ones.

then gives

$$\log \Omega(Q_a, J) \sim -\frac{\pi i N^2}{\tau^{\star 2}} \prod_{a=1}^{3} \left( \{\Delta_a^\star\}_{\tau^\star} - \frac{1+\eta_1}{2} \right) - 2\pi i (2\tau^\star J + \sum_a (\{\Delta_a^\star\}_{\tau^\star} - \frac{1+\eta_1}{2}) Q_a), \tag{3.99}$$

where we have replaced $\Delta_a Q_a$ with $(\{\Delta_a^\star\}_{\tau^\star} - \frac{1+\eta_1}{2})Q_a$ in the exponent of (3.98) assuming integer electric charges $Q_a \in \mathbb{Z}$. From here on we take the large-$N$ limit and use $N$ as a large control parameter for the saddle point approximation [20], but you may use the Cardy-like limit and use $1/|\tau|$ instead [36, 27]. The saddle points $(\tau^\star, \Delta_a^\star)$ are determined by solving the saddle point equations

$$0 \sim -\frac{\pi i N^2}{\tau^{\star 2}} \frac{\prod_{a=1}^{3} \left( \{\Delta_a^\star\}_{\tau^\star} - \frac{1+\eta_1}{2} \right)}{\{\Delta_b^\star\}_{\tau^\star} - \frac{1+\eta_1}{2}} - 2\pi i (Q_b + \Lambda) \quad (b = 1, 2, 3), \tag{3.100a}$$

$$0 \sim \frac{2\pi i N^2}{\tau^{\star 3}} \prod_{a=1}^{3} \left( \{\Delta_a^\star\}_{\tau^\star} - \frac{1+\eta_1}{2} \right) - 4\pi i (J - \Lambda), \tag{3.100b}$$

derived by taking partial derivatives of (3.99). Here $\Lambda$ is a Lagrange multiplier for the constraint

$$\sum_{a=1}^{3} \left( \{\Delta_a^\star\}_{\tau^\star} - \frac{1+\eta_1}{2} \right) - 2\tau \in \mathbb{Z}, \tag{3.101}$$

which is from (3.15). Solving the saddle point equations (3.100) for $\Lambda$ gives three solutions but only one of them yields a real, positive value for $\log \Omega$ (3.99) that physically makes sense. Refer to [27, 20] for details. The corresponding positive $\log \Omega$ is given as

$$\log \Omega(Q_a, J) = 2\pi \sqrt{Q_1 Q_2 + Q_2 Q_3 + Q_3 Q_1 - N^2 J}. \tag{3.102}$$

This matches the $AdS_5$ black hole entropy (1.50) exactly under the AdS/CFT dictionary

$$N^2 = \frac{\pi}{2 G_{(5)} g^3}. \tag{3.103}$$

Hence we conclude that the ensemble of BPS states of $\mathcal{N} = 4$ SU($N$) SYM theory counted by the SCI provides quantum origin of microstates associated with the dual $AdS_5$ black hole entropy (1.50).

### 3.4.2 Implication in the gravitational side

We have clarified that the conclusion made at the very end of the last subsection is valid only under the assumption (3.97). In the previous section 3.3, we have found that (3.97) is true only within the $M$-wing where chemical potentials satisfy the condition (3.92). Hence in the $W$-wing where chemical potentials satisfy the opposite condition (3.93), the SCI will have a different dominant contribution. According to the conjecture 2, such a dominant contribution will take the following form

$$\mathcal{I}(\tau,\Delta)\sim\exp\left[-\frac{\pi i N^2}{C^3\tau^2}\prod_{a=1}^{3}\left(\{C\Delta_a\}_\tau-\frac{1+\eta_C}{2}\right)+o(N^2|\tau|^{-2})\right] \tag{3.104}$$

in the large-$N$ after the Cardy-like limit, with some natural number $C>1$. In this subsection, we discuss what entropy the leading asymptotics (3.104) in the $W$-wing bears.

The number of BPS states of our interest can be computed from the same formula (3.98). Substituting the leading asymptotics (3.104) instead of (3.97) into (3.98) and using the saddle point approximation then gives

$$\begin{aligned}C\log\Omega(Q_a,J)\sim&-\frac{\pi i N^2}{(C\tau^\star)^2}\prod_{a=1}^{3}\left(\{C\Delta_a^\star\}_{\tau^\star}-\frac{1+\eta_C}{2}\right)-4\pi i(C\tau^\star)J\\&-2\pi i\sum_{a=1}^{3}(\{C\Delta_a^\star\}_{\tau^\star}-\frac{1+\eta_C}{2})Q_a,\end{aligned} \tag{3.105}$$

where we have replaced $\Delta_aQ_a$ with $\frac{1}{C}(\{C\Delta_a^\star\}_{\tau^\star}-\frac{1+\eta_C}{2})Q_a$ assuming $Q_a\in C\mathbb{Z}$. Since we take the large-$N$ after the Cardy-like limit, we can use any of $N$ and $1/|\tau|$ as a large control parameter for the saddle point approximation. The saddle points $(\tau^\star,\Delta_a^\star)$ are determined by solving the saddle point equations

$$0\sim-\frac{\pi i N^2}{(C\tau^\star)^2}\frac{\prod_{a=1}^{3}\left(\{C\Delta_a^\star\}_{\tau^\star}-\frac{1+\eta_C}{2}\right)}{\{C\Delta_b^\star\}_{\tau^\star}-\frac{1+\eta_C}{2}}-2\pi i(Q_b+\Lambda)\quad(b=1,2,3), \tag{3.106a}$$

$$0\sim\frac{2\pi i N^2}{(C\tau^\star)^3}\prod_{a=1}^{3}\left(\{C\Delta_a^\star\}_{\tau^\star}-\frac{1+\eta_C}{2}\right)-4\pi i(J-\Lambda), \tag{3.106b}$$

derived by taking partial derivatives of (3.105). Here $\Lambda$ is a Lagrange multiplier for the constraint

$$\sum_{a=1}^{3}\left(\{C\Delta_a^\star\}_{\tau^\star}-\frac{1+\eta_C}{2}\right)-2C\tau\in\mathbb{Z}, \tag{3.107}$$

which is from (3.15). Since (3.99) and (3.100) are equivalent to (3.105) and (3.106) respectively under simple replacements

$$\begin{aligned} \{\Delta_a^\star\}_{\tau^\star} - \frac{1+\eta_1}{2} \quad &\to \quad \{C\Delta_a^\star\}_{\tau^\star} - \frac{1+\eta_C}{2}, \\ \tau^\star \quad &\to \quad C\tau^\star, \end{aligned} \tag{3.108}$$

we do not need to solve the saddle point equations (3.106) to compute $\log\Omega$ through (3.105) this time: once you find a saddle point solution of (3.100) that yields (3.102), the map (3.108) will gives you the corresponding solution of (3.106). Substituting that solution back into (3.105) will then gives the corresponding entropy as

$$\log\Omega(Q_a, J) = \frac{2\pi}{C}\sqrt{Q_1Q_2 + Q_2Q_3 + Q_3Q_1 - N^2J}. \tag{3.109}$$

Note the the entropy is different from (3.102) that matches the $AdS_5$ black hole entropy by a factor of $1/C$ with some natural number $C > 1$. This strongly implies that there is a missing gravity dual solution whose entropy is supposed to match (3.109).

## 3.5 Concluding remarks

In this section, we summarize the main results of this chapter about the SCI of $\mathcal{N} = 4$ SU($N$) SYM theory and discuss future directions.

First, in section 3.2 and Appendix C.1, we computed contributions from $C$-center saddles (C.1) (resp. BAE solutions (C.14)) to the SCI through the saddle point approximation (3.20) (resp. BA formula (3.47)) in the Cardy-like limit (resp. large-$N$ limit) beyond the leading order. The results are given in (3.46) and (C.13) (resp. (3.81) and (C.17)). In these results, sub-leading corrections are more involved in $C$-center contributions with $C > 1$. For a complete description of the SCI, it would be important to improve these results by figuring out the full sub-leading corrections up to exponentially suppressed terms in both limits. The next question would then be understanding sub-leading corrections in the holographic side, particularly the universal $\log N$ correction we've found.

Second, in section 3.3, we conjectured the leading asymptotics of the SCI in the large-$N$ after the Cardy-like asymptotics in Conjecture 2 based on the results from section 3.2. In due process, we also confirmed that the SCI has not only a fully deconfined phase of order $\mathcal{O}(N^2)$ and a confined phase of order $\mathcal{O}(N^0)$, but also a partially deconfined phase of order $\mathcal{O}(N^2)$ but distinguished from the fully deconfined one. Proving the Conjecture 2 would be an important direction for future research. In the context of BA approach, this will include understanding the contribution from non-standard BAE solutions and comparing them with the $C$-center contributions. Looking for Euclidean holographic duals of partially deconfined phases in grand canonical ensemble where chemical potentials are not extremized could be another direction to explore. Recent discovery of new Euclidean black saddles in $AdS_4$ background [24] implies that the same scenario may work in $AdS_5$ background, and furthermore a duality between Euclidean $AdS_5$ black saddles and partially deconfined phases. Research in this direction will improve our holographic understanding of the SCI.

Lastly, in section 3.4, based on the Conjecture 2, we have shown that partially deconfined phases of the SCI implies the existence of gravity dual solutions whose entropies are different from the $AdS_5$ black hole entropy by multiplicative factors of $1/C$ with natural numbers $C > 1$. The fact that the entropy (3.109) has a multiplicative factor of $1/C$ is reminiscent of the exponentially suppressed contributions from black hole farey tails computed explicitly in [92, 10], whose concepts are first introduced in [88, 48]. Based on this observation, it would be very interesting to construct missing gravity dual solutions explicitly and to investigate the relation between those solutions and the black hole farey tails.

# Appendix A

# Elliptic Functions

Here we gather definitions and useful identities of elliptic functions.

## A.1 Definitions

The Pochhammer symbol is defined as

$$(z;q)_\infty = \prod_{k=0}^{\infty}(1 - zq^k). \tag{A.1}$$

The Dedekind eta function is then defined as

$$\eta(\tau) = q^{\frac{1}{24}}(q;q)_\infty \tag{A.2}$$

where $q = e^{2\pi i\tau}$. The elliptic theta functions used in this dissertation have the following product forms:

$$\theta_0(u;\tau) = \prod_{k=0}^{\infty}(1 - e^{2\pi i(u+k\tau)})(1 - e^{2\pi i(-u+(k+1)\tau)}), \tag{A.3a}$$

$$\begin{aligned}\theta_1(u;\tau) &= -ie^{\frac{\pi i\tau}{4}}(e^{\pi iu} - e^{-\pi iu})\prod_{k=1}^{\infty}(1 - e^{2\pi ik\tau})(1 - e^{2\pi i(k\tau+u)})(1 - e^{2\pi i(k\tau-u)}) \\ &= -i(-1)^m e^{\frac{\pi i\tau}{4}} e^{\pi i[(2m+1)u+m(m+1)\tau]}\prod_{k=1}^{\infty}(1 - e^{2\pi ik\tau})(1 - e^{2\pi i((k+m)\tau+u)})(1 - e^{2\pi i((k-m-1)\tau-u)}) \\ &= ie^{\frac{\pi i\tau}{4}} e^{-\pi iu}\theta_0(u;\tau)\prod_{k=1}^{\infty}(1 - e^{2\pi ik\tau}).\end{aligned} \tag{A.3b}$$

From the product form of $\theta_1(u;\tau)$ (A.3b), we can derive a useful identity

$$\begin{aligned}\prod_{\hat{j}=0}^{m-1}{}'\prod_{\hat{k}=0}^{n-1}{}'\frac{\theta_1(u+\frac{\hat{j}+\hat{k}\tilde{\tau}}{m};\tau)}{\eta(\tau)} &= e^{i\frac{n-1}{2}\pi}e^{-\frac{i\pi nr}{6}}e^{-\pi im(n-1)u}\tilde{q}^{-\frac{(n-1)(n-1/2)}{6}}\frac{\eta(\tau)}{\theta_1(u;\tau)}\frac{\theta_1(mu;\tilde{\tau})}{\eta(\tilde{\tau})},\\ \prod_{\hat{j}=0}^{m-1}{}'\prod_{\hat{k}=0}^{n-1}{}'\frac{\theta_1(\frac{\hat{j}+\hat{k}\tilde{\tau}}{m};\tau)}{\eta(\tau)} &= e^{i\frac{n-1}{2}\pi}e^{-\frac{i\pi nr}{6}}\tilde{q}^{-\frac{(n-1)(n-1/2)}{6}}\frac{m\eta(\tilde{\tau})^2}{\eta(\tau)^2},\end{aligned} \tag{A.4}$$

with $\tilde{q}=e^{2\pi i\tilde{\tau}}$ and $\tilde{\tau}=\frac{m\tau+r}{n}$. The elliptic gamma function and the 'tilde' elliptic gamma function are defined as

$$\Gamma(z;p,q)=\prod_{j,k=0}^{\infty}\frac{1-p^{j+1}q^{k+1}z^{-1}}{1-p^jq^kz}, \tag{A.5a}$$

$$\widetilde{\Gamma}(u;\sigma,\tau)=\prod_{j,k=0}^{\infty}\frac{1-e^{2\pi i[(j+1)\sigma+(k+1)\tau-u]}}{1-e^{2\pi i[j\sigma+k\tau+u]}}. \tag{A.5b}$$

In this paper, we are mainly interested in the case with $\sigma=\tau$ and abbreviate $\Gamma(z;q,q)$ and $\widetilde{\Gamma}(u;\tau,\tau)$ as $\Gamma(z,q)$ and $\widetilde{\Gamma}(u;\tau)$ respectively. We also define a special function $\psi(u)$ as

$$\psi(u)\equiv\exp\left[u\log\left(1-e^{-2\pi iu}\right)-\frac{1}{2\pi i}\mathrm{Li}_2(e^{-2\pi iu})\right]. \tag{A.6}$$

## A.2 Basic properties

The elliptic theta functions have quasi-double-periodicity, namely

$$\theta_0(u+m+n\tau;\tau)=(-1)^n e^{-2\pi inu}e^{-\pi in(n-1)\tau}\theta_0(u;\tau), \tag{A.7a}$$

$$\theta_1(u+m+n\tau;\tau)=(-1)^{m+n}e^{-2\pi inu}e^{-\pi in^2\tau}\theta_1(u;\tau), \tag{A.7b}$$

for $m,n\in\mathbb{Z}$. The inversion formula of $\theta_0(u;\tau)$ can be written simply as

$$\theta_0(-u;\tau)=-e^{-2\pi iu}\theta_0(u;\tau), \tag{A.8a}$$

$$\theta_1(-u;\tau)=-\theta_1(u;\tau). \tag{A.8b}$$

The elliptic gamma function also has quasi-double-periodicity, namely

$$\widetilde{\Gamma}(u;\sigma,\tau)=\widetilde{\Gamma}(u+1;\sigma,\tau)=\theta_0(u;\tau)^{-1}\widetilde{\Gamma}(u+\sigma;\sigma,\tau)=\theta_0(u;\sigma)^{-1}\widetilde{\Gamma}(u+\tau;\sigma,\tau). \tag{A.9}$$

It also satisfies the inversion formula

$$\widetilde{\Gamma}(u;\sigma,\tau)=\widetilde{\Gamma}(\sigma+\tau-u;\sigma,\tau)^{-1}. \tag{A.10}$$

The Pochhammer symbol, the Dedekind eta function, and the elliptic theta functions are transformed under the modular transformations $T:\tau\to\tau+1$ and $S:\tau\to-1/\tau$ as $(q=e^{2\pi i\tau},\ \tilde{q}=e^{-\frac{2\pi i}{\tau}})$

$$(\tilde{q};\tilde{q})_\infty=(-i\tau)^{\frac{1}{2}}e^{\frac{\pi i}{12}(\tau+\frac{1}{\tau})}(q;q)_\infty, \tag{A.11a}$$

$$\eta(\tau+1)=e^{\frac{\pi i}{12}}\eta(\tau), \tag{A.11b}$$

$$\eta(-1/\tau)=(-i\tau)^{\frac{1}{2}}\eta(\tau), \tag{A.11c}$$

$$\theta_0(u/\tau;-1/\tau)=e^{\frac{\pi i}{\tau}(u^2+u+\frac{1}{6})-\pi i(u+\frac{1}{2})+\frac{\pi i\tau}{6}}\theta_0(u;\tau), \tag{A.11d}$$

$$\theta_1(u;\tau+1)=e^{\frac{\pi i}{4}}\theta_1(u;\tau), \tag{A.11e}$$

$$\theta_1(u/\tau;-1/\tau)=-i(-i\tau)^{\frac{1}{2}}e^{\frac{\pi i u^2}{\tau}}\theta_1(u;\tau). \tag{A.11f}$$

These modular properties can be extended to general SL(2,$\mathbb{Z}$) transformations as (see [25] for example)

$$\begin{aligned}\eta\left(\frac{a\tau+b}{c\tau+d}\right)&=\xi\sqrt{c\tau+d}\,\eta(\tau),\\ \theta_1\left(\frac{u}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right)&=\xi^3\sqrt{c\tau+d}\,e^{\frac{\pi icu^2}{c\tau+d}}\theta_1(u;\tau),\end{aligned} \tag{A.12}$$

where $\xi$ is a 24-th root of unity and $a,b,c,d\in\mathbb{Z}$ with $ad-bc=1$. The SL(2,$\mathbb{Z}$) transformations of the Pochhammer symbol and $\theta_0$ can be derived similarly. The elliptic gamma function can be written in terms of these $S$-transformed elliptic theta functions and the $\psi$-function (A.6) as (see [4] for example)

$$\widetilde{\Gamma}(\Delta_a;\tau)=\frac{e^{2\pi iQ(\{\Delta_a\}_\tau;\tau)}}{\theta_0(\frac{\{\Delta_a\}_\tau-1}{\tau};-\frac{1}{\tau})}\prod_{k=0}^{\infty}\frac{\psi(\frac{k+\{\Delta_a\}_\tau}{\tau})}{\psi(\frac{k+1-\{\Delta_a\}_\tau}{\tau})}, \tag{A.13}$$

where $Q(\cdot;\cdot)$ is defined as (3.66) that we repeat here for convenience:

$$Q(u;\tau)\equiv-\frac{B_3(u)}{6\tau^2}+\frac{B_2(u)}{2\tau}-\frac{5}{12}B_1(u)+\frac{\tau}{12}. \tag{A.14}$$

## A.3 Asymptotic behaviors

For a small $|\tau|$ with fixed $0 < \arg\tau < \pi$, the Pochhammer symbol can be approximated as

$$\log(q;q)_\infty = -\frac{\pi i}{12}(\tau + \frac{1}{\tau}) - \frac{1}{2}\log(-i\tau) + \mathcal{O}(e^{-\frac{2\pi\sin(\arg\tau)}{|\tau|}}). \tag{A.15}$$

The Dedekind eta function in the same limit is then given as

$$\log\eta(\tau) = -\frac{\pi i}{12\tau} - \frac{1}{2}\log(-i\tau) + \mathcal{O}(e^{-\frac{2\pi\sin(\arg\tau)}{|\tau|}}). \tag{A.16}$$

To study asymptotic behaviors of elliptic functions, first we introduce a $\tau$-modded value of a complex number $u$, namely $\{u\}_\tau$, as

$$\{u\}_\tau \equiv u - \lfloor \mathrm{Re}\, u - \cot(\arg\tau)\,\mathrm{Im}\, u \rfloor \quad (u \in \mathbb{C}). \tag{A.17}$$

By definition, the $\tau$-modded value satisfies

$$\{u\}_\tau = \{\tilde{u}\}_\tau + \check{u}\tau, \qquad \{-u\}_\tau = \begin{cases} 1 - \{u\}_\tau & (\tilde{u} \notin \mathbb{Z}) \\ -\{u\}_\tau & (\tilde{u} \in \mathbb{Z}), \end{cases} \tag{A.18}$$

where we have defined $\tilde{u}, \check{u} \in \mathbb{R}$ as

$$u = \tilde{u} + \check{u}\tau. \tag{A.19}$$

Note that, for a real number $x$, a $\tau$-modded value $\{x\}_\tau$ reduces to a normal modded value $\{x\}$ defined as

$$\{x\} \equiv x - \lfloor x \rfloor \quad (x \in \mathbb{R}). \tag{A.20}$$

Now, the elliptic theta function $\theta_0(u;\tau)$ can be approximated for a small $|\tau|$ with fixed $0 < \arg\tau < \pi$ as

$$\begin{aligned} \log\theta_0(u;\tau) = &\frac{\pi i}{\tau}\{u\}_\tau(1 - \{u\}_\tau) + \pi i\{u\}_\tau - \frac{\pi i}{6\tau}(1 + 3\tau + \tau^2) \\ &+ \log\Big(1 - e^{-\frac{2\pi i}{\tau}(1-\{u\}_\tau)}\Big)\Big(1 - e^{-\frac{2\pi i}{\tau}\{u\}_\tau}\Big) + \mathcal{O}(e^{-\frac{2\pi\sin(\arg\tau)}{|\tau|}}), \end{aligned} \tag{A.21}$$

based on an alternative product form of $\theta_0(u;\tau)$ ($m \in \mathbb{Z}$):

$$\begin{aligned} \theta_0(u;\tau) = &-ie^{-\frac{\pi i}{6}(\tau+\frac{1}{\tau})}e^{\frac{\pi i}{\tau}(u-m)(1-u+m)}e^{\pi i(u-m)} \\ &\times\prod_{k=1}^{\infty}(1 - e^{-\frac{2\pi i}{\tau}(k-u+m)})(1 - e^{-\frac{2\pi i}{\tau}(k-1+u-m)}). \end{aligned} \tag{A.22}$$

This product form can be derived by combining (A.3a) with the $S$-transformation (A.11d).

Similarly, the elliptic theta function $\theta_1(u;\tau)$ is approximated for a small $|\tau|$ with fixed $0 < \arg\tau < \pi$ as

$$\begin{aligned}\log\theta_1(u;\tau) = &\frac{\pi i}{\tau}\{u\}_\tau(1-\{u\}_\tau) - \frac{\pi i}{4\tau}(1-\tau) + \pi i\lfloor \operatorname{Re} u - \cot(\arg\tau)\operatorname{Im} u\rfloor - \frac{1}{2}\log\tau \\ &+ \log\Big(1-e^{-\frac{2\pi i}{\tau}(1-\{u\}_\tau)}\Big)\Big(1-e^{-\frac{2\pi i}{\tau}\{u\}_\tau}\Big) + \mathcal{O}(e^{-\frac{2\pi\sin(\arg\tau)}{|\tau|}}),\end{aligned} \tag{A.23}$$

based on an alternative product form of $\theta_1(u;\tau)$ ($m \in \mathbb{Z}$):

$$\begin{aligned}\theta_1(u;\tau) = &(-i\tau)^{-\frac{1}{2}} e^{-\frac{\pi i}{4\tau}} e^{m\pi i} e^{\frac{\pi i}{\tau}(u-m)(1-u+m)} \\ &\times \prod_{k=1}^{\infty}(1-e^{-\frac{2\pi i}{\tau}k})(1-e^{-\frac{2\pi i}{\tau}(k-u+m)})(1-e^{-\frac{2\pi i}{\tau}(k-1+u-m)}).\end{aligned} \tag{A.24}$$

This product form can be derived by combining (A.3b) with the $S$-transformation (A.11f).

For a small $|\tau|$ with fixed $0 < \arg\tau < \pi$, the elliptic gamma function can be approximated as

$$\log\widetilde{\Gamma}(u;\tau) = 2\pi i\, Q(\{u\}_\tau;\tau) + \mathcal{O}(|\tau|^{-1} e^{-\frac{2\pi\sin(\arg\tau)}{|\tau|}\min(\{\tilde{u}\},1-\{\tilde{u}\})}), \tag{A.25}$$

provided $\tilde{u} \not\to \mathbb{Z}$ (see [4] for example). See (A.14) for the definition of $Q(\cdot;\cdot)$.

## A.4 Plethystic expansions

The Pochhammer symbol (A.1) and the elliptic gamma function (A.5a) have plethystic expansions:

$$(p;p)_\infty = \exp\left[\sum_{m=1}^{\infty}\frac{1}{m}\frac{-p^m}{1-p^m}\right], \tag{A.26a}$$

$$\Gamma(y;p,q) = \exp\left[\sum_{m=1}^{\infty}\frac{1}{m}\frac{y^m - p^m q^m y^{-m}}{(1-p^m)(1-q^m)}\right]. \tag{A.26b}$$

In terms of (A.26), we can derive useful identities

$$\exp\left[\sum_{a=1}^{3}\sum_{i,j=1}^{N}\sum_{m=1}^{\infty}\frac{1}{m}\frac{y_a^m - p^m q^m y_a^{-m}}{(1-p^m)(1-q^m)}(z_i/z_j)^m\right] = \prod_{a=1}^{3}\prod_{i,j=1}^{N}\Gamma(y_a z_i/z_j;p,q), \tag{A.27a}$$

$$\exp\left[\sum_{m=1}^{\infty}\frac{1}{m}\frac{2p^m q^m - p^m - q^m}{(1-p^m)(1-q^m)}\right] = (p;p)_\infty (q;q)_\infty, \tag{A.27b}$$

and

$$\begin{aligned}
&\exp\left[\sum_{i<j}^{N}\sum_{m=1}^{\infty}\frac{1}{m}\frac{2p^m q^m - p^m - q^m}{(1-p^m)(1-q^m)}((z_i/z_j)^m + (z_j/z_i)^m)\right] \\
&= \exp\left[\sum_{i<j}^{N}\sum_{m=1}^{\infty}\frac{1}{m}\frac{p^m q^m - 1}{(1-p^m)(1-q^m)}((z_i/z_j)^m + (z_j/z_i)^m) + \sum_{i<j}^{N}\sum_{m=1}^{\infty}\frac{1}{m}((z_i/z_j)^m + (z_j/z_i)^m)\right] \\
&= \prod_{i<j}^{N}\frac{1}{(1-z_i/z_j)(1-z_j/z_i)\Gamma(z_i/z_j;p,q)\Gamma(z_j/z_i;p,q)} \\
&= \prod_{i<j}^{N}\frac{1}{|z_i-z_j|^2}\prod_{1\le i,j\le N}^{i\ne j}\frac{1}{\Gamma(z_i/z_j;p,q)}.
\end{aligned} \tag{A.28}$$

Here we have used $(1-z_i/z_j)(1-z_j/z_i) = |z_i-z_j|^2$ from $z_i^{-1} = z_i^*$. These identities will be used to derive the elliptic hypergeometric integral formula of the SCI (3.13).

# Appendix B

# The Topologically Twisted Index

## B.1 Proof of Lemma 1

Here we prove Lemma 1 in the main text.[1] Using the identity

$$\theta_1'(n;\tau) = \theta_2(n;\tau)\theta_3(n;\tau)\theta_4(n;\tau) \tag{B.1}$$

for an arbitrary integer $n \in \mathbb{Z}$, we can generalize Theorem 2.1 of [90] as

$$\sum_{a=1}^{3} \frac{\theta_I'(\Delta_a;\tau)}{\theta_I(\Delta_a;\tau)} = \begin{cases} +\frac{\theta_1'(n;\tau)}{\theta_I(n;\tau)} \prod_{a=1}^{3} \frac{\theta_1(\Delta_a;\tau)}{\theta_I(\Delta_a;\tau)} & (I=3); \\ -\frac{\theta_1'(n;\tau)}{\theta_I(n;\tau)} \prod_{a=1}^{3} \frac{\theta_1(\Delta_a;\tau)}{\theta_I(\Delta_a;\tau)} & (I=2,4), \end{cases} \tag{B.2}$$

where $\sum_a \Delta_a = n \in \mathbb{Z}$. Here $\theta_{2,3,4}$ are related to $\theta_1(u;\tau)$ defined in (A.3b) via

$$\theta_2(u;\tau) = \theta_1(u+1/2;\tau), \tag{B.3a}$$

$$\theta_3(u;\tau) = e^{\frac{\pi i\tau}{4}} e^{\pi i u}\theta_1(u+(1+\tau)/2;\tau), \tag{B.3b}$$

$$\theta_4(u;\tau) = -ie^{\frac{\pi i\tau}{4}} e^{\pi i u}\theta_1(u+\tau/2;\tau). \tag{B.3c}$$

These theta functions are basically obtained from $\theta_1(u;\tau)$ by shifting the first argument $u$ by three different half-periods $\frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2}$ respectively. They satisfy the so-called Jacobi's formula together with $\theta_1(u;\tau)$, namely [110]

$$\begin{aligned} 2\theta_1(u_0,u_1,u_2,u_3;\tau) = {}& \theta_1(u_0',u_1',u_2',u_3';\tau) + \theta_2(u_0',u_1',u_2',u_3';\tau) \\ & - \theta_3(u_0',u_1',u_2',u_3';\tau) + \theta_4(u_0',u_1',u_2',u_3';\tau), \end{aligned} \tag{B.4}$$

[1] We are indebted to Hjalmar Rosengren, a mathematician at Chalmers University, for an instrumental correspondence regarding the proof.

where $2u'_\alpha = \sum_{\beta=0}^{3} u_\beta - 2u_\alpha$ $(\alpha = 0, 1, 2, 3)$, and we have used the abbreviations

$$\theta_I(u_0, u_1, u_2, u_3; \tau) \equiv \prod_{\alpha=0}^{3} \theta_I(u_\alpha; \tau). \tag{B.5}$$

Since $\theta_1(n; \tau) = 0$ for an arbitrary integer $n \in \mathbb{Z}$, the following special case of Jacobi's formula (B.4) is valid for $\sum_a \Delta_a = n \in \mathbb{Z}$:

$$0 = \theta_2(n, \Delta_1, \tilde{\Delta}_2, \tilde{\Delta}_3; \tau) - \theta_3(n, \tilde{\Delta}_1, \tilde{\Delta}_2, \tilde{\Delta}_3; \tau) + \theta_4(n, \tilde{\Delta}_1, \tilde{\Delta}_2, \tilde{\Delta}_3; \tau). \tag{B.6}$$

Combining (B.6) and (B.2) establishes the Lemma 1.

## B.2 Proof that the map (2.83) is bijective

First we prove that (2.83) is one-to-one, i.e.

$$\hat{j}'_1 = \hat{j}'_2 \ \& \ \hat{k}'_1 = \hat{k}'_2 \quad \Rightarrow \quad \hat{j}_1 = \hat{j}_2 \ \& \ \hat{k}_1 = \hat{k}_2. \tag{B.7}$$

To begin with, note that (2.83a) implies

$$\hat{j}'_1 = \hat{j}'_2 \ \& \ \hat{k}'_1 = \hat{k}'_2 \quad \Rightarrow \quad \hat{k}_1 = \hat{k}_2 \pmod{n}, \tag{B.8}$$

which means $\hat{k}_1 = \hat{k}_2$ in fact. Combined with this fact, (2.83b) implies

$$\hat{j}'_1 = \hat{j}'_2 \ \& \ \hat{k}'_1 = \hat{k}'_2 \quad \Rightarrow \quad \hat{j}_1 = \hat{j}_2 \pmod{m}, \tag{B.9}$$

and therefore we have $\hat{j}_1 = \hat{j}_2$. Hence (2.83) is one-to one.

Next we prove that (2.83) is onto, i.e. there exists $(\hat{j}, \hat{k}) \in \mathbb{Z}_m \times \mathbb{Z}_n$ satisfying (2.83) for any given $(\hat{j}', \hat{k}') \in \mathbb{Z}_{m'} \times \mathbb{Z}_{n'}$ where $m' = g$ and $n' = N/g$ with $g \equiv \gcd(n, r)$. To begin with, recall that we have

$$\frac{n}{g}(-b) + \frac{r}{g}(d) = 1 \tag{B.10}$$

from (2.78). Then for any given $(\hat{j}', \hat{k}') \in \mathbb{Z}_{m'} \times \mathbb{Z}_{n'}$, we have

$$\frac{n}{g}\left(-b\hat{k}' + \frac{r}{g}\hat{j}'\right) + \frac{r}{g}\left(d\hat{k}' - \frac{n}{g}\hat{j}'\right) = \hat{k}'. \tag{B.11}$$

This can be solved for $\hat{k}'$ as

$$\hat{k}' = \left\{\frac{n}{g}\left\{-b\hat{k}' + \frac{r}{g}\hat{j}' + r\left\lfloor\frac{d\hat{k}' - \frac{n}{g}\hat{j}'}{n}\right\rfloor, m\right\} + \frac{r}{g}\left\{d\hat{k}' - \frac{n}{g}\hat{j}', n\right\}, \frac{N}{g}\right\}, \tag{B.12}$$

where $\{A, B\}$ denotes $A$ mod $B$ ($0 \leq A < B$). Now it is straightforward to check that

$$\hat{j} = \left\{-b\hat{k}' + \frac{r}{g}\hat{j}' + r\left\lfloor\frac{d\hat{k}' - \frac{n}{g}\hat{j}'}{n}\right\rfloor, m\right\} \in \mathbb{Z}_m, \tag{B.13a}$$

$$\hat{k} = \left\{d\hat{k}' - \frac{n}{g}\hat{j}', n\right\} \in \mathbb{Z}_n, \tag{B.13b}$$

indeed satisfy (2.83), so (2.83) is onto.

## B.3 Invariance of the Jacobian determinant under modular transformations

Here we demonstrate that $\mathcal{H}_{\{m,n,r\}}$ transforms according to (2.74) and (2.84) under $T$ and $S$ transformations, respectively. We first note that the holonomies for the BAE solution denoted by $\{m, n, r\}$ are canonically ordered according to (2.18). The key step here is then to order the holonomies for the BAE solution denoted by $\{m', n', r'\}$ differently, according to

$$\{m, n, r\} \text{ sector}: \; u_{n\hat{j}+\hat{k}} \quad \to \quad u_{\hat{j}\hat{k}}, \tag{B.14a}$$

$$\{m', n', r'\} \text{ sector}: \; u_{n\hat{j}+\hat{k}} \quad \to \quad u_{\hat{j}'\hat{k}'}. \tag{B.14b}$$

Note that $(\hat{j}, \hat{k}) \to (\hat{j}', \hat{k}')$ is a bijective map from $\mathbb{Z}_m \times \mathbb{Z}_n$ to $\mathbb{Z}_{m'} \times \mathbb{Z}_{n'}$ for both $T$ and $S$ transformations, namely (2.73) and (2.83), so the above ordering for the $\{m', n', r'\}$ sector is valid. Furthermore, it does not affect the value of $\mathcal{H}_{\{m',n',r'\}}$ as the determinant does not depend on the ordering of holonomies.

Now, with respect to the above ordering, we prove

$$H_{\{m,n,r\}}(\tau + 1, \Delta_a) = H_{\{m',n',r'\}}(\tau, \Delta_a), \tag{B.15a}$$

$$H_{\{m,n,r\}}(-1/\tau, \Delta_a/\tau) = \tau^{N-1} H_{\{m',n',r'\}}(\tau, \Delta_a). \tag{B.15b}$$

at a matrix level, which then yields (2.74) and (2.84) automatically. Note that $\{m', n', r'\}$ are different for $T$ and $S$ cases as (2.73) and (2.83) respectively. From (2.46), the $(j, N)$

entries of the LHS and the RHS are the same as unity for $j \in \{1, \cdots, N\}$. To prove that the other matrix elements also match, it suffices to show

$$\mathcal{G}_{\{m,n,r\}}(\hat{j}-\hat{j}_0, \hat{k}-\hat{k}_0; \Delta_a, \tau+1) = \mathcal{G}_{\{m',n',r'\}}(\hat{j}'-\hat{j}'_0, \hat{k}'-\hat{k}'_0; \Delta_a, \tau), \tag{B.16a}$$

$$\mathcal{G}_{\{m,n,r\}}(\hat{j}-\hat{j}_0, \hat{k}-\hat{k}_0; \Delta_a/\tau, -1/\tau) = \tau\mathcal{G}_{\{m',n',r'\}}(\hat{j}'-\hat{j}'_0, \hat{k}'-\hat{k}'_0; \Delta_a, \tau), \tag{B.16b}$$

for any $\hat{j}, \hat{j}_0 \in \mathbb{Z}_m$ and $\hat{k}, \hat{k}_0 \in \mathbb{Z}_n$ and the primed indices given from (2.73) and (2.83) for (B.16a) and (B.16b) respectively. See (2.49) for the definition of the $\mathcal{G}$-function. Note that these are not trivial from (2.72) and (2.82) but can be proved based on those relations and the following properties of the $\mathcal{G}$-function:

$$\mathcal{G}_{\{m,n,r\}}(\hat{j}+m, \hat{k}; \tau, \Delta_a) = \mathcal{G}_{\{m,n,r\}}(\hat{j}, \hat{k}; \tau, \Delta_a), \tag{B.17a}$$

$$\mathcal{G}_{\{m,n,r\}}(\hat{j}, \hat{k}+n; \tau, \Delta_a) = \mathcal{G}_{\{m,n,r\}}(\hat{j}+r, \hat{k}; \tau, \Delta_a). \tag{B.17b}$$

## Proof of (B.16a)

$$\begin{aligned}
LHS &= \mathcal{G}_{\{m,n,r\}}\left(\left\{\hat{j}-\hat{j}_0+r\left\lfloor\frac{\hat{k}-\hat{k}_0}{n}\right\rfloor, m\right\}, \{\hat{k}-\hat{k}_0, n\}; \Delta_a, \tau+1\right) \\
&= \mathcal{G}_{\{m',n',r'\}}\left(\left\{\hat{j}-\hat{j}_0+r\left\lfloor\frac{\hat{k}-\hat{k}_0}{n}\right\rfloor + \{\hat{k}-\hat{k}_0, n\}\left\lfloor\frac{m+r}{n}\right\rfloor, m\right\}, \{\hat{k}-\hat{k}_0, n\}; \Delta_a, \tau\right) \\
&= \mathcal{G}_{\{m',n',r'\}}\left(\left\{\hat{j}+\hat{k}\left\lfloor\frac{m+r}{n}\right\rfloor, m\right\} - \left\{\hat{j}_0+\hat{k}_0\left\lfloor\frac{m+r}{n}\right\rfloor, m\right\}, \hat{k}-\hat{k}_0; \Delta_a, \tau\right) \\
&= RHS.
\end{aligned} \tag{B.18}$$

Here $\{A, B\}$ denotes $A$ mod $B$ $(0 \leq A < B)$. Note that (B.17) has been used in the 1st and the 3rd lines. The 2nd line comes from (2.72).

## Proof of (B.16b)

$$
\begin{aligned}
LHS &= \mathcal{G}_{\{m,n,r\}}\left(\left\{\hat{j}-\hat{j}_0+r\left\lfloor\frac{\hat{k}-\hat{k}_0}{n}\right\rfloor,m\right\},\{\hat{k}-\hat{k}_0,n\};\frac{\Delta_a}{\tau},-\frac{1}{\tau}\right)\\
&= \tau\mathcal{G}_{\{m',n',r'\}}\Bigg(\left\{-\frac{g}{n}\left((\hat{k}-\hat{k}_0)+d\left\{\frac{n}{g}(\hat{j}-\hat{j}_0)+\frac{r}{g}(\hat{k}-\hat{k}_0),\frac{N}{g}\right\}\right),g\right\},\\
&\qquad\left\{\frac{n}{g}(\hat{j}-\hat{j}_0)+\frac{r}{g}(\hat{k}-\hat{k}_0),\frac{N}{g}\right\};\Delta_a,\tau\Bigg)\\
&= \tau\mathcal{G}_{\{m',n',r'\}}\Bigg(\left\{-\frac{g}{n}\left(\hat{k}+d\left\{\frac{n}{g}\hat{j}+\frac{r}{g}\hat{k},\frac{N}{g}\right\}\right),g\right\}\\
&\qquad-\left\{-\frac{g}{n}\left(\hat{k}_0+d\left\{\frac{n}{g}\hat{j}_0+\frac{r}{g}\hat{k}_0,\frac{N}{g}\right\}\right),g\right\},\\
&\qquad\left\{\frac{n}{g}\hat{j}+\frac{r}{g}\hat{k},\frac{N}{g}\right\}-\left\{\frac{n}{g}\hat{j}_0+\frac{r}{g}\hat{k}_0,\frac{N}{g}\right\};\Delta_a,\tau\Bigg)=RHS.
\end{aligned}
\tag{B.19}
$$

Note that (B.17) has been used in the 1st and the 4th lines. The 2nd line comes from (2.72) followed by the identity $M\{A,B\}=\{MA,MB\}$.

# Appendix C

# The Superconformal Index

## C.1 Contribution from $C$-center solutions

In this Appendix we repeat the same procedures in 3.2.1 and 3.2.2 for a more general class of saddle point solutions and the BA solutions respectively. Both solutions are denoted by a finite, positive divisor of $N$, namely $C$, and the solution with $C = 1$ corresponds to what we have discussed in the main text. We will call them $C$-center saddles and $C$-center BAE solutions respectively.

The final results of this Appendix, namely (C.13) and (C.18), are consistent with each other for the first three terms. The remaining pure imaginary or order $\mathcal{O}(N^0)$ terms do not match apparently: more detailed analysis on contour deformations in the saddle point approach and on the Jacobian contribution in the BA approach would be required for a perfect match and we leave it for future research.

Another important implication of this Appendix is that 3D SU($N$) Chern-Simons theory arises from $\mathcal{N} = 4$ SU($N$) SYM on $S^1 \times S^3$ in the Cardy-like limit independently of saddle point solutions. In the main text we have observed it for a basic saddle point (3.33). The following subsection C.1.1 will generalize this result to $C$-center saddles (C.1).

Lastly, it is worth highlighting the robustness of the universal $\log N$ term. We will demonstrate that these $C$-center saddles or BAE solutions, which can be dominant in certain domain of chemical potentials $\Delta_a$ as we discussed in 3.3, still contribute $\log\left(\frac{N}{C}\right)$ to the SCI. This is compatible with the result for $C = 1$ in the main text.

### C.1.1 Saddle point approach

In 3.2.1, we have investigated the contribution from a basic saddle point ansatz (3.33) to the SCI through the saddle point approximation (3.20). Here we repeat the same procedure but with a more general ansatz for $C$-center saddles [5], namely

$$\{u_j\}^{(C,m)} = \left\{ u_j^{(C,m)} = \frac{m}{N} + \frac{\lfloor \frac{j-1}{N/C} \rfloor - \frac{C-1}{2}}{C} + v_j\tau \,\middle|\, v_j \sim \mathcal{O}(|\tau|^0),\ \sum_{j=1}^{N} v_j = 0 \right\} \tag{C.1}$$

with $m \in \{0, 1, \cdots, \frac{N}{C} - 1\}$. The range of $m$ is determined from the integration contour deformed from (3.22) as

$$\bigcup_{\mu=1}^{N-1} (v_\mu\tau - \frac{1}{2N} - \frac{C-1}{2C},\ v_\mu\tau + 1 - \frac{1}{2N} - \frac{C-1}{2C}], \tag{C.2}$$

which passes through the above $C$-center saddle $\{u_j\}^{(C,m)}$. The $C$-center ansatz (C.1) and the corresponding deformed contour (C.2) reduce to the ones in the main text (3.33) and (3.37) respectively for $C = 1$.

In the strict Cardy-like limit $|\tau| \to 0$, the $C$-center ansatz (C.1) reduces to $C$ groups of holonomies, where each group has equal number $(N/C)$ of condensed holonomies and is separated from adjacent groups by $1/C$ along the domain $(0, 1]$ with 0 identified with 1. The name '$C$-center' comes from its symmetry breaking pattern $\mathbb{Z}_N \to \mathbb{Z}_C$ [5].

Following 3.2.1 and using the following identity of Bernoulli polynomials

$$\sum_{J=0}^{C-1} B_n(\{\frac{J}{C} + u\}_\tau) = \frac{1}{C^{n-1}} B_n(\{Cu\}_\tau) \quad (u \in \mathbb{C}), \tag{C.3}$$

we simplify the effective action (3.31) <u>near</u> the $C$-center ansatz (C.1) up to exponentially suppressed terms as

$$\begin{aligned} N^2 S_{\text{eff}}(\{u_i\}; \tau, \Delta) \sim \sum_{I=0}^{C-1} &\left( -\frac{\pi i \eta_C N}{C\tau^2} \sum_{i=1}^{N/C} (u_{I,i} - \bar{u}_I)^2 + \sum_{i \neq j}^{N/C} \log\left( 2 \sin \frac{\pi(u_{I,i} - u_{I,j})}{\tau} \right) \right) \\ &- \frac{\pi i}{2\tau^2} \frac{N^2}{C^2} \sum_{I,J=0}^{C-1} \xi_{I-J} (\bar{u}_{IJ})^2 - \frac{\pi i N^2}{C^3 \tau^2} \prod_{a=1}^{3} \left( \{C\Delta_a\}_\tau - \frac{1+\eta_C}{2} \right) \\ &+ \frac{\pi i}{\tau^2} \prod_{a=1}^{3} \left( \{\Delta_a\}_\tau - \frac{1+\eta_1}{2} \right) - \frac{5\pi i \eta_C N^2}{12C} + \frac{\pi i N}{2} - \frac{\pi i (6 - 5\eta_1)}{12} \\ &- (N-1) \log \tau, \end{aligned} \tag{C.4}$$

where we have introduced $u_{I,i}$ and $\bar{u}_I$ as

$$\begin{aligned} u_i &= \frac{m}{N} + \frac{I - \frac{C-1}{2}}{C} + u_{I,i-(N/C)I} \quad \left(I = \left\lfloor \frac{i-1}{N/C} \right\rfloor,\ i = 1, \cdots, N\right), \\ \bar{u}_I &= \frac{1}{N/C} \sum_{i=1}^{N/C} u_{I,i}. \end{aligned} \tag{C.5}$$

Note that $\sum_{I=0}^{C-1} \bar{u}_I = 0$ from the SU($N$) constraint (3.14). We have also defined $\xi_J$ as

$$\sum_{a=1}^{3} \{\frac{J}{C} + \Delta_a\}_\tau = 2\tau + \frac{3 + \xi_J}{2}, \tag{C.6}$$

which is related to $\eta_C$ defined in (3.28) as $\eta_C = \sum_{J=0}^{C-1} \xi_J \in \{\pm 1\}$ under the assumption $C\tilde{\Delta}_a \notin \mathbb{Z}$ (3.29).

Substituting the effective action (C.4) into the saddle point evaluation (3.20) gives the SCI as

$$\begin{aligned} \mathcal{I}(\tau, \Delta) \sim &\sum_{m=0}^{N/C-1} \frac{\mathcal{A}}{((N/C)!)^C} \int_{D_{\{u_i\}^{(C,m)}}} \prod_{\mu=1}^{N-1} du_\mu \, e^{N^2 S_{\text{eff},u\text{-dept}}(\{u_i\};\Delta_a,\tau)} \\ &+ (\text{contribution from the other saddles}), \end{aligned} \tag{C.7}$$

where $D_{\{u_i\}^{(C,m)}}$ denotes a small neighborhood of a $C$-center saddle $\{u_i\}^{(C,m)}$ (C.1) on the contour (C.2), namely

$$D_{\{u_i\}^{(C,m)}} = \bigcup_{\mu=1}^{N-1} (v_\mu\tau + \frac{m}{N} + \frac{\lfloor \frac{\mu-1}{N/C} \rfloor - \frac{C-1}{2}}{C} - \epsilon, v_\mu\tau + \frac{m}{N} + \frac{\lfloor \frac{\mu-1}{N/C} \rfloor - \frac{C-1}{2}}{C} + \epsilon], \tag{C.8}$$

with some small positive number $\epsilon$. The $u$-dependent part of the effective action, namely $N^2 S_{\text{eff},u\text{-dept}}(\{u_i\}; \tau, \Delta)$, denotes the first three terms of (C.4) and the prefactor $\mathcal{A}$ is related to the remaining $u$-independent part of (C.4) as

$$\begin{aligned} \mathcal{A} = \exp\Bigg[ &-\frac{\pi i N^2}{C^3\tau^2} \prod_{a=1}^{3} \left( \{C\Delta_a\}_\tau - \frac{1+\eta_C}{2} \right) + \frac{\pi i}{\tau^2} \prod_{a=1}^{3} \left( \{\Delta_a\}_\tau - \frac{1+\eta_1}{2} \right) \\ &- \frac{5\pi i \eta_C N^2}{12C} + \frac{\pi i N}{2} - \frac{\pi i (6 - 5\eta_1)}{12} - (N-1)\log\tau \Bigg]. \end{aligned} \tag{C.9}$$

Note that we have $((N/C)!)^C$ instead of the original $N!$ in the denominator of (C.7) taking an extra factor of $\frac{N!}{((N/C)!)^C}$ into account, which corresponds to the number of evenly distributing

$N$ holonomies into $C$ groups as (C.5).

Introducing new integration variables $\lambda_{I,i}$ and $\bar{\lambda}_I$ as

$$\begin{aligned} -i\lambda_{I,i}\tau &= u_{I,i} - \bar{u}_I \qquad (I = 0, \cdots, C-1 \text{ and } i = 1, \cdots, N/C - 1), \\ -i\bar{\lambda}_I\tau &= \bar{u}_I \qquad (I = 0, \cdots, C-2), \end{aligned} \tag{C.10}$$

whose Jacobian is given as

$$\left| \frac{\partial(u_1, \cdots, u_{N-1})}{\partial(\lambda_{0,1}, \cdots, \lambda_{0,N/C-1}, \bar{\lambda}_0, \lambda_{1,1}, \cdots, \lambda_{C-1,N/C-1})} \right| = e^{-\frac{\pi i(N-1)}{2}} \left(\frac{N}{C}\right)^{C-1} \tau^{N-1}, \tag{C.11}$$

the SCI (C.7) can be rewritten as

$$\begin{aligned} \mathcal{I}(\tau, \Delta) = &\frac{N}{C}\tau^{N-1}\mathcal{A}\, e^{-\frac{\pi i(N^2/C-1)}{2}} \left(Z^{CS}_{\mathrm{SU}(N/C)}\right)^C \int \prod_{I=0}^{C-2} d\bar{\lambda}_I\, e^{\frac{\pi i}{2}\sum_{I,J=0}^{C-1}\xi_{I-J}(\bar{\lambda}_{IJ})^2} \\ &+ (\text{contribution from other saddles}). \end{aligned} \tag{C.12}$$

Here we have assumed smooth deformations of contours as we have done from (3.43) to real lines in the main text. Note that the original SU($N$) group breaks down into $C$ copies of SU($N/C$) groups and the remaining $C-1$ copies of U(1) groups. As a result, we obtained $C$ copies of the SU($N/C$) Chern-Simons partition function together with an extra ($C-1$)-dimensional integral for U(1) terms. We denote the latter simply as $Z_{\mathrm{U(1)'s}}$. Finally, substituting the partition function of SU($N$) CS theory (C.60) with $N \to N/C$ into (C.12) gives

$$\begin{aligned} \mathcal{I}(\tau, \Delta) = \frac{N}{C} \exp\Bigg[ &-\frac{\pi i N^2}{C^3\tau^2}\prod_{a=1}^{3}\left(\{C\Delta_a\}_\tau - \frac{1+\eta_C}{2}\right) + \frac{\pi i}{\tau^2}\prod_{a=1}^{3}\left(\{\Delta_a\}_\tau - \frac{1+\eta_1}{2}\right) \\ &+ \frac{5\pi i(\eta_1 - C\eta_C)}{12}\Bigg] \times Z_{\mathrm{U(1)'s}} \\ &+ (\text{contribution from other saddles}). \end{aligned} \tag{C.13}$$

### C.1.2 Bethe Ansatz approach

In 3.2.2, we have investigated the contribution from basic solutions (3.56) to the SCI through the Bethe Ansatz formula (3.47). Here we generalize it with a larger set of BAE solutions denoted by a positive divisor of $N$, namely $C$, as

$$\{u_i\}_C = \left\{ u_i = \bar{u} + \frac{I}{C} + \frac{i - (N/C)I}{N/C}\tau + m_i + n_i\tau \,\middle|\, I = \left\lfloor \frac{i}{N/C} \right\rfloor,\ i = 1, \cdots, N \right\} \tag{C.14}$$

with arbitrary integers $m_i$'s and $n_i$'s. We call them $C$-center BAE solutions. Note that this solution is equivalent to the $\{C, N/C, 0\}$ solution in [67] and the $(C, N/C)$ saddle in [30]. Since the calculation is parallel to the one in the main text, we summarize the key intermediate results only.

First, the degeneracy gives $\log N! + \log N$ by the same token we discussed in the beginning of 3.2.2. The prefactor contribution also remains the same as (3.63). Calculating the contribution from $\log \mathcal{Z}(\{u_i\}_C; \tau, \Delta)$ is more involved but does not require extra techniques other than using (3.69) and (C.3). The result is given as

$$\begin{aligned}
\log \mathcal{Z}(\{u_i\}_C; \tau, \Delta) &= C \sum_{J=0}^{C-1} \sum_{i,j=0}^{N/C-1} \sum_{a=1}^{3} \log \tilde{\Gamma}(\frac{i-j}{N/C}\tau + \frac{J}{C} + \Delta_a; \tau) - N \sum_{a=1}^{3} \log \tilde{\Gamma}(\Delta_a; \tau) \\
&\quad - C \sum_{J=1}^{C-1} \sum_{i,j=0}^{N/C-1} \log \tilde{\Gamma}(\frac{i-j}{N/C}\tau + \frac{J}{C}; \tau) - C \sum_{i,j=0\,(i\neq j)}^{N/C-1} \log \tilde{\Gamma}(\frac{i-j}{N/C}\tau; \tau) \\
&= -\frac{\pi i N^2}{C^3 \tau^2} \prod_{a=1}^{3} \left( \{C\Delta_a\}_\tau - \frac{1+\eta_C}{2} \right) + \frac{\pi i (1-\eta_C) N^2}{2C} + \frac{\pi i (1 - 3\tau + \tau^2) N}{6\tau} \\
&\quad + \frac{\pi i \eta_C C}{12} - N \sum_{a=1}^{3} \log \tilde{\Gamma}(\Delta_a; \tau) + N \log \frac{N}{C} - 2N \log(\tilde{q}; \tilde{q})_\infty \\
&\quad + \mathcal{O}(N e^{-\frac{2\pi N \sin(\arg \tau)}{|\tau|} \min(\{\frac{J}{C} + \tilde{\Delta}_a\}, 1 - \{\frac{J}{C} + \tilde{\Delta}_a\} |\, J = 0,1,\cdots,C-1)}),
\end{aligned} \tag{C.15}$$

where $\eta_C$ is defined as (3.28). The Jacobian contribution $-\log \mathcal{H}(\{u_i\}_C; \tau, \Delta)$ can also be obtained by following 3.2.2 and 2.2.2 as

$$\begin{aligned}
-\log H(\{u_i\}_C; \tau, \Delta) &= -\log N - (N-1) \log \left( \frac{i}{\pi} \sum_{\Delta} \partial_\Delta \log \theta_1(C\Delta; \frac{\tau}{N/C^2}) \right) - \log \det \Big( I_{N-1} + \widetilde{H} \Big) \\
&= -N \log N + (N-1) \log \frac{C\tau}{\eta_C} - \log \det \Big( I_{N-1} + \widetilde{H} \Big) \\
&\quad + \mathcal{O}(e^{-\frac{2\pi N \sin(\arg \tau)}{|\tau|} \min(\{\frac{J}{C} + \tilde{\Delta}_a\}, 1 - \{\frac{J}{C} + \tilde{\Delta}_a\} |\, J = 0,1,\cdots,C-1)}).
\end{aligned} \tag{C.16}$$

Substituting all the contributions to the BA formula of the SCI (3.47) and using (A.11a), finally we obtain the contribution from $C$-center BAE solutions (C.14) in the large-$N$ limit as

$$\log \mathcal{I}(\tau,\Delta)\big|_{C\text{-center}} = -\frac{\pi i N^2}{C^3\tau^2}\prod_{a=1}^{3}\Big(\{C\Delta_a\}_\tau - \frac{1+\eta_C}{2}\Big) + \log\frac{N}{C} + \frac{\pi i(1-\eta_C)N^2}{2C} + \frac{\pi i\eta_C C}{12} - \frac{\pi i(1-\eta_C)(N-1)}{2} - \sum_{a=1}^{3}\log\tilde{\Gamma}(\Delta_a;\tau) - 2\log(q;q)_\infty - \log\tau - \log\det\Big(I_{N-1}+\widetilde{H}\Big) + \mathcal{O}(Ne^{-\frac{2\pi N\sin(\arg\tau)}{|\tau|}\min(\{\frac{J}{C}+\tilde{\Delta}_a\},1-\{\frac{J}{C}+\tilde{\Delta}_a\}|\,J=0,1,\cdots,C-1)}). \tag{C.17}$$

In the Cardy-like limit that imposes $|\tau|\ll 1$, this reads

$$\log \mathcal{I}(\tau,\Delta)\big|_{C\text{-center}} \sim -\frac{\pi i N^2}{C^3\tau^2}\prod_{a=1}^{3}\Big(\{C\Delta_a\}_\tau - \frac{1+\eta_C}{2}\Big) + \frac{\pi i}{\tau^2}\prod_{a=1}^{3}\Big(\{\Delta_a\}_\tau - \frac{1+\eta_1}{2}\Big) + \log\frac{N}{C} + \frac{\pi i(1-\eta_C)N^2}{2C} - \frac{\pi i(1-\eta_C)(N-1)}{2} + \frac{\pi i\eta_C C}{12} - \frac{\pi i(6-5\eta_1)}{12} - \log\det\Big(I_{N-1}+\widetilde{H}\Big) \tag{C.18}$$

up to exponentially suppressed terms. Based on the comments below (2.103), we may simplify this further as

$$\log \mathcal{I}(\tau,\Delta)\big|_{C\text{-center}} \sim -\frac{\pi i N^2}{C^3\tau^2}\prod_{a=1}^{3}\Big(\{C\Delta_a\}_\tau - \frac{1+\eta_C}{2}\Big) + \frac{\pi i}{\tau^2}\prod_{a=1}^{3}\Big(\{\Delta_a\}_\tau - \frac{1+\eta_1}{2}\Big) + \mathcal{O}(|\tau|^{-1}). \tag{C.19}$$

## C.2 Saddle point solutions of 3D Chern-Simons theory

In this Appendix, we investigate the saddle point equation (3.36) from the effective action of the $\mathcal{N}=4$ SU($N$) SYM theory in the Cardy-like expansion (3.35), namely

$$i\eta_1\, v_j = \frac{1}{N}\sum_{k=1\,(\neq j)}^{N}\cot\pi v_{jk} \quad (i=1,\cdots,N). \tag{C.20}$$

This equation is in fact equivalent to the saddle point equation of 3D Chern-Simons theory with a 't Hooft coupling $t$ [1, 63],

$$\frac{1}{t}u_j = \frac{1}{N}\sum_{k=1\,(\neq j)}^{N}\coth\frac{u_{jk}}{2}, \tag{C.21}$$

under $v_j \to iu_j/2\pi$ and $t = 2\pi i/\eta_1$. We solve this saddle point equation in the planar limit, or equivalently in the large-$N$ limit.

The partition function of 3D Chern-Simons theory on $S^3$ can be written as [1, 63]

$$Z = \frac{1}{N!}\int \prod_i du_i \prod_{i<j}\left(2\sinh\frac{u_{ij}}{2}\right)^2 \exp\left(-\frac{1}{2g_s}\sum_i u_i^2\right), \tag{C.22}$$

where $g_s = 2\pi i/\hat{k}$ and $\hat{k}$ is the effective Chern-Simons level. As we have seen in (3.35), the fluctuations around the dominant saddle point of the $\mathcal{N} = 4$ SYM theory are described by such a Chern-Simons theory, provide we make the identification $t = 2\pi i/\eta_1$ where $t = g_s N$ is the 't Hooft coupling. Although this partition function can be evaluated directly [79] as detailed in Appendix C.3, it is important to note that our starting point is a saddle point evaluation of the $\mathcal{N} = 4$ SYM index. Hence, in principle, we should seek a saddle point evaluation of the 3D Chern-Simons partition function. As we demonstrate in this Appendix, the saddle point result coincides with the exact partition function in the large-$N$ limit, so in practice this distinction is immaterial. However, we highlight an interesting observation that there are, in fact, multiple saddle point solutions to the Chern-Simons model and that it is important to properly identify the dominant saddle in order to find agreement.

### C.2.1 The dominant saddle point

The saddle point equation obtained by varying the action in (C.22) takes the form

$$\frac{1}{t}u_j = \frac{1}{N}\sum_{k=1\,(\neq j)}^{N} \coth\frac{u_{jk}}{2}. \tag{C.23}$$

As in [89], it is convenient to introduce the exponentiated eigenvalues $X_j = e^{u_j}$, so that the saddle point equation becomes

$$\log X_j = \frac{t}{N}\sum_{k=1\,(\neq j)}^{N}\left(-1 + \frac{2X_j}{X_j - X_k}\right). \tag{C.24}$$

As usual, in the large-$N$ limit, we assume the eigenvalues condense along a single cut, $x \in [a, b]$ on the real axis, provided the 't Hooft parameter $t$ is real. (Later on we will analytically continue to complex $t$.) We then introduce the density of eigenvalues $\rho(x)$ such that

$$\sum_i f(x_i) \quad \longrightarrow \quad N\int_a^b dx\,\rho(x) f(x). \tag{C.25}$$

The important properties of the matrix partition function are now encoded in the eigenvalue density. In order to determine $\rho(x)$, we introduce the resolvent

$$\omega(X) \equiv -t + 2t \int_a^b dy\, \rho(y) \frac{X}{X - Y} \qquad (X \in \mathbb{C} \setminus \mathcal{C}). \tag{C.26}$$

This function is analytic in the complex $X$ plane except for a cut $\mathcal{C}$ from $[e^a, e^b]$ on the positive real axis. By studying $\omega(X)$ on both sides of the cut, we can reproduce the saddle point equation

$$\log X = \frac{1}{2}[\omega_+(X) + \omega_-(X)] \qquad (X \in \mathcal{C}), \tag{C.27}$$

and also recover the eigenvalue density

$$\rho(x) = -\frac{1}{4\pi i t}[\omega_+(X) - \omega_-(X)] \qquad (X \in \mathcal{C}). \tag{C.28}$$

Here we have defined

$$\omega_\pm(X) = \omega(e^{x \pm i\epsilon}) = \omega(X \pm i\epsilon) \qquad (X \in \mathcal{C}). \tag{C.29}$$

Following [89], we can use the following trick to derive the resolvent $\omega(X)$. Recall that $\omega(X)$ is analytic on $X \in \mathbb{C} \setminus \mathcal{C}$. Then it is straightforward to check that the function $g(X)$ defined as

$$g(X) \equiv e^{\omega(X)/2} + X e^{-\omega(X)/2} \qquad (X \in \mathbb{C} \setminus \mathcal{C}) \tag{C.30}$$

can be analytically continued to the entire complex plane including $\mathcal{C}$ since

$$g_+(X) = e^{\omega_+(X)/2} + X e^{-\omega_+(X)} = X e^{-\omega_-(X)} + e^{\omega_-(X)} = g_-(X) \quad (X \in \mathcal{C}), \tag{C.31}$$

where the equality in the middle corresponds to the saddle point equation (C.27). Furthermore, using the asymptotic behavior of (C.26)

$$\lim_{X \to 0} \omega(X) = -t, \qquad \lim_{|X| \to \infty} \omega(X) = t, \tag{C.32}$$

we deduce the form of $g(X)$ as

$$g(X) = e^{-t/2}(X + 1) \qquad (X \in \mathbb{C}). \tag{C.33}$$

Substituting this into (C.30) then gives

$$e^{\omega(X)/2} = \frac{1}{2}\left(g(X) \pm \sqrt{g(X)^2 - 4X}\right). \tag{C.34}$$

Consistency of this solution demands that the branch cut of the square root is along $\mathcal{C}$. In particular, note that the branch points of the square root are given by

$$X_\pm = 2e^t - 1 \pm 2(e^{2t} - e^t)^{\frac{1}{2}}, \tag{C.35}$$

with the product $X_+X_- = 1$.

**The solution for $t > 0$**

Although we have assumed that the eigenvalues condense along the real line, the endpoints $X_\pm$ are only real for real $t > 0$. Assuming this to be the case, the resolvent (C.34) can be written as

$$e^{\omega(X)/2} = \frac{1}{2}\left(e^{-t/2}(X+1) - e^{-t/2}(X - X_+)^{\frac{1}{2}}(X - X_-)^{\frac{1}{2}}\right), \tag{C.36}$$

where the principal branch is taken for both square roots. The eigenvalue density can then be recovered from the discontinuity across the cut using (C.28), with the result [89]

$$\rho(x) = \frac{1}{\pi t}\tan^{-1}\frac{\sqrt{e^t - \cosh^2\frac{x}{2}}}{\cosh\frac{x}{2}} \quad (x \in [a,b]), \tag{C.37}$$

where the endpoints are given by $-a = b = 2\cosh^{-1}(e^{t/2})$.

Substituting this eigenvalue density into the saddle point action is non-trivial, but can be shown to give the genus-zero free energy (see *e.g.* Appendix A of [63])

$$\log Z = N^2\left(\frac{\zeta(3) - \mathrm{Li}_3(e^{-t})}{t^2} + \frac{t}{6} - \frac{\pi^2}{6t}\right) + o(N^2). \tag{C.38}$$

This has a simple expansion in the large-$t$ limit

$$\log Z/N^2 \sim \frac{t}{6} - \frac{\pi^2}{6t} + \frac{\zeta(3)}{t^2} + \mathcal{O}(e^{-t}) \qquad (t \gg 1), \tag{C.39}$$

but remains valid for real $t > 0$. For small $t$, it has an expansion

$$\log Z/N^2 = \frac{1}{2}\log t - \frac{3}{4} + \frac{t}{12} + \frac{t^2}{288} + \cdots \qquad (t \to 0^+), \tag{C.40}$$

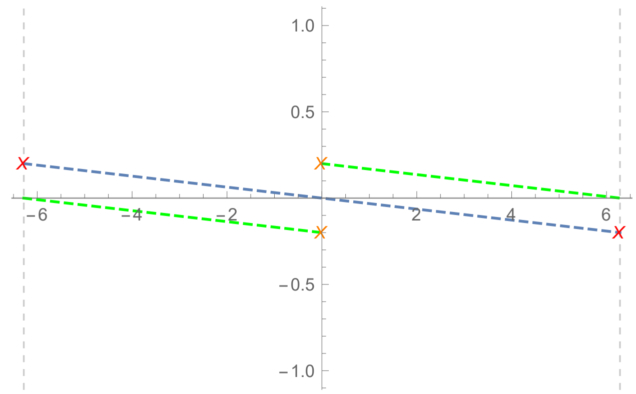


Figure C.1: Orange (red) crosses are branch points and green (blue) lines are branch cuts of $h_+(x)$ and $h_-(x)$, respectively. Here we chose $\epsilon = 1/10$ for presentation.

which diverges logarithmically as $t \to 0$.

### The solution for $t = 2\pi i/\eta$ with $\eta_1 = \pm 1$

While we have worked with real $t$ above, in order to connect to $\mathcal{N} = 4$ SYM, we want to analytically continue to a purely imaginary value $t = 2\pi i/\eta_1$ where $\eta_1 = \pm 1$. However, this continuation is subtle, since $\eta_1 = \pm 1$ turns out to be the endpoints of a singular region of the Chern-Simons matrix model. In particular, there is a divergence for $t = 2\pi i/\eta_1$ with $-1 < \eta < 1$ [91]. This subtlety can also be seen by noting that the endpoints of the cut, $X_\pm$ in (C.35), collapse to $X_\pm = 1$ when $\eta_1 = \pm 1$.

To avoid this singularity issue for $\eta_1 = \pm 1$, we take $t = 2\pi i/\eta_1 + \epsilon^2$ where $\epsilon$ is a small positive number. Although we have assumed real $t$ above, it was not strictly needed in order to obtain the resolvent (C.34). We thus start from there and analytically continue to imaginary eigenvalues, $x \to ix$. In particular, we take $X = e^{ix}$, in which case the resolvent takes the form

$$e^{\omega(X)/2} = \frac{1}{2}\left(e^{-t/2}(e^{ix}+1) + (e^{-t/2}(e^{ix}+1) + 2e^{ix/2})^{\frac{1}{2}}(e^{-t/2}(e^{ix}+1) - 2e^{ix/2})^{\frac{1}{2}}\right). \quad \text{(C.41)}$$

For $t = \pm 2\pi i + \epsilon^2$, the square root factors have the following branch cuts:

$$\begin{aligned} h_+(x) &\equiv (e^{-t/2}(e^{ix}+1) + 2e^{ix/2})^{\frac{1}{2}} : \quad \bigcup_{n\in\mathbb{Z}}[(4n+2)\pi - x_*, (4n+2)\pi + x_*], \\ h_-(x) &\equiv (e^{-t/2}(e^{ix}+1) - 2e^{ix/2})^{\frac{1}{2}} : \quad \bigcup_{n\in\mathbb{Z}}[4n\pi - x_*, 4n\pi + x_*], \end{aligned} \quad \text{(C.42)}$$

where $x_* = 2\pi - 2i\epsilon + \mathcal{O}(\epsilon^2)$ (see Figure C.1). Using[1]

$$h_\pm(x) = (-(1 \mp e^{ix/2})^2)^{\frac{1}{2}} + \mathcal{O}(\epsilon^2), \tag{C.43}$$

we can write down $h_\pm(x)$ more explicitly with the above specified branch cuts as

$$h_+(x) = \begin{cases} \pm i(1 - e^{ix/2}) + \mathcal{O}(\epsilon^2) & \text{(above the cuts of } h_+(x)) \\ \mp i(1 - e^{ix/2}) + \mathcal{O}(\epsilon^2) & \text{(below the cuts of } h_+(x)), \end{cases} \tag{C.44a}$$

$$h_-(x) = \begin{cases} \pm i(1 + e^{ix/2}) + \mathcal{O}(\epsilon^2) & \text{(above the cuts of } h_-(x)) \\ \mp i(1 + e^{ix/2}) + \mathcal{O}(\epsilon^2) & \text{(below the cuts of } h_-(x)). \end{cases} \tag{C.44b}$$

We now rewrite the resolvent (C.41) using (C.44) within the strip $\operatorname{Re} x \in (-2\pi, 2\pi)$ explicitly as

$$e^{\omega(X)/2} = \begin{cases} -1 + \mathcal{O}(\epsilon^2) & \text{(above the cuts of } h_\pm(x)) \\ -e^{ix} + \mathcal{O}(\epsilon^2) & \text{(between the cuts of } h_\pm(x)) \\ -1 + \mathcal{O}(\epsilon^2) & \text{(below the cuts of } h_\pm(x)), \end{cases} \tag{C.45a}$$

$$\to \quad \omega(X) = \begin{cases} -\frac{2\pi i}{\eta} + \mathcal{O}(\epsilon^2) & \text{(above the cuts of } h_\pm(x)) \\ -\frac{2\pi i}{\eta}(1 - \frac{x}{\pi}) + \mathcal{O}(\epsilon^2) & \text{(between the cuts of } h_\pm(x),\ \operatorname{Re} x \in [0, 2\pi)) \\ \frac{2\pi i}{\eta}(1 + \frac{x}{\pi}) + \mathcal{O}(\epsilon^2) & \text{(between the cuts of } h_\pm(x),\ \operatorname{Re} x \in (-2\pi, 0)) \\ \frac{2\pi i}{\eta} + \mathcal{O}(\epsilon^2) & \text{(below the cuts of } h_\pm(x)). \end{cases} \tag{C.45b}$$

Since (C.45a) determines $\omega(X)$ only up to $4\pi i\mathbb{Z}$, we have used the asymptotic conditions from (C.32) along with continuity outside of the branch cuts to fix $\omega(X)$. Finally, the eigenvalue density can be obtained by substituting (C.45b) into (C.28)

$$\rho(x) = \begin{cases} \frac{1}{2\pi}\left(1 - \frac{x}{2\pi}\right) + \mathcal{O}(\epsilon^2) & x \in [0, x_*) \\ \frac{1}{2\pi}\left(1 + \frac{x}{2\pi}\right) + \mathcal{O}(\epsilon^2) & x \in (-x_*, 0). \end{cases} \tag{C.46}$$

[1]This Taylor expansion becomes subtle as $x \to 2\pi\mathbb{Z}$ where the leading order vanishes. So we focus on the bulk and ignore this subtle issue near the endpoints $x = 2\pi\mathbb{Z}$.

Taking the limit $\epsilon \to 0$ then gives the simple expression

$$\rho(x) = \frac{1}{2\pi}\left(1 - \frac{|x|}{2\pi}\right) \qquad x \in (-2\pi, 2\pi), \tag{C.47}$$

which satisfies the normalization condition $\int_{-2\pi}^{2\pi} dx\, \rho(x) = 1$ as expected. Recall that, since we have analytically continued, the actual eigenvalues $u = ix$ are now distributed between $\pm 2\pi i$ along the imaginary axis.

The genus-zero free energy can be obtained by evaluating the saddle point action

$$S_{\text{eff}}/N^2 = \left[ -\frac{1}{2t}\int dx \rho(x) u^2 + \frac{1}{2}\int \rho(x)\rho(\tilde{x}) dx\, d\tilde{x} \log\left(4\sinh^2 \frac{u - \tilde{u}}{2}\right)^2 \right]_{u=ix,\, \tilde{u}=i\tilde{x}} \tag{C.48}$$

on the solution given by (C.46). Here some care must be taken in keeping the $\epsilon$ regulator while integrating the log term because of branch issues. The result is simply

$$\log Z/N^2\Big|_{t=\pm 2\pi i} = \frac{5\pi i}{12}\eta_1, \tag{C.49}$$

which is purely imaginary. This result can also be obtained directly by analytic continuation, namely by inserting $t = 2\pi i/\eta_1$ into (C.38) but our careful analysis provides some direct insight into the structure of eigenvalues.

### C.2.2 The sub-leading saddle point

In deriving the resolvent (C.34), we assumed a one-cut solution with the cut extending along $[X_-, X_+]$. The function $g(X)$ defined in (C.30) is then argued to be analytic in the complex plane. For $t > 0$, this picture is evident as the cut is on the positive real axis in the $X$ plane. However, for $t = \pm 2\pi i$, the cut starts at $1 - 2\epsilon$, wraps twice around the unit circle, and ends at $1 + 2\epsilon$, where $\epsilon$ prevents the cut from overlapping with itself.

This picture of a cut wrapping twice around the unit circle in the $X$ plane suggests the possibility of another solution where the cut extends only once around the circle. We have in fact identified such a solution where the cut starts at $X = -1$, goes around the circle, and ends again at $X = -1$. What is special about this solution is that the double endpoint $X = -1$ may be singular, and this allows for $g(X)$ defined in (C.30) to have a pole at $X = -1$. In particular, we find that

$$g(X) = e^{-t/2}(X+1) + e^{t/2}\frac{X}{X+1} \qquad (X \in \mathbb{C} \setminus \{-1\}), \tag{C.50}$$

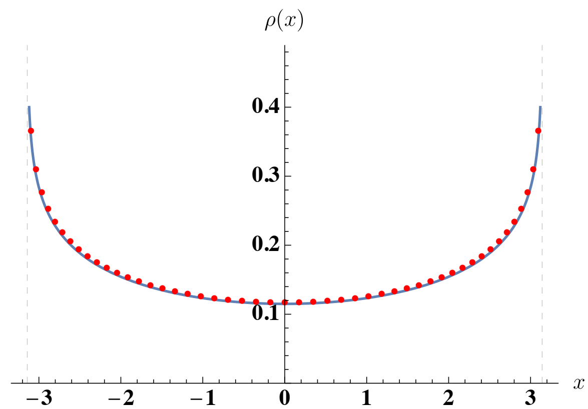


Figure C.2: The numerically determined eigenvalue density, $\rho(x)$, for $N = 50$ and $t = 5$ (red dots) along with the large-$N$ analytic solution (blue line), (C.52). The numerical density is obtained by finite differencing.

is consistent with analyticity except for a pole at $X = -1$. The regular (first) term is identical to that of the standard solution, (C.33), while the pole (second) term is new but does not modify the asymptotic conditions (C.32).

**The solution for $t > 0$**

For $t > 0$, we choose the cut to lie along the unit circle, starting and ending at the singular point $X = -1$. Using (C.34), we obtain the resolvent

$$\omega(X) = \begin{cases} -t + 2\log(1 + X) & (|X| < 1) \\ t - 2\log(1 + 1/X) & (|X| > 1), \end{cases} \tag{C.51}$$

where the principal branch is taken for the log. Here the 'inside' and 'outside' solutions are chosen to satisfy the asymptotic conditions (C.32). In this case, the matrix eigenvalues are imaginary and lie in the interval $(-i\pi, i\pi)$. The eigenvalue density is obtained from (C.28), and is given by

$$\rho(x) = \frac{1}{2\pi}\left(1 - \frac{1}{t}\log\Big(4\cos^2\frac{x}{2}\Big)\right) \qquad (x \in (-\pi, \pi)), \tag{C.52}$$

and the eigenvalues themselves are $u = ix$. Although the 't Hooft coupling multiplies the log term, it averages to zero over the interval $(-\pi, \pi)$, so the normalization condition is satisfied with an average eigenvalue density of $1/2\pi$. This sub-leading solution is somewhat unusual as $\rho(x)$ diverges logarithmically at the endpoints, as highlighted in Figure C.2.

The genus-zero free energy can be obtained by using the above eigenvalue density in

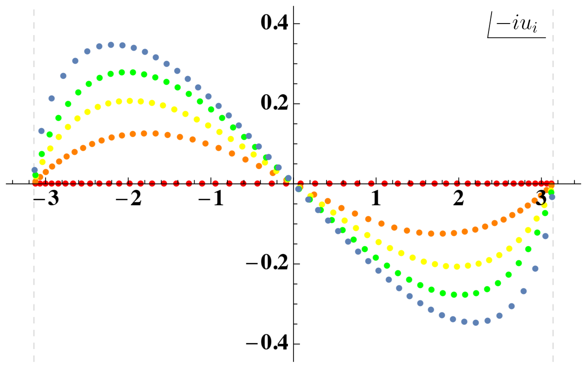


Figure C.3: The numerically determined eigenvalues, $-iu_j$ for $N = 50$. The family of solutions correspond to $t = 5$ (red), $t = 5 + \pi i/2$ (orange), $t = 5 + 2\pi i$ (yellow), $t = 3 + 2\pi i$ (green), and $t = 2\pi i$ (blue), respectively.

(C.48), with the result

$$\log Z/N^2 = \frac{\zeta(3)}{t^2} + \frac{\pi^2}{6t} + (t\text{-independent imaginary term}), \tag{C.53}$$

where we have not been careful enough to keep track of the log branch issues that go into computing the imaginary term. Note that, even though here we have taken real $t > 0$, the saddle point free energy is complex since this sub-leading saddle itself is complex.

### The solution for $t = 2\pi i/\eta$ with $\eta_1 = \pm 1$

For connection to the $\mathcal{N} = 4$ SYM saddle, we are interested in analytically continuing to $t = 2\pi i/\eta_1$ with $\eta_1 = \pm 1$. While in the previous cases the eigenvalues either lie entirely on the real or imaginary axis, this is no longer the case for the sub-leading saddle with $t = 2\pi i/\eta_1$. Instead, from numerical observations, the eigenvalues lie along a curve connecting $u \in (-i\pi, i\pi)$. We have been unable to obtain an analytic form of this curve. However, it can be examined numerically, as shown in Figure C.3, where the 't Hooft coupling is analytically continued from $t = 5$ to $t = 2\pi i$.

The genus-zero free energy for the sub-leading saddle with $t = \pm 2\pi i$ may be obtained by analytic continuation of (C.53)

$$\log Z/N^2\big|_{t=\pm 2\pi i} = -\frac{\zeta(3)}{4\pi^2} + (\text{imaginary}). \tag{C.54}$$

Since this has a negative real part, it is always sub-dominant to the leading saddle whose free energy (C.49) which has vanishing real part.

### C.2.3 Saddle point solutions of $\mathcal{N} = 4$ SYM from direct numerical evaluation

We now return to the original problem at hand, namely the saddle point evaluation of the $\mathcal{N} = 4$ SYM index in the Cardy-like limit. As we have shown in (3.42), the effective action reduces to that of 3D SU($N$) Chern-Simons theory. As a result, we may simply apply the saddle point solution of the latter theory to the $\mathcal{N} = 4$ SYM index. However, it is instructive to see how this works in practice. To do so, we have numerically solved the saddle point equation arising from the effective action in (3.18). This was performed using FindRoot in Mathematica, where the elliptic gamma function was approximated by truncating its product representation (A.5a).

We find that numerical solutions to the saddle point equation for the $\mathcal{N} = 4$ SYM index are sensitive to the initial trial configuration for the eigenvalues. Based on large-$N$ investigations of the SCI that suggest the eigenvalues are distributed along the 'thermal' circle [30, 5], it is natural to start with an initial configuration distributed uniformly along the interval $(-\tau/2, \tau/2)$. This starting point, however, converges to the sub-leading saddle point solution discussed in section C.2.2. To find the dominant saddle point studied in section C.2.1, we have to instead start with an initial configuration mirroring (C.47) of the 3D Chern-Simons theory. Here the initial eigenvalues wraps twice around the 'thermal' circle, and are distributed non-uniformly in the interval $(-\tau, \tau)$.

As an example, we compare the numerical solution to the $\mathcal{N} = 4$ SYM saddle point equations with those from the 3D Chern-Simons theory in Figure C.4 for the leading saddle and Figure C.5 for the sub-dominant saddle. For $\mathcal{N} = 4$ SYM, we take $\tau = ie^{i\pi/6}$ and chemical potentials such that $\eta_1 = 1$, so that $t = 2\pi i$ in the Chern-Simons theory. Since $|\tau| = 1$, the numerical results are not taken in the Cardy-like limit. Nevertheless, the similarity of the full SYM solution with that of the corresponding Chern-Simons theory is apparent. We have observed numerically that the sub-leading saddle point solution becomes indistinguishable from that of the Chern-Simons theory in the Cardy-like limit. However, the leading order saddle is more sensitive to $1/N$ effects arising from the repulsion between eigenvalues on the inner and outer circles of Figure C.4. In any case, the distinction between $\mathcal{N} = 4$ SYM and 3D Chern-Simons solutions is small compared to the difference between the dominant and sub-leading saddles, which is clearly evident when comparing Figure C.4 with Figure C.5.

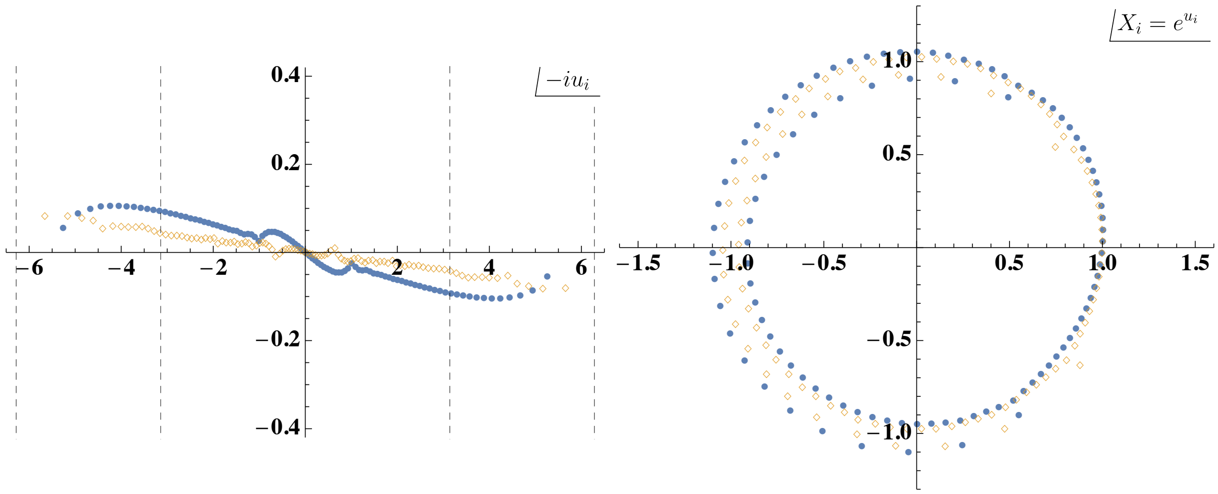


Figure C.4: Comparison between the $\mathcal{N} = 4$ SYM (blue dots) and 3D Chern-Simons (orange diamonds) solutions for the dominant saddle point. Here we have taken $N = 100$ along with $\tau = ie^{i\pi/6}$ and $\Delta_a = (2/3, 2/3, 2/3 + 2\tau)$, which maps to $t = 2\pi i$ in the Chern-Simons theory. As seen in the figure on the right, the exponentiated eigenvalues wrap twice around the circle. The 3D Chern-Simons eigenvalues $u_i$ are given as in (C.22), while the $\mathcal{N} = 4$ SYM eigenvalues $\tilde{u}_i$ are mapped according to $u_i = 2\pi i \tilde{u}_i/\tau$.

## C.3 The $S^3$ partition function of SU($N$) Chern-Simons theory

Here we compute the $S^3$ partition function of SU($N$) Chern-Simons theory, namely

$$Z^{CS}_{\mathrm{SU}(N)} = \frac{1}{N!}\int_{-\infty}^{\infty}(\prod_{j=1}^{N-1} d\lambda_j)\, e^{-ik\pi\sum_{j=1}^{N}\lambda_j^2}\prod_{i\neq j} 2\sinh\pi\lambda_{ij} \tag{C.55}$$

with the constraint $\sum_{j=1}^{N}\lambda_j = 0$, where $k = -\eta N$ ($\eta_1 = \pm 1$).

Recall that the $S^3$ partition function of U($N$) Chern-Simons theory is given in Appendix B of [79] as

$$\begin{aligned} Z^{CS}_{\mathrm{U(N)}} &= \frac{1}{N!}\int_{-\infty}^{\infty}(\prod_{j=1}^{N} d\lambda_j)\, e^{-ik\pi\sum_{j=1}^{N}\lambda_j^2}\prod_{i\neq j} 2\sinh\pi\lambda_{ij} \\ &= \frac{(-1)^{\frac{N(N-1)}{2}} e^{-\frac{\pi i N(N-1)}{4}} e^{-\frac{\pi i}{6k}N(N^2-1)}}{(ik)^{N/2}}\prod_{m=1}^{N-1}\left(2\sin\frac{\pi m}{k}\right)^{N-m}. \end{aligned} \tag{C.56}$$

Under the change of variables $\lambda_\mu \to \lambda_\mu + \sum_{j=1}^{N}\lambda_j$ ($\mu = 1, \cdots, N-1$), whose Jacobian is

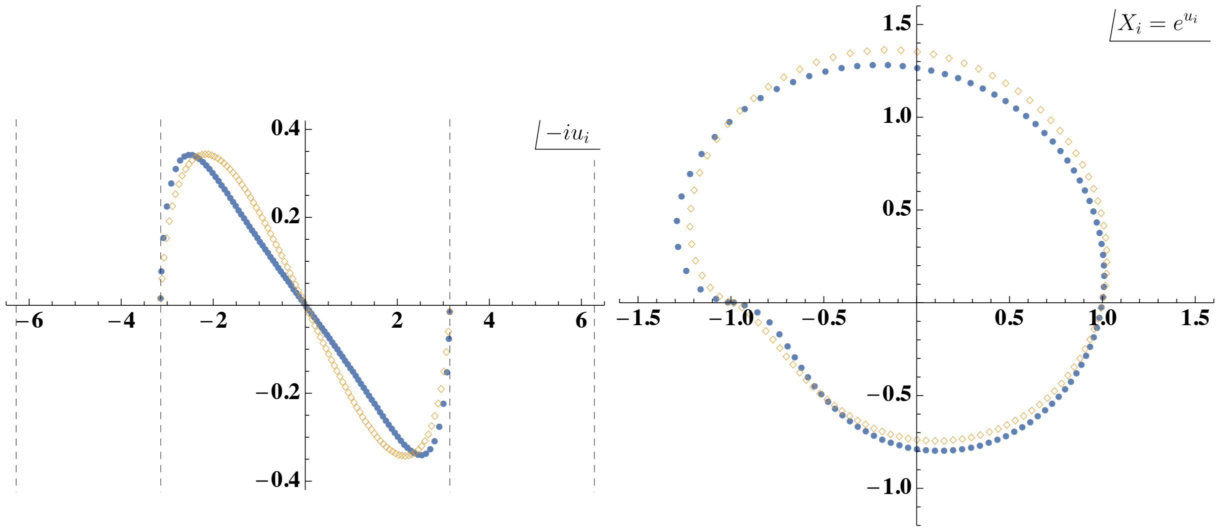


Figure C.5: Comparison between the $\mathcal{N}=4$ SYM (blue dots) and 3D Chern-Simons (orange diamonds) solutions for the sub-leading saddle point. The parameters are the same as in Figure C.4, but for the sub-leading saddle the exponentiated eigenvalues go only once around the (distorted) circle.

given as

$$\prod_{j=1}^{N} d\lambda_j \quad \to \quad N\prod_{j=1}^{N} d\lambda_j, \tag{C.57}$$

the U(N) partition function (C.56) can be rewritten as

$$\begin{aligned} Z^{CS}_{\mathrm{U(N)}} &= \frac{1}{(N-1)!}\int_{-\infty}^{\infty}(\prod_{j=1}^{N} d\lambda_j)\, e^{-ik\pi\sum_{\mu=1}^{N-1}(\lambda_\mu+\sum_{j=1}^{N}\lambda_j)^2-ik\pi\lambda_N^2}\prod_{\mu\neq\nu}^{N-1} 2\sinh\pi\lambda_{\mu\nu} \\ &\quad\times\prod_{\mu=1}^{N-1} 2\sinh\pi(\lambda_\mu+\Sigma_j\lambda_j-\lambda_N)2\sinh\pi(\lambda_N-\lambda_\mu-\Sigma_j\lambda_j) \\ &= \frac{1}{(N-1)!}\int_{-\infty}^{\infty}(\prod_{\mu=1}^{N-1} d\lambda_\mu)\prod_{i\neq j}^{N} 2\sinh\pi\lambda_{ij}\Big|_{\lambda_N=-\sum_{\mu=1}^{N-1}\lambda_\mu} \\ &\quad\times\int_{-\infty}^{\infty} d\lambda_N\, e^{-ik\pi N(\lambda_N+\sum_{\mu=1}^{N-1}\lambda_\mu)^2} e^{-ik\pi(\sum_{\mu=1}^{N-1}\lambda_\mu^2+(\sum_{\mu=1}^{N-1}\lambda_\mu)^2)} \\ &= \left(\frac{N}{ik}\right)^{\frac{1}{2}} Z^{CS}_{\mathrm{SU}(N)}. \end{aligned} \tag{C.58}$$

For $k=-\eta N$ with $\eta_1=\pm1$, substituting the identity

$$\prod_{m=1}^{N-1}\left(2\sin\frac{\pi m}{N}\right)^{N-m} = N^{N/2} \tag{C.59}$$

into the U($N$) partition function (C.56) and using the relation (C.58), we have

$$\begin{aligned} Z^{CS}_{\mathrm{SU}(N)} &= e^{-\frac{\pi i \eta}{4}} \frac{e^{\frac{\pi i N(N-1)}{2}} e^{-\frac{\pi i N(N-1)}{4}} e^{\frac{\pi i \eta}{6}(N^2-1)}}{e^{-\frac{\pi i \eta N}{4}}} e^{\frac{\pi i (1+\eta) N(N-1)}{4}} \\ &= \exp\left[\frac{\pi i N(N-1)}{2} + \frac{5\pi i \eta (N^2-1)}{12}\right]. \end{aligned} \tag{C.60}$$

## C.4 Proof of Lemma 2

Cover the torus $\mathbb{R}^2/\mathbb{Z}^2$ with balls of radius $\varepsilon/2$. By the pigeonhole principle, there are two integers $A < B$ such that $(\{Ax\}, \{Ay\})$ and $(\{Bx\}, \{By\})$ are in the same ball. Then $(\{(B-A)x\}, \{(B-A)y\})$ is in the ball of radius $\varepsilon$ around 0 (mod $\mathbb{Z}^2$).

Now, if $\{(B-A)x\} + \{(B-A)y\} > 1$, we are done by taking $C = B - A$.

If on the other hand $\{(B-A)x\} + \{(B-A)y\} < 1$, then the relation

$$\{\alpha - \beta\} = \begin{cases} \{\alpha\} - \{\beta\}, & \{\alpha\} \geq \{\beta\}; \\ \{\alpha\} - \{\beta\} + 1, & \{\alpha\} < \{\beta\}, \end{cases} \tag{C.61}$$

guarantees that for $\varepsilon$ small enough we have $(\{(B-A-1)x\} + \{(B-A-1)y\}) > 1$, so we are done by taking $C = B - A - 1$. *Q.E.D.*

Let us see how things work in an example. Take $x = y = 1/3$. The two values $B = 6$ and $A = 3$ are acceptable. Now, since $\{(B-A)x\} + \{(B-A)y\} = \{1\} + \{1\} = 0 < 1$, we can take $C = B - A - 1 = 2$. Indeed $\{2 \cdot \frac{1}{3}\} + \{2 \cdot \frac{1}{3}\} = \frac{4}{3} > 1$ as desired. This implies that for $0 < \arg\tau < \frac{\pi}{2}$, at the point $(\{\tilde{\Delta}_1\}, \{\tilde{\Delta}_2\}) = (1/3, 1/3)$, $C = 2$ satisfies the condition (3.94).

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